

Chapter 2

Artifact-Centric Activity Theory—A Framework for the Analysis of the Design and Use of Virtual Manipulatives

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Abstract It is a challenge to analyze the design and the use of Virtual Manipulatives due to their high complexity. As it is possible to create entirely new virtual worlds that can host objects that behave differently than any real objects, allowing for new and unprecedented actions in learning processes, we are in need of tools that enable us to focus on those aspects that are important for our analyses. In this chapter we show how ACAT, Artifact-Centric Activity Theory, can be used to analyze the design and use of a virtual manipulative place value chart.

2.1 Introduction

Manipulatives play a very important role in learning mathematics. Operations with manipulatives are the basis for further mathematical learning processes. Operations are purposeful and understood in their internal structure (Aebli 1983, p. 182) and it is essential that the student is aware of the relations of the objects. In this regard, manipulatives do not necessarily have to be of a physical nature but can be virtual as well (Clements 1999, p. 47; Ladel 2013, p. 59). This fact creates a lot of potential for the technological support of mathematical learning processes. However, the use and benefit of virtual manipulatives¹ is very complex as there are many influencing

¹We follow the definition of virtual manipulatives by Moyer et al. (2002), but extend it to the category of “apps” on mobile devices (which was unforeseen at the time), that can host virtual manipulatives in a similar way as web sites do.

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factors: the student with prior knowledge; the teacher with mathematical, didactical, educational and media competence; the class; the environment; etc.

Furthermore, mathematical, didactical and design principles have to be considered when designing, analyzing or even selecting a virtual manipulative for use in the classroom. The complexity of the situation and the number of influencing factors require instruments that help analyze the beneficial design and use of virtual manipulatives. In recent years, Human-Computer-Interaction (HCI) research and practice has identified Activity Theory as a helpful theoretical framework, since it is crucial to understand human activity for the design and analysis of technology: “*Understanding and designing technology in the context of purposeful, meaningful activities is now a central concern of HCI research and practice*” (Kaptelinin 2014, foreword). With regard to our requirements in the overlapping context of school, we adapted Activity Theory and developed it further to the Artifact-Centric Activity Theory (ACAT). In this chapter, we will present the theoretical framework of ACAT as an instrument for the design and analysis of virtual manipulatives. Furthermore we will illustrate its application with a virtual manipulative place value chart (Kortenkamp 2015).

2.2 Theoretical Framework

In the following sections, we recall several theories and notions that have influenced ACAT.

2.2.1 Activity Theory

Activity as the purposeful, transformative and developing interaction between *subject* and *object* is the key concept of Activity Theory. It originates from the socio-cultural tradition in Russian psychology and was developed by Leontiev (1978).

In today’s knowledge society, *learning* is a defining feature of our society. It is not only the acquisition of knowledge and skills but also the responsibility of the individual to learn and to strengthen the autonomy of each person (Giest and Lompscher 2004). A learning culture must enable people to shape their own educational biography and to take responsibility for their educational processes. Thus, high learning motivation, as well as a positive attitude towards lifelong learning, is very important. Therefore it is an essential condition that learners are the subjects of their learning and educational processes.

The world around us is structured and comprised of *objects*—objectively existing matter. Objects exist independent of the observing human and have ‘objective meanings’. However, there is a second aspect of each object, the image of the object as a product of psychological reflection realized as an activity of the

subject. This psychological reflection does not have to be consistent with the objective meaning. There is, for example, an objective meaning of ‘addition’ that is socially and culturally defined and can be described as $n + m = (n + m)$. But there are different basic concepts (“*Grundvorstellungen*” as described by Vom Hofe 1995) of addition (e.g., addition as the union of two amounts or as adding an amount to an existing one). The child may also have a concept of addition as ‘counting on’. Thus the meaning attributed to an object can differ depending on the perception of the individual.

Subjects in activity theory have needs. In order to meet those needs and to interact with objects, the subjects have to carry out *activities*. An activity is the process of relating the subject to the object (Fig. 2.1). The attributes of the subject and the object influence these activities. Prior knowledge, for example the student’s abilities and skills, influences the student’s actions and the way the student solves a mathematical task. The student’s experience while solving a mathematical problem, on the other hand, influences the student’s abilities.

