Mathematics Education in the Digital Era

Patricia S. Moyer-Packenham *Editor*

International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives

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Foreword by Douglas McDougall

As an elementary school teacher, my students played with cubes, pattern blocks, and other concrete materials. The link between the objects and the learning was made by students making drawings of the manipulatives on paper and then discussing the drawings. In order to tap into student's current understanding and lifestyle, we need to find ways to help them take those experiences and make sense of the physical world through experiences in the virtual world. The experiences of young children provide a context for learning mathematics. Children as young as two years old are playing with virtual manipulatives on iPads and other portable devices, to communicate and manipulate their world.

In 1995, I was working on my doctoral dissertation in geometry. I chose to replace the compass and protractor in middle school mathematics with Geometer's Sketchpad. As an early adopter of dynamic geometric technology, I learned much about the role virtual manipulatives play on the role of the teacher. The examples in this book will help the reader to better understand the role of virtual manipulatives and how they relate to student learning and the wider field of teacher knowledge. This book helps to trace possible trajectories of teachers and students learning in the use of virtual manipulatives.

For many years, when mathematics teachers were using manipulatives, they were using physical materials that might represent or model mathematical ideas and concepts. In 2002, Moyer, Bolyard and Spikell defined a virtual manipulative. There has been much written and explored about virtual manipulatives before and after 2002 in this area. This book helps researchers to distinguish and then use virtual manipulatives in their work. It also helps the educational and research communities to have a common understanding of the language around manipulatives.

As you will read in this book, the representation of a dynamic mathematical object provides learning opportunities for constructing mathematical knowledge. As with physical manipulatives, we need to manipulate the object. However, an important question to pose is "what do we need to do to something to capture the mathematical ideas?" I suggest that the reader first answer this question and then reflect on their own understanding of the role of virtual manipulatives as they navigate the key messages in each chapter.

There are many conflicting views about what the research has shown us about virtual manipulatives. This book helps us to frame the research possibilities as well as provide a framework to guide the data collection and analysis. It also provides a foundation for enhancing the development of additional frameworks.

I am impressed with the interplay between dynamic interactions with the computer and the way that Dewey described the use of manipulatives. The story throughout the book about virtual manipulatives and the natural need of humans to visualize and touch their world is powerful. These authors bring alive the physical world and the virtual world and the human interactions that will help readers to make sense of this important work.

I think this is a very important and timely book. Patricia Moyer-Packenham has selected key researchers in the area and the names are recognized around the world as being experts in the field. I also think that beginning with a frameworks and definitions section sets up the book as a serious collection of research and practical contexts that will help the reader fully capture the characteristics of virtual manipulatives and the growing common understanding of the field. Many authors describe various frameworks that are in use in the study of virtual manipulatives. It is particularly interesting that every chapter contains a theoretical perspective. Many authors extend their work to suggest practical suggestions for how to use virtual manipulatives.

I was intrigued by the various educational contexts presented in this book. I have had experience teaching in elementary and secondary schools as well as with pre-service teachers and graduate students. In this book, we learn about the use of virtual manipulatives in many different classrooms: early childhood education, primary, Grade 5, and secondary school. In addition, we learn about how pre-service teachers can learn to teach using virtual manipulatives as well as increasing their own understanding of geometry and algebra. This broad spectrum of contexts provides a valuable resource for researchers, emerging scholars, pre-service teachers, in-service teachers, school leaders, and university professors.

Douglas McDougall
University of Toronto

Contents

Part I Frameworks and Definitions

1 **Revisiting the Definition of a Virtual Manipulative** 3
Patricia S. Moyer-Packenham and Johnna J. Bolyard

2 **Artifact-Centric Activity Theory—A Framework for the Analysis of the Design and Use of Virtual Manipulatives** 25
Silke Ladel and Ulrich Kortenkamp

3 **The Modification of Attributes, Affordances, Abilities, and Distance for Learning Framework and Its Applications to Interactions with Mathematics Virtual Manipulatives** 41
Stephen I. Tucker

4 **Physical and Virtual Manipulatives: What Is “Concrete”?** 71
Julie Sarama and Douglas H. Clements

5 **Manipulatives, Diagrams, and Mathematics: A Framework for Future Research on Virtual Manipulatives** 95
Helena P. Osana and Nathalie Duponsel

Part II Research and Design

6 **Fingu—A Game to Support Children’s Development of Arithmetic Competence: Theory, Design and Empirical Research** 123
Ingemar Holgersson, Wolmet Barendregt, Jonas Emanuelsson, Torgny Ottosson, Elisabeth Rietz and Berner Lindström

7 **Developing Virtual Mathematics Manipulatives: The SAMAP Project** 147
Erol Karakırık

8 Enhancing Mathematical Skills Through Interventions with Virtual Manipulatives 171
Annemie Desoete, Magda Praet, Claire Van de Velde, Brigitte De Craene and Edwin Hantson

9 Influence of Prior Knowledge and Teaching Approaches Integrating Non-routine Worked Examples and Virtual Manipulatives on the Performance and Attitude of Fifth-Graders in Learning Equivalent Fractions 189
Chun-Yi Lee and Ming-Jang Chen

10 The Role of Virtual Manipulatives in High School Students' Understanding of Geometric Transformations 213
Hilal Gulkilik

Part III Tools for Research

11 Geometry and iPads in Primary Schools: Does Their Usefulness Extend Beyond Tracing an Oblong? 247
Kevin Larkin

12 Selection of Apps for Teaching Difficult Mathematics Topics: An Instrument to Evaluate Touch-Screen Tablet and Smartphone Mathematics Apps 275
I.K. Namukasa, G. Gadanidis, V. Sarina, S. Scucuglia and K. Aryee

13 Ambitious Teaching: Designing Practice-Based Assignments for Integrating Virtual Manipulatives into Mathematics Lessons 301
Jennifer M. Suh

14 Developing an Interactive Instrument for Evaluating Teachers' Professionally Situated Knowledge in Geometry and Measurement 323
Agida G. Manizade and Dragana Martinovic

Index 343

Part I
Frameworks and Definitions

Chapter 1
Revisiting the Definition of a Virtual Manipulative

Patricia S. Moyer-Packenham and Johnna J. Bolyard

Abstract In 2002, Moyer, Bolyard and Spikell defined a virtual manipulative as an "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). The purpose of this chapter is to revisit, clarify and update the definition of a virtual manipulative. After clarifying what a virtual manipulative is and what it is not, we propose an updated definition for virtual manipulative: *an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge*. The chapter describes the characteristics of five of the most common virtual manipulative environments in use in education: single-representation, multi-representation, tutorial, gaming and simulation.

Fifteen years ago, colleagues Moyer et al. (2002) proposed a definition for a virtual manipulative. They defined a virtual manipulative as an "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). The term "interactive" was used in the definition to distinguish tools that users could interact with from those that were simply static images viewed on the screen. The term "Web-based" was used in the definition to distinguish easily accessible tools on the Internet from those that were being commercially produced as computer programs. The term "visual representation" was used in the definition to highlight that a pictorial image had the potential to accurately represent some mathematical idea. The term "dynamic" was used in the definition to focus on the manipulability of the image representation that could be moved by the user. The term "object" was used to refer to the idealized mathematical object, beyond its physical inscription, that the two-dimensional

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image was used to represent (Kirby 2013). The terms “presents opportunities for constructing mathematical knowledge” were used in the definition to distinguish that virtual manipulatives are designed for the purpose of facilitating the opportunity for mathematical learning.

Since this definition was published in 2002 in *Teaching Children Mathematics*, it has been referenced and cited over 280 times (source: Google Scholar), demonstrating its usefulness to the educational and research communities. Because of the widespread use of the term *virtual manipulative* and its definition, a number of questions have arisen as new technologies have been developed that include technology tools with virtual manipulatives. What is and what is not a virtual manipulative? Are all virtual manipulatives “web-based” as described in the 2002 definition? Is a virtual manipulative simply the representation, alone, or does the virtual manipulative include some or all of the features that are designed in the environment around it? What is the relationship between games and virtual manipulatives? What is the difference between virtual manipulatives designed as Java-based apps and the newer touch-screen apps?

At the time of the release of the original definition, Moyer et al. (2002) described virtual manipulatives as “a new class of manipulatives” (p. 372). In the 2002 publication, the authors described virtual manipulatives being manipulated by a computer mouse. Today, virtual manipulatives are presented on computer screens, on touch screens of all sizes (e.g., tablets, phones, white boards), as holographs, and via a variety of different viewing and manipulation devices. The virtual manipulatives on these devices will likely be manipulated by a mouse, stylus, fingers, lasers, and a variety of other manipulation modalities in the years to come. Several collections of virtual manipulatives have been developed over the years including the National Library of Virtual Manipulatives (NLVM) (http://nlvm.usu.edu), National Council of Teachers of Mathematics (NCTM) Illuminations (http://illuminations.nctm.org), and Shodor Interactivate Curriculum Materials (http://shodor.com/curriculum/). There are also new libraries of virtual manipulatives being developed for the touch-screen environment, although to date, there are none as extensive as those developed for the computer.

As new technologies have developed and questions arose in the field, we believed it was time to revisit the definition of a virtual manipulative and to discuss some of the most common environments for the educational setting in which virtual manipulatives appear. The purpose of this chapter is to address questions that have arisen in the field since the publication of the original definition; revisit, clarify and update the definition of a virtual manipulative; and to describe the characteristics of five of the most common virtual manipulative environments in use in education. Describing examples of different environments in which users may find a virtual manipulative allows educators and researchers to have a common language and understanding of these important technology tools for teaching and learning mathematics.

1.1 What Is and What Is Not a Virtual Manipulative?

Moyer et al. (2002) clarified the difference between technology tools that are and are not virtual manipulatives. One of the most important distinctions made in the 2002 publication was that the virtual manipulative user needs to be able to interact with a dynamic object in such a way that these interactions provide opportunities for constructing mathematical knowledge. Therefore, as described in the 2002 article, filling in worksheets on the screen or simply answering questions in the presence of a pictorial object does not fit the definition of a virtual manipulative.

A key defining feature of a virtual manipulative is the difference between static images of the representation and dynamic images of the representation on the screen. The user needs to be able to interact with, move, or manipulate the dynamic mathematical representation in some way that accurately represents a mathematical concept, relationship, procedure, and/or students' thinking about mathematical concepts, relationships, and procedures. This movement could take place using a mouse, stylus, fingers, lasers, and a variety of other manipulation devices yet to be developed (see Fig. 1.1). This interactive feature of the visual representation of the dynamic mathematical object distinguishes a virtual manipulative from other mathematics technology tools.

Child using a mouse to move a virtual manipulative on a computer screen

Child using fingers to move a virtual manipulative on a touch-screen

Fig. 1.1 Users can interact with, move, or manipulate the virtual manipulative using a mouse, fingers, or other interaction modalities

1.2 What Is the History of the Term "Virtual Manipulative"?

In the late 1990s different developers proposed the creation of a new class of manipulatives, which they referred to as digital manipulatives and virtual manipulatives. For example, Resnick et al. (1998) proposed the creation of digital manipulatives. The goal of these digital manipulatives, as described by Resnick and colleagues, was to:

> …embed computational and communications capabilities in traditional children's toys. By using traditional toys as a starting point, we hope to take advantage of children's deep familiarity with (and deep passion for) these objects. At the same time, by endowing these toys with computational and communications capabilities, we hope to highlight a new set of ideas for children to think about. (Resnick et al. 1998, p. 282)

Also, in the late 1990s, colleagues Jim Dorward, Bob Heal, Larry Cannon and Joel Duffin at Utah State University proposed the creation of a library of virtual manipulatives (Dorward and Heal 1999; Heal et al. 2002). They were funded by the National Science Foundation and, in 1999, created the National Library of Virtual Manipulatives (NLVM) (http://nlvm.usu.edu/), a collection of Java-based applets for K-12 mathematics teaching and learning. The NLVM is still in use today and is available in four different languages (Chinese, English, French, and Spanish). Throughout the years, the terms *digital manipulatives* (Manches and O'Malley 2012; Resnick et al. 1998), *computer manipulatives* (Sarama and Clements 2009), and *virtual manipulatives* (Dorward and Heal 1999; Heal et al. 2002) have been used most commonly as synonyms.

1.3 Are All Virtual Manipulatives Web-Based?

Technologic innovations have exploded over the past decade. This innovation has caused virtual manipulatives to appear in a variety of forms beyond the World Wide Web. So perhaps now is the time to amend the original definition, which defined virtual manipulatives as "web-based", and revise the definition to say "technology-enabled". Currently, virtual manipulatives are available through multiple technological means; thus, the term "web-based" no longer encompasses all of the forms of virtual manipulatives that are available. It is also important to recognize the shift from "based" to "enabled". In the future it is very likely that virtual manipulatives will no longer be based in any technology (e.g., they may be projected 3D objects or holographic images). Describing virtual manipulatives as technology-enabled allows for changes in future iterations of these tools.

1.4 Is a Virtual Manipulative Simply the Representation, Alone, or Does the Virtual Manipulative Include Some or All of the Features that Are Designed in the Environment Around It?

Some researchers make a subtle distinction between the visual representation (i.e., the image, the inscription) of a virtual manipulative and the features of the representation, which enable it to be acted upon as a dynamic mathematical object. Because the original definition of a virtual manipulative says "an interactive … visual representation of a dynamic object" some have interpreted this to mean that the virtual manipulative is the inscription of the representation only, while others have interpreted this to mean that the virtual manipulative is the representation including its dynamic and programmable features. In the original definition by Moyer et al. (2002), the intention of the authors was that a virtual manipulative includes the representation and its dynamic and programmable features that allow the user to come to understand it as a representation of the idealized mathematical object (Kirby 2013). The representation portion of the virtual manipulative is only "interactive" and "dynamic" when its programmable features enable capabilities for knowledge construction.

As Kirby (2013) explains, "the properties of the object derive from the relevant definition, not the inscription itself…" (p. 1). For example, in Fig. 1.2, we can see an inscription or representation of an icosahedron. From the idealized mathematical object for an icosahedron, developers created this technology representation. The representation that appears on the computer screen only represents the icosahedron. Yet the representation, because of its limitations and constraints, can never be the idealized mathematical object with all of its properties and relationships. Through an individual's mathematical development, learners begin to understand the properties and relationships of the icosahedron as an idealized mathematical object beyond the representation. This goes beyond the simple images and limited inscriptions that appear in two dimensions on the screen. Most importantly, it is the interactive and dynamic programmable features that allow the user to explore with the representation and develop the concept of the icosahedron beyond its two-dimensional screen inscription. Therefore, in a virtual manipulative, the representation cannot be separated from its interactive and dynamic programmable features.

Further, the potential of the virtual manipulative to provide opportunities for constructing mathematical knowledge is dependent upon the representation's potential to accurately provide an interaction with the mathematics and for the user to be able to perceive the mathematics through this interactivity (Simon 2013). Goldin (2003) describes representation as process and product. Representational systems are both internal (within the individual) and external (outside the individual) and it is the interaction between these two systems that is the key to learning (Goldin and Shteingold 2001).

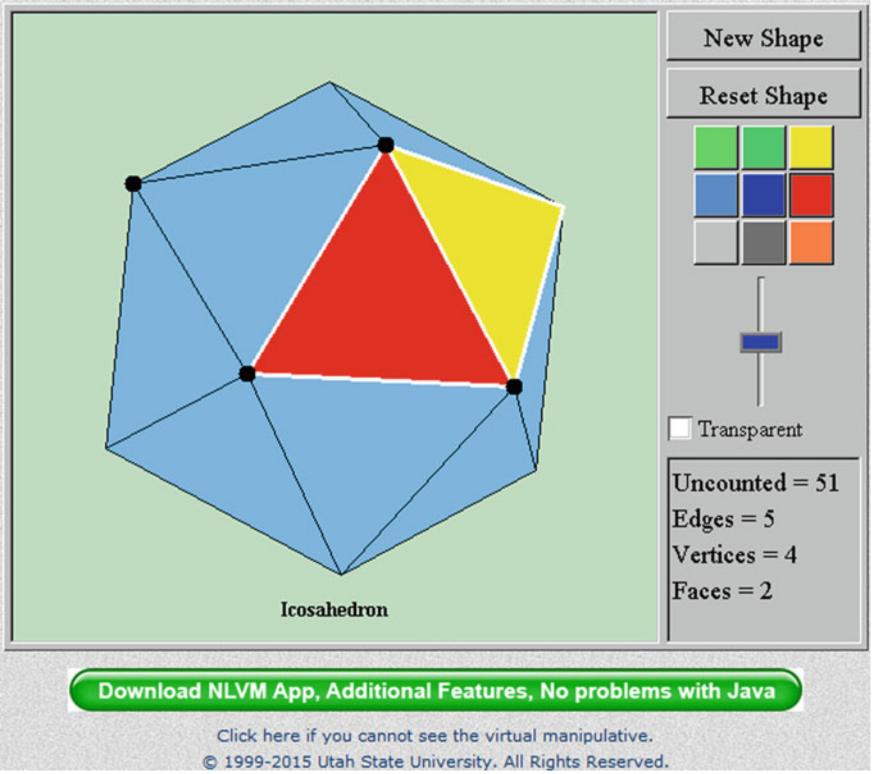


Fig. 1.2 Icosahedron virtual manipulative with marked faces, edges and vertices

Research has shown that the dynamic and interactive features of a virtual manipulative facilitate interactions between representational systems (Moyer-Packenham and Westenskow 2013). The dynamic movements of the visual representations and observation of the resulting outcomes support the structuring of the user's internal representation of the mathematics under study; likewise, the same movements and outcome observations can represent the user's current mathematical thinking, allowing the user to test and refine ideas.

Opportunities for constructing mathematical knowledge consist of more than the visual representation. The use of a virtual manipulative has maximum potential to support learning by behaving in a way that represents the idealized mathematical object when manipulated by the user and by accurately representing the user's mathematical thinking. Consequently, the manipulative representation alone, is not the virtual manipulative. It is the interactive and dynamic capabilities of the manipulative representation that makes it a virtual manipulative. Therefore, the programmable features of the application that support its interactivity are part of the virtual manipulative. The features that allow the representation to be manipulated, to be interactive, and to be dynamic are an inherent part of the virtual manipulative. Without these features, it is simply a static inscription.

To clarify the original definition, it could be amended to say a "representation of a dynamic object, including all of the programmable features that allow it to be manipulated; or that allow it to be dynamic; or that allow it to be interactive". For example, in Fig. 1.2 which shows the three-dimensional representation of an icosahedron, the features of the app that allow the user to change the color of the faces, mark the vertices with black dots, mark the edges with white lines, move the slider to change the object's size and change the solid to a transparent view are all part of the interactivity and manipulability of the virtual manipulative that can be acted upon by the user to draw attention to or highlight the relevant properties of the solid. In addition, using a mouse to click on and drag the icosahedron or using fingers to swipe the icosahedron allows the user to move and rotate it.

All of these actions take the user beyond the simple representation of the object to a greater understanding of the properties and relationships imposed by the definitions and theorems of the idealized icosahedron. Therefore, the virtual manipulative is not simply the visual representation of the icosahedron, the virtual manipulative is the visual representation of the icosahedron and all of the programmable features surrounding it that allow it to be dynamic, interactive and manipulated by the user to explore and observe its properties. These programmable features allow it to be manipulated and are an inherent part of it being classified as a virtual manipulative. Without these programmable features, the icosahedron is simply a visual/pictorial representation on a computer screen. With these programmable features, it is a virtual manipulative because it is an interactive and dynamic representation that can be manipulated.

1.5 What Is the Relationship Between Games and Virtual Manipulatives?

There are some virtual manipulatives that are embedded within gaming environments. When virtual manipulatives are embedded within a gaming environment, the environment is designed to host the virtual manipulative with its dynamic features. Some gaming environments are very basic, while other gaming environments can be highly developed and multi-layered. The game may have increasing levels, points, goals, timers, and other elements of game design (Deterding et al. 2011). Therefore, the entire gaming environment and everything in it is not a virtual manipulative, but there are often virtual manipulatives embedded in gaming environments. This could be the result of a designer taking a virtual manipulative and gamifying it to make it more appealing to learners.

Deterding et al. (2011) define *gamification* as "the use of game design elements in non-game contexts" (p. 10). For example, in the Motion Math Zoom app, a virtual manipulative is housed in a gaming environment (see Fig. 1.3, Zoom app).

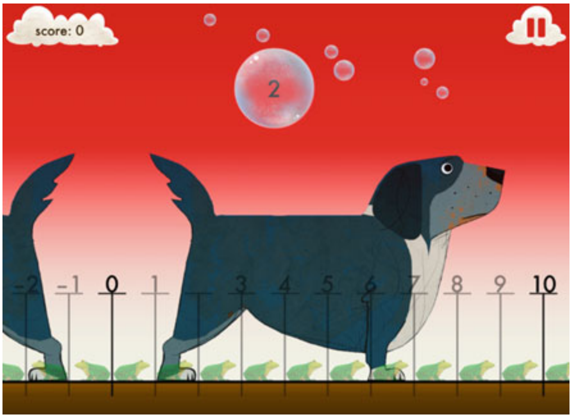


Fig. 1.3 Motion math zoom game app

The virtual manipulative is the dynamic number line that can be expanded, contracted and swiped by the user. This dynamic number line is placed inside a gaming environment where there are levels for the user to achieve using the virtual manipulative number line.

The gaming environment in which the virtual manipulative number line is housed could be changed; however, the dynamic number line remains the virtual manipulative for the learner to manipulate. For example, the virtual manipulative number line that is used in the Motion Math Zoom app could be placed in a different environment where the user is not playing a game. The environment could have number line tasks for the user to complete. Therefore, the relationship between games and virtual manipulatives is that virtual manipulatives are sometimes embedded in gaming environments.

1.6 What Is the Difference Between Virtual Manipulatives Designed as Java-Based Apps and the Newer Touch-Screen Apps?

Virtual manipulatives have been developed over the years in a variety of different formats from Java- and Flash-based applications, largely for Windows computers and Android devices to Swift-based applications for Apple iOS products (e.g., iPads). Whether these dynamic objects are Java-based, Swift-based, or developed using a host of available programming languages and tools, they are still virtual manipulatives. The programming language or tool used to develop the virtual manipulative or the platform through which it is delivered does not change the essence of the virtual manipulative. As long as the product that is created is a dynamic representation of a mathematical object, having the characteristics of interactivity and manipulability that presents opportunities for constructing mathematical knowledge, it is a virtual manipulative. New programming languages may allow new and different capabilities, but these capabilities simply allow the virtual manipulative to have different kinds of interactivity and manipulability.

1.7 How Is the Term "Virtual Manipulative" Confused with Other Technology Terminology?

Over the years, there have been subtle, yet important, distinctions made in the literature among the terminology used to describe technologies for mathematics teaching and learning. Some of the terminology related to *virtual manipulatives* includes: *cognitive technology tools* (Pea 1985), *learning objects* (Kay 2012), *virtual math objects* (Bos 2009b), and *computer-based mathematical cognitive tools* (Sedig and Liang 2006). This similar terminology has led to confusion about virtual manipulatives. Some publications have used terminology other than the term *virtual manipulative* to refer to technologies that actually fit the definition of a virtual manipulative; conversely, the term virtual manipulative has been used to refer to technologies that do not fit the definition of a virtual manipulative.

Using a term other than virtual manipulative to refer to a virtual manipulative in a research study makes it challenging for researchers to determine what mathematics technologies were actually used in the study, to identify if the tools investigated meet the definition of a virtual manipulative, and to conduct rigorous evaluations and meta-analyses (Moyer-Packenham and Westenskow 2013) that summarize the effects of virtual manipulatives on student achievement and learning. When a term other than virtual manipulative is used in a research publication, it is unclear if the authors are simply using another term when they actually mean virtual manipulative, or if the authors are actually referring to something different than a virtual manipulative. These distinctions among terminology warrant some clarification.

Pea (1985) defined *cognitive technology tools* as "any medium that helps transcend the limitations of the mind, such as memory, in activities of thinking, learning, and problem solving" (p. 168). Because cognitive technology tools include the broad class of "any medium," we consider virtual manipulatives as a sub-category of the term cognitive technology tools because there are also many other types of medium that can be considered cognitive technology tools. Therefore, cognitive technology tools and virtual manipulatives are not synonymous.

Kay (2012) defines *learning objects* as "interactive Web-based tools that support the learning of specific concepts by enhancing, amplifying, and/or guiding cognitive processes of learners" (p. 351). Kay (2012) gives two examples of learning objects in his study: "adding integers with virtual colored tiles" and "three-dimensional objects transform to two-dimensional nets in order to examine surface area" (p. 351). Based on Kay's definition of a learning object, virtual manipulatives would be considered learning objects because the examples of the learning objects he describes in his study fit the definition of a virtual manipulative. However, if learning objects include other tools, beyond those described in the study that do not fit the definition of a virtual manipulative, then learning objects and virtual manipulatives are not synonymous.

Bos (2009b) writes about *virtual math objects*: “A math object enhanced with technology offers manipulations, multiple representations, multiple entry points, and provides opportunity to test, revisit, revise, and apply mathematical patterns” (p. 522). “The *math object* uses multiple representations that are interactive and change with the given input” (Bos 2009a, p. 110). Given this description, virtual manipulatives may be the same as *virtual math objects* or one type of *math object* because virtual manipulatives contain “multiple representations that are interactive and change with the given input.” Although Bos (2009b) wrote, “Virtual manipulatives…are often mistaken as math objects…” (p. 522), the description of virtual math objects in these publications implies that virtual math objects and virtual manipulatives may be synonymous.

Sedig and Liang (2006) describe *computer-based mathematical cognitive tools* (MCTs) as “a category of external aids intended to support and enhance learning and cognitive processes of learners. MCTs often contain interactive visual mathematical representations…” (p. 179). Sedig and Liang (2006) go on to describe these visual mathematical representations as “graphical representations that encode causal, functional, structural, logical, and semantic properties and relationships of mathematical structures, objects, concepts, problems, patterns, and ideas” (p. 180). Based on these definitions, virtual manipulatives are a subcategory of computer-based mathematical cognitive tools because there are some tools that would be considered computer-based mathematical cognitive tools but that would not fit the definition of a virtual manipulative.

An additional source of confusion comes from the science literature in which virtual science materials are sometimes referred to as virtual manipulatives. In some studies, the research uses the terms *physical and virtual manipulatives* and *physical and virtual material* interchangeably. For example, Triona and Klahr (2003) compared the effectiveness of two instructional conditions, which they called the “physical, manipulable materials” condition and the “virtual, computer-based materials” condition (p. 152). Olympiou and Zacharia (2012) compared the effectiveness of three instructional conditions which they called experimenting with physical manipulatives (PM), with virtual manipulatives (VM), and with a blended combination of PM and VM, to determine students’ understanding of concepts in the domain of Light and Color. Zacharia and deJong (2014) compared the effectiveness of five instructional conditions that included “virtual material” and a “Virtual Labs Electricity environment” in which students manipulated “virtual objects and virtual instruments” to develop an understanding of electric circuits (p. 112). In another comparison study, Lazonder and Ehrenhard (2013) compared the effectiveness of physical and virtual manipulatives in an inquiry task about falling objects. Just like the mathematics literature, it is unclear how closely aligned the “virtual manipulatives” being used in these science studies are with the 2002 definition of virtual manipulatives for mathematics. It may be important for the science education community to define virtual manipulatives and virtual materials in the context of science.

1.8 An Updated Definition for Virtual Manipulatives

As these questions posed over the past decade show, there is a need for greater clarification of the definition of a virtual manipulative. Based on the discussion in the preceding sections, which included proposed revisions, here we suggest an updated definition of a virtual manipulative: *an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge*. This updated definition preserves the term "interactive" in the definition because this is a defining characteristic of a virtual manipulative. The updated definition takes into account that all virtual manipulatives do not have to be "web-based", and replaces this terminology with the term "technology-enabled". The updated definition also preserves the terms "visual representation of a dynamic object" and adds the term "mathematical" to clarify that we are referring to a representation of a mathematical object.

The updated definition clarifies that the visual representation of a dynamic object is accompanied by all of its programmable features, because without these features it would not be interactive and dynamic. Implied in this updated definition is that a virtual manipulative may: (a) appear in many different technology-enabled environments; (b) be created in any programming language; and (c) be delivered via any technology-enabled device.

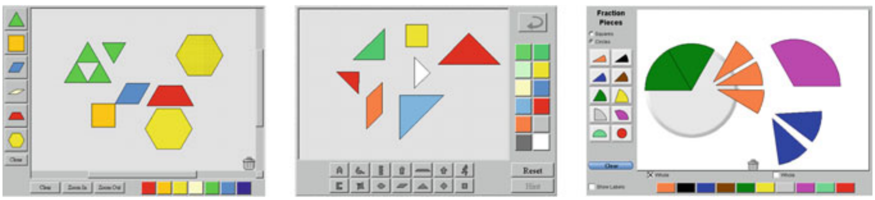
1.9 Common Virtual Manipulative Environments

One source of confusion about what is and what is not a virtual manipulative has been that virtual manipulatives have been designed to be housed in various technological environments. Other authors have outlined categories of computer-based learning technologies for mathematics education. For example, Handal and Herrington (2003) reported that there are six categories of computer-based learning in mathematics and these include: drills, tutorials, games, simulations, hypermedia, and tools. Kurz et al. (2005) reported that there are five categories of tool-based mathematics software and these include: review and practice, general, specific, environment, and communication. Although there are some commonalities between these categories and virtual manipulative environments, the categories are not specific to virtual manipulatives. In an NCTM conference presentation, Bolyard and Moyer (2007) discussed four virtual manipulative environments. However, there has been no publication that has described these environments.

This section of the chapter seeks to put that discussion into print by describing the common environments in which virtual manipulatives frequently appear. Currently, there are five common virtual manipulative environments that have been used by developers. These environments include: single-representation, multi-representation, tutorial, gaming and simulation. While other environments may exist and new environments may be developed, these five environments have stood the test of time and can be found most commonly among the virtual manipulatives currently available to users.

The single-representation virtual manipulative environment. The single-representation virtual manipulative environment contains an interactive pictorial/visual representation (i.e., image) of the dynamic mathematical object and is not accompanied by any numerical or text information. Bolyard and Moyer (2007) referred to this as "pictorial-only" in their NCTM presentation. The single-representation environment typically relies on only one type of representation of the mathematics and, most commonly, that single representation is a pictorial image. In some cases, the pictorial image is based on a physical manipulative, and in some cases the virtual manipulative image has no physical counterpart. Some publications mistake this notion, which implies that all virtual manipulatives are patterned after physical manipulatives: "Virtual manipulatives are screen-based instantiations of physical manipulatives…" (Manches and O'Malley 2012, p. 406).

Three examples of the single-representation environment are the Pattern Blocks, the Tangrams, and the Fraction Pieces found at the National Library of Virtual Manipulatives (NLVM; nlvm.usu.edu) website (see Fig. 1.4). The virtual manipulative pattern blocks contain six different geometric shapes that users can move and alter (e.g., change color, change location, and change the orientation). The tangrams also contain several different geometric shapes that users can move and alter (e.g., change color, change location, and change the orientation). The fraction pieces contain different fractional portions of a circle region that users can move and compare with the whole. In the single-representation environment, the pictorial image is the predominant representation, with limited information provided in numerical or text form. As can be seen in Fig. 1.4, this environment simply includes the pictorial representation of the objects for the user to manipulate along with all of the accompanying programmable features.



Pattern Blocks Tangrams Fraction Pieces

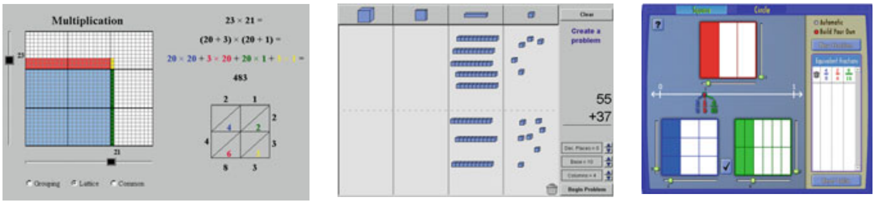
Fig. 1.4 Examples of the single-representation virtual manipulatives environment found at the nlvm.usu.edu

The single-representation environment requires the teacher to design specific tasks for learners that will help draw their attention to the mathematical ideas under study. However, this environment also allows the teacher more flexibility with the tools to design specific tasks that meet the needs and goals of the curriculum. Because of its open-ended nature, the single-representation environment can easily be used as the basis for independent practice activities (Wight and Kitchenham 2015). Anderson-Pence (2014) reported that, because the single-representation environment relies only on pictorial images, this environment is more versatile for use in teaching because the pictorial images can be used for many different types of mathematical explorations.

The single-representation environment also places responsibility on the student for attending to and making sense of connections between the pictorial representations and numeric representations of the mathematics, because the numeric representations do not appear simultaneously with the pictorial images, as is the case in other virtual manipulative environments. Anderson-Pence (2014) reported that, when student pairs worked with the single-representation environment (which she called "pictorial"), they had the largest amount of discussion and the highest use of gestures (both physical gestures and computer-based gestures). However, these discussions were not at a high level that would lead to mathematical generalizations.

Other reports on the single-representation environment have noted that this environment leads to more creative variation during problem solving (Moyer-Packenham and Westenskow 2013). For example, Moyer et al. (2005) reported that children using the virtual manipulative pattern blocks (a single-representation environment) exhibited more creative behaviors with the blocks. Because this environment contains only visual images, students working in pairs must put forth more effort in communicating how to manipulate the objects, how to solve problems, and what mathematics these activities represent.

The multi-representation virtual manipulative environment. The multi-representation virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object and is accompanied by numerical and, sometimes, text information. Therefore, the multi-representation environment typically relies on two or more forms of representations, and these are often pictorial and numeric representations. Bolyard and Moyer (2007) referred to this as "combined pictorial and numeric" in their NCTM presentation. Three examples of the multi-representation environment are the Rectangle Multiplication of Fractions and Base Blocks Addition found at the NLVM and Equivalent Fractions found at the NCTM Illuminations website (nctm.org; see Fig. 1.5). The Rectangle Multiplication of Fractions app shows a pictorial image of a grid with numerical information to accompany the visual changes in the amounts in the grid. The Base Blocks Addition app shows a pictorial image of base-10 blocks with numerical information that represents the changing amounts displayed by the blocks. The Equivalent Fractions app shows a pictorial image of three rectangular regions that can be divided and shaded to show fraction amounts that are displayed on a number line and recorded in a table for the user.



Rectangle Multiplication of Fractions | Base Blocks Addition | Equivalent Fractions

Fig. 1.5 Examples of the multi-representation virtual manipulatives environment found at the nlvm.usu.edu and nctm.org

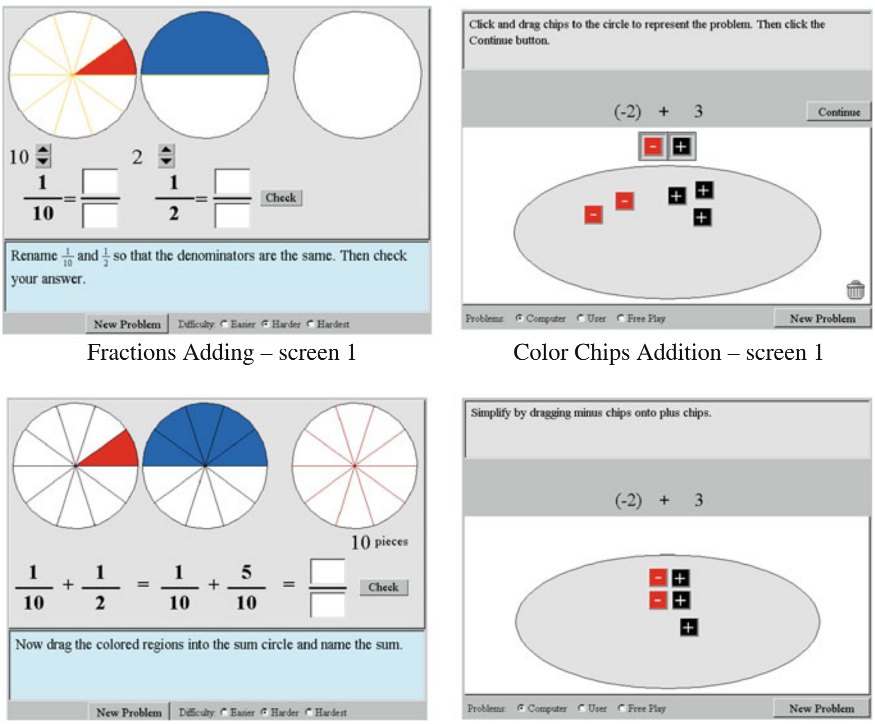
In each of these applications, the environment contains multiple representations and the pictorial images are commonly linked simultaneously with the numeric information. As the user interacts with the pictorial images, the numeric information provides an abstract model that accompanies the images. The presentation of two or more different representations (e.g., pictorial, numeric, text) simultaneously enables the user to link images with abstractions in numeric mathematical form. As can be seen in Fig. 1.5, the multi-representation environment often contains primarily pictorial representations and numerical representations in a linked form along with all of the accompanying programmable features.

For many years, researchers have recognized the importance of linking features in computational media to promote *representational fluency* and learners' ability to see relationships among representations (Kaput 1986). Sarama and Clements (2009) describe this as "linking the concrete and the symbolic with feedback" (p. 147). A meta-analysis of the research on virtual manipulatives shows that simultaneous linking of representations has positive impacts on students' mathematics achievement (Moyer-Packenham and Westenskow 2013). For example, Suh and Moyer (2007) reported that their students observed the links between the algebra symbols and the movement of a balance scale. Haistings (2009) reported that her students preferred the linked pictorial/symbolic apps because the mathematical information appeared for them on the screen and they did not have to remember or recount the blocks during problem solving. Additionally, the numbers changed as they performed actions with the blocks allowing them to see the result of their actions.

Viewing numeric and pictorial information that changes simultaneously allows the user to adapt and reinterpret the representations (Martin and Schwartz 2005). Anderson-Pence (2014) reported that, when students worked in pairs using the multi-representation environment (which she referred to as "combined"), students' discussions reflected higher levels of mathematical generalization, justification, and collaboration. The multiple representations encouraged students to make connections, make comparisons among the representations, and see patterns more easily. A similar finding was also reported by Ares et al. (2008), who noted that interacting with multiple representations promoted mathematical discourse among students.

The tutorial virtual manipulative environment. The tutorial virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object and is accompanied by numerical and text information in a format that guides the user through a tutorial of the mathematical procedures and processes being presented. Therefore, the tutorial environment provides a guiding and tutoring support structure for the user and relies on multiple forms of representation—pictorial, numeric, and text. The guiding and tutoring features are what make the tutorial environment different from the multi-representation environment.

Two examples of the tutorial environment are Fractions Adding and Color Chips Addition found at the National Library of Virtual Manipulatives (see Fig. 1.6). The Fractions Adding app presents the user with two fractions that have unlike denominators. The prompt in the tutorial guides the user to rename the two fractions so that they have a denominator that is common to both fractions. As students use the arrow button to change the number of pieces of each fraction, they can see how the total number of pieces changes on each fraction region until they find divisions of the regions that are common. Once the common denominator is found, students are prompted to rename the two fractions and check to see if their answer is correct.



Fractions Adding – screen 1 Color Chips Addition – screen 1

Fractions Adding – screen 2 Color Chips Addition – screen 2

Fig. 1.6 Examples of the tutorial virtual manipulatives environment found at the nlvm.usu.edu

When they have created correct common denominators, they continue to the next screen and are guided to add the renamed fractions by dragging the fraction pieces into a sum region. When students type the answer in symbolic form that represents the pictorial image they have created, they receive feedback that tells them if their response is correct or that guides them to make an adjustment to their answer if it is incorrect.

The Color Chips Addition app presents the user with a numeric expression and prompts the user to use the positive and negative chips to build the expression. Students continue to the next screen where they are prompted to simplify the expression and type in a solution. The tutorial environment generally follows this format of guiding and tutoring students to understand a process in a step-by-step manner. As can be seen in Fig. 1.6, this environment can include multiple steps that guide students through a process or procedure using a variety of representations.

Anderson-Pence (2014) reported that the tutorial environment is better suited to students working individually because the tutorial essentially serves as an individual tutor that walks students through the steps of solving a problem or learning a mathematical procedure. This environment discourages communication among student pairs because of the step-by-step format that allows little exploration or deviation from the tutoring process.

While this environment is not as useful for students working in pairs, the tutorial environment has been shown to have significant positive effects in classroom studies where students were working individually at their own computers (Reimer and Moyer 2005; Steen et al. 2006; Suh and Moyer 2007). For example, in one study with low, average and high achievement groups, researchers reported that the low achievers benefited from the treatment because of the step-by-step presentation format in the tutorial environment. Researchers stated: "The low achieving group used a step-by-step methodical process to find multiples and common denominators…" (Moyer-Packenham and Suh 2012, p. 53). The step-by-step tutorial environment led the low achieving group through this process to successfully complete the mathematical procedures.

The gaming virtual manipulative environment. The gaming virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object that is embedded in a format that allows the user to play a game with the objective to reach goals that are reflected in the game play. Therefore, the gaming environment relies on multiple forms of representation embedded in an environment with a variety of gaming features that might include levels, badges, time constraints, clear goals, challenge and play-centric design (Deterding et al. 2011).

Three examples of the gaming environment are Motion Math Zoom, Dragon Box Algebra, and Hungry Guppy found on the Apple iTunes store (see Fig. 1.7). The Motion Math Zoom app is an interactive number line that users can swipe left and right to view higher numbers and lower numbers on the number line, respectively. To quickly move from ones to tens to hundreds to thousands, users employ a

Motion Math Zoom   Dragon Box Algebra   Hungry Guppy

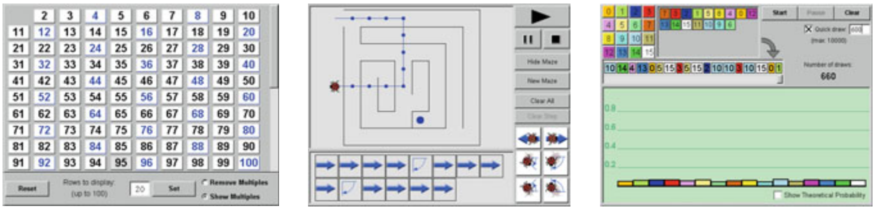
Fig. 1.7 Examples of the gaming virtual manipulatives environment

two-finger pinching and stretching motion to “zoom in and out” on the number line. In the game, numbers appear in bubbles above the number line. The user must move the number line to the correct location so that it is below the number in the bubble and then pop the bubble so that the number lands at the correct placement on the number line. The game has 24 levels, with multiple tasks in each level, that increase in difficulty. There is a needle that can be turned on or off that acts as a timer to encourage the user to become increasingly more efficient at identifying where the numbers go on the interactive number line.

The Dragon Box Algebra app engages the user with operations, additive and multiplicative thinking, solving expressions and equations, and fractions. The game has ten 20-level chapters where the user moves game pieces to solve expressions or equations to complete the game levels. The Hungry Guppy app requires the user to combine bubbles of different numbered dots to create a target number and feed the hungry fish. When the correct number of dots is fed to the fish, the fish gets larger and the user completes the level. As can be seen in Fig. 1.7, the gaming environment typically has multiple representations and a more developed background design and visual images that enhance the appearance of the app when compared with the other virtual manipulative environments.

Tucker (2015) reported that a user’s mathematical and technological distance (with *distance* defined as the “degree of difficulty in understanding how to act upon [something] and interpret its responses” (Sedig and Liang 2006, p. 184)) changed as they interacted with the Zoom app. Other studies have reported that virtual manipulatives in gaming environments can have positive effects on the development of mathematics learning (Carr 2012). For example, Barendregt et al. (2012) reported that, when five- and six-year-old children played the Fingu game during a three-week period, it supported the development of their subitizing and arithmetic skills. Riconscente’s (2013) research using the Motion Math Fractions game for the iPad with 122 fourth-grade students showed that when the students played the game for 20 min daily for a 5-day period, there was a 15 % improvement in students’ fraction test scores.

The simulation virtual manipulative environment. The simulation virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object along with other representations (e.g., numeric,



Sieve of Eratosthenes Lady Bug Maze Box Model

Fig. 1.8 Examples of the simulation virtual manipulatives environment

text) that are embedded in a format that allows the user to run a simulation intended to represent or draw attention to embedded mathematics concepts. Therefore, the simulation environment may rely on one or multiple forms of representation that can be used to run the simulation. Three examples of the simulation environment are the Sieve of Eratosthenes, Lady Bug Maze, and the Box Model found at the National Library of Virtual Manipulatives (see Fig. 1.8).

The Sieve of Eratosthenes app allows users to run a simulation showing the multiples of the numbers on a number board. Running the simulation of each successive number on the board (e.g., the multiples of 2, 3, 4, 5, etc.) reveals patterns in the multiples and helps users to identify the prime numbers on the number board. The Lady Bug Maze allows the user to create a program for the path of a lady bug in order to help the lady bug reach a point within the maze. Each time the user creates and modifies the program, there is a “play” button that allows the user to run the simulation to see if the programing commands that they have created allow the lady bug to successfully navigate the maze. By repeatedly running the simulation, the user can make adjustments to their programing commands until the lady bug is successful.

The Box Model app simulates multiple random draws of numbers from a box and plots the numbers on a chart comparing actual probability to theoretical probability. The simulation environment allows the user to efficiently perform and model multiple trials over and over again. Clements et al. (2001) research with a virtual manipulative in the simulation environment used Logo Geometry (which has a similar design to the Lady Bug Maze pictured in Fig. 1.8) to simulate geometric shapes, paths and motions. In a study of 1624 Kindergarten through 6th grade students, those who used the Logo Geometry curriculum made significant gains, which were almost double the gains of those students who participated in traditional geometry instruction. This study of the simulation virtual manipulative environment showed that Logo Geometry helped students link symbolic and visual representations, demanded greater precision in geometric thinking from students, and encouraged students to make and test geometric conjectures.

1.10 Concluding Remarks

This chapter provided an update to the definition of a virtual manipulative. This new definition reflects attention to technology developments and clarification about what is and is not included in the technology for it to be defined as a virtual manipulative. The chapter also described five different environments in which virtual manipulatives are commonly embedded and provided examples of each to show the structure of the most common designs of virtual manipulative environments. As these examples demonstrate, there are a variety of virtual manipulative environments currently in use today. This updated definition and the descriptions of the five environments provide guidance for educators and researchers on a common language and understanding of the meaning of a virtual manipulative for teaching and learning mathematics.

The potential of virtual manipulatives to support students' developing mathematical ideas relies on judicious, appropriate, and effective use. Learners must experience the virtual manipulative and interact with its characteristics and features in ways that represent the relevant mathematics. Virtual manipulatives are technologies, and like any technology, virtual manipulatives do not create learning; rather, it is the quality of the engagement with the technology that presents opportunities for learning mathematics.

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Chapter 2
Artifact-Centric Activity Theory—A Framework for the Analysis of the Design and Use of Virtual Manipulatives

Silke Ladel and Ulrich Kortenkamp

Abstract It is a challenge to analyze the design and the use of Virtual Manipulatives due to their high complexity. As it is possible to create entirely new virtual worlds that can host objects that behave differently than any real objects, allowing for new and unprecedented actions in learning processes, we are in need of tools that enable us to focus on those aspects that are important for our analyses. In this chapter we show how ACAT, Artifact-Centric Activity Theory, can be used to analyze the design and use of a virtual manipulative place value chart.

2.1 Introduction

Manipulatives play a very important role in learning mathematics. Operations with manipulatives are the basis for further mathematical learning processes. Operations are purposeful and understood in their internal structure (Aebli 1983, p. 182) and it is essential that the student is aware of the relations of the objects. In this regard, manipulatives do not necessarily have to be of a physical nature but can be virtual as well (Clements 1999, p. 47; Ladel 2013, p. 59). This fact creates a lot of potential for the technological support of mathematical learning processes. However, the use and benefit of virtual manipulatives[1] is very complex as there are many influencing

[1]We follow the definition of virtual manipulatives by Moyer et al. (2002), but extend it to the category of "apps" on mobile devices (which was unforeseen at the time), that can host virtual manipulatives in a similar way as web sites do.

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factors: the student with prior knowledge; the teacher with mathematical, didactical, educational and media competence; the class; the environment; etc.

Furthermore, mathematical, didactical and design principles have to be considered when designing, analyzing or even selecting a virtual manipulative for use in the classroom. The complexity of the situation and the number of influencing factors require instruments that help analyze the beneficial design and use of virtual manipulatives. In recent years, Human-Computer-Interaction (HCI) research and practice has identified Activity Theory as a helpful theoretical framework, since it is crucial to understand human activity for the design and analysis of technology: "*Understanding and designing technology in the context of purposeful, meaningful activities is now a central concern of HCI research and practice*" (Kaptelinin 2014, foreword). With regard to our requirements in the overlapping context of school, we adapted Activity Theory and developed it further to the Artifact-Centric Activity Theory (ACAT). In this chapter, we will present the theoretical framework of ACAT as an instrument for the design and analysis of virtual manipulatives. Furthermore we will illustrate its application with a virtual manipulative place value chart (Kortenkamp 2015).

2.2 Theoretical Framework

In the following sections, we recall several theories and notions that have influenced ACAT.

2.2.1 Activity Theory

Activity as the purposeful, transformative and developing interaction between *subject* and *object* is the key concept of Activity Theory. It originates from the socio-cultural tradition in Russian psychology and was developed by Leontiev (1978).

In today's knowledge society, *learning* is a defining feature of our society. It is not only the acquisition of knowledge and skills but also the responsibility of the individual to learn and to strengthen the autonomy of each person (Giest and Lompscher 2004). A learning culture must enable people to shape their own educational biography and to take responsibility for their educational processes. Thus, high learning motivation, as well as a positive attitude towards lifelong learning, is very important. Therefore it is an essential condition that learners are the subjects of their learning and educational processes.

The world around us is structured and comprised of *objects*—objectively existing matter. Objects exist independent of the observing human and have 'objective meanings'. However, there is a second aspect of each object, the image of the object as a product of psychological reflection realized as an activity of the subject. This psychological reflection does not have to be consistent with the objective meaning. There is, for example, an objective meaning of 'addition' that is socially and culturally defined and can be described as $n + m = (n + m)$. But there are different basic concepts ("*Grundvorstellungen*" as described by Vom Hofe 1995) of addition (e.g., addition as the union of two amounts or as adding an amount to an existing one). The child may also have a concept of addition as 'counting on'. Thus the meaning attributed to an object can differ depending on the perception of the individual.

Subjects in activity theory have needs. In order to meet those needs and to interact with objects, the subjects have to carry out *activities*. An activity is the process of relating the subject to the object (Fig. 2.1). The attributes of the subject and the object influence these activities. Prior knowledge, for example the student's abilities and skills, influences the student's actions and the way the student solves a mathematical task. The student's experience while solving a mathematical problem, on the other hand, influences the student's abilities.

2.2.2 *Instrumental Act*

Vygotsky (1997, p. 87) focused on the use of tools as the key characteristic of human mental activity. He characterized the process that combines subject, tool and object as 'the instrumental act' (Van Oers et al. 2008). Even though his theoretical focus was primarily on mental tools such as language, his theory can also be adapted to technical devices and, in our case, especially to virtual manipulatives. The tool or the *artifact* mediates between the subject and the object (Fig. 2.2).

Vygotsky's cultural-historical psychology influenced Leontiev's framework fundamentally, in particular, the mediation of the tool between subject and object.

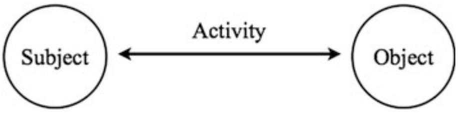

Fig. 2.1 Subject—Object—Activity

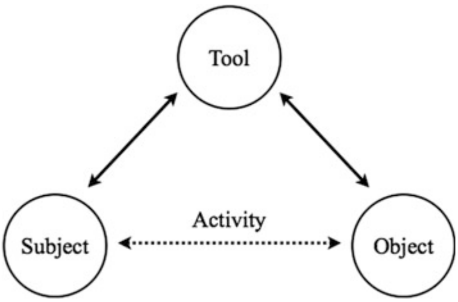

Fig. 2.2 Instrumental act

People encode their experience in the structural properties of tools, as well as their knowledge of how the tool should be used. In that way tools always reflect the previous experience of other people. In other words, virtual manipulatives always contain and reflect the experience and the knowledge of the programmer or designer. Therefore, we have to design virtual manipulatives very carefully and pay attention to the knowledge that we encode in the design of the virtual manipulatives. Similarly, we must also carefully select virtual manipulatives for instruction in mathematics and consider the knowledge we want students to construct.

Through the use of tools, processes of externalization and internalization emerge both for the subject and object. The internal and external components of an activity are mutually transformative: during the process of internalization, external components become internal; during the process of externalization, internal components of an activity become external. Higher physical functions, in the sense of Vygotsky, always arise in social interaction and communication in common activity. Phenomena that were previously interpsychological, become intrapsychological. This means, for example, that conventions used to communicate about mathematics can be learned by individuals and create an internal representation that can be used for further reflection and the creation of new knowledge.

2.2.3 Instrumental Genesis

Instrumentation theory (Rabardel 2002) distinguishes between *artifact* and *instrument*. An instrument is "more" than an artifact. While an artifact is the object that is used as a tool, it only becomes an instrument if the relationship that exists between the artifact and the user is meaningful for a specific type of task (Verillion and Rabardel 1995). The process of an artifact becoming an instrument is called *instrumental genesis*. The instrument also includes the techniques and mental schemes that the user develops and applies when using the artifact. While the artifact is just the material, the instrument involves artifact-type components as well as schematic components, the *utilization schemes*. In that way, an instrument is strictly related to the *context*.

We will illustrate this with a place value chart as an instrument. In the context of checking if a number can be divided by 9, we can just add all digits of the number, not considering their place and check whether the sum is divisible by 9. This is correct because we know that the value of a number changes by multiples of nine when digits change the place (e.g., when we move one counter in the place value chart from the hundreds to the tens, the value of the number changes by −90). However, in another context, for example the context of written algorithms, the change of place should not change the value but demands for (re-) bundling and unbundling (e.g., when there are 15 ones, we bundle them to 1 ten and 5 ones).

Both operations are carried out with the same (physical) artifact, but only the utilization scheme that is a component of the instrument tells us how to work with it. In both cases, the artifact is used as an instrument, but with different utilization schemes.

The so-called *instrumental orchestration*, introduced by Trouche (2004), describes the management of the individual instruments in the collective learning process by the teacher and it has been applied to digital media successfully. "*An* instrumental orchestration *is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a—in this case computerised—learning environment in a given mathematical task situation, in order to guide students' instrumental genesis* (Trouche 2004)" (Drijvers et al. 2011, p. 1350). According to Giest and Lompscher (2004), the subject is, however, not only a single student, but a number of students as individuals embedded into social structures. The activity of the individual is subject to conditions on interaction, communication and cooperation, and these conditions also hold for the relations between teachers and learners as well as further participants. Giest and Lompscher refer to a *pedagogical collective subject* that is acting during instruction.

Children have to elaborate instruments in the process of instrumental genesis to use an artifact to accomplish a particular task. However, besides this semiotic link between the artifact and a task, there is another semiotic link between the artifact and its mathematical meanings that emerges from the epistemological analysis made by teachers and experts. Rabardel (2002) speaks in this context of a two-fold entity—artifactual and psychological. In that way, "*the artifact is not a mediator of mathematical meanings per se […]. It becomes a mediator when used in a teaching-learning situation*" (Bartolini 2011, p. 97).

2.2.4 Task-Artifact Cycle

The inclusion of an artifact changes the activity between the subject and the object. First of all, it sets to work a number of new functions that are connected to the use and control of the given tool. As the artifact undertakes some tasks, it also abolishes a number of previously necessary processes. Thirdly, the artifact modifies various aspects (e.g., intensity, duration, etc.) of all mental processes as it replaces some functions with others (Van Oers et al. 2008). Considering these modifications of the activity we also have to focus on the development of tasks. The task-artifact cycle (Carroll et al. 1991) captures the idea that tasks and artifacts coevolve (see Fig. 2.3).

To perform a given task, an artifact needs to meet certain requirements. The artifact that has been designed for the task, in turn, creates new or unexpected possibilities. It poses new constraints on the performance of the tasks that may suggest a revision of the original task for which the artifact was made. This creates an iterative process of continuous development between the task and the artifact.

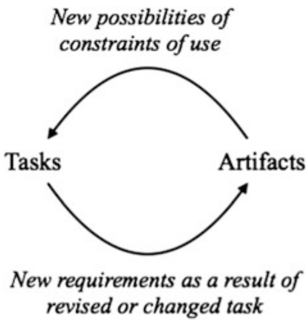

Fig. 2.3 Task-artifact cycle

2.2.5 *Artifact-Centric Activity Theory*

Considering this background, the design and the analysis of manipulatives are necessarily very complex. This applies even more to the design and analysis of virtual manipulatives because they allow for many more affordances (in the HCI sense, that is *action possibilities*) than their physical counterparts. In a way, virtual manipulatives even allow for *miraculous mathematical transformations*, that is, they can make things happen that would not be possible in the physical world. That is why there is a need for an instrument that helps when analyzing and designing virtual manipulatives. This is the purpose of the development of ACAT (Fig. 2.4).

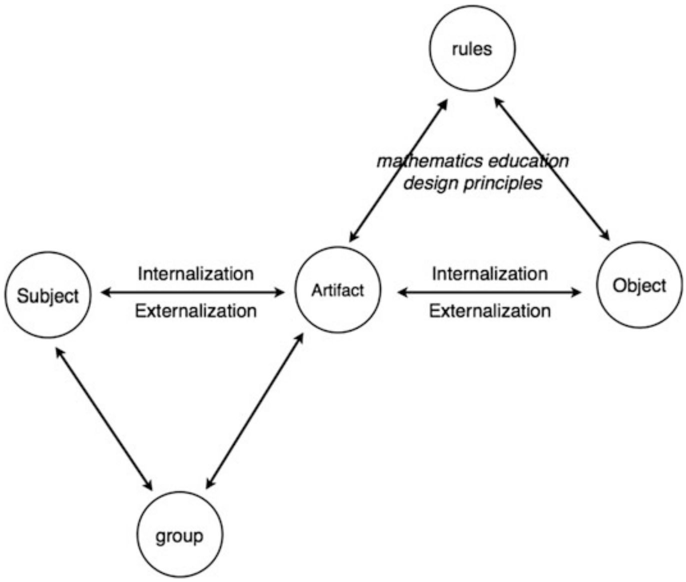

Fig. 2.4 The Artifact-Centric Activity Theory

As the artifact is being developed into an instrument through instrumental genesis, some may argue that we could use the terminology of Instrument-Centric Activity Theory. However, the framework also captures the process of designing, and in that stage the artifact could not yet develop into an instrument. As such, we deem the name ACAT better suited.

ACAT takes into account the student (here: subject) in the classroom with peers and the teacher (lower left triangle), as well as the mathematics (here: object) and the rules that arise from different disciplines (mathematics, mathematic didactics, design principles, multimedia learning, etc.) (upper right triangle). Whereas the design question "*How can we design the artifact?*" has to be asked in advance, the analysis question "*How could we have designed it?*" is asked afterwards. We try to tackle the complexity of this balance by breaking the overall design and analysis down into numerous individual decisions and tests.

ACAT can be divided into three components: the main axis, the upper right triangle and the lower left triangle. Whereas the main axis focuses on the subject—artifact—object relations, with the artifact as its main component, the lower left triangle focuses on the use and benefit of the artifact in classroom situations and the upper right triangle takes into account the rules and principles that help to design the artifact according to the object.

Being based on Activity Theory, the fundamental concept of ACAT (Ladel and Kortenkamp 2013) is the activity between subject and object. We moved the artifact into the center of this relationship and into the center of the activity, as we want to analyze the impact of the artifact as the mediator between subject and object. The children's use of an artifact, in our case a virtual manipulative, influences their activity and the processes of internalization and externalization. The subject externalizes mental representation through the artifact and in turn internalizes specific knowledge that is represented by the artifact as feedback to the subject's actions. The artifact (i.e., the virtual manipulative) itself externalizes the object (i.e., the mathematics) as a psychological reflection of the programmer's (or designer's) knowledge. The programmer in turn designs the artifact according to the programmer's knowledge about the object.

Neither the mental representation nor the visual representation(s) of the object are predefined through ACAT. The only requirement is that there is some kind of internal/mental representation of the object in question and that there is at least one way to visualize the object.

The lower left triangle focuses on classroom situations. The integration of technological tools into mathematics education is a non-trivial issue. The use of tools involves the process of instrumental genesis during which the artifact turns into an instrument. However, it is also important to observe the instrumental genesis by the teacher through the teacher's orchestration of mathematical situations.

The upper right triangle takes into account the "objective" object, the rules and the artifact. Before designing, analyzing or using the artifact, we have to analyze the object, its properties and its structures. What is it that we want the children to learn? What knowledge do we "put into the artifact"? There are rules resulting from the object and there are also rules on how to design the artifact that are the result of instructional principles and research (e.g., mathematics education or Gestalt psychology). When designing or analyzing an artifact, we have to keep those rules in mind. Also, concentrating on the upper right triangle emphasizes the fact that artifacts are usually not designed for each subject on an individual basis, but for a number of subjects.

2.3 ACAT by Example: The Virtual Place Value Chart

In the following section, we will elaborate the Artifact-Centric Activity Theory and emphasize its importance by using the example of a virtual place value chart (Kortenkamp and Ladel 2013). This virtual manipulative was created as an app for iPads and iPhones (Kortenkamp 2015).

The virtual place value chart uses the full screen for up to four columns that can be filled with counters by touching the screen. These counters can be moved around with the finger after being created. Every column has a header showing the number of counters in that column and the word describing the value of those counters (e.g., ones, tens, hundreds…). Counters can be removed from the chart by moving them to the top, and they can be moved from one column to the other. In the latter case, either the counter is automatically unbundled into the correct number of lower-valued counters, or, if possible, the missing number of same-value counters is also moved to create a new higher-valued bundle.[2]

2.3.1 The Main Axis: Subject—Artifact—Object

Numbers can be represented in many different ways (e.g., with base ten blocks or with counters in a place value chart). It is, however, only important to consider the principle of place value, if the numbers are represented as digits such as “734” (see Ladel and Kortenkamp 2015a, b). If a child solves the equation 7 + 8 = __ by counting “eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen” the special role of “ten” is not apparent. “Ten” is just another word and does not play a special role. In that way language is an *artifact* that does not promote place value—on the contrary: language can even prevent place value understanding, in cases where there are irregularities in the word formation of numbers (Sarama and Clements 2009). The number 24 in German is spoken as “four-and-twenty”. This inversion of numbers leads to the problem that a lot of children write 42 instead of 24. Therefore, it is important to use a good artifact to teach and learn about place value. The place value chart could be such an artifact. In the place value chart, the column in which a certain number or amount of counters are placed indicates its value.

[2]At http://kortenkamps.net/placevalue we provide a screen recording showing the app in action.

If we consider place value only with regard to change among the different modes of representation (*intermodal transfer*)—representing numbers in a place value chart or reading numbers from a place value chart—it is unambiguous. However, when we start working with the place value chart and operating in it to explore its properties, there are different possibilities. There are two facets of the *object* ‘*place value*’: the objective, mathematically defined view of place value as it exists in the world per se and the subjective view of place value of the student.

In the objective view, our numeration system is based on five properties: the positional property, the base-ten property, the multiplicative property, the additive property and the principle of continued bundling (Ladel and Kortenkamp 2014, 2015a, b; Ross 1989). According to the positional property, the place where a digit is positioned gives us information about its value. For example, the 3 in the number 734 has the value of 3 tens, and the 3 in the number 473 has the value of 3 ones. One way to teach children place value is to visualize numbers as similar (undistinguishable) counters in a place value chart. The children can represent numbers with the counters in the place value chart and have to pay attention to the place or column they lay the counter, or they can ‘read’ the number and write it down in digits.

Another action is to move tokens in the place value chart and to discover that the value of a counter changes depending on the column in which it is placed. Thus the value of a number (e.g., 243) changes if we move one token from the tens to the ones (243 becomes 234) as in the example on the left side of Fig. 2.5. To enrich the concept of place value it is important to establish the relationships to the other principles. The principle of continued bundling is about creating new bundles until it is no longer possible. For example, 243 ones = 24 tens and 3 ones = 2 hundreds, 4 tens and 3 ones. In this context, moving a counter from the tens to the ones has the meaning of unbundling 1 “ten” to 10 “ones,” as in the example on the right side of Fig. 2.5. In that way, there are two different meanings for moving a counter from the tens to the ones, and a designer of a virtual place value chart has to decide how the counters in the application should behave to reflect that meaning (Fig. 2.5).

The *subject* (the student) has certain concepts of numbers. Depending on the artifacts the teacher has used for place value instruction, the student may have internalized the meaning of the counters as a change of value or the student may have internalized the meaning of the counters as bundling and unbundling. In this

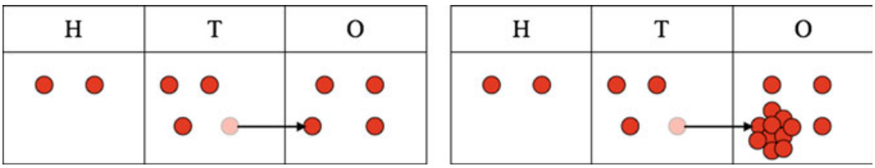

Fig. 2.5 Two different behaviors of one action. *On the left* The action of moving one counter from the tens to the ones changes the total value. *On the right* The action of moving one counter from the tens to the ones emphasizes the necessary unbundling of one ten to ten ones, which preserves the value

case, the design of a particular virtual manipulative application may lead a child to a cognitive conflict because the reaction of the current virtual manipulative application does not match the child's mental schema (Ladel and Kortenkamp 2014).

2.3.2 The Upper Right Triangle

The upper right triangle provides rules and principles that determine the design of the artifact. We have already mentioned the impact of the object on the artifact. There are, however, also principles for the artifact's design that are guided by mathematics education and psychology. Concerning principles from mathematics education, we would like to focus on two aspects: the *intermodal transfer* and the *spiral curriculum*.

According to Bruner et al. (1971), we distinguish three *modes of representation*—enactive, iconic and symbolic—whereas the latter has to be distinguished into verbal- and nonverbal-symbolic representations. The comprehension of operations is only fully developed when the child is able to change among the different modes of representation. This is called *intermodal transfer*. Although children have to perform the intermodal transfer by themselves, it is important to support them during the learning process so that they understand the meaning of the symbols and the operations. In this regard, multiple representations with automatic linking features can be very helpful (Ladel 2009).

The virtual place value chart provides all three modes of representation. The counters that can be moved create the enactive representation; the counters drawn in various places create the iconic representation; and the number written in digits and words is the symbolic representation. Children are able to interact with the app, to create counters, and to delete them, as well as move counters from one column to another. At the same time, they see the iconic representation of the counters. The nonverbal-symbolic mode is automatically given in the way that the amount of counters in each column is written in the title row and, optionally, the whole number is written in words above the columns. In that way 2 hundreds, 4 tens and 3 ones are represented as well as the number 243 along with representation of the counters. It is even possible to show the numeral (i.e., the verbal-symbolic representation) (Fig. 2.6).

With Bruner's *principle of a spiral curriculum* in mind, the designers valued the (vertical) compatibility of the virtual manipulative within the curriculum and beyond. In this particular app there are several possibilities of modification the user can apply within the settings (see Fig. 2.7). For example, in Montessori-mode, one can chose between homogeneous or multicolored counters. Thus it is possible to go back to a lower level of abstraction and to work with multicolored counters in accordance with the place value. Also the number of places before and after the decimal point can be configured. So the virtual manipulative can be used from first grade, starting with ones and tens only, through the upper grades. The app allows features that expand the chart to the left (for hundreds, thousands…) and to the right

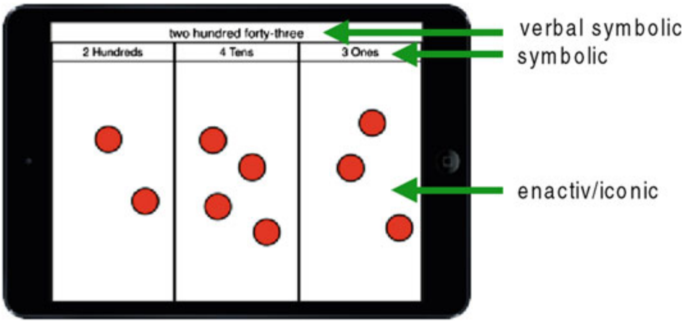

Fig. 2.6 Multiple external linked representation linked in the virtual place value chart

(for tenths, hundredths…) to make visible the decimal principle of our number system.

Working with decimal numbers, designers' paid attention to the decimal point. The separator between the ones and the tenths is shown as an asymmetrical pair of a thick and a thin line, in order to work against the illusion of symmetry at the decimal point. For more advanced work with place value (e.g., with university students), it is also possible to change the base (that is, the size of the bundles) and the used base for counting. In that way, this virtual manipulative is suitable for first graders up to students at the university level (Fig. 2.7).

Another design question was whether the counters should be laid in a structured order or not. The aim of this virtual manipulative is not to support the children in their quasi-simultaneous subitizing. Actually, there is no need for quasi-simultaneous subitizing as the numbers of counters are written in the title row. Furthermore, an additional design feature is that if there are more than nine counters in a column, the written number is colored red. The reason that designers decided to write numbers more than nine in red was for students to be subtly warned that the representation is non-standard, while not being stopped from creating such non-standard representations.

As these examples from just one app demonstrate, there are numerous design decisions to be considered when creating a virtual manipulative and these are highly influenced by the upper right triangle of ACAT—the interplay of the artifact, the mathematical object and the findings from mathematics education, psychology and other relevant fields.

2.3.3 The Lower Left Triangle

Any teaching and learning must consider the context—the individuals, the group, the tools, the conditions, the social structures, etc. The lower left triangle of ACAT is related to the use of the artifact in the classroom. A virtual manipulative does not

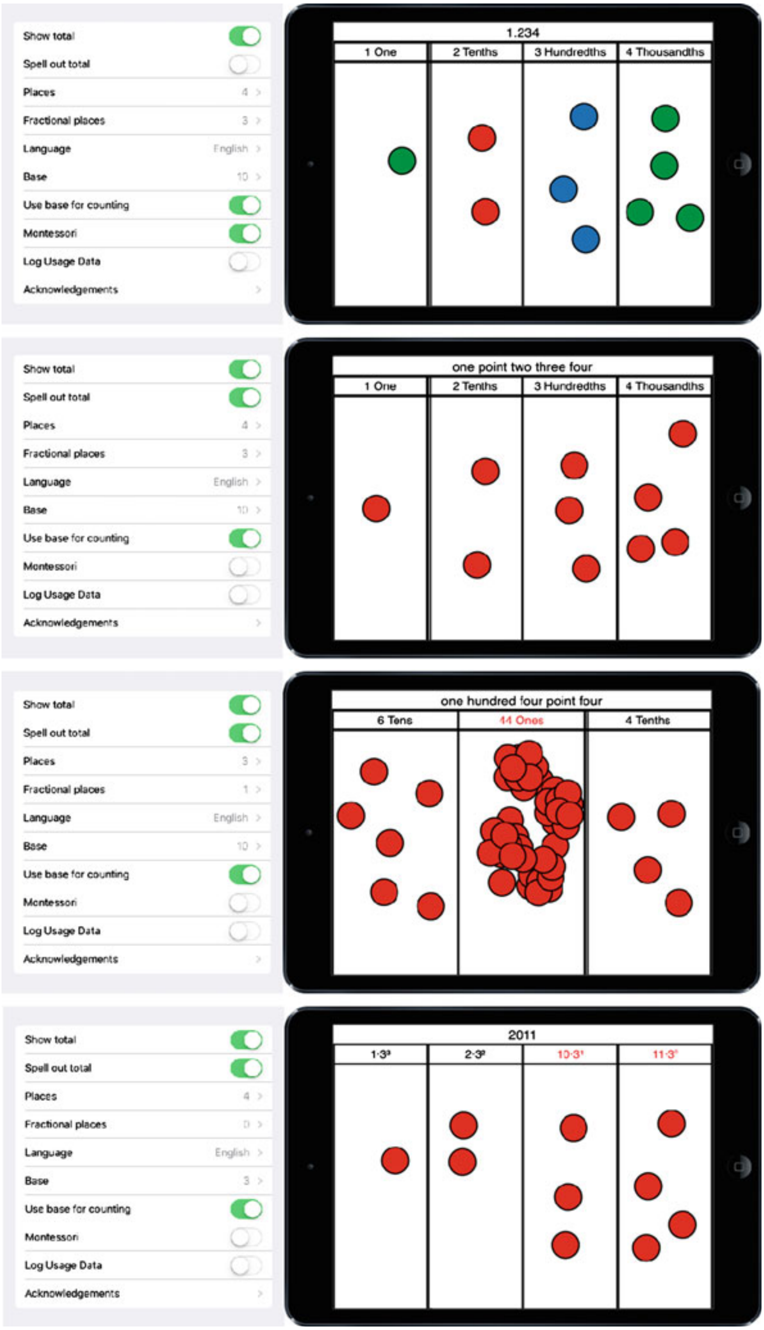

Fig. 2.7 Possible modifications of the virtual place value chart in the settings

just work, but its effects are determined by the way it is employed in the classroom. Of importance are the individual and socio-cultural factors and the orchestration of the virtual manipulative in the class. The theoretical analysis of the full situation is beyond the scope of the ACAT framework. The lower left triangle in the diagram is a reminder that it is necessary to analyze the circumstances and conditions in the classroom (or other learning situation), in the same way as the artifact itself is being analyzed, for effective use of virtual manipulatives in teaching and learning situations. One suitable theory for that analysis is instrumental genesis (Artigue 2002), and in particular it is necessary to find a suitable *instrumental orchestration* (Trouche 2004; Drijvers et al. 2011) that is an intentional and systematic organization and use of the artifact in the learning situation.

One way to use virtual manipulatives in the classroom is to use them for demonstration and visualization. In our example, the teacher could use the virtual place value chart to show students how a counter in the tens place can be unbundled into ten counters in the one's place. Observing this miraculous mathematical transformation could help students to better understand that the same counters have different values depending on their location.[3] Using a virtual manipulative in that manner, though, misses the opportunity for the students to experience the major affordances that the virtual manipulative has to offer. Students can use the artifact themselves and experience the built-in mathematics.

By designing suitable tasks, we can steer the activities of the students such that they will be exposed to the externalization of the mathematical objects and interact with them. Some examples for good and productive tasks are:

- Find as many ways to represent the number 132 in the virtual place value chart as you can! Explain what you did to find them!
- Which numbers have only 2 (3, 4, …) ways of representation? Justify!
- Divide 1505 by 7. Start by representing 1505 in the standard way and change the representation until the division becomes easy!

By working on tasks of these types, the students themselves can experience the miraculous mathematical transformation that is provided by the virtual manipulative. For example, here is what a student might do to divide 1505 by 7: Place counters for 1Th 5H 5O (standard representation—1 Thousands 5 Hundreds 5 Ones). Move the thousands counter to the hundred's place, as it cannot be divided easily by 7, so you now have 15H 5O. Observe that you can divide 14H of the 15H by 7 easily, so move the extra H one place to the right, to the Tens (T), and you get 14H 10T 5O. Move 3T of the 10T to the right to get 14H 7T 35O. The student ends up with a representation of 1505 that is easy to divide, and the solution is 2H 1T 5O = 215. This is just one example of an exploration that could be conducted by a student to investigate this question. At each step the exchange of counters from

[3]There is no research so far that could prove or disprove this assumption—this expected help for learning is just hypothetical and based on the theoretic considerations behind the design of the artifact.

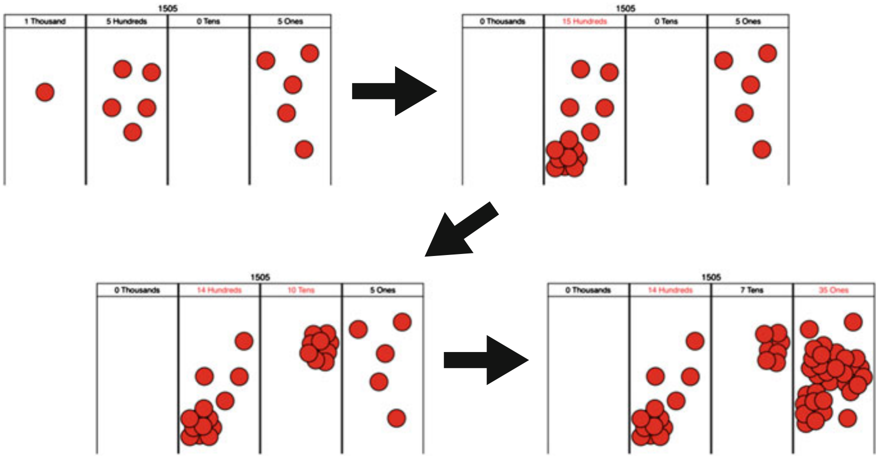

Fig. 2.8 Solving 1505 divided by 7 by flexible use of place value

higher values to lower values is visualized and experienced by the student (Fig. 2.8).

Again, suitable research has to be carried out to gather data that justifies our claim that these and similar tasks are indeed good and productive tasks. But the ACAT framework helps us to identify these research questions and to design virtual manipulatives that are integrated into learning environments. We invite the reader to try ACAT with their favorite virtual manipulative for analysis, but highlight the fact that ACAT can also be used to design new virtual manipulatives, as shown in Ladel and Kortenkamp (2013).

2.4 Conclusion

Virtual manipulatives have the potential to support students' mathematics learning, but in order to turn this into affordances and learning opportunities, it is necessary to have a theoretical tool that can guide both the design and the analysis of these artifacts. In this chapter we presented ACAT, a theoretical framework that helps to structure the complexity of this design task. As ACAT is based on Activity Theory, it focuses on the interaction of subjects (students) with objects (mathematical concepts) mediated through artifacts (the virtual manipulatives). The design of an artifact is usually not for an individual student but for a larger audience. The mathematical concepts are also independent of the students. Therefore, the upper right triangle in our model can be used for designing (or analyzing) the virtual manipulative. The lower left triangle adds the necessary context for teaching and learning and helps to focus on the specific tasks or usage scenarios of the virtual manipulative by educators in teaching and learning environments.

While ACAT has been created with digital artifacts and virtual manipulatives in mind, it is not restricted to these tools alone. The framework can be applied to any artifact used in teaching. Still, we suggest that ACAT is most suitable for characterizing the miraculous mathematics transformations that can be created through virtual manipulatives, and that can be experienced by the students indirectly in demonstrations, and also directly by doing miraculous mathematical transformations themselves.

