Chapter 10
The Role of Virtual Manipulatives in High School Students' Understanding of Geometric Transformations

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Abstract Although there is widespread research that articulates their importance in mathematics education, manipulatives are pushed aside in high school learning settings as something inappropriate or frivolous. The purpose of this study was to identify the role of virtual manipulatives in high school students' mathematical understanding about geometric transformations, which included translations, reflections, rotations, and dilations. The main data sources for this study were semi-structured task-based interviews that were conducted after each weekly transformation lesson. The mathematical understanding of students was analyzed using representation theory and the Pirie-Kieren model. This was presented using a two-dimensional model that shows the students using virtual manipulatives within different levels of mathematical understanding (Pirie and Kieren in Educ Stud Math 26(2):165–190, 1994). Results of the study revealed that virtual manipulatives helped students to apply distinct representations of geometric transformations and translate among them. As interventions in the environment, virtual manipulatives strengthened students' mathematical understanding in terms of progressing from inner to outer levels of the Pirie-Kieren model, folding back movements, and acting-expressing activities.

Keywords Mathematical understanding · Virtual manipulatives · Representation · Pirie-Kieren model · High school students

Mathematical manipulatives were recently defined by Bartolini and Martignone (2014) in the Encyclopedia of Mathematics Education as "artifacts used in mathematics education: they are handled by students in order to explore, acquire, or

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investigate mathematical concepts or processes and to perform problem-solving activities drawing on perceptual (visual, tactile, or, more generally, sensory) evidence" (p. 365). As concrete learning objects, manipulatives have been used to support children's mathematical development since the 1800s. They gained a theoretical basis through learning theorists such as Jerome Bruner, Zoltan Dienes and Jean Piaget in the 1900s (McNeil and Jarvin 2007). At the end of the 1900s, innovations in computer and information technology, the accessibility of Internet media, and increases in the number of computers led to a new trend in mathematics education that introduced virtual manipulatives (Moyer et al. 2002).

A virtual manipulative is "an interactive, Web-based, visual representation of a dynamic object that provides opportunities for constructing mathematical knowledge" (Moyer et al. 2002). Most virtual manipulatives are on the computer screen and are interacted with using a mouse or keyboard whereas others are on touch screen devices controlled by finger movements. Unlike the physical versions, virtual manipulatives contain the verbal, graphical and notational representations of a concept together, which helps students to make connections among different representations (Suh and Moyer-Packenham 2007). Moyer-Packenham and Westenskow (2013) carried out a meta-analysis of 32 research reports using 83 effect size scores that investigated the effects of virtual manipulatives in students' mathematics achievement. According to this meta-analysis, instruction with virtual manipulatives produces moderate effects when compared to instruction with all other instructional treatments (0.37; 0.44, with one outlier). The qualitative results of the study revealed that there are five key affordance categories that positively affect students' learning: (a) focused constraint, (b) creative variation, (c) simultaneous linking, (d) efficient precision, and (e) motivation.

Even though there is extensive literature supporting manipulative use for all grade levels, there is a lack of research on high school learning environments (Gibbons 2012; Gordon 1996; Jones 2010; Marshall and Paul 2008). The reasons for this gap in the literature may be high school teachers' knowledge, experience, beliefs, and attitudes about manipulatives. Teachers hesitate to use manipulatives in secondary level classrooms because they believe that students in these grades should work with symbolic and abstract knowledge (Jones 2010). In view of the limited research at the high school level, there is a need to examine high school students' manipulative use in learning mathematics.

Mathematics education researchers collectively emphasize the importance of learning mathematics with understanding. There are several researchers, who have characterized mathematical understanding from different perspectives (see Dubinsky 1991; Herscovics 1989; Hiebert and Carpenter 1992; Sfard 1991; Sierpinska 1994; Skemp 1978). Pirie and Kieren (1989) make a classification of perspectives that describe mathematical understanding as "acquisition" or "process" (p. 7). They assert that mathematical understanding is more than categories of knowing and define it as a "whole, dynamic, leveled but non-linear, transcendently recursive process" (Pirie and Kieren 1994, p. 166). Figure 10.1, consisting of eight embedded circles, models their theory. These circles show the potential levels that one goes through during the growth of mathematical understanding. Pirie and

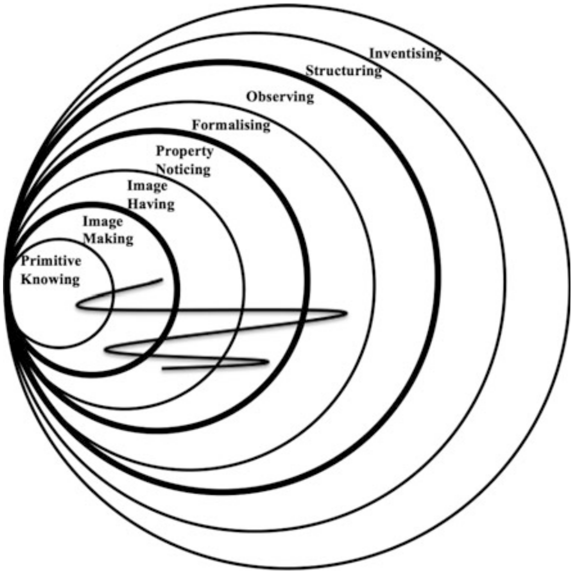

Fig. 10.1 The Pirie-Kieren model for the growth of mathematical understanding (Pirie and Kieren 1994)

Kieren (1989) state that "each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further is constrained by those without" (p. 8).

The *Primitive Knowing* level includes a student's previously developed knowledge about the topic. Using her/his primitive knowledge, s/he engages in mental or physical activities in order to develop an idea about the topic at the *Image Making* level. At the *Image Having* level, the student has an image about the concept, which may contain verbal, visual, written or any other representations. The student realizes the different properties of the concepts at the *Property Noticing* level by thinking about the differences and similarities of the images s/he has. At the level of *Formalising*, with the help of different properties based on various images, s/he makes general statements or develops common ideas about the concept that are similar to the mathematical definition of concept. S/he considers the constructed formal structures to develop theorem-like ideas about the related concept at the *Observing* level while s/he is able to verify these ideas logically at the *Structuring* level. At the *Inventising* level, with a structured understanding about the concept, the student asks questions that lead her/him to invent "a totally new concept" (Pirie and Kieren 1994, p. 171).

The theory describes one's understanding of mathematics through four different features: folding back, the 'don't need' boundaries, complementary aspects, and interventions. When a student folds back, it means that s/he cannot handle the mathematical task by working in the present level and needs to go back to an inner level to elaborate on the understanding. The 'don't need' boundaries are the darker lines in the model (see Fig. 10.1). These are the boundaries that "convey the idea that beyond the boundary one does not need the specific inner understanding that gave rise to the outer knowing" (Pirie and Kieren 1994, p. 173). The third feature of the theory is related to complementary aspects of acting and expressing. Acting maintains the continuity of a particular level with inner levels, while expressing strengthens the understanding at this level. Interventions pertain to another feature of the theory. Provocative interventions make students move towards the outer levels whereas invocative interventions result in students folding back to the inner levels to review the current process. Validating interventions are the stimulants that influence students to think about their existing understanding. The Pirie-Kieren model, by means of mathematical understanding levels and these features, serves as an efficient research tool to observe, understand and model students' growth of mathematical understanding (Gibbons 2012; Martin 1999; Nillas 2010; Towers 1998).

Another theory that views mathematical understanding as a process is representation theory, which emphasizes the role of dynamically networked mathematical representations of concepts (see Goldin 2003; Hiebert and Carpenter 1992; Janvier 1987). Representations in mathematical learning environments come in two forms, external and internal. External representation is used "to refer to physically embodied, observable configurations such as words, graphs, pictures, equations, or computer micro worlds" whereas internal representation is used "to refer to possible mental configurations of individuals, such as learners or problem solvers" (Goldin and Kaput 1996, pp. 399–400). The interaction between the two kinds of representations is two-way; an internal representation may be transformed to an external representation by an externalization process while an external representation may be transformed into an internal representation by an internalization process (Goldin 2003; Zhang 1997).

A consensus on the importance of providing learning environments that allow students to engage in these processes highlights the key role of multiple representations of mathematical concepts (Ainsworth et al. 2002; Duval 2006; Even 1998). Students can develop an appropriate mathematical understanding by using multiple representations appropriately and making connections among these representations (Kaput 1989; Lesh et al. 1987; Renkl et al. 2013). One of the multiple representations that students may use to develop internal representations of mathematical concepts is manipulative models (see Fig. 10.2). Representation theory gives us the opportunity to examine the role of manipulatives in mathematical understanding in terms of the relationships with other representations. The Pirie-Kieren model offers the possibility to analyze the understanding of students in a dynamic way in conjunction with representation theory. The model highlights non-linear movements through the understanding levels, as a student reconstructs and strengthens her/his understanding by using representations of concepts. In this chapter, the Pirie-Kieren model and representation theory were used together as a lens to determine the role of virtual manipulatives in 10th-grade students' mathematical understandings about geometric transformations.