2.2.2 Instrumental Act

Vygotsky (1997, p. 87) focused on the use of tools as the key characteristic of human mental activity. He characterized the process that combines subject, tool and object as ‘the instrumental act’ (Van Oers et al. 2008). Even though his theoretical focus was primarily on mental tools such as language, his theory can also be adapted to technical devices and, in our case, especially to virtual manipulatives. The tool or the *artifact* mediates between the subject and the object (Fig. 2.2).

Vygotsky’s cultural-historical psychology influenced Leontiev’s framework fundamentally, in particular, the mediation of the tool between subject and object.

Fig. 2.1 Subject—Object—Activity

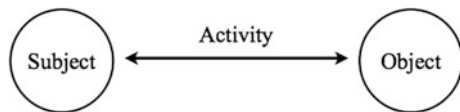
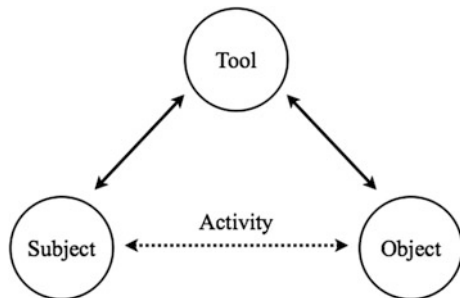


Fig. 2.2 Instrumental act



People encode their experience in the structural properties of tools, as well as their knowledge of how the tool should be used. In that way tools always reflect the previous experience of other people. In other words, virtual manipulatives always contain and reflect the experience and the knowledge of the programmer or designer. Therefore, we have to design virtual manipulatives very carefully and pay attention to the knowledge that we encode in the design of the virtual manipulatives. Similarly, we must also carefully select virtual manipulatives for instruction in mathematics and consider the knowledge we want students to construct.

Through the use of tools, processes of externalization and internalization emerge both for the subject and object. The internal and external components of an activity are mutually transformative: during the process of internalization, external components become internal; during the process of externalization, internal components of an activity become external. Higher physical functions, in the sense of Vygotsky, always arise in social interaction and communication in common activity. Phenomena that were previously interpsychological, become intrapsychological. This means, for example, that conventions used to communicate about mathematics can be learned by individuals and create an internal representation that can be used for further reflection and the creation of new knowledge.

2.2.3 *Instrumental Genesis*

Instrumentation theory (Rabardel 2002) distinguishes between *artifact* and *instrument*. An instrument is “more” than an artifact. While an artifact is the object that is used as a tool, it only becomes an instrument if the relationship that exists between the artifact and the user is meaningful for a specific type of task (Verillion and Rabardel 1995). The process of an artifact becoming an instrument is called *instrumental genesis*. The instrument also includes the techniques and mental schemes that the user develops and applies when using the artifact. While the artifact is just the material, the instrument involves artifact-type components as well as schematic components, the *utilization schemes*. In that way, an instrument is strictly related to the *context*.

We will illustrate this with a place value chart as an instrument. In the context of checking if a number can be divided by 9, we can just add all digits of the number, not considering their place and check whether the sum is divisible by 9. This is correct because we know that the value of a number changes by multiples of nine when digits change the place (e.g., when we move one counter in the place value chart from the hundreds to the tens, the value of the number changes by -90). However, in another context, for example the context of written algorithms, the change of place should not change the value but demands for (re-) bundling and unbundling (e.g., when there are 15 ones, we bundle them to 1 ten and 5 ones).

Both operations are carried out with the same (physical) artifact, but only the utilization scheme that is a component of the instrument tells us how to work with it. In both cases, the artifact is used as an instrument, but with different utilization schemes.

The so-called *instrumental orchestration*, introduced by Trouche (2004), describes the management of the individual instruments in the collective learning process by the teacher and it has been applied to digital media successfully. “An instrumental orchestration is defined as the teacher’s intentional and systematic organisation and use of the various artefacts available in a—in this case computerised—learning environment in a given mathematical task situation, in order to guide students’ instrumental genesis (Trouche 2004)” (Drijvers et al. 2011, p. 1350). According to Giest and Lompscher (2004), the subject is, however, not only a single student, but a number of students as individuals embedded into social structures. The activity of the individual is subject to conditions on interaction, communication and cooperation, and these conditions also hold for the relations between teachers and learners as well as further participants. Giest and Lompscher refer to a *pedagogical collective subject* that is acting during instruction.