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Chapter 3
The Modification of Attributes, Affordances, Abilities, and Distance for Learning Framework and Its Applications to Interactions with Mathematics Virtual Manipulatives

Stephen I. Tucker

Abstract While extensive research has examined the outcomes of interacting with virtual manipulatives, less research has focused on constructs and relationships among constructs involved in user-tool interactions. This chapter presents the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning framework, which conceptualizes the relationships among these constructs to describe user-tool interactions, including those involving virtual manipulatives. The framework is primarily grounded in theories of representation and embodied cognition, as user-tool interactions in mathematics involve internalizing and externalizing representations through physically embodied mathematical practices. In the framework, attributes, affordance-ability relationships, and distance are interrelated, and modification of one construct contributes to modification of the other constructs. Each attribute can contribute to many affordance-ability relationships and to distance. Attribute modification can change the approach or degree of affordance access and alter the degree of distance present, which can, in turn, lead to attribute modification. This chapter illustrates the constructs and relationships among constructs that form the framework in the context of user-tool interactions in mathematics. The chapter then applies the framework to examples of children's interactions with mathematics virtual manipulative touchscreen tablet apps. The MAAAD for Learning framework has implications and applications relevant to theory, development, implementation, and research concerning technology tools, including virtual manipulatives.

Keywords Virtual manipulatives · Mathematics · Tools · Learning

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A menagerie of digital tools exist for learning mathematics and other content, including virtual manipulatives (Moyer et al. 2002), learning objects (Kay and Knaack 2007), mathematical cognitive tools (Sedig 2004), visual mathematical representations (Sedig and Liang 2006), and many others. Definitions of these tools often overlap, but a virtual manipulative is "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al. 2002, p. 373). One interacts with content presented by a virtual manipulative via the interface (e.g., iPad touchscreen platform) that presents the virtual manipulative as part of a digital environment (e.g., app). Thus, characteristics contributing to interactions with a virtual manipulative, the digital environment, and the interface are interrelated.

Continued development and implementation has seen virtual manipulatives become important tools for learning. Recent research suggests that virtual manipulatives need not be web-based (Tucker et al. 2014) and can be used to construct knowledge in other content areas (e.g., Zacharia and de Jong 2014; Zacharia et al. 2008). Although a substantial body of research has indicated that virtual manipulatives can be effective tools for learning (e.g., Moyer-Packenham et al. 2015; Olympiou and Zacharia 2012; Satsangi and Bouck 2014), less research has examined why these tools are effective (e.g., Durmuş and Karakırık 2006; Moyer-Packenham and Westenskow 2013). In particular, little research has identified constructs and relationships among constructs that contribute to interactions with virtual manipulatives. Thus, the purpose of this chapter is threefold: (a) to introduce the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning Framework, which models constructs and relationships involved in user-tool interactions, (b) to apply the framework to describe interactions with virtual manipulatives, and c) to discuss potential implications and further applications of the framework.

3.1 Theoretical Grounding

The MAAAD for Learning framework is grounded in theories of representation and embodied cognition set in the context of interaction with technology tools, including virtual manipulatives. These tools offer varying levels of embodiment and fidelity, which also influence user-tool interactions.

3.1.1 Representing Mathematics

Learning mathematics involves interactions among and development of internal and external representations. Internal representations are individuals' mental configurations of mathematics and cannot be directly observed (Goldin and Kaput 1996). External representations are observable, physically embodied configurations of mathematics (i.e., pictures, words, equations, digital environments) which one can access with sufficient understanding of the representations. Interplay among representations includes internalizing external representations (e.g., interpreting graphs, symbols, and pictures) and externalizing internal representations (e.g., writing, speaking, manipulating concrete objects). Importantly, interactions with appropriate combinations of multiple external representations can enhance learning (Ainsworth 2006). Understanding of representations and connections among multiple representations is representational fluency (Zbiek et al. 2007) which influences interactions among and development of internal and external representations. Representational fluency can both facilitate and result from mathematical learning (Heinze et al. 2009; Nathan and Kim 2007) and thus both contributes to, and results from, learning mathematics.

3.1.2 Embodied Cognition: Physical Interactions with Representations

Mathematical practices that include physically interacting with external representations involve embodied cognition, as cognitive processes are part of bodily interactions with the environment. From an embodied cognition lens, human cognition is rooted in sensorimotor processing (Wilson 2002), which integrates perception of the environment with actions upon the environment. Thus, human cognition is based in action and perception, and is grounded in the physical environment (Alibali and Nathan 2012). Nemirovsky et al. (2013) suggested that "the intertwining of perceptual and motor aspects of tool use [is] *perceptuomotor integration,*" allowing one to perceive and interact with representations in such a way that integrates action and thought (p. 373, emphasis in original).

Applied to mathematics, mathematical thinking is equivalent to physical engagement in mathematical practices, and mathematical learning involves changes in these physically embodied practices (cf. Lakoff and Núñez 2000). Therefore, perceptuomotor integration is the way in which one uses bodily activity to facilitate interplay between internal and external representations and develop representational fluency. Thus, bodily engagement (external) in mathematical practices, such as interactions with mathematics virtual manipulatives, can provide evidence of (internal) representations of mathematics, and changes in these practices provide evidence of learning.

3.1.3 Technology for Interactions with Representations

Technology tools offer varying degrees of embodiment. Bodily engagement includes representational gestures, which bodily actions use in interplay among internal and external representations (Hostetter and Alibali 2008; Segal 2011). Gestures can help children retain and apply knowledge within similar contexts (Cook et al. 2008) when developmentally appropriate (Ginsburg et al. 2013; Shuler 2009) and mapped to the content (Segal 2011; Segal et al. 2014). Many technology tools feature multi-touch interfaces (e.g., iPads), which can support a wide variety of input gestures (e.g., Hamon et al. 2013) for user-tool interactions.

Although relatively few apps effectively incorporate multi-touch capabilities (Byers and Hadley 2013), apps that do use multi-touch capabilities may uniquely influence children's mathematical understandings and strategy development (Baccaglini-Frank and Maracci 2015). Multi-touch technology can thus afford users greater embodiment, relatively direct control over the manipulation of representations, and a wider range of mathematically meaningful gestures than mouse-based interaction, when tasks and the tools are appropriately designed.

3.1.4 Faithfulness of Technology Tools for Interacting with Representations

Researchers have theorized ways to design educational tools that facilitate mathematics learning (e.g., Ginsburg et al. 2013; Pelton and Francis Pelton 2011). Many guidelines originate with Dick (2008), who recommended that technology tool designers insure high levels of cognitive, pedagogical, and mathematical fidelity. Cognitive fidelity is the degree of alignment of the mathematical representations of the tool with the cognitive processes of the student. Pedagogical fidelity is the degree of alignment of the tool with design principles. Mathematical fidelity is the degree of mathematical appropriateness of the representations of the content.

Tools and tasks, such as those that involve virtual manipulatives, vary in fidelity (Moyer-Packenham et al. 2008). Some researchers consider the greatest challenge in designing digital tools for learning mathematics to be insuring cognitive fidelity by allowing effective externalization of a child's mathematical thinking (Olive 2013). For many concepts, digital tools can offer "idealized" representations that are more mathematically faithful than concrete representations (de Kirby 2013), allowing users embodied interactions with visual models of concepts that formerly were only accessible in mental models (Carpenter 2013). Discussions of pedagogical fidelity often include pedagogical approaches of digital tools (e.g., instructive, manipulable, and constructive: Highfield and Goodwin 2013; self-leveling, collaborative, and sandbox: Zanchi et al. 2013). Each type of fidelity influences the design of the tool and the users' perception of and interactions with the tool, thereby influencing the internalization and externalization of representations via perceptuomotor integration.

3.1.5 *Summary of the Theoretical Framework*

Embodied cognition and representation, in the form of perceptuomotor integration and representational fluency, influence the transformation of internal representations, while gestures assist the externalization and internalization of representations. Technology affords embodiment in human-computer interaction, and cognitive, pedagogical, and mathematical fidelity influence how users interact with technology tools. These theories provide theoretical grounding for the MAAAD for Learning framework.

3.2 Building Toward the Conceptual Framework

The MAAAD for Learning framework integrates attributes, affordances-ability relationships, and distance to model user-tool interactions. Thus, the conceptual framework emerges from a synthesis of empirical and theoretical research involving these constructs and relationships among these constructs (Tucker 2015).

3.2.1 *Roles of the Constructs*

Attributes, affordance-ability relationships, and distance each play roles in children's interactions with technology tools, including virtual manipulatives.

Attributes. Attributes are characteristics of people or things (Attribute [Def. 5] 2014). Relevant attributes of tools (e.g., virtual manipulatives) and users are involved in user-tool interactions. Using an embodied cognition perspective of learning mathematics, attributes contribute to physical engagement in mathematical practices (i.e., mathematical thinking) and changes in the physically embodied practices (i.e., mathematical learning). Users and tools both have attributes related to content (e.g., mathematics), technology (i.e., physical interactions with the tool), and other aspects of user-tool interactions (e.g., user: personal characteristics; tool: structural characteristics).

In a study examining children's interactions with mathematics virtual manipulative iPad apps, Tucker (2015) categorized app attributes and user attributes involved in physical interactions with mathematical representations. Both apps and users had mathematical attributes (i.e., content attributes), which were characteristics involved in representing mathematical content. Apps and users had subcategories of mathematical attributes related to content (e.g., decimals) and representation (e.g., number line). Users also had a subcategory of mathematical attributes related to flexibility (e.g., transfer from shaded rectangles to shaded circles). Other literature also implies evidence of content attributes of technology tools (e.g., fraction models: Rick 2012) and content attributes of users (e.g., understandings of fraction models: Moyer-Packenham et al. 2014b).

Research has identified empirical evidence of technological attributes pertaining to physical interactions between user and tool in user-app interactions (Tucker 2015). For apps, technological attribute subcategories included input range (i.e., scope of gestures accepted by the [tool] for a given function) and input complexity (i.e., intricacy of the required gestures). For users, technological attribute subcategories included motor skills (i.e., facility with which a user performed the relevant physical actions) and input familiarity (i.e., how conversant a user was in a given input). Other literature also implies the presence of technological attributes of technology tools (e.g., touch input types: Lao et al. 2009) and technological attributes of users (e.g., motor skills: Dejonckheere et al. 2014). Users and tools each have an additional, unique category of attributes: personal and structural, respectively (Tucker 2015). Personal attributes are characteristics of one's personality that influence how one interacts with a tool, including affect, persistence, and goals (e.g., goal of accuracy or speed). Structural attributes are non-content presentation features, including feedback, context, and scaffolding (e.g., hint scaffold reveals worked example). Other literature also implies evidence of personal attributes of users (e.g., affect: Goldin et al. 2011) and structural attributes of technology tools (e.g., scaffolding: Belland and Drake 2013).

Attributes are not static, and attribute alignment influences attribute modification (Tucker 2015). Alignment of content (e.g., mathematical) or technological attributes varies. Finding a missing addend by adding on objects is developmentally appropriate for many 4–5 year-old children (i.e., relatively aligned mathematical attributes), but is likely to be developmentally inappropriate for 2–3 year-old children (i.e., misaligned mathematical attributes) (Sarama and Clements 2009). When interacting with the mathematics virtual manipulative iPad app Motion Math: Zoom, some children efficiently performed a pinching input gesture (i.e., aligned technological attributes), while other children struggled to do so (i.e., misaligned technological attributes) (Tucker et al. 2016a). Users and tools influence attribute alignment through attribute modification, which can be proactive or reactive (Tucker 2015). Reactive modification occurs when tools modify tool attributes and in response, users apply and modify user attributes.

When attributes align and the user successfully completes the task, the tool may in turn modify tool attributes. The user responds by applying and modifying user attributes, continuing the cycle. Proactive modification occurs when tools modify tool attributes, users apply and modify user attributes, and users modify tool attributes. For example, some users repeatedly attempted the same level of mathematics virtual manipulative iPad apps despite consistently poor performance (i.e., reactively modified attributes), whereas other users returned to a previous level with related content to build back toward the more challenging content (i.e., proactively modified attributes). From an embodied cognition perspective of learning mathematics, attribute modification can contribute to and result from changes in physically embodied mathematical practices (i.e., learning).

Affordance-ability relationships. Research suggests that affordance-ability relationships play a complex but key role in how children learn while interacting with technology (e.g., Tucker 2015; Tucker et al. 2016b). Greeno (1994), drawing on Gibson (e.g., 1986), posited that an affordance related attributes of an object in the environment (e.g., tool) to an interactive activity undertaken by an agent (e.g., user). The agent applied an ability based on its attributes as part of this interactive activity. Thus, an interactive activity links an affordance of a tool with the ability of a user. Each affordance exists only in relation to an ability, and vice versa (Greeno 1994), and the two are coupled in a continuous system (Chemero 2003).

Some authors discuss the idea of constraints, which one can consider part of what the app affords. However, widely varying conceptions of affordances led Burlamaqui and Dong (2014) to state that "the only uncontroversial claim about affordances is that they are about action possibilities relative to the agent" (p. 13). From an embodied cognition perspective of learning mathematics, affordance-ability relationships are interactive links between user and tool as part of physically embodied mathematical practices.

Authors have applied the concept of affordances to technology (e.g., Gaver 1991; Sedig and Liang 2006), such as virtual manipulatives for various content areas (e.g., fractions: Moyer-Packenham et al. 2014a) and as part of multiple technology tools (e.g., mouse-controlled computer applets: Moyer-Packenham et al. 2013; touch-screen tablet apps: Tucker et al. 2016b). Although meta-analyses of affordances of virtual manipulatives show that instruction using virtual manipulatives has positive effects on learning (Moyer-Packenham and Westenskow 2013, 2016), accession of the same affordance can vary greatly by user ability and by context (Moyer-Packenham et al. 2016; Tucker and Moyer-Packenham 2014).

Furthermore, applying the same approach to affordance access may still result in different outcomes (e.g., Tucker et al. 2016b) For example, DragonBox Algebra 12+ affords efficient precision by guiding completion of the additive equality and additive identity properties (Tucker 2015). Some children used the same approach, efficiently combining these properties for the first part of a task when possible without combining these properties for later stages of the same task. However, outcomes varied, as some of these children correctly completed both properties, while others only completed the additive equality property.

Research also indicates that affordance-ability relationships are not static and can interact with one another. For example, children can access simultaneous linking of pictorial representations, symbolic representations, and actions in different ways, and some children use different approaches as their relevant ability changes (Tucker et al. 2016b). From an embodied cognition perspective of learning mathematics, changes in physically embodied mathematical practices (i.e., learning) can both contribute to and result from changes in affordance-ability relationships.

Furthermore, multiple affordance-ability relationships can influence one another, such as efficient precision, creative variation, and focused constraint (Tucker 2015). For example, while interacting with a mathematics virtual manipulative iPad app, a child attempted efficient, mathematically correct input that the app disallowed, constraining focus on another mathematical concept. During a more advanced level, the app permitted the mathematical input it had previously disallowed, emphasizing efficiency. The child then creatively applied this input by combining it with other mathematical input. Thus, affordance-ability relationships play a role in user-tool interactions.

Distance. Sedig and Liang (2006) define distance as the "degree of difficulty in understanding how to act upon [something] and interpret its responses" (p. 184). From an embodied cognition perspective of learning mathematics, distance characterizes the degree of difficulty in engaging in mathematical practices through user-tool interactions. Cognitive, pedagogical, and mathematical fidelity (Dick 2008) may contribute to distance, as they influence tool design and user perception of the tool. Distance can be reduced if one designs the tool to fit the learner's understandings or if the learner determines how to use the tool, and maintaining appropriate distance through purposeful, stepwise distance modification by a tool can facilitate learning (Sedig et al. 2001). Maintaining appropriate distance relates to Vygotsky's (1978) Zone of Proximal Development (ZPD), applied to technology as when tasks remain developmentally appropriate while users progressively master instructional objectives (Murray and Arroyo 2002). Progressive mastery implies that users also change to maintain an appropriate degree of distance. Thus, both users and technology tools change during interactions to maintain distance, which facilitates the learning process.

There are multiple types of distance, including mathematical (i.e., content) and technological (c.f., Sedig and Liang 2006). *Mathematical distance* is "the degree of difficulty of the mathematical aspects of interactions between the user and the tool (e.g., a mathematics virtual manipulative iPad app)" (Tucker 2015, p. 82). A high degree of mathematical distance is evident when one struggles to complete the mathematical aspects of a task, whereas a low degree of mathematical distance is evident when one has less difficulty completing the mathematical aspects of a task. For example, when navigating a number line displaying intervals of one tenth to find the range in which 0.05 is located (i.e., 0.0–0.1), choosing 0.5–0.6 shows evidence of a higher degree of distance than choosing the range of 0.0–0.1. *Technological distance* is "the degree of difficulty of the technological aspects of interactions between the user and the tool" (Tucker 2015, p. 83). A high degree of technological distance is evident when one struggles to complete the technological aspects of a task, whereas a low degree of technological distance is evident when one has less difficulty completing the technological aspects of a task. For example, some children had difficulty controlling mouse input when interacting with computer-based virtual manipulatives (i.e., higher degree of mathematical distance), whereas other children found these interactions less difficult (i.e., lower degree of mathematical distance) (Highfield and Mulligan 2007).

Distance is not static and distance types can influence each other (Tucker 2015). Research implies that distance differs by context. Distance can decrease, as when children initially struggled to accurately complete a task involving the splitting model of fractions, yet successfully completed the task after additional experience with the content (i.e., decreasing mathematical distance) (Martin et al. 2013).

Distance can also increase, as when children initially appropriately used input gestures but later chose inappropriate gestures that hindered task completion (i.e., increasing technological distance) (Tucker 2015). Using an embodied cognition perspective of learning mathematics, changes in distance can both lead to and result from changes in physically embodied mathematical practices (i.e., learning).

Research also implies that these distance types can interact. Difficulty performing required input such as controlled mouse movements (technological distance) while interacting with virtual Pattern Blocks can lead to unintended mathematical outcomes such as unintentionally rotating shapes instead of sliding them (mathematical distance) (Highfield and Mulligan 2007). A high degree of technological distance in the form of difficulty using appropriate input can also lead to a user focusing attention on performing the gestures (i.e., decreasing technological distance) rather than attending to the mathematical content (i.e., high degree of mathematical distance) (Rick 2012). Thus, distance plays a role in user-tool interactions.

3.2.2 Relationships Among the Constructs

Relationships among attributes, affordance-ability relationships, and distance also play roles in children's interactions with technology tools, including virtual manipulatives.

Attributes and affordance-ability relationships. In this context, (a) tools have attributes that combine to provide affordances, (b) users have attributes that combine to create abilities, and (c) an affordance-ability relationship exists between user and tool (see Fig. 3.1).

Modification of attributes can lead to modification of affordance-ability relationships and vice versa (Tucker 2015). Modifying user attributes can lead to modification of ability as part of affordance-ability relationships. For example, research indicates that children may be less likely to access audio feedback as part of efficient precision as they become more proficient at completing tasks while

Fig. 3.1 Relationship between attributes and affordance-ability relationships set within user-tool interactions (adapted from Tucker 2015, p. 19)

interacting with mathematics iPad apps (Bartoschek et al. 2013; Paek 2012). Modification to affordance-ability relationships can lead to modification of app attributes, such as when children access efficient precision by placing the appropriate number of fingers on the screen to indicate an answer while interacting with the mathematics virtual manipulative app Fingu, contributing to advancement to a level featuring different content (Barendregt et al. 2012).

Modifying app attributes can modify an affordance as part of the affordance-ability relationship, which can lead to modification of user attributes (Tucker 2015). For example, during interactions with the mathematics virtual manipulative iPad app DragonBox Algebra 12+, the app initially prohibits children from combining the additive inverse property and the additive equality property to efficiently move a variable from one side of an equation to the other, focusing their attention on separately applying these properties. In a more advanced level, the app removed this constraint, permitting the “drag across” move. After this, some children creatively and efficiently combined the properties when possible, providing evidence of a change in user mathematical attributes and abilities as part of the corresponding affordance-ability relationships. Each attribute can contribute to multiple affordance-ability relationships, such as coordination (user technological: motor skills) contributing to accession of both planning (efficient precision) and navigation restrictions (focused constraint). Improving the coordination attribute could lead to a different approach to planning and fewer encounters with navigation restrictions. Thus, there are relationships between attributes and affordance-ability relationships during user-app interactions.

Attributes and distance. Attributes also relate to distance, and attribute modification can lead to modification of distance, while modification of distance can contribute to modification of attributes (Tucker 2015). Distance is the degree of alignment (i.e., difference) between clusters of relevant user attributes and app attributes (see Fig. 3.2).

Evidence of relationships between attributes and distance is present in research literature. Specifically, distance can be conceived of as the degree of alignment between clusters of relevant attributes of the tool and user (Tucker 2015). For example, during the *Pop the Bubble* activity in the *MathemAntics* app suite, children compare schools of fish by counting each set and popping the bubble that

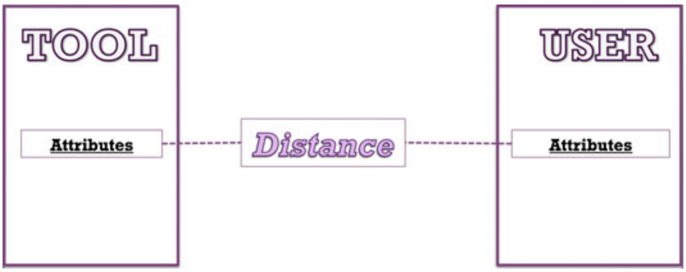

Fig. 3.2 Relationship between attributes and distance set within user-tool interactions

contains more fish (Ginsburg et al. 2013). The app represents quantity and comparison using sets of objects (fish grouped in bubbles) and the user must have sufficient knowledge of this representation of quantity, as well as counting and quantity comparison skills to decide which set has more objects in it. The degree of alignment of these attributes is the mathematical distance. The app requires the user to click on the bubble to indicate a response, so the user must have sufficient familiarity with this input method and sufficient motor skills to perform this gesture. The degree of alignment of these attributes is the technological distance.

Patterns related to attributes and distance are also evident in user-app interactions. For example, when children advanced to a new level while interacting with an app, mathematical distance often increased (Tucker 2015). Children applied and modified user mathematical attributes in an attempt to decrease mathematical distance. Some children decreased mathematical distance by proactively modifying app attributes to select different mathematical content. This provided an environment in which they could strengthen relevant user attributes, leading to decreased distance when returning to levels with content that was initially too difficult. Users can also modify user technological attributes to align with requirements for interacting with apps, such as by using gestures that the tool can recognize after initial attempts are unsuccessful (Ladel and Kortenkamp 2012).

Modification of structural attributes and personal attributes can also influence distance. For example, upon first encountering the needle timer (structural) during interactions with Motion Math: Zoom, children struggled to accurately complete mathematical tasks (increased mathematical distance) or efficiently perform appropriate gestures (increased technological distance) (Tucker 2015). One child proactively chose when to activate and deactivate the needle timer while as a way to increase or decrease the degree of difficulty (i.e., goal led to changing distance by modifying app attributes). Thus, there are relationships between attributes and distance during user-app interactions.

Distance and affordance-ability relationships. Distance also relates to affordance-ability relationships, as accession of affordances can influence distance, and distance can influence accession of affordances (Tucker 2015) (see Fig. 3.3).

Fig. 3.3 Relationship between distance and affordance-ability relationships set within user-tool interactions

Research also provides evidence of relationships between distance and affordance-ability relationships, as seen in Fig. 3.3. Affordance-ability relationships can influence distance, such as when a child stated that each group of levels in a mathematics virtual manipulative iPad app “starts off easy and then gets harder and it tells you what [math] to do at first and then you do that on your own on the next one” (Tucker 2015, p. 116). The group of levels began with a level that focused attention on one mathematical property, before modifying constraints by providing a level that required the user to apply the newest property with others to complete the task, often resulting in increased mathematical distance. Distance also influences affordance-ability relationships, such as when high achieving students (i.e., implied lower degree of distance) ignored pictorial models as part of simultaneous linking, whereas lower achieving students (i.e., implied higher degree of distance) relied on the linked pictorial models (Moyer-Packenham and Suh 2012).

Interactions between mathematical distance and technological distance can also influence accession of motivation. For example, a child described interactions with a mathematics virtual manipulative iPad app as, “easy, but it wouldn’t give me enough time to do stuff because it was super-hard to get to areas you wanted to go to” (Tucker 2015, p. 116). This implied the perception that the mathematical distance was not worth overcoming due to the degree of technological distance, which led to a high degree of access to negative motivation and the decision to stop interacting with the app.

However, other research indicated that children who struggled to overcome technological difficulties that interfered with mathematical accuracy were still motivated to persist with mathematical activities (Paek and Hoffman 2014). Thus, there are relationships between distance and affordance-ability relationships during user-app interactions. Research provided evidence of attributes, affordance-ability relationships, distance, and relationships among these constructs in user-tool interactions. Integration of these constructs and relationships among these constructs led to the development of a conceptual framework to model the roles they play during user-tool interactions.

3.3 The Modification of Attributes, Affordances, Abilities, and Distance for Learning Framework

Syntheses of theoretical and empirical research on user-tool interactions, such as those involving virtual manipulatives, provides evidence of the interconnected relationships among attributes, affordance-ability relationships, and distance that form the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning framework (see Fig. 3.4).

As seen in Fig. 3.4, the MAAAD for Learning framework for user-tool interactions begins with attributes (Tucker 2015). The difference between clusters of relevant tool and user attributes forms distance. Modification of attributes may

Fig. 3.4 Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning framework (Tucker 2015, p. 117)

bring attributes into alignment (e.g., the user masters the content the tool presents) leading to decreased distance, or it may misalign attributes (e.g., the tool presents relatively challenging content after a user successfully completes tasks) leading to increased distance. Abilities, stemming from clusters of related user attributes, relate to specific affordances, which are based on related clusters of app attributes. A variety of approaches or degrees of affordance access may emerge from variations in user attributes. A particular attribute can contribute to a multitude of affordance-ability relationships and to distance. Distance influences affordance-ability relationships; for example, a high degree of distance due to misaligned attributes may induce different affordance access than when a low degree of distance is present because attributes are aligned. Figure 3.5 presents a version of the MAAAD for Learning framework applied to learning mathematics through user-app interactions.

As illustrated in Fig. 3.5, the MAAAD for Learning framework can be applied to user interactions with mathematics virtual manipulative apps, and includes distance types and attribute categories and subcategories identified in prior research (Tucker 2015). In this application of the framework, mathematical distance is the difference between clusters of app mathematical attributes and the corresponding clusters of user mathematical attributes, while technological distance is the difference between clusters of app technological attributes and the corresponding clusters of user technological attributes. Clusters of user attributes (mathematical, technological, and personal) form abilities to access app affordances, which stem from clusters of app attributes (mathematical, technological, and structural). Each attribute can contribute to multiple affordance-ability relationships (e.g., user: mathematical: content: comparison contributes to accessing efficient precision of range contents and focused constraint of navigation restrictions); therefore, an affordance-ability relationship can influence another affordance-ability relationship if they share contributing attributes. Distance types (mathematical and technological) can interact, as well as influence affordance-ability relationships, which contribute to variations in accession of affordances.

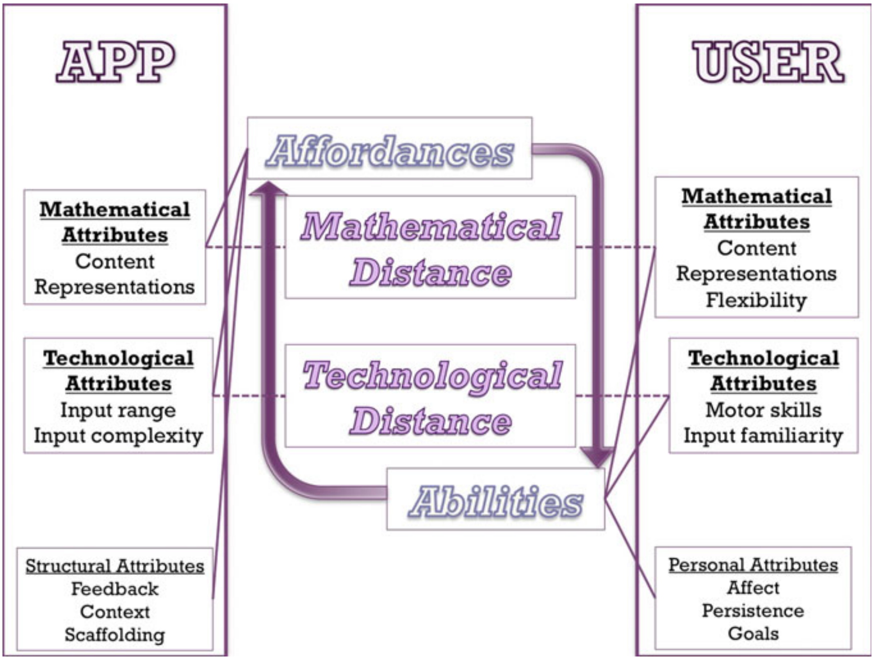

Fig. 3.5 Expanded version of the Modification of Attributes, Affordances, Abilities, and Distance for Learning framework applied to learning mathematics through user-app interactions (Tucker 2015, p. 119)

3.4 Applying the MAAAD for Learning Framework

Researchers have applied the MAAAD for Learning framework to describe user-tool interactions. The following examples are drawn from a study in which Tucker (2015) examined fifth-grade children's interactions with two mathematics virtual manipulative iPad apps during one-to-one semi-structured interviews.

Applying the framework to interactions with Motion Math: Zoom. Children's interactions with the mathematics virtual manipulative tablet app Motion Math: Zoom provides evidence of the MAAAD for Learning framework. Motion Math: Zoom includes content related to number comparisons, magnitude, and estimation on an idealized number line that is navigable and scalable. This representation is more mathematically faithful to the infinite number line than a static physical representation. The user employs single-touch and multi-touch input to change scales and navigate the number line and place target numbers (see Fig. 3.6). Figures 3.7, 3.8, 3.9 and 3.10 illustrate the MAAAD for Learning framework as applied to an excerpt from a sequence of a child's interactions with Motion Math: Zoom.

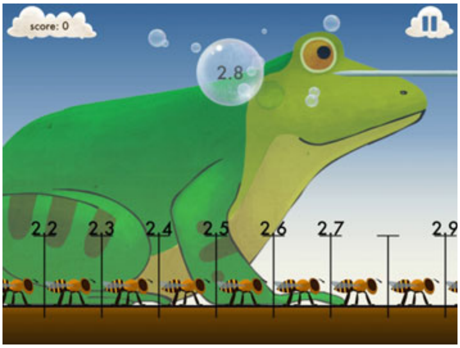

Fig. 3.6 Screenshot of Motion Math: Zoom (Tucker 2015, p. 33)

As Figs. 3.7, 3.8, 3.9 and 3.10 demonstrate, relationships within the MAAAD for Learning framework can be found throughout the child's interactions with Motion Math: Zoom. Throughout these interactions, technological attributes remained aligned and there was a low degree of technological distance. For much of the time, the child consistently attempted planning when possible. Initial

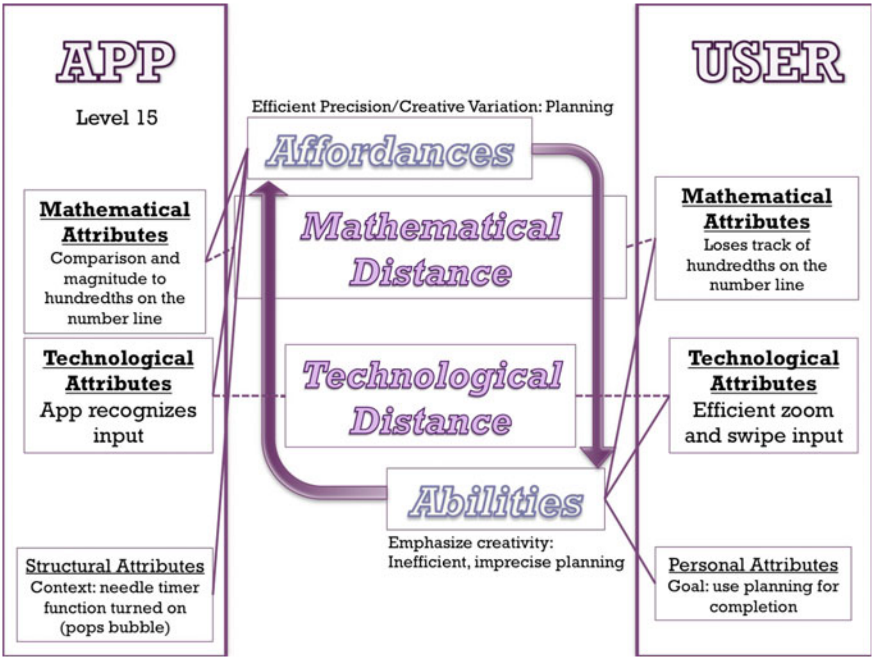

Fig. 3.7 The child experimented with planning as part of completing the level but there was a high degree of mathematical distance as the child lost track of hundredths on the idealized number line. The needle popped the bubble as time for task completion expired. The child made efficient input gestures that the app recognized, so there was a low degree of technological distance. The child restarted the level (adapted from Tucker 2015, pp. 125–127)

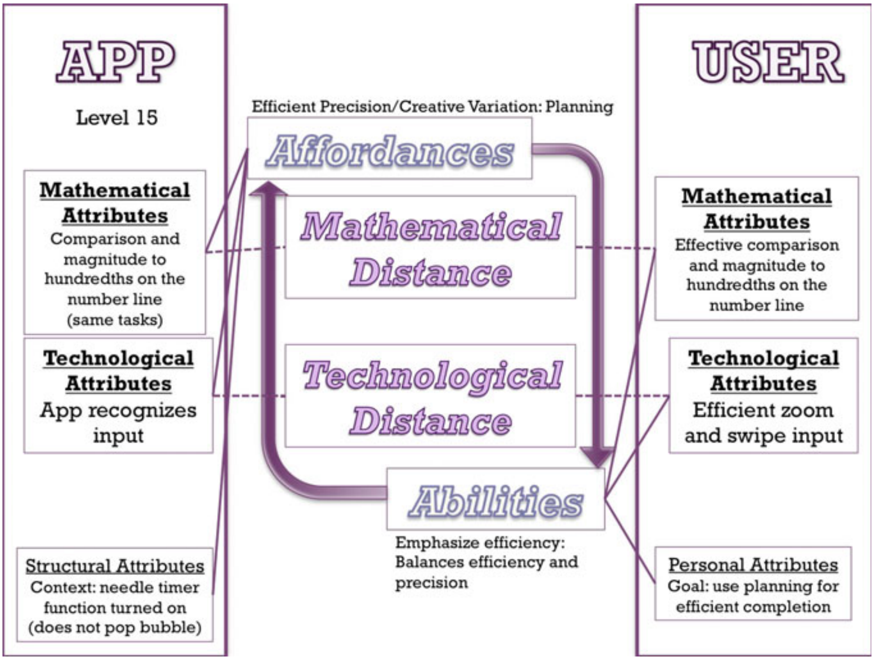

Fig. 3.8 The child modified user mathematical attributes, effectively navigating hundredths on the number line, thus decreasing mathematical distance. The child changed approach to accessing planning, emphasizing efficiency and precision instead of creatively experimenting. Technological distance remained minimal (adapted from Tucker 2015, pp. 125–127)

experimentation with a creative approach to planning contributed to failure to complete the level in a timely manner (Fig. 3.7). The child then modified the approach to planning to focus on efficiency, which in tandem with modification of mathematical attributes led to decreased mathematical distance and successful completion of the level (Fig. 3.8).

On the following level, which presented similar but slightly more advanced tasks, the degree of mathematical distance was so great that the child did not successfully complete even the first task and thus did not have the opportunity to plan (Fig. 3.9). During this attempt, the child struggled to access the affordance of efficient precision in the form of consistent range contents (i.e., 0–10 contains 0, 1, 2–10). In the final attempt in the excerpt, the child proactively changed app attributes by choosing to attempt a different level with related but easier content, decreasing mathematical distance (Fig. 3.10). The child could again access the planning affordance, but discontinued planning after realizing it was not more efficient to do so in that context.

Applying the framework to interactions with DragonBox Algebra 12+. Children's interactions with the mathematics virtual manipulative tablet app DragonBox Algebra 12+ also provide evidence of the MAAAD for Learning

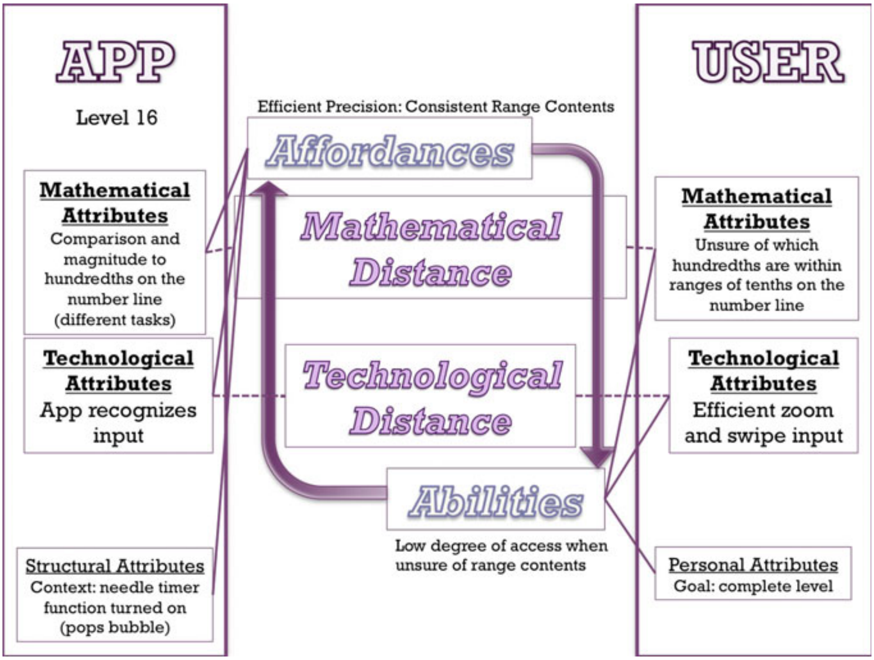

Fig. 3.9 The app presented different tasks based on similar content, showing a change in mathematical attributes. The child did not effectively transfer understanding of ranges to the new tasks, and mathematical distance increased. The child struggled to access efficient precision in the form of consistent range contents. Technological distance remained minimal (adapted from Tucker 2015, pp. 125–127)

framework. DragonBox Algebra 12+ includes content related to solving expressions and equations, operations, negative and positive values, additive and multiplicative thinking, and fractions, presented using pictorial and symbolic tiles (see Fig. 3.11). The user employs single-touch input to drag or tap tiles to complete each expression or equation. Figures 3.12, 3.13, 3.14 and 3.15 illustrate the MAAAD for Learning framework as applied to an excerpt from a sequence of a child's interactions with DragonBox Algebra 12+.

As Figs. 3.12, 3.13, 3.14 and 3.15 demonstrate, relationships within the MAAAD for Learning framework can be found throughout children's interactions with DragonBox Algebra 12+. While accessing a high degree of negative motivation during struggles to complete level 2:13, the child proactively used the solution scaffold to model the steps to complete the level (Fig. 3.12). However, the child rushed to replicate the solution, blurring gestures and increasing technological distance (Fig. 3.13). During this failure to replicate the solution, the child showed signs of frustration and a high degree of access to negative motivation.

During the next attempt, the child's goal changed to accurate completion (Fig. 3.14). The use of relatively precise gestures decreased technological distance,

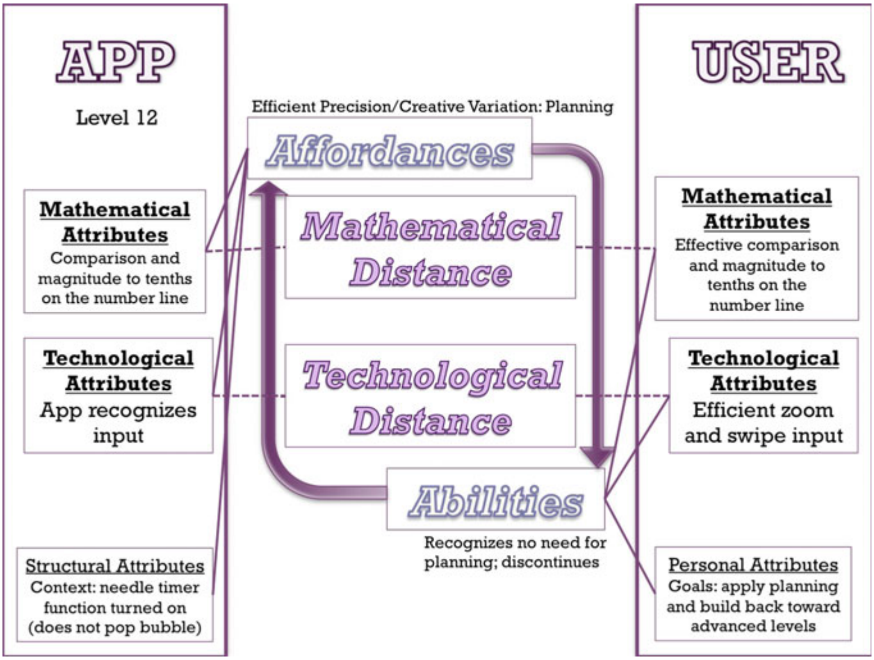

Fig. 3.10 The child proactively modified app attributes, reducing the level and changing to less advanced mathematical content, decreasing mathematical distance. Technological distance remained minimal. During this attempt, the child stopped planning after recognizing it did not contribute to efficiency in this level (adapted from Tucker 2015, pp. 125–127)

and the child recognized the need to apply the reverse order of operations. After restarting the level, the child correctly applied the reverse order of operations, demonstrating changes in mathematical attributes contributing to a decrease in

Fig. 3.11 Screenshot of DragonBox Algebra 12+ (Tucker 2015, p. 35)

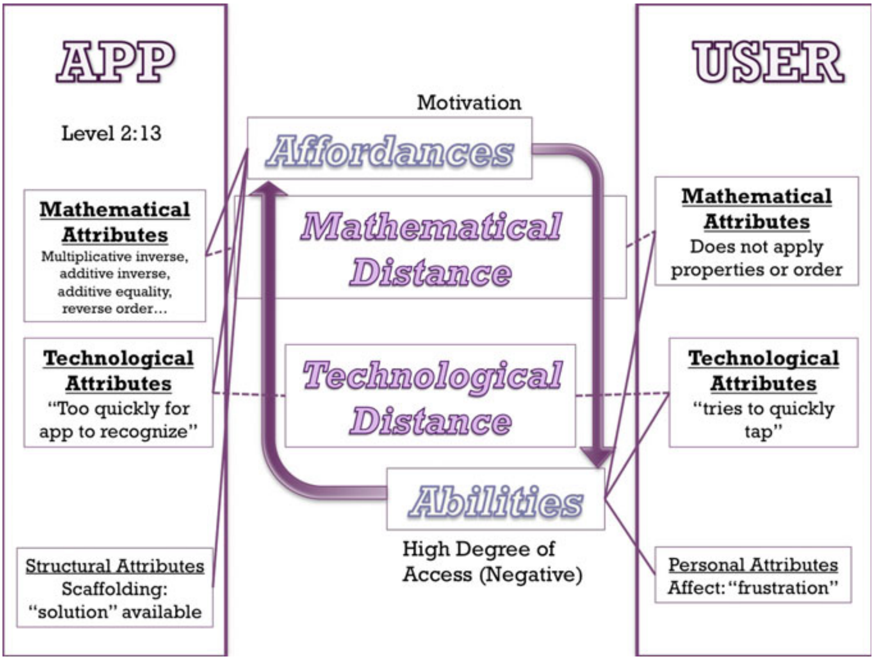

Fig. 3.12 The child attempted to decrease mathematical distance due to unaligned mathematical attributes through proactive modification of app structural attributes (solution scaffolding). The child showed a high degree of access to negative motivation (adapted from Tucker 2015, pp. 121–123)

mathematical distance. The child also honed input gestures, contributing to both a decrease in technological distance and a change in ability to access the affordance of simultaneously linking mathematical representations with actions. These examples show that the MAAAD for Learning framework models relationships among attributes, affordance-ability relationships, and distance in user-tool interactions, such as children's interactions with mathematics virtual manipulative iPad apps.

3.5 Implications and Applications

The MAAAD for Learning framework has implications and applications relevant to theory, development, implementation, and research concerning interactions with technology tools, including virtual manipulatives. These implications and applications build on the descriptive power of the framework (e.g., analyzing user-tool interactions), which has not been applied for prescriptive purposes (e.g., hypothesizing specific user-tool interactions).

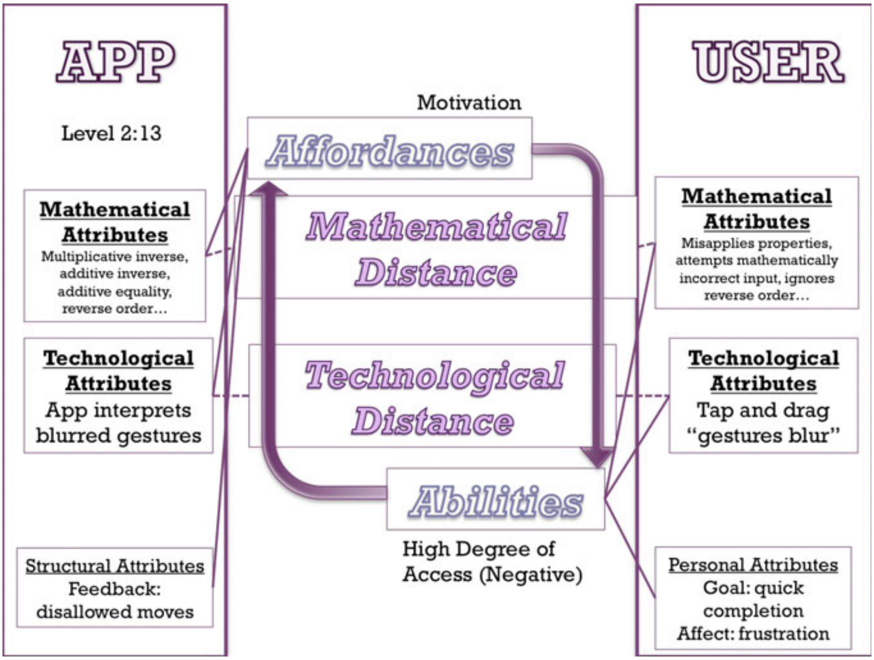

Fig. 3.13 The child failed to correctly replicate solution while attempting to quickly complete the level. A high degree of mathematical distance remained and technological distance increased as the child struggled to make the app recognize some input gestures. The child continued to have a high degree of access to negative motivation and reset the level (adapted from Tucker 2015, pp. 121–123)

3.5.1 Theory

One can view the MAAAD for Learning framework through a variety of theoretical lenses, but embodied cognition and representation provided a context for the developmental phases (Tucker 2015). The framework models specific constructs and relationships among constructs that contribute to bodily engagement in mathematical practices (i.e., physically embodied interactions with representations) that constitute mathematical thinking and learning. However, researchers could consider MAAAD for Learning using other theoretical lenses and conceptual frameworks. Potential approaches include: (a) different theories of affordances (see Burlamaqui and Dong 2014), (b) multimedia learning (e.g., Interactive Multimedia Model for Cognitive Learning: Daghestani 2013), (c) complex cognitive activities (e.g., EDIFICE-AP: Sedig and Parsons 2013), and (d) activity theory (e.g., Artifact-centric activity theory: Ladel and Kortenkamp 2013). These and other theoretical discussions could continue development of the framework, such as by locating the teacher, interviewer, or peer. Along with embodied cognition and representation,

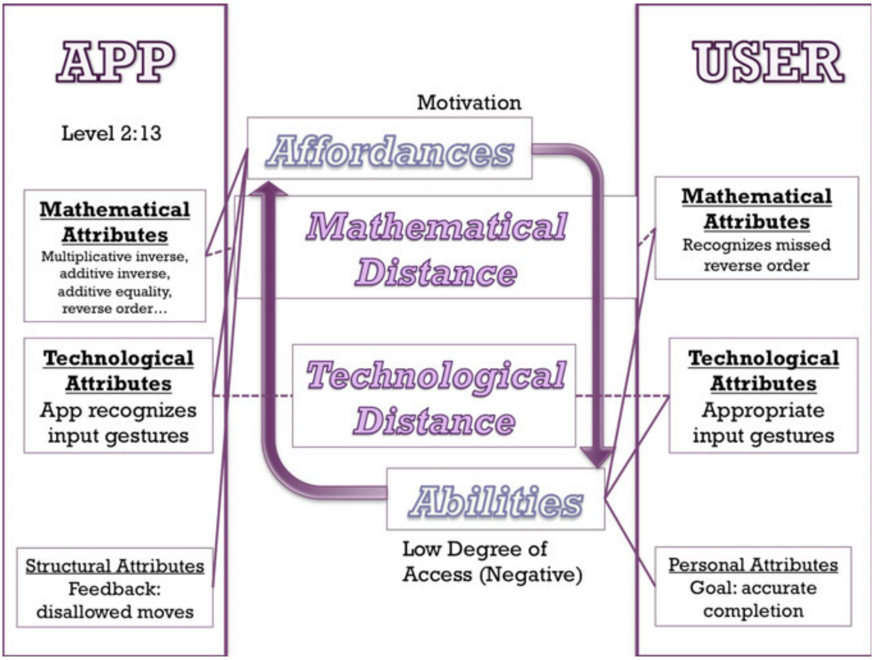

Fig. 3.14 The child attempted to increase accuracy, but failed to correctly replicate solution. However, the child noticed the missed use of the reverse order of operations for solving. The degree of mathematical distance remained high but technological distance decreased as the child produced recognizable input gestures. The child reduced the degree of access to negative motivation and reset the level (adapted from Tucker 2015, pp. 121–123)

various theories could inform further development and application of the framework.

3.5.2 *Design*

The MAAAD for Learning framework has implications and applications for those who design virtual manipulatives and other technology tools. During research and development, designers could use MAAAD for Learning to examine and organize the attributes that contribute to affordance-ability relationships involved in the user-tool interactions, as well as the myriad of potential manifestations of these relationships. Within the framework, designers could also consider purposeful modification of the constructs, including when and how the tool could modify attributes that in turn modify distance and affordance-ability relationships, as well as possible outcomes of these modifications. Additionally, by clarifying for users which tool attributes are modifiable, designers could encourage proactive modification. The

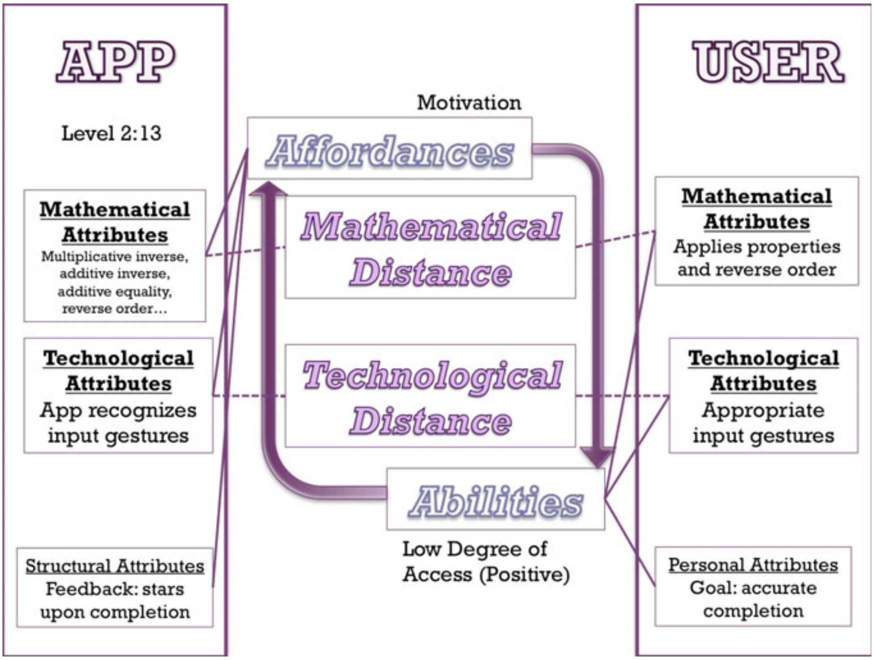

Fig. 3.15 The child slowed interactions and accurately completed the level, having changed mathematical attributes and reduced mathematical distance by correctly applying the properties in the correct (reverse) order. The child also reduced technological distance, completing the level without struggling to perform recognizable input gestures. The child showed a low degree of access to positive motivation (adapted from Tucker 2015, pp. 121–123)

framework may be of use to technology research and development groups when examined in relation to human-computer interaction research in the technology design field, including decision making, information visualization, and adaptive systems (e.g., Jacko 2012).

3.5.3 Implementation

The MAAAD for Learning framework also has implications and applications for implementers of virtual manipulatives and other technology tools and for those who train others to implement these tools. Teacher educators could consider the framework in relation to literature about teachers' use of technology tools, such as teacher beliefs about technology integration (e.g., Ertmer 2005) and Technological Pedagogical Content Knowledge (Mishra and Koehler 2006). Teacher educators could also develop a practitioner presentation of the framework that would permit teachers to use MAAAD for Learning to evaluate the appropriateness of a given tool for a particular child, including the alignment of user attributes and tool attributes. This may help teachers decide when to provide targeted external scaffolding to encourage appropriate proactive attribute modification, such as by helping users recognize opportunities to modify tool attributes. Although mathematical thinking and learning can occur throughout these interactions, children may not be aware they are engaged in mathematical practices. Thus, teachers could also use the framework to examine user-tool interactions for evidence of mathematical thinking and learning as part of informal assessment, supporting facilitation of intentional discussions of these mathematical interactions that could aid recognition of the mathematical thinking and learning.

3.5.4 Research

The MAAAD for Learning framework has potential implications and applications for those who research learning and technology tools, including virtual manipulatives. Fine-grained applications of the framework, such as using it to analyze user-tool interactions during specific mathematical learning trajectories (e.g., Sarama and Clements 2009) could aid research into the potential influences of multi-touch technology on the ways that children learn mathematics (e.g., Baccaglini-Frank and Maracci 2015). Researchers could also investigate manifestations of specific attributes (e.g., representing Base 10) or attribute categories (e.g., personal, structural). Lateral applications of MAAAD for Learning include applying it to other user-tool interactions. These studies could involve different subject matter (e.g., science) to develop subject-specific variants (e.g., science attributes and scientific distance). Additional investigations could apply the framework to interactions with other technology tools (e.g., video games) in a variety of settings (e.g., classroom) involving various users (e.g., diverse learners). These applications would inform research on user-tool interactions in multiple contexts, such as using virtual manipulatives to teach children with learning disabilities in mathematics in general education classes (e.g., Satsangi and Bouck 2014).

Broader applications of the framework are also possible. Researchers could investigate MAAAD for Learning in relation to specific outcomes, such as achievement on learning assessments, particularly when conducting longitudinal examinations of user-tool interactions. This could build on research that indicates use of virtual manipulative touchscreen apps can positively influence performance on mathematical tasks (e.g., Riconscente 2013; Zhang et al. 2015). Extensions of this research could identify long-term patterns in user-tool interactions that correlate with learning outcomes.

3.6 Conclusion

The MAAAD for Learning framework models relationships among attributes, affordance-ability relationships, and distance in the context of user-tool interactions, and primarily emerges from studies focusing on interactions with mathematics virtual manipulatives. The framework can be a useful tool for developers, educators, and researchers whose work involves technology tools. Developers of technology tools can use the framework to model relationships among constructs that play a role in user-tool interactions and the resulting experiences. Educators who implement technology tools to support learning can use the framework to evaluate learning that occurs during children's classroom-based interactions with technology tools. Researchers can apply the framework to investigate constructs contributing to children's learning during interactions with technology, in addition to potential outcomes of these interactions.

Importantly, consistent use of the MAAAD for Learning framework across these applications could provide a common language for modeling and discussing user-tool interactions. These applications may also lead to further development of the framework, such as through clarification of constructs, relationships, and emergent themes, or creation of versions for different content areas. To aid both aims, it may be beneficial to clarify potential differences between a tool and an object embedded within the tool (e.g., attributes of the touchscreen device, the app, and each mathematics virtual manipulative), or if the entire tool should be considered as one when interacting with the embedded object (i.e., attributes of a mathematics virtual manipulative touchscreen app). Future research involving connections to learning outcomes, diverse populations, various contexts, different content areas, and additional technology tools will advance the literature concerning user-tool interactions and contribute to the development and application of the MAAAD for Learning framework.

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Chapter 4
Physical and Virtual Manipulatives: What Is "Concrete"?

Julie Sarama and Douglas H. Clements

Abstract We discuss research on both physical manipulatives and virtual manipulatives to provide a framework for understanding, creating, implementing, and evaluating efficacious manipulatives—physical, virtual, and a combination of these two. We provide a theoretical framework and a discussion of empirical evidence supporting that framework, for the use of manipulatives in learning and teaching mathematics, from early childhood through the elementary years. From this reformulation, we re-consider the role virtual manipulatives may play in helping students learn mathematics. We conclude that manipulatives are meaningful for learning only with respect to learners' activities and thinking and that both physical and virtual manipulatives can be useful. When used in comprehensive, well planned, instructional settings, both physical and virtual manipulatives can encourage students to make their knowledge explicit, which helps them build Integrated-Concrete knowledge.

4.1 Physical and Virtual Manipulatives: What Is "Concrete"?

"Young students need to learn concretely." "Concrete manipulatives should be used in early and elementary education to teach mathematics." "Teaching and learning should proceed from the concrete to the abstract." These are commonly accepted generalizations in most educational circles. Although they capture a good deal of wisdom, unreflective application of them can lead not only to missing important nuances, but even to ineffective educational practices. This is not a new concern. Consider the caution from John Dewey, written nearly a century ago.

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> The maxim enjoined upon teachers, "proceed from the concrete to the abstract," is familiar rather than wholly intelligible. Few who read and hear it gain a clear conception of the starting point, the concrete; of the nature of the goal, the abstract; and of the exact nature of the path to be traversed in going from one to the other. At times the injunction is positively misunderstood, being taken to mean that education should advance from things to thought—as if any dealing with things in which thinking is not involved could possibly be educative. So understood, the maxim encourages mechanical routine or sensuous excitation at one end of the educational scale—the lower—and academic and unapplied learning at the upper end. (Dewey 1933, p. 220)

In this chapter, we discuss research on both physical manipulatives and virtual manipulatives to provide a framework for understanding, creating, implementing, and evaluating efficacious manipulatives—physical, virtual, and a combination of these two. We provide a theoretical framework and a discussion of empirical evidence supporting that framework, for the use of manipulatives in learning and teaching mathematics, from early childhood through the elementary years. From this reformulation, we reconsider the role virtual manipulatives may play in helping students learn mathematics.

4.2 Research on Manipulatives in Mathematics Education

Research on using manipulatives in learning and teaching mathematics generally indicates that students who use manipulatives in their mathematics classes learn more than those who do not (Butler et al. 2003; Driscoll 1983; Greabell 1978; Guarino et al. 2013; Johnson 2000; Lamon and Huber 1971; Lane 2010; Lesh and Johnson 1976; Raphael and Wahlstrom 1989; Sowell 1989; Suydam 1986; Thompson 2012). Some studies showed an increase on assessments of retention and problem solving.

Even some early studies, however, found that students *not* using Cuisenaire rods to learn multiplication as repeated addition scored higher than students using manipulatives on a transfer test (Fennema 1972). Second graders were taught multiplication as repeated addition with manipulatives or symbolically (e.g., 2 + 2 + 2). Both groups learned multiplication but the symbolic group scored higher on a transfer test. All teachers in this study emphasized learning with understanding whether using manipulatives, mental math, or paper and pencil. A study on geometry similarly showed lower achievement for those using manipulatives (Palardy and Rumberger 2008).

One possibility is that instruction does not adequately promote *connection* between children's representations based on manipulatives and those based on paper and pencil (e.g., Carnine et al. 1997; Sherman and Bisanz 2009). For example, in one study, students who performed subtraction well with manipulatives performed the worst with paper and pencil, and vice versa (Resnick and Omanson 1987). So, the researchers tried "mapping instruction," designed to help children to connect their "concrete" knowledge shown by their use of manipulatives to symbolic work with numerals. Unfortunately, this did not work particularly well. The only children who benefited were those who received extensive instruction and used that time to make more correct verbalizations of the quantities involved in renaming. Thus, it was not simple “concrete” experience that helped but rather attention to quantities. Concrete objects may play an important role but they need to be used carefully to create a strong understanding and justification for each step of a procedure (Resnick and Omanson 1987; see also Thompson and Thompson 1990).

In a similar vein, it is not just symbols that are too often learned by rote. Students often learn to use manipulatives by rote (Miura and Okamoto 2003). They perform the correct steps but have learned little more about the manipulation of quantities. One student used beans and beansticks to model place value, but used the (one) bean as ten and the beanstick (with ten beans on it) as one (Hiebert and Wearne 1992). In a similar study, kindergarten students could not use simple cubes to help them solve simple addition and subtraction problems. They did not have a strategy to use the cubes to solve the problems. Using a number line was even more difficult (Skoumpourdi 2010). These and other studies support an essential point: Although they provide support and mediation, manipulatives do not “carry” mathematical ideas directly to the learner. “Although kinesthetic experience can enhance perception and thinking, understanding does not travel through the fingertips and up the arm” (Ball 1992, p. 47).

Given these cautions and nuances, a final concern is that teachers frequently use manipulatives as a main way of *reforming* their teaching. However, they often do not reflect on their use of representations of ideas or on the other aspects of their instruction that must be changed (Grant et al. 1996)—manipulatives are an end in themselves. Both teachers and parents often believe that reform in mathematics education indicates that “concrete” is good and “abstract” is bad. Infrequently discussed in early childhood are the opposite view of some mathematicians and mathematics teachers of older students, who argue that “abstract” mathematics is worthwhile, and instruction with concrete (or “real-world”) representations is a waste of time.[1]

A similar debate emerges when we consider *virtual* manipulatives—those on screens. If they are not concrete in the physical sense, how can they serve the expected role? Surprisingly, contrary to our intuition, technology might provide representations that are just as personally meaningful to students as physical manipulatives (Yerushalmy 2005). Research indicates that technology-based representations may even be more manageable, “clean,” flexible, and extensible than their physical counterparts.[2] For example, a computer base ten environment offered

[1]This view was expressed by several members of the National Mathematics Advisory Panel (2008), of which Clements was a member.

[2]This is why we prefer the term “technological” instead of “virtual” manipulatives. Although we use the latter to be consistent with this book, “virtual” means “not physically existing.” Although of course they are not physical in the same way, technological screens *do* physically exist. More important, children’s *phenomenological experience with and actions on them* are what matters, and we find few differences to call one physically more “real.”

greater control and flexibility to students than base ten materials (Char 1989). Similarly, students who used physical and software manipulatives demonstrated a greater sophistication in classification and logical thinking than did a control group that used physical manipulatives (Olson 1988). A single-group pre-post design revealed that third graders working with virtual manipulatives made statistically significant gains learning fractional concepts (Reimer and Moyer 2004). Qualitative evidence indicated that the virtual manipulatives helped students because they were easier and faster to use and because they provided immediate and specific feedback. A case study revealed that third graders using either virtual manipulatives or concrete manipulative had greater increase in test results than those who did not (Lane 2010).

Most convincing, a meta-analysis of 66 research studies reported a moderate effect of virtual manipulatives on mathematics learning (Moyer-Packenham and Westenskow 2013). Effects varied by counterfactual, mathematical topic, grade level, and study duration. The largest effects were those that compared virtual manipulatives alone or in combination with physical manipulatives with textbook use. Even in comparison with physical manipulatives alone, virtual manipulatives alone or in combination yielded at least small effects, suggesting that they have unique advantages (Moyer-Packenham and Westenskow 2013).

Thus, research suggests that instruction should start off with "concrete" representations, but also that manipulatives do not guarantee meaningful learning (e.g., Gagatsis 2003; Johnson 2000; MacDonald et al. 2012; Martin et al. 2007; Sherman and Bisanz 2009; Skoumpourdi 2010) and may not need to be physical. To understand the role of so-called concrete manipulatives and any concrete-to-abstract pedagogical sequence (cf. Martin 2009) we need to discuss what we *mean* by "concrete."

4.3 The Meaning of "Concrete" in Education

When they discuss "concrete manipulatives," many teachers mean physical objects that students can hold (Rao et al. 2009). This sensory nature is assumed to make manipulatives "real," connected with one's experience in the physical world, and therefore foundational to learning. There are, however, problems with these assumptions (Metz 1995).

One problem is assuming that concepts can be "read off" manipulatives. That is, students may hold, move, and arrange physical objects without thinking about the concepts. Consider the experiences of the sensitive and insightful teacher-author John Holt. Working with Cuisenaire rods, Holt and his fellow teacher "were excited about the rods because we could see strong connections between the world of rods and the world of numbers. We therefore assumed that children, looking at the rods and doing things with them, could see how the world of numbers and numerical operations worked. The trouble with this theory is that [my colleague] and I already knew how the numbers worked. We could say, 'Oh, the rods behaved just the way numbers do.' But if we hadn't known how numbers behaved, would looking at the rods enable us to find out? Maybe so, maybe not" (Holt 1982, p. 138). To return to John Dewey:

> Instruction in number is not concrete merely because…beans or dots are employed…. If the physical things used in teaching number … do not leave the mind illuminated with recognition of a *meaning* beyond themselves, the instruction that uses them is as abstruse as that which doles out ready-made definitions and rules, for it distracts attention from ideas to mere physical excitations. (cf. Bana and Nelson 1978; Dewey 1933, p. 224, emphasis in original)

Another problem with this perspective is that, even if children begin to link between manipulatives and nascent ideas, physical actions with certain manipulatives may suggest different mental actions than students are to learn. For example, when students use number lines to add 5 + 2, students may locate 5, count "one, two" and read the answer, "7." However, this does not help them solve the problem mentally, because they would have to count "six, seven" and at the same time *count the counts*—6 is one, 7 is two (Gravemeijer 1991; see also Sarama and Clements 2009b). Similarly, students' external actions on an abacus are not consistent with the mental activity we want to develop. Indeed, some authors believe that the number line model does not help young children learn addition and subtraction, and that, certainly, using the number line model to assess children's knowledge of arithmetic makes several important assumptions about what else they know (Ernest 1985). In any case, the number line cannot be viewed as a "transparent" model (Núãez et al. 2012); if used, it must be taught. Similarly, second graders did not learn more sophisticated strategies (e.g., adding 63 and 26 by counting by tens: "63, …, 73, 83, 84, 85, …, up to 89") using a hundreds board, because it did not correspond to students' activity or help them to build useful figural imagery that supported creation of abstract composite units of ten (Cobb 1995).

Although manipulatives may often play an important role in learning, their *physicality* does not carry the meaning of the mathematical idea. Students may benefit from physically concrete materials—or even their virtual counterparts—to build meaning initially, but they must *reflect on and talk about their actions* with manipulatives to do so. "Concrete" may not equal "physical." Consider that we eventually appreciate students having a kind of "concrete" understanding that goes beyond physicality. For example, good students often mentally manipulate quantities *as if* they were concrete. A child who is mentally subtracting 72 − 37 might operate on mental representations that sound physical—"I counted to find out how much to add to 37 to make 72. So I put 3 ones on the 37 to make 40, 2 more to make 42, that's umm, 5 ones I have. Then, I stuck on 3 tens to make 72, so I put on 35 in all." A child with a different "concrete" understanding of the quantities may "break apart" the number itself, as in "I took the tens and ones apart. Then I separated the 3 tens away from the 7 tens and that left 4 tens. Then I took 2 off the 42, and I got 40, but I had to take 5 more off, so 35" (Cobb et al. 1996; Sarama and Clements 2009b). Such "concrete" understanding results from complex networks of knowledge that connect numbers and number relations to meaningful experiences with sensory, active, and social contexts. The connections in networks like these support arithmetic and algebraic thinking (Schliemann et al. 2007).

This is why we say that there are *different types* of concrete knowledge. Students with *Sensory-Concrete* knowledge need sensory material to make sense of a concept or procedure (Clements 1999; Clements and McMillen 1996; Sarama and Clements 2009a). For example, most children do not solve larger-number problems without the support of concrete objects until 5.5 years of age (Levine et al. 1992). Young children can solve complex equivalence problems with manipulatives when presented in a non-symbolic context (Sherman and Bisanz 2009), and may not be able to solve even the simplest of problems without such sensorial, concrete support (Baroody et al. 2007). Manipulating the sensory material (physical or virtual) supports students' thinking (Correa et al. 1998). *But these action schemes are themselves abstractions* (cf. Dewey 1933).

Integrated-Concrete knowledge is knowledge that is *connected* in specific ways (Clements and McMillen 1996). Consider the root of the term. Concrete sidewalks are strong due to the combination of separate particles in an interconnected mass ("concrete" means *to grow together*). What gives Integrated-Concrete thinking its strength is the combination of separate ideas in an interconnected knowledge network. For students with this type of interconnected knowledge, knowledge of physical objects, actions performed on them, and symbolic representations are all interrelated in a strong mental structure (cf. Martin 2009).

An idea is not either concrete or not concrete. Rather, depending on how you think about it, on what kind of *relationship* you have with it (Wilensky 1991), it might be Sensory-Concrete, abstract-only, or Integrated-Concrete (Clements 1999). What ultimately makes mathematical ideas Integrated-Concrete is not their physical characteristics, but how "meaningful"—connected to other ideas and situations—they are. *Good manipulatives and good education with manipulatives provides students with meaningful material from which students can build, strengthen, and connect powerful representations of mathematical ideas.*

Comparing the two levels of concrete knowledge, we see a transformation in what "concrete" describes. In *Sensory-Concrete*, it refers to the support of sensory (physical or virtual) objects and their manipulation to cognitive actions. In *Integrated-Concrete, it* refers to knowledge that is "concrete" at a higher level because they are connected to other knowledge, both physical/sensory knowledge that has been abstracted and thus distanced from sensory objects and abstract knowledge. Such knowledge consists of units that "are primarily *concrete*, embodied, incorporated, lived" (Varela 1999, p. 7). Ultimately, these are descriptions of changes in the configuration of knowledge as children develop. Consistent with other theoreticians (Anderson 1993), we do not believe there are fundamentally different types of knowledge, such as "concrete" versus "abstract" or "concrete" versus "symbolic" (Sarama and Clements 2009b). However, one tenet of the theory of *hierarchic interactionalism is* "*cyclic concretization*"—*that the* developmental progressions of learning trajectories proceed from Sensory-Concrete and implicit levels at which perceptual Sensory-Concrete supports are necessary and reasoning is restricted to limited cases (such as small numbers) to more explicit, verbally-based (or enhanced) generalizations and abstractions that are tenuous to Integrated-Concrete understandings relying on internalized mental representations that serve as mental models for operations and abstractions that are increasingly sophisticated and powerful. Such progressions can cycle within domains and contexts (Sarama and Clements 2009b).

Perhaps most surprisingly, then, manipulatives do not have to be physical objects. Teachers of later grades expect students to have a “concrete understanding” that goes beyond manipulatives. For example, we like to see that numbers—as mental objects (“I can think of 43 + 26”)—are “concrete” for older students.

> Whenever the use and bearing of number relations are clearly perceived, a number idea is concrete even if figures alone are used. Just what sort of symbol it is best to use at a given time—whether blocks, or lines, or figures—is entirely a matter of adjustment to the given case. (Dewey 1933, p. 224)

Thus, people have Sensory-Concrete knowledge when they need to use sensory material to make sense of an idea. For example, at early stages, children cannot count, add, or subtract meaningfully unless they have actual things (usually, these are young children, but also older students with mathematical difficulties or disabilities). Consider Brenda, a primary grade student in a teaching experiment (Steffe and Cobb 1988). The interviewer had covered four of seven squares with a cloth, told Brenda that four were covered, and asked how many in all. Brenda tried to raise the cloth but was thwarted by the interviewer. She then counted the three visible squares.

B 1, 2, 3 (touches each visible item in turn)
I There’s four here (taps the cloth)
B (Lifts the cloth, revealing two squares) 4, 5. (touches each and puts cloth back)
I OK, I’ll show you two of them (shows two). There’s four here, you count them
B 1, 2 (then counts each visible): 3, 4, 5
I There’s two more here (taps the cloth)
B (Attempts to lift the cloth.)
I (Pulls back the cloth.)
B 6, 7 (touches the last two squares) (Steffe and Cobb 1988).

Brenda’s attempt to lift the cloth indicates that she was aware of the hidden squares and wanted to count the collection. This did not lead to counting because she could not yet coordinate saying the number word sequence with items that she only imagined. She needed physically present items to count. Note that this does not mean that manipulatives were the original root of the idea. Research tends to indicate that is not the case. However, there appears to be a level of thinking when children can solve tasks with physical objects that they cannot solve without such objects. For example, consider asking a girl who just turned 4 years of age to add small numbers with and without blocks (“bricks”) (Hughes 1981).

Examiner Let's just put one more in (does so). Ten and one more, how many's that?
Child Err … (thinks) … eleven!
E Yes, very good. Let's just put one more in (does so). Eleven and one more, how many's that?
C Twelve!

Five minutes later, with the bricks put away:

E I'm just going to ask you some questions, OK? How many is two and one more?
C (No response.)
E Two and one more, how many's that?
C Err … makes
E Makes … how many?
C Err … fifteen (in couldn't-care-less tone of voice) (Hughes 1981, p. 216).

The following involved a slightly older boy.

E What's three and one more? How many is three and one more?
C Three and what? One what? Letter—I mean number?
(We had earlier been playing a game with magnetic numbers, and he is presumably referring to them here.)
E How many is three and one more?
C One more what?
E Just one more, you know?
C I *don't* know (disgruntled) (Hughes 1981, p. 218).

This is consistent with research showing that most children do not solve larger-number problems without the support of concrete objects until 5.5 years of age (Levine et al. 1992), but have also developed the ability to convert verbal number words to quantitative meaning (Fuson 1992a, b). Preschoolers are more successful solving arithmetic problems when they have blocks available (Carpenter and Moser 1982) and may not be able to solve even the simplest of problems without such sensory, concrete support (Baroody et al. 2007). At an even younger age, researchers argue that children have a relatively concrete understanding of number until they learn number words. At that point, they gain a more abstract understanding (Spelke 2003). In summary, those with Sensory-Concrete knowledge need to use or at least refer directly to sensory material to make sense of a concept or procedure (Jordan et al. 1994). Such material often facilitates children's development of mathematical operations by serving as material support for children's action schemes (Correa et al. 1998). This does not mean that their understanding is only concrete; even infants make and use abstractions in thinking (Gelman 1994).

As another example, preschoolers understand—at least as "theories-in-action"—principles of geometric distance and do not need to depend on concrete, perceptual experience to judge distances (Bartsch and Wellman 1988).

Concrete "versus" abstract? Then what of abstraction? Some decry limited abstract knowledge. This can occur: "Direct teaching of concepts is impossible and fruitless. A teacher who tries to do this usually accomplishes nothing but empty verbalism, a parrot-like repetition of words by the child, simulating a knowledge of the corresponding concepts but actually covering up a vacuum" (Vygotsky 1934/1986, p. 150). This is abstract-only knowledge.

However, abstraction is not to be avoided at any age (Dewey 1933). Mathematics is about abstraction and generalization. "Two"—as a concept—is an abstraction. Further, even infants use conceptual categories that are abstract as they classify things (Lehtinen and Hannula 2006; Mandler 2004), including by quantity. These are enabled by innately specified knowledge-rich predispositions that give children a head start in constructing knowledge. These are "abstractions-in-action," not represented explicitly by the child but used to build knowledge (Karmiloff-Smith 1992). When an infant says "two doggies," she is using abstraction structures of numerosity to label a concrete situation. Thus, the situation is analogical to Vygotsky's (1934/1986) formulation of spontaneous versus scientific ("abstract") concepts in that abstractions-in-action guide the development of concrete knowledge and eventually, depending largely on social mediation, become explicated as linguistic abstractions. The result is Integrated-Concrete knowledge. Ideas such as "4," "3/7," and "rhombus" become as real, tangible, and as strong as a concrete sidewalk. Each idea is as concrete as a pliers is to a carpenter—an accessible and useful tool. An idea is not simply concrete or not concrete. We as educators cannot engineer mathematics into Sensory-Concrete materials because ideas such as number are not "out there." As Piaget has shown us, they are constructions—reinventions—of each human mind. "Fourness" is no more "in" four blocks than it is "in" a picture of four blocks. The child creates "four" by building a representation of number and connecting it with either physical or pictured blocks (Clements 1989; Clements and Battista 1989; Kamii 1986).

As Piaget's collaborator Hermine Sinclair says, "… numbers are made by children, not found (as they may find some pretty rocks, for example) or accepted from adults (as they may accept and use a toy)" (Sinclair, Forward, in Steffe and Cobb 1988, p. v). How does one help children build such representations? We often assume that more able or older students' greater facility with mathematics stems from their greater knowledge of mathematical procedures or strategies. However, it is more often true that younger children possess the relevant knowledge but cannot effectively create a mental representation of the necessary information (Greeno and Riley 1987). This is where good manipulatives can play a role.

Are abstract ("only") ideas ever worthwhile? Yes, there is a level of abstraction above Integrated-Concrete that intentionally strips concrete connections away, moving close to a level of "pure thought"—thinking as a means to generate more thinking—developing conceptual generalities that can serve critical analytic and theoretical purposes. Such levels are common in the work of mathematicians and are recognized in educational writings (Dewey 1933; van Hiele 1986). Students can engage in interesting work with mathematical systems with symbols and rules that lack palpable connections with real-world situations. However, such successful examples are usually built upon a foundation of Integrated-Concrete knowledge (cf. De Lange 1987).

4.4 Research-Based Guidelines for Teaching with Physical and Virtual Manipulatives

What role should manipulatives play in supporting the development of such knowledge? Research offers some guidelines.