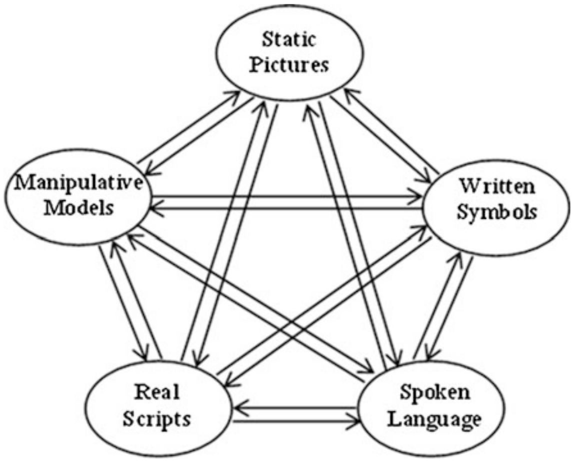

Fig. 10.2 Multiple representations of a concept (Lesh et al. 1987, p. 34)

10.1 Methods

The method used for this study was a teaching experiment (Cobb 2000; Steffe and Thompson 2000). Ms. Yilmaz (a pseudonym), a teacher with 13 years of experience, taught a transformational geometry unit to a 10th-grade class in a high school in Turkey. She conducted eight lessons in one month, two for each of the transformations: translation, rotation, reflection, and dilation. The lessons were enriched with virtual and physical manipulatives in addition to verbal, graphical, and algebraic representations of these transformations. The researcher observed the class during these lessons and performed task-based interviews (Goldin 2000) with the participants after lessons conducted about each transformation.

10.1.1 Participants

There were 32 10th-grade students (17 females and 15 males) in Ms. Yilmaz's geometry class. The researcher entered the class two months before the main study began and carried out a pilot study to interact with students to develop a level of comfort. During the pilot study, Ms. Yilmaz used physical and virtual manipulatives to teach the concepts about triangles. Prior to the first week of the study, the researcher administered a pretest, which covered translation, rotation, reflection, and dilation, and a spatial ability test, which was used to purposefully select four participants for in-depth analyses of growth of mathematical understanding. The pretest had 26 mathematical tasks about translation, rotation, reflection, and dilation. These tasks examined students' understandings about identifying the transformations in verbal, graphical and algebraic representations. The Spatial Ability Test (SAT) was adopted from the tasks in the Kit of Factor-Referenced Cognitive

Table 10.1 Characteristics of participants

Name	Age	Spatial ability test score	Pretest score	Previous geometry class score
Defne	17	67.5	28	71.75
Elif	16	150.25	55	78.50
Metin	17	161.50	40	55.00
Selim	16	48.00	32	87.00

Note The highest score possible was 282 for the Spatial Ability Test, 100 for the pretest, and 100 for the previous geometry class score

Tests (Ekstrom et al. 1976) to determine spatial ability, which is an important component in students' understanding of geometric concepts (Battista 1990). The test was translated to Turkish by Delialioğlu (1996). There were test items including mental rotation of 2-D figures, mental rotation of cubes, imagination of the folding and unfolding of a paper, and mentally folding given 2-D figures to obtain 3-D objects. The four participants selected for the study described in this chapter were two female and two male students who represented different levels of classroom participation and manipulative engagement in the pilot study, geometry class scores, and spatial abilities (see Table 10.1). Their pseudonyms are Defne, Elif, Selim, and Metin.

10.1.2 Instructional Settings and Manipulatives

Ms. Yilmaz conducted lessons in the computer lab of the school during the study. There were 30 computers and a teacher computer station with a display screen in the lab (see Fig. 10.3).

Lessons began with an introduction to the current geometric transformation using pictures, animations, or videos as visual representations and real-life examples. This was followed by mathematical tasks in which the teacher and students used virtual or physical manipulatives. The teacher modeled how to use the manipulatives before students worked with them. For every lesson, students also

Fig. 10.3 The front and back view of the computer lab

had instructions explaining how to perform the manipulative activities. At the end of the lessons, the teacher shared the algebraic representations and guided discussions with the students to construct a shared language and to make connections among multiple representations of the concept. The researcher video-recorded all of the lessons by positioning herself near the participants and took field notes during instruction.

The teacher and students in the class used virtual manipulatives from the National Library of Virtual Manipulatives website (www.nlvm.usu.edu). They interacted with the dynamic applets to identify and connect the mathematical properties of transformations in different kinds of representations. The applets allowed students to manipulate objects on the plane using the four geometric transformations (see Fig. 10.4).

In addition, students and the teacher used physical manipulatives that were designed by mathematics educators at a university under the guidance of the researcher and with the help of preservice secondary mathematics teachers (see Fig. 10.5).

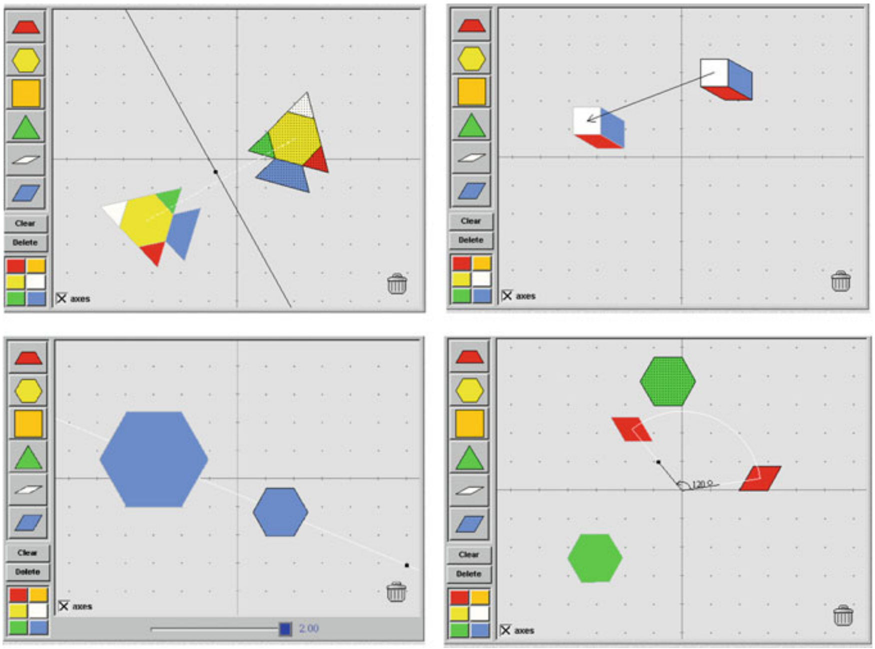

Fig. 10.4 Samples from virtual manipulatives used during instruction

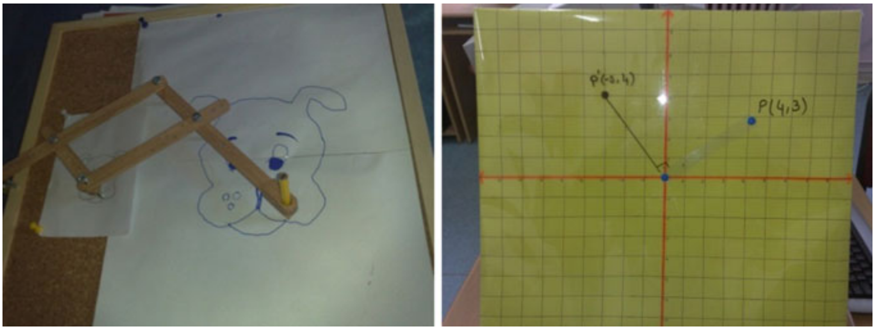

Fig. 10.5 Samples of physical manipulatives used during instruction

10.1.3 Collecting Data

Data collection occurred in two main phases. Before the first lesson of the study began, two data sources were used, the Spatial Ability Test (SAT) and the pretest. Prior to the first week of the study, the researcher interviewed each of the participants to discuss their responses to pretest tasks. The purpose of this interview was to determine the participants' primitive knowledge about geometric transformations in detail.

After the lessons began, the main data source was participants' responses during videotaped task-based interviews that were conducted weekly. The field notes that were taken during the classroom observations were used to support the interview data. The researcher conducted four task-based interviews with each of the participants about each of the transformations after the lessons. Tasks were designed to probe students' understandings of geometric transformations in the context of representations. Because high school students were expected to construct a formal level of understanding and make formal observations in the curriculum, the nature of the tasks was based on the knowledge that is required to go through to the Observing level. Several studies on students' understanding of geometric transformations (e.g., Flanagan 2001; Yanik 2006) were used to design the tasks. First, questions asked participants to provide descriptions for the geometric transformation, give examples and non-examples of the transformation, and clarify the properties of the transformation. There were tasks using verbal, visual and algebraic representations of the transformations. The researcher prompted the collection of information about participants' growth of mathematical understanding by posing questions about identifying, applying, and translating among multiple representations.