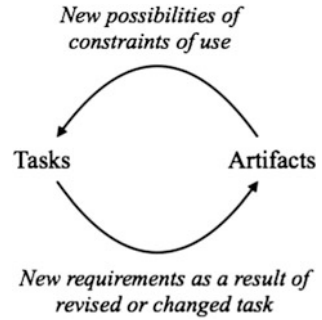
Children have to elaborate instruments in the process of instrumental genesis to use an artifact to accomplish a particular task. However, besides this semiotic link between the artifact and a task, there is another semiotic link between the artifact and its mathematical meanings that emerges from the epistemological analysis made by teachers and experts. Rabardel (2002) speaks in this context of a two-fold entity—artifactual and psychological. In that way, “the artifact is not a mediator of mathematical meanings per se [...]. It becomes a mediator when used in a teaching-learning situation” (Bartolini 2011, p. 97).

2.2.4 Task-Artifact Cycle

The inclusion of an artifact changes the activity between the subject and the object. First of all, it sets to work a number of new functions that are connected to the use and control of the given tool. As the artifact undertakes some tasks, it also abolishes a number of previously necessary processes. Thirdly, the artifact modifies various aspects (e.g., intensity, duration, etc.) of all mental processes as it replaces some functions with others (Van Oers et al. 2008). Considering these modifications of the activity we also have to focus on the development of tasks. The task-artifact cycle (Carroll et al. 1991) captures the idea that tasks and artifacts coevolve (see Fig. 2.3).

To perform a given task, an artifact needs to meet certain requirements. The artifact that has been designed for the task, in turn, creates new or unexpected possibilities. It poses new constraints on the performance of the tasks that may suggest a revision of the original task for which the artifact was made. This creates an iterative process of continuous development between the task and the artifact.

Fig. 2.3 Task-artifact cycle



2.2.5 Artifact-Centric Activity Theory

Considering this background, the design and the analysis of manipulatives are necessarily very complex. This applies even more to the design and analysis of virtual manipulatives because they allow for many more affordances (in the HCI sense, that is *action possibilities*) than their physical counterparts. In a way, virtual manipulatives even allow for *miraculous mathematical transformations*, that is, they can make things happen that would not be possible in the physical world. That is why there is a need for an instrument that helps when analyzing and designing virtual manipulatives. This is the purpose of the development of ACAT (Fig. 2.4).

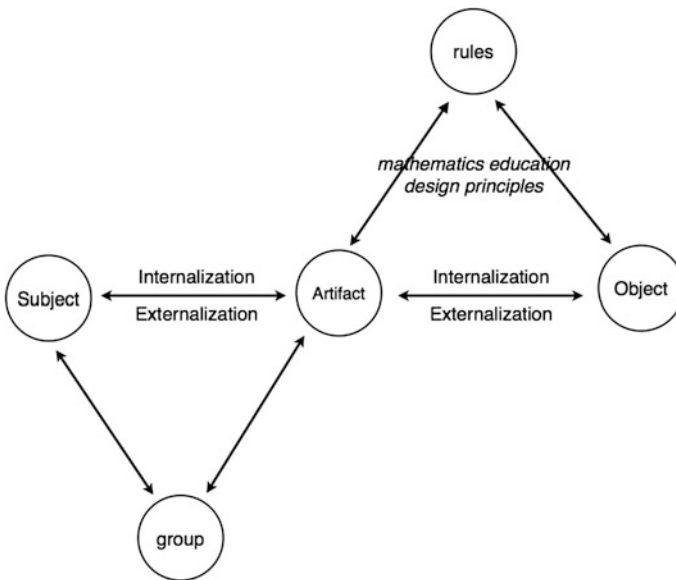


Fig. 2.4 The Artifact-Centric Activity Theory

As the artifact is being developed into an instrument through instrumental genesis, some may argue that we could use the terminology of Instrument-Centric Activity Theory. However, the framework also captures the process of designing, and in that stage the artifact could not yet develop into an instrument. As such, we deem the name ACAT better suited.