- *Model with manipulatives*. We noted that young children can solve problems and, at the earliest ages, appear to need concrete manipulatives—or, more precisely, Sensory-Concrete support—to do so. One study showed higher achievement in children who used manipulatives for counting tasks (Guarino et al. 2013). However, the key is that they are successful because they can model the situation (Carpenter et al. 1993; Outhred and Sardelich 1997). Nevertheless, early number recognition, counting, and arithmetic may require (recall Brenda), or benefit from, the use of Sensory-Concrete support, if they help children investigate and understand the mathematical structures and processes. For example, children benefitted more from using chenille sticks than pictures to make non-triangles into triangles (Martin et al. 2007). They merely drew on top of the pictures but they transformed the non-triangles made with chenille sticks, which is more likely to expand the actions and their thinking. Again, though, the key is not necessarily that the sticks were physical and the pictures were not, but instead that the sticks were "manipulable." Recall that manipulative base 10 representations on computer screens were just as or more supportive of children's learning (Char 1989). One sequence of studies showed that 3-year-olds who used more "interesting" manipulatives (fruit instead of plain blocks) were more likely to accurately identify numbers in a recall task and answer subtraction questions correctly. There was no difference in children's attentiveness to the lesson. The authors provide no additional interpretations, but connections to children's existing experiences, perhaps building more elaborated mental models, may have accounted for the difference (Nishida and Lillard 2007a, b).
- *Use manipulatives to represent mathematical ideas*. Too often, manipulatives are used to "make math fun," where "manipulative math" and "real math" are seen as different enterprises (Moyer 2000). Manipulatives are used as a diversion, frequently because teachers and software designers may sometimes not understand their role as representations of mathematical ideas. Justification for their instructional role is often that they are "concrete" and thus "understandable." We have seen, however, that—like beauty—"concrete" is, quite literally, in the mind of the beholder.

- *Ensure manipulatives serve as symbols*. This is closely linked to the previous point. Multiple studies (Munn 1998; Uttal et al. 1997) support this guideline: Physical "concreteness" is not necessarily an instructional advantage. It can make it difficult for children to use a manipulative as a symbol. To be useful, children must interpret the manipulative as representing a mathematical idea. A second example comes from early introduction of algebraic thinking. When the goal is abstraction, concrete materials may not be particularly helpful. For example, working with differences in children's heights (e.g., Mary is 4 in. taller than Tom), agreeing that Tom's height would be T, children resisted representing Mary's height as "T + 4," preferring "M" (Schliemann et al. 2007). Others solved some problems but still said "T" stood for "tall" or "ten." Also, students tended to think of the differences in height as the (absolute) heights. Part of their difficulty was thinking of any letter as a variable amount when the concrete situations used in the instruction implied that there was a particular quantity—unknown, perhaps, but not one that varies. That is, children could think of the value of a height, or the amount of money in a wallet as unknown, or a "surprise," but had difficulty thinking of it as a range of values. In contrast, they learned more from playing activities such as "guess my rule," in which the context was simply mathematics, not with physical manipulatives, objects, or settings. The pure number activities were meaningful and had advantages in helping children from a low-performing school to think about numerical relationships and to use algebraic notations. In summary, the relationship of manipulatives to the concepts they are to represent is not transparent to children (Uttal et al. 1997). Children must be able to see the manipulative as a symbol for a mathematical idea. This may be why using "bland," compared to realistic, manipulatives are more likely to serve as symbols, even for children as young as preschoolers (Carbonneau 2015; Nishida and Lillard 2007a, b). In addition, in some contexts the physicality of a manipulative may interfere with students' mathematical development, and other representations, including virtual manipulatives, may be more effective for learning. Further, active teaching must guide children to make, maintain, and use manipulatives as symbols or tools for doing mathematics (Carbonneau and Marley 2015). Connecting manipulative work (e.g., place value blocks) with verbalizations and representations can build both concepts and skills successfully (Brownell and Moser 1949; Fuson and Briars 1990; Fyfe et al. 2014; Hiebert and Wearne 1993; Murata 2008; Vitale et al. 2014). Thus, children must construct, understand, and use the structural similarities between any representation and the problem situation to use objects as tools for thinking. When children do not see those similarities, manipulatives may fail to help, and many even hinder, problem solving and learning (Outhred and Sardelich 1997). As we saw in the previous section, if they do not mirror the mental actions we wish children to develop, their use could be a waste of time or even counterproductive. Manipulatives, drawings, and other representations should as much as possible, be used instructionally in ways consistent with the mental actions on objects that students are to develop.

- *Encourage appropriate play with manipulatives*. Is it good to let children play with manipulatives? Usually yes, sometimes no. Most teachers recognize that if young children have not explored a manipulative on their own (say, toy dinosaurs), getting them to address the teachers agenda (say, counting) can be at best inefficient, and at worst, a serious struggle. Further, children can and do learn pre-mathematical foundations through their self-directed play, especially with structured manipulatives, such as pattern blocks or building blocks (Seo and Ginsburg 2004). However, these experiences are rarely mathematical without teacher guidance. Children, as young as preschoolers, learn less without guidance (Carbonneau and Marley 2015). Further, counter intuitively, play can sometimes be counterproductive. When a sensory object is intended to serve as a symbol, playing with the object can interfere with understanding. For example, having children play with a model of a room decreased young children's success in using it as a symbol in a map search task, and eliminating such play increased their success (DeLoache et al. 1997). Thus, the purpose and intended learning with the manipulatives must be considered carefully within each context.
- *Use few manipulatives well*. Some research indicates the more manipulatives, the better. However, U.S. teachers tend to use different manipulatives to increase "motivation" and "make math more fun" (Moyer 2000; Uttal 1997). Further, Diénès' (1971) "multiple embodiment" theory suggests that to truly abstract a mathematical concept, students need to experience it in more than one context. However, there are opposing practices and evidence. Successful teachers in Japan tend to reuse the same manipulatives repeatedly (Uttal 1997). Research indicates that, indeed, deeper experience with one manipulative is more productive than equivalent experiences using various manipulatives (Hiebert and Wearne 1996). The former is what successful Chinese teachers do (Ng and Rao 2010). A synthesis may be that multiple representations are useful (e.g., a physical manipulative and corresponding virtual manipulative, drawings, verbalizations, symbols), but many different manipulatives may be less useful. These few manipulatives should be used for multiple tasks, so children do not view them as objects to play with but tools for thinking (Sowell 1989). Do not neglect fingers as manipulatives, as they play a fundamental role (Crollen and Noël 2015).
- *Use caution in beginning with "prestructured" manipulatives*. We must be wary of using "prestructured" manipulatives—ones where the mathematics is built in by the manufacturer, such as base-ten blocks (as opposed to interlocking cubes). They can be as colored rods for John Holt's students—"another kind of numeral, symbols made of colored wood rather than marks on paper" (Holt 1982). Sometimes the simpler the better. For example, educators from the Netherlands found students did not learn well using base-ten blocks and other structured base-ten materials. There may have been a mismatch between trading one base-ten block for another and the actions of mentally separating a ten into ten ones or thinking of the same quantity simultaneously as "one ten" and "ten ones." The Netherlands' students were more successful hearing a story of a sultan who often wants to count his gold. The setting of the story gave students a reason for counting and grouping: The gold had to be counted, packed, and sometimes

unwrapped—and an inventory constantly maintained (Gravemeijer 1991). So, students might best start using manipulatives with which they create and break up groups of tens into ones (e.g., interlocking cubes) rather than base-ten blocks (Baroody 1990). Settings that provide reasons for grouping are ideal.

- *Use drawings and symbols—move away from manipulatives* (Fyfe et al. 2014). Children using manipulatives in second grade to do arithmetic tend to do so even in fourth grade (Carr and Alexeev 2011). That is a failure to move along the learning trajectory. Although modeling necessitates manipulatives at some early levels of thinking, even preschoolers and kindergartners can use other representations, such as drawings and symbols, with, or instead of, manipulatives (Carpenter et al. 1993; Outhred and Sardelich 1997; van Oers 1994). Even for children as young as 5 years of age, physical manipulatives may play a surprisingly small role. For example, in one study there was no significant difference between kindergartners accuracy in the discovery of arithmetic strategies when they were given and not given manipulatives (Grupe and Bray 1999). The similarities go on: Children without manipulatives used their fingers on 30 % of all trials, while children with manipulatives used the bears on 9 % of the trials but used their fingers on 19 % of trials for a combined total of 28 %. Finally, children stopped using external aids approximately halfway through the 12-week study. Physical objects can make an important contribution, but are not guaranteed to help (Baroody 1989; Brown et al. 2009; Clements 1999; Skoumpourdi 2010). Drawings can include models, such as the "empty number line" approach (Beishuizen 1993). Another consideration here is children's use of images. High-achieving children build images that have a spectrum of quality and a more conceptual and relational core. They are able to link different experiences and abstract similarities. Low-achieving children's images tended to be dominated by surface features. Instruction might help them develop more sophisticated images (Gray and Pitta 1999). We should choose meaningful representations in which the objects and actions available to the student parallel the mathematical objects (ideas) and actions (processes or algorithms) we wish the students to learn. We then need to guide students to make connections between these representations (Fuson and Briars 1990; Lesh 1990). Virtual manipulatives may make a special contribution to these connections and to the move towards more Integrated-Concrete representations, as we see in the following section.

4.5 The Unique Affordances of Virtual Manipulatives

As previously discussed, even if we agree that "concrete" does not equal "physical," we might have difficulty accepting objects on a tablet or computer screen as valid manipulatives. How can we explain the success of virtual manipulatives (Moyer-Packenham and Westenskow 2013), which can be just as

(Moyer-Packenham et al. 2013; Yerushalmy 2005) or more (Char 1989; Johnson-Gentile et al. 1994) effective than physical manipulatives? We believe that an overarching but underemphasized reason for the positive effects of virtual manipulatives in such studies is that virtual manipulatives provide unique affordances for the development of Integrated-Concrete knowledge. The following is an update of our theoretical framework (see Clements 1999; Clements and McMillen 1996; Moyer-Packenham and Westenskow 2013; Sarama and Clements 2006, 2009a; Sarama et al. 1996). Moyer-Packenham and Westenskow (2013) similarly found that specific researcher-reported affordances promoted learning and their results will be related to our framework in the discussion below. Perhaps the most powerful is embodying the processes we want children to develop and internalize as mental actions. Seven hypothesized, interrelated affordances follow (summarized from Sarama and Clements 2009a).

Bringing mathematical ideas and processes to conscious awareness. Most students can use physical manipulatives to perform motions such as slides, flips and turns; however, they make intuitive movements and corrections without being aware of these geometric motions. Even young children can move puzzle pieces into place without any attention to the geometric motions that can describe these physical movements. Using virtual tools to manipulate shapes can bring those geometric motions to an explicit level of awareness (Clements and Sarama 2007a). For example, preschool children were unable to explain the motions needed to make the pieces fit in a physical puzzle. These children were able to adapt to using computer tools within one session and were able to explain their actions to peers. The children in this study did not use the same tools to manipulate shapes in an exploratory environment. Therefore, without a specific task or guidance, this potential benefit was not realized (cf. Moyer-Packenham and Westenskow 2013, "focused constraint"). This brings us to an important issue: Using a mouse to perform specific geometric motions vs. a touch screen, or even holographic images, are all different ways of interacting with virtual manipulatives. In some instances, the direct manipulation of the latter two may be helpful, such as direct touching of screens with fingers to enact numbers in different ways (Barendregt et al. 2012). However, in this case, manipulation of an icon representing the mathematical process was more beneficial. As direct manipulation environments increasingly approach physical manipulation, this advantage, if not considered in the design, could be lost.

Encouraging and facilitating complete, precise, explanations. Compared to students using paper and pencil, students using computers work with more precision and exactness (Clements et al. 2001; Gallou-Dumiel 1989; Johnson-Gentile et al. 1994). One study included two treatments to teach geometric transformations, symmetry, and congruence. One of the treatments used specially-designed Logo computer environments to provide computer actions (geometric motions) on virtual manipulatives (geometric figures). The other treatment group used physical manipulatives and paper and pencil. Otherwise, the curriculum and tasks were identical. Pre- and post-treatment interviews revealed that both treatment groups, especially the Logo group, performed at a higher level of geometric thinking than did a non-treatment control group. Although the two treatment groups did not significantly differ on the immediate posttest, the Logo group outperformed the nonLogo group on the delayed posttest. The Logo-based version enhanced the construction of higher-level conceptualizations of motion geometry, aiding retention (Johnson-Gentile et al. 1994; cf. Moyer-Packenham and Westenskow 2013, “efficient precision”). Such higher-level concepts are at the core of Integrated-Concrete knowledge.

Supporting mental “actions on objects.” The flexibility of virtual manipulatives allows them to mirror mental “actions on objects” better than physical manipulatives. For example, physical base-ten blocks can be so clumsy and the manipulations so disconnected one from the other, that students tended to act on physical pieces but not the place value ideas. In virtual manipulatives, students can break computer base-ten blocks into ones, or glue ones together to form tens. Such actions are more in line with the *mental actions* that we want students to learn (cf. Thompson 1992) and are at the heart of what we mean by Integrated-Concrete knowledge.

Geometric tools can encourage mental composition and decomposition of shapes (Clements and Sarama 2007b; Sarama et al. 1996). In an observational study of young children’s use of physical and virtual manipulatives, kindergartner Mitchell started making a hexagon out of triangles on the computer (Sarama et al. 1996). After placing two, he counted with his finger on the screen around the center of the incomplete hexagon, imaging the other triangles. Whereas off-computer, Mitchell had to check each placement with a physical hexagon, the intentional and deliberate actions on the computer lead him to form mental images (decomposing the hexagon mentally) and predict each succeeding placement.

Actions on virtual manipulatives can include precise decompositions that cannot easily be duplicated with manipulatives; for example, cutting a shape (e.g., a regular hexagon) into other shapes (e.g., not only into two trapezoids but also two pentagons and variety of other combinations). Virtual manipulatives have supported dramatic gains in this competency (Clements and Sarama 2007b; cf. Moyer-Packenham and Westenskow 2013, “focused constraint” and “creative variation”; Sarama et al. 1996; Spitler et al. 2003)

Changing the nature of the manipulative. In a similar vein, virtual manipulatives’ flexibility allows children to explore geometric figures in ways not available with physical shape sets. For example, children can change the size of the computer shapes, altering all shapes or only some. One study compared how a linguistically and economically diverse population of kindergarten and second grade students worked and learned with physical, compared to virtual, manipulatives. Researchers stated that the virtual manipulative’s flexibility had several positive effects on kindergartners’ patterning (Moyer et al. 2005). They made a greater number of patterns and used more elements in their patterns when using virtual manipulatives than when using physical manipulatives or drawing. Finally, only when working on the computer did they create new shapes (e.g., by partial occluding one shape with another). Thus, this develops, potentially more than physical manipulatives, a richer network of Integrated-Concrete knowledge.

Symbolizing and making connections. Virtual manipulatives can also serve as symbols for mathematical ideas, often better than physical manipulatives. For example, the manipulative can have just the mathematical features that we wish it to have, and just the actions on it that we wish to promote, and not additional properties that may be distracting. An example is a computer game to teach motion geometry. Three modes were compared (Sedighian and Klawe 1996; Sedighian and Sedighian 1996): Direct Manipulation (DM), in which a student might drag a shape to turn it; Direct Concept Manipulation (DCM), in which the student manipulated a representation of turning and angle measure, not the shape direction; and, Reflective DCM (RDCM), which included faded scaffolding (teaching help that is gradually withdrawn). Students using RDCM were significantly and substantially better than students using DCM versions, which were again significantly better than students using versions with conventional direct manipulation. This indicates the primary importance of careful design in the development of virtual manipulatives.

Linking the concrete and the symbolic with feedback. Closely related, the computer can link manipulatives to symbols—the notion of multiple linked representations. For example, the number represented by the base-ten blocks is dynamically linked to the students' actions on the blocks, so that when the student changes the blocks the number displayed is automatically changed as well. This helps students make sense of their activity and the numbers.

Virtual manipulatives can also *connect* objects that you make, move, and change to other representations. For example, students can draw rectangles by hand, but never go further thinking about them in a mathematical way. In Logo, however, students direct the movements of an on-screen object (often a "turtle") with geometric commands such as "FORWARD 100" and "RIGHT 90" (degrees). In so doing, students must analyze the figure to construct a sequence of commands to draw a rectangle. So, they have to apply numbers to the measures of the sides and angles (turns). This helps them become explicitly aware of such characteristics as "opposite sides equal in length." The link between the symbols, the actions of the turtle object, and the figure are direct and immediate (Clements and Battista 1989, 1992; Clements and Meredith 1993; cf. Moyer-Packenham and Westenskow 2013, "simultaneous linking").

Is it too restrictive or too hard to have to operate on symbols rather than directly on the manipulatives? Ironically, less "freedom" might be *more* helpful. In a study of place value, one group of students worked with a computer base-ten manipulative. The students could not move the computer blocks directly. Instead, they had to operate on symbols (Thompson 1992; Thompson and Thompson 1990). Another group of students used physical base-ten blocks. Although teachers frequently guided students to see the connection between what they did with the blocks and what they wrote on paper, the physical blocks group did not feel constrained to write something that represented what they did with blocks. Instead, they appeared to look at the two as separate activities. In comparison, the computer group used symbols more meaningfully, tending to connect them to the base-ten blocks.

In technology environments such as computer base-tens blocks or computer programming, students may not be able to overlook the consequences of their actions because the computer offers immediate direct feedback, whereas such feedback is absent from most work with physical manipulatives. So, virtual manipulatives can help students build on their physical experiences, tying them tightly to symbolic representations. In this way, computers help students link Sensory-Concrete and abstract knowledge so they can build Integrated-Concrete knowledge.

Recording and replaying students' actions. Computers allow us to store more than static configurations. Once we finish a series of actions, it is often difficult to reflect on them. But computers have the power to record and replay *sequences* of our actions on manipulatives. We can record our actions and later replay, change, and view them.

As previously mentioned, tablets have opened a new way of interacting with and using virtual manipulatives. Children of different ages interact with and benefit from them in different ways (Moyer-Packenham et al. 2015). Research and development in this area should be supported.

4.6 Final Words: Manipulatives and Integrated-Concrete Ideas

Manipulatives are meaningful for learning only *with respect to learners' activities and thinking*. Physical and virtual manipulatives can be useful, but will be more so when used in comprehensive, well planned, instructional settings. Their physicality is not important—their *manipulability* and *meaningfulness* make them educationally effective. In addition, some studies suggest that virtual manipulatives can encourage students to make their knowledge explicit, which helps them build Integrated-Concrete knowledge. Such knowledge includes both "concrete" and "abstract" thinking. "A person who has at command both types of thinking is of a higher order than he who possesses only one" (Dewey 1933). Moreover, Integrated-Concrete knowledge is the *synergistic combination* of these types and should be a main goal of the use of manipulatives.

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Chapter 5
Manipulatives, Diagrams, and Mathematics: A Framework for Future Research on Virtual Manipulatives

Helena P. Osana and Nathalie Duponsel

Abstract Our objective in this chapter is to present a framework that can be used as a guide for designers of virtual manipulatives and for researchers who study their effects on student learning in mathematics. Because a significant amount of research has been devoted to the effects of concrete manipulatives on student learning, the crux of the framework is based on the existing literature in this area. Specifically, the framework consists of three interrelated components that align with the research on students' learning with external representations: the surface features of the representations themselves, the pedagogical contexts that support students' meaning making, and the students' perceptions and interpretations of the representations. Where applicable, we integrate the research on virtual manipulatives to support the validity of the framework itself and its applicability for researchers of virtual mathematics tools.

Many teachers in elementary grades use concrete objects, or "manipulatives," in their classroom to reify concepts that are often difficult for children to grasp. The goal of using manipulatives is for students to appropriate the mathematical ideas and actions that are the referents for the objects and their manipulations (Beishuizen 1993; Chao et al. 2000; Clements 1999; Thompson 1994). Some have argued that when children become proficient in their interactions with objects, they create mental images of the concepts that the objects are intended to signify (Stigler 1984). Others, such as Dienes (1963), and more recently English (2004), suggested that by working with a variety of manipulatives, students abstract their common underlying conceptual structure. This idea was echoed by Richland et al. (2012), who referred to analogical reasoning across a number of different mathematical systems.

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A number of different types of concrete manipulatives have been used in mathematics classrooms for many years for several purposes, and researchers have examined their effects with students at both elementary and secondary levels (e.g., Carbonneau et al. 2013; Fujimura 2001; Gürbüz 2010; McNeil et al. 2009). Often used in the early grades, for example, are Base ten blocks (Dienes 1963), which are objects designed to concretely represent groups of ones, tens, hundreds, and thousands with “units” (individual blocks), “longs” (sticks of 10 units stuck together in a rod), “flats” (flat blocks made up of ten longs stuck together), and “blocks” (10 flats stacked one on top of the other). Other manipulatives, such as colored chips and Digi-Blocks™ (Corbiere 2003; www.Digi-Block.com), can also be used to help students gain a conceptual understanding of the mathematical structure of number and place value. Manipulatives also exist for teaching and learning mathematical concepts other than place value. Cuisinaire™ rods, for example, can be used to highlight concepts behind mathematical properties, such as commutativity and distributivity. Pattern blocks have been used to explore fractions concepts, and tangrams to gain conceptual proficiency in transformational geometry.

In recent years, a variety of computer programs and mobile-device applications have been developed that contain digital objects and tools aimed at improving children’s mathematics and science learning (e.g., Burns and Hamm 2011; Moyer-Packenham and Westenskow 2013; Moyer-Packenham et al. 2013, 2015). Known as “virtual manipulatives,” such digital objects have been described by Moyer-Packenham and Westenskow (2013) as “movable pictorial representations in the form of applets…or apps” (p. 35). Many environments that contain virtual manipulatives allow the user to interact with them in the context of a target mathematical activity, including representing, estimating, reasoning, problem solving, and experimenting (see Moyer-Packenham and Westenskow 2013, for a review). The ultimate goal is to provide the user opportunities for learning mathematics, which is an element retained from earlier definitions of virtual manipulatives (e.g., Moyer et al. 2002).

Although “many theories have yet to clarify what aspects of manipulative materials enhance learning” (Namukasa et al. 2009, p. 285), what has emerged from the research on concrete manipulatives is that children do not spontaneously appropriate the abstract concepts they are intended to represent, and that simply interacting with manipulatives will not guarantee learning (Ambrose 2002; Ball 1992; Osana et al. 2013; Uttal et al. 2006). Several accounts of children’s failure to construct useful meanings of manipulatives exist, and our contention in this chapter is that these explanations have implications for the design and use of virtual manipulatives. In some cases, virtual manipulatives are digital replicas of existing concrete objects (Moyer-Packenham and Westenskow 2013), such as counters and Base ten blocks; in other cases, no real-world counterparts exist for the virtual manipulatives because they are designed to capitalize on the affordances unique to the specific technology (Carpenter 2013; Namukasa et al. 2009). In either case, the object on the screen is meant to be interpreted in some way or another by the learner, and we contend that what is known about the conditions that affect students’ interpretations of concrete objects has useful implications for the design and research on virtual manipulatives.

Our objective in this chapter is to present a framework that can be used as a guide for designers of virtual manipulatives and for researchers who study their effects on student learning in mathematics. Because a significant amount of research has been devoted to the effects of concrete manipulatives on student learning, the crux of the framework is based on the existing literature in this area. Where applicable, we integrate the research on virtual manipulatives to support the validity of the framework itself and its applicability for researchers of virtual mathematics tools.

5.1 Manipulatives as Symbols in Mathematics Teaching and Learning

Concrete manipulatives vary on a number of dimensions, including their physical features, the concepts they are intended to represent, and the extent to which they "resemble" the concepts they target (Reys et al. 2014; Richland 2011). Despite the number of materials available for teachers of mathematics and the variety in their features, it is becoming increasingly clear that *the use of manipulatives is beneficial to the extent that students make clear associations between the objects themselves and the mathematical concepts they are intended to represent* (English 2004; Hiebert and Wearne 1992; Sarama and Clements 2009). More specifically, students must acquire what DeLoache et al. (1997) and Uttal et al. (2006) called dual representation, namely the understanding that manipulatives are more than objects with their own physical and perceptual features—they also represent something more abstract, such as quantities and the relationships among them. In this sense, therefore, manipulatives can be considered symbols in that they are concrete instantiations of targeted abstractions in mathematics (Goldin 1998; Nührenbörger and Steinbring 2008; Pimm 1995; Uttal et al. 2006). English (2004) maintained that at the heart of mathematical reasoning is the ability to perceive the structural relationships in the attributes of external representations and to make "mappings" to the abstract concepts they signify.

One way to conceptualize the representational role of manipulatives, and dual representation in particular, is through the lens of children's symbolic development. In their now seminal work on the development of symbolization, DeLoache and colleagues (DeLoache 1987, 1989, 1995; DeLoache and Sharon 2005; Marzolf and DeLoache 1994) demonstrated that children's ability to view objects as symbols undergoes a shift between the ages of two and a half and three years old. In a well-known study, DeLoache (1987) introduced 2½- and 3-year-old children to a room, complete with items and pieces of furniture, and a scale model of the room that was identical in every way except size. She found that compared to the younger children, the 3-year-olds were able to use the scale model as a symbol that referred to where a toy was hidden in the actual room. The children's dual representation assisted them to view the scale model as more than an object in its own right, but also one that represented something else (in this case, the actual room; DeLoache et al. 1997).

The notion of dual representation, although not typically referred to as such, has been cited as a central explanation for children's disconnected understanding of school mathematics as well (e.g., Richland 2011; Richland et al. 2012; Uttal et al. 2006). Resnick and Omanson (1987), for example, observed that children could operate with Base ten blocks quite independently of the quantitative concepts they targeted, using them procedurally without attaching meaning to the objects or understanding what their actions meant (observed more recently by Kamii et al. 2001; Puchner et al. 2008). In his theory of symbolic competence in mathematics, Hiebert (1992) proposed that children must construct meaningful and appropriate connections between symbols and their referents and come to understand how the referents support the actions performed on the symbols (e.g., algorithms). If the student fails to make a meaningful association between the symbol (e.g., "2/3") and its referent (e.g., a conceptual understanding of two-thirds as a quantity), the learning that occurs in the symbolic world stays detached from any concepts or experience that would give it meaning (see also Osana and Pitsolantis 2013). Thus, the appropriate use of manipulatives entails a shift from the object itself to seeing beyond the symbol to its referent (Uttal et al. 2006).

The framework for the design and study of virtual manipulatives we present in this chapter is based on three factors currently known to impact children's ability to view concrete objects as entities in their own right and as objects that "stand for" intended conceptual referents. More specifically, the components of the framework are (a) the perceptual features of manipulatives, (b) different types of pedagogical support, and (c) students' own interpretations of the manipulatives themselves. Despite the observation that teachers often seek out "fun" and attractive manipulatives on the assumption that these will keep students motivated (Moyer 2001; Uttal et al. 1997), we will show that perceptually rich manipulatives can, in fact, detract from the intended target concepts. Furthermore, there is evidence that teachers often do not pay enough instructional attention to the conceptual correspondences between manipulatives and their referents because the relations are obvious to them (Goswami 2004), although they are often not obvious to children. The relationship between a manipulative and its intended quantitative referent often does not occur to children automatically and is likely to remain obscure without appropriate intervention. Finally, we will review evidence that shows that children's own interpretations of manipulatives are predictive of the ways they are used and the mathematics that is learned.

The research presented in the chapter, while not exhaustive, will focus on critical aspects of knowledge representation (e.g., Markman 1999) as opposed to components of the information processing system, such as selection, attention, working memory, and long-term memory (e.g., Mayer 2011; Sweller et al. 1998). To varying degrees, we focus our analysis on the following elements of knowledge representation: (a) the "represented world" (Palmer 1978), which in the context of mathematics learning is often conceptualized as the target learning objective; (b) the "representing world" (Palmer 1978), such as manipulatives and images; (c) correspondences between the two systems (Gentner and Colhoun 2010), or "representing rules" (Markman 1999); and (d) the individual (in our case, the child or student), who engages in the process of creating a representation, by, for example, making mappings between the represented and the representing worlds (Gentner and Colhoun 2010). While Mayer's (2011) cognitive theory of multimedia learning can explain learning with the use of manipulatives, and is not incompatible with the representational lens we use here, our attention is focused on the individual's representation as the inputs to a cognitive system rather than the manipulatives and images per se (Ainsworth 2006).

In sum, there is a general consensus in both the educational psychology and mathematics education communities that concrete manipulatives are intended to be symbolic representations of abstract ideas or principles, and that certain conditions must be in place for students to appropriate the concepts that are targeted. Because mathematical proficiency, including conceptual understanding and mathematical reasoning, is ultimately the end goal for students who work with any type of tool—concrete and virtual manipulatives alike (e.g., Carbonneau et al. 2013; Ginsburg et al. 2013; Ladel and Kortenkamp 2013; Moyer et al. 2002; Martin and Schwartz 2005; Moyer-Packenham and Westenskow 2013; Segal et al. 2014; Tucker et al. 2014)—we argue that there is ample rationale for connecting the research on concrete manipulatives to the study of virtual ones. Given that many virtual manipulatives often appear to the student as two-dimensional images on the screen, we will also review some of the research on students' representations of diagrams in mathematics and science in support of the framework.

5.2 A Framework for Future Research on Virtual Manipulatives

The framework is depicted in Fig. 5.1. Its three primary components align with factors predictive of students' acquisition of dual representation and learning in mathematics: (a) characteristics of the representation itself; (b) the nature of the pedagogical support on the connection between the representation and its referent; and (c) students' knowledge and cognitions, including students' perceptions about objects and diagrams. The three components influence each other: External factors related to the representation itself and the types of pedagogical supports present in the learning environment exist in a bidirectional relationship with students' interpretations of the manipulatives, all of which together influences student learning. Each component is discussed in turn, with parallel discussions to integrate existing findings on virtual manipulatives and other digital environments.

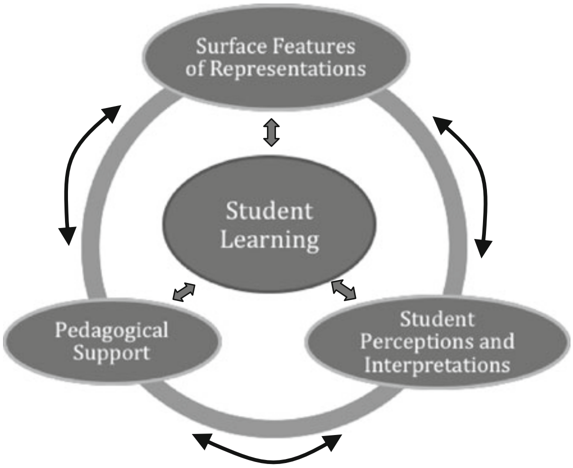

Fig. 5.1 Framework for research on virtual manipulatives

5.2.1 Surface Features of Representations

Recently, researchers have focused on the ways perceptual features of representations impact students' ability to "see through" (Callanan et al. 2002; DeLoache 2000) to their associated referents. Evidence suggests that representations with salient superficial features, such as those designed to resemble real-world objects (e.g., pizza manipulatives when teaching fractions), can actually distract the student from seeing the abstract referent the object is intended to signify (Gravemeijer 2002; Uttal et al. 1997). Uttal et al. (2009) explained that children have more difficulty creating dual representation for such realistic objects: They are attracted to the color or shape of, say, pizza slices, which then reduces the likelihood that the objects are seen as symbols representing concepts related to fractions.

The negative effects of salience (defined as an object's "transparency" as a representation; Callanan et al. 2002) on the ability to use objects as symbols appear early in life. When DeLoache (2000) allowed 3-year-old children to play with the scale model, they were less likely to use it as a symbol to find a hidden object in the actual room. She concluded that when the children played with the scale model, they interacted with it in such a way that its perceptual features were highlighted, which decreased their attention to what the object represented. When the model was placed behind a transparent barrier, however, preventing a group of 2.5-year-old children from physically interacting with it, their ability to use the model as a symbol for the actual room increased. Consistent with Uttal et al. (2009), DeLoache interpreted these findings to mean that increasing an object's salience reduces its representational status.

Along the same lines, Gelman et al. (2005) examined the types of conversations mothers had with their 2- to 3-year-old children about photographs and concrete objects (e.g., toys in the shape of fruits, miniature toy animals). The authors found that both children and mothers talked more often about the photographs as general kinds (e.g., they spoke about the objects as general categories, such as elephants in general) and more often about the concrete objects as individual things in a category (e.g., this particular elephant looks like Babar). In a second study, the authors placed the objects behind a glass case, thereby reducing their salience. With the objects behind the case, the participants made more "kind" type utterances (though still less so than when talking about the photographs) than they did with the same objects that could be manipulated in the first study. The authors explained their findings using the dual representation hypothesis: When the objects were encased, the participants were not drawn to their perceptual features and could more easily treat them as kinds and not as individuals of a kind.

The effects of the perceptual features of representations have been seen, albeit less consistently, in the mathematics context as well. For example, McNeil et al. (2009) found that fourth- and sixth-graders who used perceptually rich money manipulatives were less accurate in their problem solutions than those who used bland money manipulatives, but at the same time, produced fewer conceptual errors. Thus, it appears that perceptually rich manipulatives can, on one level, reduce students' ability to acquire dual representation—that is, their ability to see beyond the manipulative to the concepts they symbolize. On another level, however, there appear to be factors that moderate the effects of salience. In this particular study, the participants may have benefited from the activation of prior knowledge from the perceptually rich manipulatives to the extent that they could apply more conceptually sound strategies than those students who used the bland manipulatives.

Indeed, in a subsequent study, Petersen and McNeil (2013) found that the effects of the manipulatives' perceptual richness interacted with established prior knowledge. More specifically, preschoolers' performance on counting tasks using concrete counters that were colorful and attractive (i.e., rich in salience) was higher than the performance of children who used bland manipulatives, but only when the objects were unfamiliar to them (e.g., miniature pinwheels, colored plastic chips). In contrast, when the manipulatives were attractive and when they aligned with the children's prior knowledge (e.g., colorful toy strawberries or giraffes), they were less likely to think about their quantitative referents and were distracted by the objects' known purpose. Thus, it is clear that the perceptual richness of manipulatives has an effect on children's ability to see them as symbols of mathematical ideas, but there are moderating factors, such as prior knowledge, that are not yet well understood. Additionally, several questions remain about the types of mathematical thinking that are impacted by the physical characteristics of the manipulatives, which open up several avenues for future research.

In the world of virtual manipulatives, colorful and perceptually attractive representations are commonplace (see Carpenter 2013; Ginsburg et al. 2013). We conducted an informal review of educational iPad apps that incorporated the number line to teach some aspect of elementary mathematics. In the first 40 apps that appeared in our initial search, we found that 31 (77.5%) incorporated attractive perceptual features, often on the number line itself. The studies we reviewed above on children's responses to perceptually attractive manipulatives suggest, however, that it is important for developers of virtual manipulatives to attend to the digital representations themselves, but this design principle is often ignored. An object's superficial features, such as its color and other physical attributes, can draw students' attention away from the underlying idea that it is intended to target. Mirroring the research in children's cognitive development, Ladel and Kortenkamp (2013) presented a theoretical analysis of multi-touch tools in the context of early mathematics learning and argued that the features of the representation itself will govern how children interpret and use it (see also Hiebert et al. 1997).

Some empirical evidence on the perceptual features of computerized objects has recently emerged to support Ladel and Kortenkamp's (2013) analysis. Kaminski et al. (2009), for example, demonstrated that attractive, perceptually rich objects that appear in a computerized environment could also distract the learner from inducing target concepts. More specifically, the authors found that concrete representations on a computer screen (e.g., perceptually rich instantiations of elements in a mathematical system, such as pitchers of water and pizzas) hindered the learning of a mathematical rule when compared to isomorphic generic instantiations of the same rule. They concluded that perceptually rich features of representations interfere with students' internalization of the target mathematical structure.

Also in a computerized environment, Goldstone and Sakamoto (2003) again demonstrated that a representation's perceptual features can have an effect on learning and transfer, and moreover, that this effect is moderated by learners' prior knowledge. The researchers tested the effects of varying the level of similarity (i.e., the degree to which a representation resembles its referent; Gentner and Markman 1997; Richland et al. 2006) of the visual displays used in computer simulations. One group of participants explored the basic governing rules of a scientific principle known as competitive specialization by interacting with a computer simulation of ants (i.e., moving drawings of ants) seeking out food (i.e., drawings of fruit). Another group of participants explored an identical simulation, but with abstract representations of the ants and food (e.g., dots and blobs, respectively).

The authors found a relationship between level of similarity and participants' performance on a learning task that assessed their knowledge of the underlying scientific principles that governed the simulation. An overall analysis revealed that, relative to those who had interacted with the abstract representations, those who had explored the "similar" representations—those that looked like ants and food—were at an advantage on the learning task. In contrast, those same participants were hindered on a subsequent transfer task. A different picture emerged, however, when the participants were separated into groups according to their performance on the learning task: The authors found that similarity hindered transfer for the poor performers, but not for the high performers. In sum, these studies reveal that the deleterious effects of superficial features of virtual representations appear to be similar to those found in concrete environments, and that such effects are likely moderated by factors related to the learner herself, such as initial knowledge.

5.2.2 Pedagogical Support

The role of the social context and the nature of external support as factors in children's learning and development has been well documented (e.g., Callanan et al. 2002; DeLoache 1995; Gelman et al. 2005; Hiebert and Grouws 2007; Osana and Pitsolantis 2013; Osana et al. 2013; Rogoff 1990; Sarama and Clements 2002). With respect to the development of children's symbolization specifically, similar patterns emerge. For example, young children are better able to use symbols when they are given explicit explanations about the relations between the objects and their referents (e.g., DeLoache 1989). Children as young as 3 years can use concrete objects as symbols, but their performance is enhanced after having received explanations on how they relate to their referents (DeLoache et al. 1999). The research in analogical reasoning has established that explicit instruction on the relationships between source and target analogs greatly augments the likelihood that the common underlying conceptual structure will be abstracted and transferred (Gentner and Colhoun 2010; Gentner et al. 2003; Gick and Holyoak 1983; Goldstone and Day 2012; Perkins and Salomon 2012; Richland et al. 2012).

In mathematics, the level of instructional guidance that is needed for students to use manipulatives and other representations in meaningful ways is considerably less clear. One perspective on the role of pedagogical support is that explicit instruction on the relations between concrete objects and their referents is effective, even required (Goswami 2004; Sarama and Clements 2009; Uttal et al. 2006). In our own work with first-graders, we came to a similar conclusion (Osana et al. 2013; Przednowek et al. 2013). We introduced first-graders to colored plastic chips in three different ways: One group of students were explicitly told that a blue chip represented the quantity "1" and a red chip "10" (quantitative condition); a second group encoded the chips in non-quantitative ways by using them as game pieces in a board game (non-quantitative condition); and a third group encoded the chips in any way they wished in a free play setting (play condition). A control group was not introduced to the chips at all. We found that those who were explicitly told the quantitative representations of the chips were better able to use them as symbols for the intended quantities than the students in the other three conditions.

More importantly, even after a quantitative activity using the blue chips as one and the red chips as ten, the students who received explanations on the chips' quantitative meanings were at an advantage compared to the other groups on various dual representation tasks. Those who played with the chips initially were no better at appropriating their quantitative meaning than the control group, who were not exposed at all to the chips before the quantitative activity. In fact, when asked to show how to represent certain quantities with the chips, students in the play condition used the chips to physically draw out the numerals involved in the task rather than represent the quantity (see Fig. 5.2). These data illustrate how children can operate with manipulatives in ways that are seemingly entirely divorced from their quantitative meaning (see Hughes 1986), but they also underscore the importance

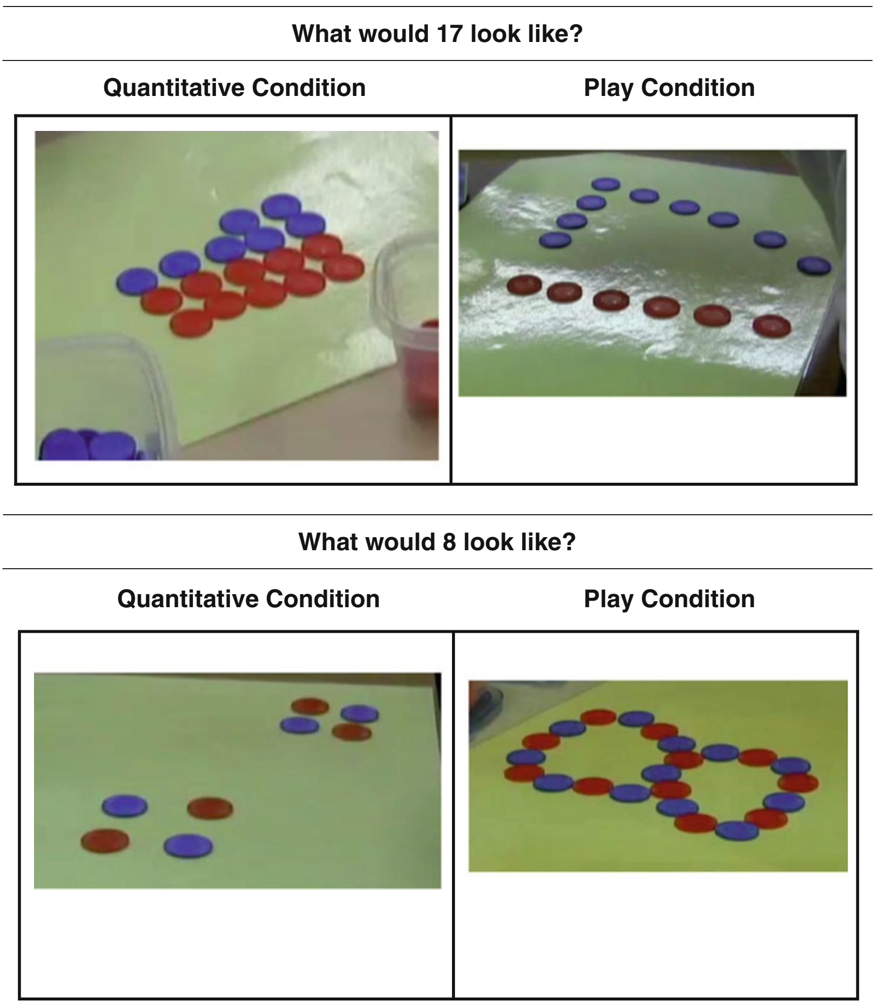

Fig. 5.2 Data from Przednowek et al. (2013) illustrating children's uses of concrete manipulatives in quantitative and non-quantitative ways

of at least some level of instructional guidance on the conceptual referents targeted by the concrete objects.

In another study from our lab (Adrien et al. 2015), we presented two different representations of quantities to second-graders: Base ten blocks and written notation. Children were asked to solve a number of multi-digit addition problems using the two representations in small groups in their classrooms. We varied the sequencing of the representations: In the manipulatives first condition, the students solved ten problems with manipulatives and then ten isomorphic problems with written symbols. In the symbols condition, the sets of problems were reversed.

In the iterative condition, the students solved problems using the representations in alternating order. We also manipulated the presence of explicit explanations on the relationships between the two representations: Half of the students in each of the sequence conditions received explanations and the other half did not.

The data revealed that, regardless of representation sequence, those students who received direct instruction on the relationship between the concrete and written representations were better able to show their conceptual knowledge on measures assessing regrouping knowledge. In the case of place value knowledge, however, sequence had an effect (both manipulatives-first and symbols-first conditions improved significantly from pretest to posttest, whereas the iterative sequence did not), but explanation was a moderating variable on sequencing. Explicit explanations resulted in improved place value knowledge, but only for students in the symbols-first condition; students in the manipulatives-first condition improved without explicit explanations, and those in the iterative group did not improve regardless of whether explanations were provided (see Fyfe et al. 2014, for a theoretical account of this finding). Together, these results reveal the benefits of explicit instruction on the meaning of the concrete materials, but also that other aspects of the learning environment, such as representation sequencing, have a role to play in students' learning with different mathematical representations.

In many environments that contain virtual manipulatives, a number of different representations, such as static and moving pictures of real-world objects, micro-world simulations, and interactive tools, are presented simultaneously on the screen (Namukasa et al. 2009). Several researchers have argued that one of the benefits of the virtual environment is that more than one representation can be placed on the same screen at the same time, and that the technology allows for students to see in real time the results of interacting with one on the other (Clements 1999; Namukasa et al. 2009; Sedig and Liang 2006; Suh and Moyer 2007). The assumption underlying this argument is that such affordances would allow students to make the connections between representations on their own, resulting in the abstraction of the underlying conceptual structure. In one study, Suh and Moyer (2007) allowed third-grade students to work with either virtual manipulatives or concrete manipulatives to promote relational thinking. In both virtual and physical environments, the children used several different representations, such as pictures, virtual manipulatives, and symbolic notation, in the context of solving simple linear equations. Students in both environments demonstrated large gains in performance on algebra problems after working with the manipulatives, both concrete and virtual. Suh and Moyer concluded that the exposure to multiple representations in each environment promoted students' representational fluency, which then contributed to the development of their relational thinking.

Our own research with concrete manipulatives, as well as a number of studies in analogical reasoning (Gentner and Colhoun 2010; Goswami 2004; Richland et al. 2012), would indicate, however, that children will not necessarily make the required connections spontaneously. Indeed, Quintana et al. (2004) argued that specific forms of scaffolding are needed to support students' scientific reasoning and proposed a number of "scaffolding guidelines" for the design of science inquiry software tools. Their guidelines are prompted by extensive research in educational and cognitive psychology on the benefits of scaffolded learning (e.g., Bransford et al. 2000) as well as previous work on the types of software supports that can be offered to the learner (Bell and Davis 2000; Toth et al. 2002). Yet, little is understood about the types of scaffolds required for learning with virtual manipulatives (Carpenter 2013), and as such, we argue for greater research attention on the types of pedagogical supports needed for students to make sense of the virtual worlds in which they operate.

A contrasting perspective on the role of pedagogical supports on children's learning with manipulatives emanates from those who caution against too much prescription in instruction. Gravemeijer (2002) maintained that children need to actively construct their own meaning of manipulatives through exploration and physical interaction (see also Martin 2009). He cautioned that when students are given prescriptions for how to use manipulatives, their use of the objects can become highly mechanical, which is not conducive to mathematical understanding. Martin and Schwartz (2005) provided evidence to show that children make sense of mathematical concepts by physically interacting with manipulatives in the context of problem solving, but acting on objects is not sufficient for learning. The authors found that students' physical actions with manipulatives "co-evolve" with their ideas about what the objects represent and what the actions mean. In particular, acting on objects constrains the ideas that are generated from those actions; in turn, the resulting ideas constrain subsequent actions, influencing learning in a "hand-over-hand" fashion (see also Rittle-Johnson and Alibali 1999; Rittle-Johnson and Koedinger 2009). By solving multiple problems, children eventually construct "action-interpretation sequences," which allow for the abstraction of key mathematical concepts and ultimately transfer.

A parallel line of research highlighting the relationship between action and thought is emerging in the context of virtual environments as well (see Chap. 3, this volume). For instance, the important role of embodied cognition—ways in which bodily perceptions and actions can affect thought (Glenberg et al. 2007; Wilson 2002)—has been demonstrated in virtual environments as well. For example, Segal et al. (2014) compared the effects of congruent and incongruent actions with computerized objects, either through a mouse or a touch screen device, on first- and second-grade students' estimation and addition performance. Children made fewer errors on the addition and estimation tasks when the actions built into the tool's design were congruent with adding (i.e., discrete counting actions) and estimating (i.e., smooth sweeps) than when they were incongruent with them. Segal et al. concluded that interfaces that require the learner to use actions that are congruent with thought are desirable, and can even take the place of explicit instruction.

Visual cues in concrete learning environments may also act as pedagogical supports for students making connections to appropriate mathematical concepts. In a pilot study, we presented undergraduates with "bar diagrams," models that have been used as visual tools to help students perceive the structure of mathematical word problems (Englard 2010; Ng and Lee 2009; Parker and Baldridge 2008). We presented undergraduates with a series of mock textbook pages (see Fig. 5.3) that

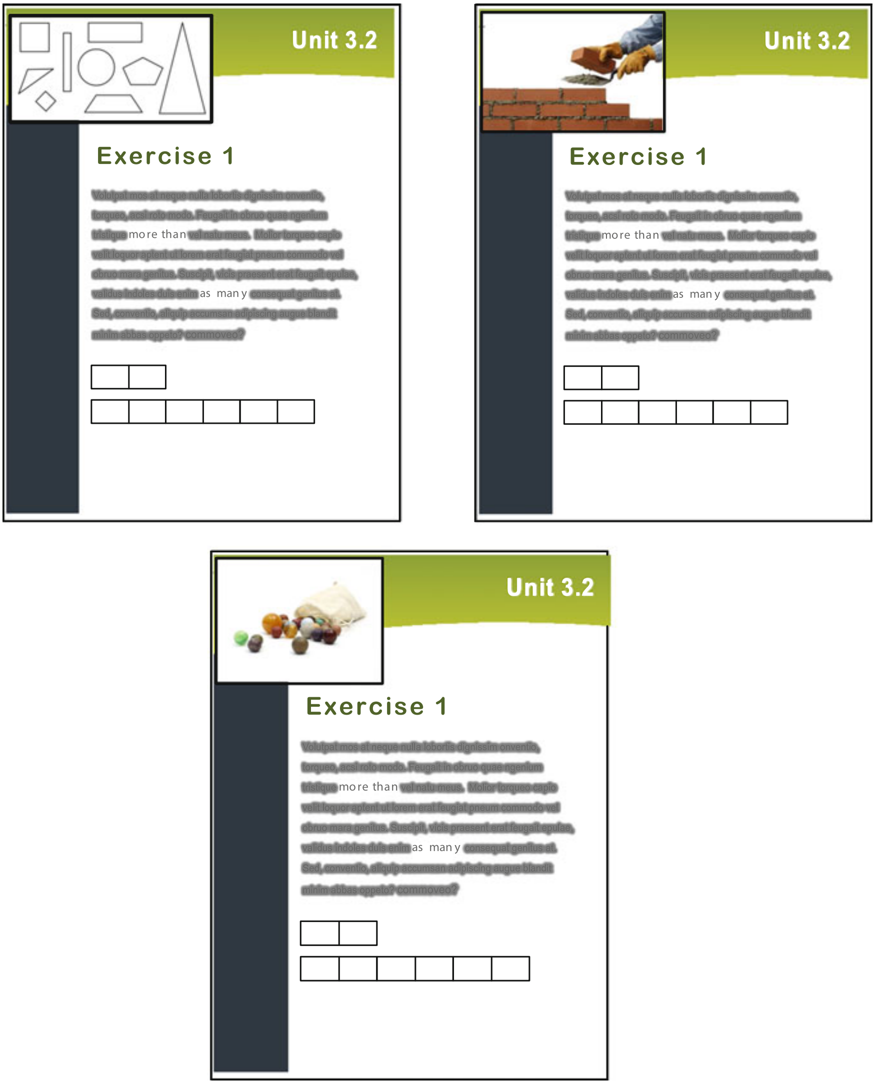

Fig. 5.3 Mock textbook pages with visual cues: Geometric shapes (*top-left*); bricks (*top-right*); marbles (*bottom*)

had three components to them: a visual cue (an image at the top-left corner of the page), textual cues (e.g., “as many as” or “fewer”), and a set of unlabeled bar diagrams. The visual cues were, in one condition, geometric shapes, some of which were similar to the bar diagrams themselves; in a second condition, a brick wall, intended to act as a real-world referent for the diagrams; and in a third condition, a sack of marbles, intended as a quantitative referent. An arrow pointed to the bar diagrams on the page, and the participants were asked to write responses to the questions, "What are these?" and "Why do you think they are here?" The design of the materials allowed us to observe the effects of the visual and textual cues on the participants' ability to perceive the bar diagrams as representations of quantity.

Preliminary analyses indicate that the visual cues (shapes, bricks, marbles) had an effect on the acquisition of dual representation. The majority of students who were presented with a mock textbook page with the shapes (60 %) or the bricks (67 %) perceived the bar diagrams as representations (e.g., "bricks" or "quantities in a ratio"). In contrast, three quarters of the students who were presented with the marbles did not perceive the bar diagrams as symbols at any point (e.g., "boxes to put the answer in" or "further instructions for the exercise"). We speculate that for the students in both the shapes and the bricks condition, the perceptual similarity of the visual cues and the bar diagrams triggered the students' interpretations of the diagrams as representational. The bricks may have cued real-world knowledge, and the shapes may have cued "this is about math," both of which triggered the participants' representational insight (i.e., "the realization of the existence of a symbol-referent relation," DeLoache 1995, p. 110). The marbles, on the other hand, had no superficial features in common with the bar diagrams, and as such, participants were less likely to see the diagrams' representational role. We find the differences between conditions particularly striking given that all participants were exposed to textual cues related to mathematics on the textbook page. In sum, although our data are limited because of the small number of participants in each condition, visual cues in the learning environment appear to impact students' perceptions of symbols in important ways.

To summarize, while the research is as yet inconclusive on the appropriate levels of instructional support for the development of students' dual representation of concrete manipulatives, most researchers, even those who study the effects of virtual manipulatives, would agree that students need to make the cognitive connection between a representation and its conceptual referent (Clements 1999; Moyer-Packenham et al. 2013; Namukasa et al. 2009; Suh and Moyer 2007). To our knowledge, however, little is known about supporting students' efforts to make such connections in virtual environments. Several discussions exist in the literature on the nature and effects of feedback in computerized environments, but they often focus on assisting the user to complete a given task successfully (Carpenter 2013; Rick 2012; Sedig and Liang 2006) rather than on ways to support students' constructions of the object's meaning. With respect to virtual manipulatives, then, effective ways to support students' interpretations of the representations themselves remain untested.

Likewise, little research has examined the effects of visual cues in virtual environments on students' learning and cognition. In fact, as described above, Namukasa et al. argued that one of the advantages of virtual manipulatives over concrete ones is that multiple representations can be presented simultaneously on one screen, which is assumed to make the link between them explicit. Based on our research, we argue, however, that there are conditions that would make simultaneous representations more effective than others and that more research is needed on ways to design virtual environments so that the visual cues constrain and direct the learner's interpretations in targeted ways.

5.2.3 Students' Perceptions and Interpretations

To this point, we have seen that students' prior knowledge impacts the learning that arises from interactions with external representations (e.g., Goldstone and Sakamoto 2003; Osana et al. 2013; Petersen and McNeil 2013). It seems therefore reasonable to hypothesize that the effects of manipulative use is not only dependent on external factors, such as the characteristics of the object itself or the pedagogical support offered, but are also contingent on students' cognitions, including the interpretations they themselves create of the representations in question. Sarama and Clements (2009) further argued that the appropriate use of manipulatives and subsequent learning are contingent on the nature of the meanings students construct of the objects: Whether they are concrete or virtual, "manipulatives are meaningful for learning only *with respect to learners' activities and thinking*" (italics in original, p. 148).

A detailed depiction of the role of meaning making as students work with new mathematical ideas and representations is presented by Carraher and Schliemann (2002). In their study, two fifth-grade students worked with visual representations on a computer screen to explore ideas related to integer arithmetic. The authors concluded that students do not take a "monolithic" piece of knowledge learned in one previous experience and bring it over intact to solve a new problem (i.e., in their view, an antiquated conception of transfer; see Singley and Anderson 1989); instead, the students in their study engaged actively in accommodating their knowledge to the demands of a new situation, which involved continually revising their interpretations of the representations used. Indeed, as suggested by Dufour-Janvier et al. (1987), "representations will be useful to the child to the extent that they have been 'grasped' by him" (p. 116).

Simply put, ways in which children invoke their previous knowledge and experiences to make sense of unfamiliar representations are important factors to consider when investigating the meanings that children construct (Martin and Schwartz 2005). Uttal and O'Doherty (2008) argued that regardless of the type of representation—concrete, pictorial, or virtual—the relationship between students' initial and developing interpretations and the acquisition of dual representation is not well understood. With respect to concrete manipulatives in particular, important questions also remain, including the nature of students' initial interpretations of manipulatives; the ways in which their interpretations develop through interaction, reflection, and instruction; and how prior interpretations may affect the development of dual representation, learning, and transfer over time.

Some of our own data provide a window on the initial perceptions of concrete objects held by young children and adults. In one of our studies led by Nicole Pitsolantis (Osana and Pitsolantis 2015), a doctoral student in our research laboratory, we examined the effects of different levels of scaffolding on Kindergarteners' dual representation of Base ten blocks. As the students' teacher, Nicole purposely designed her teaching for the year so that students had no prior exposure to the Base ten blocks at the beginning of the study. She delivered six lessons to 25 students in two classrooms on various aspects of the blocks, including their quantitative referents, how to count with them, and how to use the blocks to display given quantities. After each lesson, we individually interviewed 12 students on their dual representation of the blocks using tasks we developed in previous studies (Osana et al. 2014). The goal of the study was to examine the point at which the students acquired dual representation as a function of the scaffolds provided in each lesson and their incoming number knowledge.

Before the lessons began, we individually interviewed 23 of the 25 students in both classrooms about their initial perceptions of the blocks. Specifically, we asked the students "What are these?" and "What do you think they are used for?" A majority of the students (19 students or 83 %) indicated that the blocks were objects with which to build things, such as castles, graveyards, and televisions. A considerably smaller proportion (5 students or 22 %) indicated that the blocks could be used to perform some sort of mathematical activity, such as counting or measuring, but none of these students saw them exclusively as mathematical tools—in other words, all five students also invoked non-quantitative aspects of the manipulatives in their responses. Three of the five students who invoked mathematical uses for the blocks referred to them as measurement tools, which likely stemmed from previous experiences in their classroom measuring objects with arbitrary units.

Two important conclusions can be drawn from these preliminary data. First, all but two students ascribed some meaning to the blocks, which were all based on their previous experiences—either play-based or mathematical. This underscores their efforts to make sense of unfamiliar objects using their prior knowledge, and is important because their understandings are likely to evolve as a function of such initial perceptions (Carraher and Schliemann 2002). In the study with first graders described above (Osana et al. 2013), we demonstrated that students' initial perceptions of the manipulatives impacted the subsequent development of their dual representation. Not surprisingly, the students who were encouraged to play with the chips perceived them as toys "to make stuff with." More importantly, however, compared to students in the quantitative group, who were explicitly told the intended quantitative referents of the chips, the students who interpreted the chips as toys did not develop dual representation, even after engaging in a quantitative task that used the manipulatives as ones and tens. Their perceptions of the objects as toys appeared to have prevented their ability to subsequently see them as representing quantities, which points to the importance of children's initial and evolving interpretations of manipulatives.

Another important conclusion that we draw from the Osana and Pitsolantis (2015) data is that although a small number of children indicated that the blocks could be used as tools in a mathematical activity, none of them alluded to their role as symbols of quantity *or even symbols at all*. This suggests that, at least with respect to manipulatives in mathematics, children tend to see them as objects to play with, and their representational role is not invoked spontaneously. Seeing mathematical manipulatives as "standing for" quantities (in this case) appears to be contingent on some type of instructional support, whether in the form of explicit instruction, scaffolding from a more knowledgeable peer, or task design. Indeed, our preliminary analyses of the development of the Kindergarteners' dual representation appear to bear this out. This conclusion is summarized well by Carraher and Schliemann (2002) when they stated, "…there is no reason to expect that … students will understand the concepts [represented by external representations] merely by inspecting the diagrams and notation" (p. 6). As noted above, however, the appropriate type and amount of support remains an open question.

In another study from our research laboratory, this time with preservice teachers in an elementary mathematics methods course, we obtained similar results. In a project led by Danielle Houstoun, we showed undergraduates how to solve problems with bar diagrams in three instructional sessions and then gave them practice problems after each session. We were interested in students' interpretations of the bar diagrams, their ability to use them, and their problem solving performance as a function of whether or not explicit explanations on the bar diagrams' quantitative meaning were provided (links condition vs. no links condition).

Before the instruction, fewer than 20 % of all participants indicated that the diagrams represented quantities or amounts. In much the same way as the Kindergarteners' in Pitsolantis' classroom, the bar diagrams were rarely associated with mathematics, and this despite the fact that the participants were enrolled in a mathematics methods course and that they had solved word problems on a pretest immediately before their perceptions were elicited. How the data are distinct from the Kindergarten study, however, is that the preservice teachers were clearly aware of the representational role of the diagrams: Three-quarters of them attached real-world referents to the bar diagrams that matched their perceptual features (e.g., patches of grass, train cars). Having acquired a certain amount of "meta-representational competence" (diSessa 2004; diSessa and Sherin 2000) or "symbolic sensitivity" (DeLoache 1995), adults are typically more aware of the representational purposes of diagrams than children. Furthermore, the presence of explicit links between the bar diagrams and their conceptual referents during instruction resulted in a considerable shift in the preservice teachers' perceptions. After instruction, almost 80 % of the participants in the links condition ascribed quantitative referents to the bar diagrams and only 14 % in the no links condition did so, even though the same bar diagrams and problem solving tasks were used in both conditions.

What Houstoun's study suggests is that a potential factor in the construction of meaningful interpretations is the degree of symbolic awareness that the student brings to bear on the task. There may be a developmental shift in such metarepresentational skill (diSessa 2004), but more research is needed to support such a contention. More specifically related to mathematics learning, however, the studies from our lab also show that even for adults, quantitative interpretations for symbols are not generated spontaneously; in fact, the undergraduates benefited from explicit instruction on the associations between the diagrams and their conceptual referents to construct interpretations that aligned with the target mathematical activity. We conclude by highlighting the importance of taking students' interpretations of manipulatives into account in the context of mathematics learning.

It should be evident from our discussion in this section that we conceive of students' perceptions of manipulatives as a cognitive construct, one that is akin to the mental representations they construct of the objects. From this perspective, we know of no research that examines the various types of representations students construct of virtual manipulatives. Numerous investigations exist that document the affective component of students' perceptions, such as their acceptance of the technology (Ozel et al. 2014) and their perceptions of the usefulness of the manipulatives (Lee and Yuan 2010). Thus, because of the paucity of work on students' cognitions about virtual manipulatives themselves, it features prominently in our recommendations for future research in this area.

5.3 A Research Agenda for Virtual Manipulatives

The use of concrete manipulatives in mathematics has a long history (Kim and Albert 2014). Early research on their effects focused on examining outcomes of use relative to no use (e.g., Raphael and Wahlstrom 1989; Suydam 1986), but such comparisons have not provided definitive answers on whether or not manipulatives should be used in mathematics classrooms (Carbonneau et al. 2013). Because the research is inconclusive, recent attention has turned instead to the conditions under which they are effective in students' learning. What has emerged from this shift in focus is widespread agreement that the mere presence of concrete objects does not guarantee learning (Ball 1992; Uttal 2003), and the same is apparent for other symbolic representations in mathematics, such as diagrams and formal symbols, such as ">" and "=" (Carraher and Schliemann 2002; Dufour-Janvier et al. 1987; Sherman and Bisanz 2009). The use of manipulatives in the mathematics classroom can have clear benefits for student learning, and the research points to the conditions that are necessary for such success.

In this chapter, we reviewed what is currently known about how concrete manipulatives and two-dimensional visualizations can support students' internalizations of intended mathematical concepts. A review of existing research brought several themes to the fore, which we used in constructing a framework for future research on students' learning of mathematics with virtual manipulatives. The framework consists of three components: (a) the characteristics of the representation itself, (b) the amount and type of pedagogical support that is present in the learning environment, and (c) the types of interpretations and conceptions students hold of the manipulatives and diagrams.

One way the research on concrete and two-dimensional representations is relevant to virtual manipulatives is for the design of the computer images themselves. To the extent that the computer environment is designed to augment learning in mathematics, the image on the screen is meant to stand for a concept in the "represented world." Circles on a place value chart app, rectangular bars in a problem solving app, and computerized length models to represent quantities on a real number line are all entities in the "representing world" that are intended to signify target concepts in mathematics. As such, the research suggests that such virtual objects be stripped of as many extraneous superficial features as possible so that students' attention is focused on the mathematics the objects are designed to signify. Indeed, some research on computerized images can attest to the soundness of this recommendation (e.g., Goldstone and Sakamoto 2003; Kaminski et al. 2009), but more investigations are needed, particularly because the vast majority of virtual manipulatives are designed specifically to be visually appealing to the user (Ginsburg et al. 2013).

Further, the nature of the pedagogical support provided to the student, whether in the form of explicit explanations, visual cues, or other scaffolds, is an overarching theme in the research on concrete manipulatives. In virtual environments, parallels have been found. Supports in the form of representational sequence (Goldstone and Son 2005) and "concrete" scaffolds (Sedig et al. 2001) have been shown to have positive influences on students' ability to make sense of the objects and actions on the screen. Aside from research on how to embed virtual manipulatives in intelligent tutoring systems, more research is needed on how the design of a virtual environment can take the place of a human being, such as a teacher, who can provide useful scaffolds to the student who is learning about the correspondences between the manipulative and its referent.

Little work has been conducted on students' initial and developing perceptions of concrete manipulatives, and even less so of virtual manipulatives, which leaves considerable room for future research. For any representation, virtual or not, there is nothing inherent in the manipulative or image that makes it a symbol for a mathematical concept; there are any number of aspects belonging to the manipulative (e.g., color, shininess, weight) that could signify something entirely different, such as the temperature of the coffee in a mug. In fact, the features of the manipulative are likely to govern how students construct their initial representations of the object, which in turn impacts how such representations are processed during mathematics learning. Some open-ended interview data from Goldstone and Sakamoto (2003) appear to support this contention in a computerized environment. When asked to explain the ants' behavior after they had explored the simulation, the participants who had explored the concrete instantiations placed anthropocentric interpretations on the ants that did not correspond to the rules of the system (e.g., "scaring other ants away," "getting tired," p. 453). The authors used the participants' perceptions of the ants to account, in part, for their inability to transfer the rules to another context.

What also lies ahead in terms of research are ways in which the three components of the framework interact to produce student learning in mathematics, both in virtual environments and in "real" ones. Virtual environments are not identical to concrete ones, however, and aspects related to the system's design and rules that govern them must be taken into account. For instance, investigations should address how the characteristics of a virtual manipulative interact with types and levels of scaffolds in a system's design (e.g., Quintana et al. 2004); how the superficial features of a virtual manipulative influence the interpretations made by learners, how those interpretations develop over time, and how they impact subsequent learning in mathematics; and the ways students can be supported to see beyond the superficial aspects of the manipulative to their intended referents.

Aside from guiding future research on virtual manipulatives, the framework can also assist teachers in their evaluation of digital tools for learning mathematics. Moyer et al. (2002) presented a number of questions to consider when determining the likelihood that a specific set of virtual manipulatives will be useful in the classroom. One of these questions is, "Are the images dynamic and interesting?" (p. 375). The framework allows us to ask additional questions that we think would provide a more focused evaluation of the manipulatives, such as "to whom are the images dynamic and interesting?", "what types of previous experience and knowledge are my students bringing into the classroom that might impact how the manipulatives are perceived?," and "will the perceptual features that make the images interesting detract from the learning that is targeted?" Another question suggested by Moyer et al. is, "Do [the images] represent the target mathematics?" (p. 375). Again, the framework can assist a teacher in asking further questions that would aid in the evaluation. To a teacher, virtual Base ten blocks clearly represent the concepts of numeration and place value, but to a child, they may not. Thus, asking the questions, "will my students see the appropriate mathematical referent when using these manipulatives?" and "how can I ensure that my students make the appropriate connection between the manipulatives and the abstract referents they are designed to symbolize?" are critical to students' developing learning and understanding.

Although the questions offered by Moyer et al. (2002) appeared almost 15 years ago, our framework allows us to argue that they are still relevant today, particularly because the issues they raise have not yet been adequately addressed in the research. Indeed, more recently Ginsburg et al. (2013) reiterated many of the same concerns about the potential of mathematics software to enhance the mathematical development of young children. In much the same way as we suggest here, they proposed a decidedly cognitive focus in the analysis of mathematical software tools and called for the design of software that best supports children's representations of abstract ideas. We propose that the concerns raised by Moyer et al. and Ginsburg et al. can be answered through research that focuses more squarely on the virtual representations themselves and the types of scaffolds that will have a direct impact on students' interpretations.

We caution, however, that the framework presented in this chapter is useful only to the extent that there are sufficient parallels between the representations reviewed in the chapter (i.e., concrete objects and two-dimensional representations) and virtual manipulatives. One obvious parallel between the two environments is the fact that many virtual manipulatives appear as two-dimensional images on a computer screen. As such, clear correspondences exist between diagrams in the concrete world and in the virtual world, even if the user often has greater control of the images in the latter. Second, according to Moyer et al. (2002), virtual manipulatives are only "true" virtual manipulatives if the user can engage and control physical actions with them for the purposes of constructing mathematical knowledge (see also Chap. 1, this volume). To the extent that virtual manipulatives are similar to concrete ones on this level, all three components of the framework are arguably relevant for future research.

Finally, from a cognitive perspective, the research suggests that manipulatives are useful to the extent that children make them personally meaningful for the task at hand (Sarama and Clements 2009). Students' "efforts after meaning" (Bartlett 1932) are based on their initial and developing interpretations of the representations as well as their prior knowledge and experience about the target domain. The same processes may be occurring in a virtual environment as well: We provided preliminary evidence that students attempt to construct meaning of images on the screen just as they attempt to make meaning of concrete objects and two-dimensional representations off screen (Goldstone and Sakamoto 2003). As with concrete objects, however, students require appropriate supports to interpret virtual manipulatives, acquire dual representation, and ultimately transfer their knowledge to other mathematical contexts.

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Part II
Research and Design

Chapter 6
Fingu—A Game to Support Children's Development of Arithmetic Competence: Theory, Design and Empirical Research

Ingemar Holgersson, Wolmet Barendregt, Jonas Emanuelsson, Torgny Ottosson, Elisabeth Rietz and Berner Lindström

Abstract This chapter aims at describing research on Fingu, a virtual manipulative housed in a game environment, which is designed to support young children's learning and development of number concepts and flexible arithmetic competence. More specifically Fingu targets the understanding and mastering of the basic numbers 1–10 as part-whole relations, which according to the literature on early mathematics learning is critical for this development. In the chapter, we provide an overview of the theoretical grounding of the design, development and research of Fingu as well as the theoretical and practical design rationale and principles. We point out the potential of Fingu as a research platform and present examples of some of the empirical research conducted to demonstrate the learning potential of Fingu. Methodologically, the research adopts a design-based research approach. This approach combines theory-driven design of learning environments with empirical research in educational settings, in a series of iterations. In a first series of iterations, a computer game—the Number Practice Game—was designed and researched, based on phenomenographic theory and empirical studies. In a second series of iterations, Fingu was designed and researched, based on ecological psychology in a

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socio-cultural framing. The design trajectory of NPG/Fingu thus involves both theoretical development and (re)design and development of specific educational technologies.

6.1 Introduction

This chapter describes research on Fingu, a virtual manipulative housed in a game environment, which is designed to support young children's learning and development of number concepts and flexible arithmetic competence. More specifically, Fingu targets the understanding and mastering of the basic numbers 1–10 as part-whole relations. Our primary aim is to give an overview of the theoretical grounding of the design, development and research of Fingu as well as the theoretical and practical design rationale and principles. The potential of Fingu as a research platform is pointed out. We also present examples of some of the empirical research conducted, with the aim of demonstrating the learning potential of Fingu.

6.2 The Fingu Game

In the Fingu game, the player is exposed to two moving sets of objects on the touch screen (in the case shown in Fig. 6.1, four apples on the left and three apples on the right), and is supposed to tell how many objects there are in total (seven) by touching the screen with the corresponding number of fingers. Fingers can be placed anywhere on the screen and there are no restrictions concerning which fingers are used, but they must be pressed down roughly at the same time, as a representation of the total number.

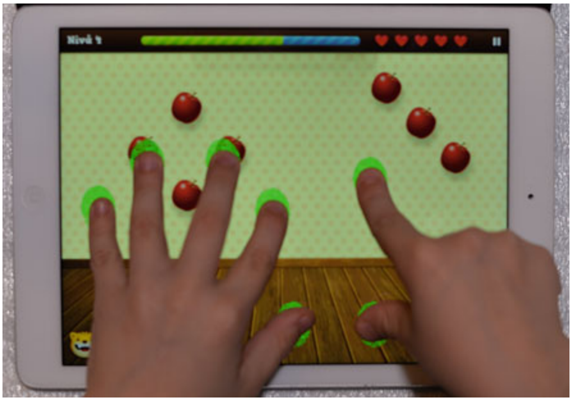


Fig. 6.1 A screen view of a child playing Fingu

The objective is to progress through different levels of the game, making as few wrong answers as possible. Each level comprises a set of tasks and the levels are progressively more demanding. In order to advance to the next level, the player is not allowed to give more than a certain number of wrong answers (represented by hearts in the upper right of the screen in Fig. 6.1).

6.3 Young Children's Learning of Mathematics

One of the key goals for mathematics education around the world is to ensure all children's proficiency with numbers. This means to have a flexible and adaptive understanding and knowledge of arithmetic that can be used with confidence and fluency in a variety of situations occurring in everyday life and at work (Kilpatrick et al. 2001). Learning to become proficient with numbers is a process that starts early on and then becomes a part of the growing child for several years (Baroody et al. 2013; Sarama and Clements 2009). One of the foundations in this process is to develop flexible arithmetic competence, that includes mastering number concepts for the basic numbers 1–10, and to this end, the use of reasoning strategies, based on using known sums and relations between them to deduce unknown sums (Baroody et al. 2013). For many children, this is a straightforward process, but some children seem to have trouble learning and using such reasoning strategies, as observed by Gray and Tall (1994), who found that children aged 7–12 with below average mathematical performance dominantly relied on counting strategies while performing basic sums. Neuman (1987, 2013) also found that counting one by one, as a dominant strategy for managing single-digit sums, is common among children with mathematical difficulties.

There are different views on the role of counting for the development of conceptual knowledge about numbers. A dominant idea is that basic sums are learned in a process, where children begin by using counting one by one to determine the sums of single-digit additions. They then are supposed to memorize these facts. But in order to develop fluency with basic sums (and not just basic facts), which includes the ability of using them both appropriately and in adaptable ways, children "typically progress through three overlapping phases in the meaningful learning of a particular basic sum or family of sums: (a) Phase 1 (counting strategies), (b) Phase 2 (conscious or deliberate reasoning strategies), and (c) Phase 3 (fluent retrieval)" (Baroody et al. 2013, p. 536). Here, the counting strategies of Phase 1 can be seen as the first steps in determining and exploring different sums, which in Phase 2 can turn into knowledge of the part-whole relations that constitute the sums. Continuing in Phase 2 to explore these relations further, may then lay the ground for a network of relations, which in Phase 3 will constitute itself in a more or less fluent performance. So Phase 2 will act as a mediator between the less efficient counting strategies in Phase 1 and the more efficient retrieval strategies in Phase 3. Note that these phases must be understood as overlapping, and that there is no absolute succession from one phase to another.

This network of relations within the first ten natural numbers is built upon a network of quantitative part-whole relations, which, in the terminology of Resnick (1989), is grounded in the principle of additive composition—that numbers are composed of other numbers, and any number can be decomposed into parts—combined with more qualitative knowledge of part-whole relations. Thus the ability to make use of part-whole number relations is foundational for developing the kind of number networks that are necessary for becoming fluent with mental computations (Baroody et al. 2009; Sarama and Clements 2009). Later, in the development of arithmetic fluency, the ability to use these part-whole relations in a flexible and adaptive way is also fundamental in understanding fractions (McMullen et al. 2014, 2015).

One of the basic arithmetic competences that support the development of good number sense is the ability to quickly and reliably determine the numerosity of a collection of objects. This can be done by counting all the objects, but can also be speeded up either by different forms of skip counting or by using some kind of subitizing. Humans are born with an innate ability to *subitize*, that is “the direct and rapid perceptual apprehension of the numerosity of a group” (Sarama and Clements 2009, p. 29). Kaufman et al. (1949) invented the term subitizing to distinguish it from estimating, which is also a rapid but only approximate way of quantifying. They emphasized that subitizing goes together with a high degree of accuracy and confidence. Thus subitizing appears to be a distinct way of quantifying, different from counting and estimating. From the neuroscientific literature the conclusion is that “true” subitizing only occurs in adults for the numerosities 1, 2, and 3 (Dehaene 2011). For some young children, the number 3 may, according to Sarama and Clements (2009), be questionable. This fact does however not exclude the possibility of rapid recognition of higher numerosities, but this is limited to identifying certain familiar configurations such as those used on dice or dominoes.

Noting that “subitizing develops considerably as children grow and combines with other mental processes”, Sarama and Clements (2009, p. 44) distinguish between perceptual and conceptual subitizing. They define perceptual subitizing as “recognizing a number without consciously using other mental or mathematical processes and then naming it” (p. 44). This ability, limited to the quantities 1, 2, and 3, includes the decomposing of 2 items as 1 and 1, and 3 items as 2 and 1 without having to count. This ability then provides a foundation for conceptually experiencing somewhat larger numbers as composed of smaller parts (e.g., 4 as 2 + 2 or 3 + 1, 5 as 2 + 3, or 2 + 2+1, and 6 as 3 + 3 or 2 + 2 + 2). This process of expanding perceptual subitizing into immediate recognition of combinations of smaller units they call conceptual subitizing (Sarama and Clements 2009). Conceptual subitizing thus presupposes a network of part-whole relations for small numbers, and the richer it is the more effectively, flexibly and adaptively, it can be used for performing different computations more rapidly.

Today there is a growing awareness that the concept of embodiment is an important dimension of learning mathematics. As stated by Edwards and Robutti (2014), “although mathematics may be socially constructed, this construction is not arbitrary or unconstrained but rather is rooted in and shaped by the body” (p. 2). The body then becomes “an important resource in the construction and communication of meaning". Therefore, in addition to the traditional view of learning modalities such as visual and auditory, an expanded view will include motor modalities such as gesture and touch. A growing neuroscientific literature connecting fingers to numerical cognition (Fischer et al. 2012) is an example of this embodiment. From a mathematics education perspective, it is well-known that fingers play a central role in learning arithmetic. Children can use them in two ways, either by making a finger pattern displaying a certain number, for instance showing the index, middle, ring and pinkie fingers on one hand to denote the number four, or as a tool, while performing a calculation, keeping track of their counting, giving attention to one finger at a time. Historically, fingers as a means of representing numbers have also played important roles in the early development of computational methods found in commerce and administration (Ifrah 2000). So, from an embodied perspective, using fingers to form different number representations is a way to enrich the learning of basic numbers.

6.4 Theoretical Framework

Methodologically, the research on Fingu adopts a design-based research (DBR) approach (Brown 1992; Cobb et al. 2003). This approach combines theory-driven design of learning environments with empirical research in educational settings, in a series of iterations. In a first series of iterations a computer game—the Number Practice Game—was designed and researched, based on phenomenographic theory and empirical studies (Lindström et al. 2011; Marton and Booth 1997; Neuman 1987). In a second series of iterations Fingu was designed and researched, based on ecological psychology (Gibson 1986, 2000) in a socio-cultural framing. The design trajectory of NPG/Fingu thus involves both theoretical development and (re)design and development of specific educational technologies.