She provided paper, pencil, and related physical and virtual manipulatives during each of the interviews. Participants were told they were free to use any of these instruments while they were working on the tasks. A video camera was positioned behind the participants and captured audio and video of participants' engagement with the tasks and manipulatives. The researcher wanted students to read the questions and think aloud as much as possible during the interviews. Each of the four weekly interviews lasted approximately one hour.

10.1.4 Data Analysis

The units of analysis for this case study were the participants (Miles and Huberman 1994). First each case was analyzed separately, and then a cross case analysis was conducted to compare the cases in terms of the role of virtual manipulatives in developing mathematical understanding of transformations. The constant comparison method (Strauss and Corbin 1990) was used to analyze the data.

Before starting the data analysis, the researcher prepared a coding protocol, based on multiple representations of transformations and mathematical understanding levels of the Pirie-Kieren model. The protocol included the possible components of each mathematical understanding level that students developed about each transformation by using distinct representations. The participants' mathematical understanding about each geometric transformation was traced according to this protocol. For example, when a student was using formal understanding, but for some reason s/he needed to revise her/his image and began to use virtual manipulatives at the Image Making level, it was coded as *using virtual manipulatives to fold back.*

The whole data set was analyzed twice. First, the line-by-line coding of each interview was performed using Pirie and Kieren's understanding levels and characteristics of the theory. Second, the data were coded to determine students' multiple representation usage by focusing on the virtual manipulatives. After coding each interview for mathematical understanding levels and engagement with virtual manipulatives, codes from the two sets were associated with each other by axial coding and emergent themes were identified looking at these associations.

10.2 Results

Table 10.2 summarizes participants' virtual manipulative use during the weekly interviews. Participants differed in their use of virtual manipulatives as they developed mathematical understandings about the geometric transformations.

Elif used virtual manipulatives during all four weeks consistently, Defne used them during two of the four weeks, whereas Selim and Metin used virtual manipulatives only during one of the four weeks. Elif used virtual manipulatives while she was working at almost every mathematical understanding level from Image Having to Observing. Defne used virtual manipulatives mostly to express her images and to notice some properties about transformations. Selim preferred to use them during the rotation interview while he was engaging with some activities at the Property Noticing and Observing levels. Metin used virtual manipulatives only

Table 10.2 The number of times participants used virtual manipulatives during the interviews

	Elif	Defne	Selim	Metin
Week 1: Translation	2 (IH, F)	0	0	0
Week 2: Rotation	5 (IH, PN, F)	0	2 (PN, O)	0
Week 3: Reflection	5 (F, O)	2 (IH, PN)	0	0
Week 4: Dilation	6 (PN, F, O)	2 (PN)	0	2 (F, O)

Note IH Image Having; *PN* Property Noticing; *F* Formalising; *O* Observing

during the dilation interview. He worked with these manipulatives at the Formalising and Observing levels to express the formal ideas and inferences that he constructed after the lesson. In the following section, each participant's virtual manipulative usage will be examined deeply as their mathematical understanding process is traced within the Pirie and Kieren's levels.

The mapping of the participants' mathematical understandings about each transformation is presented below in a two-dimensional model that maps the representations with Pirie and Kieren's mathematical understanding levels. The horizontal axis shows the levels from the Pirie-Kieren model with acting and expressing aspects, and the vertical axis shows the three areas of the tasks presented during the interviews (verbal, graphical, and algebraic) and the five types of representations used (physical manipulatives, virtual manipulatives, verbal, graphical, and algebraic representations). To guide the reader to track the mathematical understanding process of the students, the numbers were added to the model to show how participants traversed the levels. The same numbers appear in parentheses in the text to help the reader to follow participants' mathematical understanding explicitly. Participants' engagement with distinct representations through their use of virtual manipulatives will be focused on during the interviews to analyze their role in the mathematical understanding process.

Case 1 (*Elif*)

Elif was a 16-year-old girl in the second year of her high school education. She was a successful student in her previous geometry classes and had good spatial ability (see Table 10.1).

Elif's mathematical understanding about translation and virtual manipulatives. Figure 10.6 shows the development of Elif's mathematical understanding about translation as she worked with distinct representations within the different levels. She used virtual manipulatives two times during this interview, first at the Image Having level and second at the Formalising level.

The first question of the interview asked Elif to give an example and a non-example of a translation. She preferred to use virtual manipulatives to express her image by giving an example of the transformation (1a). She drew two congruent triangles on the virtual manipulative and said that she drew the translated image of the triangle by moving the original triangle "three units to right and four units to down" (see Fig. 10.7).

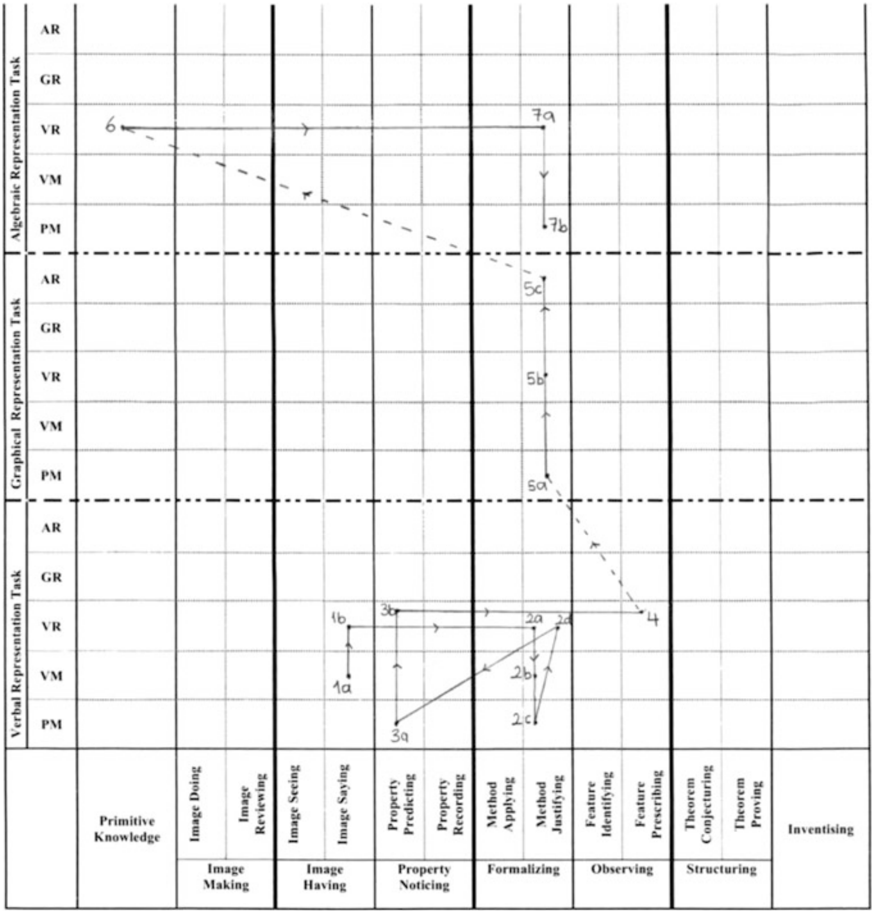

Fig. 10.6 The mapping of Elif's mathematical understanding about translation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

In the following phase of the interview, the researcher wanted Elif to explain what she understood about translating an object on the plane by using the idea of a vector. She answered that question by saying that "it was moving the object through the length of the vector" while she supported this idea with an example on the virtual manipulative (2b). This time she preferred to use virtual manipulatives to express her formal idea about the transformation at the Formalising level.

Elif was able to understand and apply the verbal, graphical and algebraic representations of translation at the formal levels and she successfully connected these representations with each other. At the end of the interview, she explained that she "could work virtual manipulatives properly and could make faster drawings with them". She preferred to use virtual manipulatives to express the images she had from

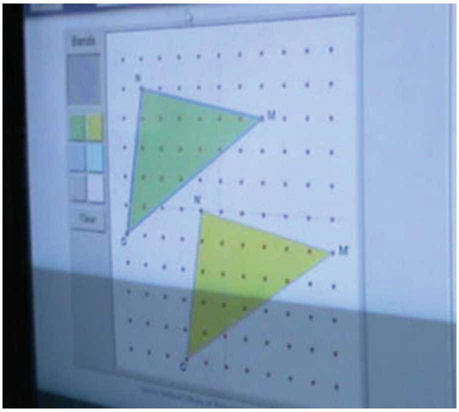

Fig. 10.7 The congruent triangles Elif drew on the virtual manipulative

her previous experiences and the formal ideas she constructed after the lessons during the interview. Virtual manipulatives were validating tools in the environment that helped her to verify the level of understanding she was working on at that time.