ACAT takes into account the student (here: subject) in the classroom with peers and the teacher (lower left triangle), as well as the mathematics (here: object) and the rules that arise from different disciplines (mathematics, mathematic didactics, design principles, multimedia learning, etc.) (upper right triangle). Whereas the design question “*How can we design the artifact?*” has to be asked in advance, the analysis question “*How could we have designed it?*” is asked afterwards. We try to tackle the complexity of this balance by breaking the overall design and analysis down into numerous individual decisions and tests.

ACAT can be divided into three components: the main axis, the upper right triangle and the lower left triangle. Whereas the main axis focuses on the subject—artifact—object relations, with the artifact as its main component, the lower left triangle focuses on the use and benefit of the artifact in classroom situations and the upper right triangle takes into account the rules and principles that help to design the artifact according to the object.

Being based on Activity Theory, the fundamental concept of ACAT (Ladel and Kortenkamp 2013) is the activity between subject and object. We moved the artifact into the center of this relationship and into the center of the activity, as we want to analyze the impact of the artifact as the mediator between subject and object. The children’s use of an artifact, in our case a virtual manipulative, influences their activity and the processes of internalization and externalization. The subject externalizes mental representation through the artifact and in turn internalizes specific knowledge that is represented by the artifact as feedback to the subject’s actions. The artifact (i.e., the virtual manipulative) itself externalizes the object (i.e., the mathematics) as a psychological reflection of the programmer’s (or designer’s) knowledge. The programmer in turn designs the artifact according to the programmer’s knowledge about the object.

Neither the mental representation nor the visual representation(s) of the object are predefined through ACAT. The only requirement is that there is some kind of internal/mental representation of the object in question and that there is at least one way to visualize the object.

The lower left triangle focuses on classroom situations. The integration of technological tools into mathematics education is a non-trivial issue. The use of tools involves the process of instrumental genesis during which the artifact turns into an instrument. However, it is also important to observe the instrumental genesis by the teacher through the teacher’s orchestration of mathematical situations.

The upper right triangle takes into account the “objective” object, the rules and the artifact. Before designing, analyzing or using the artifact, we have to analyze the object, its properties and its structures. What is it that we want the children to learn? What knowledge do we “put into the artifact”? There are rules resulting from the object and there are also rules on how to design the artifact that are the result of

instructional principles and research (e.g., mathematics education or Gestalt psychology). When designing or analyzing an artifact, we have to keep those rules in mind. Also, concentrating on the upper right triangle emphasizes the fact that artifacts are usually not designed for each subject on an individual basis, but for a number of subjects.

2.3 ACAT by Example: The Virtual Place Value Chart

In the following section, we will elaborate the Artifact-Centric Activity Theory and emphasize its importance by using the example of a virtual place value chart (Kortenkamp and Ladel 2013). This virtual manipulative was created as an app for iPads and iPhones (Kortenkamp 2015).

The virtual place value chart uses the full screen for up to four columns that can be filled with counters by touching the screen. These counters can be moved around with the finger after being created. Every column has a header showing the number of counters in that column and the word describing the value of those counters (e.g., ones, tens, hundreds...). Counters can be removed from the chart by moving them to the top, and they can be moved from one column to the other. In the latter case, either the counter is automatically unbundled into the correct number of lower-valued counters, or, if possible, the missing number of same-value counters is also moved to create a new higher-valued bundle.²

2.3.1 *The Main Axis: Subject—Artifact—Object*

Numbers can be represented in many different ways (e.g., with base ten blocks or with counters in a place value chart). It is, however, only important to consider the principle of place value, if the numbers are represented as digits such as “734” (see Ladel and Kortenkamp 2015a, b). If a child solves the equation $7 + 8 = _$ by counting “eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen” the special role of “ten” is not apparent. “Ten” is just another word and does not play a special role. In that way language is an *artifact* that does not promote place value—on the contrary: language can even prevent place value understanding, in cases where there are irregularities in the word formation of numbers (Sarama and Clements 2009). The number 24 in German is spoken as “four-and-twenty”. This inversion of numbers leads to the problem that a lot of children write 42 instead of 24. Therefore, it is important to use a good artifact to teach and learn about place value. The place value chart could be such an artifact. In the place value chart, the column in which a certain number or amount of counters are placed indicates its value.