The design-based research methodology is grounded in cultural-historical activity theory (CHAT). There are different versions of CHAT. The work by Engeström (Engeström and Sannino 2010) emphasizes structural and systemic aspects, while the foundational work of Vygotsky and Leontiev (cf. Hodkinson et al. 2008) focuses more on human agency. As a foundation for the design of environments and tools for learning, recent developments of CHAT come closer to the originators (cf. Kaptelinin and Nardi 2006).

In CHAT, human activity is the core unit of analysis. Activities should be understood as socially and historically situated. For one thing, this means that they are multi-layered. For example, the activity of playing a mathematics computer game is inevitably a part of a larger activity system or practice, of institutional schooling or day-care/preschool or play or family upbringing. At the same time, an activity is realized by the actions performed by the participating individuals. Actions in turn are composed of operations, which in the knowledgeable individual (expert) are unconscious.

An activity unfolds over time, realized by human actions and operation. Humans change (i.e., learn) by participating in the activity. Learning can then be understood as a by-product of participation, appropriating whatever patterns of actions and operations are (deemed) functional given the activity. Human abilities, knowledge and skill, in general, have this dynamic and relational character. They are not to be regarded as something static "in the head" or, for that matter, "in the body", but only realized when acted out "in situ".

An activity is understood as historically situated on different levels. In our case, not only the individuals have a specific history with respect to playing computer games, mathematics learning activities, money counting, etc. This is clearly demonstrated in our empirical studies of Fingu in pre-school and school. The game playing activity became a part of a larger process of teaching and learning arithmetic. Despite our attempts to offer the game as a mathematics activity not belonging to the regular practice, the Fingu activities were organized and fitted into the everyday classroom scheme. Thus, how children frame the activity (Goffman 1974), for example as gaming or as mathematics learning or both, is to be understood in a historical perspective, both on the collective sociogenetic level and on an individual ontogenetic level.

Thus, a specific activity, such as playing a mathematics game, does not necessarily belong only to one larger activity system. Sometimes it belongs to several and even larger activity systems. As pointed out, playing a mathematics game can be both a gaming activity and/or a school mathematics activity. The activity then becomes a boundary activity and the game can be considered a boundary object (cf. Akkerman and Bakker 2011). This boundary character of game playing is important to consider when evaluating the general argument that game playing offers a learning potential that breaks away from institutional schooling (Gee 2003).

A basic tenet in CHAT is that teaching and learning are intrinsically related, that is, two aspects of an unfolding activity. Teaching should then be understood in a generic sense. It might be an active involvement by a teacher, but a game playing activity involving a single player and a mathematics learning game also involves a teaching or instructional component. Vygotsky had a specific term for this learning/teaching process, *obuchenie* (Cole 2009). Learning/teaching is thus a two-sided process and understanding learning is thus a matter of understanding a teaching/learning activity in relational terms, as a relation between the individual and the social and material environment.

CHAT, or more generally socio-cultural theory, is often used as a foundation for designing learning environments with collaborative activities. Whereas not incorrect, an epistemology that premises human learning and development as culturally and socially situated does not necessarily imply this kind of educational or instructional model. Any activity, collaborative or not, is socially and culturally situated. Reading a book or playing a computer game in private is also a socially and culturally situated activity.

In summary, CHAT provides a framework for both design and analysis. Designing a mathematics computer game is a design for certain teaching/learning activities to take place, just like designing a set of tasks in a mathematics workbook. However, the actual use of the game—the teaching/learning activity—is also highly dependent upon the overall system of activities it is part of. This means that the context of use is always an issue to be considered. This is also an analytic consideration in doing research on game use. How do the game playing activities relate to other mathematical activities in which children are involved? How do children perceive the game playing activities?

Given this general framework, the theoretical underpinnings for designing and studying Fingu as a virtual manipulative are given by Gibsonian ecological psychology (Gibson, 1986), and in particular the theory of perceptual learning (cf. Gibson and Pick 2000). Acknowledging the contributions of both James and Eleanor Gibson, Gibsonian theory is intrinsically relational and non-dualistic. As pointed out above, this is something that is shared with cultural-historical activity theory. It is also non-representational; it rejects the idea that cognition and learning are about constructing inner representations of the world "outside" the individual.

Ecological psychology is grounded in a "realist" ontology, which like CHAT, acknowledge that human activity is grounded in the material world. Perception, for example, is the selection and "picking up" of (invariant) information in the course of acting in a concrete physical environment. It should be noted that, when it comes to social and intellectual activities, for example, teaching/learning, these are constituted in interactions unfolding over time.

The Gibsons (Gibson and Gibson 1955; Gibson and Pick 2000) argue against "enrichment theories", where perception or "sensory reception is enriched and supplemented by the addition of something" (Gibson and Pick 2000, p. 7). Instead "perception begins as unrefined, vague impressions and is progressively *differentiated* into more specific percepts" (p. 7). In development, through perceptual learning, the individual becomes more and more apt to learn the specific affordances of the environment.

The concept of affordance, which takes on a number of different definitions in contemporary social and behavioral sciences, is pivotal to Gibsonian theory and was developed to account for the relational nature between the individual and the environment. It "refers to the 'fit' between an individual's capabilities and the scaffolds/support and opportunities that makes a certain activity possible" (Gibson and Pick 2000, p. 15). An affordance can be thought of as an offering for meaning in a given situation, or put in more general terms, an offering for action.

That differentiation is fundamental to learning and development has a number of consequences. In the present context, it means that the development of flexible and adaptive competence in dealing with part-whole-relations in the range from 1 to 10 is a matter of successive refinement of the understanding of numbers, from a more undifferentiated whole (for example the number 7) to grasping a differentiated network of relations between parts that can make up the whole (i.e., 6|1; 5|2; 4|3; 3|3|1 etc.). The whole can take on different forms, for example number words of a more symbolic nature; sets or constellations of concrete objects; visually presented patterns (of objects); or even procedures (of which the counting sequence is an example).

Gibsonian theory also emphasizes that perception is not static but dynamic; thus, building on our actions in the environment, in which individuals engage in activities that are extended in time (e.g., moving around). The construct "perception-action cycles" captures this dynamic, emphasizing that perception is not a prerequisite for action. Rather, action is foundational for perception and perception is foundational for action, making up a perceptual system. Thus, perceptual learning is

> the means of discovering distinctive features and invariant properties of things and events ..... Learning to distinguish faces from one another or to distinguish letters of the alphabet are such cases. Discovering a repeated theme in a symphony and the variations on it is another. Discovering distinctiveness and invariance is another kind of meaning, also a product of perceptual learning. (Gibson 2000, p. 295)

The idea of discovering and retaining information as invariant features of the environment is central in James Gibson's seminal work (Gibson 1986). That perceiving invariance(s) (which presumes variation) is fundamental to perception, action and learning, is similar to what variation theory proposes (Marton and Pang 2006). This is particularly important since pre-cursors of the Fingu game were designed from variations of theoretical ideas (cf. Lindström et al. 2011).

An important aspect of perceptual (and cognitive) systems is that they are typically not uni-modal, but multi-modal, building on the use of several sensory modalities (Neisser 1976). This is important in the present context, where children are exploring a game environment that explicitly draws on both visual and kinesthetic modalities. Embodiment, then, is in this view building on multi-modal agency.

Going further, perceptual learning builds on two complementary processes: exploratory activity and performatory activity. Gibson (2000) states that exploratory activity

> is itself an event, a perception–action sequence that has consequences. It brings about new information of two kinds: information about changes in the world that the action produces and information about what the active perceiver is doing. (p. 296)

This kind of learning tends toward flexibility and is geared to maintain an adaptive relation with the environment. But learning is also geared towards economy and efficiency, and this results in a tendency for specificity, resulting in actions that from the beginning of an encounter can be more varied and exploratory. This then develops into more specifically limited actions that effectively fulfill the goals of a task. This is what the theory of perceptual learning conceptualizes as performatory activity: "Activity that starts as exploratory can become performatory as an affordance is discovered. This shift is marked by making contact with the environment and ensuing control of it" (Gibson 2000, p. 297). This idea resonates with CHAT in that operations that build up actions (for example "seeing" or counting number patterns) can become unconscious in the course of learning.

Building on the theory of perceptual learning, Kellman and Garrigan (2009) formulated design principles for interventions aimed at developing expertise with some key areas in mathematics learning. These principles include many and varied short tasks, where the child has the opportunity to develop rapid selection of task-relevant information, and the pick-up of higher-order relations and invariances in different modalities such as visual, auditory and kinesthetic. The latter principle corresponds with the core idea in variation theory and phenomenography (Marton and Booth 1997; Marton 2015).

6.5 Design Principles for Fingu

In this section, we describe the design principles of Fingu and how these principles are realized/materialized in the game. As previously discussed, these design principles are grounded in empirical research on mathematics learning and instruction and in more general theories of learning and instruction (Neuman 1987; Lindström et al. 2011).

Design of a computer game for learning is not a straightforward derivation from theory. The design is influenced further by contemporary and historical mathematics education practice and by more general game design. Furthermore, Fingu is a second-generation implementation of design ideas for a new technological platform (i.e., tablets). This platform offers possibilities to implement new design elements (e.g., using finger patterns and thus multiple fingers to manipulate the game) that were not available in earlier generations of the technology (such as laptops).

Since the design of Fingu is encapsulated in a design-based research process, the design principles and design elements outlined in this section can, to some extent, also be regarded as results.

6.5.1 Overall Design

When designing a mathematics game for children, there are a number of alternatives for framing the mathematics tasks. A common practice is to embody the mathematics in a cover story or activity, for example using a route metaphor. When advancing from START to END, the player meets a number of obstacles that have to be handled with specific tools (for example collecting a number of keys to open a door). This type of design was rejected for Fingu. Recent research shows children might be focusing on completing the gaming elements rather than engaging in the desired content (cf. Linderoth 2012). Since the goal was to design a learning environment that maximized children's attention to part-whole relations, we chose a design that resembles a microworld (cf. Papert 1980). In this case, it is a world of all possible part-whole relations in the number range from 1 to 10.

As argued above, mastering part-whole relations in the number range from 1 to 10 is pivotal to the development of flexible and adaptive arithmetic competence. From an arithmetic point of view being able to decompose every number in the range into two parts in all the possible ways ($2 = 1|1$; $3 = 2|1$; $4 = 3|1 = 2|2$; $5 = 4|1 = 3|2$; …) and, conversely, to construct larger numbers by combining these parts into new numbers, is an important developmental milestone.

In accordance with the theory of perceptual learning and variation theory, children are presented with a set of tasks covering all the possible part-whole relations in the number range from 1 to 10. Each task targets a specific part-whole relation (for example $5 = 3|2$). Playing Fingu makes it possible to discern invariances (for example that 5 is 5 regardless of if the parts are $2|3$ or $4|1$ and regardless of modality) and extract information about higher order relations (a network of part-whole relations).

Fingu is designed as a game in order to encourage extensive experience with many and varied tasks, as the theory of perceptual learning prescribes (Kellman and Garrigan 2009). A basic game design element is a progression of mastery from an introductory level comprised of tasks with small numbers (1–5 and canonical visual patterns) to an end level with tasks with large numbers (6–10) and non-canonical visual patterns. Mastery is represented by the number of lives preserved while going through the tasks on a level. With mastery of one level, it is then possible to advance to the next higher level. Another game design element is immediate and simple feedback on correctness. This feedback is presented both auditory and visually, to be clearly recognized by the child before taking on the next task. Fingu is also packaged as a computer game, in terms of graphical layout and in terms of vocabulary. For example, the child has to pick an icon for player and name the player.

As pointed out above, a problem in developing arithmetic competence might be that children develop non-productive or even counter-productive counting procedures that are not used in a flexible and adaptive way. Fingu is designed to afford conceptual learning (with a focus on part-whole relations) and to minimize or even prevent counting. The main design element is to put time-constraints on the individual tasks. Even if the task design affords both counting and perception of part-whole relations using subitizing, the latter is a more efficient approach to solve the task. In the next section we elaborate more on how time-constraints are used in the design of tasks.

A distinctive feature is that the design draws on the embodied nature of arithmetic. As discussed above, there are two distinct aspects of this. One is making subitizing (both perceptual and conceptual) a basic design element in the development of the understanding of part-whole relations. The other is making children use fingers as tools in dealing with the tasks. More specifically, Fingu affords the use of finger patterns. Essentially, it is the latter characteristic that makes Fingu a virtual manipulative. Fingu is thus fundamentally a multi-modal learning environment. First, it allows children to find out invariances across modalities, as pointed out by variation theory and perceptual learning theory. Second, it affords transformations across modalities, for example making a visually presented part-whole relation (for example $6 = 4|2$) into another relation ($6 = 5|1$) using the fingers and preserving the whole.

Of note, Fingu was not designed as a symbolic activity (i.e., invoking the use of number symbols). There is however not any principled reason for this. On the contrary, from a CHAT perspective, language is a critically important tool in human activity. Thus, the game, as currently envisioned, does not capitalize on children's communication through number vocabulary. Furthermore, Fingu is used in educational contexts where language and number symbols play a decisive role. In our empirical research the educational use of the game was embedded in language. However, the present version of Fingu aims primarily at developing non-symbolic aspects of arithmetic competence, partly for the reason of avoiding the bias on using simple counting procedures and rote learning that might come with language.

From a methodological point of view, however, the fact that Fingu is a non-symbolic game, meaning that children learn part-whole relations not explicitly coupled to number symbols, might pose a problem. It might be difficult to capture the embodied form of knowledge and understanding that Fingu affords with the common practice of testing children's understanding of arithmetic by using interviews, which heavily rely on language.

6.5.2 *Design of Tasks*

The basic structure of a single task in Fingu resembles an IRE-sequence (Initiation—Response—Evaluation, Mehan 1979), with a problem presentation in visual mode (I); an answer given by the fingers (R); and feedback about the correctness of the answer (E). However, it is critically important to appreciate that the task activity comprises all phases in the sequence. The affordances for learning are tied to the whole activity sequence.

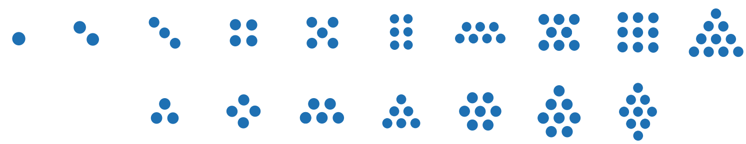
6.5.2.1 Visually Presented Collections of Objects

In the problem presentation phase, collections or sets of objects (e.g., pieces of fruit) are presented visually. Either one collection is presented, which is assumed perceivable as an undifferentiated whole (e.g., 5), or two collections of objects are presented that together make up a differentiated whole (e.g., 5 = 3|2). Fingu then essentially affords building up differentiated wholes, drawing on the ability to subitize the parts, and develop a conceptual subitizing of the whole.

The spatial arrangements of these collections of objects are regular and symmetric and often have the same configuration as a dice pattern, although there are many less familiar configurations (see Table 6.1). We call the familiar dice configurations canonical and the rest of the configurations non-canonical, since only the dice patterns (which is supported by our empirical results), are familiar to

Table 6.1 Visual number configurations used in Fingu

N	1	2	3	4	5	6	7	8	9	10
Variant a										
Variant b										



Swedish 5–7-year-old children. All configurations were chosen to be possible to subitize either perceptually or conceptually.

As can be seen in Table 6.1, using our definition strictly, there are 6 canonical configurations and 11 non-canonical. There are two configurations for each of the numbers 3–9 (allowing for invariance across visual patterns), but only one configuration for the numbers 1, 2 and 10. We use the nomenclature of 3a and 3b etc., to refer to the different representations of a given number.

In total, there are 60 tasks with different combinations of configurations and sums ranging from 1 to 10. As an example, 5a + 5b stands for the task where the canonical die-5 configuration (5a) is combined with the non-canonical configuration (5b).

A progression, or trajectory of learning, is built into the game design with seven levels of difficulty involving increasing sums and more unfamiliar visual patterns of objects. On all levels tasks are presented in random order and each task is presented twice.

As mentioned previously, time-constraints are important to the game design. The visual part-whole patterns are therefore displayed for a limited period of time. This design can be seen as a modernized version of Kühnel's (1916) flash card activities. In order to afford perception of part-whole relations, rather than counting the whole or the parts, the second presentation of a task is of shorter duration than the first presentation, given that the child has arrived at the correct number. This affords counting in examining the numerosity of a part, something that typically is done when new and complex patterns are met. However, the shorter duration of the second presentation should encourage more use of subitizing.

6.5.2.2 Answering with Coordinated Finger Patterns

A key part of the design is that children are forced to use a coordinated finger pattern to complete each task. The child/player cannot sequentially touch the screen with one finger at a time, but has to touch it with all the fingers that constitute the patterned response *at the same time*. A limited touch input latency (default 0.25 s, adjustable) is used to assure this. Thus, the player is stimulated to focus on the parts of the presented problem and the total sum instead of resorting to counting one by one in presenting the total sum.

This limited touch input latency can make it inconvenient in the initiation phase due to the risk of the game interpreting a multi-touch response as a single finger response. The task, which on the surface level may appear a simple skill-focused activity, in this way becomes more of a problem solving activity focusing on part-whole relations.

This is amplified by the player's freedom of choosing which fingers and which partition to use in the completion of the task. Learning to manage the fingers to express sums is in this way (as structured finger patterns) an essential part of what *Fingu* affords, coupled with the time constraint to reduce the likelihood that the fingers are used as tools for counting.

6.6 Fingu as a Research Platform

Fingu is designed for use in different contexts and for different purposes. The default version is a mathematical computer game to be used by children as is, preferably introduced by a more knowledgeable person (teacher/parent/sibling/peer), but not necessarily so. This is also the basic or default mode of usage.

Fingu is also designed as a research platform, essentially allowing the researcher to tailor the game for different research purposes. Part of this functionality is made available in the Settings menu of the game where game parameters can be changed. Examples of game parameters are exposure time for individual tasks (ExposureTime), time allowed to give an answer (AnswerTime), number of errors allowed on each level (Lives) and how long the player has to hold down a stable number of fingers before the answer is registered (TouchInputLatency). The default values of these parameters are based on earlier research of the forerunner of Fingu, NPG (Lindström et al. 2011) and pilot studies of Fingu (Barendregt et al. 2012).

Another level of flexibility is the ability to change the game by re-designing the content, structure and sequence of tasks in the game. This can be done by defining new individual tasks and collections of tasks for the different levels. Even the number of levels can be altered. In this way, it is possible to change the task trajectory of the game, and potentially different learning trajectories. It is, for example, possible to make different versions of the game for different age groups or to adapt the game to children's individual needs. In our empirical research we have developed re-designs in order to make the game less challenging for weak children. These re-designs are implemented in xml-code and have been locally entered into the game as customized game behavior files.

In addition, Fingu provides tools for logging and real-time playback of children's game playing behavior including answering latency and finger placement for further analysis through a visualization function built into the program. Log-files can be saved in the system (as xml-files) and can be exported for further analyses. However, logging is optional, with no logging as the default.

The functionalities that make Fingu a flexible research platform are also accessible for a teacher or a parent. The settings are easy to alter and experiment with. However, re-designing the game by developing new xml-files is more demanding. It presupposes knowledge about the architecture of the game and basic XML-coding.

Furthermore, in order to analyze progress in the game, there is a simple statistics function available that presents the success rates summed for each quantity in the range from 1 to 10. There are records for the current session and the accumulated results over all sessions. It is also possible for a researcher, teacher or parent to use the replay function as an audit trail in a debriefing session with a child.

6.7 Software

Fingu was developed in collaboration with a company developing game software and built on an open source platform. The research group, in discussion with the software company, made the design and the company did the technical implementation. Revisions to the design and development of new versions were driven by empirical research (Barendregt et al. 2012).

Fingu is available for free on the App Store. One reason for this open access was to have a stable method for distribution of the program in our naturalistic research settings. Another reason was to make Fingu available for teachers, parents, children and other researchers, as an output of our research. The app comes in different languages (presently Swedish and English).

6.8 Empirical Research

In this section, we will give examples of the empirical research we have conducted based on a larger study involving children aged 5–7 years. First, we describe the design of the study and empirical data generated. The first example is a quantitative analysis of game playing. The second example is an analysis of effects of playing on children's arithmetic abilities based on group data. The third example is an analysis of different ways of playing the game, based on group data. The fourth example is an analysis of individual development in playing the game, based on a single case.

6.8.1 Study Design and Empirical Data

In order to investigate if and how Fingu is a productive learning tool, we carried out a larger study in pre-schools and in schools, educational settings with high ecological validity, where children were given opportunities to play Fingu extensively. The study was designed with pre-, post-, and delayed tests. We gathered data from 112 children (approximately equally as many children in each age group; see Table 6.2), and with equal numbers of girls and boys. Before playing Fingu children were tested with a set of arithmetic tests. Then they played the game for 8 weeks as a part of their ordinary practice. Immediately after the playing period, children completed post-tests and 8 weeks later they completed delayed tests. The same set of test instruments were used as pre-, post-, and delayed tests. Children were not allowed to play Fingu between post- and delayed testing.

The tests were administered through individual interviews, and focused on general or more specific mathematical abilities, to measure changes in the arithmetic knowledge of the children. We used: The Test of Early Mathematics Ability, version 3 (TEMA-3) (Ginsburg and Baroody 2003); A test of part-whole knowledge, PWK, using tasks focusing on part-whole relations (some with finger patterns, some with other patterns), most of them inspired from the Early Numeracy Research Project, ENRP, in Australia (ENRP 2015); A problem solving test, PS, consisting of arithmetic problems of change or combine type with sums less than or equal to 10, and similar to the problems used by Neuman (1987); and a pattern recognition test, PR, where single configurations from the Fingu design were exposed in random order for half a second each, and children responded verbally (Holgersson et al. 2016).

During the study, the Fingu log function was used to gather log data, including data on tasks, answering times, and childrens' responses (including how many fingers were registered and their coordinates). To complement these data, we also video-recorded the children three times when they played the game: first when introduced to the game, secondly after a few weeks, and thirdly towards the end of the intervention period. For the study of a child's success in playing the game and how it develops, analyses of answering times and correctness of different trials are important, and in our third and fourth examples of analysis, we use individual and task specific median answering times (MATs) of the correctly answered trials, and mean proportions of correct answers (PCAs) to reach our conclusions. In our fourth example of analysis, we also complement this information with regression analyses (linear and binary for answering times and correctness respectively), and a study of the finger patterns used in responding to different tasks.

6.8.2 Example 1—Playing Fingu

Here we present overall data on Fingu playing time and children's success. Although the teachers were instructed to give the children opportunities to play at least three times a week, the variation in children's playing was substantial (see Table 6.2). The 5-year-olds have a median number of trials equal to around 1100 and an IQR equal to about 1200 trials, which is much bigger than the IQRs for the 6- and 7-year-olds who have medians equal to about 900 trials with IQRs equal to around 600 and 950 respectively.

Table 6.2 Number of trials made during the intervention period, disaggregated by age

Age	N	Median	Q_1	Q_3	IQR	Min	Max
5	35	1114	683	1896	1213	186	4572
6	38	905	630	1214	584	279	3189
7	39	916	654	1599	945	320	2445
All	112	953	654	1620	966	186	4572

Note Q_1 Lower quartile, Q_3 Upper quartile, and *IQR* Interquartile range

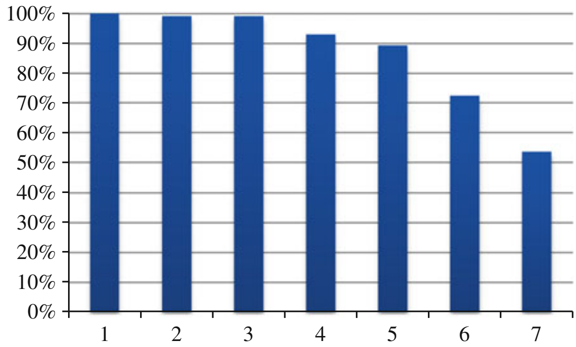


Fig. 6.2 The percentage of children playing on different levels

There is also a large variation in how many of the children played on different levels, where level 5 and 6 were the hardest to pass (see Fig. 6.2). Only 54 % (60 out of 112) of the children played on all the levels.

6.8.3 Example 2—Learning by Playing Fingu

To study the effects on children's arithmetic abilities, we performed paired-samples t-tests on the mean results on the separate arithmetic tests. The results (see Table 6.3) were that between the pre- and post-tests, there is a small effect on the TEMA-3, small to moderate effects on the PWK and the PS tests, and a large effect on the PR test.

Between the post- and the delayed post-tests (see Table 6.4) the only significant effects were found on the TEMA-3 and the PWK test. Thus, playing Fingu did have an immediate and delayed effect on children's mathematics knowledge. The effects we observed varied with age group and were largest for the 7-year-olds while the effects for the 5- and 6-year-olds were similar and comparable to the overall effects.

Table 6.3 Results from a paired-samples T-test of the mean values of the different pre- and post-tests complemented by effect sizes

Test	N	Pre-test	Post-test	Difference	Effect size	t	*p*-value
Tema3	81	28.15	31.40	3.25	0.34	7.54	< 0.001
PWK	82	15.88	18.65	2.77	0.42	6.33	< 0.001
PS	74	4.08	5.28	1.20	0.46	5.17	< 0.001
PR	75	11.64	14.44	2.80	0.79	8.54	< 0.001

Table 6.4 Results from a paired-samples T-test of the mean values of the different post- and delayed-post-tests complemented by effect sizes

Test	N	Post-test	Delayed test	Difference	Effect size	t	*p*-value
Tema3	82	31.43	33.16	1.73	0.18	4.64	< 0.001
PWK	80	18.65	19.68	1.03	0.16	2.38	0.020
PS	81	5.12	5.09	−0.04	−0.01	−0.22	0.829
PR	78	14.46	13.97	−0.49	−0.15	−1.76	0.082

6.8.4 Example 3—Two Different Ways of Playing Fingu

Our third example focuses on how two different groups of children, identified by their tendency to take shorter or longer amounts of time to answer the different tasks, differ in their ability to make use of the structural affordances that become available when playing the game.

A core idea in the design of Fingu was that children would develop subitizing strategies rather than relying on only counting strategies. Signs that such strategies were used should be evident as shorter MATs and/or greater PCAs. When we analyzed the answering times for different tasks in the game, there were some tasks (e.g., 2 + 0, 5a + 0, and 5a + 5a), where almost 100 % of the answering times were shorter than 3.0 s. In other tasks (e.g. 7a + 1 and 7a + 2), the majority of the answering times were longer than 4.0 s. However, there were a few tasks (e.g., 9a + 1 and 10 + 0), where the distribution of answering times displayed a clear bimodal form, with one bump between 1 and 3 s, and the other between 4 and 6 s (see Fig. 6.3). An interpretation of these bumps in the data is that the first bump is the result of strategies using retrieval or subitizing, whereas the second bump is the result of counting all the objects using a one-by-one method. Of course the answering time is the result of the joint time it takes to determine the sum and to form the finger response pattern.

Since the tasks 10 + 0 and 9a + 1 only appear on level 7, we restricted our analysis of this question to those 60 children, that is 54 % of all children, (see Fig. 6.2), who played all the levels of Fingu. Using the individual MATs from only three special tasks, 10 + 0, 9a + 1 and 5a + 5b, we identified three different groups (with roughly equally numbers of subjects in each): F (Fast) individuals (with MATs shorter than 3.0 s on all three tasks); S (Slow) individuals (with MATs longer than 4.0 s on all three tasks); and M (Medium) individuals (individuals with mixed answering time patterns).

To compare the performance of these three groups, we performed independent samples t-tests between the F and S groups' mean MATs on each of the 60 tasks. What we found is that the F and S groups had significantly different mean MATs on 52 of the 60 tasks (with $p < 0.001$ for 26 of the tasks and p-values between 0.001 and 0.05 for the remaining 26 tasks). The tasks that, divided the F and S groups most significantly were tasks where the configuration 5b was one of the elements.

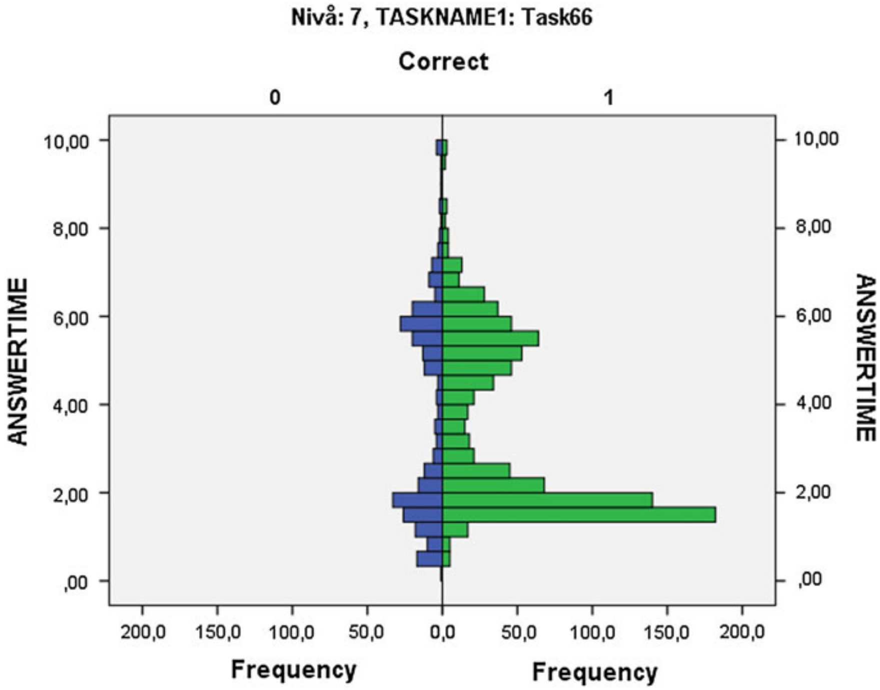


Fig. 6.3 Distribution of answer time for correct and non-correct answers from 60 players that answered the task 10 + 0

The ability to perceive configuration 5b as the quantity five, in the same way as 5a is perceived by almost all of the players, thus seems to be a crucial difference between the F and the S groups.

Other tasks that showed strong significant differences between the F and S groups included tasks with one large (>5) single element, or tasks of the N + 1 type, where N was larger than 3. These tasks represented at least two different types of increases in complexity. The first was to learn the affordance to subitize a non-canonical configuration as 6b, 7b, or 9b. The other type of complexity was to be able to quickly add +1, an ability that presupposes the ability to conceptually subitize the larger part.

In summary, we identified two types of strategies that children developed as they played Fingu. The first was to use counting to find out how many fingers to use and then respond accordingly. The other was to pick up some of the affordances that the game offers to use some kind of subitizing (either perceptual or conceptual), to directly recognize either (a) the single configuration of a task, (b) the two configurations separately, or (c) the totality of these tasks, all resulting in shorter answering times. These strategies were used by all of the different groups, but in very different proportions. There were tasks such as 1 + 0, 5a + 0 and 5a + 5a, which both the F and the S groups solved by subitizing and there were tasks such as

6b + 3a and 7b + 2, which the majority in both groups solved by enumeration (although the F group was faster on these tasks). The biggest observed difference between the groups was in how they managed to utilize the affordances of Fingu. As a sign of an individual's ability to take advantage of these affordances, his or her relation to configuration 5b seems to be indicative.

6.8.5 Example 4—An Individual Developmental Trajectory

Our last example is an analysis of individual development while playing the game based on a single case.

Adam is a five-year old boy attending pre-school, who made large improvements between pre- and post-testing on all four tests, with delayed test results about the same as the post-test results. Among the participants he completed the most trials, with a total of 4572 (IRE-sequences). These trials fell into two periods. In the first 5 weeks of the intervention, he completed 2140 trials during 195 attempts in levels 1–6. In the last three weeks of the intervention, he completed 2432 trials during 141 attempts in levels 6 and 7 while also replaying other levels (see Table 6.5).

In Period 1, Adam was busy trying to advance in the game. What is striking in Table 6.5 is the display of endurance and persistence even with low proportions of correct answers. He completed Levels 1 and 2 quickly but, on Levels 3–5, he made a number of attempts before he succeeded. On Level 6, he made 17 unsuccessful attempts. In Period 2, Adam started by successfully replaying all the levels that he had completed before, and then he continued with his attempts to succeed on Levels 6 and 7. This took another 12 attempts on Level 6 and 28 attempts on Level 7. After that, there was a long period where he repeatedly (mostly successfully) replayed Levels 1–5. Levels 6 and 7 remained difficult with only 2 successes on Level 6 and 4 successes on Level 7. Altogether, what emerged was Adam's perseverance in the game and his willingness to replay it, becoming a more confident Fingu player.

What kind of mathematics has Adam learned? Looking at the PCAs and MATs of the different tasks on different levels reveals several patterns. Since the MATs for correct answers were only greater than 4.0 s for 5 of the tasks and less than 3.0 s for 53 of the tasks, we concluded that Adam had a clear tendency of answering quickly and avoiding counting methods. Regression analyses showed that Adam significantly improved the PCA on 30 of the 60 tasks, and became significantly faster in giving correct answers on 25 of the tasks.

Table 6.5 Adam's number of attempts on different levels and success rates

	Level	1	2	3	4	5	6	7	Sum
Period 1	Attempts	5	7	32	46	88	17	0	195
	Successful (%)	0.60	0.29	0.03	0.02	0.01	0.00		0.04
Period 2	Attempts	21	9	10	9	8	34	50	141
	Successful (%)	0.95	0.78	0.80	0.78	0.88	0.06	0.08	0.39

Using the playback system (see Fig. 6.4 for an illustration), we analyzed the finger patterns that Adam used. During the first period, there were often flashes of dots flickering by and many of the patterns that emerged were interpreted as missing one finger. The first two videos of Adam's play, which showed that he was very impulsive and kinesthetically imprecise, confirmed this observation. An illustrative example is Adam's trajectory for the task 7b + 0. In the first period, he usually answered with the pattern 3 + 5 (3 fingers on the left hand +5 fingers on the right) together with occasional correct answers with the pattern 3 + 4. The most reasonable interpretation of this observation is that he used the pattern 3 + 5 all the time, but occasionally the pinky of his right hand was placed on the touch-insensitive frame of the iPad, which resulted in a correct answer. In this example, the feedback must have been confusing to him, most often being negative, but occasionally positive, for what may have appeared to Adam as the same response. However, as he continued playing, he resolved this dilemma during period 2 by changing his response to 2 + 5.

Another observation made from Adam's response patterns is his tendency to rely on subitizing in solving the tasks. From the beginning, Adam quickly responded with the pattern 0 + 5 on the task 5a + 0 and it did not take long before he used the same pattern on the task 5b + 0. On the tasks 6a + 0 and 6b + 0, however, he quickly developed the response pattern 3 + 3. In the configuration 6a, it was easy for Adam to recognize a 3 + 3 pattern, but in the configuration 6b, this was not as easy. However, it is possible that Adam recognized the 3b triangular configuration on the top of a linear 3-dot pattern. To explain why the task 8b + 0 was harder for Adam to learn than the task 8a + 0, there is the possibility he saw the configuration 8a as composed of the configurations 5b and the same linear 3-dot pattern.

The 3 + 3 response pattern for 6a + 0 and 6b + 0 from the later part of Period 1 into the whole of Period 2 becomes the dominating response pattern for all tasks with 6 as the total sum, with the exception of the task 5a + 1, where the response pattern 1 + 5 persists. In the first part of Period 1 he uses the semi-decimal response pattern 1 + 5 also for tasks 4a + 2, 4b + 2, and 3a + 3a. Our interpretation of these observations is that in the beginning of his playing Adam uses the response pattern 1 + 5 to represent the number 6, while he later establishes the response pattern 3 + 3 as a form of mapping, and as his favorite representation of the number 6. The exception is the task 5a + 1 where he uses the pattern 1 + 5, because it is a direct mapping of the task.

6.8.6 Additional Empirical Observations

In summing up, our analyses have shown that individuals exhibited large variations, not only in the amount of time they played, but also in the strategies they developed to manage the game. Our experience was that on tasks where there was a configuration that children did not immediately recognize, they either explicitly or tacitly counted in order to determine the number of objects that were presented on the

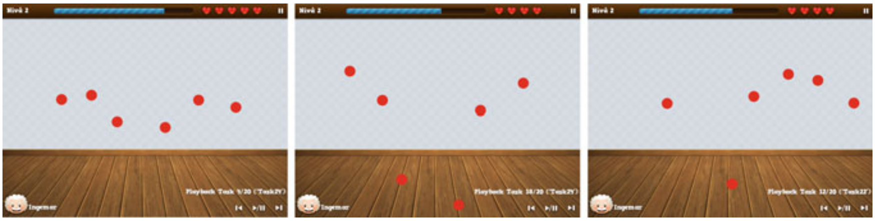


Fig. 6.4 Adam's different ways of answering using six fingers

screen. However, they were much less inclined to count their fingers to determine the number of fingers to put down in responding to the task. Instead, they seemed to develop a kind of personal canonical finger pattern for each of the ten basic numbers. For total quantities of 1–5 these patterns were most often formed by one hand with 1–5 adjacent fingers. For total quantities of 6–10, these patterns could be semi-decimal (i.e., consisting of all the fingers on one hand complemented by fingers on the other hand), or they could be a symmetrical pattern (e.g., 4 fingers on each hand representing 8, or 3 fingers on each hand representing 6). Another way children determined the number of fingers to use, as part of their response was to map each set of the presented objects separately. When this strategy was used, most of the children did not count the number of objects in either of the sets. Instead, they seemed to subitize these numbers. In this way, their strategy was more efficient than counting. In Fig. 6.4, both mapping patterns and a semi-decimal pattern is illustrated. All these analyses of different finger patterns have made use of the built-in replay function of Fingu. As the figure shows, due to the pattern of the red spots, it is very often possible to be almost certain if one or two hands have been used, or even which fingers are used. This interpretation becomes much stronger when compared to the information from the corresponding videos.

6.9 Concluding Remarks

In this chapter, we have described the Fingu game as a virtual manipulative, outlined the design principles, and discussed the underlying theoretical rationale. We have also illustrated some of the affordances of Fingu and the potential effects of playing the game. Our conclusion is that Fingu is a game that offers valuable experience for teaching and learning early numeracy, whether in school or in home settings.

As pointed out above, design, development, use and research of Fingu are part of a research program adopting a DBR-approach. Fingu has gone through several iterations of revision of a number of its design elements, including layout changes and enhancements of the game packaging. We also aim to develop other versions of Fingu, with designs for other affordances than the present version.

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Chapter 7
Developing Virtual Mathematics Manipulatives: The SAMAP Project

Erol Karakırık

Abstract Online educational activities providing interactive environments where users can investigate the properties of abstract concepts and reflect on them are in great demand for all subjects in primary and secondary schools. However, the ubiquitous nature of these activities does not always guarantee students' conceptual development if enough consideration was not given to the design and implementation of the system and an appropriate role was not defined for the technology used. A computer system could play a wide range of roles changing from a 'tutor' acting as "a decision-making" subject to a 'tool' acting as an "auxiliary" object. One can also interpret this classification of roles as a system having total control of flow or a system allowing free explorations. A computer system is regarded as suitable to be used in education when it provides facilities that promote the student's conceptual development through engaging him/her in meaningful and authentic tasks. The new Turkish mathematics curriculum is based on constructivist educational approaches and advocates the wide usage of educational activities that help to make mathematical concepts and relations meaningful. The purpose of this chapter is to report the findings of a research project, SAMAP, funded by the Turkish National Science Foundation (TUBITAK), which aimed to develop virtual mathematics manipulatives in Turkish for the primary and secondary school curriculum.

7.1 Introduction

Recent developments in Information Communication Technologies (ICT) offer many new possibilities to enhance students' comprehension during the learning-teaching process. Hence, online educational activities providing interactive environments where users can investigate the properties of concepts and reflect on them are in great demand for all subjects in primary and secondary schools.

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However, the ubiquitous nature of these activities does not always guarantee students' conceptual development if enough consideration was not given to the design and implementation of the system and an appropriate role was not defined for the technology used (Durmuş and Karakırık 2006). A computer system could play a wide range of roles changing from a '*tutor*' acting as "*a decision-making*" subject to a 'tool' acting as an "*auxiliary*" object (O'Shea and Self 1983; Crook 1994). One can also interpret this classification as a system having total control of flow or a system allowing free explorations. In Jonassen's (1996) cognitive tool metaphor, an appropriate role for computers in line with the constructivist approaches is defined. A computer system is regarded as suitable to be used in education when it provides facilities that promote a student's conceptual development through engaging him/her in meaningful and authentic tasks.

Mathematics is rightly regarded as one of the most important subjects for the primary school primary school curriculum. Students at the concrete operational stage are introduced to fundamental abstract mathematical concepts and relations for the first time at this stage. Hence, it is vital to make abstract mathematical concepts and relations concrete with different models in order for students to be able to grasp them. Lack of such models leads students to focus on arithmetic and procedural skills rather than mathematical concepts and relations. Hence, there are many projects in this regard, such as National Library of Virtual Manipulatives (NLVM) (http://nlvm.usu.edu), WisWeb (http://www.fi.uu.nl/wisweb/en/) and the National Council of Teachers of Mathematics (NCTM) Illuminations (http://illuminations.nctm.org), to provide comprehensive sets of mathematics manipulatives to be used from kindergartens to the graduate studies to promote students' mathematical skills and understandings. The new Turkish mathematics curriculum, updated in 2005 (MEB 2005), is based on constructivist educational approaches and advocates the wide usage of educational activities that help to make mathematical concepts and relations meaningful. It also aims to promote the use of ICT and the Internet and to remove the digital gap among primary and secondary school students through the FATİH project (fatihprojesi.meb.gov.tr), which employs physical and virtual manipulatives.

Manipulatives are physical objects or concrete models that can make abstract ideas and symbols more meaningful and understandable to students (e.g., base-ten blocks and algebra tiles). A virtual manipulative is "an interactive, Web-based, visual representation visual representation of a dynamic object that provides opportunities for constructing mathematical knowledge" (Moyer et al. 2002). Virtual manipulatives are distinguished from other digital resources used for learning in their dynamic nature and provision of interactive experiences. The importance of using play and manipulation to grasp abstract mathematical concepts or using tools or concrete objects to mediate learning is emphasized by many educators for constructivist learning environments (Bruner 2003; Dienes 1971; Duffy and Cunningham 1996; Piaget 1952; Pea 1985; Vygotsky 1978). Many studies also confirm virtual and physical manipulatives physical manipulatives as effective tools of instruction (Butler et al. 2003; Sowell 1989; Suh and Moyer 2007). However, the provision of tools alone is not sufficient without adequately clarifying their place and their usage in the teaching-learning process. Hence, it is necessary to develop virtual manipulative sets that specifically highlight certain mathematical concepts and relations in the mathematics curriculum.

7.2 The SAMAP Project and Manipulatives Development Process

SAMAP is a Turkish acronym composed by the initial letters of the Turkish phrase "virtual mathematics manipulative project". The SAMAP project was launched to develop an interactive, comprehensive and multi-lingual mathematical manipulative set, primarily focusing on the Turkish audience, for the primary and secondary school curriculum (Grades 1–8) in five strands of mathematics (numbers, geometry, measurement, data analysis and algebra). It was implemented by the author at Abant İzzet Baysal University, Bolu, and sponsored by the Turkish National Science Foundation, TUBITAK (Karakırık 2008, 2010). The SAMAP project included a graphic designer who was responsible for designing graphical elements (e.g., icons and images) displayed on the manipulatives and the website. The design and coding of the manipulatives and instructions and explanations provided in the manipulatives and on website were all managed by the author. SAMAP could be regarded as the first attempt to produce the Turkish version of the National Library of Virtual Manipulatives (NLVM) (NLVM) (Nlvm.usu.edu 1999). The general outline of the NLVM was adopted for SAMAP's implementation. Many novel manipulatives as well as modified versions of available manipulatives were implemented in SAMAP. Most SAMAP manipulatives were designed with a direct reference to a mathematical objective from the Turkish mathematics curriculum.

SAMAP manipulatives were coded by the author in an object-oriented manner using JAVA programming language. All SAMAP manipulatives were derived from the same JAVA code, which allowed for the creation of both an *applet* version, running on a webpage, and a stand-alone *application* version which could be downloaded. The SAMAP project initially employed the applet versions of the manipulatives on a website and later a SAMAP CD was produced with the application versions of the manipulatives. Figure 7.1 shows the outline of a typical SAMAP manipulative. Each SAMAP manipulative screen was divided into certain areas to provide a consistent and user-friendly environment: the *main working area* holds the actual implementation of the manipulative, *title bar* displays the manipulative's title, *information panel* gives specific information about the manipulative or shows the latest feedback based on a user action and the *command panel* provides an interactive area where all graphical items providing user interaction (such as buttons, textboxes etc.) are placed. Since communication among panels is achieved through a special messaging service, many components seen on the screen are independent of each other and can be reused in the design of different manipulatives.

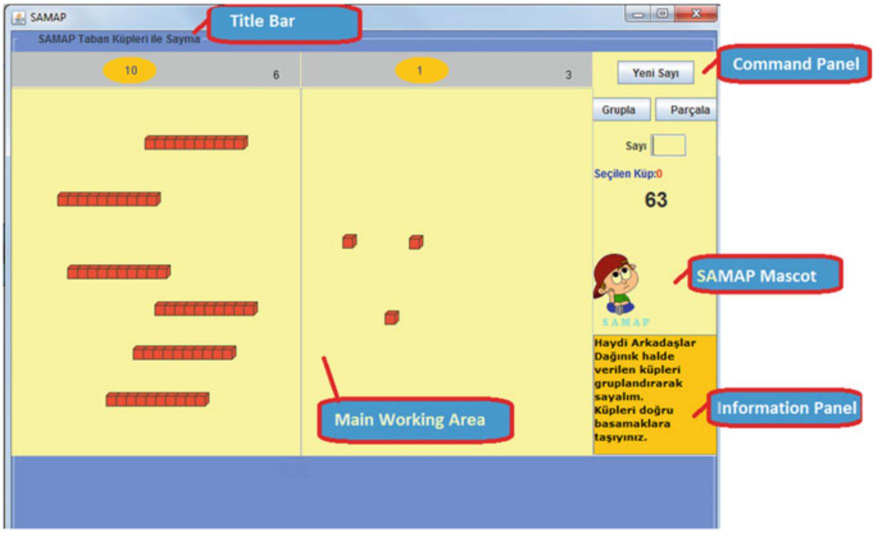


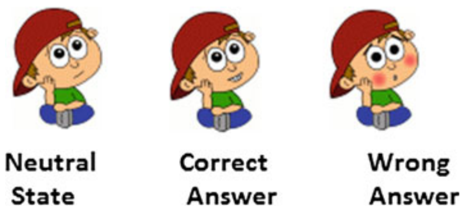
Fig. 7.1 An example SAMAP manipulative layout

All panels are optimized to 800 × 600 screen resolution and are adjusted automatically for lower resolutions and provide scrollbars if necessary. The command panel and the information panel are automatically placed at the bottom of the screen whenever a manipulative requires a larger horizontal working area. SAMAP's default language is Turkish but can detect and adapt itself to the user's local language. However, a property file holding language specific data needs to be updated to use SAMAP in another language. Manipulatives could interact with the website through JavaScript and display instructions and help pages specific to a manipulative.

The information panel also includes the SAMAP mascot, which provides instant graphical and audio feedback about the latest user action as portrayed in Fig. 7.2. The SAMAP mascot smiles for a correct action and it blushes for a wrong one. It is interesting to note that this mascot is known as SAMAP especially among young users.

Educational technologists need to determine the educational and technical specifications of a manipulative before implementation. Objectives from the Turkish

Fig. 7.2 States of a SAMAP mascot



primary mathematical curriculum were thoroughly investigated and activities that could be adapted to the computer were selected by the author and educational aims and specifications were determined. Technical specifications of the manipulatives were determined with both the requirements of the task at hand and the restrictions of the programming environment or the expertise of the programmers.

Virtual manipulatives, in essence, provide novel virtual artifacts (objects or tools) to be used to reflect or play with certain mathematics concepts and relations. The design of any artifact is determined by the specifications of its affordances. An affordance defines the relationship between the properties of an object and the capabilities of the agent that determine just how the object could possibly be used (Norman 2013). Hence, unique affordances for virtual manipulatives need to be identified to help students to learn specific mathematics concepts and relations. Moyer-Packenham and Westenskow's (2013) meta-analysis of virtual manipulatives suggest 5 affordance categories of virtual manipulatives, namely focused constraint, creative variation, simultaneous linking, efficient precision and motivation. The SAMAP manipulatives were designed in a way to support these pedagogical affordances to impact student learning.

The SAMAP manipulatives were designed to be very flexible but constrain student actions and restrict input when necessary and have *focused constraints*. For instance, drop down boxes and radio-buttons were widely employed to limit the user inputs to correct values. If a textbox was used to prompt user input, validity of the input was confirmed by necessary checks before the input was accepted. Students were allowed free explorations and solved specific problems with restricted actions depending on the running mode of the manipulatives.

The SAMAP manipulatives were designed to support *simultaneous linking* by using multiple representations of mathematics concepts in symbolic, graphical, textual or other forms. For instance, the set manipulations dynamically supports textual, pictorial, and symbolic forms for set operations.

The SAMAP manipulatives were designed to be attractive and entertaining and afford *motivation*. A specific icon was designed for each manipulative and a special title was chosen for each activity to attract students' attention. The SAMAP mascot was very motivating for younger students. Each manipulative was designed to have a specific meaningful task. The tasks were also designed to be challenging enough to increase motivation.

The SAMAP manipulatives were designed in a way to support *efficient precision*, by employing precise representations and simulating real behaviors. For instance, the *hit the target* manipulative uses an aircraft and cannon to demonstrate the concept of angle. The *finding symmetry* manipulative allows users to split an image flexibly and test whether they are comparable by dragging one piece to another and rotating them. However, efficient precision is sometimes disregarded on purpose to make students focus on the concept at hand. For instance, the right angle symbol is not used when the angle is 90° to emphasize their interchangeable usage in the *hit the target* manipulative. However, most students regarded this as an error and requested the regular right angle graphical symbol be used to denote that an angle having a value of 90° is a right angle.

The SAMAP manipulatives were designed to have open-ended tasks that encourage *creativity* and enable students' multiple solutions. For instance, the *logo* manipulative allows users to produce their own drawings (e.g., cars and houses). Many manipulatives require subject-specific problem-solving knowledge to solve the presented problem in a task and allow multiple solutions for the problem. For instance, the *set* manipulative could interpret any set operations and allow different symbolic representations for any area on the Venn diagram beside symbolic expressions provided in the drop down boxes.

The SAMAP project included different types of manipulatives such as manipulatives for solving certain mathematics problems, exemplifying or simulating certain mathematical concepts or relations, doing certain mathematical calculations and procedures, and innovative applications of mathematical concepts and activities that could provoke discussions in the class. Manipulatives that could be employed for both individualized instruction and collaborative work were implemented. Each manipulative is automatically adapted to different class levels through certain parameters. For instance, base ten blocks were adapted to employ numbers up to 20, 99, 999 and 9999 for first, second, third and fourth grade students respectively, thereby employing the focused constraint affordance. Furthermore, setting the base to a number other than 10 is also permitted for grades 5–8.

Around 75 distinct (100 with variations) mathematical manipulativewere implemented by the author at Abant İzzet Baysal University during the three years of the project between the years of 2005 and 2008. The number of manipulatives can be roughly categorized with respect to the five strands of mathematics as follows:

- numbers (28),
- geometry (20),
- data analysis and probability (14),
- measurement (8),
- algebra (5).

These numbers roughly correspond to the weights each mathematical strand occupies in the primary level. These numbers are regarded as in line with the project aims since numerical activities focusing on only arithmetic and numerical operations without any problem solving tasks were generally avoided. For instance, activities requiring arithmetic operations but not attaching any specific meaning to the numbers used were avoided. In fact, activities of this type were designed only after specific requests by in service teachers (e.g., one such example is the *number pyramid* manipulative in number strand). The relevant files for each manipulative were brought together in a JAR file. Then deployment, instruction and web pages for the manipulatives were designed. Figure 7.3 shows the sequence of web pages to reach specific manipulatives in a strand.

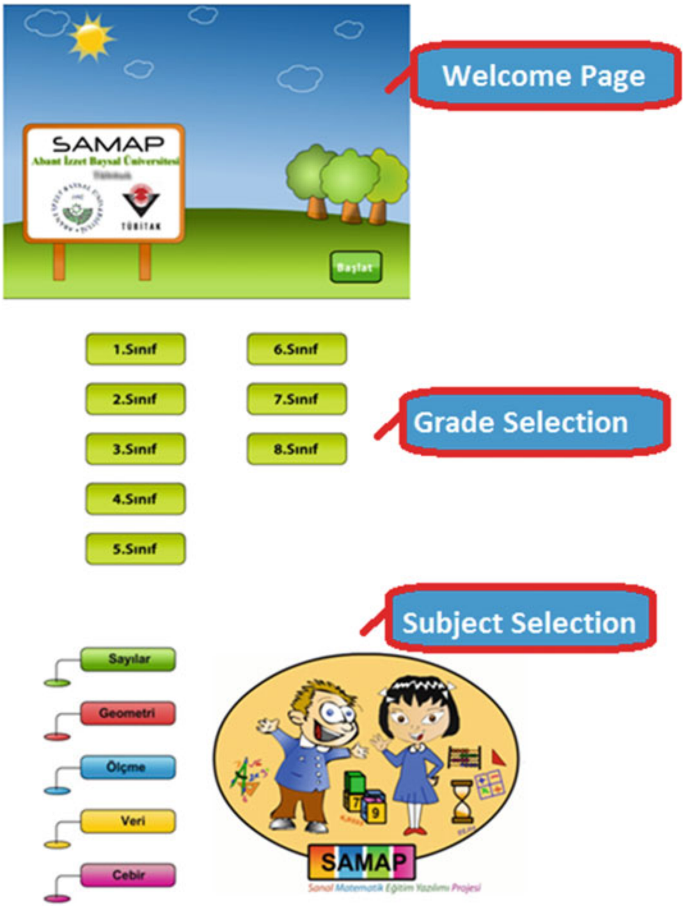


Fig. 7.3 Web pages to reach specific manipulatives in a strand

7.3 Examples of SAMAP Manipulatives

Here are two examples of SAMAP manipulatives exemplifying individualized and collaborative usage in the class to demonstrate the functionality and nature of the manipulatives.

Figure 7.4 shows a SAMAP manipulative for operations on sets and shows its results dynamically in multiple representations namely symbolic, verbal, and graphical forms. The manipulative has many options in the form of radio-buttons on the bottom-left of the screen to change the appearance and the functionality of the set operations such as:

- number of sets available (one, two, or three),
- types of the sets (intersected sets, disjoint sets and subsets), and

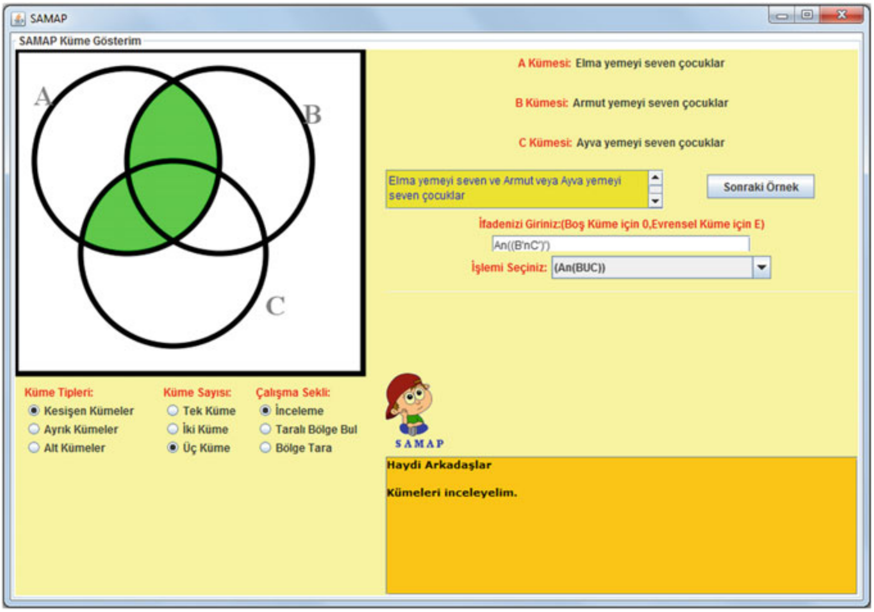


Fig. 7.4 The set manipulation of SAMAP

- running mode (free investigation, finding the shaded region, and shading a specified region).

Furthermore, the manipulative has the following components to provide further display options:

- interactive graphical display of the results of the set operations on the top right of the screen,
- a drop down box showing all of the available results of set operations in symbolic form $(A \cap (B \cup C))$ on the bottom-right of the screen,
- a textbox to enter any set operations in symbolic form,
- a button to change the text examples resembling the current set operations (there are 7 different text examples for three intersected sets such as children eating apples, pears, and quinces to show the sets A, B, and C respectively for the current selection), and
- a text area where the results of the current set operations are displayed with respect to the selected text example.

As Fig. 7.4 shows, this manipulative requires subject-specific problem-solving knowledge for solving all relevant problems for the sets. Some of the capabilities of the system, such as finding all different symbolic representations for the current selection, such as $A \cap ((B' \cap C'))'$ or $\left(A' \cup \left(B' \cap C'\right)\right)'$ which could be found by applying De Morgan's Laws for the current selection, were hidden to reduce the complexity of the manipulative. In fact, one could argue that this manipulative has the potential to be turned into more distinct manipulatives by just focusing on one of the representations, on one of the set operations, or on one of the running modes. Hence, in an individualized or collaborative teaching scenario, this manipulative has the functionality to display the results of set operations dynamically in different forms.

Although this manipulative resembles the *Venn Diagrams* manipulative available at the NLVM, it has novel and flexible features to distinguish it as a completely different manipulative. For instance, one could use various real life situations such as reading (newspapers, magazines, and books), attending lectures (math, physics, and chemistry) and learning languages (English, French, and German) as well as eating various foods to concretize the meaning of the set operations in the example while the NLVM Venn Diagrams does not have this capability.

Figure 7.5 shows another SAMAP manipulative, which focuses on the concept of arithmetic mean that dynamically computes and displays the arithmetic mean of a set of random or user-defined data between 0 and 100. Unlike the previous example, this manipulative has neither a comprehensive problem-solving capability nor range of options for user-interaction. It only accepts random or user-defined data and displays the data points and their arithmetic mean on the screen. In other words, it enables users to see how the arithmetic mean changes when data set

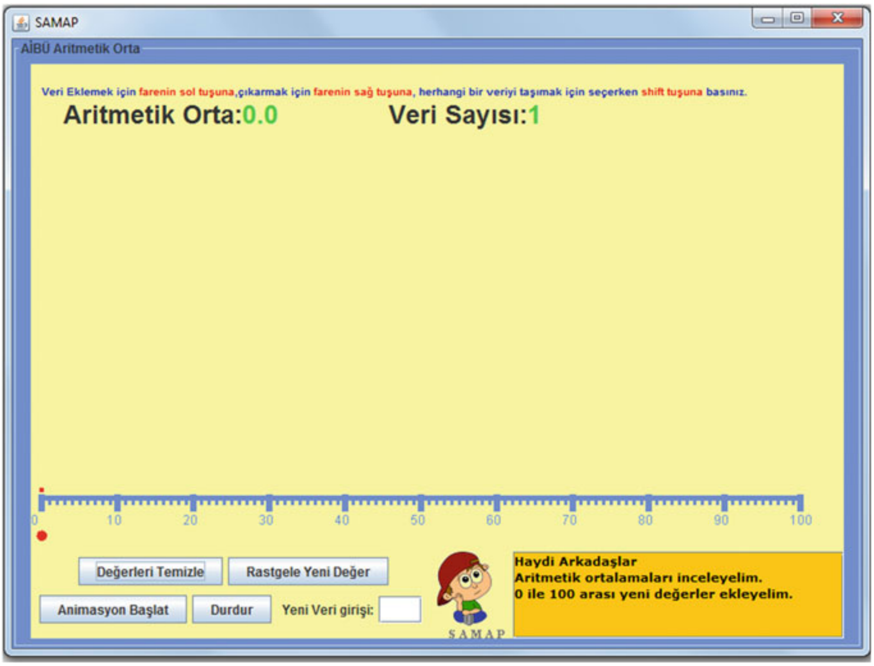


Fig. 7.5 The arithmetic mean manipulative of SAMAP

changes. Many users, including teachers, who seem to be inclined to press buttons randomly or change some parameters in a manipulative to see what it leads to had difficulty seeing or predicting the rationale for this manipulative since it has neither a specific problem-solving tool nor a didactic style to teach something.

This does not make this manipulative less effective since it is purportedly designed to be used with the guidance of an instructor in a teaching setting to provoke discussions about the arithmetic mean. The manipulative enables users to enter new data or remove or change existing data to resolve some of their hypotheses about the arithmetic mean. However, the instructor needs to ask leading questions to help users to grasp the subtleties of the concept of arithmetic mean. For instance, one could ask the following leading questions that could be answered with the help of this manipulative to demonstrate different aspects of arithmetic mean:

- state an observation about how arithmetic mean changes when new random data are inserted;
- enter new data in order to increase/decrease the current arithmetic mean;
- remove any data on the screen in order to increase/decrease the current arithmetic mean; and,
- change any data on the screen in order to increase/decrease the current arithmetic mean.

All of the above questions point to some aspect of the concept of the arithmetic mean. In fact, this manipulative never requires users to compute the arithmetic mean by hand. Rather it forces users to think and comment on the influence of data on the arithmetic mean.

7.4 Evaluation of SAMAP Manipulatives

The main aim of the SAMAP project was to be able to develop a comprehensive virtual manipulative set for mathematics. The technical side of the project was overwhelming and the project was understaffed for the evaluation part since one of the team members responsible for evaluating the manipulatives had to leave at the beginning of the project. Hence, the effectiveness of the manipulatives was mainly evaluated through online questionnaires (e.g., the SAMAP assessment scale surveys) and users' comments on the technical and educational aspects of the manipulatives. Suggestions for improvements were collected through the website during the project. There were no quantitative or qualitative studies conducted to make comparisons of SAMAP manipulatives with concrete manipulatives or other instructional treatments because of the author's belief that virtual manipulatives need to be developed to promote students' conceptual understanding rather than to improve students' performance on tasks and tests. However, SAMAP manipulatives were widely used in primary schools and were introduced to pre-service teachers in educational technology courses in several Turkish Universities. The next section includes findings from implementation efforts, online evaluation, and reports from pre-service teachers.

7.4.1 Online Evaluation of SAMAP Manipulatives

A project website was launched at the beginning of the project and all SAMAP manipulatives were gradually integrated to the website. Users were required to register to be able to use the site and participate in the study on voluntarily basis. Hence, all numbers reported in the study refer to unique users. The study was conducted between years 2006 and 2009. The online users of the SAMAP website were classified into 9 different categories with respect to different class levels and experience as shown in Table 7.1. Users that did not fall into one of these categories, such as parents and non-students, were classified as other and not included.

Nearly ten thousand unique users accessed the website during the project. A special web page was prepared for each manipulative containing the instructions on how to use the manipulative and online questionnaires depending on a particular user's classification. Various bits of statistical information about the users, such as their locations and their computer screen resolutions, were collected using the Google analytics tool. The number of users and completed questionnaires for different numbers of manipulatives in each strand are shown in Table 7.2. Five strands of mathematics, namely numbers, geometry, measurement, data analysis and algebra, are decoded using the abbreviation ST1 to ST5, respectively, in all the results that follow.

Online questionnaires using Likert scale responses were administered to every group for each manipulative. Different questionnaires having different numbers of items were prepared with a measurement expert. Questions about technical and educational aspects of the manipulatives, as well as users' attitudes, were asked in the questionnaires. Table 7.3 shows the number of questionnaire items and the aspects of questions for each group. Participants were asked to complete the

Table 7.1 Classification of online users of SAMAP website

Group No	Group members
Group 1	1st grade, 2nd grade and 3rd grade primary school students
Group 2	4th grade and 5th grade primary school students
Group 3	6th grade, 7th grade and 8th grade secondary school students
Group 4	First and second year pre-service teachers
Group 5	Third year pre-service teachers
Group 6	Fourth year pre-service teachers
Group 7	Graduate students
Group 8	Teachers
Group 9	Academicians

Table 7.2 The number of questionnaires completed for each strand of mathematics

	Size (N)	Number of manipulatives	ST1	ST2	ST3	ST4	ST5	Total
Group 1	43	34	32	23	14	4	–	73
Group 2	20	23	10	5	8	3	–	26
Group 3	16	23	5	10	2	1	2	20
Group 4	44	43	63	23	9	9	2	106
Group 5	51	62	116	74	54	42	1	287
Group 6	33	36	22	16	4	4	12	58
Group 7	16	64	57	44	31	38	34	204
Group 8	45	55	57	22	16	4	3	102
Group 9	12	30	14	8	7	5	8	42
Total	280	370	376	225	145	110	62	918

Table 7.3 The number of questionnaire items for each Group

	Attitude		Educational		Technical		Total
	#	%	#	%	#	%	Total
Group 1	3	60	1	20	1	20	5
Group 2	3	27	2	18	6	55	11
Group 3	4	27	4	27	7	47	15
Group 4	4	20	5	25	11	55	20
Group 5	4	15	8	31	14	54	26
Group 6–7	4	10	20	49	17	41	41
Group 8–9	4	8	28	55	19	37	51

questionnaires for any manipulative they preferred. User comments were also collected through a text area. The full analysis of the questionnaires (Karakırık and Cakmak 2009) is beyond the scope of this chapter. We will just summarize the results of each group and focus on primary school students in Group 1 and Group 2.

7.4.2 *Group 1*

Group 1 included 1st to 3rd grade students. A simple questionnaire consisting of five three-level Likert items was prepared and pilot-tested with 10 children of this level to ensure comprehensibility of questionnaire items. Students were required to choose one of the three emoticons, ☺, 😐 and ☹ to answer an item to denote "agree" (A), "no opinion" (N) and "do not agree" (D). The results of questionnaires for Group 1 are displayed in Table 7.4.

Table 7.4 The results of questionnaires from Group 1

Items	Opinion	ST1	ST2	ST3	ST4	Total	%
1. I liked this game	A	29	20	11	2	62	0.85
	N	2	2	1	0	5	0.07
	D	1	1	2	2	6	0.08
2. I like to play this game again	A	26	17	10	3	56	0.77
	N	5	4	2	1	12	0.16
	D	1	2	2	0	5	0.07
3. It is easy to play this game	A	28	18	12	1	59	0.81
	N	4	4	1	3	12	0.16
	D	0	1	1	0	2	0.03
4. I can play this game on my own	A	28	20	13	3	64	0.88
	N	3	1	0	1	5	0.07
	D	1	2	1	0	4	0.05
5. I learned something new in this game	A	23	18	10	2	53	0.73
	N	6	4	0	2	12	0.16
	D	3	1	4	0	8	0.11

Note A agree; *N* no opinion; *D* do not agree

Forty-three students completed 73 questionnaires for 34 different manipulatives in Group 1. Students in this group stated that they loved the manipulatives (85 %), wanted to play again (77 %), found them easy (81 %), were able to play the games themselves (88 %), and learned something new (73 %). Students' comments also confirmed that they liked SAMAP manipulatives and found them easy to use.

7.4.3 Group 2

Group 2 included 4th and 5th grade students. A pilot study was performed on 5th-grade students to determine the comprehensibility of 29 questionnaire items. Based on the pilot study, a final questionnaire, consisting of eleven three-level Likert items, was prepared like Group 1 but written expressions were used instead of emoticons. The questionnaire administered to Group 2 and the results are displayed in Tables 7.5 and 7.6 respectively.

Twenty students completed 26 questionnaires for 23 different manipulatives in Group 2. Students in Group 2 loved the manipulatives (88 %), found them easy (96 %), thought the descriptions were clear (92 %), liked the screen layouts (77 %), wanted to play again (85 %), thought they drew attention to the mathematics (73 %), and reported that they learned something new (69 %). Many students stated

Table 7.5 The questionnaire administered to Group 2

Q1. I like this activity
Q2. I want to use this activity again
Q3. It is easy to use this activity
Q4. I can use this activity on my own
Q5. This activity has increased my interest in mathematics
Q6. The screen layout of this activity is nice
Q7. This activity has the opportunity to correct my mistakes
Q8. This activity gives adequate warning and information
Q9. This activity has the opportunity to test myself
Q10. I learned new topics in this activity
Q11. The description of this activity is clear

Table 7.6 The results of questionnaires from Group 2

	ST1			ST2			ST3			ST4			Total			%		
	A	N	D	A	N	D	A	N	D	A	N	D	A	N	D	A	N	D
Q1	10	0	0	4	0	1	6	0	2	3	0	0	23	0	3	0.9	0	0.1
Q2	10	0	0	4	1	0	6	1	1	2	1	0	22	3	1	0.9	0.1	0
Q3	10	0	0	5	0	0	7	0	1	3	0	0	25	0	1	1	0	0
Q4	9	1	0	5	0	0	8	0	0	3	0	0	25	1	0	1	0	0
Q5	9	1	0	3	2	0	5	3	0	2	1	0	19	7	0	0.7	0.3	0
Q6	8	1	1	4	0	1	5	2	1	3	0	0	20	3	3	0.8	0.1	0.1
Q7	9	1	0	3	1	1	6	0	2	3	0	0	21	2	3	0.8	0.1	0.1
Q8	8	1	1	4	1	0	4	1	3	2	1	0	18	4	4	0.7	0.2	0.2
Q9	9	1	0	4	0	1	5	3	0	3	0	0	21	4	1	0.8	0.2	0
Q10	5	5	0	4	1	0	6	1	1	3	0	0	18	7	1	0.7	0.3	0
Q11	9	1	0	5	0	0	7	0	1	3	0	0	24	1	1	0.9	0	0

that they found SAMAP manipulatives useful and entertaining. Some wanted more challenging activities with time restrictions. Furthermore, manipulatives related to numbers and geometry areas were used more frequently by students in Group 1 and Group 2 than measurement and data analysis since they are more likely to include arithmetic operations. Students preferred to employ manipulatives involving certain procedural skills and mathematical calculations such as the number pyramid manipulative in Fig. 7.6.

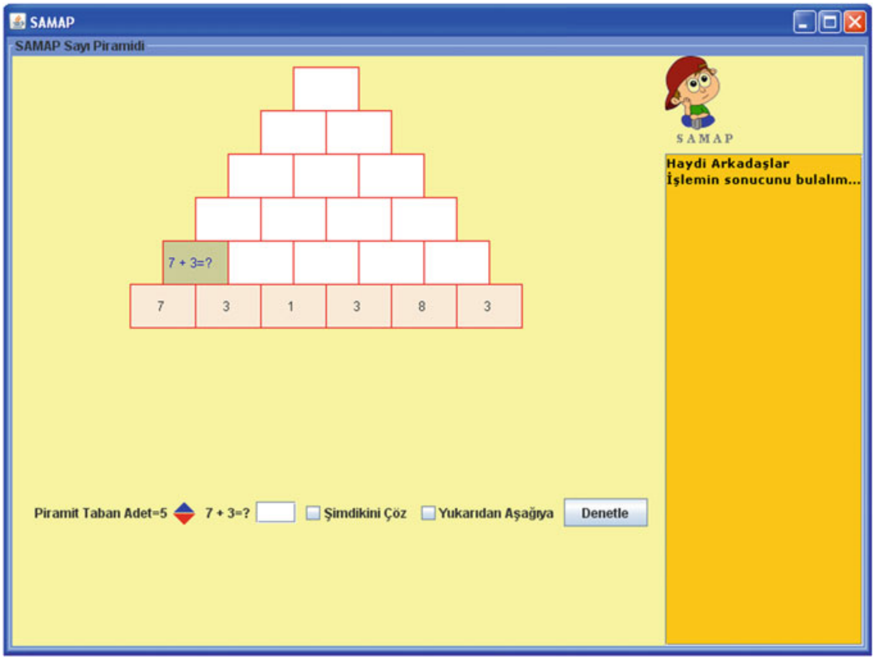


Fig. 7.6 The number pyramid manipulative of SAMAP

7.4.4 Group 3

Group 3 included secondary school students. Sixteen students completed 20 questionnaires for 23 manipulatives. There were positive and negative opinions in this group about the manipulatives. Some stated that manipulatives were not interesting enough for them to use since they found the activities simple. Others stated that activities were nice and useful. The analysis of the questionnaire items revealed that students wanted to use the activities individually but did not have much desire to re-use the activities.

7.4.5 Groups 4–7

Groups 4–7 included undergraduate and graduate students. The highest numbers of questionnaires were completed by these groups. One-hundred forty-four students completed 655 questionnaires for 205 manipulatives. These groups completed similar questionnaires, except first and second year undergraduate students were asked fewer questions on educational aspects. Participants in these groups complained about the number of questions in the questionnaires and were careless about completing test items since it was mainly introduced in a course. However, many useful comments were gathered from these groups especially about errors found in the manipulatives and suggestions for modifications for better interactivity. Furthermore, most of the participants stated that they liked the manipulatives and were optimistic about their possible usage in classes to improve mathematics education.

7.4.6 Group 8

Group 8 included primary school teachers. A comprehensive questionnaire, consisting of 51 five-level Likert items, was used with the range of options from “strongly agree” to “strongly disagree.” Teachers were asked detailed questions about the technical and educational aspects of the manipulatives. Forty-five teachers completed 102 questionnaires for 55 manipulatives. Analysis of the questionnaires revealed that most teachers thought that students would like the manipulatives but they would be unable to use them on their own. Teachers found the manipulatives in-line with the curriculum with specific learning objectives but they were generally pessimistic about their possible contributions to students’ mathematics achievement.

Teachers described manipulatives working as desired, but found some manipulatives were inflexible and did not enable them to construct their own problems, objects or movements. For instance, they wanted to change the analog clock by dragging the minute hand using the mouse rather than using the keyboard or the digital clock. One teacher commented that she used the manipulatives in the class and students liked the manipulatives. In fact, many teachers sent emails to the author to express their positive feelings about the manipulatives during the evaluation process. Some teachers demanded more explanations and instructions to be provided to direct students and teachers on how to use the manipulatives.

7.4.7 Group 9

Group 9 included academicians working actively in universities. Twelve academicians completed 42 questionnaires for 30 manipulatives. Academicians also found the manipulatives promising and useful. Some stated that manipulatives could be used as complementary materials in the class and may help students to reflect on mathematics concepts. Some demanded more feedback be provided and suggested improvements in manipulative design.

7.4.8 Overall

In summary, the results of the online study reveal that participants liked and enjoyed SAMAP manipulatives. Manipulatives were generally regarded as having potential to be useful for teaching mathematics concepts. The level of interactivity provided in SAMAP manipulatives was generally found satisfactory, despite the many suggestions for modifications that were put forward. Furthermore, many schools utilized SAMAP manipulatives in classroom settings for teaching mathematics all over Turkey.

7.5 Analysis of Pre-service Teachers' Reports of SAMAP Manipulatives

It is not easy to evaluate the effectiveness of manipulatives through only questionnaires or statistical means since ICT applications may not always lead to significant changes as measured by classical evaluation methods. Hence, SAMAP manipulatives were incorporated into educational technology courses in several Turkish Universities between 2009 and 2014, including Abant İzzet Baysal and Selcuk Universities. Pre-service primary school and mathematics teachers and graduate students were required to submit detailed reports on the technical and educational aspects of SAMAP manipulatives.

SAMAP manipulatives were introduced to the senior pre-service primary school mathematics teachers and classroom teachers within a mathematics methods course that met three hours per week. Some of the manipulatives were implemented in a classroom setting and pre-service teachers were required to use and investigate the manipulatives for two weeks and prepare a report on their effectiveness and state their opinions for improvement. More than a thousand pre-service teachers' reports were collected in five years in two universities containing pre-service teachers' own words and reflections about many SAMAP manipulatives. Many encouraging comments and feedback were received from these reports. These comments were taken into account to revise some of the manipulatives. Most of the participants had optimistic opinions about the effectiveness of the manipulatives. Major issues highlighted in the reports about virtual manipulatives could be summarized as follows:

SAMAP manipulatives were seen as entertaining, easy to learn and useful activities by most of the pre-service teachers. Many reports talk about the motivational aspect of many manipulatives by referring their entertaining nature. The manipulatives were thought to be attractive for children and they were thought to be appropriate to be implemented in class. Mathematics was said to be more enjoyable through these manipulatives. They were also generally found to be appropriate for the Turkish curriculum and class levels. Furthermore, manipulatives were said to have the potential to change students' perception of mathematics after seeing mathematical models of some real life problems.

SAMAP manipulatives are successful in modeling mathematical concepts for primary school, which may contribute to the persistence of learning mathematics. For instance, the *function machine* manipulative in which users were required to guess a function definition is said to make the function concept very concrete. The input and output areas of the manipulative that models a function were thought to be very explanatory.

Likewise, the *counting scales* manipulative is said to be useful in making addition and subtraction operations concrete. It was found very helpful to teach the properties of positive and negative numbers by showing different colored scales and requiring users to hide scales until they have scales of just one color. A scale was hidden by dragging a different colored scale over it. Users quickly get the idea that one cannot hide a positive or a negative scale by another positive or negative scale respectively.

Virtual manipulatives have been viewed as helping to develop students' thinking skills, making connections among mathematical concepts and improving their estimation skills. For instance, it was stated that the *fill and pour* manipulative helped students to make connections between different numbers and improved their estimation skills in an entertaining way.

Virtual manipulatives may contribute students' problem-solving skills. For instance, several measurement activities involving real-life situations were found useful to teach practical problem solving skills and helpful for students to realize how to apply mathematical concepts to real life.

SAMAP manipulatives were generally found to be very useful. For instance, the "*tossing coins*" manipulative was seen as useful in exemplifying the probability concept despite its simple design and functionality. It was especially regarded helpful for showing the difference between theoretical and empirical probability. It helps users to discover that this difference gets smaller when the number of tossed coins increases.

SAMAP manipulatives reinforce what was learned in class. For instance, the *number decipher* manipulative was found very useful in consolidating arithmetic operations on natural numbers. Students need to make use of higher-order thinking skills in this activity and take into consideration the next operation to apply. This kind of manipulative involving a puzzle may encourage students to enjoy solving novel problems.

The manipulatives have a pedagogically appropriate design. For instance, the "*set manipulations*" manipulative covers all set operations in a manner from simple to difficult.