Elif's mathematical understanding about rotation and virtual manipulatives. Elif preferred to use virtual manipulatives during the rotation interview more than she did in the translation interview. As seen in Fig. 10.8, she used the virtual manipulative mostly at the beginning of the interview. She drew a triangle on the manipulative and rotated it easily 90°, 180°, and 360° while she was giving examples of rotation (1). She continued to express her ideas on the virtual manipulative at the Property Noticing level while the following dialogue took place between her and the researcher (2):

Elif	For example in this rotation, the figure goes on that circle (pointing to the circle on the virtual manipulative) but in translation it would move through a given vector.
Researcher	I see. What would you add if one of your classmates would ask you about rotation?
Elif	I would say it is spinning a figure around a point that is not on the figure without changing the distance.
Researcher	Distance?
Elif	I mean the distance from the figure to the point.
Researcher	You mean the point that is not on figure?
Elif	Yes, rotation center I mean. It may be on the figure also, sorry. But the distance between the rotation center and the turning point will not change.
Researcher	Ok. How will you rotate the figure without changing that distance?
Elif	Around the point. I mean it will make a circle around the rotation center.

Elif used formal explanations while she was saying that rotating a figure meant turning a figure around a point by any angle measure on the plane (3). She made a

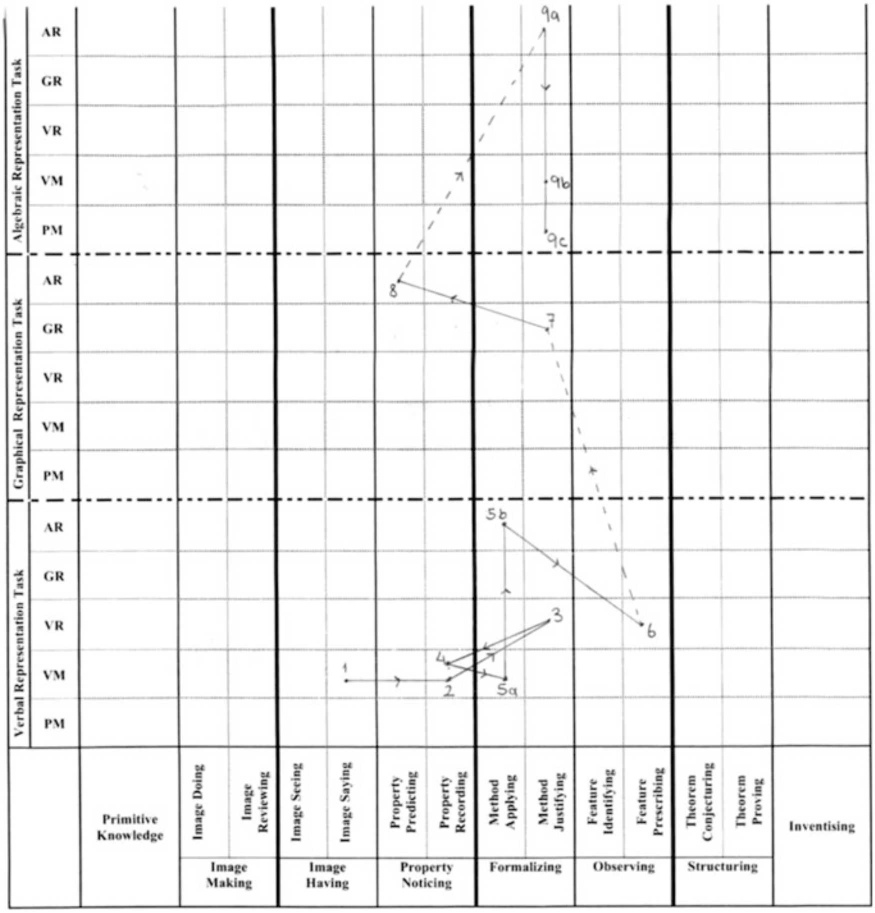

Fig. 10.8 The mapping of Elif's mathematical understanding about rotation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

folding back movement to the Property Noticing level and chose a rocket figure on the virtual manipulative while she was explaining further (4) (see Fig. 10.9).

> First I determine the distance between the corner points of this figure (showing the original figure) and rotation center, then look at the rotation angle. I turn the figure around the center by that angle. Like this, I am identifying the distance (showing the length of line segment between one of the corner points of the figure and rotation center), then, rotation center is the origin point. I am looking at the distance between this point (showing the same corner point) and center. I will turn the figure around the center without changing the distance I determined. It will make a circle around the center and the radius of a circle is equal to this distance everywhere.

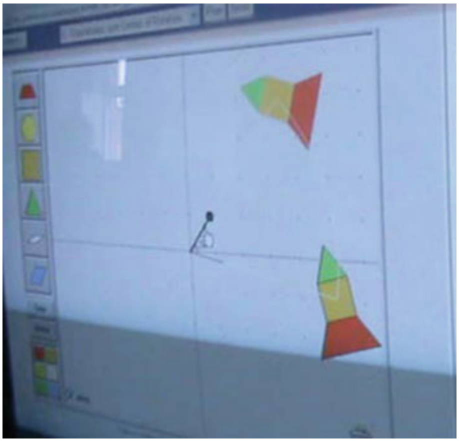

Fig. 10.9 The figure Elif used to express some properties about rotation

Elif used the virtual manipulative as an invocative intervention that caused her to work in an inner level to strengthen her formal level of understanding. She continued to use the virtual manipulative when she was asked to rotate a triangle around the origin by 90° in the verbal representation task. She quickly drew the triangle on the virtual manipulative and found the image of the triangle under a rotation of 90° around the origin by herself, without clicking the rotate button (5a). To understand if she was able to translate from verbal representations to algebraic representations of rotation, she was asked to reanswer this question by using a mathematical formula this time. First she wrote the expression $(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$ without any equality and then looked at the virtual manipulative for a while (5b). The researcher observed that Elif was checking the formula with the solution that she performed on the virtual manipulative by putting the corner points of the triangle into the expression. For this situation, the virtual manipulative was playing the role of a validating intervention for her to confirm the formal level of mathematical understanding she had.

To analyze Elif's mathematical understanding of rotation in detail, she was asked to find $R_{60°}(P)$ without giving any mathematical formula, where P was (1, 0). Elif was successful in explaining the meaning of the mathematical notation $R_{60°}(P)$ in the task by stating that it was asking for the image of point P under the rotation around the origin by 60° (9a). When she was asked to resolve the problem by using visual representations she opened the virtual manipulative again and turned the point around the origin by 60° and marked an estimated point on the screen as the image (9b). She stated that she could find the image on the virtual manipulative approximately, but to identify the exact location of this point she would need to use the physical manipulative that was designed to rotate a point around the origin by a protractor.

As seen on Fig. 10.8, Elif was successful understanding and applying distinct representations of rotation at the Formalising level and translating among verbal, graphical and algebraic representations. As interactive and dynamic external visual representations, virtual manipulatives seemed to be helping her to apply these transformations in mathematical situations and make connections between representations. In contrast with the translation interview, Elif preferred to use virtual manipulatives more than physical manipulatives in the rotation interview. At the end of the interview, she clarified this situation by stating that "virtual manipulatives were ready to use quickly" at that time.

Elif's mathematical understanding about reflection and virtual manipulatives. Elif continued to use the virtual manipulatives during the reflection interview. As seen in Fig. 10.10, she began the interview by creating an example on the virtual manipulative at the Formalising level (1).

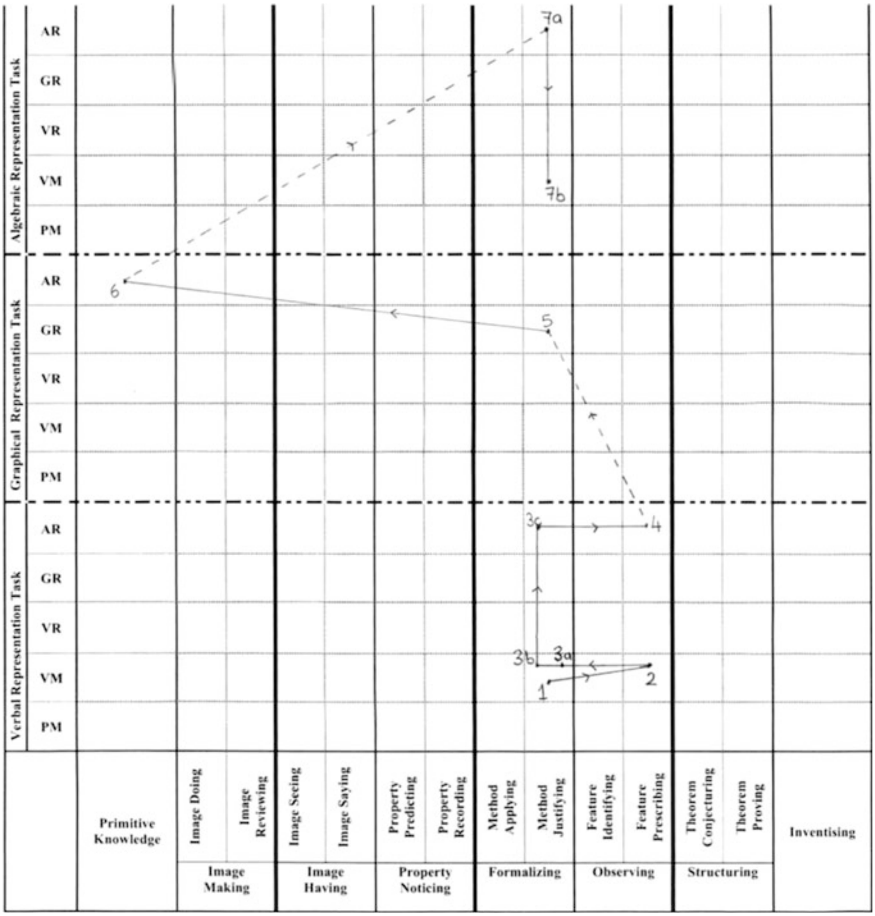

Fig. 10.10 The mapping of Elif's mathematical understanding about reflection. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