²At <http://kortenkamps.net/placevalue> we provide a screen recording showing the app in action.

If we consider place value only with regard to change among the different modes of representation (*intermodal transfer*)—representing numbers in a place value chart or reading numbers from a place value chart—it is unambiguous. However, when we start working with the place value chart and operating in it to explore its properties, there are different possibilities. There are two facets of the *object* ‘place value’: the objective, mathematically defined view of place value as it exists in the world per se and the subjective view of place value of the student.

In the objective view, our numeration system is based on five properties: the positional property, the base-ten property, the multiplicative property, the additive property and the principle of continued bundling (Ladel and Kortenkamp 2014, 2015a, b; Ross 1989). According to the positional property, the place where a digit is positioned gives us information about its value. For example, the 3 in the number 734 has the value of 3 tens, and the 3 in the number 473 has the value of 3 ones. One way to teach children place value is to visualize numbers as similar (undistinguishable) counters in a place value chart. The children can represent numbers with the counters in the place value chart and have to pay attention to the place or column they lay the counter, or they can ‘read’ the number and write it down in digits.

Another action is to move tokens in the place value chart and to discover that the value of a counter changes depending on the column in which it is placed. Thus the value of a number (e.g., 243) changes if we move one token from the tens to the ones (243 becomes 234) as in the example on the left side of Fig. 2.5. To enrich the concept of place value it is important to establish the relationships to the other principles. The principle of continued bundling is about creating new bundles until it is no longer possible. For example, 243 ones = 24 tens and 3 ones = 2 hundreds, 4 tens and 3 ones. In this context, moving a counter from the tens to the ones has the meaning of unbundling 1 “ten” to 10 “ones,” as in the example on the right side of Fig. 2.5. In that way, there are two different meanings for moving a counter from the tens to the ones, and a designer of a virtual place value chart has to decide how the counters in the application should behave to reflect that meaning (Fig. 2.5).

The *subject* (the student) has certain concepts of numbers. Depending on the artifacts the teacher has used for place value instruction, the student may have internalized the meaning of the counters as a change of value or the student may have internalized the meaning of the counters as bundling and unbundling. In this

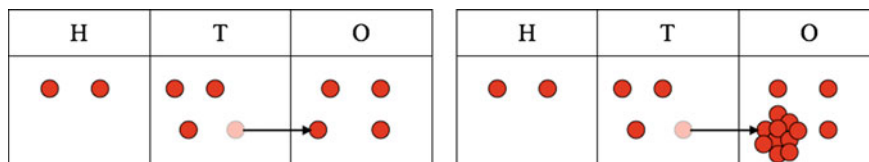


Fig. 2.5 Two different behaviors of one action. *On the left* The action of moving one counter from the tens to the ones changes the total value. *On the right* The action of moving one counter from the tens to the ones emphasizes the necessary unbundling of one ten to ten ones, which preserves the value

case, the design of a particular virtual manipulative application may lead a child to a cognitive conflict because the reaction of the current virtual manipulative application does not match the child's mental schema (Ladel and Kortenkamp 2014).

2.3.2 *The Upper Right Triangle*

The upper right triangle provides rules and principles that determine the design of the artifact. We have already mentioned the impact of the object on the artifact. There are, however, also principles for the artifact's design that are guided by mathematics education and psychology. Concerning principles from mathematics education, we would like to focus on two aspects: the *intermodal transfer* and the *spiral curriculum*.

According to Bruner et al. (1971), we distinguish three *modes of representation*—enactive, iconic and symbolic—whereas the latter has to be distinguished into verbal- and nonverbal-symbolic representations. The comprehension of operations is only fully developed when the child is able to change among the different modes of representation. This is called *intermodal transfer*. Although children have to perform the intermodal transfer by themselves, it is important to support them during the learning process so that they understand the meaning of the symbols and the operations. In this regard, multiple representations with automatic linking features can be very helpful (Ladel 2009).