The manipulatives were seen as appropriate for individualized learning since the design takes student differences into account. One can set several parameters in the manipulatives in accordance with his/her speed of learning. This encourages students' active participation and use of technology in the class. For instance, the *taking symmetry* manipulative exemplifies the symmetries of figures or lines and provides a good pretext for using technology in mathematics classes.

The manipulatives provide valuable feedback and have potential to remove students' misconceptions. For instance, it was reported that some students make mistakes in multiplication procedures even at the secondary school level. Virtual manipulatives are useful in correcting students' misconceptions.

The dynamic nature and interactivity of SAMAP manipulatives were appreciated by most pre-service teachers. For instance, the *hit the target* and *finding symmetries* manipulatives were found very interactive and visually demonstrated the angle and symmetry concepts respectively.

Some SAMAP manipulatives were found very ordinary and not interesting enough to motivate students (e.g., *factor tree).* It was criticized for employing big numbers in the activity for 6th grade and for the provision of all factors when one is entered by the user. Similarly, the *finding balance* manipulative was found simple and inflexible since it contains a limited number of weights.

It was concluded from the reports that manipulatives, including animations and multiple representations to illustrate the properties of certain mathematical concepts, were more popular and seen as educationally effective by the participants. It was interesting to note that many preferred manipulatives to be used on an individual basis rather than collaboratively. This reflects the perception of the participants about the place of technology in classes. Many aim to employ manipulatives to concretize some mathematical concepts, which may be difficult to achieve by physical manipulatives or other means. Manipulatives requiring teachers' guidance and whose aims and functionalities are not so apparent though operational instructions were given, such as the *Arithmetic mean* manipulative in Fig. 7.5, are found to be troublesome by most participants. This may reflect teachers' expectations to find ready-made activities and their unwillingness to invest time and energy to prepare their own activities through flexible manipulatives and problem-solving tools.

7.6 Discussion

A recent meta-analysis on virtual manipulatives reported that virtual manipulatives have produced overall moderate effects in favor of the virtual manipulatives when compared with other instructional treatments (Moyer-Packenham et al. 2014). Moyer-Packenham and Suh (2012) reported that virtual manipulatives have significantly different effects on different achievement groups because of students' different types of experiences with the virtual manipulative. All virtual manipulatives cannot provide similar learning performance and efficiency and children in different age groups might respond in different ways to virtual manipulatives (Moyer-Packenham et al. 2015). Virtual manipulatives in different mathematical strands also produce varying results. The quality of interactions provided or the number of different representations employed in these manipulatives greatly affect these results. For instance, virtual manipulatives prepared for the geometry strand are thought to produce the biggest effect for conceptual understanding since the enigmatic nature of geometry allows easy investigation of the concepts through more representations.

It is also well-documented that adequately designed virtual manipulatives could be used interchangeably with physical ones (Burns and Hamm 2011; Ozgün-Koca and Edwards 2011) as in our earlier prediction (Durmuş and Karakırık 2006) and even support further affordances. These affordances might include bringing out emotional connections to learning and self-regulatory behaviors (McLeod et al. 2012) and highlighting the visual and kinesthetic senses (Namukasa et al. 2009) and mathematical discourse (Anderson-Pence and Moyer-Packenham 2015). Özgün-Koca et al. (2013) advocates implementing activities that make use of both physical and virtual manipulatives. Using virtual manipulatives has also been reported to minimize the impact of extraneous demographic variables on learning (Moyer-Packenham et al. 2013). The focus of the recent studies that compare the effectiveness of physical and virtual manipulatives seems to shift from comparing students' performances to evaluating manipulatives from the perspective of the students (McLeod et al. 2012) by taking into account the quality of interaction provided to improve conceptual knowledge. Virtual manipulatives are far more effective in demonstrating and teaching conceptual knowledge (Suh and Moyer 2007), which is difficult to measure by classical standardized tests that compare students' performances. They are also used as an effective practice to teach mathematical concepts to students with learning disabilities (Satsangi and Bouck, 2015).

The aim of the SAMAP manipulatives was to promote higher order thinking skills by providing an environment for investigating mathematical concepts and relations rather than focusing on simple calculations and mathematical operations. However, many teachers and students were uncertain how to make use of the manipulatives since they were used to repetitive calculations instead of investigating mathematical concepts. Furthermore, many teachers only make use of the SAMAP manipulatives as extracurricular activities since they lack the experience to structure a mathematics lesson with virtual manipulatives (Reimer and Moyer 2005). The author argues that SAMAP manipulatives could be used to promote discussions, to increase students' participation and enhance their conceptual understanding in mathematics classrooms (Karakırık 2011). It was reported that linked virtual manipulatives enhance students' collaboration and mathematical discourse (Anderson-Pence and Moyer-Packenham 2015). As Wegerif and Dawes (2004) rightly points out a dialogical approach is required rather than dialectic one and dialogue among students and between students and the teacher is important in creating learning environments that promote thinking and productive learning.

SAMAP manipulatives were designed both for individual use and collaborative use in the classroom with the guidance of teachers. Most of the SAMAP manipulatives are related to the embodiment or concretization of some mathematical concepts and the creation of meaningful problem-solving scenarios, and avoid activities focused solely on procedural skills (Kaput 1992). A flexible design that enabled the most user interaction through expert systems was preferred. SAMAP manipulatives attach great importance to student interaction and the active participation of students are taken into account in designing manipulatives by adopting a constructivist approach. Repetitive tasks or animations are mostly avoided and open-ended tasks are preferred. A virtual manipulative may provide direct or indirect feedback either in structurally didactic manner or without providing guidance for any solution path (Anderson-Pence and Moyer-Packenham 2015). Implementing classroom activities highlighting the mathematical concepts is argued to be the most outstanding characteristics of a constructivist classroom. In this respect, manipulatives could be regarded as effective and successful to the extent they foster reflective and deep thought among students and teachers and introduce them new mathematical knowledge. This could be regarded as guided discovery activities that both involve discovery and guidance as suggested by Kuhn (2007).

Two main types of manipulatives emerged in this project for the purpose of demonstrating mathematical concepts of an abstract nature which implements both ‘learning with models’ and ‘learning to model’ approaches (Durmuş and Karakırık 2006). The first type illustrates different aspects of a mathematical concept through multiple representations. These manipulatives include more exploration activities and help students to see and investigate the mathematical relations and concepts. The second type provides problem-solving or modeling tools by which students were able to express themselves and devise their own models or problem solutions. These manipulatives included meaningful problem situations where students can internalize the mathematical concepts and problem solving strategies. Each SAMAP manipulative is designed by taking both types into consideration and having both demonstration and interaction modes where applicable.

Different representations of a concept have different inferential powers (Cox and Brna 1995) and may highlight different computational properties of a concept (Larkin and Simon 1987). The design parameters, unique functions supporting learning and the cognitive tasks to interact with multiple representations, needs to be taken into consideration to improve the effectiveness of multiple representations (Ainsworth 2006). The SAMAP manipulatives extensively use multiple representations of mathematics concepts such as graphical, algebraic, and written representations and allow users to change the representations dynamically. It is argued that virtual manipulatives are far more flexible and helpful for conceptual learning since they help students to visualize different interpretations and properties of a concept through dynamic representations.

7.7 Conclusion

SAMAP manipulatives covering objectives in the Turkish mathematics curriculum have been widely used by Turkish primary school teachers and students. In light of feedback from SAMAP users, it could be argued that many students are satisfied with the resources provided by the manipulatives. Therefore, the author argues that SAMAP manipulatives offer a favorable environment for discussion of mathematical concepts. Students accustomed to procedural operations in mathematics classes do not seem to fully understand the purpose of the manipulative at first, but can benefit from them over time with proper guidance of teachers. This emphasizes the leading roles of teachers in conceptual activities where it is not easy to see how to apply certain mathematical concepts in a problem situation. Considering the failure of early optimistic artificial intelligence studies that aimed to create a virtual teacher, no manipulative is expected to be smart enough to create a discussion platform for students. It is believed that manipulatives having special problem-solving expertise in well-defined scenarios should be developed to encourage students to investigate certain mathematical concepts: SAMAP activities aim to serve this purpose.

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Chapter 8
Enhancing Mathematical Skills Through Interventions with Virtual Manipulatives

Annemie Desoete, Magda Praet, Claire Van de Velde, Brigitte De Craene and Edwin Hantson

Abstract In this chapter, we report the findings from a randomized controlled trial investigating the effect of using virtual manipulatives to improve preschool students' early mathematics skills. One hundred thirty-two preschool children were randomly assigned to nine sessions of adaptive computerized counting or comparison with virtual manipulatives, or to a typical instruction control group. Children in both experimental intervention groups, including children with poor calculation skills at the start of the intervention, performed better than controls not using virtual manipulatives on early mathematics tasks at the posttest. In addition, the effects of the training held six months later with the counting intervention improving number knowledge and mental arithmetic performance, and the comparison intervention only enhancing the number knowledge proficiency in Grade 1. The effect of virtual manipulatives was present in empathic children, thinkers, persisters, dreamers, rebels and promoters according to the Kahler Process Communication Model. In addition children in both experimental groups became more adventurous after the training. We discuss the value of these short interventions with virtual manipulatives in preschool as a forward-looking approach to enhance arithmetic proficiency.

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8.1 Introduction

Children begin formal education with a very positive view of mathematics and with good feelings about their own abilities. However both interest and motivation decline as children grow older and the positive affect and 'good feelings' fade away (Moore et al. 2015). Given the high social and individual cost associated with poorly developed mathematical skills, it is essential to tackle this underperformance by gaining insight into the processes of decline leading to a suboptimal mathematical development and by trying to enhance motivation and young children's numeracy.

Early numerical foundations have been receiving ongoing attention, because researchers hope that by pinpointing the core deficits at an early age, the problems might be reduced or even solved (Aunio et al. 2005; Dowker 2005, 2015; Van Luit and Schopman 2000; Wilson and Räsänen 2008). Interventions could prevent children from falling further behind or close the achievement gap between low-performing children and their age-related peers. Longitudinal studies have shown that early numeracy skills are accurate predictors of later mathematics achievement (Jordan et al. 2012; Stock et al. 2010).

There are arguments for the claim that number comparison and counting skills can be considered as cognitive foundations or early numeracy skills associated with later proficient mathematics skills (Desoete 2014, 2015; Gallistel and Gellman 1992; Geary 2011a, b; LeFevre et al. 2006).

Moreover, a transactional analysis and closer look at the communication (Bradley and Pauley 2001; Kahler 2004) between teachers and children seems indicated to broaden the picture and include the motivational foundations of young children learning mathematics. Kahler (2008) described a differentiation among six types of children with different 'drivers' or habits to deal with challenges. All of these children seem to have different underlying motivations. The Kahler-model can be visualized on three axes (see Fig. 8.1).

'Empathic children' have a strong 'please'-driver, with pleasantness, compliance to others' wishes and generosity as assets. These children (30 % of the population) are attentive to others and sensitive. They are motivated by a well-willing management style, work in groups with a lot of sensory stimulation and appreciate getting recognized and acknowledged as a person.

'Thinkers' and 'persisters' have a 'be perfect'-driver with wisdom, purposefulness and high standards as assets. Thinkers (25 % of the population) are children that are responsible, logical and organized, performing best in a democratic management style where they can work alone. They are motivated by recognition of their work and time structure. Persisters (10 % of the population) are devoted, good

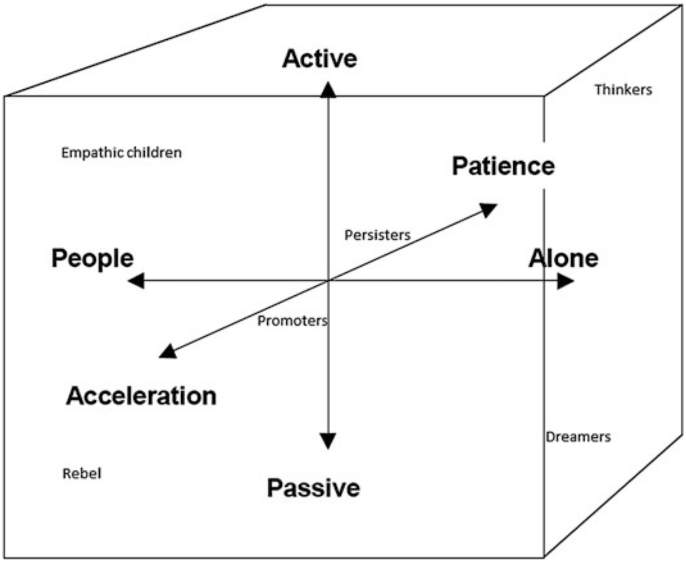

Fig. 8.1 A visualization of the axes in the Kahler model (inspired by Heasman 2000)

observers and conscientious children. These children are motivated by a democratic management style where they can work alone with recognition of work done and respect for opinions.

'Dreamers' and 'promoters' have a 'be strong'-driver with self-sufficiency, consideration of others' needs, reliability and resilience as assets. Dreamers (10 % of the population) are imaginative, reflective and calm. These children are motivated within an autocratic management style respecting their need of solitude and inviting them to act after a period of reflection. Promoters (5 % of the population) are convincing, adaptive and capable to realize things. These children need strong sensations and actions and an autocratic management style to be motivated.

'Rebels' have a 'try hard'-driver and passionate commitment, sympathy for the underdog and persistence as assets. These children (20 % of the population) are spontaneous, creative and playful and enjoy the here and now. They are motivated by playful 'contact' with teachers with a 'laissez faire' management style inviting them to work in a group-to-group environment.

Finally the model has a 'hurry-up'-driver with as assets adventurousness, responsiveness and sensitivity to others' feelings. According to the Process Communication Model (PCM) model (Hantson et al. 2015), this driver can be present in all types of children (empathic children, thinkers, persisters, dreamers, promoters and rebels).

Several (non-computerized) types of instruction have been developed to enhance early numeracy or cognitive skills in young children (e.g. Bloete et al. 2006; Kroesbergen and Van Luit 2003; Wilson et al. 2006). These studies reveal that active instruction is effective in the enhancement of early numeracy in young children. In addition, Computer Assisted Interventions (CAI) have received growing interest and proved to be effective as arithmetic support (Butterworth and Laurillard 2010; Räsänen et al. 2009). However, most of these interventions focus on primary school children (Codding et al. 2009; Kroesbergen and Van Luit 2003; Räsänen et al. 2009; Slavin et al. 2009; Templeton et al. 2008; Wilson et al. 2006). Moreover, it remains unclear whether one should target children's counting or comparison skills as specific cognitive components of early numeracy. Moreover, although low performing children were found to benefit especially from long and intensive, supplemental instruction (Aunio et al. 2009; Dyson et al. 2011; Haseler 2008; Jordan et al. 2009, 2012), it remains unclear if they benefit from less intensive computerized interventions using virtual manipulatives. Finally, the role of motivation in the understanding of mathematical variation remains unclear.

In the present chapter, we report the findings of a randomized controlled trial with two groups using CAI with virtual manipulatives to improve preschool students' counting and quantity comparison skills and one control group. A pretest, posttest and delayed posttest design was used to investigate the growth of early numerical cognitive foundations. Dependent variables included an assessment of early calculations in preschool (posttest) and mental arithmetic and number knowledge as conventional measures of mathematics achievement in Grade 1 (delayed posttest). In addition we investigated if virtual manipulatives worked for all types of children, namely for low and average performers and for all types of children (empathic, thinkers, persisters, rebels, promoters and dreamers), thus taking cognitive and motivational variables of mathematical variation into account.

The general aim of the present study was fourfold. First, we investigated the impact of the three different teaching approaches (namely virtual manipulatives with counting content, virtual manipulatives with number comparison and a control group) on numeracy in young children. We expected positive outcomes, since early numeracy skills have been found to be trainable in other studies (e.g. Baker et al. 2002; Codding et al. 2009). However, we were interested if counting and number comparison strategy approaches with virutal manipulatives were capable of modifying preschoolers' early numeracy skills in all children (empathic children,-thinkers, persisters, dreamers, rebels and promoters.

Second, we used two groups—a counting and number comparison group—to explore to what extent those approaches differed and if one was more effective than the other with both groups using virtual manipulatives. The control group played on the computer without doing mathematics. We expected that both interventions would contribute to improving the children's arithmetic level in Grade 1, but we were especially interested in the differential effects of both CAIs and on whether comparison or counting was the most efficient to support children's learning of arithmetic . Both groups were hypothesized to be capable of improving the early numeracy (posttest in preschool) and the performance on conventional measures of mathematical achievement (delayed posttest in Grade 1) in young children.

Third, we investigated the potential of the interventions on high-risk kindergartners with below average performance on early numeracy measures. Especially the effect of a preschool intervention on children at risk to develop mathematical problems was studied and compared to the effect of peers with higher preparatory or early numerical skills in preschool. We expected the interventions to be effective on posttests and were interested in the delayed posttest results.

Fourth, we investigated if all children (empathic children, thinkers, persisters, dreamers, rebels and promoters) benefitted from virtual manipulatives in preschool. It might be that only some types of children benefitted from the experimental interventions in preschool using virtual manipulatives.

8.2 Method

8.2.1 Participants

Participants were 132 (53 % male) full-day kindergartners with a mean age of 68 months (*SD* = 4.01) from five schools in Belgium. Forty-six of these participants were low mathematical performers (<pc25), assessed on early numeracy with the TEDI-MATH (Grégoire et al. 2004).

The children had an average intelligence TIQ = 101.39 (*SD* = 12.73), VIQ = 102.9 (*SD* = 11.97), PIQ = 99.30 (*SD* = 11.68) on the WPPSI. Most parents had working and middle-class-socio-economic backgrounds. Dutch was the only language spoken at home.

8.2.2 Interventions

The interventions took place in nine individual computerized instruction sessions using virtual manipulatives in a separate classroom during five weeks, 25 min each session. Each session consisted of solving problems in accordance with the instructions given in the computer program. The number comparison and counting experimental groups using virtual manipulatives were compared with a control group just playing on the computer.

Four paraprofessionals were trained to teach in the two experimental groups (number comparison and counting intervention with virtual manipulatives) and to administer the pretest, posttest and delayed post-test measures. All paraprofessionals were skilled therapists with experience with children with mathematical learning problems. During the training, there was one teacher per classroom as well as a paraprofessional present. Initial paraprofessional training took place one month prior to the start of the interventions. Systematic ongoing supervision and training was provided during the interventions.

Number comparison. Each of the number comparison sessions involved a non-intensive, but individualized and adaptive computer assisted number comparison task. Children learned through virtual manipulatives to focus on quantity,

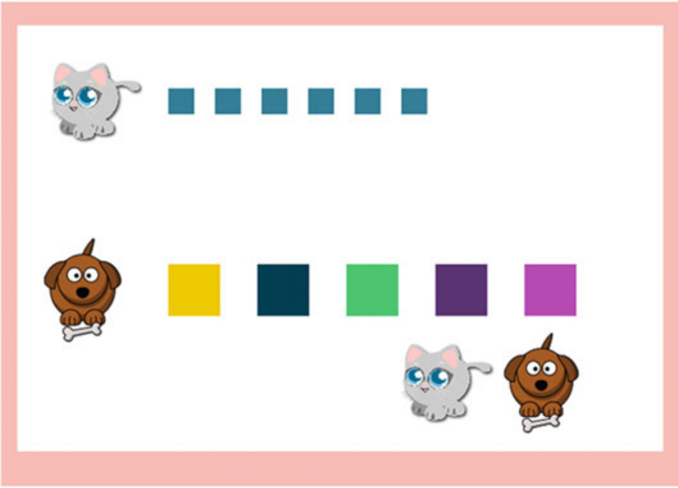

Fig. 8.2 Screenshot comparison of quantity groups

rather than size of objects. They learned to compare the number of animals, by pointing the mouse to the group of animals that had the greater number, making abstraction of the number of animals. In addition children had to compare two different kinds of stimuli (animals/dots—see Fig. 8.2 where the cat had the most squares).

There were exercises with organized and non-organized objects. Moreover, children learned to compare visual and auditory quantities and to compare quantities (squares) with number words or Arabic numerals and number words. The adaptive nature of the CAI program provided children with additional exercises on the questions and quantities that proved difficult for them.

Counting training. In the experimental computer assistant counting training, children worked on exercises based on procedural and conceptual counting knowledge using virtual manipulatives. They learned to count synchronously and learned to count without mistakes experiencing the cardinality principle. By clicking on a symbol, a quantity of that symbol with an upper bound of 6 was generated. The child was asked to count and register the amount by tapping the number on the keyboard. Auditory feedback was given. Children were shown an image of various objects and asked: "How many animals are there?" while on the screen there were objects, plants and animals (see Fig. 8.3). A second question followed: "How many of these animals can bark?" The instruction was read aloud and an answer was expected from the child by tapping the number of stars at the bottom of the screen.

Visual feedback was given by a happy-face or a sad-face icon. Auditory feedback was given by a sob when they made a mistake or an applause when they succeeded. There were exercises with the emphasis on adding, subtracting, leaving only a certain quantity. All children started basically at the same level. As CAI has

Fig. 8.3 Screenshot of a counting group

an adaptive structure additional exercises were provided for children who experienced difficulties.

Children in the control group received the same amount of instructional time, as did the children in the two experimental groups. However, instead of counting or comparison instruction, the control group received nine sessions in regular preschool activities (intervention as usual and had the opportunity to do some ordinary language games on the computer). An example of a non-math game would be that children had to sort sounds.

8.2.3 Procedures

Parents received a letter with the explanation of the research and submitted informed consent in order to allow their children to participate. All children were assessed individually , outside the classroom setting. Within each school and preschool class, children were randomly assigned to participate in the study in the virtual manipulatives counting group, the virtual manipulatives number comparison group, or a business-as-usual control group. Preliminary comparisons revealed that the children in the three groups did not differ significantly on pretest measures. In addition, teachers were blinded to the research questions of this study.

8.2.4 Measures

The study involved three waves of data collection: pretest, posttest and delayed posttest. The pretest took place while the children were in preschool before the children were randomly assigned to one of the three groups. The posttest took place just after the training. The delayed posttest took place in November when the children were enrolled in Grade 1. Children enter first grade at the age of seven in Belgium.

Pretest and posttest mathematics measures (assessed in preschool). Children's early numeracy achievement was measured (age 5–6) using the counting task, comparison task, Piagetian logical thinking task and the calculation task of the TEDI-MATH (Grégoire et al. 2004). In addition their intelligence was assessed. Children met one-on-one with the experimenter for six sessions of about three hours in total for these measures.

Procedural knowledge of counting was assessed with the TEDI-MATH using accuracy in counting numbers, counting forward to an upper bound (e.g., 'count up to 6'), counting forward from a lower bound (e.g., 'count from 3'), counting forward with an upper and lower bound (e.g., 'count from 5 up to 9'). One point was given for a correct answer. The internal consistency of this task was strong (Cronbach's alpha = 0.73).

Conceptual knowledge of counting was assessed with the TEDI-MATH using judgments about the validity of counting procedures. Children had to judge the counting of linear and random patterns of drawings and counters. To assess the abstraction principle, children had to count different kinds of objects that were presented in a heap. Furthermore, a child who counted a set of objects was asked, 'How many objects are there in total?' or 'How many objects are there if you start counting with the leftmost object in the array?' When children have to count again to answer, this is considered to represent good procedural knowledge, but this shows a lack of understanding of counting principles so they earn no points. One point was given for a correct answer (e.g., 'You did not add objects so the number of objects has not changed'). The internal consistency of this task was strong (Cronbach's Alpha = 0.85).

In addition, the early arithmetic abilities subtest of the TEDI-MATH was used as a pretest and posttest measure. This subtest consisted of a series of simple arithmetic operations. The child was presented with six arithmetic operations on pictures (e.g., "Here you see two red balloons and three blue balloons. How many balloons are there together?"). The internal consistency of this subtest was strong (Cronbach's alpha = 0.84).

The TEDI-MATH has been used and tested for conceptual accuracy and clinical relevance in previous studies (e.g., Stock et al. 2010). The psychometric value has been demonstrated on a sample of 550 Dutch speaking Belgian children from the second year of preschool to the third grade of primary school.

Delayed posttest mathematics measures in Grade 1 (assessed in January). In order to have a comprehensive assessment of the mathematical abilities of children, all children completed the Kortrijk Arithmetic Test Revised (Kortrijkse Rekentest Revision, KRT-R, Baudonck et al. 2006). The Kortrijk Arithmetic Test Revision (Kortrijkse Rekentest Revision, KRT-R; Baudonck et al. 2006) is a standardized test on arithmetical achievement which requires that children solve 30 mental arithmetic (e.g., '16–12 = _') and 30 number knowledge tasks (e.g., '1 more than 3 is _'). The KRT-R is frequently used in Flemish education as a measure of arithmetic achievement (e.g., Desoete et al. 2004; Desoete and Grégoire 2007). The test results in a score for mental computation, number system knowledge and a total score. The row item scores were converted to percentile scores. The psychometric value of the KRT-R has been demonstrated on a sample of 3246 children. A validity coefficient (correlation with school results) and reliability coefficient (Cronbach's alpha) of 0.50 and 0.92 respectively were found for first grade.

Intelligence. Intelligence was assessed (as pretest) in preschool with the WIPPSI-NL (Hendriksen and Hurks 2009; Wechsler et al. 2002). Children completed the three core verbal tests (information, vocabulary and word reasoning) and the three performance tests (block patterns, matrix reasoning and concept drawing). We also took into account the item substitution as being a core-subtest.

Motivation. Motivation and personality type was assessed after the intervention (when all children were seen again for another study) with the 'driver' construct of Kahler. In his Process Communication Model (PCM), drivers are defined as scripts being repeated over and over under stress, having positive and negative aspects. With a cluster analysis, Kahler described five drivers typical for six types of persons (see Fig. 8.1). Empathic children were found to have a tense for a 'please me'-driver. The 'be strong'-driver could be identified in dreamers and promoters. The 'be perfect'-driver was especially present in persisters and thinkers. The 'try hard'-driver was observed in the behavior of rebels. The 'hurry up' (or racing) driver appeared to be present in all types of children.

In this study, we identified the motivation of children indirectly, based on the Kahler drivers of the PCM-model. A 25-item questionnaire with a Likert scale was adapted from the Kreyenberg (2003) study. Children had to answer questions like (I like to work with others; In stress I remain calm and think logically; I want to work precisely without making mistakes; I like to work on new tasks; I like to work fast) on a 1–5 Likert scale (1 = totally not agree, 5 = totally agree). The questionnaire was tested in previous studies in order to determine the usefulness for this age group and for the sensitivity in measuring individual differences. Analyses showed that children and observers/coders could handle the instrument well. In addition, children were asked why they answered that way and what they thought while performing the task. The given answers all referred to the driver-constructs in question. The validity and reliability of the PCM-model and the driver construct was demonstrated in several studies (Ampaw et al. 2013; Gilbert 1996, 1999; Journal of Process Communication 2013; McGuire et al. 1990). In addition, the psychometric value of the model has been demonstrated in Flanders and Belgium (Hantson et al. 2015), where this study took place.

8.3 Results

8.3.1 Preliminary Comparisons in Preschool

The three groups of children were matched on pretest skills in preschool. No significant differences were found (*F*(*2128*) = *0.05; p* = *0.949*) for preschool calculation skills assessed with the TEDI-MATH (Grégoire et al. 2004) before the intervention in preschool. Moreover the groups did not differ on the WPPSI-III (*F*(*2128*) = *0.73; p* = *0.484*). See Table 8.1 for a summary of the descriptive statistics.

8.4 Treatment Effects

In order to investigate the research hypotheses on the modifiability of early numeracy as well as on the value of counting versus number comparison instruction on the learning of mathematical skills, a posttest and a delayed posttest were included. Dependent measures were analyzed by a univariate analysis of variance (ANOVA) or multivariate analysis of variance (MANOVA) with experimental group (counting experimental group, number comparison experimental group, control group) as group. Each (M)ANOVA determined whether significance existed among the three groups, when compared on the dependent measure at pretesting, posttesting and delayed posttesting. In addition, posthoc tests were performed on the posttest and delayed posttest scores, using an appropriate posthoc procedure (Tukey if equal variance could be assumed from the Levene test and Tamhane if equal variance could not be assumed from the Levene test). In addition, the observed power and the effect sizes ($ή^2$) were calculated.

Table 8.1 Means and standard deviations of the pretest skills in preschool

	Control group *N* = 49	Counting *N* = 44	Number comparison *N* = 39	F(2, 129)=
Mean age	67.67 (*4.05*)	68.50 (*3.83*)	68.28 (*3.96*)	0.08
VIQ	102.21 (11.11)	102.50 (12.68)	103.67 (12.42)	0.45
PIQ	97.21 (12.73)	99.41 (10.10)	101.72 (11.79)	0.79
Procedural counting	6.31 (1.58)	6.30 (1.74)	6.49 (1.71)	0.21
Conceptual counting	9.98 (3.07)	9.75 (3.38)	10.41 (2.31)	2.42
Arithmetic	7.57 (5.07)	7.72 (5.49)	7.64 (4.94)	0.05

$^{*}p \leq 0.05$

8.4.1 Posttest in Preschool

Significant differences between both experimental groups using virtual manipulatives were found ($F(2128) = 21.86; p = 0.001,$ $ή^2 = 26$) on the calculation skills assessed with the TEDI-MATH after the intervention took place. Children in the counting group using virtual manipulatives did better than children in the number comparison intervention group using virtual manipulatives. Children in both experimental intervention groups had significantly higher calculation scores than the children in the control group who did not use virtual manipulatives in preschool.

8.4.2 Delayed Posttest in Grade 1

The MANOVA, with group (counting, comparison, control group) as the independent variable, and number knowledge and mental arithmetic assessed, with the KRT-R in Grade 1 as the dependent variable, was significant at the multivariate level ($F(4, 246) = 3.95$; $p = 0.004$; $\eta^2 = 0.06$). Significant differences were found among the groups for number knowledge ($F(2, 124) = 6.29$; $p = 0.003$, $\eta^2 = 0.09$) and mental arithmetic ($F(2, 124) = 6.04$; $p = 0.003$; $\eta^2 = 0.09$).

The Tukey posthoc analysis revealed that both experimental groups using virtual manipulatives had a better number knowledge compared to the control group. For mental arithmetic there was a significant difference between the experimental counting group using virtual manipulatives and the control group.

8.4.3 Motivation

Although there was no difference at the multivariate level ($F(10, 204) = 1.36;$ $p = 0.203,$ $ή^2 = 68$), on the univariate level children in the groups that used manipulatives differed from the control children ($F(2, 109) = 3.87; p = 0.024,$ $ή^2 = 07$) on the 'hurry up' or racing-driver. The control group ($M = 14.00$; $SD = 2.39$) had significantly lower hurry-up scores compared to the children that used counting related virtual manipulatives ($M = 15.50$; $SD = 3.62$) or the group that used comparison-related virtual manipulatives ($M = 15.97$; $SD = 3.62$), meaning means that children that used virtual manipulatives in preschool became more adventurous and responsive to challenges.

On the other drivers, the groups did not differ. Thus, there were no significant differences for the 'please me'-driver ($F(2, 109) = 0.47; p = 0.626,$ $ή^2 = 00$), the 'be strong'-driver ($F(2, 109) = 0.87; p = 0.424,$ $ή^2 = 0.02$), the 'be perfect'-driver ($F(2, 109) = 0.60; p = 0.548,$ $ή^2 = 0.01$), and the 'try hard'-driver ($F(2, 109) = 2.13; p = 0.124,$ $ή^2 = 0.04$). These results mean that there was no significant difference among the groups on purposefulness (be perfect driver), resilience (be strong driver), pleasantness (please driver) or persistence (try hard driver).

8.4.4 Effect on Low Versus Average Performing Children

A 2 × 3 MANOVA was conducted to investigate if there was a difference among low and average mathematics achievers in the three experimental groups. The MANOVA had performance group (low achievers, average achievers) and experimental group (control, counting, number comparison) as independent variables and posttest as the dependent variable. There was a significant main effect for experimental group ($F(2, 121) = 24.41$; $p < 0.001$; $\eta^2 = 0.29$) and performance group ($F(2, 121) = 26.45$; $p < 0.001$; $\eta^2 = 0.18$) but no significant experimental group x performance group interaction effect ($F(2, 121) = 0.71$; $p = 0.493$), meaning that both groups (low achievers and average achievers) benefitted from the intervention using virtual manipulatives.

The MANOVA on the delayed posttest, revealed a significant main effect for the experimental group ($F(4, 238) = 4.42$; $p = 0.002$; $\eta^2 = 0.07$) and for the performance group ($F(2, 119) = 11.69$; $p < 0.001$; $\eta^2 = 0.16$), and again, no significant experimental group x performance group interaction effect ($F(4, 238) = 1.23$; $p = 0.297$). See Table 8.2 for a summary of the descriptive statistics (*M* and *SD*) and posthoc analyses (abc) among the groups.

Table 8.2 reveals that early numeracy can be enhanced, even in low performers, by a short computerized (number comparison or counting) intervention using virtual manipulatives in preschool, with a sustained effect on arithmetic in Grade 1.

Table 8.2 Posttest and delayed posttest of low and average achievers

	Control Low *M* (*SD*)	Control AA *M* (*SD*)	Counting Low *M* (*SD*)	Counting AA *M* (*SD*)	Comparison Low *M* (*SD*)	Comparison AA *M* (*SD*)
Arithmetic	7.18(*c*) (*2.81*)	10.64 (*3.30*)	12.16(*a*) (*2.63*)	14.23 (*2.32*)	9.94(*b*) (*2.74*)	12.34 (*3.21*)
Number knowledge	17.00 (*b*) (*5.91*)	22.10 (*4.70*)	22.08(*a*) (*4.20*)	23.41 (*4.50*)	20.74(*a*) (*4.19*)	24.80 (*3.59*)
Mental arithmetic	15.23 (*b*) (*5.95*)	21.85 (*5.53*)	21.76(*a*) (*4.88*)	23.53 (*4.90*)	18.65 (*4.71*)	23.73 (*5.05*)

*$p \leq 0.05$; *abc* posthoc indexes at $p \leq 0.005$; *AA* average achievers

8.5 Discussion

Early arithmetic abilities have been found to be strong predictors for later school achievement (Jordan et al. 2009). In addition, studies have reported large individual differences among children even before the onset of formal education (e.g., Aunio et al. 2009). If signs for mathematics difficulties can be identified and if effective early intervention and adaptations can be set up for children having additional educational needs based on their achievement status, it might be possible to diminish later learning difficulties and prevent some children from falling further behind.

The central question behind this study was whether or not an intervention using virtual manipulatives in preschool could modify children's numeracy skills and facilitate instruction of arithmetic in Grade 1, as already found in older children (Räsänen et al. 2009; Wilson et al. 2006). Therefore, children in this study were randomly assigned to the experimental virtual manipulative number comparison, virtual manipulative experimental counting or control group. The intervention on number comparison (using images, number words and Arabic numbers) or counting (using number words and Arabic Numbers to count) took place at the end of preschool.

Both interventions using virtual manipulatives had a sustained effect on children's arithmetic in Grade 1 as indicated on the delayed posttest, six months after the training took place. Children in both experimental groups performed better than the control group (taking into account that the groups were matched on their pretest scores) for number knowledge. Moreover, the counting group had better mental arithmetic skills than the comparison and control group.

The use of virtual manipulatives seemed to work for all types of children, independent of their type of motivation needed. The experimental training had positive effects on students' learning regardless their type identified by the Kahler Model. All children (empathic, thinkers, persisters, dreamers, rebels and promotors) improved in mathematics learning using virtual manipulatives. Moreover the groups that used virtual manipulatives in preschool seemed to have a higher 'hurry up' driver, meaning that they set up higher standards on the speed and involvement in mathematics tasks compared to children in the control group. They were more adventurous than their peers that did not participate in the intervention with virtual manipulatives in preschool.

Especially interesting is that early numeracy could be enhanced in preschool children with additional educational needs based on their achievement status. The present data indicated that even a short (at the most 9 sessions of 25 min) period of playing adaptive educational computer games with virtual manipulatives in preschool enhanced mathematics learning in Grade 1, even in vulnerable children at-risk for mathematics difficulties. In line with Aunio et al. (2009) this is good news for siblings of children with learning disabilities having an enhanced risk to develop a disability themselves (Desoete et al. 2013; Shalev et al. 2001). Perhaps didactic methods, including educational counting games with virtual manipulatives as Universal Design for Learning (UDL), can prevent children at risk from falling behind, avoiding mathematics or even developing mathematics anxieties. In addition, virtual manipulatives seemed to identify strengths and weaknesses in all low performing children in preschool. Such educational computer games as supports in regular preschool classes can contribute to the realization of inclusive education in elementary schools.

These results should be interpreted with care, since there are some limitations to the present study. We only assessed a small group of low performing children in preschool. Research with larger groups of children at-risk for mathematics difficulties and disabilities is necessary. Moreover, context variables such as home and school environment should be included in order to obtain a complete overview of the development of these children.

Nevertheless, this study highlights that early intervention with virtual manipulatives can enhance students mathematics skills. In addition mathematics learning can be enhanced in children at-risk for mathematics difficulties by playing adaptive computer games supporting the counting skills in preschool. Finally all types of children benefitted from the training with this training making them set higher standards on the speed and involvement than peers who did not participate in this intervention. Thus, the findings demonstrate that even non-intensive and computerized adaptive interventions using virtual manipulatives in preschool can enhance children's early numeracy in young children with a delayed effect on arithmetic performances in Grade 1.

8.6 Conclusion

In this chapter, we described the results of a study conducted with 132 Dutch-speaking children from five preschools that served children from families with working and middle-class socio-economic backgrounds. Children were randomly assigned to adaptive computerized counting or comparison interventions using virtual manipulatives , or to a control group in preschool.

Children in both experimental groups using virtual manipulatives performed better than children in the control groups on the posttest. The effects of training held in Grade 1.Playing adaptive serious counting games improving number knowledge and mental arithmetic performances, and playing adaptive serious comparison games, enhanced children's number knowledge proficiency in Grade 1. Similar to the results of Ramani and Siegler (2008, 2011), the results of this research revealed that early numeracy can be stimulated by using virtual manipulatives in preschool, even in low-performers, with a sustained effect on arithmetic in Grade 1. Moreover motivation was taken into account revealing that virtual manipulatives enhanced the adventurousness, speed and involvement of the children to work on mathematical tasks.

This is good news for children at risk of developing mathematical learning difficulties. Playing educational counting games using virtual manipulatives (see

Wilson et al. 2006; Räsänen et al. 2009) might create a buffer against poor mathematical outcomes. An important finding from this study is that it is possible to use computer software and virtual manipulatives in an entertaining game-like format for providing learning experiences with an effect on later arithmetic proficiency.

The discovery of the key role of counting reminds us that, in particular, exposure to counting games seems applicable in preschool. There is value in these short periods of e-gaming using virtual manipulatives in preschool as a way to improve young children's numeracy, to enhance the speed and motivation or involvement in learning, and to promote the success in arithmetic proficiency in first grade and beyond.

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Chapter 9
Influence of Prior Knowledge and Teaching Approaches Integrating Non-routine Worked Examples and Virtual Manipulatives on the Performance and Attitude of Fifth-Graders in Learning Equivalent Fractions

Chun-Yi Lee and Ming-Jang Chen

Abstract The objective of this study was to investigate the influence of prior knowledge and non-routine worked examples, integrated with manipulatives, on the performance and attitude of fifth-graders learning of equivalent fractions. The participants were divided into three groups based on the teaching method to which they were exposed: continuous examples paired with physical manipulatives (continuous-physical group), continuous examples paired with virtual manipulatives (continuous-virtual group), and integrated examples paired with virtual manipulatives (integrated-virtual group). The results indicate the following: (1) The integrated-virtual group displayed better performance than the continuous-physical and continuous-virtual groups in basic and advanced flexible thinking, whereas the continuous-physical and continuous-virtual groups showed no significant differences; (2) The students with high prior knowledge in the continuous-virtual group presented higher scores in learning enjoyment and mathematics anxiety than those in the integrated-virtual and continuous-physical groups, and those in the continuous-virtual group also displayed greater learning motivation than those in the integrated-virtual group; and, (3) The students with low prior knowledge in the continuous-virtual and integrated-virtual groups presented higher scores in learning enjoyment than those in the continuous-physical group. However, the students with low prior knowledge in the three groups displayed no significant differences in learning motivation and mathematics anxiety.

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9.1 Research Background and Objectives

The concept of equivalent fractions plays a significant role in learning mathematics. Students must possess a basic knowledge of fractions in order to understand equivalent fractions, and they must master equivalent fractions before they can understand rational numbers. Equivalent fractions imply comparison of fractions and are fundamental for arithmetic involving fractions with different denominators. It is also one of the most difficult sub-concepts in fractions because it requires students to think flexibly and progress from concrete operations to formal operations to solve problems. Inadequate understanding of equivalent fractions is common among students (Kamii and Clark 1995; Yu and Leu 2002). This is an important issue because if students' understanding of fractions is flawed or their approach to them is too rigid, they will be unable to solve equivalent fraction problems correctly.

The majority of elementary school students are in the stage of concrete operations; that is, they require implementation to establish physical and pictorial representations. Physical manipulatives are often used in mathematics education to make abstract concepts and symbols more meaningful and understandable to students (Durmuş and Karakırık 2006). Developing the concept of equivalent fractions requires an object to be divided into multiple equal parts for interpretation. However, it is difficult to divide or combine concrete manipulatives freely. In recent years, Moyer et al. (2002) introduced the concept of virtual manipulatives, which are defined as dynamic object representations that assist students in formulating mathematical concepts. Such representations must be operable and accessible on the Internet. Virtual manipulatives allow students to divide objects to help them understand the concept of equivalent fractions. This enables students to visualize the concept with which they are engaged (Chang et al. 2013). Virtual manipulatives therefore have the potential to become an important tool for teaching equivalent fraction concepts. A worked example is a step-by-step demonstration of how to perform a task or how to solve a problem (Clark et al. 2006). Worked examples are another teaching aid with high potential. They have been successfully used to teach computer programming, algebra, and geometry (Carroll 1994; Paas and van Merrienboer 1994). Students that have access to worked examples exhibit better performance in problem-solving than those without this access (Chandler and Sweller 1991).

Two types of worked examples, continuous examples and integrated examples, are used in our experiment. In the continuous quantity scenario, continuous examples refer to colored blocks that are continuous and non-scattered, such that the learners can refer to the same fraction using different names through visualization and by ignoring dividing lines. In the discrete quantity scenario, continuous examples refer to the equivalent fraction problem in group mode, in which re-splitting or re-combination is conducted using a group of identical objects to demonstrate that two fractions are equal. In addition to providing learners with examples of continuous equivalent fractions, integrated examples can also provide non-routine examples that cannot be seen in traditional teaching materials. In the continuous quantity scenario, non-routine examples refer to colored blocks that are not continuous, in which the learner must first arrange discontinuous blocks into continuous blocks and refer to the same fraction using different names through visualization and by ignoring dividing lines. In the discrete quantity scenarios, non-routine examples refer to situations in which there are more than two kinds of objects in the equivalent fraction problem in group mode. The learner must first re-arrange the objects to determine an appropriate splitting approach in order to realize that the two fractions are equal.

Although both virtual manipulatives and worked examples have been shown to possess promise as teaching aids, there exists a notable lack of research on the effectiveness of these strategies, particularly on methods combining the two. Understanding the characteristics of this combination and how they are associated with learning is crucial, as it can help teachers select the most effective manipulatives and provide inspiration for the design of new manipulatives.

Prior knowledge has been shown to have considerable influence on how teachers and students interact with manipulatives (Lin and Huang 2013). For instance, Rittle-Johnson et al. (2009) found that students require sufficient prior knowledge for solving equations to be able to weigh the benefits of different solution methods. The issue of whether prior knowledge influences the effect of worked examples with virtual manipulatives when learning equivalent fractions has rarely been studied. The purpose of this study was, therefore, to investigate whether prior knowledge and different teaching methods, combining examples and manipulatives, interact with the learning performance and attitude of fifth-graders with regard to equivalent fractions.

9.2 Literature Review

9.2.1 Prior Knowledge

Prior knowledge is considered a vital element of learning effectiveness. Previous research has shown that prior knowledge impacts how teachers and students interact with the manipulatives that they use (Lin and Huang 2013; Rittle-Johnson et al. 2009). Kim and Rehder (2011) examined the influence of prior knowledge on selective attention during category learning. Using an eye tracker, they discovered that prior knowledge affects the features, which are seen and the features associated with prior knowledge are more frequently fixated. This outcome is not because of an initial bias in attention towards certain features, but rather a gradual result of category observation. After the explanation of learning criterion, this effect was even more prominent.

Van Loon et al. (2013) found that incorrect prior knowledge causes elementary school students to formulate erroneous concepts during learning and even be overconfident about them. Their results indicated that incorrect prior knowledge affects learning and calibration of younger students. Once their incorrect prior knowledge was activated, they expressed extremely inaccurate judgment in their retained learning responses. Moreover, they were overconfident in their judgment of quality in posttest retained responses. This overconfidence often discourages students from studying the target concepts further. Rittle-Johnson et al. (2009) discovered that prior knowledge is crucial in the use of comparison in learning. They had 236 seventh- and eighth-graders compare different solutions to the same problem, compare different types of problems that use the same method of solution, or sequentially examine examples. The results revealed that students with less prior knowledge displayed greater learning effectiveness in the comparison of different types of problems that used the same method of solution and in the sequential examination of examples. In contrast, students with greater prior knowledge benefited more from comparing different solutions to the same problem.

Prior knowledge has also been found to interact with the form of teaching, the form of manipulative representation, and the method of practice (Kalyuga 2007; Rittle-Johnson and Kmicikewycz 2008). Teaching procedures that are more effective on first-time learners may become ineffective on learners with more related knowledge. For example, beginners learn more effectively from worked examples than from proceeding directly to problem solving themselves. However, as they gain knowledge, direct self-organized problem solving becomes a more effective learning activity (Renkl and Atkinson 2003).

Cognitive load theory can be used to explain why more detailed instruction and guidance and bounded tasks generally effectively facilitate learning for beginners but not for more experienced learners (Sweller et al. 2011). For beginners, a task that requires the processing of numerous new elements of information at once can easily overload their working memory. In contrast, learners with some experience in the field can use the knowledge structures that they already possess to explain and complete the task, so their working memory is not overloaded. Nevertheless, few studies have examined the minute interactions between prior knowledge and manipulatives with worked examples with regard to learning performance and student attitudes.

Students with greater prior knowledge should be more capable of organizing and integrating unfamiliar virtual resources and more efficient in processing non-routine examples. Therefore, it seems reasonable that prior knowledge should play a crucial role in learning involving manipulatives with worked examples. This study therefore investigated the influence of prior knowledge and manipulatives with worked examples on the mathematics learning of elementary students.

9.2.2 Factors Influencing Students Learning Equivalent Fractions

A number of factors influence students in learning equivalent fractions: (1) their ability to think flexibly, (2) their combination ability, (3) their operative thinking ability, and (4) their unitization ability. Ineffective learning of equivalent fractions is caused by inflexible thinking in students (Peng and Leu 1998). We introduce these separately below.

In terms of graphical representations with continuous amounts, the ability to think flexibly means that the learner can refer to the same fraction in different ways and imagine or ignore dividing lines. The number of dividing lines and whether the divided blocks are connected often influence the learning outcome as well. Some students may insist that the denominator be the same as the number of divided blocks and that all of the blocks be connected before accepting that another fraction is of equivalent value (Behr et al. 1984; Behr and Post 1992). Booth (1987) interviewed eleven-year-old students on equivalent fractions and found that 95 % of the interviewees deemed that Picture A in Fig. 9.1 was $\frac{1}{3}$, whereas only 73 % of the interviewees deemed that Picture B was $\frac{1}{3}$. The 22 % point difference was a result of students believing that the $\frac{2}{6}$ depicted in Picture B was not equivalent to $\frac{1}{3}$. This shows that if learners can imagine or ignore dividing lines in graphical representations, they will be able to generate multiple labels for equivalent fractions. In graphical representations with discrete amounts, the ability to think flexibly means that the learner is able to re-divide or recombine discrete blocks to solve problems (Behr et al. 1984). Suppose that a student uses small circles to solve the problem $\frac{?}{3} = \frac{4}{6}$; the learner first needs to convert the representation, regarding two circles as a portion, as shown in Fig. 9.2. The six circles are thus grouped into three portions, and filled in black as necessary. This kind of representation allows the student to infer that $\frac{4}{6} = \frac{2}{3}$.

Combination ability indicates that a student can use a certain problem-solving strategy when solving an equivalent fraction problem or dividing the remainder of a problem. This strategy involves dividing unit amounts into several portions and recombining correctly processed portions into designated fractions with the unit amounts (Peng and Leu 1998; Yu and Leu 2002). Operative thinking refers to the

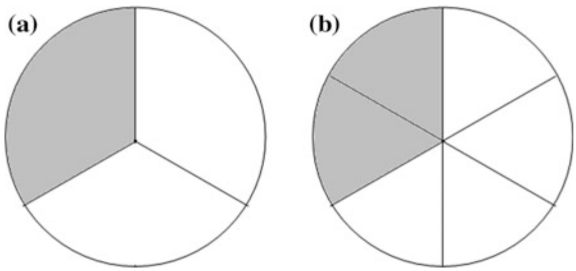
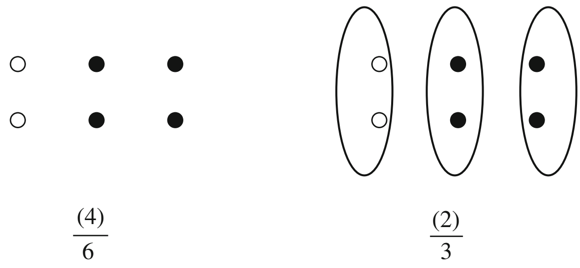


Fig. 9.1 Flexible thinking with equivalent fractions using continuous objects

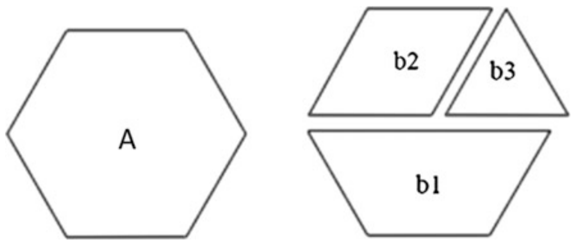
Fig. 9.2 Flexible thinking with equivalent fractions using discrete objects



ability to apply different methods of division to the same figure, which falls under the concept of area conservation (Kamii and Clark 1995). Knowledge contains a figurative aspect and an operative aspect. The figurative aspect is based on shapes, which are observable, while the operative aspect is based on association, which is not observable. For instance, there are a number of ways to divide a rectangle into two halves; the results may be rectangles or triangles. From the figurative aspect, the rectangles are different from the triangles in shape and thus do not look the same in area. However, from the operative aspect, knowing how the triangles and rectangles are associated with the original rectangle means that the learner can infer that the triangles and rectangles are the same in area, that is, half of the original rectangle, without being affected by the visual indications.

Another ability that is crucial to successful learning of equivalent fractions is the ability to find a suitable unit by which to divide a designated portion without remainder. This ability, which involves first identifying an appropriate unit by which to divide the whole and then using the unit to recombine the parts into the whole or a set, is called unitization ability (Saenz-Ludlow 1994, 1995). For example, Saenz-Ludlow (1994, 1995) showed the picture in Fig. 9.3 to third-graders. Of the three blocks in the picture, the students were first asked to compare block b1 with the whole (Picture A), and then compare blocks b2 and b3 with the whole. The three blocks can completely divide the whole; block b2, for example, equals one-third of Picture A. The students were also asked to compare blocks b1 and b2. At first, they could not determine the relationship between the two, but once block b3 was given as a hint, they quickly realized that block b2 equaled two thirds of block b1. The results showed that the students could adjust the smallest unit to meet the needs of the problem.

Fig. 9.3 Unitization ability for equivalent fractions



9.2.3 Current Manipulatives for Equivalent Fractions

The Principles and Standards for School Mathematics in the United States (NCTM 2000) suggest that students should develop rich number concepts from kindergarten to twelfth grade. In terms of fractions, students begin to learn concepts related to fractions in second grade. From third grade to fifth grade, they learn how to (1) use models, reference points, and equivalent forms to determine the magnitudes of fractions, (2) recognize and find equivalent forms of decimals and percentages, and (3) use visual models, reference points, and equivalent forms to add and subtract fractions and decimals. The curriculum from sixth grade to eighth grade emphasizes the flexible use of fractions to solve problems in addition to calculation abilities (NCTM 2000). The mathematics course principles in the Grade 1–9 Curriculum Guidelines of Taiwan stipulates that basic fraction concepts must be introduced between first and third grade, while equivalent fractions and the reduction, expansion, and four operations of fractions are introduced in fourth and fifth grade (Ministry of Education 2004).

The textbooks from various publishers in Taiwan do not mention the terms reduction and expansion directly with regard to equivalent fractions and do not explain that multiplying or dividing the numerator and denominator by the same number does not change the value of the fraction. Rather, the textbooks present examples where parts of the whole are re-grouped into smaller or larger units, and these newly-formed units are then compared to show that they are equivalent (Tsai 2003). In examples involving continuous objects, the textbooks often resort to strategies alluding to intuitive experience or divided shares. The former generally divides shapes such as blocks or circles into various portions, and then compares them in terms of length or area. Strategies that resort to divided shares directly indicate the number of divided shares and then use number lines to compare whether the two fractions are equivalent. In examples involving discrete objects, it is easier to directly count the contents of the group or groups. The former involves comparing the number of objects in the contents to determine whether the fractions are equivalent, and the latter involves viewing multiple objects as one entity to determine whether the numbers of objects are the same and then finding other names for the fractions (Tsai 2003).

Analysis of the textbook materials on equivalent fractions from the Kang Hsuan Educational Publishing Group revealed that the examples given required the ability to think flexibly about both continuous and discrete objects. Continuous objects are presented as closed shapes marked with dividing lines and discrete objects are presented as groups of identical objects. In mathematics learning, worked examples can help learners develop appropriate mental models to achieve their learning objectives and are a beneficial way of learning basic cognitive skills. Worked examples are a preferable learning model for beginners and an effective learning model for experienced learners (Atkinson et al. 2000). Provision of relevant worked examples is therefore a significant aspect of teaching for conceptual learning.

9.2.4 Virtual Manipulatives

The objective of incorporating information technology into teaching is to create a quality teaching environment that enhances teaching and learning effectiveness. It should be diverse and highly interactive while prompting students to actively explore problems, thereby contributing to the creation of a problem-solving environment (Lee and Chen 2008). Mathematics can sometimes be abstract and therefore difficult to understand or require actual operations to experience the concepts involved, and as a result, students can lose their motivation to learn. To increase motivation and enhance learning effectiveness, visualizing is necessary to give abstract content a concrete form. With conventional teaching methods, a substantial amount of time and manpower is often needed to achieve this, but information technology today can do so easily. Multimedia offers rich visual and sound effects, gives students more opportunities to do and learn, and enables them to make connections with prior learning experiences, thereby increasing student interest and motivation and contributing to better learning effectiveness (Lee and Chen 2009, 2014). Understanding equivalent fractions requires objects to be divided into equal parts. However, physical manipulatives cannot be divided into any given number of parts, and they cannot fully explain fractional number sense and unit concepts. Multimedia interactions on computers provide semi-physical manipulatives and visual access to concepts that are difficult to experience. Abstract concepts thus become concrete and visible and students have ample opportunity to use manipulatives and experience the mathematical concepts within. Such manipulatives effectively present learning content and help students to understand the target material (Lee and Chen 2014, 2015).

Virtual manipulatives are similar to physical manipulatives but possess dynamic interactive features and can be made widely accessible through placement on websites. The virtual representation of these dynamic objects provides students with unique opportunities to acquire mathematical knowledge (Moyer et al. 2002, 2005). Yuan and Lee (2012) listed the following as advantages of virtual manipulatives: (1) variability—learners can color portions of an object and increase or decrease the number of objects; (2) unlimited supply—the problems of insufficient physical manipulatives in classrooms as well as time-consuming distribution and organization are resolved; (3) conceptual connections—figures and symbols can be simultaneously displayed on the screen to enhance the connection among different representations.

A comprehensive review of empirical research focused on virtual manipulatives in mathematics classrooms (Moyer-Packenham and Westenskow 2013; Reimer and Moyer 2005; Steen et al. 2006; Suh et al. 2005) revealed the following features of virtual manipulatives: (1) they provide learners with the opportunity to achieve self-discovery as virtual manipulatives can improve learners' visual and conceptual abilities, (2) they encourage students to explore mathematical relationships as the dynamic opportunities for interaction enable students to focus on the task at hand, (3) they link figures to symbols as these can be displayed simultaneously, (4) they prevent students from making typical mistakes when they learn addition with fractions, (5) they give students immediate feedback, (6) they offer easier and quicker methods of manipulation than do pen and paper, and (7) they are accessible to learners over wide geographical areas and socioeconomic divides.

Moyer-Packenham and Westenskow (2013) conducted a meta-analysis that synthesizes the findings from 66 research reports examining the effects of virtual manipulatives. The results of the averaged effect size scores yielded a moderate effect for virtual manipulatives compared with other instructional treatments. The results of the conceptual analysis revealed empirical evidence that five specific interrelated affordances of virtual manipulatives promoted mathematics learning. These five specific affordances included: (1) virtual manipulatives focus and constrain student attention on mathematical objects and processes; (2) virtual manipulatives encourage creativity and increase the variety of students' actions; (3) virtual manipulatives simultaneously link representations with each other and with students' actions; (4) virtual manipulatives contain precise representations allowing accurate and efficient use; and (5) virtual manipulatives motivate students to persist at mathematical tasks.

9.3 Methodology

This study adopted a quasi-experimental approach. The independent variables included prior knowledge and teaching approaches combining examples and manipulatives. When we use the term prior knowledge, we mean the basic knowledge associated with fractions that the students possessed before learning equivalent fractions. We first administered a prior knowledge test and ranked the students based on their scores. The top and bottom 50 % of the students were designated as the high and low prior knowledge groups. Based on the worked examples and manipulatives used, the teaching approaches were divided into three types: continuous examples paired with physical manipulatives (continuous-physical group), continuous examples paired with virtual manipulatives (continuous-virtual group), and integrated examples paired with virtual manipulatives (integrated-virtual group). The dependent variables were test scores and mathematics learning attitude. The test scores assessed the learning performance of the students with regard to equivalent fractions after the teaching experiment, including (1) basic flexible thinking, which encompasses basic drawing and segmentation abilities, and (2) advanced flexible thinking, which comprises advanced drawing, combination, operative thinking, and unitization abilities. Mathematics learning attitude refers to the opinions of the learners toward learning mathematics after the teaching experiment, including learning enjoyment, learning motivation, and mathematics anxiety.

The participants of this study were fifth-graders at an elementary school in Taipei City. Before the teaching experiment, the participants were taught the basic concepts of equipartitioning, simple fractions, and unit amounts, which constitute the prior knowledge of equivalent fractions, but had not yet been taught. In coordination with their original class schedules, we randomly selected six classes out of the 13 fifth-grade classes at the school. Containing roughly 180 students, the six classes were randomly designated as the continuous-physical group, continuous-virtual group, and integrated-virtual group. All of the students took the prior knowledge test and were then categorized into the high and low prior knowledge groups based on their scores.

The research instruments of this study included the prior knowledge test, the equivalent fraction manipulatives, the equivalent fraction achievement test, and the mathematics learning attitude questionnaire, the details of which are as follows.

9.3.1 Prior Knowledge Test

The contents of the prior knowledge test included the concepts of equipartition, simple fractions, unit amounts, and equivalent fractions, which are all vital aspects of the equivalent fraction curriculum. We used the scores of this test to group the participants. The contents of the prior knowledge test were prepared by two professors and three experienced elementary school mathematics teachers, so the test has expert validity. In terms of reliability, the internal consistency of the question items was measured using Cronbach's α. This measure returned the following values: 0.75 for equipartition, 0.78 for simple fractions, 0.75 for unit amounts, 0.83 for equivalent fractions, and 0.93 overall. Thus, the reliability is acceptable. The difficulty values of the problems ranged from 0.53 to 0.84, the degree of item discrimination ranged from 0.32 to 0.85, and the item discrimination coefficients all reached the level of significance. Thus, the prior knowledge test displays appropriate levels of difficulty and item discrimination.

9.3.2 Equivalent Fraction Manipulatives

One primary objective of this study was to determine the influence of different teaching approaches on fifth-graders learning of equivalent fraction concepts. Based on the teaching approach, we divided the participants into a continuous-physical group, a continuous-virtual group, and an integrated-virtual group. The difference between the continuous-physical and continuous-virtual groups was the type of manipulative used; the continuous-physical group learned with physical manipulatives, whereas the continuous-virtual group used virtual manipulatives. The worked examples and concept explanations for the two groups were the same. The worked examples were divided into continuous examples and integrated examples. The former presents problems with continuous equivalent fractions, whereas integrated examples included problems with non-continuous equivalent fractions in addition to problems with continuous equivalent fractions. Based on the type of manipulatives used, the teaching approaches were divided into physical teaching and virtual teaching. In the former, the teacher used physical manipulatives when teaching equivalent fractions and gave students opportunities to use the manipulatives to verify their understanding of relevant mathematical concepts. In contrast, the teacher used virtual manipulatives in virtual teaching, also giving students opportunities to use the virtual manipulatives. The virtual manipulative used in this study was the Magic Board developed by Professor Yuan of Chung Yuan Christian University (Yuan and Lee 2012). The Magic Board retains the functions of physical manipulatives while surpassing the limits of physical manipulatives (Chang et al. 2013). At present, this tool has been set up on the Internet and offers free access (http://magicboard.cycu.edu.tw/). Below, we explain the difference between the continuous-virtual group and the integrated-virtual group.

9.3.2.1 Continuous-Virtual Group

In continuous examples, the colored blocks are connected. The learners must be able to recognize a fraction by different names and be able to imagine or ignore division lines. The continuous objects in these equivalent fraction problems included length models and area models. Figure 9.4 displays an example of continuous equivalent fractions using the length model, showing the fractions equivalent to one half. Learners can use the fraction bar to find appropriate ways of dividing the blocks and identifying the portions that are equal to one half. Coloring can only be achieved by dragging, emphasizing the continuous nature of the

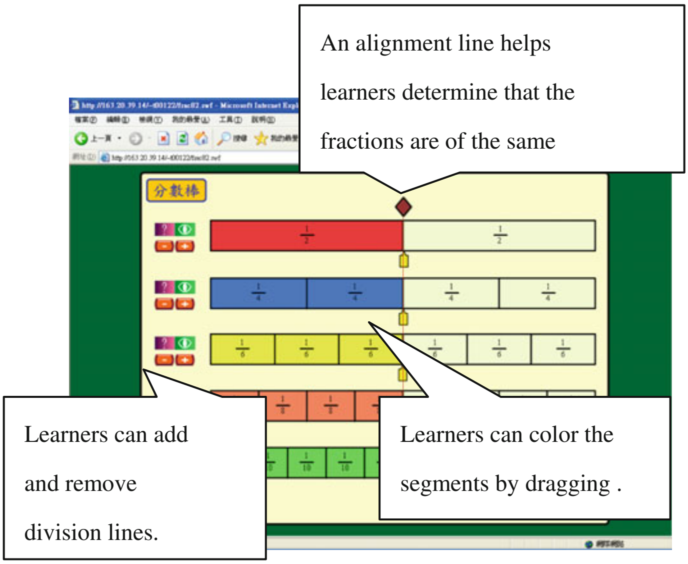


Fig. 9.4 Example of continuous equivalent fractions (using length)

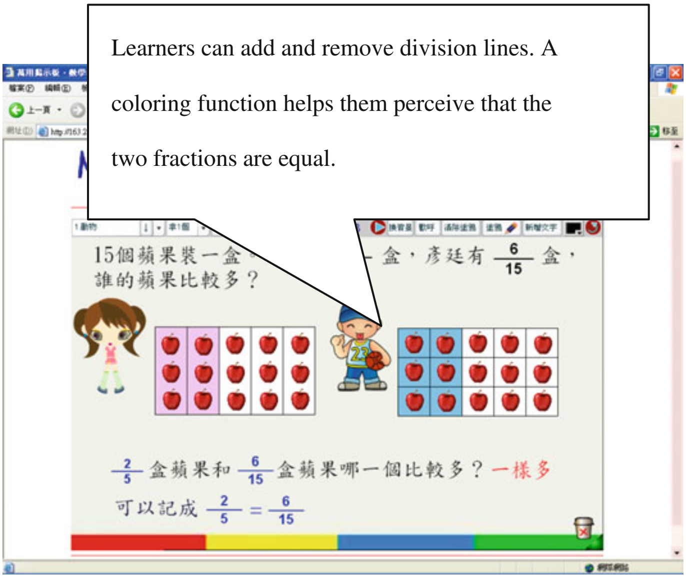


Fig. 9.5 Example of discrete equivalent fractions (using groups)

objects. Learners can determine whether the colored portions are the same length using an alignment line.

Examples featuring discrete objects use the group model. For these learners must re-divide or recombine a group of the same objects to understand that the two fractions are equal. The problem in Fig. 9.5 asks learners to determine which scenario indicates more apples: two fifths of a box of apples or six fifteenths of a box of apples. The virtual manipulative allows learners to add and remove division lines so that they can find the most appropriate segmentation method. A coloring function is also available to help students further visualize the problem.

9.3.2.2 Integrated-Virtual Group

The integrated examples include not only routine examples but also non-routine examples. In non-routine examples of continuous objects, the colored blocks are not continuous; learners must rearrange them into continuous blocks, be able to imagine or ignore division lines, and refer to the same fraction in different ways. Both length models and area models are used. Figure 9.6 shows a non-routine example using the length model, the objective of which is to determine how many sixths one third is equal to. Learners can use the virtual manipulative to add and remove division lines to find an appropriate method of segmentation. Any of the blocks can be colored, and the colored blocks can then be moved to form a

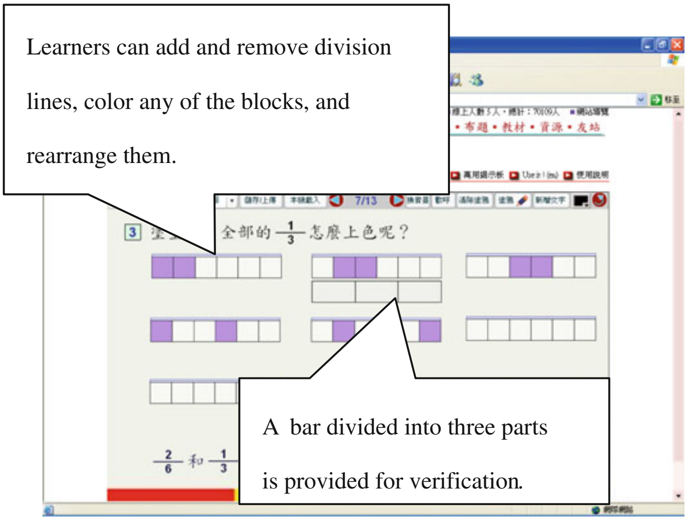


Fig. 9.6 Non-routine example of continuous equivalent fractions (using length)

continuous block. A bar divided into three parts is provided so that learners can verify that two sixths equals one third.

Non-routine examples of discrete objects use the group model with two or more types of objects. Learners must rearrange the objects and find an appropriate segmentation method to obtain equivalent fractions. The objective of the example in Fig. 9.7 is to find a fraction equivalent to sixteen twenty-fourths. The objects in the

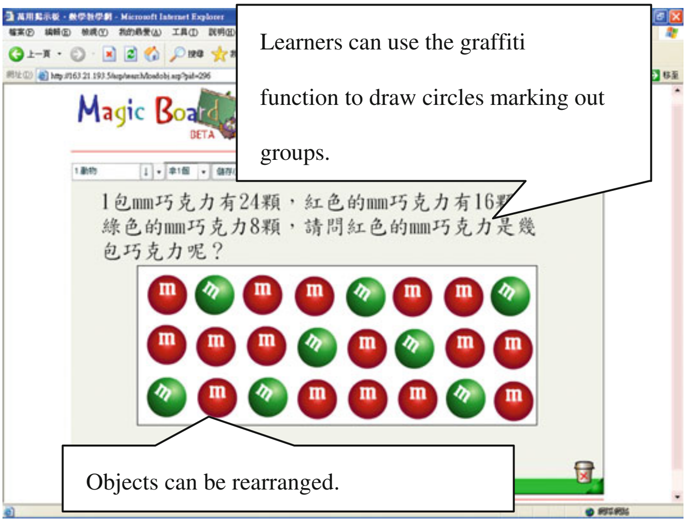


Fig. 9.7 Non-routine example of discrete equivalent fractions (using groups)

picture are scattered, so learners must rearrange them and gather the same objects together before using the graffiti function to divide them appropriately. They must be able to view multiple objects as a group to find the equivalent fraction.

9.3.3 Equivalent Fraction Achievement Test

The purpose of the achievement test was to evaluate the performance of the learners in flexible thinking with regard to equivalent fractions after the teaching experiment. The test problems were based on the unit content taught in the experiment and the studies conducted by Lee and Chen (2015). The content was divided into basic flexible thinking and advanced flexible thinking. The former type assessed the performance of learners in solving continuous equivalent fraction problems, which required more fundamental flexible thinking abilities regarding equivalent fractions. Basic flexible thinking was further divided into two categories: basic drawing ability and segmentation ability. Advanced flexible thinking involves the more advanced thinking skills that learners apply when solving discrete equivalent fraction problems; it is further divided into four categories: advanced drawing ability, combination ability, operative thinking ability, and unitization ability.

The two categories associated with basic thinking and four associated with advanced thinking were each allocated four problems, resulting in a total of 24 problems. Each of the problems was worth 1 point, bringing the total score to 24 points. With regard to the internal consistency of the problems, the Cronbach's α values were as follows: 0.70 for basic drawing ability, 0.75 for segmentation ability, 0.65 for advanced drawing ability, 0.83 for combination ability, 0.60 for operative thinking ability, 0.84 for unitization ability, and 0.92 overall. Thus, the achievement test presented acceptable reliability.

9.3.4 Mathematics Learning Attitude Questionnaire

The purpose of the mathematics learning attitude questionnaire was to understand the feelings of the participants as they learned the concept of equivalent fractions using different teaching approaches. The questionnaire was discussed and designed by two university professors and three experienced elementary school mathematics teachers, so the test has expert validity. The questionnaire included three portions regarding learning enjoyment, learning motivation, and mathematics anxiety. The portion on learning enjoyment measures to what degree the different teaching approaches were enjoyed by the participants. One example of a question focused on learning enjoyment is "Learning mathematics is fun". The portion on learning motivation measures the degree to which different teaching approaches increased participants' motivation to learn. One example of a question focused on learning motivation is "I will try my best to learn math well". The portion on mathematics anxiety examines whether different teaching approaches alleviate the level of anxiety that the participants may feel when learning mathematics. One example of a question focused on mathematics anxiety is "I will worry about my bad math learning". Each portion included 5 question items, with 15 question items in total. A five-point Likert scale was adopted for the question items, the answers scoring from 1 to 5 points each. The question items in the portions on learning enjoyment and learning motivation were positive items, whereas those in the portion on mathematics anxiety were negative items. For positive items, strong disagreement scored 1 point and strong agreement scored 5 points, while the opposite was true for the negative items. A higher score thus indicated a more positive mathematics learning attitude. The reliability of this questionnaire was verified using an internal consistency test, the values of which are as follows: Cronbach's alpha for learning enjoyment ($\alpha = 0.83$), learning motivation ($\alpha = 0.85$), mathematics anxiety ($\alpha = 0.72$), and for the entire questionnaire ($\alpha = 0.90$), indicating excellent internal consistency.

9.4 Results

9.4.1 Analysis of Equivalent Fraction Achievement Test

Table 9.1 shows that the mean score of the integrated-virtual group ($M = 6.60$) in basic flexible thinking was higher than those of the continuous-physical group ($M = 5.67$) and the continuous-virtual group ($M = 5.77$). In addition, the high prior knowledge group presented a higher mean score ($M = 7.28$) than the low prior knowledge group ($M = 4.63$).

Two-way ANOVA with the teaching approach and prior knowledge as the independent variables and the basic flexible thinking posttest score as the dependent variable (Table 9.2) revealed that the interaction effect between the teaching approach and prior knowledge, with regard to the basic flexible thinking posttest score, was not significant ($F = 0.367$, $p = 0.694$). We therefore looked at the main effects of teaching approach and prior knowledge and discovered that the main effect of the teaching approach was significant ($F = 3.845$, $p = 0.023$). Scheffe's post hoc test showed that the integrated-virtual group displayed better performance in basic flexible thinking than the continuous-physical and continuous-virtual groups, while the continuous-physical and continuous-virtual groups presented no significant differences. The main effect of prior knowledge also reached the level of significance ($F = 99.259$, $p = 0.000$), meaning that the students in the high prior knowledge group performed better in basic flexible thinking than the students in the low prior knowledge group.

As shown in Table 9.3, the mean score of the integrated-virtual group in advanced flexible thinking ($M = 12.93$) was higher than the mean score of the continuous-physical group ($M = 10.40$) and the continuous-virtual group ($M = 10.87$).

Table 9.1 Mean scores in different teaching approach and prior knowledge groups on basic flexible thinking posttest

Teaching approach		Continuous-physical	Continuous-virtual	Integrated-virtual	Total
Prior knowledge		(60)	(60)	(60)	(180)
High (94)	(M)	6.94	7.33	7.56	7.28
	(SD)	1.41	1.13	0.77	1.15
	(N)	34	24	36	94
Low (86)	(M)	4.00	4.72	5.17	4.63
	(SD)	1.96	2.46	2.20	2.27
	(N)	26	36	24	86
Total (180)	(M)	5.67	5.77	6.60	
	(SD)	2.21	2.40	1.91	
	(N)	60	60	60	

Table 9.2 Two-way ANOVA of prior knowledge and teaching approach with regard to basic flexible thinking posttest scores

Source of variation	SS	Df	MS	F	Sig
Prior knowledge	304.999	1	304.999	99.259***	0.000
Teaching approach	23.628	2	11.814	3.845*	0.023
Teaching approach *Prior knowledge	2.253	2	1.126	0.367	0.694
Error	534.660	174	3.073		
Total	7378.000	180			

Note *<0.05, ***<0.001

Table 9.3 Mean scores in different teaching approach and prior knowledge groups on advanced flexible thinking posttest

Teaching approach		Continuous-physical	Continuous-virtual	Integrated-virtual	Total
Prior knowledge		(60)	(60)	(60)	(180)
High (94)	(M)	12.76	13.50	14.67	13.68
	(SD)	4.08	3.72	1.66	3.33
	(N)	34	24	36	94
Low (86)	(M)	7.31	9.11	10.33	8.91
	(SD)	3.08	3.88	3.35	3.66
	(N)	26	36	24	86
Total (180)	(M)	10.40	10.87	12.93	
	(SD)	4.56	4.36	3.25	
	(N)	60	60	60	

Table 9.4 Two-way ANOVA of prior knowledge and teaching approach with regard to advanced flexible thinking posttest scores

Source of variation	SS	Df	MS	F	Sig
Prior knowledge	972.377	1	972.377	85.170***	0.000
Teaching approach	176.928	2	88.464	7.748**	0.001
Teaching approach *Prior knowledge	11.728	2	5.864	0.514	0.599
Error	1986.545	174	11.417		
Total	26584.000	180			

Note *<0.05, **<0.01, ***<0.001;

Furthermore, the high prior knowledge group presented a higher mean score on advanced flexible thinking ($M = 13.68$) than the low prior knowledge group ($M = 8.91$).

Two-way ANOVA with the teaching approach and prior knowledge as the independent variables and the advanced flexible thinking posttest score as the dependent variable (Table 9.4) revealed that the interaction effect between the teaching approach and prior knowledge with regard to the advanced flexible thinking posttest score was not significant ($F = 0.514$, $p = 0.599$). We therefore looked at the main effects of teaching approach and prior knowledge and discovered that the main effect of the teaching approach was significant ($F = 7.748$, $p = 0.001$). Scheffe's post hoc test showed that the integrated-virtual group displayed better performance in advanced flexible thinking than the continuous-physical and continuous-virtual groups, while the continuous-physical and continuous-virtual groups presented no significant differences. The main effect of prior knowledge also reached significance ($F = 85.170$, $p = 0.000$), meaning that the students in the high prior knowledge group performed better in advanced flexible thinking than the students in the low prior knowledge group.

9.4.2 Analysis of Mathematics Learning Attitude

Table 9.5 shows that the mean scores of the continuous-virtual group in the three aspects of mathematics learning attitude (enjoyment $M = 4.03$; motivation $M = 3.87$; anxiety $M = 3.65$) were higher than those of the continuous-physical group (enjoyment $M = 3.43$; motivation $M = 3.61$; anxiety $M = 3.34$) and the integrated-virtual group (enjoyment $M = 3.59$; motivation $M = 3.61$; anxiety $M = 3.47$).

Two-way ANOVA with the teaching approach and prior knowledge as the independent variables and mathematics learning attitude as the dependent variable revealed that the interaction effect between the teaching approach and prior knowledge was significant with regard to learning enjoyment ($F = 5.627$,

Table 9.5 Mean scores in different teaching approach and prior knowledge groups in mathematics learning attitude

Teaching approach		Continuous-physical	Continuous-virtual	Integrated-virtual	Total
Prior knowledge		(60)	(60)	(60)	(180)
High (94)	Enjoyment	3.69 (0.67)	4.52 (0.52)	3.51 (1.13)	3.83 (0.94)
	Motivation	3.81 (0.66)	4.28 (0.66)	3.59 (0.92)	3.85 (0.81)
	Anxiety	3.58 (0.86)	4.27 (0.55)	3.53 (0.99)	3.74 (0.90)
Low (86)	Enjoyment	3.08 (0.66)	3.71 (1.03)	3.72 (0.81)	3.52 (0.91)
	Motivation	3.34 (0.44)	3.60 (0.88)	3.65 (0.94)	3.53 (0.80)
	Anxiety	3.03 (0.59)	3.23 (0.85)	3.38 (0.96)	3.21 (0.82)
Total (180)	Enjoyment	3.43 (0.73)	4.03 (0.95)	3.59 (1.01)	
	Motivation	3.61 (0.62)	3.87 (0.86)	3.61 (0.92)	
	Anxiety	3.34 (0.80)	3.65 (0.90)	3.47 (0.97)	

Note The values in brackets stand for standard deviations

$p = 0.004$), learning motivation ($F = 3.473$, $p = 0.033$), and mathematics anxiety ($F = 4.057$, $p = 0.019$). We then analyzed the simple main effects.