She found the image of a triangle that she drew on the virtual manipulative under the reflection across the line $y = x$. She said it would be an example of a reflection because "each point of the original triangle has a corresponding point on the reflected triangle that is the same distance from the line of reflection as the original point". When she was asked to give a non-example of reflection she said that a translation would not be a reflection because the direction of a figure might be changed in reflection but not in translation. She marked one of the corresponding sides of the triangles that she previously drew on the virtual manipulative and said that the direction of both sides did not stay the same after a reflection (2). As a provocative intervention, the virtual manipulative helped her to carry her understanding to the next level. Elif began to work at the Formalising level again while she was explaining how to reflect a figure across a line. She expressed that "all of the corresponding points would be the same distance from the line of reflection" by looking at the example she drew on the virtual manipulative (3a). Her understanding seemed to maintain that level while she was engaging in the verbal representation task in which three points as $A(1, 1)$, $B(0, 2)$ and $C(2, 3)$ on the plane were given and she was asked to reflect the ABC triangle in the origin. She was successful in finding the image with the virtual manipulative by placing it directly on an opposite point on the other side of the center for each of the corner points on the triangle. The point of reflection was the midpoint of the segment joining each point of the triangle with its image (3b) (see Fig. 10.11).

The next time she used virtual manipulatives was the session that she was resolving the algebraic representation task by using graphical representations. She was asked to find $S_M(P)$ where P was $(1, 0)$ and M was $(2, 2)$ in the task. She said that $S_M(P)$ was the image of point P under the reflection in point $M(2, 2)$ and found the image by using $S_M(P) = 2M - P$ (7a). She was also able to show the graphical solution on the virtual manipulative that confirmed that she understood and applied distinct representations of the transformation (7b). Elif used virtual manipulatives to clarify her formal understandings during the reflection interview. As seen in her

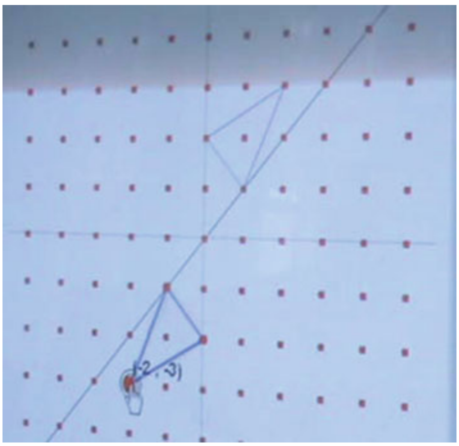

Fig. 10.11 Elif was finding the image of a triangle on the manipulative

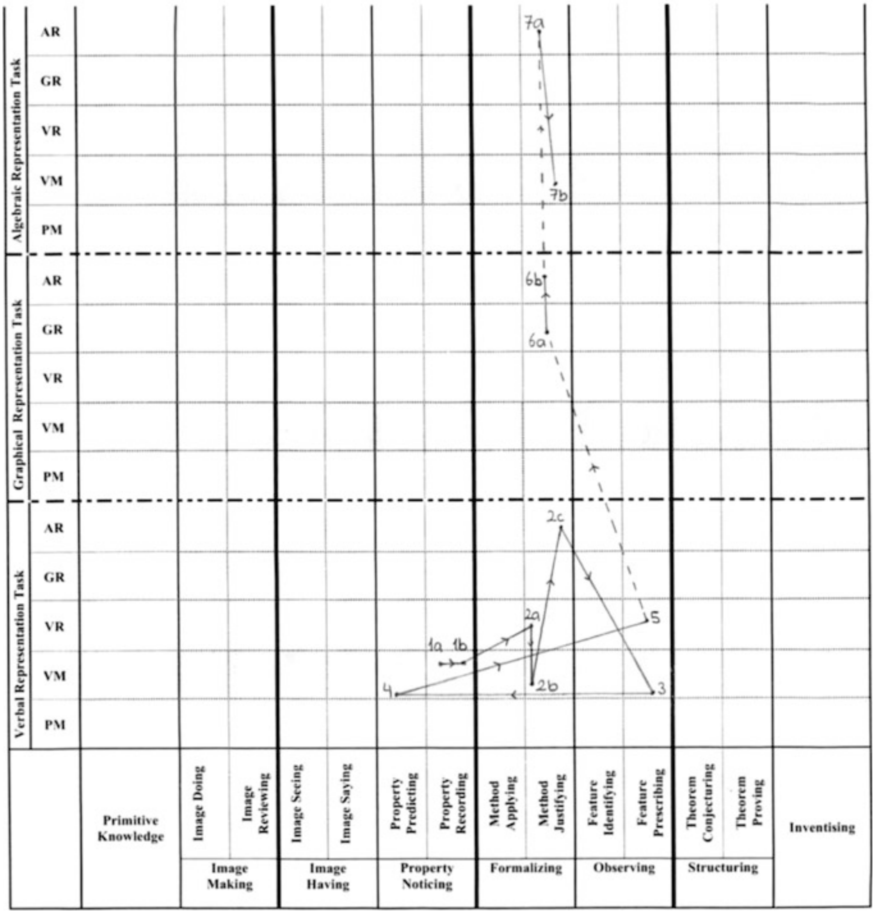

Fig. 10.12 The mapping of Elif's mathematical understanding about dilation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

previous explanations, the virtual manipulative played the *bridge* role in her understanding to use distinct representations and translate among them.

Elif's mathematical understanding about dilation and virtual manipulatives. The dilation interview was the session that Elif primarily used virtual manipulatives (see Fig. 10.12). She began the interview by expressing some properties of dilation in an example on the virtual manipulative (1a) (see Fig. 10.13). She continued to work at the Property Noticing level while she was recording some properties of dilation on the same manipulative (1b). Next, Elif was expected to find the image of a triangle after a dilation with the center at point (0, 2) and a scale factor of 2. She continued to use the virtual manipulative and found the image of the triangle easily (2b) (see Fig. 10.14).

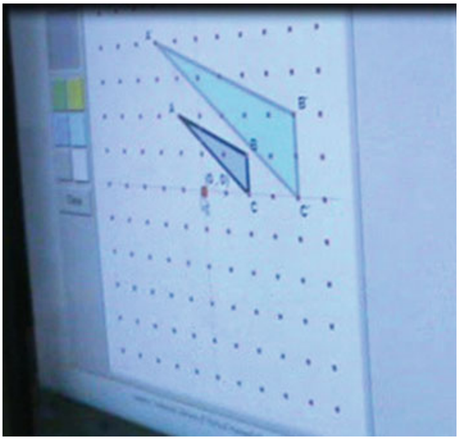

Fig. 10.13 The example of a dilation Elif drew using the virtual manipulative

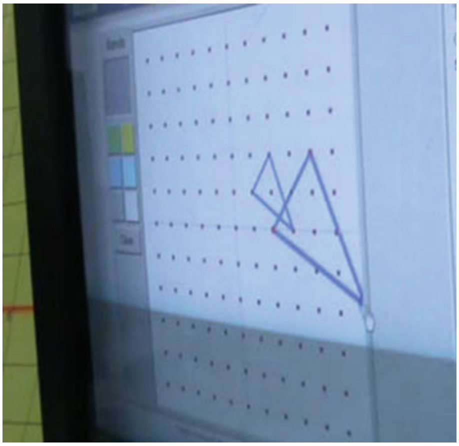

Fig. 10.14 The example of dilation Elif drew on the virtual manipulative while she was checking some properties

Elif was working at the Formalising level and she was able to connect the verbal representation and the graphical representation of the concept together using the virtual manipulative. When she was asked to resolve the problem by using algebraic representations she could not remember the whole mathematical formula, $H(P) = P' = M + k(P - M)$, which gives the image of point P under the dilation with a center at the point M and a scale factor of k. She used virtual manipulatives to confirm the $P - M$ part of the formula. She picked some corresponding points from the original and image triangles and put these points in the formula to see if the notation was right or not. After trying several points, she stated that $P - M$ was the distance between the original point and the center and if she multiplied this distance with k and added the new distance to the center she would find the image of the point (2c). The virtual manipulative, again as a validating intervention, helped her to build a connection between the graphical and algebraic representations of the concept.