The virtual place value chart provides all three modes of representation. The counters that can be moved create the enactive representation; the counters drawn in various places create the iconic representation; and the number written in digits and words is the symbolic representation. Children are able to interact with the app, to create counters, and to delete them, as well as move counters from one column to another. At the same time, they see the iconic representation of the counters. The nonverbal-symbolic mode is automatically given in the way that the amount of counters in each column is written in the title row and, optionally, the whole number is written in words above the columns. In that way 2 hundreds, 4 tens and 3 ones are represented as well as the number 243 along with representation of the counters. It is even possible to show the numeral (i.e., the verbal-symbolic representation) (Fig. 2.6).

With Bruner's *principle of a spiral curriculum* in mind, the designers valued the (vertical) compatibility of the virtual manipulative within the curriculum and beyond. In this particular app there are several possibilities of modification the user can apply within the settings (see Fig. 2.7). For example, in Montessori-mode, one can choose between homogeneous or multicolored counters. Thus it is possible to go back to a lower level of abstraction and to work with multicolored counters in accordance with the place value. Also the number of places before and after the decimal point can be configured. So the virtual manipulative can be used from first grade, starting with ones and tens only, through the upper grades. The app allows features that expand the chart to the left (for hundreds, thousands...) and to the right

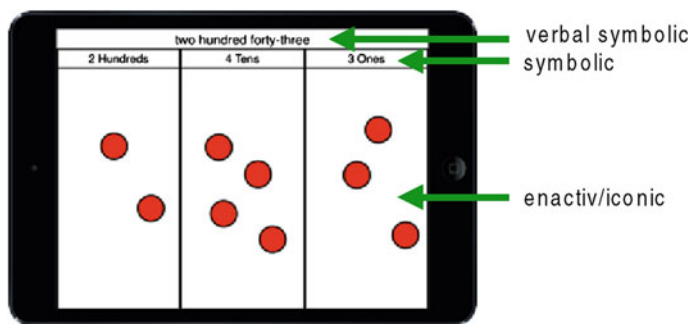


Fig. 2.6 Multiple external linked representation linked in the virtual place value chart

(for tenths, hundredths...) to make visible the decimal principle of our number system.

Working with decimal numbers, designers' paid attention to the decimal point. The separator between the ones and the tenths is shown as an asymmetrical pair of a thick and a thin line, in order to work against the illusion of symmetry at the decimal point. For more advanced work with place value (e.g., with university students), it is also possible to change the base (that is, the size of the bundles) and the used base for counting. In that way, this virtual manipulative is suitable for first graders up to students at the university level (Fig. 2.7).

Another design question was whether the counters should be laid in a structured order or not. The aim of this virtual manipulative is not to support the children in their quasi-simultaneous subitizing. Actually, there is no need for quasi-simultaneous subitizing as the numbers of counters are written in the title row. Furthermore, an additional design feature is that if there are more than nine counters in a column, the written number is colored red. The reason that designers decided to write numbers more than nine in red was for students to be subtly warned that the representation is non-standard, while not being stopped from creating such non-standard representations.

As these examples from just one app demonstrate, there are numerous design decisions to be considered when creating a virtual manipulative and these are highly influenced by the upper right triangle of ACAT—the interplay of the artifact, the mathematical object and the findings from mathematics education, psychology and other relevant fields.

2.3.3 *The Lower Left Triangle*

Any teaching and learning must consider the context—the individuals, the group, the tools, the conditions, the social structures, etc. The lower left triangle of ACAT is related to the use of the artifact in the classroom. A virtual manipulative does not