As shown in Table 9.6, students with high prior knowledge in the continuous-virtual group (enjoyment $M = 4.52$; anxiety $M = 4.27$) displayed a higher score on learning enjoyment and a higher score on mathematics anxiety than those in the integrated-virtual group (enjoyment $M = 3.51$; anxiety $M = 3.53$) and in the continuous-physical group (enjoyment $M = 3.69$; anxiety $M = 3.58$). Those in the continuous-virtual group ($M = 4.28$) also displayed better learning motivation than those in the integrated-virtual group ($M = 3.59$). This means that the students with high prior knowledge in the continuous-virtual group possessed more positive views on learning enjoyment, mathematics anxiety, and learning motivation. In contrast, students with low prior knowledge in the continuous-virtual group ($M = 3.71$) and the integrated-virtual group ($M = 3.72$) displayed a higher score on learning enjoyment than those in the continuous-physical group ($M = 3.08$). This means that students with low prior knowledge had more enjoyment when the virtual manipulatives were used. However, the three groups displayed no significant differences in learning motivation and mathematics anxiety. Thus, in terms of the continuous-physical and continuous-virtual groups, the students with high prior knowledge presented higher scores in learning enjoyment, learning motivation, and mathematics anxiety than the students with low prior knowledge. However, in the integrated-virtual group, the students with high prior knowledge showed no significant differences from those with low prior knowledge in the three aspects of mathematics learning attitude.

Table 9.6 Two-way ANOVA of simple main effects of prior knowledge and teaching approach on mathematics learning attitude

Source of variation	SS	df	MS	F	Sig	Post hoc comparison
Teaching approach						
High						
Enjoyment	15.603	2	7.802	10.801	.000	Continuous-virtual > Integrated-virtual = Continuous-physical
Motivation	7.010	2	3.505	5.908	0.004	Continuous-virtual > Integrated-virtual
Anxiety	9.103	2	4.551	6.310	0.003	Continuous-virtual > Integrated-virtual = Continuous-physical
Low						
Enjoyment	7.347	2	3.674	4.822	0.010	Continuous-virtual = Integrated-virtual > Continuous-physical
Motivation	1.474	2	0.737	1.167	0.316	No significant differences
Anxiety	1.575	2	0.787	1.182	0.312	No significant differences
Prior knowledge						
Continuous-physical enjoyment	5.612	1	5.612	12.664	0.001	High > Low
Motivation	3.301	1	3.301	10.066	0.002	High > Low
Anxiety	4.387	1	4.387	7.656	0.008	High > Low
Continuous-virtual enjoyment	9.344	1	9.344	12.468	0.001	High > Low
Motivation	6.724	1	6.724	10.486	0.002	High > Low
Anxiety	15.376	1	15.376	27.514	0.000	High > Low
Integrated-virtual enjoyment	0.608	1	0.608	0.590	0.445	No significant differences
Motivation	0.054	1	0.054	0.062	0.804	No significant differences
Anxiety	0.324	1	0.324	0.340	0.562	No significant differences

9.5 Discussion and Suggestions

9.5.1 Learning Performance

This study investigated the influence of prior knowledge and teaching approach on learning performance and mathematics learning attitude. The students in the integrated-virtual group displayed better performance than the continuous-physical and continuous-virtual groups in both basic and advanced flexible thinking. This supports the results derived by Lee and Chen (2009, 2015), in which integrated examples can improve students' learning performance. This is probably because non-routine examples stimulate students' thinking and arouse their curiosity, which prompts them to integrate various strategies, solutions, and representations to develop more sophisticated understanding and inference processes. Unfortunately, the current equivalent fraction units in elementary school textbooks in Taiwan only contain routine examples. Publishers should therefore consider adding non-routine examples into the curriculum to improve students' learning performance with regard to equivalent fractions, and teachers should encourage students to think about such non-routine examples.

The continuous-physical and continuous-virtual groups displayed no significant differences in basic or advanced flexible thinking. This is consistent with the findings of Yuan et al. (2010) in that both virtual and physical manipulatives can effectively facilitate learning. In other words, if the teaching content and method are the same, the form of the manipulatives does not significantly impact learning, and thus, changing physical manipulatives into virtual manipulatives does not change students' learning performance. In fact, the results of this study indicate that the use of non-routine examples was the crucial factor influencing equivalent fraction learning. Regardless of the type of manipulative, teachers should invest more effort in the design of effective examples and their appropriate coordination with physical or virtual manipulatives to maximize learning effectiveness. Inappropriate example designs can affect the performance of students in solving equivalent fraction problems.

The high prior knowledge group displayed significantly better performance than the low prior knowledge in both basic and advanced flexible thinking. This may be due to the high prior knowledge group recognizing patterns quickly and transitioning to the use of symbols (Moyer-Packenham and Suh 2012). However, students with less prior knowledge do not have a large amount of mathematical knowledge to activate schema or adequate cognitive resources to perform activities such as self-explanation. Consequently, they cannot integrate their learning experiences when needed. An attempt at doing so may even lead to cognitive overload and thereby failure to integrate. It is therefore reasonable that they may not perform as well as students with high prior knowledge.

9.5.2 Mathematics Attitude

In terms of students with high prior knowledge, those in the continuous-virtual group displayed more positive views in learning enjoyment and mathematics anxiety than those in the continuous-physical group. This is consistent with the findings of past studies on virtual manipulatives (Reimer and Moyer 2005; Lee and Chen 2014; Yuan et al. 2010), which have been found to increase learning enjoyment for mathematics and reduce mathematics anxiety. However, the scores provided by the students with high prior knowledge in the integrated-virtual group with regard to learning enjoyment and mathematics anxiety showed no significant differences from those in the continuous-physical group but were better than those in the continuous-virtual group. It may be that the greater difficulty and complexity of the non-routine examples required that the students convert the non-routine examples into routine examples before solving them, and this increase in complexity rendered the activity less enjoyable and more likely to induce anxiety.

With regard to the students with low prior knowledge, those in the continuous-virtual group and the integrated-virtual group presented higher scores in learning enjoyment than those in the continuous-physical group. This means that using virtual manipulatives is fun for students with low prior knowledge no matter what type of examples are used. In contrast, the type of example and the type of manipulative did not show significant influence on the learning motivation or mathematics anxiety of students with low prior knowledge in the three groups. We suggest that future research consider how extrinsic motivation can be used to increase the intrinsic motivation of students in the design of manipulatives and thereby enhance mathematics learning attitudes.

In the continuous-physical and continuous-virtual groups, the students with high prior knowledge presented higher scores in learning enjoyment, learning motivation, and mathematics anxiety than the students with low prior knowledge, which supports the results of previous research (Lee and Chen 2010; Lee and Yuan 2010). Students with high prior knowledge are generally more confident in mathematics learning, and therefore, they showed better performance in the three aspects of mathematics learning attitude than the students with low prior knowledge. In contrast, the students with high and low prior knowledge in the integrated-virtual group displayed no significant differences in learning enjoyment, learning motivation, or mathematics anxiety, which is inconsistent with previous findings (Lee and Chen 2010; Lee and Yuan 2010). We speculate that the students with high prior knowledge may have felt impatient with the time-consuming nature of the non-routine examples (which required conversion into routine examples before solving) and they felt that they had not learned any new mathematical concepts. As a result, their mathematics learning attitude did not improve. In contrast, the students with low prior knowledge were more indifferent towards the non-routine example conversion process but were happy because they got to use the virtual manipulatives, and therefore, they presented better mathematics learning attitudes. For these reasons, the students with high and low prior knowledge in the integrated-virtual group displayed no significant differences in learning enjoyment, learning motivation, or mathematics anxiety. This phenomenon of students with high prior knowledge in the integrated-virtual group displaying better learning performance but poorer mathematics learning attitude has been found in our study. Future research can further investigate the major factors and causes of this phenomenon.

In short, we found that the effectiveness of learning equivalent fractions can be enhanced using non-routine examples. However, in terms of mathematics learning attitude, the students with high prior knowledge in the continuous-virtual group displayed higher scores in learning enjoyment, learning motivation, and mathematics anxiety than those in the integrated-virtual group. This indicates that non-routine examples are aggravating to students with high prior knowledge in manipulative operation, which affects their mathematics learning attitude. In contrast, virtual manipulatives have greater impact on the mathematics learning attitude of students with low prior knowledge; they got higher scores in learning enjoyment as long as they had virtual manipulatives to use. However, virtual manipulatives had less influence on learning motivation and mathematics anxiety. In the integrated-virtual group, prior knowledge did not influence the mathematics learning attitude of the students. This implies that the more complicated manipulative operations of non-routine examples had a more adverse effect on the mathematics learning attitude of students with high prior knowledge, and as a result, the scores obtained by students with high and low prior knowledge displayed no significant differences in mathematics learning attitude.

9.5.3 Limitations

This study is subject to the following limitations. First, the sample size 180 for a two-way (2×3) factorial design was small, so the research results may not be applicable to students with educational and cultural backgrounds that are different from those of the students in this study. Furthermore, learning equivalent fractions is considerably different from that of other fields, such as biology or social science, so the results of this study may not be applicable to other academic subjects or other themes in mathematics. Thirdly, the prior experience of the students, such as their view of mathematics or their self-efficacy with computers, may also influence their use of virtual or physical manipulatives, which in turn impacts their learning effectiveness. Therefore, future studies could investigate the role of prior experience in learning with virtual manipulatives. Finally, the teacher who participated in this experiment did not have more free time to teach eight classes in his schedule. Therefore, only six classes joined our experiment. Considering the variable "prior knowledge", we only designed three groups (each group contained two classes) so that each cell could have 30 students. Future studies could add one group using integrated examples paired with physical manipulatives and examine the effects of the four groups on different prior knowledge groups.

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Chapter 10
The Role of Virtual Manipulatives in High School Students' Understanding of Geometric Transformations

Hilal Gulkilik

Abstract Although there is widespread research that articulates their importance in mathematics education, manipulatives are pushed aside in high school learning settings as something inappropriate or frivolous. The purpose of this study was to identify the role of virtual manipulatives in high school students' mathematical understanding about geometric transformations, which included translations, reflections, rotations, and dilations. The main data sources for this study were semi-structured task-based interviews that were conducted after each weekly transformation lesson. The mathematical understanding of students was analyzed using representation theory and the Pirie-Kieren model. This was presented using a two-dimensional model that shows the students using virtual manipulatives within different levels of mathematical understanding (Pirie and Kieren in Educ Stud Math 26(2):165–190, 1994). Results of the study revealed that virtual manipulatives helped students to apply distinct representations of geometric transformations and translate among them. As interventions in the environment, virtual manipulatives strengthened students' mathematical understanding in terms of progressing from inner to outer levels of the Pirie-Kieren model, folding back movements, and acting-expressing activities.

Keywords Mathematical understanding · Virtual manipulatives · Representation · Pirie-Kieren model · High school students

Mathematical manipulatives were recently defined by Bartolini and Martignone (2014) in the Encyclopedia of Mathematics Education as "artifacts used in mathematics education: they are handled by students in order to explore, acquire, or

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investigate mathematical concepts or processes and to perform problem-solving activities drawing on perceptual (visual, tactile, or, more generally, sensory) evidence" (p. 365). As concrete learning objects, manipulatives have been used to support children's mathematical development since the 1800s. They gained a theoretical basis through learning theorists such as Jerome Bruner, Zoltan Dienes and Jean Piaget in the 1900s (McNeil and Jarvin 2007). At the end of the 1900s, innovations in computer and information technology, the accessibility of Internet media, and increases in the number of computers led to a new trend in mathematics education that introduced virtual manipulatives (Moyer et al. 2002).

A virtual manipulative is "an interactive, Web-based, visual representation of a dynamic object that provides opportunities for constructing mathematical knowledge" (Moyer et al. 2002). Most virtual manipulatives are on the computer screen and are interacted with using a mouse or keyboard whereas others are on touch screen devices controlled by finger movements. Unlike the physical versions, virtual manipulatives contain the verbal, graphical and notational representations of a concept together, which helps students to make connections among different representations (Suh and Moyer-Packenham 2007). Moyer-Packenham and Westenskow (2013) carried out a meta-analysis of 32 research reports using 83 effect size scores that investigated the effects of virtual manipulatives in students' mathematics achievement. According to this meta-analysis, instruction with virtual manipulatives produces moderate effects when compared to instruction with all other instructional treatments (0.37; 0.44, with one outlier). The qualitative results of the study revealed that there are five key affordance categories that positively affect students' learning: (a) focused constraint, (b) creative variation, (c) simultaneous linking, (d) efficient precision, and (e) motivation.

Even though there is extensive literature supporting manipulative use for all grade levels, there is a lack of research on high school learning environments (Gibbons 2012; Gordon 1996; Jones 2010; Marshall and Paul 2008). The reasons for this gap in the literature may be high school teachers' knowledge, experience, beliefs, and attitudes about manipulatives. Teachers hesitate to use manipulatives in secondary level classrooms because they believe that students in these grades should work with symbolic and abstract knowledge (Jones 2010). In view of the limited research at the high school level, there is a need to examine high school students' manipulative use in learning mathematics.

Mathematics education researchers collectively emphasize the importance of learning mathematics with understanding. There are several researchers, who have characterized mathematical understanding from different perspectives (see Dubinsky 1991; Herscovics 1989; Hiebert and Carpenter 1992; Sfard 1991; Sierpinska 1994; Skemp 1978). Pirie and Kieren (1989) make a classification of perspectives that describe mathematical understanding as "acquisition" or "process" (p. 7). They assert that mathematical understanding is more than categories of knowing and define it as a "whole, dynamic, leveled but non-linear, transcendently recursive process" (Pirie and Kieren 1994, p. 166). Figure 10.1, consisting of eight embedded circles, models their theory. These circles show the potential levels that one goes through during the growth of mathematical understanding. Pirie and

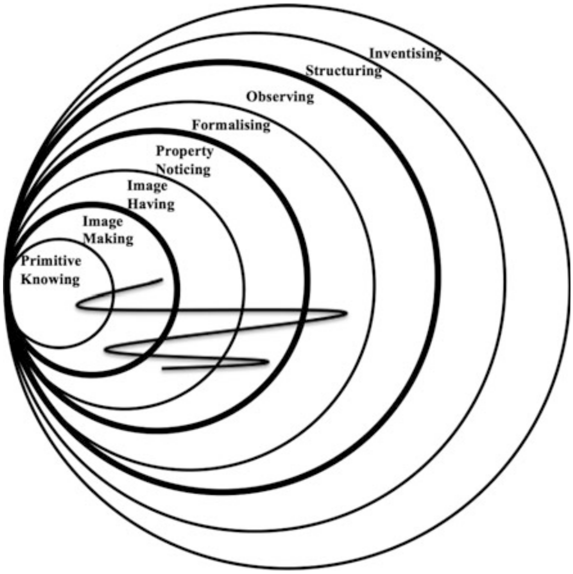


Fig. 10.1 The Pirie-Kieren model for the growth of mathematical understanding (Pirie and Kieren 1994)

Kieren (1989) state that "each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further is constrained by those without" (p. 8).

The *Primitive Knowing* level includes a student's previously developed knowledge about the topic. Using her/his primitive knowledge, s/he engages in mental or physical activities in order to develop an idea about the topic at the *Image Making* level. At the *Image Having* level, the student has an image about the concept, which may contain verbal, visual, written or any other representations. The student realizes the different properties of the concepts at the *Property Noticing* level by thinking about the differences and similarities of the images s/he has. At the level of *Formalising*, with the help of different properties based on various images, s/he makes general statements or develops common ideas about the concept that are similar to the mathematical definition of concept. S/he considers the constructed formal structures to develop theorem-like ideas about the related concept at the *Observing* level while s/he is able to verify these ideas logically at the *Structuring* level. At the *Inventising* level, with a structured understanding about the concept, the student asks questions that lead her/him to invent "a totally new concept" (Pirie and Kieren 1994, p. 171).

The theory describes one's understanding of mathematics through four different features: folding back, the 'don't need' boundaries, complementary aspects, and interventions. When a student folds back, it means that s/he cannot handle the mathematical task by working in the present level and needs to go back to an inner level to elaborate on the understanding. The 'don't need' boundaries are the darker lines in the model (see Fig. 10.1). These are the boundaries that "convey the idea that beyond the boundary one does not need the specific inner understanding that gave rise to the outer knowing" (Pirie and Kieren 1994, p. 173). The third feature of the theory is related to complementary aspects of acting and expressing. Acting maintains the continuity of a particular level with inner levels, while expressing strengthens the understanding at this level. Interventions pertain to another feature of the theory. Provocative interventions make students move towards the outer levels whereas invocative interventions result in students folding back to the inner levels to review the current process. Validating interventions are the stimulants that influence students to think about their existing understanding. The Pirie-Kieren model, by means of mathematical understanding levels and these features, serves as an efficient research tool to observe, understand and model students' growth of mathematical understanding (Gibbons 2012; Martin 1999; Nillas 2010; Towers 1998).

Another theory that views mathematical understanding as a process is representation theory, which emphasizes the role of dynamically networked mathematical representations of concepts (see Goldin 2003; Hiebert and Carpenter 1992; Janvier 1987). Representations in mathematical learning environments come in two forms, external and internal. External representation is used "to refer to physically embodied, observable configurations such as words, graphs, pictures, equations, or computer micro worlds" whereas internal representation is used "to refer to possible mental configurations of individuals, such as learners or problem solvers" (Goldin and Kaput 1996, pp. 399–400). The interaction between the two kinds of representations is two-way; an internal representation may be transformed to an external representation by an externalization process while an external representation may be transformed into an internal representation by an internalization process (Goldin 2003; Zhang 1997).

A consensus on the importance of providing learning environments that allow students to engage in these processes highlights the key role of multiple representations of mathematical concepts (Ainsworth et al. 2002; Duval 2006; Even 1998). Students can develop an appropriate mathematical understanding by using multiple representations appropriately and making connections among these representations (Kaput 1989; Lesh et al. 1987; Renkl et al. 2013). One of the multiple representations that students may use to develop internal representations of mathematical concepts is manipulative models (see Fig. 10.2). Representation theory gives us the opportunity to examine the role of manipulatives in mathematical understanding in terms of the relationships with other representations. The Pirie-Kieren model offers the possibility to analyze the understanding of students in a dynamic way in conjunction with representation theory. The model highlights non-linear movements through the understanding levels, as a student reconstructs and strengthens her/his understanding by using representations of concepts. In this chapter, the Pirie-Kieren model and representation theory were used together as a lens to determine the role of virtual manipulatives in 10th-grade students' mathematical understandings about geometric transformations.

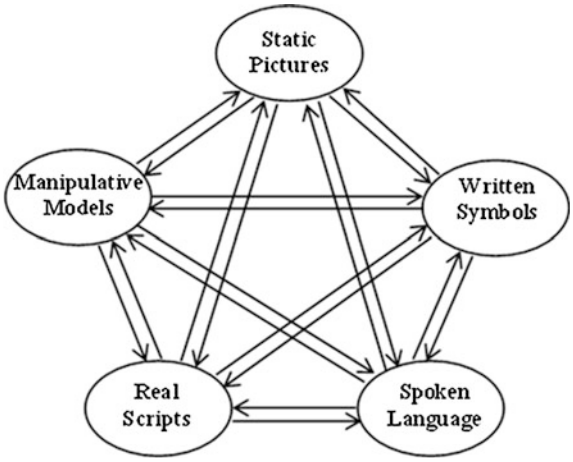


Fig. 10.2 Multiple representations of a concept (Lesh et al. 1987, p. 34)

10.1 Methods

The method used for this study was a teaching experiment (Cobb 2000; Steffe and Thompson 2000). Ms. Yilmaz (a pseudonym), a teacher with 13 years of experience, taught a transformational geometry unit to a 10th-grade class in a high school in Turkey. She conducted eight lessons in one month, two for each of the transformations: translation, rotation, reflection, and dilation. The lessons were enriched with virtual and physical manipulatives in addition to verbal, graphical, and algebraic representations of these transformations. The researcher observed the class during these lessons and performed task-based interviews (Goldin 2000) with the participants after lessons conducted about each transformation.

10.1.1 Participants

There were 32 10th-grade students (17 females and 15 males) in Ms. Yilmaz's geometry class. The researcher entered the class two months before the main study began and carried out a pilot study to interact with students to develop a level of comfort. During the pilot study, Ms. Yilmaz used physical and virtual manipulatives to teach the concepts about triangles. Prior to the first week of the study, the researcher administered a pretest, which covered translation, rotation, reflection, and dilation, and a spatial ability test, which was used to purposefully select four participants for in-depth analyses of growth of mathematical understanding. The pretest had 26 mathematical tasks about translation, rotation, reflection, and dilation. These tasks examined students' understandings about identifying the transformations in verbal, graphical and algebraic representations. The Spatial Ability Test (SAT) was adopted from the tasks in the Kit of Factor-Referenced Cognitive

Table 10.1 Characteristics of participants

Name	Age	Spatial ability test score	Pretest score	Previous geometry class score
Defne	17	67.5	28	71.75
Elif	16	150.25	55	78.50
Metin	17	161.50	40	55.00
Selim	16	48.00	32	87.00

Note The highest score possible was 282 for the Spatial Ability Test, 100 for the pretest, and 100 for the previous geometry class score

Tests (Ekstrom et al. 1976) to determine spatial ability, which is an important component in students' understanding of geometric concepts (Battista 1990). The test was translated to Turkish by Delialioğlu (1996). There were test items including mental rotation of 2-D figures, mental rotation of cubes, imagination of the folding and unfolding of a paper, and mentally folding given 2-D figures to obtain 3-D objects. The four participants selected for the study described in this chapter were two female and two male students who represented different levels of classroom participation and manipulative engagement in the pilot study, geometry class scores, and spatial abilities (see Table 10.1). Their pseudonyms are Defne, Elif, Selim, and Metin.

10.1.2 Instructional Settings and Manipulatives

Ms. Yilmaz conducted lessons in the computer lab of the school during the study. There were 30 computers and a teacher computer station with a display screen in the lab (see Fig. 10.3).

Lessons began with an introduction to the current geometric transformation using pictures, animations, or videos as visual representations and real-life examples. This was followed by mathematical tasks in which the teacher and students used virtual or physical manipulatives. The teacher modeled how to use the manipulatives before students worked with them. For every lesson, students also

Fig. 10.3 The front and back view of the computer lab

had instructions explaining how to perform the manipulative activities. At the end of the lessons, the teacher shared the algebraic representations and guided discussions with the students to construct a shared language and to make connections among multiple representations of the concept. The researcher video-recorded all of the lessons by positioning herself near the participants and took field notes during instruction.

The teacher and students in the class used virtual manipulatives from the National Library of Virtual Manipulatives website (www.nlvm.usu.edu). They interacted with the dynamic applets to identify and connect the mathematical properties of transformations in different kinds of representations. The applets allowed students to manipulate objects on the plane using the four geometric transformations (see Fig. 10.4).

In addition, students and the teacher used physical manipulatives that were designed by mathematics educators at a university under the guidance of the researcher and with the help of preservice secondary mathematics teachers (see Fig. 10.5).

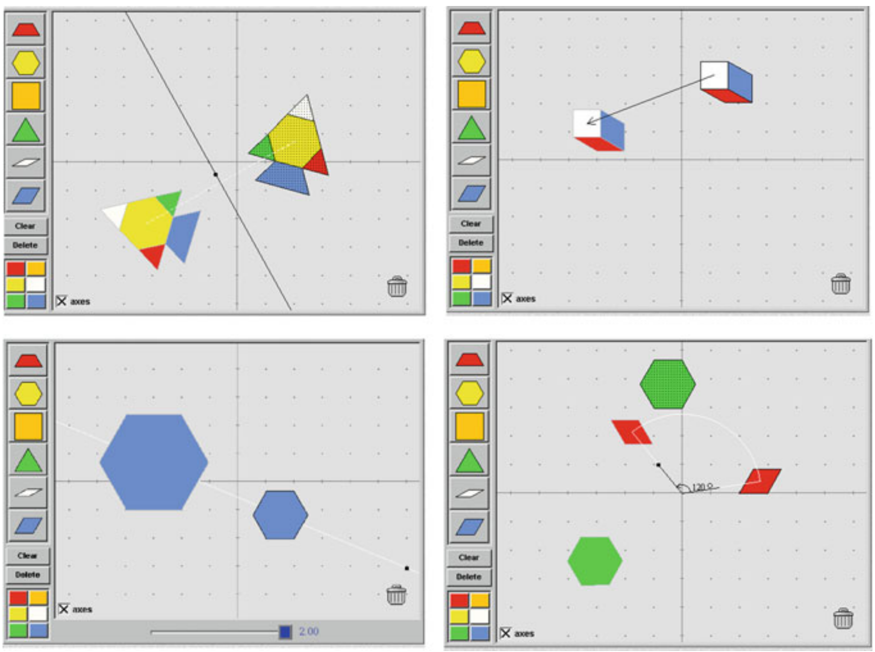


Fig. 10.4 Samples from virtual manipulatives used during instruction

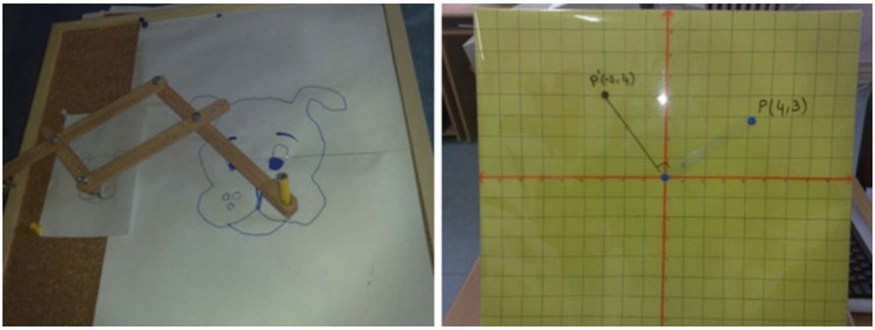


Fig. 10.5 Samples of physical manipulatives used during instruction

10.1.3 Collecting Data

Data collection occurred in two main phases. Before the first lesson of the study began, two data sources were used, the Spatial Ability Test (SAT) and the pretest. Prior to the first week of the study, the researcher interviewed each of the participants to discuss their responses to pretest tasks. The purpose of this interview was to determine the participants' primitive knowledge about geometric transformations in detail.

After the lessons began, the main data source was participants' responses during videotaped task-based interviews that were conducted weekly. The field notes that were taken during the classroom observations were used to support the interview data. The researcher conducted four task-based interviews with each of the participants about each of the transformations after the lessons. Tasks were designed to probe students' understandings of geometric transformations in the context of representations. Because high school students were expected to construct a formal level of understanding and make formal observations in the curriculum, the nature of the tasks was based on the knowledge that is required to go through to the Observing level. Several studies on students' understanding of geometric transformations (e.g., Flanagan 2001; Yanik 2006) were used to design the tasks. First, questions asked participants to provide descriptions for the geometric transformation, give examples and non-examples of the transformation, and clarify the properties of the transformation. There were tasks using verbal, visual and algebraic representations of the transformations. The researcher prompted the collection of information about participants' growth of mathematical understanding by posing questions about identifying, applying, and translating among multiple representations.

She provided paper, pencil, and related physical and virtual manipulatives during each of the interviews. Participants were told they were free to use any of these instruments while they were working on the tasks. A video camera was positioned behind the participants and captured audio and video of participants' engagement with the tasks and manipulatives. The researcher wanted students to read the questions and think aloud as much as possible during the interviews. Each of the four weekly interviews lasted approximately one hour.

10.1.4 Data Analysis

The units of analysis for this case study were the participants (Miles and Huberman 1994). First each case was analyzed separately, and then a cross case analysis was conducted to compare the cases in terms of the role of virtual manipulatives in developing mathematical understanding of transformations. The constant comparison method (Strauss and Corbin 1990) was used to analyze the data.

Before starting the data analysis, the researcher prepared a coding protocol, based on multiple representations of transformations and mathematical understanding levels of the Pirie-Kieren model. The protocol included the possible components of each mathematical understanding level that students developed about each transformation by using distinct representations. The participants' mathematical understanding about each geometric transformation was traced according to this protocol. For example, when a student was using formal understanding, but for some reason s/he needed to revise her/his image and began to use virtual manipulatives at the Image Making level, it was coded as *using virtual manipulatives to fold back.*

The whole data set was analyzed twice. First, the line-by-line coding of each interview was performed using Pirie and Kieren's understanding levels and characteristics of the theory. Second, the data were coded to determine students' multiple representation usage by focusing on the virtual manipulatives. After coding each interview for mathematical understanding levels and engagement with virtual manipulatives, codes from the two sets were associated with each other by axial coding and emergent themes were identified looking at these associations.

10.2 Results

Table 10.2 summarizes participants' virtual manipulative use during the weekly interviews. Participants differed in their use of virtual manipulatives as they developed mathematical understandings about the geometric transformations.

Elif used virtual manipulatives during all four weeks consistently, Defne used them during two of the four weeks, whereas Selim and Metin used virtual manipulatives only during one of the four weeks. Elif used virtual manipulatives while she was working at almost every mathematical understanding level from Image Having to Observing. Defne used virtual manipulatives mostly to express her images and to notice some properties about transformations. Selim preferred to use them during the rotation interview while he was engaging with some activities at the Property Noticing and Observing levels. Metin used virtual manipulatives only

Table 10.2 The number of times participants used virtual manipulatives during the interviews

	Elif	Defne	Selim	Metin
Week 1: Translation	2 (IH, F)	0	0	0
Week 2: Rotation	5 (IH, PN, F)	0	2 (PN, O)	0
Week 3: Reflection	5 (F, O)	2 (IH, PN)	0	0
Week 4: Dilation	6 (PN, F, O)	2 (PN)	0	2 (F, O)

Note IH Image Having; *PN* Property Noticing; *F* Formalising; *O* Observing

during the dilation interview. He worked with these manipulatives at the Formalising and Observing levels to express the formal ideas and inferences that he constructed after the lesson. In the following section, each participant's virtual manipulative usage will be examined deeply as their mathematical understanding process is traced within the Pirie and Kieren's levels.

The mapping of the participants' mathematical understandings about each transformation is presented below in a two-dimensional model that maps the representations with Pirie and Kieren's mathematical understanding levels. The horizontal axis shows the levels from the Pirie-Kieren model with acting and expressing aspects, and the vertical axis shows the three areas of the tasks presented during the interviews (verbal, graphical, and algebraic) and the five types of representations used (physical manipulatives, virtual manipulatives, verbal, graphical, and algebraic representations). To guide the reader to track the mathematical understanding process of the students, the numbers were added to the model to show how participants traversed the levels. The same numbers appear in parentheses in the text to help the reader to follow participants' mathematical understanding explicitly. Participants' engagement with distinct representations through their use of virtual manipulatives will be focused on during the interviews to analyze their role in the mathematical understanding process.

Case 1 (*Elif*)

Elif was a 16-year-old girl in the second year of her high school education. She was a successful student in her previous geometry classes and had good spatial ability (see Table 10.1).

Elif's mathematical understanding about translation and virtual manipulatives. Figure 10.6 shows the development of Elif's mathematical understanding about translation as she worked with distinct representations within the different levels. She used virtual manipulatives two times during this interview, first at the Image Having level and second at the Formalising level.

The first question of the interview asked Elif to give an example and a non-example of a translation. She preferred to use virtual manipulatives to express her image by giving an example of the transformation (1a). She drew two congruent triangles on the virtual manipulative and said that she drew the translated image of the triangle by moving the original triangle "three units to right and four units to down" (see Fig. 10.7).

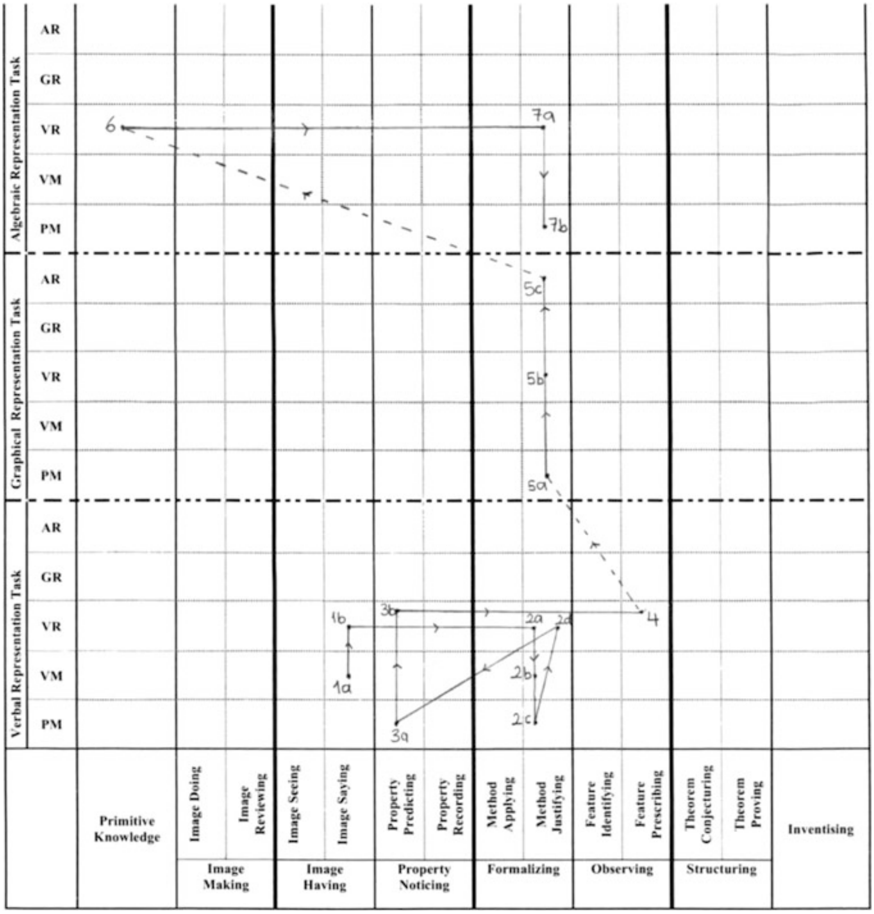


Fig. 10.6 The mapping of Elif's mathematical understanding about translation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

In the following phase of the interview, the researcher wanted Elif to explain what she understood about translating an object on the plane by using the idea of a vector. She answered that question by saying that "it was moving the object through the length of the vector" while she supported this idea with an example on the virtual manipulative (2b). This time she preferred to use virtual manipulatives to express her formal idea about the transformation at the Formalising level.

Elif was able to understand and apply the verbal, graphical and algebraic representations of translation at the formal levels and she successfully connected these representations with each other. At the end of the interview, she explained that she "could work virtual manipulatives properly and could make faster drawings with them". She preferred to use virtual manipulatives to express the images she had from

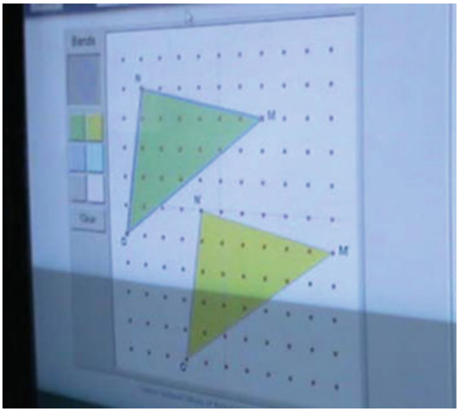


Fig. 10.7 The congruent triangles Elif drew on the virtual manipulative

her previous experiences and the formal ideas she constructed after the lessons during the interview. Virtual manipulatives were validating tools in the environment that helped her to verify the level of understanding she was working on at that time.

Elif's mathematical understanding about rotation and virtual manipulatives. Elif preferred to use virtual manipulatives during the rotation interview more than she did in the translation interview. As seen in Fig. 10.8, she used the virtual manipulative mostly at the beginning of the interview. She drew a triangle on the manipulative and rotated it easily 90°, 180°, and 360° while she was giving examples of rotation (1). She continued to express her ideas on the virtual manipulative at the Property Noticing level while the following dialogue took place between her and the researcher (2):

Elif	For example in this rotation, the figure goes on that circle (pointing to the circle on the virtual manipulative) but in translation it would move through a given vector.
Researcher	I see. What would you add if one of your classmates would ask you about rotation?
Elif	I would say it is spinning a figure around a point that is not on the figure without changing the distance.
Researcher	Distance?
Elif	I mean the distance from the figure to the point.
Researcher	You mean the point that is not on figure?
Elif	Yes, rotation center I mean. It may be on the figure also, sorry. But the distance between the rotation center and the turning point will not change.
Researcher	Ok. How will you rotate the figure without changing that distance?
Elif	Around the point. I mean it will make a circle around the rotation center.

Elif used formal explanations while she was saying that rotating a figure meant turning a figure around a point by any angle measure on the plane (3). She made a

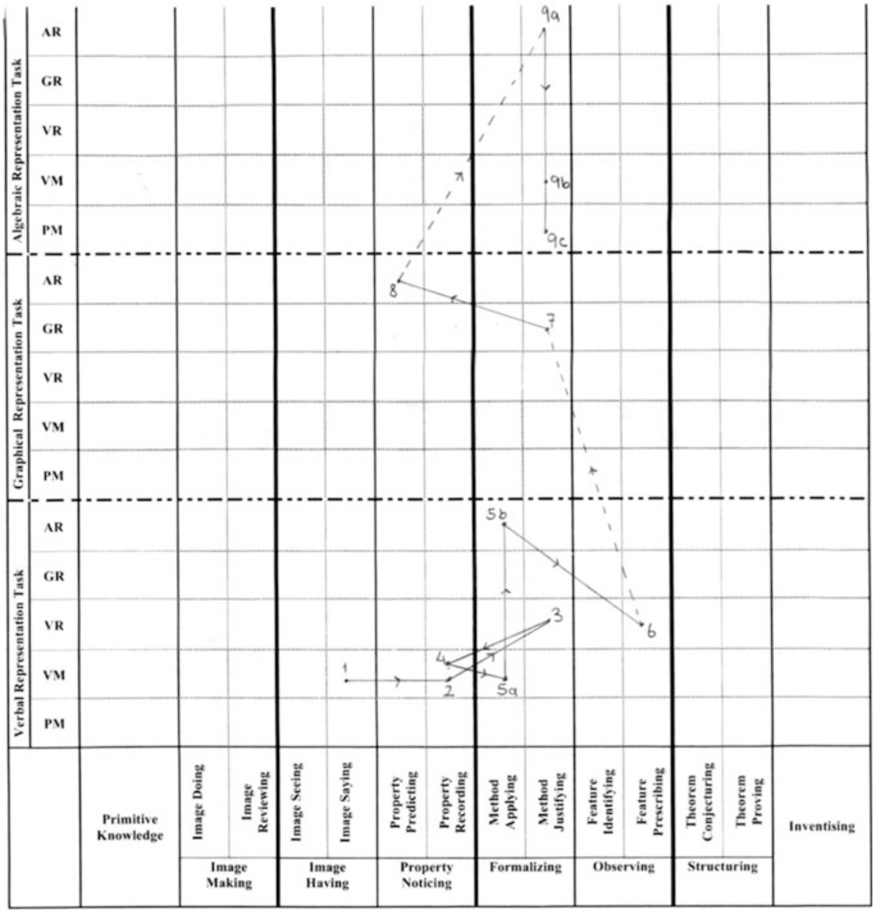


Fig. 10.8 The mapping of Elif's mathematical understanding about rotation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

folding back movement to the Property Noticing level and chose a rocket figure on the virtual manipulative while she was explaining further (4) (see Fig. 10.9).

> First I determine the distance between the corner points of this figure (showing the original figure) and rotation center, then look at the rotation angle. I turn the figure around the center by that angle. Like this, I am identifying the distance (showing the length of line segment between one of the corner points of the figure and rotation center), then, rotation center is the origin point. I am looking at the distance between this point (showing the same corner point) and center. I will turn the figure around the center without changing the distance I determined. It will make a circle around the center and the radius of a circle is equal to this distance everywhere.

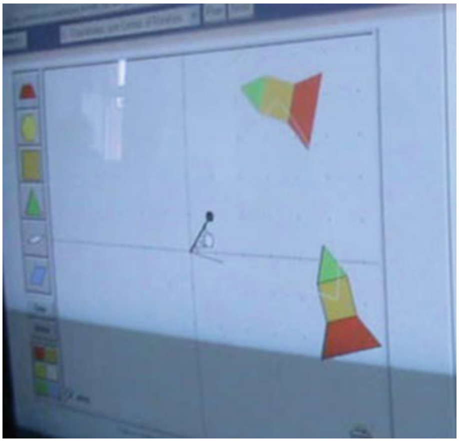


Fig. 10.9 The figure Elif used to express some properties about rotation

Elif used the virtual manipulative as an invocative intervention that caused her to work in an inner level to strengthen her formal level of understanding. She continued to use the virtual manipulative when she was asked to rotate a triangle around the origin by 90° in the verbal representation task. She quickly drew the triangle on the virtual manipulative and found the image of the triangle under a rotation of 90° around the origin by herself, without clicking the rotate button (5a). To understand if she was able to translate from verbal representations to algebraic representations of rotation, she was asked to reanswer this question by using a mathematical formula this time. First she wrote the expression $(x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$ without any equality and then looked at the virtual manipulative for a while (5b). The researcher observed that Elif was checking the formula with the solution that she performed on the virtual manipulative by putting the corner points of the triangle into the expression. For this situation, the virtual manipulative was playing the role of a validating intervention for her to confirm the formal level of mathematical understanding she had.

To analyze Elif's mathematical understanding of rotation in detail, she was asked to find $R_{60^\circ}(P)$ without giving any mathematical formula, where P was (1, 0). Elif was successful in explaining the meaning of the mathematical notation $R_{60^\circ}(P)$ in the task by stating that it was asking for the image of point P under the rotation around the origin by 60° (9a). When she was asked to resolve the problem by using visual representations she opened the virtual manipulative again and turned the point around the origin by 60° and marked an estimated point on the screen as the image (9b). She stated that she could find the image on the virtual manipulative approximately, but to identify the exact location of this point she would need to use the physical manipulative that was designed to rotate a point around the origin by a protractor.

As seen on Fig. 10.8, Elif was successful understanding and applying distinct representations of rotation at the Formalising level and translating among verbal, graphical and algebraic representations. As interactive and dynamic external visual representations, virtual manipulatives seemed to be helping her to apply these transformations in mathematical situations and make connections between representations. In contrast with the translation interview, Elif preferred to use virtual manipulatives more than physical manipulatives in the rotation interview. At the end of the interview, she clarified this situation by stating that "virtual manipulatives were ready to use quickly" at that time.

Elif's mathematical understanding about reflection and virtual manipulatives. Elif continued to use the virtual manipulatives during the reflection interview. As seen in Fig. 10.10, she began the interview by creating an example on the virtual manipulative at the Formalising level (1).

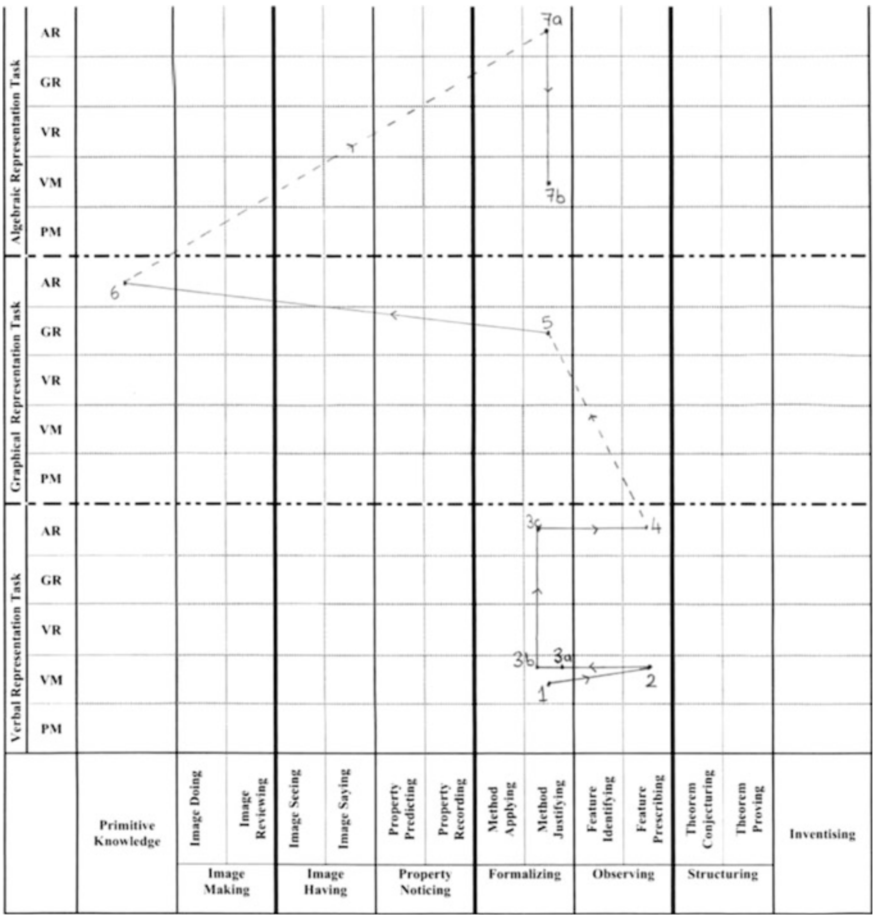
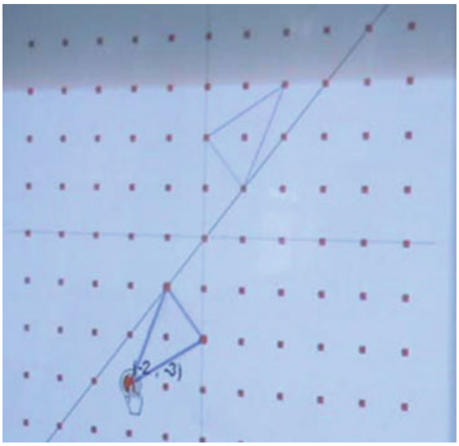


Fig. 10.10 The mapping of Elif's mathematical understanding about reflection. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

She found the image of a triangle that she drew on the virtual manipulative under the reflection across the line $y = x$. She said it would be an example of a reflection because "each point of the original triangle has a corresponding point on the reflected triangle that is the same distance from the line of reflection as the original point". When she was asked to give a non-example of reflection she said that a translation would not be a reflection because the direction of a figure might be changed in reflection but not in translation. She marked one of the corresponding sides of the triangles that she previously drew on the virtual manipulative and said that the direction of both sides did not stay the same after a reflection (2). As a provocative intervention, the virtual manipulative helped her to carry her understanding to the next level. Elif began to work at the Formalising level again while she was explaining how to reflect a figure across a line. She expressed that "all of the corresponding points would be the same distance from the line of reflection" by looking at the example she drew on the virtual manipulative (3a). Her understanding seemed to maintain that level while she was engaging in the verbal representation task in which three points as $A(1,1)$, $B(0,2)$ and $C(2,3)$ on the plane were given and she was asked to reflect the ABC triangle in the origin. She was successful in finding the image with the virtual manipulative by placing it directly on an opposite point on the other side of the center for each of the corner points on the triangle. The point of reflection was the midpoint of the segment joining each point of the triangle with its image (3b) (see Fig. 10.11).

The next time she used virtual manipulatives was the session that she was resolving the algebraic representation task by using graphical representations. She was asked to find $S_M(P)$ where P was $(1,0)$ and M was $(2,2)$ in the task. She said that $S_M(P)$ was the image of point P under the reflection in point $M(2,2)$ and found the image by using $S_M(P) = 2M - P$ (7a). She was also able to show the graphical solution on the virtual manipulative that confirmed that she understood and applied distinct representations of the transformation (7b). Elif used virtual manipulatives to clarify her formal understandings during the reflection interview. As seen in her

Fig. 10.11 Elif was finding the image of a triangle on the manipulative



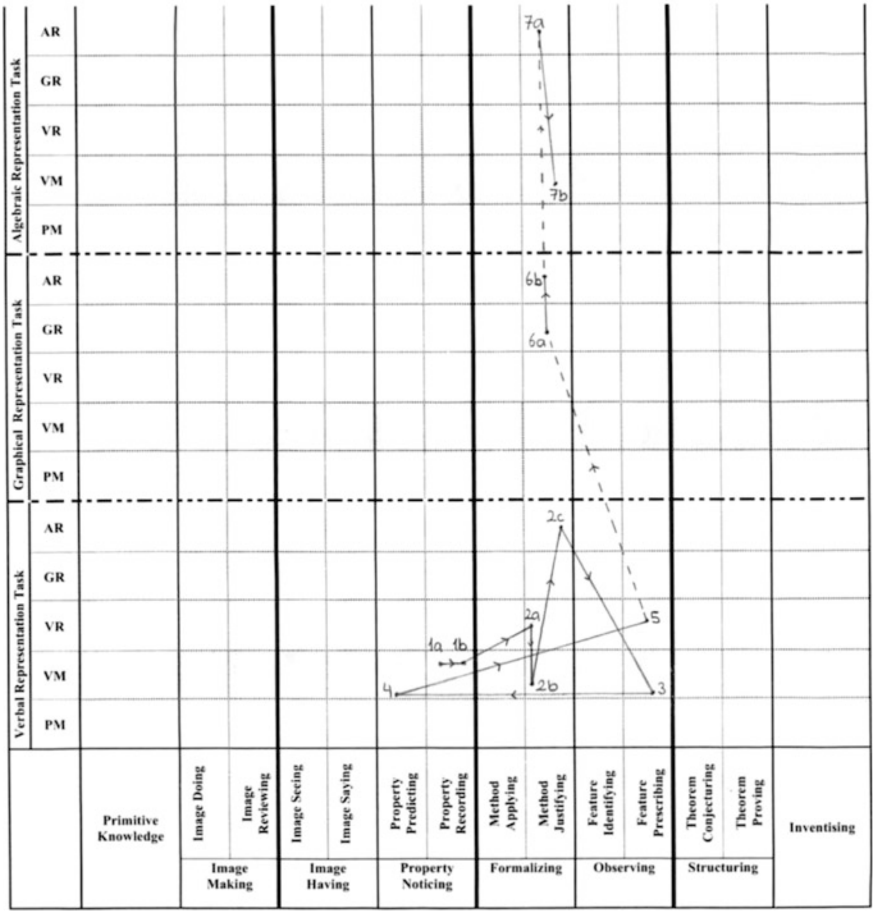


Fig. 10.12 The mapping of Elif's mathematical understanding about dilation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

previous explanations, the virtual manipulative played the *bridge* role in her understanding to use distinct representations and translate among them.

Elif's mathematical understanding about dilation and virtual manipulatives. The dilation interview was the session that Elif primarily used virtual manipulatives (see Fig. 10.12). She began the interview by expressing some properties of dilation in an example on the virtual manipulative (1a) (see Fig. 10.13). She continued to work at the Property Noticing level while she was recording some properties of dilation on the same manipulative (1b). Next, Elif was expected to find the image of a triangle after a dilation with the center at point (0, 2) and a scale factor of 2. She continued to use the virtual manipulative and found the image of the triangle easily (2b) (see Fig. 10.14).

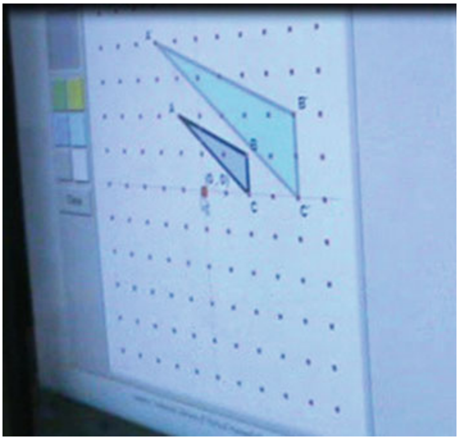


Fig. 10.13 The example of a dilation Elif drew using the virtual manipulative

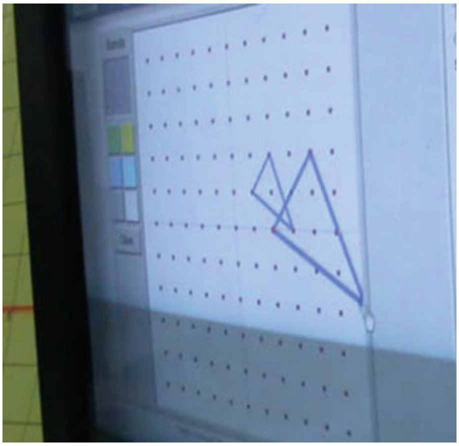


Fig. 10.14 The example of dilation Elif drew on the virtual manipulative while she was checking some properties

Elif was working at the Formalising level and she was able to connect the verbal representation and the graphical representation of the concept together using the virtual manipulative. When she was asked to resolve the problem by using algebraic representations she could not remember the whole mathematical formula, $H(P) = P' = M + k(P - M)$, which gives the image of point P under the dilation with a center at the point M and a scale factor of k. She used virtual manipulatives to confirm the $P - M$ part of the formula. She picked some corresponding points from the original and image triangles and put these points in the formula to see if the notation was right or not. After trying several points, she stated that $P - M$ was the distance between the original point and the center and if she multiplied this distance with k and added the new distance to the center she would find the image of the point (2c). The virtual manipulative, again as a validating intervention, helped her to build a connection between the graphical and algebraic representations of the concept.

She continued to work on the triangles she drew on the virtual manipulative while she was saying "because the angles, the direction of angles were the same and the sides of the image were twice the sides of the original triangle, two triangles were similar" (3). She was using the manipulative to express her formal observations about the transformation this time. She made a folding back movement to the Property Noticing level and continued to use the virtual manipulative while she was predicting some properties of the dilation in that level (4). She stated "while we were finding the image, we were thinking about the distance from the center. If the center changes, because the distance between the original point and the center will change, then the location of the image changes" while she was working on the manipulative that helped her to strengthen her mathematical understanding.

After using verbal representations to solve the algebraic representation task, she began to use the virtual manipulative again to show a graphical solution (7b). Elif was able to comprehend and apply the distinct representations of dilation and make connections among them. She used virtual manipulatives usually to express her images and formal ideas and virtual manipulatives were validating interventions in the environment that helped her to translate among the distinct representations of transformations.

Case 2 (*Defne*)

Defne was a 17-year-old girl in the second year of her high school education. She was a successful student in her previous geometry class but did not have good scores on the spatial ability test or the pretest (see Table 10.1).

Defne's mathematical understanding about reflection and virtual manipulatives. The reflection interview was the session that Defne used virtual manipulatives for the first time. She did not use them in the translation or rotation interviews. According to Fig. 10.15, which shows the mapping of Defne's mathematical understanding about reflection, she used the virtual manipulatives at the Image Having level after noticing some properties about the transformation.

When she was asked to explain what she understood from reflecting a figure across a line she made the following explanations using virtual manipulative at the Property Noticing level (2b) (see Fig. 10.16):

Defne	If I reflect a triangle over the line passing through the origin (showing the $y = x$ line) I will flip it over that line, but the distances have to be perpendicular distances.
Researcher	Perpendicular distance?
Defne	I mean it is already the distance when it comes to perpendicular.

She was trying to express that when she was reflecting a point across a line, the line was the perpendicular bisector of the segment joining the original point and its image. Her understanding moved to the Image Having level when she used her hands and made some gestures to clarify the ideas that she was explaining (see Fig. 10.17). The dialogue between the researcher and her continued (3):

Defne	Let's think about my hands. For example the reflection of this hand is my other hand, like that. If I move my hands like here (putting her

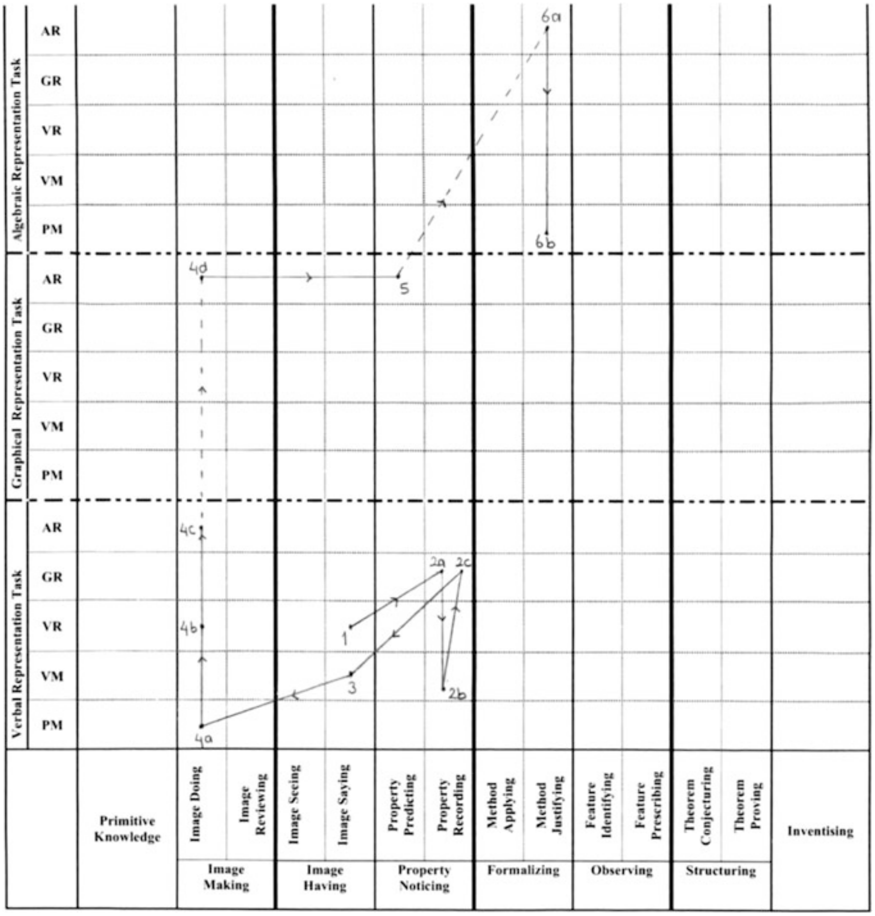


Fig. 10.15 The mapping of Defne's mathematical understanding about reflection. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

	hands on the computer screen). We can think as a whole. Reflection over a line.
Researcher	What about the reflection in a point?
Defne	Reflection in a point. This point for example (marking a point on the virtual manipulative), we will find the image around this point. It is one unit (showing the distance between the point and one corner of the figure) then it will come to here one unit (showing the point directly opposite on the other side of the center).

As an invocative intervention, Defne used virtual manipulatives to make a folding back movement to an inner level and tried to enlarge her images about reflection. From Fig. 10.15, it can be seen that she was actively working in the

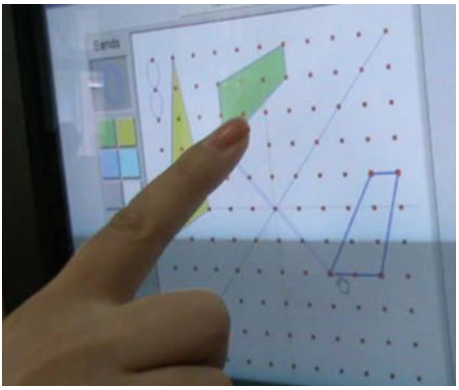


Fig. 10.16 The example Defne drew on the virtual manipulative

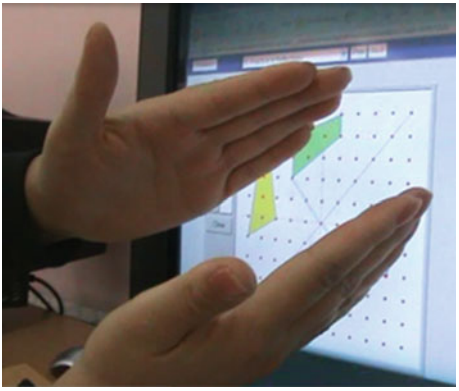


Fig. 10.17 Gesture Defne used during the interview

informal levels and could not move her understanding to the formal levels for a long time (4a-5). Because she was not able to use multiple representations at the formal levels she could not develop an appropriate understanding about the concept in that time. Moreover, she looked uncomfortable using the virtual manipulatives during the lesson. At the end of the interview, Defne stated that she lost time when she attempted to work on the virtual manipulatives because she did not like to use the computer. Her attitude seemed to be affecting her virtual manipulative use. Virtual manipulatives helped her only to express the images that she had and clarify the properties that she had already realized about reflection.

Defne's mathematical understanding about dilation and virtual manipulatives. The other interview where Defne used virtual manipulatives was the dilation interview. Figure 10.18 shows that she used virtual manipulatives two times in the acting phase of the Property Noticing level to predict some properties of dilation during the verbal representation task. She expressed the following ideas about finding the image of a figure after a dilation with a center and a scale factor on the virtual manipulative (2c):

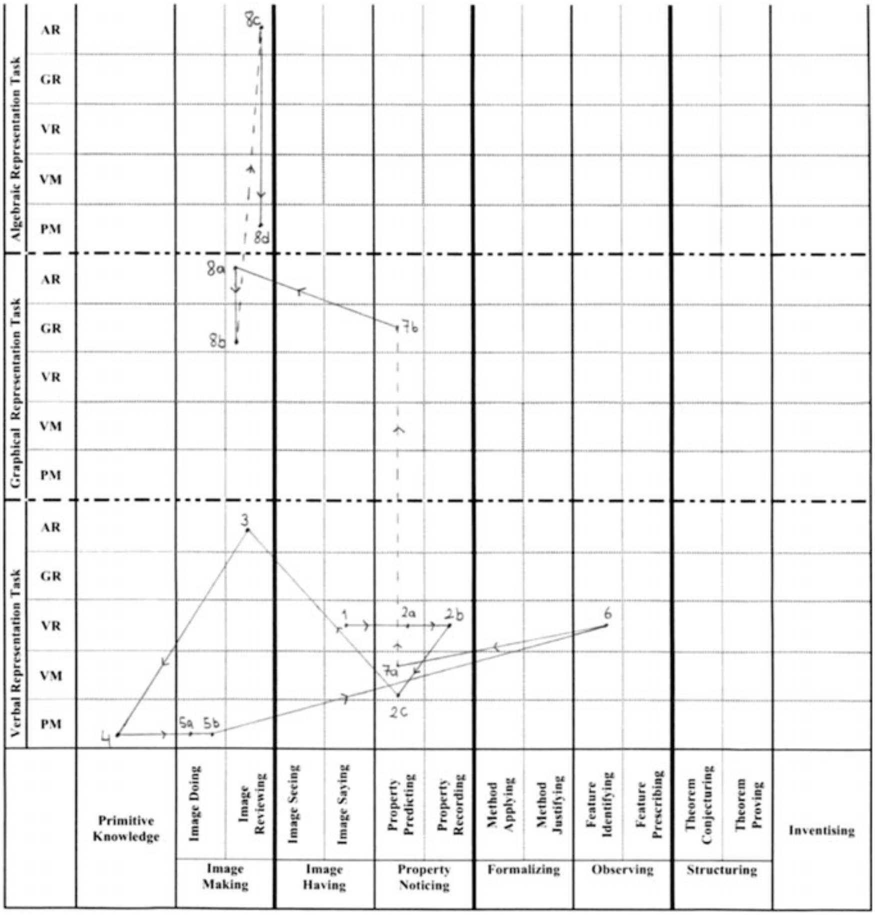


Fig. 10.18 The mapping of Defne's mathematical understanding about dilation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

> I understand that the figure is being stretched or expanded on a center. Like this example (showing an original triangle and its dilated image on manipulative). When the dilation center comes closer to the original figure, this one (the image of the triangle) comes closer to the center, too. When we reduce the scale factor the triangles come closer to the center… So, we will look at the distance to the center and the scale factor if we want to dilate a figure.

She made a folding back movement to the Property Noticing level and began to use the virtual manipulative for the second time when she was asked what would happen to the image if the dilation center was changed (7a). She expressed her ideas in the following dialogue:

Defne I will do the similar things. If I carry the center point from here to here (moving the dilation center on the virtual manipulative) but the scale

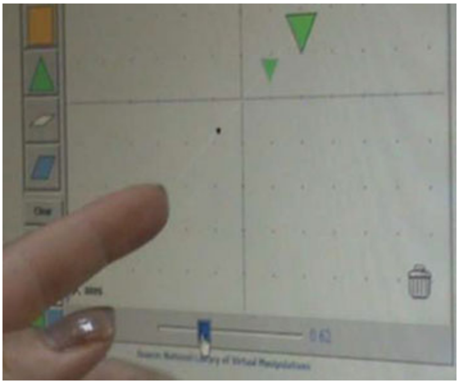


Fig. 10.19 Defne was changing the scale factor to notice some properties

	factor will be same, the size of the triangle (showing the image of triangle) will not change. Only the location of it will change.
Researcher	What about the scale factor? If I change the scale factor?
Defne	When I reduce the scale factor the image comes closer (changing the scale factor) (see Fig. 10.19).

Defne stopped using virtual manipulatives after working on these properties of the transformation. She preferred to use them to make some physical activities at the Property Noticing level. The virtual manipulatives played the role of validating interventions that helped her to justify the understanding she had at the time. Figure 10.18 shows that she continued to work mostly at informal levels as she did in the reflection interview. She could not apply the verbal, graphical and algebraic representations and was not able to connect them in formal levels as in the reflection interview.

Case 3 (*Selim*)

Selim was a 16-year-old boy in the second year of his high school education. He was a very successful student in his previous geometry class but had the lowest score on the spatial ability test and one of the lower scores on the pretest (see Table 10.1).

Selim's mathematical understanding about rotation and virtual manipulatives. Selim was the participant who preferred to use physical manipulatives during the interviews. The only session he used virtual manipulatives was the rotation interview. He used them two times during the interview; first he used them to perform some activities at the Property Noticing level, and second he used them to express his formal observations at the Observing level (see Fig. 10.20).

When he was asked to explain what he understood from rotating a figure around a point by an angle measure he began to work on a particular example on the virtual manipulative to notice some properties (4). The following dialogue took place between the researcher and Selim while he was using the virtual manipulative:

Selim	For example this figure (pointing to the figure on the virtual manipulative). Let's rotate it counter clockwise, in positive direction by 90° (rotating the figure and observing the image).

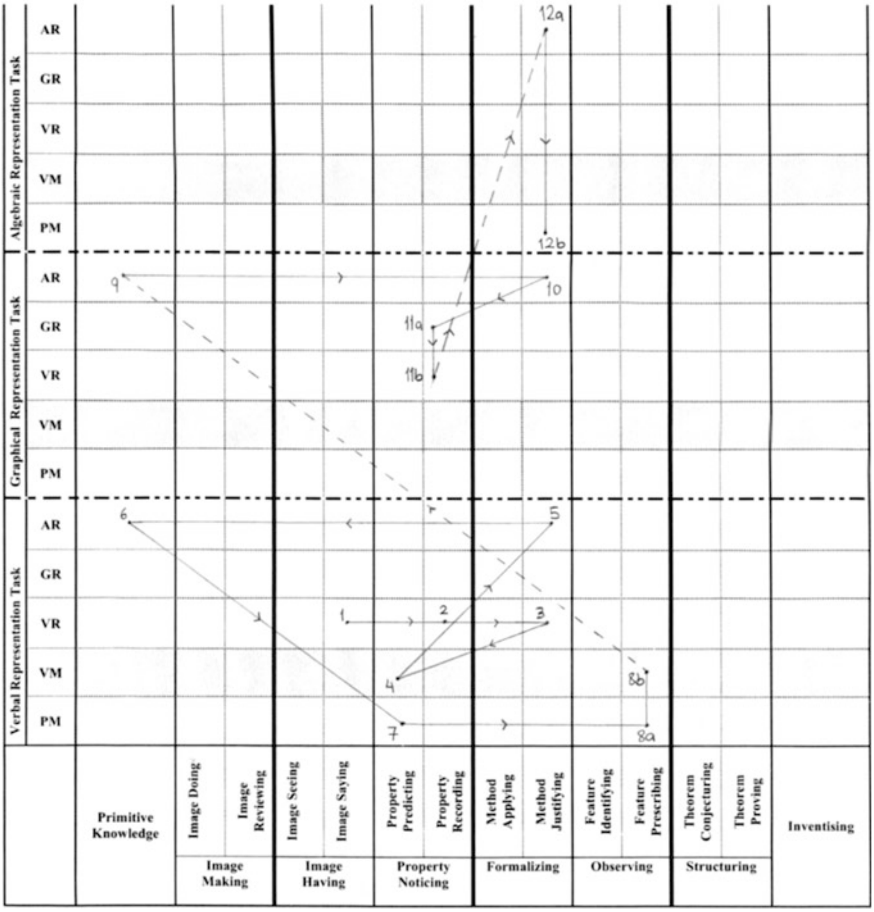


Fig. 10.20 The mapping of Selim's mathematical understanding about rotation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

Researcher It says around a point?
Selim Around the origin, the rotation center is the origin here.

The second time, he chose to work with virtual manipulatives again while he was showing the differences when the parameters of the transformation were changed (8b). Selim was able to apply verbal representations of the transformation more than graphical or algebraic representations. He was able to transition between these representations but he needed to make folding back movements to inner levels while he was working especially with algebraic representations.

Virtual manipulatives played the role of a tool for Selim to engage in physical activities and to confirm his formal observations of the transformation. He remarked he "would not trust the operations or calculations from a virtual manipulative he was using without checking these operations or calculations by mathematical formulas during problem solving".

Case 4 (*Metin*)
Metin was a 17-year-old boy in the second year of his high school education. He was not a successful student in his previous geometry class but had the highest score on the spatial ability test (see Table 10.1).

Metin's mathematical understanding about dilation and virtual manipulatives. Metin was the other participant who preferred not to use virtual manipulatives during the interviews. He used them only two times at the Formalising and Observing levels during the dilation interview (see Fig. 10.21). First he used them

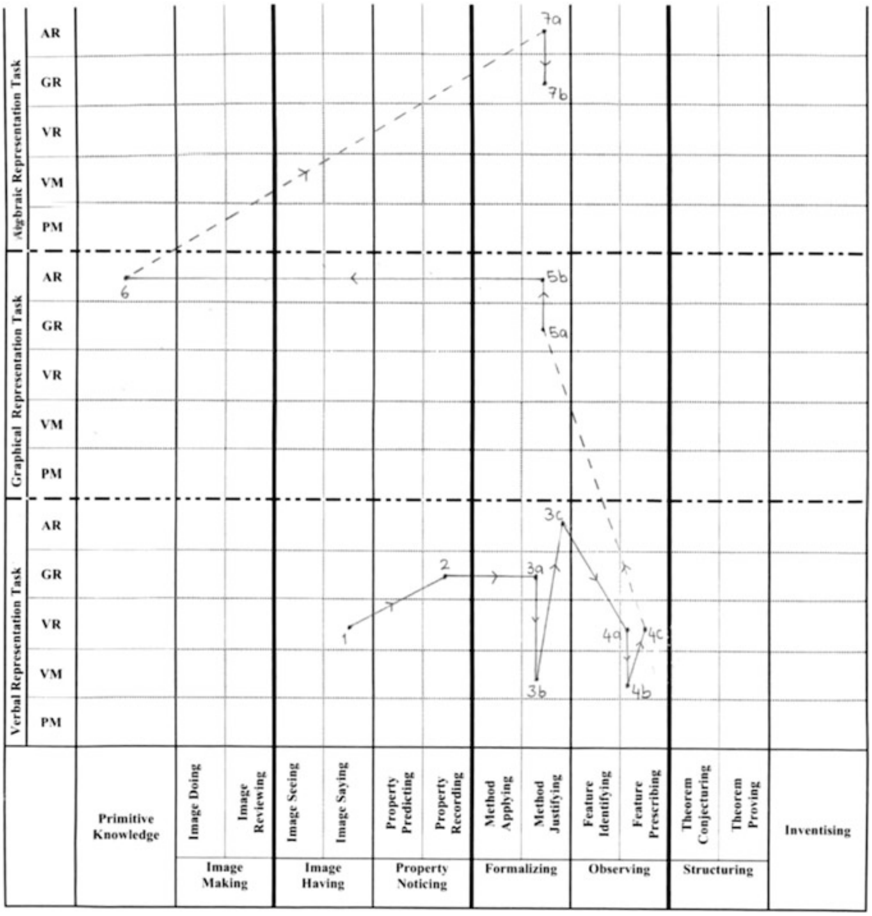


Fig. 10.21 The mapping of Metin's mathematical understanding about dilation. *PM* Physical Manipulative, *VM* Virtual Manipulative, *VR* Verbal Representation, *GR* Graphical Representation, *AR* Algebraic Representation

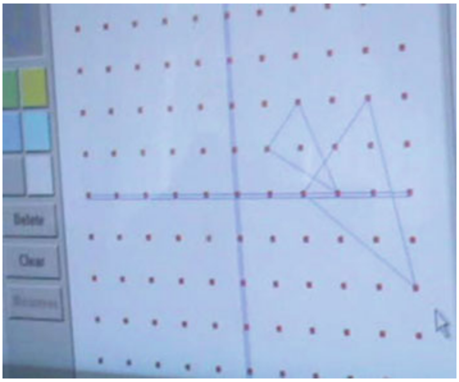


Fig. 10.22 The triangle and its' image Metin drew on the virtual manipulative

to express his formal ideas while he was dilating a triangle and second to state his observations while he was explaining the relationships between the original figure and its image under a dilation. When Metin was asked to find the image of a triangle after a dilation with the center at point (0, 2) and a scale factor of 2, he began to use the virtual manipulative and found the image of the triangle (3b) (see Fig. 10.22).

He continued to use his formal level understanding during the remaining part of the verbal representation task and used the virtual manipulative to express his theorem-like idea about dilation at the Observing level (4b). He said "the figures and their dilated images were always similar because angles were preserved and corresponding sizes of the figures had the same ratio" while he was working on the virtual manipulative.

As seen on Fig. 10.21, Metin was working mostly at the formal levels and he was able to use distinct representations of the concept. He could translate among the multiple representations of the transformation without using physical and virtual manipulatives. It was thought that because he had a proper understanding and was able to use distinct representations of dilation he did not need to engage in manipulative activities. The following sentences he used at the end of the interview supported this finding:

> I got used to computer and physical objects. I can use whichever I want. It may be more enjoyable if you work with them but using them takes your time, you have to do some extra work. Physical objects are good while you are trying to understand the concept but I would not use them if I try to solve a problem. I would use only paper and pencil.

10.3 Discussion and Conclusion

Trying to characterize students' mathematical understanding about a mathematical concept is a multifaceted process. In this study, Pirie-Kieren theory, with its understanding levels and features, and representation theory, with multiple representations, provided a tool/model to observe *how* and *what* students understood about a concept. Developing a two dimensional graph to represent students' developing understanding by combining these two theories helped the researcher to trace the change in student's mathematical understanding and interpret the role of virtual manipulatives in this understanding. According to the analysis, students' mathematical understanding levels spanned the first six levels of the Pirie-Kieren model (Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, and Observing). Students used virtual manipulatives to support their understanding in various ways not only in informal levels but also at the Formalising and Observing levels. Findings by Gibbons (2012) support this result by stating that high school students worked with physical manipulatives at formal levels like Observing and Structuring, to construct abstract meaning of mathematical concepts.

In the cross case analysis, Elif and Metin, who developed formal levels of understanding, used virtual manipulatives to express their formal ideas and observations. Expressing is a required understanding characteristic that helps students to state the structure of mental or physical activities to themselves or other observers and carry their understandings to the next levels (Borgen 2006; Nillas 2010; Pirie and Kieren 1994). On the other hand, Defne, who developed insufficient understandings, used virtual manipulatives for physical and mental activities in only the first four levels. Acting is an important understanding characteristic that helps students to manage their previous levels of understandings and realize some new features in any of the understanding levels (Pirie and Kieren 1994). Hence, it can be said that virtual manipulatives played the role of an acting/expressing medium that supported the development of students' mathematical understanding.

There were several interventions that affected students' mathematical understanding in the environment. Verbal, graphical, and algebraic representation tasks, prompts that the researcher used during the interviews, and manipulatives that were present on the table and computer screen gave direction to students' mathematical understanding processes. In particular, virtual manipulatives played the role of a provocative intervention that helped students to progress outwards (e.g., Elif's movement from the Formalising level to the Observing level in the reflection interview). These were used as validating interventions that guided students to justify their level of understanding. Lastly, virtual manipulatives were invocative interventions that caused a folding back movement to the inner levels (e.g., Selim's movement from Formalising to the Property Noticing level in the rotation interview). As invocative interventions, virtual manipulatives were the source of folding back movements where they were also helping to identify the nature of these movements. For example, Defne used virtual manipulatives in a folding back movement to "work in an inner level using existing understanding" during the rotation interview, whereas Elif used them in a folding back movement to "collect in an inner level" during the dilation interview (Martin 2008, p. 76). When we consider that the dynamic feature of folding back is essential for the mathematical understanding process (Pirie and Kieren 1994), virtual manipulatives helped students to strengthen their understanding during these movements.

Virtual manipulatives played the role of a supportive external representation that facilitated understanding of the concept. Students preferred to work with virtual manipulatives to comprehend and apply especially verbal and graphical representations. Students used them as a connection tool when they were expected to translate among distinct representations. Because applying multiple representations and making connections among distinct representations indicates a strong mathematical understanding (Goldin 2003; Hiebert and Carpenter 1992; Lesh et al. 1987; Zhang 1997) the virtual manipulatives helped students to develop a proper understanding in terms of multiple representation engagement. From the observer's perspective, virtual manipulative usage at the formal levels may be an indicator that the student can connect multiple representations of the concept. For example, Elif used virtual manipulatives to express her formal ideas while she was engaging with verbal or algebraic representations at the Formalising level. Because she could use distinct representations at the formal levels, she developed a robust understanding about the transformations.

On the other hand, virtual manipulatives supported the motion understanding of geometric transformations (Hollebrands 2003; Yanik 2006) especially if they were used without an understanding of algebraic representations of the concepts. Elif was the only participant who developed a function understanding about the transformations, which is complicated even for preservice mathematics teachers (Yanik 2011).

In terms of preferences, Elif and Metin used virtual manipulatives because they found them easy to produce proper solutions (Haistings 2009; Izydorczak 2003). Defne and Selim did not prefer to use them for different reasons. Defne was under the influence of her attitudes towards computers, while Selim did not trust the operations he did with the manipulatives. Another factor to consider was that spatial ability plays a role in effective use of virtual manipulatives in geometric transformations. Students who had good spatial ability (Elif and Metin) used them quickly and properly whereas students who had insufficient spatial ability (Defne and Selim) had some difficulties in using them.

This study was an attempt to determine the role of virtual manipulatives in high school students' mathematical understanding processes. Although the Pirie-Kieren model and representation theory give the opportunity to trace the development of mathematical understanding in detail, there is still a need for new research analyzing virtual manipulative usage in mathematical understanding processes at the high school level.