She continued to work on the triangles she drew on the virtual manipulative while she was saying “because the angles, the direction of angles were the same and the sides of the image were twice the sides of the original triangle, two triangles were similar” (3). She was using the manipulative to express her formal observations about the transformation this time. She made a folding back movement to the Property Noticing level and continued to use the virtual manipulative while she was predicting some properties of the dilation in that level (4). She stated “while we were finding the image, we were thinking about the distance from the center. If the center changes, because the distance between the original point and the center will change, then the location of the image changes” while she was working on the manipulative that helped her to strengthen her mathematical understanding.

After using verbal representations to solve the algebraic representation task, she began to use the virtual manipulative again to show a graphical solution (7b). Elif was able to comprehend and apply the distinct representations of dilation and make connections among them. She used virtual manipulatives usually to express her images and formal ideas and virtual manipulatives were validating interventions in the environment that helped her to translate among the distinct representations of transformations.

Case 2 (*Defne*)

Defne was a 17-year-old girl in the second year of her high school education. She was a successful student in her previous geometry class but did not have good scores on the spatial ability test or the pretest (see Table 10.1).

Defne’s mathematical understanding about reflection and virtual manipulatives. The reflection interview was the session that Defne used virtual manipulatives for the first time. She did not use them in the translation or rotation interviews. According to Fig. 10.15, which shows the mapping of Defne’s mathematical understanding about reflection, she used the virtual manipulatives at the Image Having level after noticing some properties about the transformation.

When she was asked to explain what she understood from reflecting a figure across a line she made the following explanations using virtual manipulative at the Property Noticing level (2b) (see Fig. 10.16):

Defne	If I reflect a triangle over the line passing through the origin (showing the $y = x$ line) I will flip it over that line, but the distances have to be perpendicular distances.
Researcher	Perpendicular distance?
Defne	I mean it is already the distance when it comes to perpendicular.

She was trying to express that when she was reflecting a point across a line, the line was the perpendicular bisector of the segment joining the original point and its image. Her understanding moved to the Image Having level when she used her hands and made some gestures to clarify the ideas that she was explaining (see Fig. 10.17). The dialogue between the researcher and her continued (3):

Defne	Let’s think about my hands. For example the reflection of this hand is my other hand, like that. If I move my hands like here (putting her

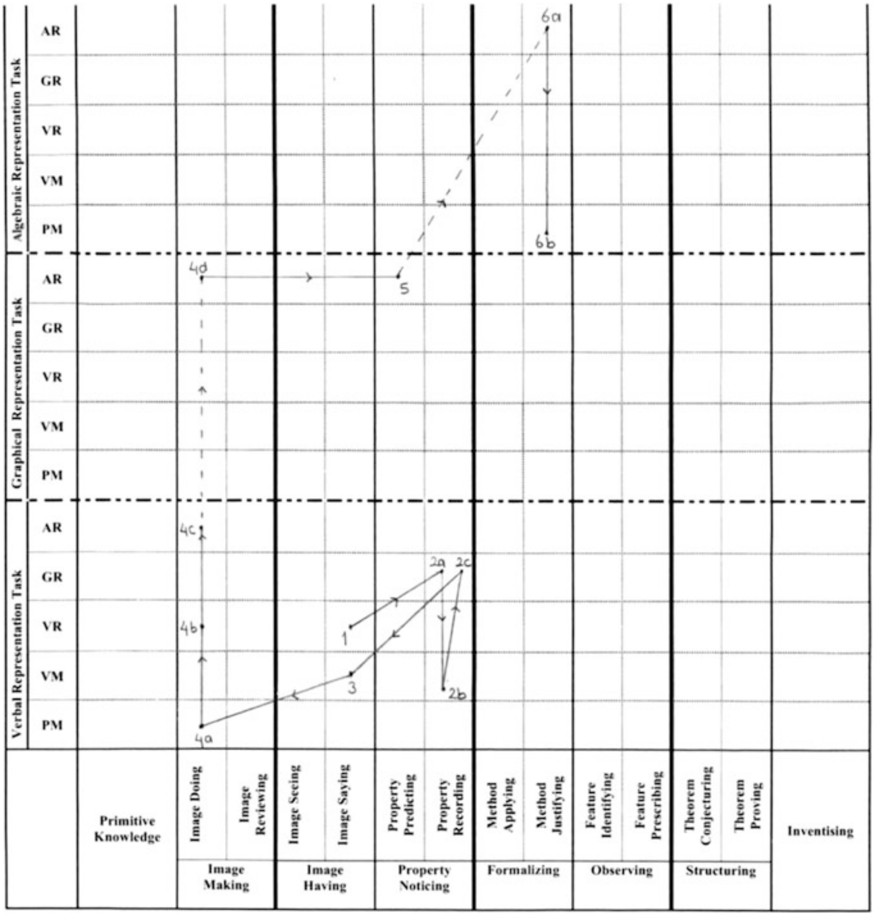

Fig. 10.15 The mapping of Defne's mathematical understanding about reflection. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

hands on the computer screen). We can think as a whole. Reflection over a line.

Researcher What about the reflection in a point?

Defne Reflection in a point. This point for example (marking a point on the virtual manipulative), we will find the image around this point. It is one unit (showing the distance between the point and one corner of the figure) then it will come to here one unit (showing the point directly opposite on the other side of the center).

As an invocative intervention, Defne used virtual manipulatives to make a folding back movement to an inner level and tried to enlarge her images about reflection. From Fig. 10.15, it can be seen that she was actively working in the

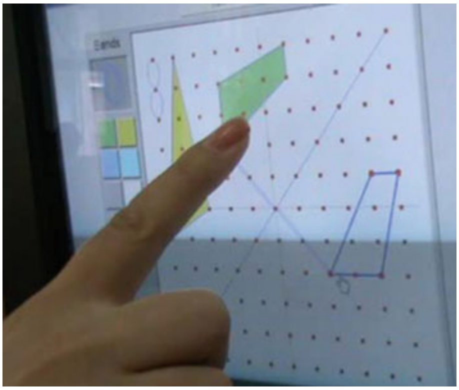

Fig. 10.16 The example Defne drew on the virtual manipulative

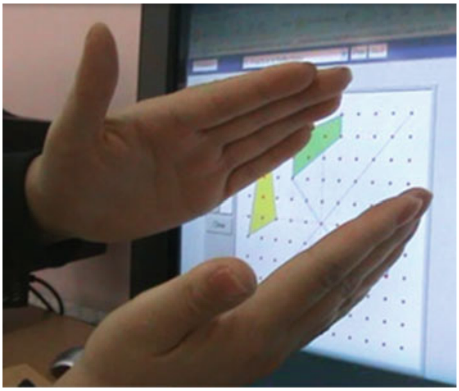

Fig. 10.17 Gesture Defne used during the interview

informal levels and could not move her understanding to the formal levels for a long time (4a-5). Because she was not able to use multiple representations at the formal levels she could not develop an appropriate understanding about the concept in that time. Moreover, she looked uncomfortable using the virtual manipulatives during the lesson. At the end of the interview, Defne stated that she lost time when she attempted to work on the virtual manipulatives because she did not like to use the computer. Her attitude seemed to be affecting her virtual manipulative use. Virtual manipulatives helped her only to express the images that she had and clarify the properties that she had already realized about reflection.