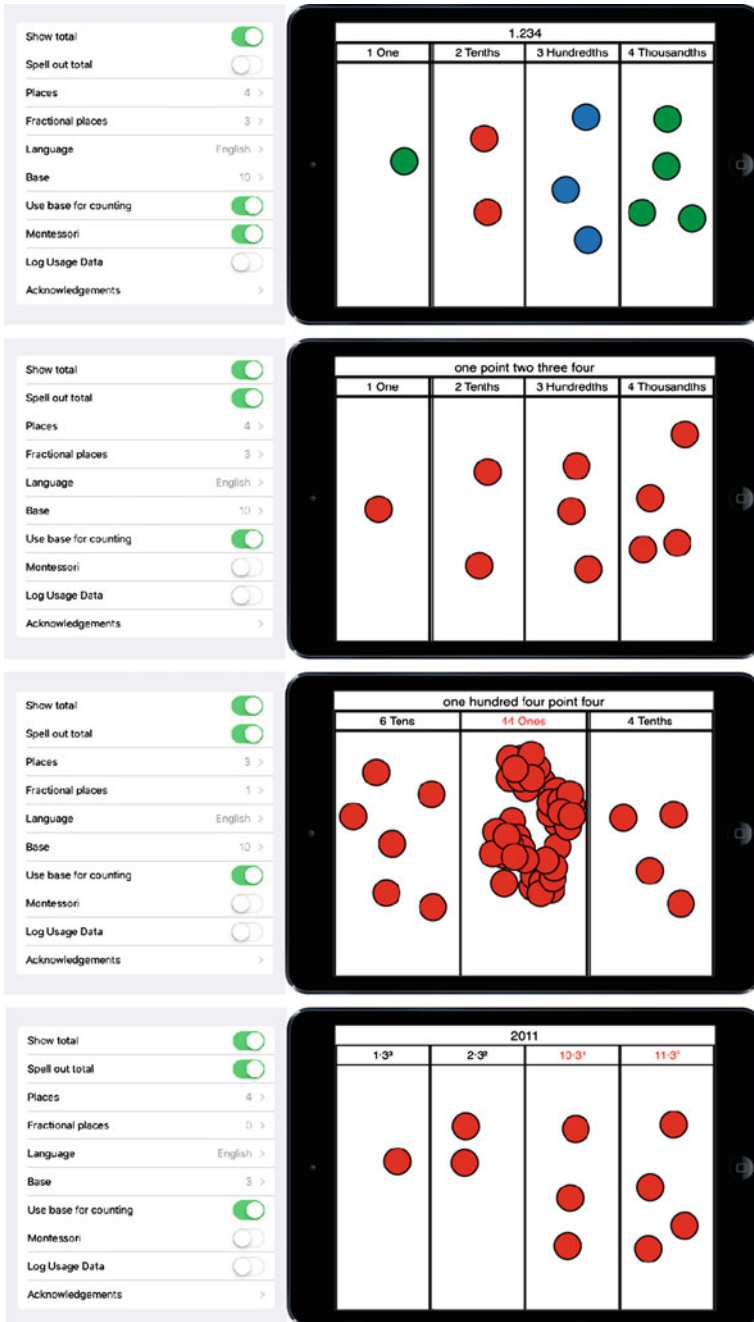


Fig. 2.7 Possible modifications of the virtual place value chart in the settings

just work, but its effects are determined by the way it is employed in the classroom. Of importance are the individual and socio-cultural factors and the orchestration of the virtual manipulative in the class. The theoretical analysis of the full situation is beyond the scope of the ACAT framework. The lower left triangle in the diagram is a reminder that it is necessary to analyze the circumstances and conditions in the classroom (or other learning situation), in the same way as the artifact itself is being analyzed, for effective use of virtual manipulatives in teaching and learning situations. One suitable theory for that analysis is instrumental genesis (Artigue 2002), and in particular it is necessary to find a suitable *instrumental orchestration* (Trouche 2004; Drijvers et al. 2011) that is an intentional and systematic organization and use of the artifact in the learning situation.

One way to use virtual manipulatives in the classroom is to use them for demonstration and visualization. In our example, the teacher could use the virtual place value chart to show students how a counter in the tens place can be unbundled into ten counters in the one's place. Observing this miraculous mathematical transformation could help students to better understand that the same counters have different values depending on their location.³ Using a virtual manipulative in that manner, though, misses the opportunity for the students to experience the major affordances that the virtual manipulative has to offer. Students can use the artifact themselves and experience the built-in mathematics.

By designing suitable tasks, we can steer the activities of the students such that they will be exposed to the externalization of the mathematical objects and interact with them. Some examples for good and productive tasks are:

- Find as many ways to represent the number 132 in the virtual place value chart as you can! Explain what you did to find them!
- Which numbers have only 2 (3, 4, ...) ways of representation? Justify!
- Divide 1505 by 7. Start by representing 1505 in the standard way and change the representation until the division becomes easy!