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Part III
Tools for Research

Chapter 11
Geometry and iPads in Primary Schools: Does Their Usefulness Extend Beyond Tracing an Oblong?

Kevin Larkin

Abstract Although research into the use of mathematics apps in classrooms is becoming more common, robust research into Geometry apps is still in its infancy. Such research is particularly necessary in the case of Geometry apps where accurate and dynamic representations are critical in enhancing mathematical learning. This chapter begins to address the lack of research in this domain and presents findings from a qualitative and quantitative analysis of 53 Geometry apps initially selected from a broader range of apps available at the iTunes App Store. These findings indicate that the majority of the 53 apps were limited in their ability to assist students in developing Geometrical conceptual understanding. While this is of concern to educators there are, however, a small number of Geometry apps which would be most useful in teaching Geometry to primary aged students.

11.1 Introduction

This chapter synthesizes the research literature concerning the use of virtual manipulatives in mathematics education and then outlines a four-step methodology for evaluating the appropriateness of Geometry apps. Research such as this is needed as there has been little to no specific research into their usefulness in developing Geometry concepts. In addition, where research has been conducted into mathematical apps, with a few exceptions (Larkin 2013, 2014, 2015a, b; Moyer-Packenham et al. 2015), such research has largely been descriptive in nature. Findings of this research indicate that, although the majority of the iPad Geometry apps utilized external representations, most were limited in assisting students in developing deepened conceptual understanding of primary-level Geometry concepts.

For the purpose of this chapter, Geometry apps are those that include content applicable to primary schooling (5–12 year olds) including 1D lines, 2D shapes, 3D

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objects, transformations, co-ordinate geometry, angles and symmetry. Determining the quality of an app is difficult not only due to a lack of current research, but also because the information that is available at the iTunes App Store is supplied by the app developer and largely serves as an infomercial. The problem of determining quality in relation to Geometry is compounded by the fact that Geometry apps are much more likely to require accurate external representations. Consequently a new methodology for evaluating the usefulness of Geometry apps was designed.

The focus of this chapter is an explanation of how the constructs of pedagogical, mathematical and cognitive fidelity (Dick 2008), the Haugland (1999) developmental scale, and a modified version of Bos' (2009) software game format were used to evaluate 53 Geometry apps. The goals of this chapter are two-fold. The first goal is to articulate a methodology for reviewing the apps such that other teachers or researchers can use the methodology to review Geometry apps as they become available. The second goal is the creation of a web-based database of Geometry apps, categorized according to how well they promote conceptual understanding in Geometry. This research recognizes that the choice and use of Geometry apps needs to be based on a deep understanding of the pedagogical, mathematical and cognitive strengths and weaknesses of the apps.

11.2 Literature Review

Research into the use of concrete manipulatives in mathematics is extensive and only indicative research is included below. Carbonneau et al. (2013) synthesize the findings of decades of research in suggesting that concrete manipulatives support the development of abstract reasoning, stimulate the real-world knowledge of learners, provide opportunities for enactment of concepts, and encourage learner-driven exploration of such concepts. Burns and Hamm (2011) indicate that students engaged in extensive use of concrete manipulatives at the early elementary levels of schooling consistently outperform students with limited to no access to such materials. Suh and Moyer (2007) argue that

> the use of manipulatives allows students to make the important linkages between conceptual and procedural knowledge, to recognize relationships among different areas of mathematics, to see mathematics as an integrated whole, to explore problems using physical models, and to relate procedures in an equivalent representation. (p. 22)

A contribution to the literature from this chapter is determining whether or not this is the case with iPad-based Geometry manipulatives. As Geometry apps rely heavily on virtual representations, it is informative to examine research related to computer-based manipulatives as touch-screen devices are likely to replicate many of the features of computer-based manipulatives in relation to external representations and physical interactions (Manches and O'Malley 2012).

11.2.1 Definitions and Findings Concerning Virtual Manipulatives

Moyer et al. (2002) define a virtual manipulative as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). Given the interactivity of virtual manipulatives, students can mimic actions applied when manipulating concrete geometric materials and flip, slide or rotate the visual representations as if they were actual 3D objects (Rosen and Hoffman 2009). Representations such as these can also be linked to symbolic notations so that concept development can be enhanced. Moyer-Packenham and Westenskow (2013) suggest that virtual manipulatives are designed to "connect pictorial representations, actions performed on them, and symbolic representations, to highlight mathematical concepts and focus the attention of the learner on the mathematics to be learned" (p. 37). For instance, they can link different forms of representation, such as symbolic, pictorial and concrete (e.g., a diagram depicting the area of a rectangle along with the formula A = L*W), or link different representational models to each other (e.g., a set model to a region model both representing ¼).

Research conducted into the use of computer-based virtual manipulatives confirms many of the positive outcomes of using concrete manipulatives. For example, Clements and Battista (1992) found that student ideas about shapes were more precise and mathematically robust after using the computer-based Logo software. Studies where virtual manipulatives were used showed positive gains in students' conceptual understanding (Reimer and Moyer 2005). Highfield and Mulligan (2007) confirmed that virtual manipulatives and dynamic interactive software were powerful mathematical tools in aiding student concept development. Moyer-Packenham and Westenskow (2013) found that virtual manipulatives have a moderate effect on student achievement (when compared against other instructional treatments) and suggest that virtual manipulatives "have unique embodiments that have positive impacts on student achievement in mathematics" (p. 46). Özel (2012) reports on some of the affective effects of the use of virtual manipulatives and notes that immediate feedback enhanced student self-efficacy.

In contrast, a number of researchers have cautioned against considering virtual manipulatives as a panacea for the much publicized woes of mathematics education. One set of concerns relates to the technological aspects of virtual manipulatives. Chang et al. (2013) suggest that the computer skills required to use virtual manipulatives can be problematic, particularly for younger students who may require significant teacher scaffolding. In addition, the use of virtual manipulatives can be distracting to some students as activities not necessarily related to mathematics are only a click away. Perhaps of greater concern is the mathematics underpinning some of the virtual manipulatives, as it cannot be automatically assumed that the use of virtual manipulatives will bring about mathematical understanding. Uribe-Flórez and Wilkins (2010) remind us that the value of virtual manipulatives lies in their ability to promote the quality of student thinking and in the extent to which their external representations assist students to generate mathematical abstractions.

These may be limited for some students who are deprived of the tactile experience of concrete manipulatives and who may thus not develop conceptual understanding as richly as might be possible with concrete materials (Chang et al. 2013). Moyer-Packenham and Westenskow's (2013) meta-analysis results also show that while virtual manipulatives have a moderate effect overall in student achievement, these effects are inconsistent across student age levels and mathematics content being taught. This suggests that a large range of contextual features need to be considered before using virtual manipulatives—for example, prior experience with computers, age, and content versus concept development. This point is supported by Uribe-Flórez and Wilkins (2010) who noted that "how teachers design their classroom activities involving manipulatives will ultimately affect the success of their use on student understanding" (p. 364). Regardless of past findings concerning the use of virtual manipulatives, it is clear that further research is required, particularly with the increasing availability of the iPad as a tool for mathematics education. The following section of the literature review concludes with a discussion on three aspects of fidelity in relation to apps, namely, pedagogical, mathematical, and cognitive fidelity (Dick 2008) and outlines how they were incorporated into a methodology for evaluating apps.

11.2.2 Pedagogical, Mathematical and Cognitive Fidelity

Pedagogical fidelity is defined by Dick (2008) as the degree to which a student can use a tool to further their learning. Zbiek et al. (2007) suggest that pedagogical fidelity also refers to "the extent to which teachers (as well as students) believe that a tool allows students to act mathematically in ways that correspond to the nature of mathematical learning that underlies a teacher's practice" (p. 1187). Dick suggests that a pedagogically faithful tool will likely be described by students in terms of how it allowed them to interact with mathematics (e.g., "I created this triangle" etc.) rather than simply as a description of procedures for use (e.g. "I set the preferences to fast" etc.). Therefore, to be an effective pedagogical tool, an app must support any action by the student that will lead to conceptual understanding of the underpinning mathematical principle.

The second of the three fidelities used to evaluate the apps is mathematical fidelity. Zbiek et al. (2007) define this as the "faithfulness of the tool in reflecting the mathematical properties, conventions, and behaviors (as would be understood or expected by the mathematical community)" (p. 1173). Thus, mathematical fidelity is present when the activity of a student is "believable, is concrete, and relates to how mathematics is a functional part of life" (Bos 2011, p. 171) and when they add strength to an understanding of mathematics as a language of patterns and order. Dick (2008) cautions, however, that the current drive for user friendliness can sometimes run contrary to mathematical fidelity. This is particularly worrisome in relation to apps as many apps are designed by non-educators for financial profit. Keeping the notion of mathematical fidelity at the forefront of decisions when selecting apps reminds teachers to avoid apps that do not deliver accuracy in terms of mathematical content or constructs, for example, incorrect scaling in transformations.

The third of the elements in evaluating apps is cognitive fidelity, which refers to "the faithfulness of the tool in reflecting the learner's thought processes or strategic choices while engaged in mathematical activity" (Zbiek et al. 2007, p. 1173). Cognitive fidelity can be viewed largely in terms of the external representations provided by the tool and Zbiek et al. argue that "if the external representations afforded by a cognitive tool are meant to provide a glimpse into the mental representations of the learner then the cognitive fidelity of the tool reflects the faithfulness of the match between the two" (p. 1176). This notion of cognitive fidelity is obviously very important in Geometry apps, which are likely to utilize many external representations. The virtual nature of app objects does allow for high degrees of cognitive fidelity; for example, 3D objects can be pulled apart and put back together, and in so doing, can reinforce the link between 3D objects and their 2D representations (i.e., nets).

11.2.3 Learnings from the Literature

It is clear from the literature that manipulatives play a special role in mathematical activity. Although external representations can never exactly represent students' internal mental representations, they are useful as "visible phenomena that can be shared and discussed with others (e.g., other learners or the teacher)" (Zbiek et al. 2007, p. 1173). However, despite positive findings, use of manipulatives by teachers is inconsistent. Reasons offered by some teachers for their lack of use include lack of time to invest in locating virtual resources, particularly those that promote mathematical understanding rather than just rote learning (Calder 2015), as well as a misunderstanding that manipulatives will, in themselves, do the teaching for them (Puchner et al. 2008). This may be because teachers tend to use manipulatives, including virtual manipulatives, in a procedural or declarative manner rather than using them to enhance conceptual development.

Although an understanding of the three types of fidelity can assist teachers in making decisions about whether or not to use apps, it is argued above that one problem for teachers is the lack of time to evaluate apps using the three fidelities (or indeed any other evaluative process). In addition, although it might be expected that some of the findings on the use of virtual manipulatives may reflect the experience of using mathematics apps, apart from a few exceptions (Larkin 2014, 2015a, b; Moyer-Packenham et al. 2015), rigorous quantitative research into Geometry apps is in its infancy and thus further research is required.

Two research questions guided this research:

1. Are the Geometry apps currently available at the iTunes store appropriate for enhancing the learning of Geometry in primary mathematics?
2. Is the methodology utilized in this research robust in terms of internal consistency and also in its “user friendliness” such that teachers and researchers can be confident in using it to evaluate new Geometry apps as they become available?

11.3 Methodology

This section will outline the process for initially finding the Geometry apps, explain how three quantitative measures were used to evaluate the apps, and discuss measures of internal coherence and inter-reliability that were deployed to maximise the accuracy of the evaluations. Teachers can also refer to the dataset generated by this research to assist them in selecting what the author considers as highly appropriate Geometry apps.

11.3.1 Locating the Apps

The evaluation process for this research commenced with a targeted search for Geometry apps at the iTunes App Store in October, 2014. The following search terms were used—Geometry Elementary Education, Geometry Junior Education, Geometry Primary Education, Symmetry Education and Transformations Education. Many of the same apps appeared in two or more of the searches. To generate a workable sample size, apps were excluded from the final review according to the following criteria.

- If both a free version and a paid version (these present as two different apps) were available, both versions were reviewed only if this were necessary to evaluate the app accurately
- Where there were a number of apps in a series, only one app was reviewed as the apps in a series share similar structural and pedagogical properties
- Whilst the author acknowledges that mathematics learning occurs via games (see Beavis et al. 2015), apps that were categorized by iTunes as Games, Entertainment or Lifestyle, rather than categorised as Education, were excluded from the sample
- Apps where mathematics was part of a bigger package of reading, writing, and spelling skills were excluded
- Apps that required additional costs for access or further online registration of students or teachers were excluded

Although a sample of the apps were also evaluated by other mathematics educators, the author was primarily responsible for generating the scores. The scores are based on the author's experience of primary school mathematics for the past 30 years and also on the findings of his doctoral research exploring the use of technology in primary school classrooms. The author extensively interacted with each app until a decision could be made about its quality. It is also acknowledged that these reviews are subjective and also that the reviews rapidly go out of date. The author is currently working with primary educators in a range of schools to correlate the review findings with the experience of classroom teachers who have used the apps. In addition, teachers have been invited, via communication through professional mathematics organisations, to contribute to the reviews so that the site remains current.

Scoring of the apps involved the use of a two-page score sheet (see Appendix). This scoresheet included a qualitative review of the apps, which was later transferred to a Google document available to teachers at the link provided later in this chapter. This qualitative review included year-level appropriateness, Australian Curriculum content covered, and a general comment regarding the usefulness of the app. The scoresheet also used a series of measures for scoring the apps: the Haugland (1999) development scale, Bos' (2009) six software formats, and Dick's (2008) three measures of fidelity. These three measures were used as they respectively evaluate the appropriateness of the apps for student use, their appropriateness as virtual manipulatives in general, and then more specifically their usefulness in developing mathematical understanding.

11.3.2 Haugland Scale—Background and Process

The Haugland software developmental scale (adapted for this research in Table 11.1) is a criterion-based tool used to evaluate the appropriateness of web-based applications and software for use by children (Haugland 1999; Haugland and Ruiz 2002).

The Haugland scale was not initially designed to evaluate mathematical apps. Consequently, two important modifications were made for this research. First, in order to analyze the data more thoroughly, the original 10 criteria were grouped into three sub-clusters (child-centered, design features, and learning features). Second, elaborations were added to the sub-indicators to emphasize the relationship of the apps to mathematics. In scoring the apps, each of the 10 criteria is worth one point and each app can thus score between 0 and 10. The scoring sheet includes a number of sub-indicators for each criterion. For apps to score a 1 for each criterion they must meet all relevant sub-indicators. If they meet 50 % or more of the indicators a score of 0.5 is recorded and if less than 50 % are met a score of 0 is recorded. For example, there are three sub-indicators in the Process Orientation criterion. If an app demonstrated all three indicators, a score of 1 was allocated; if two of the three indicators were demonstrated, a score of 0.5 was allocated; if one or none of the

Table 11.1 Adapted Haugland developmental software scale with clusters and elaborations

Cluster	Criteria	Criteria elaboration with links to mathematics
Child-centred (4 points possible)	Age appropriate	The mathematics concepts taught by the app reflect realistic expectations for the age children for which it was designed
	Child control	When using the app, children decide the flow and direction for the experience, not the device. They are navigators, determining where the experience will lead and learn the consequences of their choices
	Independence	While adults may need to assist children in loading the application, after this initial guidance and support, children operate the app with minimal adult supervision
	Non-violence	Violence in apps is of particular concern because children often initiate and control the violence. In addition, the app models appropriate societal values
Design of app (3 points possible)	Clear instructions	Verbal instructions are essential, since even children who are reading text-based instructions navigate with greater success if audio instructions are also provided. Directions are accompanied with visual prompts and/or a help option
	Technical features	The app is colorful with realistic uncluttered graphics, which enable children to focus on the learning objectives. Graphics are animated to help children attend. Whenever possible children control the animation, learning mathematics through *hands-on* experiences
	Real world model	The app provides children with concrete representations of objects found in meaningful and mathematically accurate situations or settings. The scale and color of the objects are realistic, not stereotypical
Learning app (3 points possible)	Expanding complexity	The app is an exciting world that is easy for children to enter and reflects children's current cognitive, physical, mathematical and language skills. When children use the application a logical, mathematical learning sequence emerges
	Process orientation	Intrinsic motivation; the desire to explore and experiment and discover mathematics motivates children as they use the app, not rewards. The joy of learning is the reward in using the app
	Transformations	Apps have the unique potential to give children opportunities to change objects and situations over and over and discover how different mathematical components impact their world

indicators were demonstrated, a score of 0 was allocated. A nominal rather than absolute level of scoring was used in this scale as there are differing numbers of indicators across the ten criteria.

11.3.3 Bos' Game Format—Modification for This Research and Scoring Criteria

It is important, in terms of student learning and student engagement, that teachers can efficiently and accurately make an accurate evaluation of the type (format) of app they are considering using. The work of Bos (2009) is adapted in this research to evaluate the format of Geometry apps. Bos categorized computer software into six formats: static tools, informationals, quizzes/tests, drill and practice games, virtual manipulatives (VM), and interactive maths objects (IMO). Bos' research suggested that the format greatly influences the level of fidelity present in the virtual resource. For example, static tools that generate results in symbolic or graphic representations are likely to inhibit deeper abstraction or generalizations, whereas VM, which engage students in mathematical activity, are likely to make abstract concepts more concrete and thus can be a stepping stone to a deepened conceptual understanding (Bos 2009). Table 11.2 presents a brief summary of the six formats and an indication of their purpose, strengths, and weaknesses in relation to Geometry apps. In terms of the evaluation in this research, apps which were static tools scored 1 point, informationals scored 3 points, quizzes/tests scored 4 points, drill and practice games scored 6 points, VM scored 8 points, and IMO scored 10 points.

11.3.4 Three Fidelities Score Sheet—Creation and Scoring Criteria

The final measure used in determining the quality of the Geometry apps is an evaluative tool created for this research (see Table 11.3), based on Dick's (2008) three fidelities. The three dimensions of pedagogical (including technological), mathematical and cognitive fidelity have been used by other researchers to determine the quality of mathematics manipulatives (e.g. Bos 2009; Zbiek et al. 2007). Bos (2009) went some way towards using the dimensions as a form of quantitative assessment by creating a table of the three fidelities and indicating what a *low*, *medium*, or *high* level of each dimension may look like in relation to computer software. What has not been done previously is the assigning of a numerical value to represent the degree, along a continuum, to which these three dimensions are present in software in general, let alone more specifically in Geometry apps.

In the modified schema an individual app could, for instance, score highly on mathematical fidelity yet poorly on cognitive and pedagogical fidelity. In order to make sophisticated quantitative comparisons, the nominal levels of *low*, *medium* and *high* have been replaced by a continuum ranging from 1 (no fidelity) to 10 (very high fidelity) for each of the three dimensions, resulting in a possible score of 3–30 for overall fidelity. In this manner, the observation that an individual app could score highly on mathematical fidelity yet poorly on cognitive fidelity can be

Table 11.2 Possible app formats and their strengths/weaknesses (adapted from Bos 2009)

Format of app	Purpose	Strengths	Weaknesses
Static tool (scientific calculator app)	Uses calculators or function machines to process inputs	Useful for generation and/or display of data in form of tables, charts, graphs etc.	Are discrete pieces of information and require conceptual understanding for sense making. Primarily descriptive rather than interpretive
Informational (E.G. basic geometry)	Used to convey technical and procedural information. Used for direct instruction	Can provide useful information for students. Clear, logical format	Provides facts but often lacks connectivity to other concepts. Limited to no conjectures or problem solving
Quizzes/tests (E.G. angle game)	Used to check for understanding through multiple-choice, short fill-in-the-blank, and true/false questions	Useful for checking procedural understanding and recall. More useful if error correction occurs	Focus on recall may not facilitate sense making. Focus is on correctness rather than process
Drill and practice games (E.G. Geometry 4 kids)	Used for practicing a skill and can be highly motivational for the competitive student	Motivational—students like to play games—useful for declarative knowledge	Often don't contribute to the understanding of a concept. Winning can be the aim with mathematics learning secondary
Virtual manipulatives (symmetry draw)	Used to demonstrate a conceptual understanding of a mathematical idea. Require detailed instructions and teacher monitoring	Very useful for encouraging modelling of mathematics. Can supplement concrete manipulative already in use	Often require a great deal of teacher assistance. May not always be accurate representations
Interactive maths objects (Geometry 2D Pad)	Uses multiple representations that are interactive and change with the given input. In this format, patterns can be observed and manipulated	Encourages the investigation of mathematics patterns which emerge intuitively	May not be easy to create maths objects for all mathematics concepts

represented numerically to gain a measure of how well or poorly each of the dimensions is represented. In brief, an app is considered low level (1–3) if it is generally static, is inaccurate mathematically, has limited directions, or fails to develop mathematical concepts. It is considered medium level (4–7) if more than one solution is possible, if conjectures are possible (but not testable), and transitions between different aspects of the app are possible but lack clarity. Finally, an app is considered high level (8–10) if it uses accurate representations that are easy to manipulate, transitions between app elements are logical and consistent, and multiple conjectures are possible and testable.

Table 11.3 Levels of fidelity in apps—adapted from Bos (2009)

Type of fidelity	Low level (1–3)	Medium level (4–7)	High level (8–10)
Pedagogical (including technological) The degree to which the App can be used to further student learning	App is difficult to work with. Accessing all aspects of the app is difficult. App is not appropriate for the mathematics concepts it uses. Transitions are inconsistent or illogical	Using App is not initially intuitive; but with practice becomes so. Mathematical activities presented are appropriate but could be developed without app. Transitions evident but only made via trial and error	Manipulation of App is intuitive and encourages user participation. Little or no training or instructions are required. Transitions are logical and aid sense making
Mathematical The degree to which the App reflects mathematical properties, conventions and behaviors	Mathematical concepts are underdeveloped or overly complex. Lack of patterns. Lack of connection to real world mathematics	Application of mathematics concepts unclear. Patterning is evident but lacks predictability or is unclear. Some connection to real world mathematics	Mathematics concepts developed are correct and age appropriate. Patterns are accurate and predictable. Clear connection with real world mathematics
Cognitive The degree to which the App assists the learner's thought processes while engaged in mathematical activity	No opportunities to explore or test conjectures. Static or inaccurate representations. Patterns do not connect with concept development	Limited opportunities to explore or test conjectures. Minor errors with representations but still make sense. Patterns connect in a limited way with concept development	App encourages exploration and testing of conjectures. Representations are accurate and easily manipulated. Patterns clearly aide concept development

Overall, using the three measures above, it is possible for an app to score from 0 to 10 on the Haugland scale; from 1 to 10 according to the game format; and from 1 to 10 on each of the three fidelities: resulting in a total score ranging from 4 to 50 for overall app quality.

11.3.5 Tests for Internal Coherence

In order to determine the reliability of the Haugland and fidelities scales, a Cronbach alpha—α was generated for each (see Table 11.4). A Cronbach alpha score is not appropriate for the Bos format scores. It is generally accepted that Cronbach alpha scores greater than 0.7 indicate a high degree of internal consistency (Muijs 2011).

Table 11.4 Cronbach alpha reliability scores for the two scales

Haugland Scale Reliability Scores

Case Processing Summary

		N	%
Cases	Valid	53	100.0
	Excluded[a]	0	.0
	Total	53	100.0

a. Listwise deletion based on all variables in the procedure.

Reliability Statistics

Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
.669	.676	3

Three Fidelities Reliability Scores

Case Processing Summary

		N	%
Cases	Valid	53	100.0
	Excluded[a]	0	.0
	Total	53	100.0

a. Listwise deletion based on all variables in the procedure.

Reliability Statistics

Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
.889	.893	3

Cronbach's alpha is concerned with the homogeneity of the items that make up the scale (i.e., how well the individual items consistently recognize the same level of quality). In this research, the alpha scores can be viewed in terms of the app's consistency of rating (be that high or low) across the three domains of each of the Haugland sub-clusters (child-centered, design features, and learning features) and the three fidelities (pedagogical, mathematical and cognitive). Although the Haugland scale's alpha score is slightly less than 0.7, previous research (Larkin 2015a, b) using the Haugland scale reported an alpha score of 0.768. It may be the case that the alpha score is lower in this research due to a smaller sample of apps ($N = 53$ vs. $N = 142$) and also due to this research clustering the 10 Haugland criteria, reported in the earlier research, into three sub-clusters. There is thus a high degree of confidence that the two scales are internally consistent and we can therefore be confident in their reliability to determine the quality of an app.

11.3.6 So What Does This Research Tell Us About Geometry Apps?

Initially finding apps, which might be appropriate, is not a simple process. Quality apps are difficult to locate due to the sheer number of apps (160,000 education apps at the iTunes store (148AppsBiz 2015) and this difficulty is compounded by naming mismatches or similar naming, the rapid turnover of apps at the store, and a very poor search engine. Teachers are extremely time poor and thus are likely to be guided by the description at the iTunes store. These are at best "infomercials" and may often provide misleading details about the app. For all these reasons, educationally robust reviews such as the one available here are critical if teachers are to be directed to find what amounts to a "needle in a haystack"—that is, an app that is appropriate for them to use. As the qualitative component of this research is largely self-explanatory, I include here only one example (see Table 11.5) of the qualitative information that is available to teachers regarding each of the 53 reviewed apps. Full reviews are available at http://tinyurl.com/Geometry-Apps.

11.4 Quantitative Analysis and Discussion

For ease of analysis, I have combined the findings and discussion into one overall section; however, each of the three measures is presented separately in sub-sections with an overall synthesis of the findings provided at the conclusion of the section.

Table 11.5 Example qualitative geometry app review

App name	Content	Yr. level	Generic features of the app
Montessori geometry	Shapes and objects	Years F-2	Do you remember ever wondering why you were studying geometry at school? Montessori Geometry was designed to ensure that your child will never have these doubts. Not only will this app make him/her realise that geometrical shapes are everywhere but it will also make him/her proud to be able to recognise and name them
Reviewer comments re overall quality of app: This app includes notes for parents/teachers explaining the philosophy and operation of the app. Glossary includes definitions beyond the early years at which it is targeted—e.g. curvelinear shapes. The app includes dedicated pages on various 2D shapes and a few 3D objects, sorting activities feature heavily. User is in control			 Montessori Geometry app

11.4.1 Process One—Haugland Scale Scores

Table 11.6 indicates the apps scoring 7 or more according to the Haugland scale; however, to indicate the quality of all 53 apps, overall mean scores have been included.

Figure 11.1 shows an example of one of the top scoring apps. The data indicate that the apps were strongest in the child-centred cluster (2.86/4) but weak in the other two clusters (design features 1.61/3; learning features 0.92/3) with an overall mean of 5.4/10. These are similar to the findings from earlier research on number and algebra apps (Larkin 2015a, b) which indicated that the apps were strongest in the child-centered cluster (2.96/4) but weak in the other two clusters (design features 1.35/3; learning features 0.69/3) with an overall mean of 5.01.

Further comparisons between these data and the previous data indicate that Geometry apps scored lower overall in the child-centered cluster (2.86–2.96) and higher overall in both the design features cluster (1.61–1.35) and the learning features cluster (0.92–0.69). This likely reflects the fact that the increasing complexity of the Geometry apps, in terms of external representations and the use of symbolic language, makes them less child-centered; however, the trade-off is that more consideration has gone into improving the overall design of the apps with a subsequent, marked increase in their potential to support learning.

As can be seen in Fig. 11.2, the spread of scores indicates that there is a large range of quality with roughly half of the apps scoring 5 or less. This is a disappointing result given these are apps advertised in the iTunes store as being both educational and recommended for children of primary school age. As a

Table 11.6 List of apps scoring 7 or more out of 10 on the Haugland scale

Clusters on Haugland scale	Child/4	Design/3	Learning/3	Total/10
Attribute blocks	4	2.5	2.5	**9**
Shapes (Myblee)	4	2.5	2.5	**9**
Coordinate geometry (Ventura)	4	2	2	**8**
Shapes—3D geometry	3.5	2	2.5	**8**
Shapes and colors	4	2	2	**8**
Pattern shapes	4	2	2	**8**
Montessori geometry	4	3	1	**8**
GeoEng (Patterns)	4	3	1	**8**
Jungle geometry	4	3	1	**8**
Sym shuffle	4	2.5	1.5	**8**
Isometry manipulative	3.5	2	2	**7.5**
Geoboard (Math Learning Centre)	4	2	1.5	**7.5**
Numberkiz geo	4	1.5	2	**7.5**
Geometry 4 kids	3.5	2.5	1	**7**
Symmetry draw	3	1.5	2.5	**7**
Overall Mean for 53 apps	**2.86**	**1.61**	**0.92**	**5.4**

Fig. 11.1 Hands on attribute blocks—a top scoring app on Haugland scale

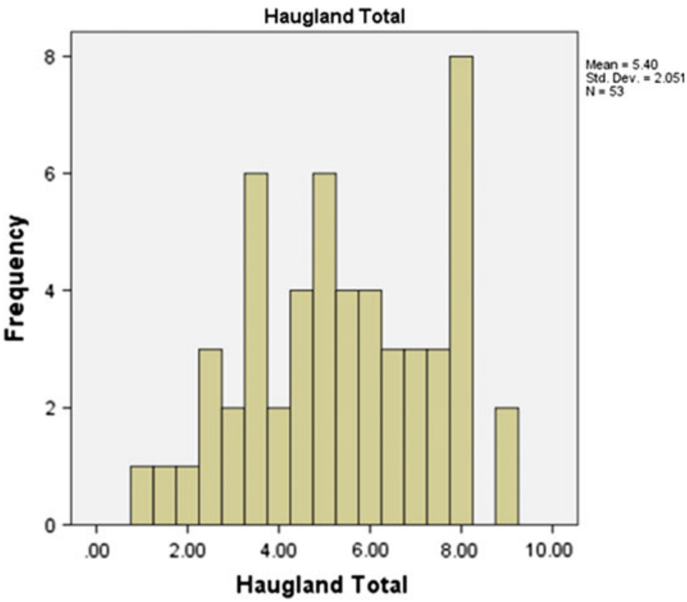

Fig. 11.2 Haugland total scores for 53 apps reviewed

consequence of these Haugland scale results, the research decision was made that any app scoring less than 50 % on the Haugland Scale is not appropriate to use in primary classrooms, regardless of whether they scored highly in terms of game format or the three fidelities. Of the 53 apps reviewed, 20 apps scored less than 50 % on the Haugland scale, and are therefore considered inappropriate for classroom use. This has implications for the potential use of one of the apps, Geometry 2D pad, which scored exceptionally well in terms of its game format (IMO) and in relation to its mathematics fidelity, but is excluded from the overall list of recommended apps as students are unlikely to be able to engage with the content it provides. Although a score below 50 % renders the app inappropriate, a score of over 50 % is a necessary, but not sufficient, condition for it to be automatically regarded as developing mathematical knowledge. Consequently, two further quantitative measures are required to determine whether or not apps are appropriate.

11.4.2 Process Two—Modified Bos Format Scores

The results from the second of the qualitative measures, the modified Bos format scale, also reflected the poor quality of most of the apps overall. Just over half of the apps (28/53) scored more than 50 % and only 17 of the apps (see Table 11.7) scored either a 10, indicating that they were IMO (two apps), or an 8, indicating they were VM (15 apps).

There were a further 11 apps which scored a 6 (drill and practice) with the resultant diminished value in terms of their usefulness. In addition, only 3 of these 11 (GeoEng, Jungle Geometry and Geometry for Kids—see Fig. 11.3) scored above 30/50 for the total score overall (see Table 11.11). This is an indication that, although many of the drill and practice games scored well on both the Haugland scale and Bos' game format, they generally scored poorly on the three measures of pedagogical, mathematical or cognitive fidelity. Consequently, the eight drill and

Table 11.7 Apps evaluated as IMO or VM

Name of app	Score	Name of app	Score
Coordinate geometry	10	Numberkiz geo	8
Geometry 2D pad	10	Symmetry draw	8
Attribute blocks	8	Transformations (investigate)	8
Shapes—3D geometry	8	Geometry—explore math	8
Shapes and colors	8	Simitri	8
Pattern shapes	8	Hands-on maths geoboard	8
Montessori geometry	8	Drawing the math	8
Isometry manipulative	8	Transformation trainer	8
Geoboard (Math Learning)	8	**Overall mean for 53 apps**	7.31

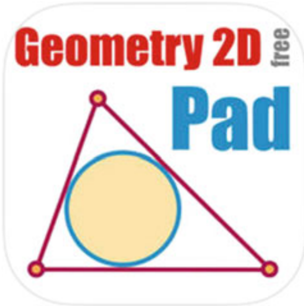

Fig. 11.3 Geometry 2D pad—a top scoring app on the Haugland scale

practice apps scoring less than 30/50 overall are not recommended for classroom use, except perhaps for review purposes, once conceptual and procedural knowledge has been well established.

It is the case that the game format categories of VM and IMO are inflating the overall score of a number of the apps. Many of the apps deemed to be VM as per Bos' (2009) definition are only minimally manipulative (i.e., only one component of the app), or are manipulatives in a way that is not likely to be conducive to student learning (e.g., rotating a shape by pushing an icon with a circular arrow on it). For example, the apps Drawing the Math and Transformation Trainer both were assessed as VM; however, they both scored unfavorably on the Haugland scale (4/10 and 5/10 respectively) and so are considered inappropriate, or only barely appropriate, for young students. This is a limitation in both Bos' categorization (VM are considered to be of medium-high fidelity) and subsequently a limitation in this research, as they have been allocated a score of 8 out of 10 in keeping with Bos' original schema of medium-high fidelity. What is needed in future research, using the game format schema, is a mechanism for identifying the degree to which an app is a VM. In this way, apps with limited opportunities for manipulation (e.g., Drawing the Math), or an app where manipulation is possible but not supportive of conceptual development due to an imprecise link between manipulation and conceptual development (e.g., Transformation Trainer), are not automatically considered as medium-high fidelity in Bos' schema or as scoring an 8 in my adaption of this schema.

11.4.3 Process Three—Three Fidelities Scores

Discussed in the following sections are (a) findings based on the levels of app quality according to each of the three fidelities, (b) an analysis of the spread of scores across the three fidelities, and finally, (c) an indication of seven apps which scored above 6/10 for each of the three fidelities indicating a high level of appropriateness. However, in order to contextualize the use of the three fidelity measures in relation to Australian content, it is worthwhile to present data on how well the apps correlated their content with the expectations of the Australian Curriculum (which largely reflects similar US and UK mathematics content). Table 11.8 indicates the number of apps that incorporated elements of the Australian Curriculum: Mathematics content.

A number of apps (e.g., Simitri) focused solely on one content area; however, many others covered content from two or more areas (e.g., EZ Geometry or Jungle Geometry). This is not always an advantage as broad coverage often meant shallow conceptual development and less classroom usefulness since only one section of the app was appropriate for any particular level. Shape content was very common as many of the apps were targeted at foundation and early years students (5–8 years old). Unfortunately, many of these "shapes apps" were very basic and only included naming of the shapes or very simple matching exercises. Reflections were the most common of the four major transformations presented in apps, perhaps because reflections are more easily represented than either rotational symmetry or translations. Angles and 1D Geometry apps were common; however, this is a result of a large number of quiz apps (largely concerning geometric reasoning) rather than the presence of apps that develop conceptual understanding of angles or 1D Geometry.

Table 11.9 provides a breakdown of the number of apps scoring 6 or more in each of the three fidelities. Although this looks like a healthy number of apps (42) scoring at least one 6, this is not the case, as many of the better apps scored a 6 or more in two or three categories and these apps are counted more than once.

Overall, 26 of the 53 apps failed to score a 6 in any category; the average score of the 53 apps was 12.9/30; and none of the three fidelity categories scored above 50 % overall. As was the case with the Haugland scale scores, these low scores are

Table 11.8 Number of apps providing different types of Australian curriculum content#

Sub-strand/concepts	No. of apps	Sub-strand | concepts	No. of apps
Lines (1D)	16	Slide (translate)	10
Shapes (2D)	31	Flip (reflect)	21
Objects (3D)	17	Turn (rotate)	16
Angles	15	Dilations	6

Note Total app count exceeds 53 as a number of apps include more than one type of content and are therefore counted more than once. # Pythagoras and trigonometry is only introduced in Australian secondary schools and so was beyond the scope of this review

Table 11.9 Number of apps scoring 6 or more in respective fidelities

Type of fidelity	Number of apps (n = 53)	Percentage (to nearest 0.1)	Average score/10
Pedagogical	21	39.6 %	4.9
Mathematical	13	24.5 %	4.3
Cognitive	8	15.1 %	3.7
Overall Average Score for apps on the three measures/30			12.9

a further indication that there are a large number of Geometry apps, categorized as educational in the iTunes store, which do not meet even a very low benchmark for classroom appropriateness. Figure 11.4 provides a visual summary of the scores of the apps on the fidelity subtotal (i.e., combined pedagogical, mathematical, and cognitive fidelity scores). As might have been anticipated, given than many apps are instructional and focus on declarative or procedural knowledge (Larkin 2014), the apps which were of some use tended to score well on the pedagogical fidelity dimension, less well in terms of the quality of the mathematics they contain, and generally poorly in their ability to assist cognitive development. This again mirrors the generally poor level of conceptual knowledge developed by apps reported in the earlier research.

The apps scored reasonably well in terms of pedagogical fidelity because this is the easiest of the categories for app designers (with likely low levels of mathematics education experience) to mimic. Many of the apps met one of the criteria, namely they were easy to use without instruction, and many of them partially met the criteria of appropriateness of activity without necessarily doing anything more than could be easily replicated with an IWB, physical manipulatives, or even pen and paper. Many of them incorporated multiple-choice quizzes, which may serve some use as review exercises. This is particularly the case where they draw from a large bank of questions, do not allow multiple guesses, or allow results to be emailed (e.g., Kids Math-Angle Geometry and Symmetry School Learning). Mathematical fidelity issues generally related to incorrect naming or classification of shapes and objects (e.g., diamonds instead of rhombuses, cubes not being considered as prisms

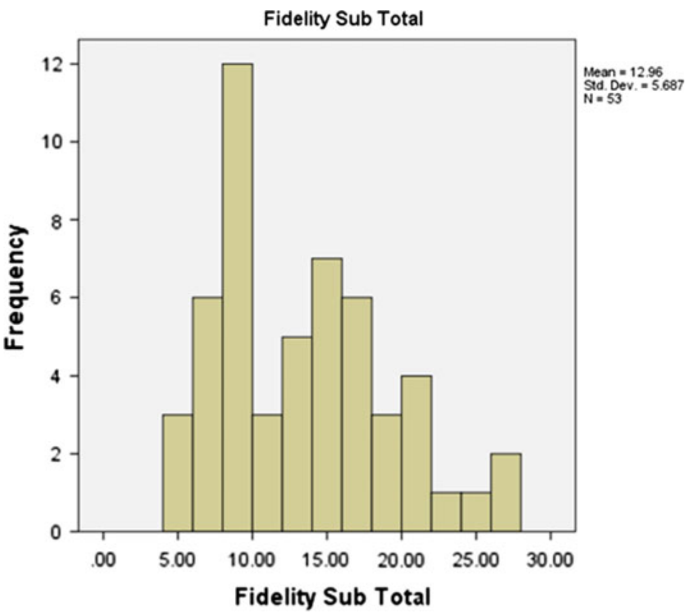

Fig. 11.4 Three fidelities subtotal

etc.); use of prototype orientations and shapes (only three apps focused on non-prototypical shapes—Cyberchase Quest, Maths Geometry, and Shapes MyBlee); and a lack of connection to any notion of real-world application of mathematics (minor exceptions to this include Geometry 4 Kids and Simitri).

Low cognitive fidelity is problematic in terms of classroom use as this relegates many of the apps to only being useful as review activities or for rote learning. The majority of apps did not meet the criteria for supporting cognitive development. Despite being technically capable, most apps only provided static representations and, where dynamic representations were used, they did not mimic the physical activity of turning or sliding or flipping but used arrows or numbers to direct the transformations (noteworthy exceptions were Squares and Colors and Shapes MyBlee). In addition, very few apps allowed opportunity for students to create patterns and develop their own conjectures regarding shapes, objects, angles or transformations. Although the technology present in the device allows for dynamic representations of shapes, objects and angles (e.g., Cyberquest and Isometry Manipulatives), by far the majority of the apps did not make use of this technology and consequently did not replicate the real-world experience of the geometry they were attempting to represent. This is a serious shortcoming in the ability of these apps to encourage Geometry conceptual development.

Despite the comments above, it is not all negative as there are some apps that perform well (see Tables 11.10 and 11.11 and Fig. 11.5).

Of the apps reviewed, seven of them (13 % of the total apps reviewed) scored 6 or more out of 10 for each of the three fidelities. These are clearly the apps that teachers should be utilizing in their classroom practice. What is interesting here is that apart from the top three, even the better apps were inconsistent in meeting the three fidelity standards as four of the seven scored at least one 6 with two of these four scoring two 6s. This level of inconsistency mirrors the findings of Moyer-Packenham et al. (2015) and Moyer-Packenham and Suh (2012) in relation to virtual manipulatives and can be seen in the wide range of scores even among the top half of the apps (see Table 11.11 and Fig. 11.6). In both of the research studies cited, the authors noted multiple affordances within each virtual manipulative such that one or more of these affordances may be more influential and beneficial for student learning. An example of this is the Isometry manipulative, where one

Table 11.10 Apps that scored 6 or more on each of the three fidelities

App name	Pedagogical	Mathematical	Cognitive	Total
Co-ordinate geometry	9	8	9	26
Transformations	9	8	9	26
Attribute blocks	8	8	8	24
Shapes—3D geometry	9	6	8	23
Shapes and colors	7	6	7	20
Pattern shapes	8	6	6	20
Isometry manipulative	7	6	6	19

Table 11.11 Ranked list of 21 apps scoring more than 50 % overall and 50 % on each of the Haugland scale, game format and three fidelities*

	App format/10	Haugland cluster/10				Fidelity/30				Total score/50
Application name		Child-centered	Design features	Maths learning	Haugland total/10	Pedagogical/10	Mathematical/10	Cognitive/10	Fidelity sub total/30	
Coordinate geometry (Ventura)	*10*	4	2	2	*8*	9	8	9	*26*	**44**
Attribute blocks	*8*	4	2.5	2.5	*9*	8	8	8	*24*	**41**
Transformations (Investigate)	*8*	2	1.5	3	*6.5*	9	8	9	*26*	**40.5**
Shapes—3D geometry	*8*	3.5	2	2.5	*8*	9	6	8	*23*	**39**
Shapes and colors	*8*	4	2	2	*8*	7	6	7	*20*	**36**
Pattern shapes	*8*	4	2	2	*8*	8	6	6	*20*	**36**
Geometry (Montessori)	*8*	4	3	1	*8*	9	6	5	*20*	**36**
Simitri	*8*	2	1.5	2.5	*6*	4	9	8	*21*	**35**
Isometry manipulative	*8*	3.5	2	2	*7.5*	7	6	6	*19*	**34.5**
GeoEng (Patterns)	*6*	4	3	1	*8*	8	6	5	*19*	**33**
Geoboard (Math Learning Centre)	*8*	4	2	1.5	*7.5*	7	5	5	*17*	**32.5**
Numberkiz geo	*8*	4	1.5	2	*7.5*	8	5	4	*17*	**32.5**
Jungle geometry	*6*	4	3	1	*8*	8	5	5	*18*	**32**
Shapes (MyBlee)	*6*	4	2.5	2.5	*9*	7	5	5	*17*	**32**
Geometry—explore math (Ventura)	*8*	3.5	2.5	0.5	*6.5*	6	6	4	*16*	**30.5**

(continued)

Table 11.11 (continued)

Application name	App format/10	Haugland cluster/10				Fidelity/30				Total score/50
		Child-centered	Design features	Maths learning	Haugland total/10	Pedagogical/10	Mathematical/10	Cognitive/10	Fidelity sub total/30	
Geometry 4 kids	*6*	3.5	2.5	1	*7*	8	6	3	*17*	**30**
Symmetry draw	*8*	3	1.5	2.5	*7*	6	4	5	*15*	**30**
Shape rotate	*6*	4	2	1	*7*	6	5	5	*16*	**29**
Sym shuffle	*6*	4	2.5	1.5	*8*	6	4	4	*14*	**28**
Symmetry School learning	*6*	4	1	1.5	*6.5*	5	5	5	*15*	**27.5**
Geometry ZD pad	*10*	1	1	2	*4*	2	7	4	*13*	**27**
Butterfly brunch	*6*	3	1	1	*5*	5	5	5	*15*	**26**
Shapes and geometry skill builders	*6*	2.5	2.5	1	*6*	1	4	3	*14*	**26**
Math geometry	*4*	3	2	1	*6*	6	5	4	*15*	**25**

* Geometry 2D Pad scored highly in the game format and in mathematics fidelity and scored 27 overall but is excluded due to a below 50 % score in the Haugland scale. Math geometry scored 25 overall but is excluded due to a below 50 % score in the Bos game format

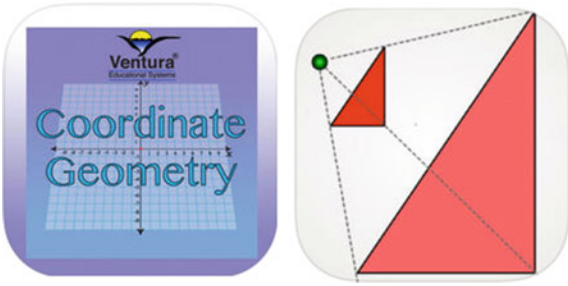

Fig. 11.5 Coordinate geometry and transformations—equal top scoring fidelity apps

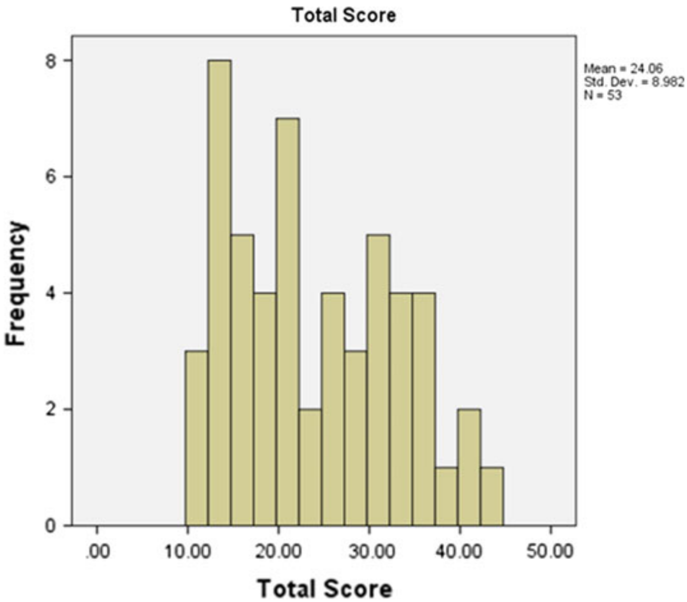

Fig. 11.6 Histogram of overall scores for 53 apps

component of the app is extremely beneficial while the second component is likely to undermine student learning. This inconsistency becomes more apparent the further down the list of scores you proceed. For example, Montessori Geometry (9, 6, 5) scored equal to or higher than three of the apps listed in the top seven but was relatively poor in terms of cognitive development. Three other apps scored highly in pedagogical and mathematical fidelity but poorly in terms of cognitive development (GeoEng—8, 6, 5; Geometry 4 Kids—8, 6, 3; and Geometry Explore—6, 6, 4). It is worth noting that only one app (Simitri—4, 9, 8) scored very poorly in pedagogical fidelity, but very highly in mathematics and cognitive fidelity. This indicates that this app, with correct scaffolding from the teacher, is potentially very useful for developing high-level mathematical and cognitive fidelity.

Table 11.11 provides further details to assist classroom teachers with the selection of appropriate apps. Each of the apps listed in Table 11.11 scored a "pass" mark of 50 % in terms of overall score and 50 % on each of the three quantitative measures. Heeding the earlier caution of Moyer-Packenham et al. (2015) and Moyer-Packenham and Suh (2012), these apps at least meet a benchmark of quality but need to be used thoughtfully by classroom teachers in their mathematics classrooms.

11.5 Limitations and Conclusion

A limitation of any research reviewing apps is an inherent consequence of the nature of the iTunes App Store. Firstly, the sheer number and method of labelling apps means that there may be useful Geometry apps not reviewed. Secondly, the iTunes store is a moveable feast as apps are generated, renamed, relocated, or removed on a daily basis. Therefore, it is not possible to claim that all quality Geometry apps have been critiqued. Furthermore, it is important for the continued currency of the reviews that other teachers and researchers add to the database of reviews.

However, within the constraints noted above, it is clearly the case that, other than the top three apps, teachers need to decide the exact instructional purpose for using the app and then look at the individual fidelity scores of the app to locate one that meets that specific purpose. In this manner, Montessori Geometry would be most appropriate to use for review purposes but not appropriate in terms of developing conceptual or mathematical fidelity.

This research has indicated that, although many Geometry apps are quite poor in terms of their fidelity, it is, to return to the question posed in the title, certainly not a futile exercise to use some of them in mathematics classrooms. Many of the apps do go beyond the rather cynical "tracing use" hinted at in the title of this chapter. The use of the Haugland scale provides an initial filter on the appropriateness of the apps for young students. In its current format, the Bos game format score provides limited information regarding quality, and is not accurate enough to be of much assistance. The key measure for teachers to use in gauging the mathematical quality of an app is the modified three fidelities scoring rubric created for this research, as apps that scored well in these measures also scored highly in the Haugland scale and game format and thus demonstrate great potential for enhancing student learning. It is hoped that this research will be useful for teachers when selecting apps to support mathematics learning. Future research will investigate the use of a selected number of quality Geometry apps in Australian and Canadian classrooms.

Acknowledgments This chapter develops further a paper presented at MERGA38, 2015. The author would like to sincerely thank Doctor Todd Milford (University of Victoria, Canada) for his assistance with the quantitative data analysis in this chapter.

Appendix—Scoring Sheet Used to Evaluate the 53 Apps

HAUGLAND DEVELOPMENTAL SOFTWARE SCALE (Adapted)

Title: **Review Date**:
Reviewer: **Recommended Age Range:**

Criteria	Total 0|0.5|1	Indicators
Age appropriate (CC)		Realistic concepts ☐ Appropriate methods ☐
Child in control (CC)		Actors not reactors ☐ Can return to start ☐ Easy navigation ☐ Trial and error ☐
Clear instructions (DF)		Icons purposes clear ☐ Simple, precise directions ☐ Verbal instructions ☐
Expanding complexity (ML)		Low entry, high ceiling ☐ Teaches powerful ideas ☐ Learning sequence is clear ☐
Independence (CC)		Adult supervision not needed after initial exposure ☐
Non-violence (CC)		Free of violent actions ☐ Models positive values ☐
Process orientation (ML)		Intrinsic motivation ☐ Process not product ☐ Discovery learning, not skill drilling ☐
Real-world model (DF)		Concrete representation ☐ Objects function ☐ Simple, reliable models ☐
Technical features (DF)		Animation ☐ Colorful ☐ Realistic graphics ☐ Operates consistently ☐ Realistic sound effects ☐ Saves work ☐
Transformations (ML)		Objects/situations change ☐ Process highlighter ☐
Learner centered *Design features* *Math/ learning* ***Total score***	/4 /3 /3 ***/10***	

Reviewer Comments:

Geometry Apps Scoring Sheet

App Name: **Date Reviewed:** **Nov 14, 014**

Mathematics Strand Content: **Year Level:**

App description from iTunes Store: This app

Reviewer summary of App: This app

App format (adapted from Bos, 2009) (Circle most relevant format)

Generates calculations **/1**	Informational **/3**	Quizzes / tests **/4**	**/10**
Drill & practice games **/6**	Virtual manipulative **/8**	Mathematics objects **/10**	

Themed Haugland Scale Score

Learner centered **/4**	Design features **/3**	Mathematical learning **/3**	**/10**

Pedagogical Fidelity (Circle appropriate score)

← → 1 2 3 4 5 6 7 8 9 10 11 12	**/12**

Mathematical Fidelity (Circle appropriate score)

← → 1 2 3 4 5 6 7 8 9 10 11 12	**/12**

Cognitive Fidelity (Circle appropriate score)

← → 1 2 3 4 5 6 7 8 9 10 11 12	**/12**
Overall Comment and Score: This app	**/50**

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Chapter 12
Selection of Apps for Teaching Difficult Mathematics Topics: An Instrument to Evaluate Touch-Screen Tablet and Smartphone Mathematics Apps

I.K. Namukasa, G. Gadanidis, V. Sarina, S. Scucuglia and K. Aryee

Abstract Manipulatives—including the more recent touch-screen mobile device apps—belong to a broader network of learning tools. As teachers continue to search for learning materials that aid children to think mathematically, they are faced with a challenge of how to select materials that meet the needs of students. The profusion of virtual learning tools available via the Internet magnifies this challenge. What criteria could teachers use when choosing useful manipulatives? In this chapter, we share an evaluation instrument for teachers to use to evaluate apps. The dimensions of the instrument include: (a) the nature of the curriculum addressed in the app—emergent, adaptable or prescriptive, and relevance to current, high quality curricula—high, medium, low; (b) degree of actions and interactions afforded by the app as a learning tool—constructive, manipulable, or instructive interface; (c) the level of interactivity and range of options offered to the user—multiple or mono, or high, moderate or low; and, (d) the quality of the design features and graphics in the app—rich, high quality or impoverished, poor quality. Using these dimensions, researchers rated the apps on a three-level scale: Levels I, II, and III. Few apps were classified as Level III apps on selected dimensions. This evaluation instrument guides teachers when selecting apps. As well, the evaluation instrument guides developers in going beyond apps that are overly prescriptive, that focus on quizzes, that are text based, and include only surface aspects of using multi-modality in learning, to apps that are more aligned with emergent curricula, that focus also on conceptual understanding, and that utilize multiple, interactive representations of mathematics concepts.

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Keywords Apps · Evaluation criteria · Integers · Learning tools · Mathematics thinking

12.1 Apps for Mathematics

Teachers continuously access learning materials that promise to assist children to think mathematically. On a lesson-to-lesson basis, teachers are faced with the challenge of how to select materials that best meet their teaching goals. The profusion of virtual learning tools available via the Internet magnifies the challenge of searching for materials. Moyer-Packenham et al. (2015) assert, "An important goal for mathematics education is the design and selection of mathematics 'apps'" (p. 42). Few studies provide educationally robust reviews on apps for mathematics (Larkin 2013, 2014, 2015a, b; Moyer-Packenham et al. 2015). Several books (e.g., Dickens and Churches 2012), web-based resources (e.g., common sense media—commonsensemedia.org, Children's technology review—childrenstech.com/), and articles in magazines offer lists of top apps and some reviews on selected apps. Reviews of apps on the app store or those Internet sources are largely based on information that advertises the apps (Larkin 2013). Few reviews are based on evaluation of the apps. For example, Larkin (2015a) shares a list of the top 20 apps (e.g., transformations), Larkin (2013) shares the top 40 Number Sense and Numeration apps (e.g., I see!! Math 1), and Larkin (no date) provides detailed reviews of 142 math apps at https://docs.google.com/file/d/0Bwd_RKnZbGDqSUtkOHZsTHdsWVE/edit. In this chapter, we share an instrument for assessing pedagogically useful apps.

Manipulatives—including the more recent touch-screen tablet/smartphone applications—belong to a broader network of learning tools. In this chapter, we refer to touch-screen tablets and smartphones as touch-screen mobile devices. The work of Namukasa et al. (2009) explore the complementary role of physical and Information Communication Technology (ICT)-based manipulatives, also referred to as virtual manipulatives. Virtual manipulatives are interactive and dynamic objects (Moyer et al. 2002). Virtual manipulatives can appear on computer screens, touch screens, holographic images, and a variety of technological environments. Apps are computer applications in which virtual manipulatives (and various end-user software) are delivered on touch-screen mobile devices. Several apps are touch-screen versions of computer and Internet-based applications. The choice of a manipulative—whether physical or virtual (i.e., a virtual manipulative on a computer, a digital board, or a touch-screen mobile device); historical or modern—is complex. It should depend on what is available, what fits the students' culture and expectations, as well as what fits the teacher's system of beliefs (Bartolini and Martignone 2014). Teachers' choices "to use virtual manipulatives in combination with physical manipulatives were influenced by familiarity with similar physical manipulatives" (Moyer-Packenham et al. 2008, p. 215). In addition, even among the same type of manipulatives, these "can be useful or useless depending on the quality of thinking they stimulate" among learners (Bartolini and Martignone 2014, p. 31). According to Hitt (2002), manipulatives are also classified by the specific meaning of a given concept they address (e.g., discrete, linear, or analogical). Educators and teachers need to pay attention to the specific representation categories (e.g., graphic, analytic, or symbolic mathematics) of a given concept that any manipulative—physical or virtual—addresses (Hitt 2002).

In the mathematics education research community, a thread of research focuses on the influence of virtual manipulatives in learning and teaching, on the design modes, and on the quality of these materials (Pepin and Gueudet 2014; Trouche et al. 2013). For a review of literature on the role of mathematics apps, see Calder (2015), Cayton-Hodges et al. (2015), Larkin (2013, 2015a), Moyer-Packenham et al. (2015), Moyer-Packenham and Westenskow (2013), Pelton and Pelton (2012), and Zhang et al. (2015). Some of this work focuses on specific apps: for example, Larkin (2013) focuses on apps for number sense and numeration, Larkin (2015a) on geometry apps, Moyer-Packenham et al. on apps for young children, Zhang et al. on multiplication and division apps.

Several articles (e.g., Peterson 1972; Skip 1990) and online forums (e.g., "negative × negative = positive" at MathForum.org) explore the use of physical, virtual, and visual strategies, among other strategies, for teaching meanings and operations of negative integers. This work builds on the long history of conversations on teaching more difficult concepts such as subtraction, fractions, and integers (e.g., Kamii et al. 2001). More recent conversations focus on how ICT-based technology (e.g., interactive whiteboard, and computer games) could be used to make difficult topics easier to learn.

12.2 Evaluation of Mathematics Apps

What evaluation criteria could teachers use when choosing the most appropriate teaching materials? The increase in the range of ICT-based materials for teaching, coupled with the emergence of a new culture of learning arising with these resources, is creating a need for quality, design, and diffusion criteria, and policies on these resources. Several studies (Calder 2015; Highfield and Goodwin 2013; Larkin 2015a, b; Pepin and Gueudet 2014; Trouche et al. 2013) voice the need for criteria for evaluating ICT-based resources. Pepin and Gueudet (2014) also maintain that the teacher, even in situations where he or she only selects the resources to use, is "a designer of his/her resources" (p. 133). Trouche et al. (2013) assert that new research and policy questions are arising: "Who designs and what do the design processes look like? How to access quality resources?" (p. 771). For Calder (2015), the question is: "What is the [major] motivation of app designers?" (p. 236). To others, the question is about the alignment between a mathematics app and mathematics curriculum for the target group. For example, Larkin (2014) examines the effectiveness of mathematics apps for the Australian curriculum.

A few studies focus on the evaluation of mathematics apps. Some studies utilize qualitative (e.g., Calder 2015; Larkin 2013, 2014, 2015b), and others quantitative, evaluation measures (Larkin 2014). Larkin (2015b) utilized two qualitative measures based on: whether the apps focused on conceptual (deep understanding related to the meaning of mathematics), procedural (following a set of sequential steps to solve a mathematics problem), or declarative (information retrieved from memory without hesitation) knowledge; and their relevance to the Australian curriculum. Of the 142 he fully reviewed, he observed that many of them "were little more than digital flash cards encouraging rote learning." Of the 40 worthwhile apps he evaluated, only 3 apps (Mathemagica, Areas of Rectangles, Maths Galaxy Fun) were exceptional; a majority of apps emphasized declarative or procedural knowledge; only 40 of the 142 apps were "worthwhile mathematical apps to support mathematics learning in primary classrooms" (p. 30); and only 12 apps involved conceptual knowledge. Several of the apps he reviewed were characterized by mismatches: between the mathematics terms in the app name and the mathematics content explored by the apps, between the description of the nature of knowledge (e.g., conceptual understanding) addressed in the app and the actual knowledge explored in the app, between targeted age levels and age levels at which the content of the app is taught in schools, and between the price of an app and the quality of an app.

Among the apps he reviewed, the Number Sense and Numeration strands were dominant. Goodwin and Highfield (2013) found that apps for toddlers, as well as science and literacy apps, dominated their top 10 apps category. Calder (2015), Larkin (2014), and Moyer-Packenham et al. (2015) noted that a variety of educational apps are available for elementary lessons. A majority of the educational apps available are, nonetheless, standalone apps, focusing on one specific content area, and many are drill and practice, only useful for rote learning of declarative and procedural knowledge (Larkin 2013, 2015b). Moyer-Packenham and Westenskow (2013) note the need for research on manipulatives with students beyond Grade 6.

Larkin (2015b) used three quantitative measures in his app evaluations: The Haugland developmental software scale (Haugland 1999); productive pedagogies (Mills et al. 2009); and Learning principles of good games (Gee 2005). The Haugland developmental software scale is based on criteria for evaluating software for young children. It consists of three dimensions: a dimension on the child (e.g., age appropriate, child control, and non-violence), on design (e.g., clear instructions, and technical features), and on learning (expanding complexity, and transformations). Larkin adopted three of the four dimensions of the productive pedagogies identified by Queensland Education (Mills et al. 2009): intellectual quality (e.g., deep understanding, and substantive conversation), supportive classroom environment (e.g., student direction, and academic engagement), and connectedness (e.g., knowledge integration, and background knowledge). The third scale is based on learning principles (e.g., active, interaction, production, customization, agency, challenge and consolidation, critical learning, probing, multiple routes, and transfer) of good video games developed by Gee (2005). Larkin's evaluation scales range from three to ten. Fullan and Donnelly (2015) offer a scale with four ratings for evaluating digital innovations: good, mixed, problematic, and off track. They identify three dimensions including pedagogy, system change (e.g., implementation support, value for money, and potential to diffuse widely), and technology. These studies show the need for instruments for evaluating apps, especially instruments that emerge from studying apps.

Bos (2009b) offers an instrument for determining the degree of fidelity on a three-point scale—low, medium, and high. Bos (2009a), Larkin (2015a), and Moyer-Packenham et al. (2008) study the fidelity—pedagogical, mathematical, and cognitive fidelity—of technology-based learning tools. Bos (2009a, b) builds on the work of Dick (2008) to further elaborate dimensions and degrees of fidelity. To her, mathematical fidelity of a mathematics tool is the tool's degree of conformity to mathematical properties, rules, and conventions of the mathematical content. A tool "should reflect accurately the mathematical characteristics and behavior that the idealized object should have" (Dick, p. 335). Mathematical fidelity is about mathematical accuracy and precision. Cognitive fidelity is about the ability of the tool to lead to learner actions, interactions, and thoughts that embody mathematics concepts or processes, and, potentially, to deeper mathematics actions, interactions, and thoughts. Pedagogical fidelity is about the elements in the tool, such as target-group appropriateness of the content and type of learning activities, that enable students to learn. Pedagogical fidelity is "evidenced… in the organization of the user interface of a technological tool" (p. 334), in features that support valued learning activities and features helpful for learners (Zbiek et al. 2007).

Larkin (2015a) reviewed 53 Geometry apps, evaluating them against the criteria on fidelity, classifying the apps as low-, medium-, or high-fidelity apps in each dimension. He found the apps to score high on pedagogical fidelity and low on cognitive fidelity. Seven (e.g., Coordinate Geometry, Transformations) of the 53 apps scored high on the three fidelities (cognitive, mathematical, pedagogical), and only the top three of these scored consistently high on all three fidelities. Calder (2015) checks to see if a mathematics learning app is appropriate in intended learning and age of users (an aspect of pedagogical fidelity), is applicable to the concepts involved, to enhancing mathematical engagement and thinking (aspects of mathematical fidelity), and whether an app utilizes "visual, sound and movement elements that learners might also find highly engaging" and appealing (an aspect of technical design features) (pp. 243–244).

12.3 Design Features of Mathematics Apps

Major design features identified in the literature on design of learning apps fall under the categories: nature of the app, content, instrumental/interface design, cognitive/intellectual, sociological, and ergonomic aspects (Gadanidis et al. 2004; Sedig et al. 2014). Human computer interactions (HCI) researchers, for instance, argue that well-designed digital tools (also referred to as visualizations or interfaces in HCI literature) are those designed with a deep understanding of cognition. They maintain that the levels of interaction afforded by digital tools vary from those involving minimal cognitive activities to those that involve higher cognitive skills. The levels of interaction afforded also vary from those evoking only physical (touch, feel, see, etc.) actions such as dragging, to interactions such as comparing, to tasks such as identifying and categorizing, and, further, to activities such as problem solving and reasoning. Several key characteristics offered by the digital tools influence higher-order cognitive activities: the range and adjustability of options—the flexibility; number and diversity of interactions; fitness of the interface to the task, to the user, and to the context; and type of transactions ranging from access only, to annotation, modification, construction, and combination of transactions (Sedig et al. 2014).

12.3.1 Digital Learning Objects and Tools

This inquiry on mathematics apps is situated within a larger framework of digital learning tools (LTs) and objects. Gadanidis and Schindler (2006) point out that the term digital learning objects (LOs) involves a variety of designs, from simple digital images or files in pdf format to complex simulations and interactive interfaces. LOs are small interactive programs that are available online and are focused on specific content topics (Gadanidis and Schindler, p. 20). Virtual manipulatives can evolve into mathematical objects (including concepts, procedures, and processes) "when acted upon," patterns perceived, and a new mathematics object emerges to deepen mathematical understanding (Bos 2009b, p. 526). Zbiek et al. (2007) use the term cognitive tools (CTs) to refer to technologies that extend the learning and thinking activities. CTs for mathematics allow the user to act on, compute and externally represent mathematical entities, and involve a variety of designs including simulation, software, micro-world, devices and tool kits. Bos (2011) uses the term *interactive mathematical objects* to refer to the digital learning tools. The tools with a high degree of fidelity enable manipulation in an intuitive way, encourage active participation of the learner, are appropriate for the age level, are mathematically correct, "provide opportunity to construct, test, and revise to understand the patterns and structure the concepts. Manipulating the patterns leads to great depth of understanding" (p. 526).

Maddux et al. (2001) identify two different types of LOs. In *Type I*, the developer determines almost everything that happens on the screen, it affords only "passive user involvement", "a limited repertoire of acceptable responses", "usually aimed at rote memory" and everything that the software is capable of doing can be observed in about 10 min or less (Gadanidis and Schindler 2006, p. 23). In Type II, the user is in charge of what happens on the screen, it affords "active intellectual involvement," the user is in charge of what happens, it is usually aimed at "creative tasks," and many hours are necessary to exhaust what the program is capable of (p. 23). Type II affords a high number of user possible inputs and a high level of interactivity between the user and object. Gadanidis and Schindler recommend LOs involving a hybrid of Type I and Type II. Godwin and Highfield (2013) refer to

Type II as constructive interfaces, with Type I as instructive, and with the manipulable interfaces lying in between. Gadanidis et al. (2004) argue “mathematical investigation, as a pedagogic tool, is not a simple undertaking. Facilitating investigations [by the learners] adds significantly to the complexity of instructional design” (p. 294). According to these researchers:

> Good design becomes possible when mathematics education and human–computer interaction design experts work together, rather than in isolation, taking into account pedagogical goals and interface design principles, and, of course, where there is commitment to test and revise based on feedback from educators in the field. (p. 295)

Bortolossi (2012) observes that factors such as the nature of the mathematical content (mathematical fidelity), pedagogical design (pedagogical fidelity), graphic design, and interface design (technical design features) are fundamental aspects in the production of educational applications. Bortolossi recommends a combination of the best features of several ICT applications to enable, in a rapid-development environment, the creation of low-cost (but richly designed) portable, dynamic, and interactive LTs with a potential for multiple didactic activities. To Fullan and Donnelly (2015), it is important to also evaluate the “underlying digital product model design” (p. 40) along the lines of ease to use, intuitive design, how data are managed, and what experiences it offers the end users.

Commonalities exist among criteria for designing high-quality apps and those for evaluating apps for learning mathematics. We, nonetheless, agree with Larkin (2013) that design criteria for apps may not directly translate to criteria for evaluating high-quality apps for learning mathematics, and with Dick (2008), that design features of learning apps should be selected to serve pedagogical, mathematical, and cognitive principles. Further, Calder (2015) adds that it helps when the motivation of the mathematics app developer is mathematical engagement, rather than profit optimization. On the question raised by Trouche et al. (2013) regarding who designs and what the design processes look like, from our interactions with the app developers on the project, it appears that some app developers are themselves teachers, educators, and educational researchers whose major motivation is pedagogic, or consult, partner with, and seek feedback (or, even, endorsement) on their products from other teachers, educators, and educational researchers. Many of these apps score lowest on cognitive and mathematics fidelity (Larkin 2015a) but higher on pedagogical fidelity. Selected iTunes apps such as Rekenrek by Mathies, Touch Counts by N. Sinclair (an app for Number Sense and Numeration for young children), and MathTappers apps by T. Pelton and Pelton are designed by mathematics educators. Pelton and Pelton (2012) explore the pedagogical practices in the MathTappers apps, some of which support concept development and consolidation of understanding, and others are for fluency building. Larkin (2015a) observes that most educational apps are designed by non-educators and for market reasons. Various publications exist on development and marketing of apps. More work is needed on the design features that influence the usefulness of apps and on how students use the apps.

Trouche et al. (2013) shares a questionnaire with nine different dimensions to measure the usefulness of any Dynamic Geometry Software (DGS), including mathematical content, pedagogical implementation, integration in a curriculum sequence, ergonomic (ease of use) aspects, instrumental content, added value (takes advantage of new possibilities of DGS), potential for use and further modification of the resource. Pepin and Gueudet (2014) illustrate how studies on quality of teaching resources in general have historically focused on mathematical, pedagogical, sociological analyses (such as analysis along the lines of patterns of class of the target audience), or on specific mathematical knowledge, skills and practices. Studies on ICT resources contribute to the dimension on technical, design features including ease of use, quality and uncluttered graphics, and interactivity of the interface (Haugland 1999; Kay and Knaack 2009).

In the early 2000s, when most digital LTs were still designed for use on desktop and laptop computers, Yerushalmy and Ben-Zaken (2004) advocated for manipulatives that could be used on cellphones, since these devices were "an easily available tool that is already part of the culture and daily life… and that is likely to become highly useful for both teachers and students" (p. 3). Mathematics apps for touch-screen mobile devices are now increasingly part of many mathematics classes. Calder asserts:

> The use of mathematics apps, across a range of contexts and age levels, enhanced learning generally, but this was determined to some extent by the appropriateness and applicability of the apps to the particular student, their learning trajectory and the suitability of the app to the particular learning situation. (p. 246)

Basham et al. (2010) voice that "to provide a highly mobile, flexible, efficient, and scalable technology experience for students that could be taken outside of a school's walls… needed to provide students with multiple means for *representation, expression, and engagement*" (p. 340).

12.3.2 Constructive, Manipulable, and Instructive Apps

Goodwin and Highfield (2013) classify digital learning tools by their design features and how the learners' interact with these features into *constructive, manipulable*, and *instructive* apps. The authors define constructive tools as LTs in which learners participate in the generation of representations, tools which are used by the learners as an expressive tool, and tools which offer learners room for higher intellectual engagement, such as for reflection and thinking processes. These tools utilize significant cognitive effort on the part of the learners. Bos (2009a), Larkin (2015b), and Moyer-Packenham et al. (2015) would refer to these as apps with both high cognitive and pedagogical fidelity. Goodwin and Highfield maintain that learning objects that are not primarily constructive may still support learning when they are manipulable.

Manipulable apps may give a predetermined context, use mostly symbolic and iconic images, but still may allow some alteration of representations through user input (i.e., they are likely to evoke moderate to high user engagement). Thus, manipulable apps offer room for experimentation and discovery. Manipulative apps use modifiable graphics. Bos (2009a, b), Larkin (2015b), and Moyer-Packenham et al. (2015) would refer to these as apps with medium cognitive and pedagogical fidelity.

On the other extreme of the spectrum of apps are learning objects that focus only on behavioural learning activities, use symbolic presentations, and present learning in a linear fashion, utilizing repetitive procedural tasks and thus involving very low cognitive investment on the part of the learner. These learning objects focus on the "learner's focus of control over the representations presented on screen" (Goodwin and Highfield 2013, p. 213). Bos (2009a, b), Larkin (2015b), and Moyer-Packenham et al. (2015) would refer to apps that only offer drill activities as apps with low cognitive and low pedagogical fidelity. Zbiek et al. (2007) classified ICT resources such as online textbooks and courses, which were cognitive in nature but only presented information and had no capabilities to offer feedback on the actions of the learner as other resources but not tools.

12.3.3 Emergent, Adaptable, and Prescriptive Apps

Heydon and Wang (2006) assert that curricula paradigms configure the teaching and learning environments in ways that can limit or expand possibilities. Heydon and Wang name three paradigms: prescriptive, adaptable, and emergent. Prescriptive curricula are in line with behavioural psychology views of learning of scripted knowledge. Adaptable curricula involve active interactions and varied roles for the learner to include tailoring of learning activities according to the learner's interests. With emergent curricula learning is co-constructed with others, and learners are also inventors. For Heydon and Wang, constructive apps would support emergent curricula. Manipulable apps would support adaptable curricular. Instructive apps would only support prescriptive curricula.

Students in Goodwin and Highfield's (2013) studies substantially benefited from constructive and manipulative multimedia in terms of depicting multiple representations of concepts and forming sophisticated concept images (Pirie and Kieren 1994). Calder (2015) agrees that the multi-modal representations provide stimulus and novelty "but it is the subsequent thinking that is key to the learning process" (p. 238). The appealing factor is secondary to appropriateness and applicability, to use Calder's terms. Goodwin and Highfield (2013) maintain that constructive apps should not be mistaken to mean "busy" apps, which include extraneous details such as animations, which place unnecessary demands on low-achieving students, and take away from the understanding of mathematics content. Bos (2011) and Calder (2015) observe that distracting animations and colors minimize mathematical engagement.

12.3.3.1 Levels I, II, and III Apps

The app evaluation criteria in this chapter consists of a three-point scale, Level I, II, and III, with Level III a classification of high-quality apps, and four dimensions. It is a qualitative instrument. Each dimension consists of degrees or categories, which lie on a continuum of increasing complexity. That is to say, apps classified as Level III, show the highest degree on a dimension and go beyond the complexity of Level II, and Level II apps go beyond Level I apps. On a given dimension, say the curricula dimension (emergent, adaptable, and prescriptive), it is possible for an app to combine some elements of the adaptable category and a few of the prescriptive category, for example. Gadanidis and Schindler (2006) refer to apps that combine elements from different categories on a dimension as hybrid LOs. Goodwin and Highfield (2013) visualize apps that combine the middle category, manipulable elements, and the top category, constructive elements, as manipulable apps approaching the constructive category. Larkin (2013) found that, whereas some apps fit only in one category on a dimension of forms of mathematical knowledge (conceptual, procedural and declarative), some apps fit in two categories (i.e., they explored both conceptual and procedural knowledge). Classifying apps by levels is in line with reviews aimed at sharing lists of top apps (e.g., Larkin 2014). After Bos (2009b) and Larkin (2015a), we present our evaluation instrument in a chart (see Table 12.1) form to show the varied degrees (or, categories) on each dimension. Level III is the highest score, Level II is the medium score and Level I is the lowest score or most impoverished category, on a dimension. Level I apps are not necessarily off track but apps with characteristics from only the lowest category.

The dimensions of the classification are: (a) the nature of the curriculum addressed in the app—emergent, adaptable or prescriptive, and relevance to current, high quality curricula—high, medium, low; (b) degree of actions and interactions afforded by the app as a learning tool—constructive, manipulable, or instructive interface; (c) the level of interactivity and range of options offered to the user—multiple or mono, or high, moderate or low; and, (d) the quality of the design features and graphics in the app—rich, high quality or impoverished, poor quality. Several of the dimensions and their categories, such as in (a) and (b), emerged from the literature we reviewed, and some, such as in (c) and (d), emerged largely from the process of analyzing the apps. The fifth row is an overall dimension speculating that apps that score high on several dimensions have the potential for intense levels of intellectual/cognitive involvement, those that score high or medium on some dimension would have a limited potential, and those that score consistently low would have the potential for only low intellectual/cognitive involvement. We present details on the dimensions with the evaluation of a selection from the 80 apps we reviewed.

12.4 The Inquiry

The evaluation instrument emerged from a broader inquiry that involved teachers, researchers, and a developer of iOS apps in three contexts. The first was a school context, in which a teacher (who team-taught a unit on integers), in collaboration with the researchers, planned, implemented, and offered feedback during a Grade 7 and 8 integer unit centred on using CTs that enhance pedagogical goals of using

Table 12.1 Classification of middle school apps

Dimension	Level III apps	Level II	Level I
Curriculum dimension			
Address	The *emergent dimension* of curriculum (e.g., building understanding, explaining why, and reflection; this on top of the adaptive dimension)	The *adaptive dimension* of curriculum (e.g., meaning making, on top of the prescriptive dimension)	Only the *prescriptive* dimension of curriculum (e.g., fact masterly)
	Current and high quality curriculum		Dated or no curriculum
Degree of interaction afforded by the App's interface			
Offer	*Modifiable, constructive* interfaces	*Manipulable* interfaces	*Non-interactive, instructive*, access only interfaces
Interactivity and range of options			
Involve	A *high number and diversity of possible user inputs* or selections	A *moderate number and diversity of possible user inputs* or selections	A *very low number and diversity of possible user inputs* or selections
	A *high level of interactivity* between the user and object and with other users *Multiple* interactions	A *moderate level of interactivity* between the user and object *Mono* interactions	The *lowest level of interactivity* between the user and object *Mono* interactions
Technical design aspects			
Utilize	*Multiple* media and *alternative* representations	*Two or three* media and *alternative* representations	*Overly symbolic, linear* interfaces
	Colour, sound, animations, or 3D effects, graphics to focus learning, eliminating those that are superfluous	Colour, sound, animations, and 3D effects graphics to focus learning	*Superfluous and extraneous* details, such as animations, which instead of focusing learning, distract students
Overall, intellectual/cognitive involvement			
Have the potential for	*Intense* (with several opportunities for) *intellectual/cognitive involvement*—also a focus on math connections, understanding, and math extensions	*Limited* (two or three opportunities for) *intellectual involvement*—also a focus on simple application of skills	*Very low* (none or one opportunity) *intellectual/cognitive involvement*—a focus on individual skills and rote learning

manipulatives in teaching. Finding that the materials she had available did not work well for her students, the teacher created the physical version and a virtual version of a manipulative that circumnavigated the errors created by some existing tools. The second context involved work with an industry partner, who provided the researchers with access to the apps his company had developed. The app developer also offered to train team members to design iOS apps for teaching integers. In the third context, the researchers developed an instrument to evaluate randomly selected apps for teaching integers. The results we share in this chapter are from this third context of studying the apps. The initial coding of the apps was based on content, nature of representations used, interactivity level in the apps, nature of the design of the task posed by the app, and relation of the app to other mathematics learning materials. The process was further informed by research literature on the evaluation of apps, resulting in refined categories and other dimensions.