Defne's mathematical understanding about dilation and virtual manipulatives. The other interview where Defne used virtual manipulatives was the dilation interview. Figure 10.18 shows that she used virtual manipulatives two times in the acting phase of the Property Noticing level to predict some properties of dilation during the verbal representation task. She expressed the following ideas about finding the image of a figure after a dilation with a center and a scale factor on the virtual manipulative (2c):

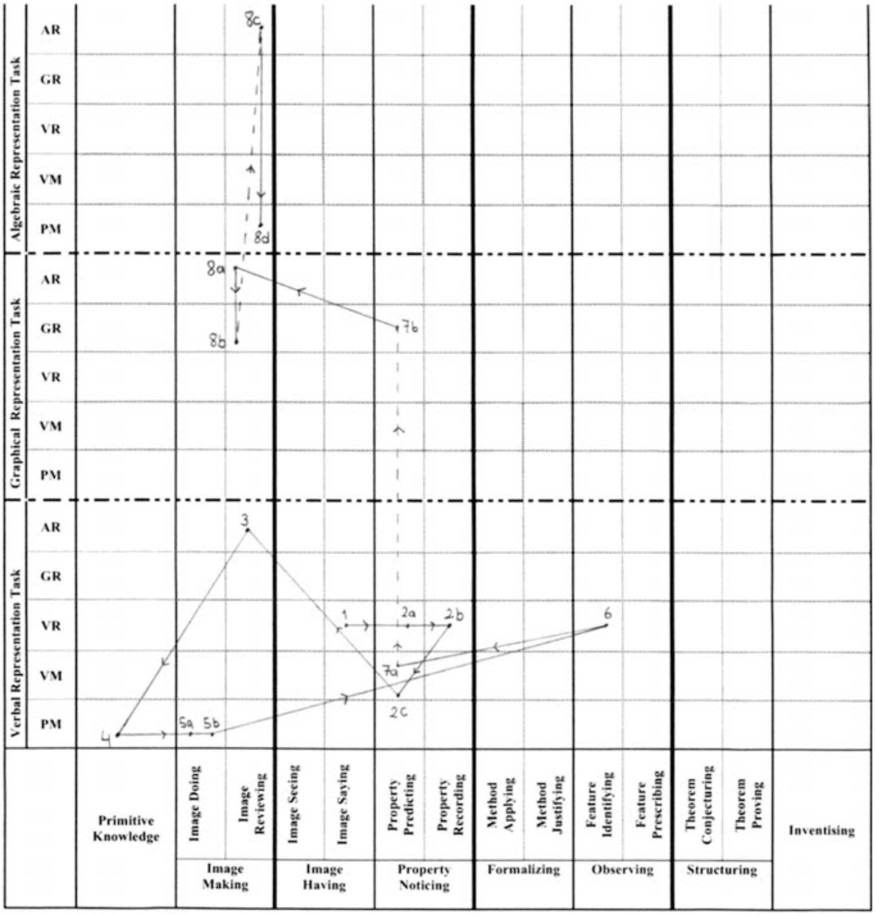

Fig. 10.18 The mapping of Defne's mathematical understanding about dilation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

> I understand that the figure is being stretched or expanded on a center. Like this example (showing an original triangle and its dilated image on manipulative). When the dilation center comes closer to the original figure, this one (the image of the triangle) comes closer to the center, too. When we reduce the scale factor the triangles come closer to the center… So, we will look at the distance to the center and the scale factor if we want to dilate a figure.

She made a folding back movement to the Property Noticing level and began to use the virtual manipulative for the second time when she was asked what would happen to the image if the dilation center was changed (7a). She expressed her ideas in the following dialogue:

Defne I will do the similar things. If I carry the center point from here to here (moving the dilation center on the virtual manipulative) but the scale

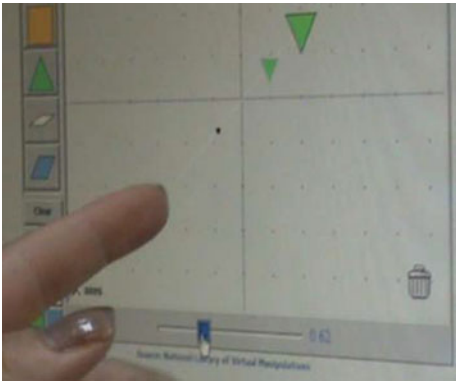

Fig. 10.19 Defne was changing the scale factor to notice some properties

	factor will be same, the size of the triangle (showing the image of triangle) will not change. Only the location of it will change.
Researcher	What about the scale factor? If I change the scale factor?
Defne	When I reduce the scale factor the image comes closer (changing the scale factor) (see Fig. 10.19).

Defne stopped using virtual manipulatives after working on these properties of the transformation. She preferred to use them to make some physical activities at the Property Noticing level. The virtual manipulatives played the role of validating interventions that helped her to justify the understanding she had at the time. Figure 10.18 shows that she continued to work mostly at informal levels as she did in the reflection interview. She could not apply the verbal, graphical and algebraic representations and was not able to connect them in formal levels as in the reflection interview.

Case 3 (*Selim*)

Selim was a 16-year-old boy in the second year of his high school education. He was a very successful student in his previous geometry class but had the lowest score on the spatial ability test and one of the lower scores on the pretest (see Table 10.1).

Selim's mathematical understanding about rotation and virtual manipulatives. Selim was the participant who preferred to use physical manipulatives during the interviews. The only session he used virtual manipulatives was the rotation interview. He used them two times during the interview; first he used them to perform some activities at the Property Noticing level, and second he used them to express his formal observations at the Observing level (see Fig. 10.20).

When he was asked to explain what he understood from rotating a figure around a point by an angle measure he began to work on a particular example on the virtual manipulative to notice some properties (4). The following dialogue took place between the researcher and Selim while he was using the virtual manipulative:

Selim	For example this figure (pointing to the figure on the virtual manipulative). Let's rotate it counter clockwise, in positive direction by 90° (rotating the figure and observing the image).

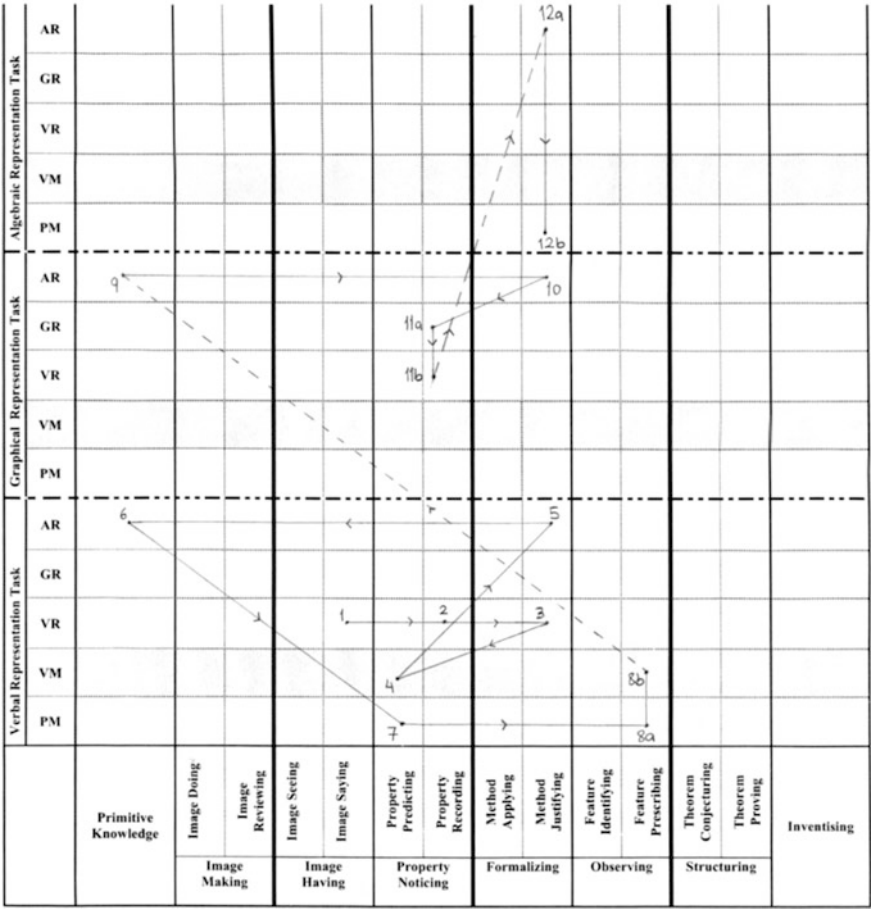

Fig. 10.20 The mapping of Selim's mathematical understanding about rotation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

Researcher It says around a point?
Selim Around the origin, the rotation center is the origin here.

The second time, he chose to work with virtual manipulatives again while he was showing the differences when the parameters of the transformation were changed (8b). Selim was able to apply verbal representations of the transformation more than graphical or algebraic representations. He was able to transition between these representations but he needed to make folding back movements to inner levels while he was working especially with algebraic representations.

Virtual manipulatives played the role of a tool for Selim to engage in physical activities and to confirm his formal observations of the transformation. He remarked he "would not trust the operations or calculations from a virtual manipulative he was using without checking these operations or calculations by mathematical formulas during problem solving".

Case 4 (*Metin*)
Metin was a 17-year-old boy in the second year of his high school education. He was not a successful student in his previous geometry class but had the highest score on the spatial ability test (see Table 10.1).