By working on tasks of these types, the students themselves can experience the miraculous mathematical transformation that is provided by the virtual manipulative. For example, here is what a student might do to divide 1505 by 7: Place counters for 1Th 5H 5O (standard representation—1 Thousands 5 Hundreds 5 Ones). Move the thousands counter to the hundred's place, as it cannot be divided easily by 7, so you now have 15H 5O. Observe that you can divide 14H of the 15H by 7 easily, so move the extra H one place to the right, to the Tens (T), and you get 14H 10T 5O. Move 3T of the 10T to the right to get 14H 7T 35O. The student ends up with a representation of 1505 that is easy to divide, and the solution is 2H 1T 5O = 215. This is just one example of an exploration that could be conducted by a student to investigate this question. At each step the exchange of counters from

³There is no research so far that could prove or disprove this assumption—this expected help for learning is just hypothetical and based on the theoretic considerations behind the design of the artifact.

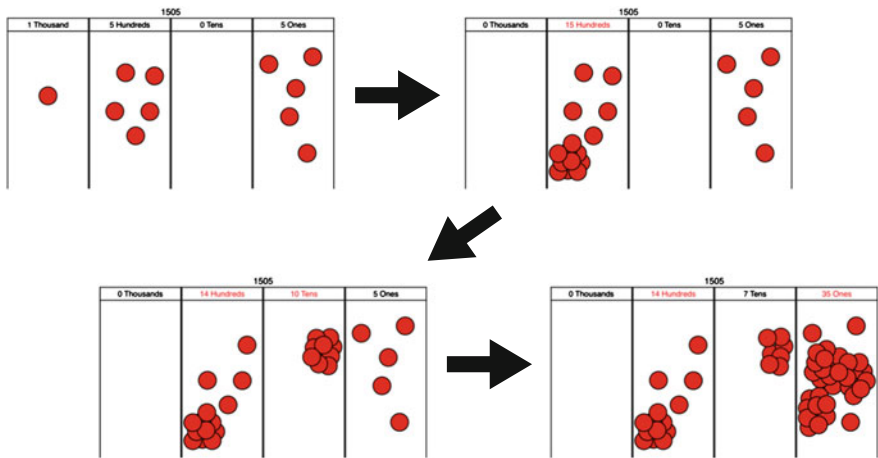


Fig. 2.8 Solving 1505 divided by 7 by flexible use of place value

higher values to lower values is visualized and experienced by the student (Fig. 2.8).

Again, suitable research has to be carried out to gather data that justifies our claim that these and similar tasks are indeed good and productive tasks. But the ACAT framework helps us to identify these research questions and to design virtual manipulatives that are integrated into learning environments. We invite the reader to try ACAT with their favorite virtual manipulative for analysis, but highlight the fact that ACAT can also be used to design new virtual manipulatives, as shown in Ladel and Kortenkamp (2013).

2.4 Conclusion

Virtual manipulatives have the potential to support students' mathematics learning, but in order to turn this into affordances and learning opportunities, it is necessary to have a theoretical tool that can guide both the design and the analysis of these artifacts. In this chapter we presented ACAT, a theoretical framework that helps to structure the complexity of this design task. As ACAT is based on Activity Theory, it focuses on the interaction of subjects (students) with objects (mathematical concepts) mediated through artifacts (the virtual manipulatives). The design of an artifact is usually not for an individual student but for a larger audience. The mathematical concepts are also independent of the students. Therefore, the upper right triangle in our model can be used for designing (or analyzing) the virtual manipulative. The lower left triangle adds the necessary context for teaching and learning and helps to focus on the specific tasks or usage scenarios of the virtual manipulative by educators in teaching and learning environments.

While ACAT has been created with digital artifacts and virtual manipulatives in mind, it is not restricted to these tools alone. The framework can be applied to any artifact used in teaching. Still, we suggest that ACAT is most suitable for characterizing the miraculous mathematics transformations that can be created through virtual manipulatives, and that can be experienced by the students indirectly in demonstrations, and also directly by doing miraculous mathematical transformations themselves.

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