Metin's mathematical understanding about dilation and virtual manipulatives. Metin was the other participant who preferred not to use virtual manipulatives during the interviews. He used them only two times at the Formalising and Observing levels during the dilation interview (see Fig. 10.21). First he used them

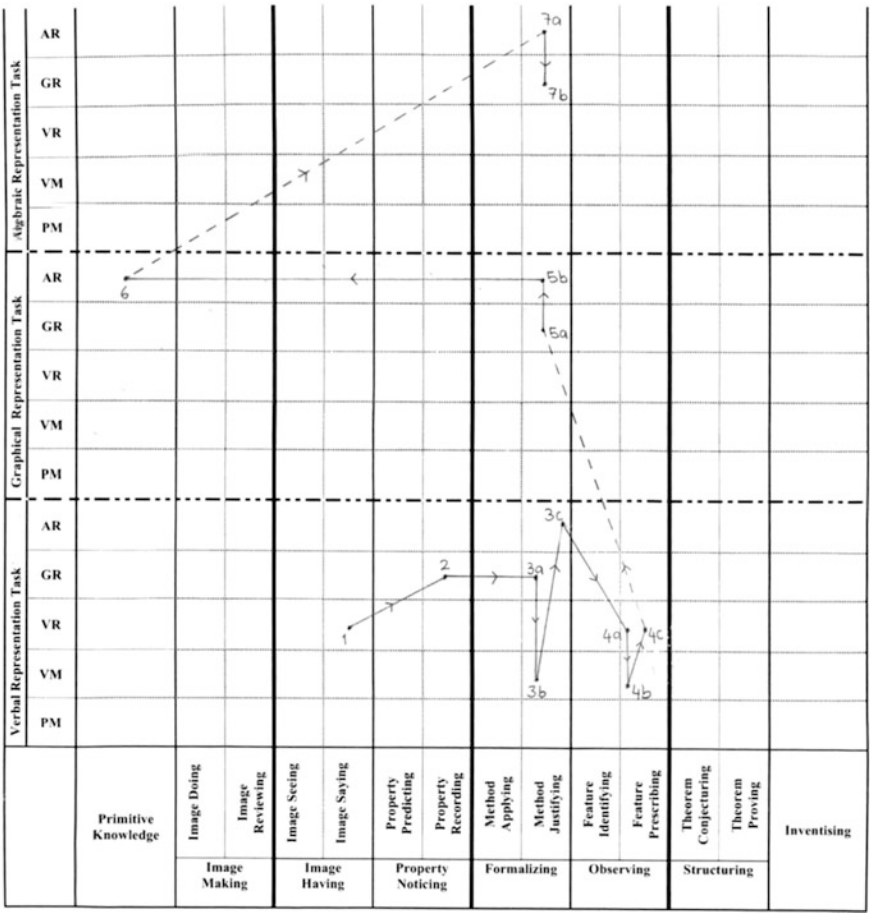

Fig. 10.21 The mapping of Metin's mathematical understanding about dilation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

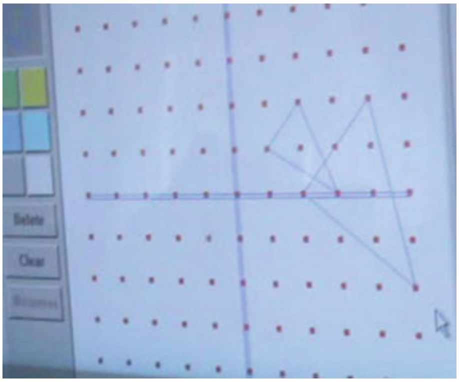

Fig. 10.22 The triangle and its' image Metin drew on the virtual manipulative

to express his formal ideas while he was dilating a triangle and second to state his observations while he was explaining the relationships between the original figure and its image under a dilation. When Metin was asked to find the image of a triangle after a dilation with the center at point (0, 2) and a scale factor of 2, he began to use the virtual manipulative and found the image of the triangle (3b) (see Fig. 10.22).

He continued to use his formal level understanding during the remaining part of the verbal representation task and used the virtual manipulative to express his theorem-like idea about dilation at the Observing level (4b). He said "the figures and their dilated images were always similar because angles were preserved and corresponding sizes of the figures had the same ratio" while he was working on the virtual manipulative.

As seen on Fig. 10.21, Metin was working mostly at the formal levels and he was able to use distinct representations of the concept. He could translate among the multiple representations of the transformation without using physical and virtual manipulatives. It was thought that because he had a proper understanding and was able to use distinct representations of dilation he did not need to engage in manipulative activities. The following sentences he used at the end of the interview supported this finding:

> I got used to computer and physical objects. I can use whichever I want. It may be more enjoyable if you work with them but using them takes your time, you have to do some extra work. Physical objects are good while you are trying to understand the concept but I would not use them if I try to solve a problem. I would use only paper and pencil.

10.3 Discussion and Conclusion

Trying to characterize students' mathematical understanding about a mathematical concept is a multifaceted process. In this study, Pirie-Kieren theory, with its understanding levels and features, and representation theory, with multiple representations, provided a tool/model to observe *how* and *what* students understood about a concept. Developing a two dimensional graph to represent students' developing understanding by combining these two theories helped the researcher to trace the change in student's mathematical understanding and interpret the role of virtual manipulatives in this understanding. According to the analysis, students' mathematical understanding levels spanned the first six levels of the Pirie-Kieren model (Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, and Observing). Students used virtual manipulatives to support their understanding in various ways not only in informal levels but also at the Formalising and Observing levels. Findings by Gibbons (2012) support this result by stating that high school students worked with physical manipulatives at formal levels like Observing and Structuring, to construct abstract meaning of mathematical concepts.

In the cross case analysis, Elif and Metin, who developed formal levels of understanding, used virtual manipulatives to express their formal ideas and observations. Expressing is a required understanding characteristic that helps students to state the structure of mental or physical activities to themselves or other observers and carry their understandings to the next levels (Borgen 2006; Nillas 2010; Pirie and Kieren 1994). On the other hand, Defne, who developed insufficient understandings, used virtual manipulatives for physical and mental activities in only the first four levels. Acting is an important understanding characteristic that helps students to manage their previous levels of understandings and realize some new features in any of the understanding levels (Pirie and Kieren 1994). Hence, it can be said that virtual manipulatives played the role of an acting/expressing medium that supported the development of students' mathematical understanding.

There were several interventions that affected students' mathematical understanding in the environment. Verbal, graphical, and algebraic representation tasks, prompts that the researcher used during the interviews, and manipulatives that were present on the table and computer screen gave direction to students' mathematical understanding processes. In particular, virtual manipulatives played the role of a provocative intervention that helped students to progress outwards (e.g., Elif's movement from the Formalising level to the Observing level in the reflection interview). These were used as validating interventions that guided students to justify their level of understanding. Lastly, virtual manipulatives were invocative interventions that caused a folding back movement to the inner levels (e.g., Selim's movement from Formalising to the Property Noticing level in the rotation interview). As invocative interventions, virtual manipulatives were the source of folding back movements where they were also helping to identify the nature of these movements. For example, Defne used virtual manipulatives in a folding back movement to "work in an inner level using existing understanding" during the rotation interview, whereas Elif used them in a folding back movement to "collect in an inner level" during the dilation interview (Martin 2008, p. 76). When we consider that the dynamic feature of folding back is essential for the mathematical understanding process (Pirie and Kieren 1994), virtual manipulatives helped students to strengthen their understanding during these movements.

Virtual manipulatives played the role of a supportive external representation that facilitated understanding of the concept. Students preferred to work with virtual manipulatives to comprehend and apply especially verbal and graphical representations. Students used them as a connection tool when they were expected to translate among distinct representations. Because applying multiple representations and making connections among distinct representations indicates a strong mathematical understanding (Goldin 2003; Hiebert and Carpenter 1992; Lesh et al. 1987; Zhang 1997) the virtual manipulatives helped students to develop a proper understanding in terms of multiple representation engagement. From the observer's perspective, virtual manipulative usage at the formal levels may be an indicator that the student can connect multiple representations of the concept. For example, Elif used virtual manipulatives to express her formal ideas while she was engaging with verbal or algebraic representations at the Formalising level. Because she could use distinct representations at the formal levels, she developed a robust understanding about the transformations.

On the other hand, virtual manipulatives supported the motion understanding of geometric transformations (Hollebrands 2003; Yanik 2006) especially if they were used without an understanding of algebraic representations of the concepts. Elif was the only participant who developed a function understanding about the transformations, which is complicated even for preservice mathematics teachers (Yanik 2011).

In terms of preferences, Elif and Metin used virtual manipulatives because they found them easy to produce proper solutions (Haistings 2009; Izydorczak 2003). Defne and Selim did not prefer to use them for different reasons. Defne was under the influence of her attitudes towards computers, while Selim did not trust the operations he did with the manipulatives. Another factor to consider was that spatial ability plays a role in effective use of virtual manipulatives in geometric transformations. Students who had good spatial ability (Elif and Metin) used them quickly and properly whereas students who had insufficient spatial ability (Defne and Selim) had some difficulties in using them.

This study was an attempt to determine the role of virtual manipulatives in high school students' mathematical understanding processes. Although the Pirie-Kieren model and representation theory give the opportunity to trace the development of mathematical understanding in detail, there is still a need for new research analyzing virtual manipulative usage in mathematical understanding processes at the high school level.

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