Larkin (2015a) observes that qualitative evaluation instruments are important "for teachers in making decisions about whether or not to use an app" (p. 344). The instrument shared in this chapter could guide teachers when selecting apps that meet the learning needs of their students. As well, it would guide app developers in going beyond apps that are overly prescriptive, focus only on quizzing students, are based on print design, and include only surface aspects of using multi-modality and play in learning, to apps that are more aligned with emergent, high-quality mathematics curricula, apps that focus also on conceptual understanding, and that utilize multiple modes and interactive representations in ways that are central to learning.

12.5 The Apps: How to Tell When an App Is a Useful App

We searched for apps on the desktop iTunes store because more information, including categories of apps, is displayed at the iTunes store as compared to the app store on a phone or tablet. We chose iOS apps because the app developer on the team created iOS apps. As noted by Larkin (2015a, b), locating relevant apps at the app store is difficult by the "sheer number of apps" and "the poor structure of the iTunes app store user interface" (p. 7); the way information on an app is largely based on the developers of the apps and is often inaccurate (e.g., app names on the app store-display names—may differ from names of apps when installed on a device); the way the results are organized and are displayed by icon, only giving the first 100 relevant results; plus the results continually change as new apps are added and old ones are removed or renamed.

We searched for both iPhone and iPad apps. We searched by keywords, including "integer" "negative," and "minus," by a combination of these keywords, such as "negative integer" "negative number", and by other relevant combinations of key words, such as "integer multiplication." The results for iPad apps were, at many times, more than for iPhone apps. Figures 12.1, 12.2 and 12.3 show screenshots of sample results. Because we are aware that app developers place apps in categories and select keywords for their apps based on market analyses, rather than on accuracy of the keywords, we also browsed the apps by categories. In the educational collections apps category, we selected the category of apps for elementary school, as well as apps for middle school and then further selected the category Math Apps. In the category Math Apps for Elementary School, we further narrowed our search by selecting the subcategories Number System/Numbers and

Fig. 12.1 iTunes store apps results for the keyword "negative integer"—iPhone apps

Fig. 12.2 iTunes store apps results for the keywords "negative integer"—iPad apps

Quantity, Early Operations, and Patterns. We also browsed apps under the categories Drill & Practice, Beyond Drill—Strategy, and Beyond Drill—Brain Busters. For middle school apps, we selected the subcategories Pre-algebra & Algebra, and Drill and Practice. Twenty or fewer apps were returned for each of these categories. We did not browse apps for subcategories such as High School Apps, nor the categories such as Geometry and Data, where we did not expect the content of negative integers to be a primary focus. Goodwin and Highfield (2012) found

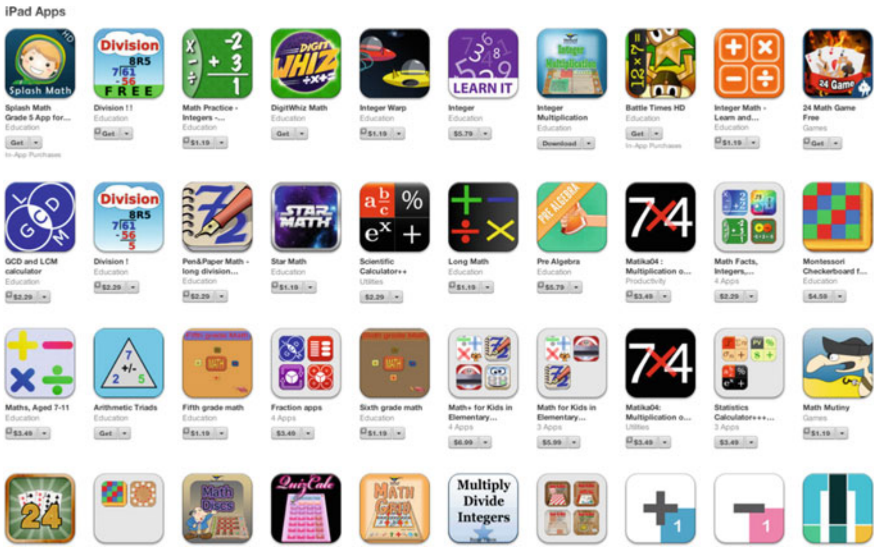


Fig. 12.3 iTunes store apps results for the keywords "integer multiplication"—iPad apps

relevant mathematics apps in other sections of the app store such as in apps for kids and edutainment. Because we were searching for apps for older children, we limited the scope of our search to the education section and to searching by key words.

We browsed all mathematics apps to select those that focused on learning negative integers as a curriculum area. We used the U.S. regional app store, although we also browsed the Canadian app store. For each of the apps in the results, from the keyword search and categories search, we examined the names and icons, as well as pulled up the iTunes App store pages of the app, to ascertain if the app fit the criteria of addressing negative integers. If an app showed a focus on positive integers, we also included it. The reason for this was because for some apps, the information available at the app store and at the app home page was not sufficient to show if an app on positive integers would extend to include negative integers. The home pages of the apps, where applicable, included more screenshots, detailed description of an app and, at times, video clips and reviews on an app. We eliminated all apps that did not focus on negative nor positive integers.

Selected searches by a keyword yielded a return of up to 100 results, the maximum possible, which pointed to the likelihood that more apps tagged with these keywords were available at the app store. To get a sense of how many more apps were left out by the app store results of the first 100 featured apps, we browsed a third-party website that offered analytics of apps at app stores—App Annie. App Annie returned 2024 iPhone apps and 1978 iPad apps for the search keyword "mathematics." It also returned 189 iPhone apps and 166 iPad apps for the keyword "integer." No apps were returned at App Annie when keywords were combined.

We selected 80 mathematics apps relevant to negative and positive integers (the Number Sense and Numeration strand) to download, try out, and review. The screenshots, descriptions, and information provided on an app were not always adequate for a review. We found that we had to download an app before we could ascertain its appropriate grade range and learning outcomes. Several app developers identified school grades, grade bands, or age groups for which the app was appropriate. Thirty-four of the 80 Number Sense and Numeration apps were found to be relevant to Grades 7 and 8; however, for many apps, the grades/ages indicated were not always accurate, at least not for the mathematics curriculum in the Canadian province where the research was conducted. Of the 34 apps that we found relevant to Grades 7 and 8 Number Sense and Numeration, only 8 were appropriately labelled as Grades 7, 8, or middle school apps. Overall, the grade bands indicated by the developers were not accurate. This is perhaps an indication that the developers are from varied countries where it is plausible that this content on negative numbers is addressed much earlier. Larkin (2013) interprets this as an indicator that the developers are not familiar with and do not consult a curriculum policy document when identifying grade fit of their app, or that the grade levels were selected from a marketing, rather than a curriculum, perspective. He found the targeted level to be 2–3 years younger than the ages specified by the app developer.

12.6 Dimensions for Selecting Appropriate Integer Apps

It was evident from the review of the 80 apps that several dimensions, including the nature of the curriculum addressed, were central when evaluating apps.

12.6.1 The Nature of the Curriculum Addressed

Emergent and adaptable activities as contrasted with overly prescriptive activities. We considered the nature of the learning that the mathematics tasks in the app could evoke. Only 3 of the apps involved what we refer to as, after Heydon and Wang (2006), emergent features (e.g., Math Alchemist Lite, and its other two versions). Math Alchemist is an example of an app that focused on a problem-solving context, the one of making 24, using any random numbers combined with number operations. A user’s response becomes part of the inputs available for use in making 24, and the level of difficulty is increased depending on the user’s success at a level. Apps with *emergent* features, ranked Level III apps on the curricula dimension, presented some rich mathematics problems that were, for instance, closely aligned with teaching through problem solving.

We labelled, again after Heydon and Wang (2006), *adaptable* apps as those 13 apps (e.g., Math Blaster Hyper Blast, Math Boosting, Interactive Integers) that posed questions or problems, which could have involved computing answers, but at least offered ways for the user to extend the problem. The Interactive Integer app posed tasks that involved conceptual understanding (see activities with coloured tiles) in addition to drill tasks for practicing integer addition and subtraction. This app was ranked Level II on the curricula dimension.

A majority of the apps, 74 out of 80, mainly offered prescriptive tasks. *Prescriptive apps* only posed traditional, prescriptive practice tasks, such as the question “3 + (−4) = ?” These focused on right/wrong responses from the learner in a manner similar to physical flash cards. These apps scored low, Level I, on the curricula dimension.

Only 4 apps (e.g., Interactive Integers, Math 24 Solver, and Math Blaster HyperBlast game) focused on building understanding of concepts, introducing a new topic, or explaining how a procedure worked, scoring high—Level III—on this dimension. A large number of apps, 58 out of 80, were for practicing earlier learned concepts, as would be the case with flash cards. That a majority of apps mainly offered prescriptive tasks was also the case in Larkin’s (2014) evaluation in which they found that procedural apps dominated.

Mathematics content aligned with more recent, higher quality curricula. Each of the 80 apps, according to their developers, was for learning, practicing, or getting quizzed on mathematical topics. The mathematics topics were listed differently, fluctuating from mentioning a single topic to listing a range of up to five topics. The topics included naming of general mathematics branches, such as arithmetic, through indicating a specific mathematics topic, such as negative numbers, to, at the highest ranking, Level III, further specifying mathematics content and learning outcomes (or, expectations), such as using models with negative integers. We view the latter focus that goes beyond naming a branch of mathematics or listing topics to specifying what is learned or practiced by using the app as a use of language consistent with that used in more contemporary, higher quality curricula of Canadian provinces and several other countries. In many curriculum documents, such as the NCTM principles and standards (NCTM 2000), the content specified goes beyond a mere mention of a topic to specifying learning expectations.

A selection of apps (e.g., Math 1st–6th Grade Digital Workbooks—Space Board) showed coverage for other strands, such as Geometry, in addition to Number Sense and Numeration. We took this focus, on connections of number sense to geometric representations of numbers, to align with the NCTM standards focus on connections among strands.

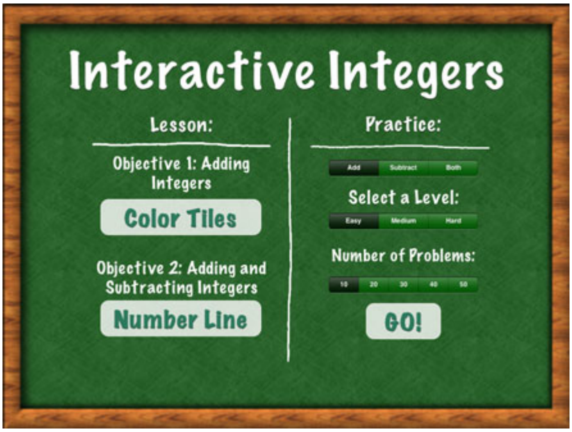
12.6.2 Actions and Interactions Afforded by the App

Constructive, Manipulable as Opposed to Largely Instructive Apps. Some adaptable apps involved interfaces with objects such as a number line that a user could act upon, or manipulate. In Figs. 12.4, 12.5, 12.6, 12.7, 12.8, 12.9 and 12.10, we show screenshots of the Interactive Integer app to illustrate how the number line and integer tiles in this app could be dragged and dropped as the user added or

Fig. 12.4 Interactive Integers app—both iPhone and iPad app. *Source* www.tictaptech.net/apps/interactive-integers/

subtracted integers including negative integers. The colored tiles in the interactive integer apps could be dragged to demonstrate the identity property (e.g., +1 + −1 = 0): When a yellow, positive tile and a red, negative tile were dragged close to each other they each disappeared. Many representations of mathematics concepts in instructive apps could not be acted on or modified. Some apps only included audio or video demonstrations of an instructor explaining a mathematics process or giving the answer. A good number of apps did not have any objects that visually represented mathematics concepts. Goodwin and Highfield's (2013) evaluation found that a majority apps were instructive.

Fig. 12.5 Interactive Integers app showing user choice on task



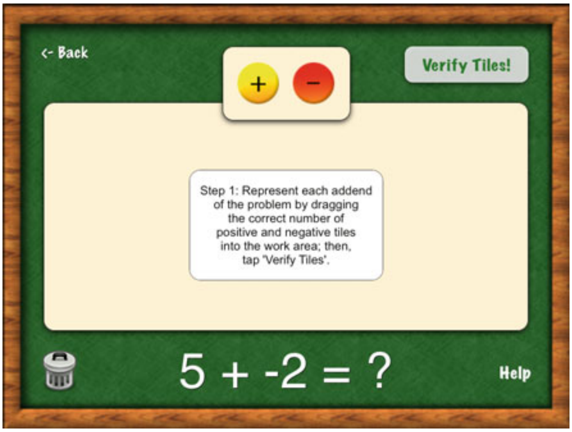


Fig. 12.6 Interactive Integers app color tile instructions

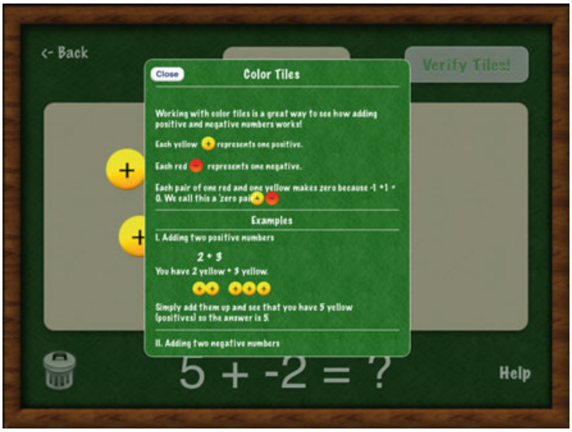


Fig. 12.7 Interactive Integers app hint on using color tiles

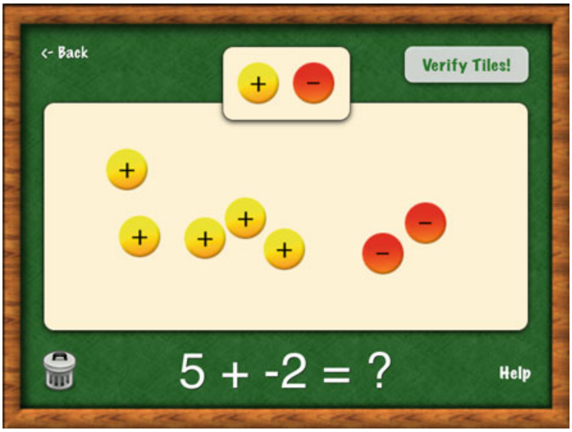


Fig. 12.8 Interactive Integers app adding 5 + −2 using dynamic counters

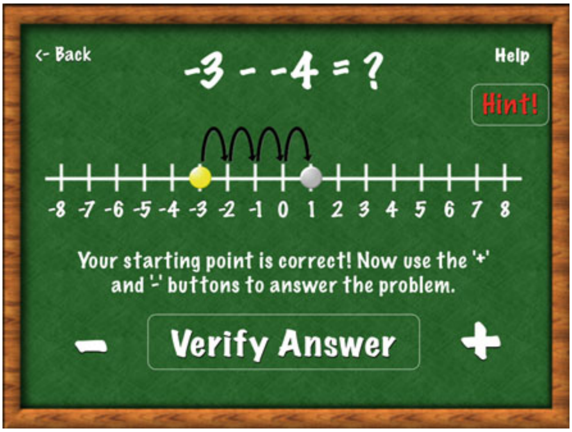


Fig. 12.9 Interactive Integers app showing the number line model

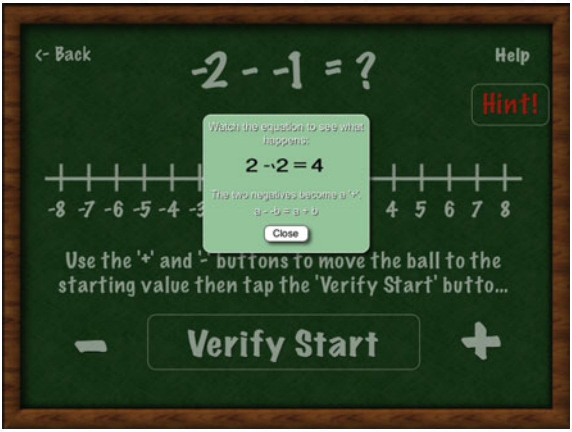


Fig. 12.10 Interactive Integers app explaining a rule on taking away a negative integer

12.6.3 The Level of Interactivity and Range of Options Offered to the User

Multiple interaction apps as opposed to mono interaction apps. Only a few apps (e.g., Math Fact Master, Math!!!, and Middle School Math Pro 7th Grade) included opportunities for multiple users, such as submission of responses or marks, and asynchronous teacher interaction with the learner. We ranked apps with multiple interactions as high, Level III, on the dimension of interactivity and range of options, to be contrasted with apps offering mono interactions. Seventy of the 80 apps, including many of the apps that ranked Level III and Level II on the other dimensions, were designed with a focus on one user—mono interaction—thus limiting interaction to one user and the interface. In reference to video games, Gee (2012) distinguishes between the piece of software together with all the social activity around it and the piece of software alone. He would refer to the social activity around an app and the app as a software as the *Big A* app in contrast to the *small a* app because the former is important for participation, production and pro-active learning.

High- and moderate-engagement and interactivity apps as contrasted with low-engagement apps. Only 3 apps (e.g., Math 1 On-Track, Math Book Pro, and Math Blaster HyperBlast) accommodated a variety of inputs and choices, and offered varied possibilities of inputs and choices so the user may insert and select options, thus ranking Level III on interactivity. We referred to apps with a range and adjustability of options as high-interactivity apps. A good number of apps (e.g., Mathopolis, Math 2112, Math 24 Solver, Math4Touch), specifically 55 of 80, involved moderate interactivity with some opportunities for the users to input values and make choices. With the Interactive Integers app a learner was offered a choice of representation—tiles or the number line; the operation—addition or subtraction; number of questions; and level of difficulty. About a quarter, 22 of the 80 apps, involved much lower-interactivity, Level I. Many apps were limited to already inputted values and allowing only up to two choices (e.g., check answer and a "next" button) for the user.

12.6.4 The Quality of the Design Interface and Graphics in the App

Multi-media, high quality apps as opposed to primarily text-based, low quality apps. Sixty-six of the 80 apps utilized visual representations and graphics in addition to numeric symbols and text. Only 20 of the apps (e.g., Math Blaster HyperBlast, Interactive Integers, Integers, and Math!!!) went beyond using numerical symbols and text to utilize other mathematical representations such as geometric, graphic, simulations, or 3D graphs. We ranked these apps as Level III apps on technical design features. Forty-four of the 80 apps utilized sound effects and music. Many of these apps utilized multiple colors. Some apps used the colors in ways that were not simple add-ons, but in ways integral to the mathematics content. For instance, in the Integer Multiplication app, an iPad only app (see Fig. 12.11), the use of colors offered ways for the learner to identify patterns and distinguish characteristics of negative and positive integers. Still, a majority of apps, over 60 out of 80, largely utilized, at the lowest rank—Level I, only numerals and text to represent mathematics concepts. Haugland (2005) warns against this "poor use of a powerful learning tool" (p. 330).

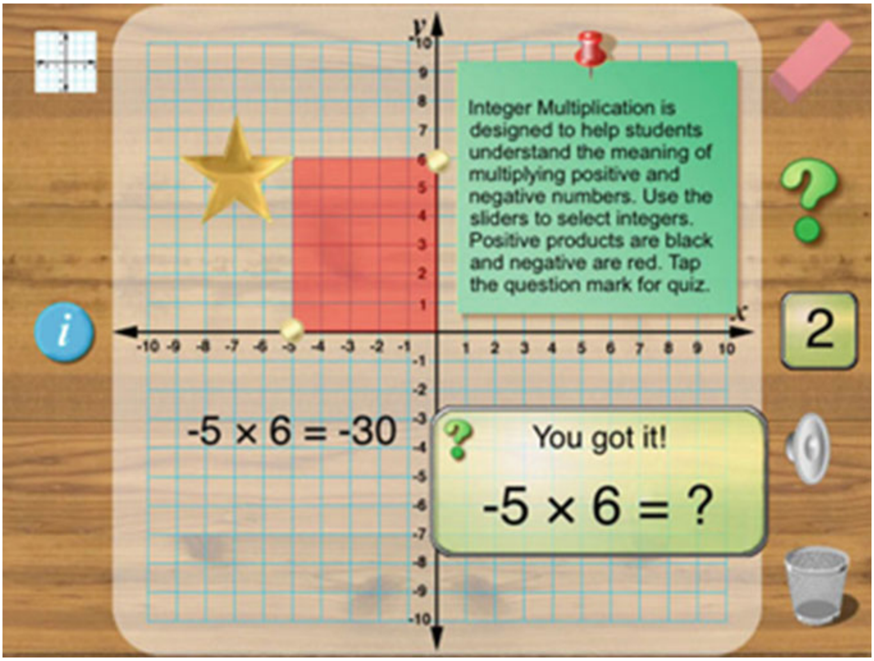


Fig. 12.11 Integer Multiplication app showing use of color

12.6.5 Instrument Content and Value Added by the Instrument

Virtual-only innovations, virtual developments of, with added value on, existing instruments as opposed to digitized images of existing materials. Because virtual and physical materials complement each other (Namukasa et al. 2009), for each of the apps, we examined the relation, if any, to existing instructional material/ resources. The team assessed if an app replicated already existing mathematics resources, such as virtual manipulatives, textbooks, or web resources, or whether an app was a digital version of these materials. This was important in assessing the app's pedagogical and cognitive elements (i.e., whether, for instance, it replicated a material that focused on developing conceptual knowledge, or on test preparation). Base-Ten Blocks replicated the physical and virtual Base-Ten Blocks manipulative. According to Bos (2009b), interactive mathematics learning tools, such as virtual manipulatives that are enhanced with technology, have a higher degree of cognitive fidelity than technology-based tools that focus on games, instructional information and quizzes. The representations of colored tiles and number lines, as seen in the Interactive Integers app, reflect the use of virtual, visual, and physical representations of integers in ways that are enhanced to represent a mathematics property. We found that many apps were designed based on mathematics puzzles (e.g., Math 24

Solver). Some apps added game contexts to paper-and-pencil mathematics puzzles. Also, many apps were game based (e.g., Mathopolis). Sixty-six apps involved some recreational features and 4 of these involved role-playing games (e.g., Math Blaster HyperBlast). Certain apps (e.g., YourTeacher, Motion Math-Zoom), in a manner similar to a mathematics textbook chapter or a lesson in a course, were part of a collection of apps focusing on varied mathematics topics for the same age level. Larkin (2013, 2015a, b) and Calder (2015) observed that many apps were stand-alone apps focusing on one particular kind of skill, knowledge, or content. Further, apps in bundles appeared to be aligned with curricula expectations.

12.6.6 The Level of Intellectual/Cognitive Involvement It Evokes—Intense, Limited or Very Low

Intense as opposed to limited or very low intellectual/cognitive involvement. Overall, apps with adaptable (or, emergent) characteristics and those with manipulable (or, modifiable) elements appeared to have the potential for intense intellectual/cognitive involvement whereas apps with instructive and prescriptive characteristics appeared to have limited to very low potential for intense intellectual/cognitive involvement. Even among prescriptive and instructive apps, some apps, because they scored high on other dimensions such as on interactivity and range of options and technical design aspects, appeared to be more engaging and thus offered potential for intellectual involvement at the procedural level.

A good number of apps combined elements on one dimension as illustrated in Goodwin and Highfield (2013) and Gadanidis et al. (2004). We did not find an app that ranked at level III for all dimensions. The Interactive Integers app combined both the adaptive and prescriptive elements on the curricular dimension, and it had a manipulable interface (level II on the actions and interactions dimension). It also offered choice and provided immediate feedback, as well as written instruction for both the lessons on understanding and for practice questions, but did not offer an opportunity for the learner to input values or make annotations by including a keypad. One of its instructions on how to take away a negative number was not mathematically accurate. Interactive Integers was limited to integer subtraction and addition. The Integer Multiplication app, that scored high on the characteristic of use of color to focus learning, covered only a single operation on integers—multiplication.

Some apps that scored low, Level I, on one dimension scored higher, Level II or III, on other dimensions. Even when it focused on right and wrong answers—Mathopolis, a prescriptive app—also involved a game context that allowed user choice on the level of difficulty and nature of operations, scoring Level II on interactivity. One could say that Mathopolis scores high among prescriptive apps because it is a Level II app on at least one other dimension. One of the apps that appeared to involve emergent features had a game context that did not appear appropriate for middle school students. We pondered the messaging and content in the apps and its appropriateness for learners. This was also the case in Larkin (2015a), where he found apps scoring high on one dimension and low on another. For instance the apps Larkin evaluated scored higher on pedagogical fidelity, followed by mathematical fidelity, and lowest on cognitive fidelity. To Haugland (1999), children's software should be evaluated on age appropriateness and non-violence.

Apps with multiple interactions—between several users (e.g., Math Fact Master which could submit scores to an email address, as well as Math!!! with the possibility of a teacher embedding messages) have promising added value of interacting with others through the cognitive tool.

12.7 Concluding Remarks

Our evaluation instrument could guide teachers when selecting apps that meet their teaching goals. As well, the evaluation instrument could guide developers in designing apps that are more aligned with emergent and adaptive curriculum, that also focus on conceptual understanding in addition to focusing on procedural and declarative knowledge, and that utilize multiple and interactive modes in ways that are central to the representation of mathematics entities.

Some teachers implement and test objects, many use objects recommended by colleagues, and yet other teachers, especially those comfortable with computer programming, increasingly approach the use of learning objects from a developer's perspective. New friendly coding programs are making it easier for more teachers, and even students, to engage in designing apps. Thus, our instrument can potentially guide students, teachers, educators, and researchers when they design apps.

When mathematics apps are thoughtfully used in ways that encourage learners to do the mathematics (i.e., explore, conjecture, test, and apply), rather than only doing procedural steps, learning apps have the potential to deepen mathematical understanding and encourage students to work at higher levels of generalization and abstraction (Bos 2009a, b). Looking to the future, with the increased focus on students of all ages learning to code, such as the mandate of coding across all grades in England's National Curriculum (UK Government News Release, 4 February 2014), we need to also consider: (1) the connection between students as coders and students as mathematics learners, and (2) the design of apps, not only as education products to be consumed, but also environments that may be edited and reprogrammed by users. For example, Gadanidis and Yiu (2014) created HTML5 apps (available at www.researchideas.ca/mathncode) that attempt to meet these conditions, respectively, by: (a) using app interfaces where users change code parameters to control a simulation or play a game, and (b) programming apps in MIT's Scratch environment, giving students full access to the code, which they can edit to create variations or new simulations and games. Explicitly incorporating coding in mathematics apps would help incorporate three pedagogical benefits of coding in mathematics learning: making concepts tangible, making relationships dynamic, and giving students more control over the learning process (Gadanidis 2014, 2015).

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Chapter 13
Ambitious Teaching: Designing Practice-Based Assignments for Integrating Virtual Manipulatives into Mathematics Lessons

Jennifer M. Suh

Abstract This chapter details a design-based research study that focused on developing effective approaches for pre-service teachers to integrate technology in the mathematics classroom. Using an iterative design cycle, the researcher developed three practice-based assignments during an elementary mathematics methods course that were designed to promote pre-service teachers' technology pedagogical content knowledge. These practice based assignments allowed the participants to (a) analyze effective technology tools for mathematics teaching and learning; (b) evaluate applets that supported a vertical learning progression on a specific mathematics concept; and (c) design, implement and reflect on a mathematics lesson where technology amplified the mathematics teaching and learning. By creating pre-service teachers' own practical image of practice after implementing the technology integrated lessons in the field, pre-service teachers gained a better "picture of practice" of ambitious teaching in the mathematics classroom where effective integration of technology helped construct students' mathematical understanding.

13.1 Introduction

In education, we know that the value of any technology tool depends on how it is used in instruction. In this digital age, teachers are inundated with educational technology. It is important for teachers to be able to judiciously evaluate the instructional worth of a technology tool. Beyond the interactive dynamic nature of mathematics applets, educators need to ensure that the content taught using the virtual manipulative in an applet is characterized by ambitious teaching and learning goals. The term "intellectually ambitious teaching and learning," has been defined as instruction that helps students "develop in-depth knowledge of subject

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matter, gain higher-order thinking skills, construct new knowledge and understanding, and effectively apply knowledge to real-world situations" (Smylie and Wenzel 2006, p. 7). Ambitious teaching, according to Lampert et al. (2013), "challenge(s) teacher educators to prepare new teachers to do a kind of teaching that most experienced teachers are not yet doing" (p. 226). Furthermore, they suggest that the challenge of preparing beginning teachers in this way is asking novice teachers to teach in a way that is more socially and intellectually ambitious than the current norm. In addition, ambitious teaching using technology adds another layer of complexity because many pre-service teachers may not have a "picture of practice" from their own learning experiences.

To help pre-service teachers understand the complexity of ambitious teaching, educators and researchers have developed a set of key instructional activities that embody core teaching practices (Lampert et al. 2010). By "chunking" some of these instructional activities, teacher educators allow pre-service teachers access to "manageable, structured routines". These routines allow teachers to practice enacting a particular instructional purpose while maintaining the associated complexity. For example, in their study, Lampert et al. (2010) focused on the following four activities to promote ambitious mathematics teaching: choral counting, strategy sharing, computation strings, and solving word problems. Using this framework for designing instructional activities to help pre-service teachers manage ambitious mathematics teaching, this chapter presents three experiences for pre-service teachers focused on integrating technology in the mathematics classroom. These experiences were designed to move pre-service teachers along a spectrum from engaging with technology, because it seems appealing, to knowledgeably selecting virtual manipulatives for their conceptual development of mathematical content, higher-order thinking skills, and problem-solving ability. The aim of the chapter is to share the high level instructional activities that helped pre-service teachers integrate technology and promote ambitious teaching.

13.2 Understanding Research on Integrating Technology in the Content Area

According to Niess and Walker (2010):

> …many digital technologies have proved useful for students learning mathematics: graphing calculators, applets or virtual manipulatives, spreadsheets, computer algebra systems, and dynamic geometry tools. Each of these technologies provides visual representations that enable students to explore mathematical ideas in more dynamic ways. (Niess and Walker 2010, p. 100)

On the other hand, a misuse of technology would be using it as merely an attention grabber. Engagement is initially high when a lesson is introduced with dynamic and animated images, but soon that novelty wears off. According to the

Technology Principle in the Principles and Standards for School Mathematics (NCTM 2000), "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances student learning" (p. 24). The phrase "influences the mathematics that is taught" is what determines the ambitious teaching goals in the mathematics classroom. The word "enhances" is what characterizes technology as a tool with high leveraging power because technology has specific affordances that can enrich learning tasks (Suh 2010).

The complexity of teaching with technology stems from the notion that teaching in itself is a complex endeavor. Shulman (1986, 1987), coined the term Pedagogical Content Knowledge (PCK) to describe the specific knowledge needed to teach effectively which includes knowledge of subject matter, knowledge of students' thinking, and knowledge of pedagogy. In mathematics education, PCK has been expanded to include Mathematical Knowledge for Teaching (MKT) as the teacher "knowledge necessary to carry out the work of teaching mathematics" (Ball et al. 2008; Hill et al. 2005) that include specific high-leverage practices such as the use of mathematical explanations and representations, interpretations of student responses, and the ability to avoid mathematics errors and imprecision. Teaching with technology adds another layer of complexity to the PCK framework. Understanding how to teach with technology, referred to as Technological, Pedagogical, and Content Knowledge (TPACK) (Mishra and Koehler 2006) integrates a third component into teachers' specialized knowledge for teaching—the integration of technology into instruction. TPACK includes understanding how technology can be used to represent concepts, knowledge of pedagogical techniques that use technology to effectively teach content, familiarity with ways technology can help students understand particularly difficult topics, and knowing how technology can be used to build on existing knowledge. Virtual manipulatives have been described as an "interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al. 2002, p. 373). The questions is how to support pre-service teachers as they take advantage of technology to elicit sense making while students construct mathematical meaning.

In working with pre-service teachers (PSTs) using the virtual manipulative environment, it is important that they situate their learning within the current thinking around TPACK. Thus, prospective elementary mathematics teachers must be prepared to teach with and through technology because of the ways in which technology enhances mathematics learning, supports effective mathematics teaching, and influences mathematical content. Due to the complex nature of teaching with technology, Mishra and Koehler (2006) recommend that understanding approaches to successful technology integration requires educators to develop new ways of comprehending and accommodating this complexity. This is the challenge I have undertaken for this study.

13.3 Motivation for the Study

The purpose of this study was to carry out design-based research producing an improved course experience for teaching PSTs technology integration. It also allowed me to engage in a faculty self-study (Samaras 2010) as I reflected on previous courses and considered better ways to help PSTs learn how to integrate technology in a mathematics classroom effectively. For these reasons, I designed a series of practice-based activities that would give PSTs opportunities to work with mathematics applets while planning for a mathematics lesson. In this way, they would be able to refine their views of effective mathematics teaching, and develop a critical lens for their own practices, while building their repertoire for teaching mathematics with technology.

13.4 Method

13.4.1 Participants and Research Questions

The participants were 26 PSTs in a mathematics methods course also enrolled in a technology integration course. The technology integration course was one credit and was taught in tandem with the three-credit mathematics methods course. The methods course included field experiences and took place the semester before their student teaching internship.

The study explored the following two research questions:

1. What technology pedagogical content knowledge was elicited by PSTs on each of the practice-based assignments?
2. How do the designed practice-based assignments, situated in the course and field experiences, better support the development of PSTs technology pedagogical content knowledge?

13.4.2 Research Design

This study used cycles of design-based research aimed at developing effective approaches for integrating technology in the mathematics classroom. The methodology in this chapter was consistent with aspects of design experiments (Brown 1992). Design-based research (DBR) was chosen because it allows for practitioner research when implementing interventions and uses an iterative analysis process in conceptualizing learning, instruction, curricular design and reform. An iterative cycle of the following steps was used: considering a framework for integrating technology in the mathematics classroom, developing a curricular design with practice-based assignments, implementing the course-based activities and analyzing feedback for improvement for the next cycle. The DBR (Design-Based Research Collective 2003) communicates relevant implications to practitioners and other educational designers. Design was the focus of the study in an effort to foster learning, create usable knowledge, and advance theories of learning and teaching using technology in the classroom.

Grounding professional learning in practice can provide teachers with opportunities to investigate authentic problems of practice and to develop knowledge and skills in the contexts of their use. I designed three related practiced-based assignments with an initial phase that included asking PSTs to observe current implementation of technology in a mathematics classroom and reflect on the current practices they observed before implementing any tasks on an electronic discussion board. The first Practice-based Assignment (PBA) #1 was called *Technology Applet and Website Evaluation*. The task sheet was modified from the *Elementary and Middle School Mathematics Field Experience Guide* (Bay-Williams and Van de Walle 2010) to include two reflective questions on how PSTs would use the applets in their classroom and how the applets promoted the Common Core Standards for Mathematical Practice (CCSS-M 2010). The second Practice-based Assignment (PBA) #2: *Sequencing Technology Applets to Reflect on Students Mathematics Learning Progressions* was designed to expose PSTs to the mathematics standards, teaching practices standards, and the learning progressions of mathematics standards. The third Practice-based Assignment (PBA) #3 was called *Planning and Integrating Technology in a Math Lesson*, which allowed PSTs to pull together what they had learned from the previous assignments to design a thoughtful mathematics lesson integrating technology. After these three PBAs, PSTs were asked to reflect on their views of integrating technology in a mathematics classroom on an electronic discussion board.

13.4.3 Data Sources

Data sources included the collection of assignments described above: Practice-based Assignment (PBA) #1 *Technology Applet and Website Evaluation*, that included two applet reviews that were one page each (see Fig. 13.1); Practice-based Assignment (PBA) #2 *Sequencing Technology Applets to Reflect on Students Mathematics Learning Progressions*, that included a one page response recording sheet (see Fig. 13.2); Practice-based Assignment (PBA) #3 *Lesson Planning and Integrating Technology in a Math Lesson,* that included a formal lesson plan with a written reflection on how the lesson went and how students responded to the lesson with screenshots of student work analyzed.

In addition, PSTs submitted an electronic discussion board entry where they reflected on integrating technology in a mathematics classroom during their field experience. These data sources were analyzed for emerging themes. I used the participants' assignments to further analyze their learning using the TPACK components for teachers' specialized knowledge for teaching—to examine how their assignments elicited specific ways technology can be used to represent

Title: NLVM Algebra Balance Scales: Negatives http://nlvm.usu.edu/en/nav/frames_asid_324_g_3_t_2.html?open=instructions&from=category_g_3_t_2.html ***Type of Tool:*** Virtual manipulative ***Grade Level:*** VA SOL 6.18 (enrichment), 7.13 and 7.14, 8.15(a) ***Math Content:*** algebra
Specific Topic: evaluating simple linear equations with one negative variable on both sides
Key Instructional Objectives: using virtual manipulatives as representations of positive and negative variables and integers to balance equations
Rating: (Rate each aspect of the tool with a 1-5, 1 being lowest and 5 being highest. When appropriate add a comment or your reasoning too.)

Criteria	Rating	Comments
The applet or website provides better opportunities to learn than alternative approaches.	5	This applet constrains the learner's use of manipulatives so s/he is not distracted by the physical objects.
Students will be engaged with the math content not the frills.	5	It is a simple layout, with a balance, operation buttons, and colored blocks that represent the variable *x*, coefficient, and constant. Students have solved the equation when only an *x* remains on one side of the equation – there is no feedback. Focus is on balancing the scale, with no distractors.
The applet provides opportunities for problem solving.	5	The user creates the equation and selects the order of procedures to solve the equation. After each step, a simplified equation is displayed below the original equation, which will help the learner develop an understanding that all the equations shown are equivalent.
The tool develops conceptual knowledge and supports student understanding of concepts.	5	Negatives, or opposites, are represented by red balloons that raise the side of the scale on which they are placed. Blue blocks and boxes represent positive variables and numbers that push down the scale. After creating their own equation, students place the manipulatives that represent the equation onto the scale. This visual allows students to better understand that they may perform any operation to solve, as long as they do the same thing to both sides of the equation to maintain balance. One drawback of this app is that it lacks the final step of substituting the *x* value back into the equation. Students benefit from closing the loop and proving why that value is a solution.
The tool develops procedural knowledge and supports student understanding of skills.	5	The student will refine strategy choice through practice and observation of the outcomes of each choice, as displayed in the equation box. This flexible approach to finding the answer supports procedural fluency.
The software or website allows the teachers to assess student learning through records and reports.	1	There is no data capability visible on the free version.
The Program is challenging for a wide range of skill levels.	5	This app builds upon a previous one that uses positive integers and variables. Challenge is limited to the student's creation of their own problems – there is no built-in feature to increase skill level.
The tool is equitable in its consideration of gender and culture.	5	This virtual manipulative app is free of any discrimination, and provides equitable access to the skills practice of balancing equations.
The tool promotes good student interaction and discussion.	2	It is a one-user app. The teacher would have to prompt student users to discuss their strategies and discoveries following the user of this tool.
The tool has quality supplementary materials such as blackline masters.	1	Blackline masters are not evident on the free version.
Your overall rating	5	

Fig. 13.1 Evaluation of an applet by a pre-service teacher

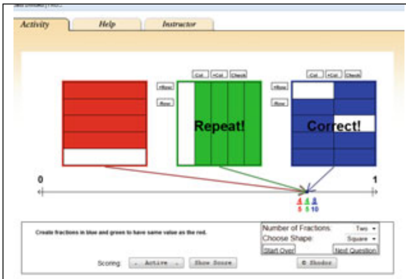
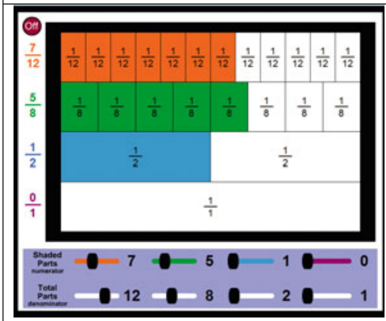
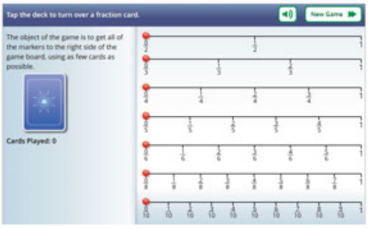
 Equivalent Fractions Finder http://www.shodor.org/interactivate/activities/EquivFractionFinder/	I selected this tool because my student struggled with fractional parts and because this tool allows the whole to be broken into fractional parts in several different ways. Students can manipulate the tool to create equivalent fractions in several different ways. I would encourage the student to create images as I have done below in the screenshot. Based upon my assessment, it seems the student is used to seeing fractional parts as the same size and shape. It's important that he understand the concept of fair shares. By creating images that are not what we normally see as fractional parts, it will deepen his understanding of this fraction concept. The student can save time by not having to color blocks repeatedly. Because the tool is online, students can divide the units in several different ways and they can do it multiple times.
 Fraction Bars http://www.mathplayground.com/Fraction_bars.html	My student seemed to have a great understanding of equivalent fractions. I chose this tool to challenge him and to move on from fraction concepts to adding and subtracting fractions with like and unlike denominators. Using this tool, students must add and subtract fractions with like and unlike denominators. So, it addresses the content well. Students must think critically about making the best choice for their move based upon their knowledge of equivalent fractions and the ability to move all tracks to one in the least amount of moves. I would use this tool to expose the students understanding of equivalent fractions by also having him keep a journal of why he chose to make each move.
 Fraction Tracks http://illuminations.nctm.org/Activity.aspx?id=4148	My student seemed to have a great understanding of equivalent fractions. I chose this tool to challenge him and to move on from fraction concepts to adding and subtracting fractions with like and unlike denominators. Using this tool, students must add and subtract fractions with like and unlike denominators. So, it addresses the content well. Students must think critically about making the best choice for their move based upon their knowledge of equivalent fractions and the ability to move all tracks to one in the least amount of moves. I would use this tool to expose the students understanding of equivalent fractions by also having him keep a journal of why he chose to make each move. The online fraction tracks game has an advantage over the paper version of the game because it can correct your move in real time. The program will not allow a student to make an incorrect move. If playing the two person version, the game also keeps track of which player's turn it will be.

Fig. 13.2 Selected applets to assess students learning progression

concepts and the specific knowledge of pedagogical techniques to effectively teach content. These data sources also provided feedback for ways to improve the designed experiences for integrating technology in the mathematics classroom. PSTs' responses from the discussion board on their beliefs about technology integration in the mathematics classroom were collected prior to and after the series of practice-based assignments to document how their views evolved after implementing these assignments. The responses to each of the three assignments also served as a way to analyze how the practice-based assignments were helping PSTs learn specific TPACK.

13.4.4 Data Analysis

Using Mishra and Koehler's description of TPACK (2006), I analyzed the learning that occurred through each assignment based on how the PSTs reflected on the knowledge of student thinking and learning, knowledge of content and instruction, and knowledge of technology. For the analysis process, I used a grounded theory approach to make sense of the data by defining codes, categories and concepts. Using the constant comparative method (Strauss and Corbin 1990), I began with open coding to develop names and categories. Then I moved to axial coding to relate the initial codes to one another. Finally, I applied selective coding to make choices on the most important codes. Using this method, I was able to analyze PSTs' assignments and reflections to display and organize categories in such a way that I was able to draw some concepts together as the emerging themes (Miles and Huberman 1994). I revisited the assignments and the reflections with the themes and categories the second time, to verify that the themes emerged across the multiple data sources.

13.5 Results

13.5.1 Initial Phase: Ideal Image of Practice and Observation in the Field

In order to help PSTs reflect on the current research on integrating technology, I asked them to read articles and the framework around TPACK. The major themes they drew from their reading were that technology is an essential tool for both learning and teaching mathematics and that the teacher's role is to select effective tools. In class, we discussed NCTM's (2000) position statement on the role of technology in teaching, which "regards technology as an *essential tool* for both learning and teaching mathematics" (p. 113).

The first reflection prompt posted for PSTs engaged them in a discussion about what they were able to observe in terms of technology integration in the mathematics classroom at their field sites. The prompt stated, "Now that you are visiting your classrooms, have you seen some ways 'real' teachers are integrating technology, particularly in the mathematics classroom? If so, share some great examples. What are some technologies available for your teachers and students? If you do not see technology being integrated in the mathematics classroom, in what ways would you integrate technology in what students are learning in your assigned classrooms?"

Common themes were the use of technology in the mathematics classroom including: (a) *focusing on visual representations*; (b) *using dynamic features to illustrate mathematics concepts; and*, (c) *incorporating collaborative learning centers.* The following is an excerpt from one of the discussion posts that illustrated the use of technology as a visual representation.

> The classroom teacher would create graphic representations for fractions where the students could come up and color sections to show their thinking. These were usually whole group lessons and students were engaged and active in working through the problems and discussing their solutions.

Another example was how the teacher used a virtual hundreds chart to keep track of patterns to facilitate a mathematics discussion about number patterns. The pre-service teacher noted how the teacher engaged students in analyzing the patterns and not just saying the numbers that were highlighted.

> The teacher pulled up a virtual 100s chart on the board, where the numbers could change color once she touched them. She used the chart to give a visual representation of counting by tens starting at various numbers on the chart. First they counted by tens starting at ten, and after every correct answer the teacher touched that number on the screen so it changed color. Once the "red tower", as the children called it, was completed on the chart, she asked the students to take a closer look at it and explain what they notice about the "red numbers" (analyzing the numbers in the ten's place and the one's place). Next she asked them to count by tens from 6. She, again, touched the number on the screen after every correct answer until another "red tower" appeared. The students really enjoyed comparing the two "red towers" and analyzing the "red numbers".

One pre-service teacher admitted to seeing benefits of virtual manipulatives despite her preference to use physical manipulatives, noticing the value of virtual base-ten blocks. She observed first graders working on a video game website and voiced how she would rather have them practice on the virtual manipulatives base ten website.

> While I prefer the hands-on nature of physical manipulatives, I can see the benefit of virtual ones. The base-10 blocks are the best and most frequent example of the virtual manipulative; these virtual blocks can show the student very effectively how to add, subtract, multiply and divide, the four key operations on which all other math is built. In the first grade class where I am currently observing I would use the base-10 blocks as a starting point rather than having the students practice using the video game website as they do now.

In addition to whole group lessons observed, several PSTs posted responses on the discussion board about ways technology allowed for collaboration and how teachers used technology as a center during guided mathematics lessons.

> The teacher mainly used the SMART board when conducting her lessons, but the students loved being able to come up to the board and manipulate the objects on the screen. Many times the teacher would put problems on the board and the students could come up and uncover the correct answers. The teacher also conducted guided math, so the students had

math centers. During the math centers, the students had the option of using to computers to play math games related to the topics being covered that week.

Some concerns and challenges were also shared in the postings. Some of the themes ranged from the *inequity of resources, anxiety caused by failure of technology, and apprehension about classroom management when using virtual manipulatives*.

Inequity of resources

In the school that I am placed in it is a Title 1 school, so from what I observed there are no computers, iPad, SMART boards, or anything in the classroom. The classroom did have a television that the teacher had hooked up to her own computer and that was the only technology I saw. From what I observed I did not see any carts that had iPad or computers on them for the teacher to reserve them. If the teacher can reserve computers the way I would use them in the classroom is by using various apps to aid students learning.

Anxiety caused by failure of technology

I have to admit that I am feeling some anxiety in using technology in one of our lesson plans. I think I feel like there is always the possibility of the computer crashing, the Internet not working, or one of the many other errors that could occur turning our lesson into a total flop. Perhaps this is the same anxiety that some of our cooperating teachers feel. It's like they know that it is important to incorporate technology but they are not comfortable with its incorporation in the world of math quite yet.

Apprehensive about classroom management

My only concern with the use of virtual manipulatives during morning work and center work is keeping track of the students proper engagement with the games, since many students may take advantage of being in front of a tablet or computer to wander off to different websites, or to fake that they are engaged when they are actually not productively thinking. I really do not know if there is a way to control that. However, balancing the use of the games and virtual manipulatives with the traditional games and worksheets would make the routine more interesting and help students who need a variety of methods to learn.

These responses served as an initial baseline for the ways PSTs were thinking about technology integration in the mathematics classroom.

13.5.2 Practice-Based Assignment #1: Technology Evaluation-Selecting a Mathematics Applet/Technology

In the first assignment, when asked to evaluate applets, PSTs freely chose applets that were available on the web. They used the evaluation form (see Fig. 13.1) categorizing the websites as drill and practice, virtual manipulatives, or investigations. Then they reflected on two reflective questions asking how they would use the applets in their classroom and how the applets promoted the Common Core State Standards for Mathematical Practice (CCSS-M 2010). Using Smith and Stein (2011) Levels of Cognitive Demand, I sorted the tasks as low level and high level cognitive demand to evaluate the potential that the tasks had for mathematics connections (see Table 13.1). According to Smith and Stein (2011), low-level tasks are characterized

Table 13.1 Analysis of applets chosen by PSTs

Low-level cognitive demand 12/40	*Applets for memorization* Examples: rehearsal of math facts-addition, subtraction, multiplication and division games that rehearse math facts tied to race or sports awards 1. Money recognition applets 2. Identifying fractions (matching exercises) 3. Ghost number sequencing	*Procedures without connections applets* 1. Online addition arcade game 2. Multiplication fluency game 3. Rounding estimation game 4. Fraction number line game 5. Garage sale money game 6. Counting Apple game 7. Place value game 8. Add like mad 9. Turtle addition game
High-level cognitive demand 28/40	*VM applets for procedures with visual representations and connections* 1. Fractions—parts of a whole (NLVM) 2. Money on NLVM 3. Base ten addition on NLVM 4. Fraction multiplication NLVM 5. Fraction feud (calculationnation) 6. Thinking blocks—Ratios 7. Okta's rescue illuminations 8. Number line bounce NLVM 9. Spin the big wheel! (explorelearning) 10. NLVM algebra balance scales 11. Deep sea duel (illuminations) 12. Adaptedmind-math 13. Factor dazzle (calculationnation) 14. Simple maze game (Shodor) 15. Base blocks addition (NLVM) 16. Equivalent fractions (Shodor)	*Doing mathematics/problem solving/logic games websites* 1. Explore learning: cannon ball 2. Thinking blocks: modeling problems: 3. Game/puzzle-circle 21 4. Kenken 5. Illuminations-bobby bear 6. Mathport 7. Proportionland 8. Math by design 9. Scale city 10. Rock and roll roadtrip 11. Explorelearning: walk the line 12. Coin logic problem: NLVM
Open-ended tools: virtual manipulatives Environment for open exploration 3/3 1. Pattern blocks on NLVM 2. Tessellate on Shodor interactivate 3. Glencoe virtual manipulatives		

by memorization and procedures without connections and high-level tasks are described as procedures with connections and doing mathematics.

This sorting exercise made me reflect that, in the next iteration of this assignment, I would ask PSTs to categorize the applets further using the levels of cognitive demand as it helped make a distinction between tools that rehearsed procedures only and tools that help build procedural understanding and problem solving. An example of a low level applet was a game and review for math facts that was procedural without connections to conceptual development. In the Math Sport game, students had to answer multiplication facts accurately to get a chance to make free throws as a reward. However, the PST rated it high, giving it a 4 out of 5. Another applet that was rated 4 was a place value game where one would have to identify the place value by clicking on the numeral. There was no assessment of the value of the digits just a recognition of the place value name. Although the PSTs noted that it was just rehearsing facts, they felt like the games added a level of engagement with rewards that would encourage the development of fact fluency. Table 13.1 shows the collection of applets compiled from the PSTs that was sorted by the levels of cognitive demand.

13.5.2.1 Opportunities to Elicit TPCK and Redesign for PBA #1: Recognizing the Need to Consider the Cognitive Demand

From the collection of applet evaluations, it was clear that PSTs needed more tailored instruction on how to select mathematics applets that offered opportunities to extend students' mathematics thinking and learning. Although I was surprised that some drill and practice games were rated as high, most of the applets chosen by PSTs were high-level mathematics applets. When I used the levels of cognitive demand to sort the forty applets evaluated by the PSTs, I categorized 12 as low-level applets and 28 as high-level applets. Three others were put in a separate category called open-ended tools because the level of the task depended on how teachers implemented the activity using the virtual tool. By sorting through the applets and websites the PSTs reviewed, it was apparent that the PSTs were able to find high cognitive demand applets. However, to increase the rigor of this activity, the next iteration of this PBA will include the levels of cognitive demand as part of the criteria when having PSTs evaluate applets. Since the choice of technologies affords and constrains the types of concepts and processes that can be taught, using Stein and Smith's framework, I determined it would help PSTs think about the level of cognitive demand present in a task and encourage the goal of ambitious teaching.

The following analysis is of the work (Fig. 13.1) of one of the participants, Kathy, a PST who explored the NLVM Algebra Balance Scale. She was placed in an upper elementary mathematics classroom teaching advanced mathematics during her field experience.

Her overall rating was a 5 even though she rated a few criteria with ratings of 1 and 2. It was obvious why she rated it high after reading her reflection. When asked, "How would you use this tool to bring out the Standards for Mathematical Practices?" she cited three specific practices: (1) Make sense of problems and persevere in solving them; (2) Model with mathematics; and, (3) Look for and express regularity in repeated reasoning. This PST showed evidence cited by Mishra and Koehler's description of TPACK (2006), where she reflected on the knowledge of student thinking and learning, knowledge of subject matter, and knowledge of technology. Kathy detailed in her reflection how the applet encouraged the following mathematical practices.

1. Make sense of problems and persevere in solving them—As students write their own equations, they will need to persevere in using the tool so that they learn to do the same thing to both sides and maintain a balanced equation each time. Linking the number representations to the manipulatives on the balanced scale will help them organize and make sense of equations with variables as well as gain a stronger conception of negatives as opposites that lift or subtract weight from the balance.

2. Model with mathematics—This app should be used to supplement teaching of variables and balancing equations. One benefit of the app is that the user cannot click "continue" until the equation has been correctly represented with blocks and balloons on the scale. Also, users may evaluate as many equations as they wish to create, which eliminates the constraints of physical materials. Students will discover the "idea of a variable as something that varies" (Van de Walle et al. 2014, p. 118). To enhance this app, I would have students then prove their solution by substituting the x value back into the equation. This is an important step in students' mathematical thinking—reflecting on and justifying their answers.
3. Look for and express regularity in repeated reasoning—Through the repeated exploration of this app, students will discover how strategies they choose work well or not. They will also link the correct manipulation of blocks and balloons to the values in their initial equation and each simplified equivalent. This will increase their fluency with variables and enhance their algebraic skills.

In this PBA, it was evident through the many other PSTs' analyses that they were thinking deeply about the important role of technology tools in education and how technology can, as Goldenberg (2000) states, "help students develop new and powerful ways of looking at problems, help them build mental models, acquire generalizable and flexible skills" (p. 6). In Kathy's reflection, she recommended improvements to the applet that would allow collaborative learning and mathematics communication:

> I would have a two-player feature, to encourage more positive social interaction as students discuss and justify their mathematical thinking and to include a printable report, so this added feature would enable teachers to collect and analyze data related to their learners' practice and thinking.

For Practice-Based Assignment #1, the themes revealed that the PSTs were developing understanding that different technology tools had specific affordances that can help develop conceptual understanding as well as procedural understanding and important mathematical practices. The thoughtful responses from the pre-service teachers demonstrated evidence that they were taking a critical look at the applets with student learning at the center of their analysis.

13.5.3 Practice-Based Assignment #2: Sequencing Technology Applets to Reflect on the Mathematics Learning Progressions

The second Practice-based Assignment was designed so that PSTs could think deeply about the learning progressions for a specific mathematics concept and select three related applets that could be used to teach and learn that concept. This assignment was also related to two of their big field assignments where they had to plan and teach a lesson using technology and assess a student's understanding about a concept using a variety of representations. The instruction for the PSTs was the following:

> Locate three different virtual manipulatives or applets that support the mathematical content you will address in the student assessment project for EDCI 552. Using the template below, analyze the models you have selected and evaluate them on their effectiveness and fidelity to the mathematical concept.

The following analysis is of the work submitted by one of the participants, Cindy, who illustrates how she interpreted this assignment and the TPACK learning that was elicited from the activity (see Fig. 13.2). Cindy chose three fraction virtual manipulative applets and discussed the affordances of each of the tools and how it could help the students she assessed as part of her student assessment assignment.

13.5.3.1 Opportunities to Elicit TPCK Through PBA #2: Mapping Along the Mathematics Learning Progression

Cindy reflected on how the different tools have different affordances. All three applets selected offer a variety of representations including using a region model, area model, and a number line model tied to the symbolic representation of the fraction notation. The concept that she focused on appears in our state's 4th-grade standard, "The student will (a) compare and order fractions and mixed numbers; (b) represent equivalent fractions" and most closely aligns to the CCSS-M (2010) Numbers Grade 4 Fractions A.1 and A.2:

Extend understanding of fraction equivalence and ordering.

> <u>CCSS.MATH.CONTENT.4.NF.A.1</u>
> Explain why a fraction a/b is equivalent to a fraction (n × a)/(n × b) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.
> <u>CCSS.MATH.CONTENT.4.NF.A.2</u>
> Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

The first applet, the Equivalent Fraction applet, ties the pictorial representation of a region model with the symbolic notation of the fraction and places the fraction on the number line. This applet allows users to compare up to three fractions with different numerators and different denominators (e.g., by creating common denominators or numerators). The Fraction Bar, the second applet, has a customizable feature for changing the numerators and denominators for four different bars. Cindy used the second applet to help her students think about comparing fractions using benchmark fractions such as ½. The final applet, Fraction Track, allowed the student to demonstrate his understanding of the fraction as a number on the number line. Not only did Cindy find appropriate applets that aligned to the mathematical learning goal, but she was able to articulate why she chose the applet and what specific mathematics the tool would highlight for the learner. By selecting these three related applets, Cindy demonstrated an understanding of the concept of representing fraction on a number line, comparing fractions by renaming fractions with like denominators, and using the benchmark of ½ to compare.

Another important mathematical practice when learning to teach mathematics is the ability to understand learning progressions to appropriately assess students' understanding. The final assignment in the mathematics methods course was to assess students' understanding. After Cindy administered this individual assessment, she revisited the applets that she had sequenced using learning progressions to make recommendations for further instruction.

> While the student was able to come to a correct answer regarding the placement of $\frac{5}{8}$ and $\frac{7}{12}$ on the number line, it did take some prodding and additional questioning. One virtual manipulative that may be helpful to this student is Math Playground's Fraction Bars found here: http://www.mathplayground.com/Fraction_bars.html. It was difficult for the student to verbalize that $\frac{1}{12}$ over the one half was different and less than $\frac{1}{8}$ over one half. I would use this tool to first allow my student to visualize the fractions of $\frac{7}{12}$ and $\frac{5}{8}$. With manipulative practice to see the relative size of fractional parts combined with practice comparing closeness to ½, the student would be better equipped to compare $\frac{5}{8}$ and $\frac{7}{12}$.

Here Cindy clearly demonstrates her TPACK and notes that the tool affords an opportunity to help students visualize the benchmark ½ and use that to compare and place 5/8 and 7/12 along the number line connecting 4/8 and 6/12 as ½ and having the student make sense of the remaining 1/8 versus 1/12.

Another recommendation Cindy makes for her student is to expose him to multiple images for fractions partitioned in equal parts, which may not always be congruent regions, by using the area model and knowing equal area (see Fig. 13.3).

> When looking to the question where the child must select which images from a group of seven are correctly partitioned into fourths, we see that the child struggles with understanding fractional parts as equal area regardless of shape. Similarly, this child can work on

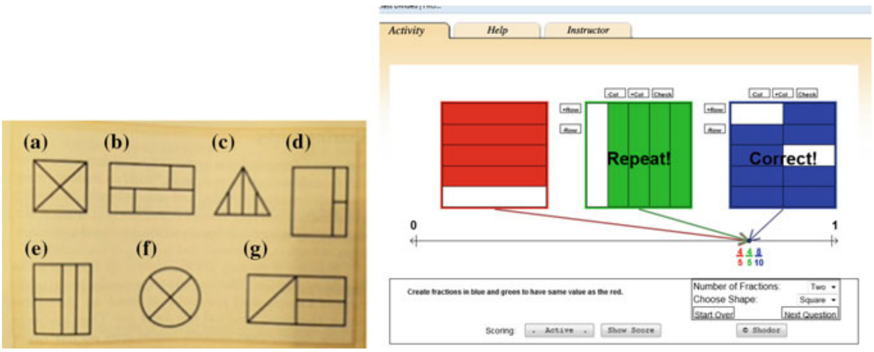


Fig. 13.3 Applets helped connect finding equivalent fractions with equal parts

finding fair shares using the "Equivalent Fractions Finder" manipulative found here: http://www.shodor.org/interactivate/activities/EquivFractionFinder/. He can use this virtual manipulative to create images that are not normally seen as fractional parts in order to deepen his understanding of this fraction concept. In the image below, the "Equivalent Fractions Finder" manipulative was used to represent the same fraction in three different ways.

For Practice-Based Assignment #2, the themes revealed that the PSTs were developing the pedagogical strategies of using learning progressions of mathematics concepts to scaffold and tier the teaching and learning sequence.

13.5.4 *Practice-Based Assignment #3: Integrating Technology in a Mathematics Lesson*

In this final case study, a PST, Linda, planned a whole group lesson where students created patterns. The focus of the lesson was to recognize, describe, extend, and create a wide variety of growing and repeating patterns. She differentiated the tasks with parallel tasks using the Open Virtual Manipulatives site from Glencoe and the National Library of Virtual Manipulatives site (see Fig. 13.4). She began the lesson working with her class as a whole group to create ABAB, ABB, AAB, and ABC patterns. Each student completed and labeled a generated pattern. For an extra challenge, Linda prompted students to turn and talk with partners asking, "What will be the 17th (next) color? What will be the 20th color?"

Linda reflected that the important idea that she learned from this lesson was that conducting a rich and focused mathematics lesson requires one to be fully prepared with content knowledge and engaging materials that motivate students to learn. She also remarked that "the objectives, tasks, and assessment must all tie together for a cohesive, standards-based learning experience."

One aspect that Linda gave particular attention to was integrating technology to facilitate the equitable access to learning for all students by differentiating the tasks. She stated,

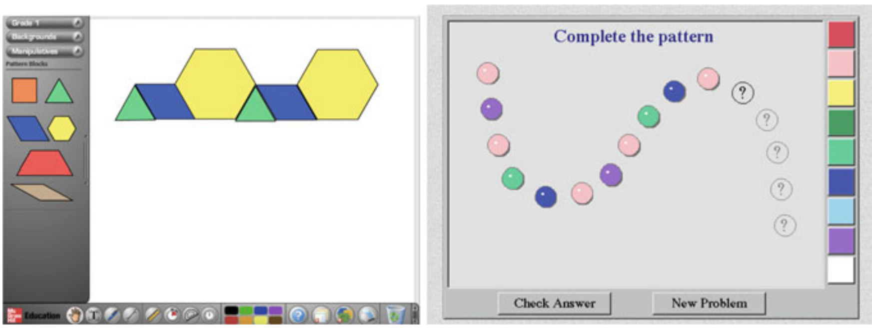


Fig. 13.4 Parallel tasks planned for a lesson on patterns

> An important skill that I must continue to work on is challenging students to develop higher-order thinking skills by posing analysis and synthesis problems. For example, students could generalize their learning about patterns through a project that helps them discover or generate patterns in their environment. Another way to stimulate higher-order thinking is to ask open-ended questions. I could pose a question such as, "How can knowing the core help you find out what comes next in the pattern?" This requires students to analyze and interpret what they know to discover the unknown. I could also include the patterns found in music, which would greatly enrich the lesson and add creativity. All of the applets used for guided practice were quicker and easier to use than paper and pencil, which allowed each student to have a turn with the technology and gave me the opportunity to probe their thinking.

13.5.5 Creating Their Own Practical Image of Practice After Performing in the Field: A Summary of PSTs' Learning

Goldenberg (2000) stated, "We must also provide time and opportunity for teachers to become fluent with the tools so that they can be flexible, use spur-of-the-moment good judgment in their classrooms, and not feel constrained by the tools or stilted by a lack of confidence in their ability to use them" (p. 7). After providing PSTs time and space to work with technology tools in the mathematics classroom, I was interested to examine how their own beliefs of teaching through technology may have evolved. In their final discussion prompt, I asked them: "On the discussion board, post your ideas for how you plan to incorporate technology into your math lessons. Cite Van de Walle's recommendations as you reflect on the implication to how you plan to integrate technology in your math lessons."

After their field experience, where PSTs planned and taught a mathematics lesson integrating technology, the most common theme was recognizing the need for more TPACK. This was a discussion that was not evident in the initial posting.

> Before I am able to effectively use digital tools for math instruction, I will seek out professional development opportunities to become more fluent in their functions and applications. Specifically, I need to strengthen my ability to use and teach using the graphing calculator. Its capabilities are many, like computing large quantities, applying a mathematical representation to model a real-world situation, testing a solution to check if it makes sense in context, estimating values to examine relationships, and selecting a strategy to solve a problem, to name just a few. I want to become more knowledgeable about its functions and teach it correctly so that students may use it confidently during their explorations of math concepts.

The finding that TPACK was the most common theme in their reflections was revealing because PSTs recognized that their knowledge of how to integrate technology was key.

Another pre-service teacher shared how she will need more time to familiarize herself with ways technology can help students understand particularly difficult topics and knowing how technology can be used to build on existing knowledge.

> I agree that there are so many useful tools out there and that it is our responsibility as educators to become experts in TPACK components. In order for us to assist students with exploring the latest technology tools we need to be able to navigate them first. I am looking forward to having a little bit of time this summer to explore more tools and also plan on taking advantage of any professional development opportunities that become available in order to build up my fluency in the available technology out there.

The importance of integrating technology in purposeful ways was voiced in this pre-service teacher's response as she commented on going beyond her technology apprehensions.

> I want to integrate technology when appropriate in my classroom and ensure that it is effective, not just incorporated to say it was. It should be beneficial to the objective of the lesson and not just an add-on to give the students something to do. It should have a well thought out purpose and be an integral part of the lesson. On the other hand, you also have to be prepared in the case that something happens and you cannot use the technology tools as we have all experienced at some point that some things just don't work out the way we have planned!

The belief that technology is a tool to bring equity to the classroom came up in this final response from one PST to another.

> I also agree with you that virtual tools help to bridge the achievement and economic gap when it comes to math content. What I mean by this is that free virtual manipulatives make math instruction equitable for all students. I did not make that connection in my post, but I am glad that you did so in yours. The fact that these tools are free, interactive and reinforce mathematic concepts is great (and truly valuable) for students in low socioeconomic schools. This is just another way to show that math can be fun and engaging when used in the right/appropriate way.

For Practice-Based Assignment #3, the themes revealed that the PSTs recognized the importance of understanding ways to use technology to provide more equity and access for diverse learners while helping students bridge a gap when they have specific learning difficulties in mathematics. This also related to their need to learn more about ways to build on students' existing knowledge.

Through these final responses, it was evident that PSTs were thinking more deeply about the importance of TPACK in their development and in their practice. As a result of the sequence of practice-based assignments (see Fig. 13.5), PSTs had more time and space to explore virtual manipulatives as learners themselves and then to use them as teachers in the classroom. The sequence of practice-based assignments allowed scaffolding needed for PSTs to explore, analyze and plan for virtual manipulatives to become an important tool for teaching and assessing student learning.

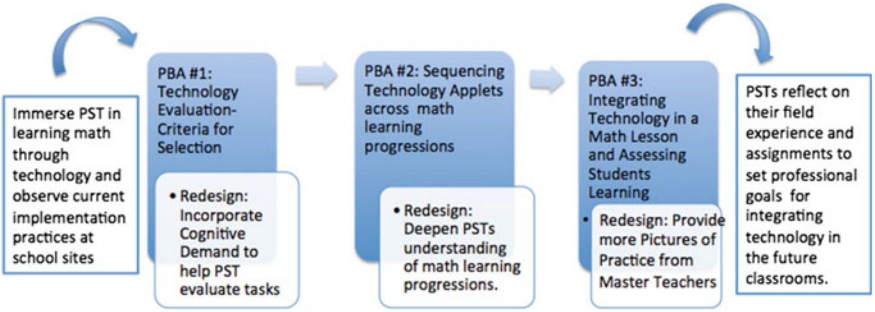


Fig. 13.5 Sequence of practice-based assignments

13.6 Conclusion and Consideration for Teacher Educators: Using Feedback from Assignments to Tailor the Next Cycle

The original purpose for the design of the practice-based assignments was to create manageable, structured routines that would allow PSTs to learn how to evaluate applets, select a few applets that aligned to their mathematics learning goals, and plan, teach, and assess student learning. The initial observation at their field site was designed so that they could have a "picture of practice" for integrating technology in the mathematics classroom. After reading and analyzing the first posts on the discussion board, it was clear that some PSTs observed best practices integrating technology while others did not. Although what they observed is the reality of today's classroom, as an instructor, I want to create an opportunity for PSTs to see several exemplars of best practices for integrating technology in the mathematics classroom. In this way, PSTs can have a "picture of practice" of ambitious teaching in the mathematics classroom that showcases effective integration of technology for learning. One way to create this opportunity is to plan instructional rounds to a classroom where the teacher is effectively integrating technology. By asking some master teachers to model instructional practice, the PSTs and I will have a collective experience of observing an exemplar lesson. Another way researchers and educators have offered this exemplary practice is through rehearsals. Lampert et al. (2013) share a method that they call rehearsal where the teacher educator and novice teachers conduct "run-throughs" or microteaching in methods courses. Here, the novice teacher teaches an instructional activity, while the teacher educator and other novice teachers are in the role of simulated classroom students, who "act back" in a way that students might in an actual classroom.

In addition to providing rehearsals and tailored observations with masterful teachers, I learned through *PBA #1: Technology Applet and Website Evaluation* that PSTs need to evaluate applets with cognitive demand in mind. The analysis I experienced when sorting through the PST selected applets allowed me to consider what level of cognitive demand the applets offered. Was the applet allowing for the user to make sense of a procedure with connections or was the applet offering opportunities for problem solving and doing mathematics? These are good ways to evaluate technology tools beyond examining external features and would forward ambitious teaching and learning goals. Another lesson I learned from *PBA #2 Sequencing Technology Applets to Reflect on the Mathematics Learning Progressions* was the need to sequence the variety of applets that are worthwhile using the mathematics learning progression more explicitly. PSTs were exceptional at aligning the applets to a standard, but more instruction could be provided to look across the vertical strands to map out the learning trajectory. This vertical articulation would be beneficial for PSTs as a thread for frequent classroom discussions when planning, designing and conducting diagnostic assessments for the course. Finally, I learned from *PBA #3 Integrating Technology in a Math Lesson* that PSTs may need more rehearsal with supportive co-teachers modeling ideal ways to integrate technology in the mathematics classroom. As mentioned in the introduction, using technology in the mathematics classroom may be a novel approach for many in-service teachers who may not be using technology in their mathematics classrooms. There is a need to work with both in-service teachers and pre-service teachers to develop their repertoire for integrating technology effectively in the mathematics classroom. Providing more "pictures of practice" from master teachers who can serve as exemplary models of ambitious teaching using technology in the mathematics classrooms may help build PSTs repertoire as they are enter the teaching profession.

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Chapter 14
Developing an Interactive Instrument for Evaluating Teachers' Professionally Situated Knowledge in Geometry and Measurement

Agida G. Manizade and Dragana Martinovic

Abstract In this study, we propose a content specific, short, interactive, on-line, scenario-based instrument that incorporates virtual manipulatives developed in GeoGebra, as one of the many ways for evaluating and describing teachers' professionally situated knowledge (PSK) in the domains of geometry and measurement. To define PSK of mathematics teachers, we use a combination of Shulman's Pedagogical Content Knowledge (PCK) and its corresponding mathematical knowledge. We describe the methodology used to develop the instrument as well as the corresponding rubrics. The study design followed a concurrent mixed-methods approach, in which the quantitative and qualitative phases of data collection were intermingled to explore the research questions related to identifying components of the PSK. As a case study, we used PSK of the area of a trapezoid, since (a) this topic is familiar to most middle school and secondary mathematics teachers; and (b) a narrow focus was most advantageous when using Delphi methodology to draft an instrument, which was then fully developed using methods of grounded theory. This comprehensive approach led to a deep investigation of multiple and diverse data sources collected from practicing teachers, which we used to create their PSK profiles in the area of trapezoid.

14.1 Introduction

The purpose of this paper is to propose a content specific, short, interactive, online, scenario-based instrument that incorporates virtual manipulatives developed in GeoGebra as one of the many ways for evaluating and describing teachers'

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professionally situated knowledge (PSK) in the domains of geometry and measurement. To define the PSK of mathematics teachers, we use a combination of Shulman's Pedagogical Content Knowledge (PCK) and its corresponding mathematical knowledge (e.g., knowledge of area concept, units of measurement, areas of polygons, geometric transformations, congruence, congruent triangle conditions, similar triangles, parallel lines and angle relations, concept of altitude in triangles and quadrilaterals, etc.).

There are several assumptions we are making in our work. We assume that, in order to possess and develop PSK, teachers have to have corresponding mathematical content knowledge (subject matter knowledge). Teachers develop PSK for each mathematics topic individually. In other words, just because a teacher has a very strong PSK for teaching angles, does not mean that he or she will effectively teach the surface area of a cone. We also assume that one of the ways to gather information on teachers' PSK is to observe their interaction with a virtual manipulative and collect their reflections on such an experience, as well as obtain data on their analysis of student work samples. Finally, we make an assumption that in order to design an effective professional development (PD) for teachers, specific information on subject matter knowledge as well as PCK has to be collected and considered prior to PD.

In this chapter, we propose designing online, interactive, dynamic, short, scenario-based instruments that would allow mathematics educators to gather information needed to design differentiated PD for mathematics teachers. These electronic instruments may also allow us to develop an understanding of the relationships and connections between different types of teachers' knowledge, which would be important for the design and development of mathematics teacher training programs.

Currently, there are many different types of instruments used in the field for measuring teachers' professionally situated knowledge (PSK), and other related constructs. These tools have their own strengths and limitations, and range from lesson observation protocols to paper and pencil tests (Hill et al. 2008; Knowing Mathematics for Teaching Algebra Project 2006; University of Louisville 2004). Several research teams have spent significant time and money to develop such instruments (Hill et al. 2008; Silverman and Thompson 2008). The major limitations of these instruments are that they are too long, costly, and do not cover all of the mathematics applicable for the content background of any group of mathematics teachers, such as geometry teachers. That is why we propose developing content specific, short, interactive, online, scenario-based instruments, which are adaptable to multiple mathematics topics. These instruments would incorporate virtual manipulatives developed in GeoGebra or another software, as one of the many ways to measure and describe teachers' PSK in the domains of geometry and measurement. These types of instruments could be coupled with other existing standardized measures to gather information about teachers' backgrounds.

In this chapter, we specifically address the following research questions:

1. In what ways, can we incorporate virtual manipulatives into designing measures of mathematics teachers' PSK?
2. What are some affordances of virtual manipulatives developed in GeoGebra that may enhance or hinder their use in a PD for geometry and measurement?
3. How can we develop profiles of mathematics teachers' PSK in geometry and measurement?

14.1.1 Virtual Manipulatives as Special Mathematics Machines or Digital Objects

Our decision to employ virtual manipulatives in this work was not made arbitrarily. The literature on this subject confirms that virtual manipulatives can be thought of as mathematical machines, which are suitable for laboratory sessions (see Bartolini Bussi and Maschietto 2008) with mathematics teachers. If used in this way, virtual manipulatives could reveal different aspects of mathematics teachers' PSK.

In our research, we extend Bartolini and Maschietto's (2008) ideas about the "specific professional competences" needed by teachers to effectively use artifacts in the mathematics classroom. For example, Bartolini and Maschietto use concrete artifacts (compass, Mira mirrors, etc.) with student teachers, because through guided use the mathematical meanings of these tools become more transparent. In our view, practicing teachers have already passed through this stage where they unpack the concrete artifact. School districts currently provide teachers with concrete tools that are well known and that have been part of mathematics culture for some time already (e.g., base 10 blocks, interlocking cubes, scales, and kits for 3-D geometry).

On the other hand, virtual tools (e.g., calculators and dynamic geometry software) change over time and teachers need to be trained in their use. Digital objects created in virtual tools come in a multitude of forms for different purposes. Such objects become virtual manipulatives when their mathematical meaning is uncovered through activity. Because the digital objects described in this chapter are claimed to have been created by students, they could be used to invite teachers to implement their formative assessment skills to connect them to their pedagogy. In addition, the teachers are asked to interpret and improve these objects to uncover and build upon their potentially limited mathematical meaning. Through this process the digital objects we developed, become virtual manipulatives. In conjunction with targeted questions, these virtual manipulatives are grounded in the multifaceted nature of teachers' PSK. We defined PSK as the possession of the following qualities: (1) specialized geometric content knowledge; (2) knowledge of student challenges and understandings; (3) ability to ask appropriate diagnostic questions; (4) pedagogical knowledge of appropriate instructional strategies and knowledge of proper use of manipulatives and technology; and (5) knowledge of geometric extensions designed to deepen students' understanding of the problem.

Mathematics manipulatives are usually used in demonstrations of a concept or in explorations (Marshall and Paul 2008). Accordingly, a virtual manipulative is defined as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al. 2002, p. 373). Virtual manipulatives created in GeoGebra have the common characteristics of all digitally created objects in what is called dynamic geometry software: (a) users can move images to change their position on the screen or to zoom in or out; (b) some points on these objects could be dragged while keeping interdependencies intact (a vertex of the triangle can be dragged to resize the triangle and change its shape, unless the triangle was constructed to have specific features, e.g., as equilateral or right, in which case these features remain during dragging); and, (c) users have control over the software and the object (e.g., to develop, animate, and change objects on the screen). While the objects created in software are not material they act like material objects by being responsive to users' actions. When the user manipulates these objects, s/he moves between the physical, the perceptual, and the conceptual domains, which may ultimately bring mathematical ideas and processes to the conscious level (Sarama and Clements 2009). This process may be useful for students as well as teachers as they develop their understanding of the presented mathematical concepts.

Stacey and Wiliam (2013) state that dynamic geometry programs allow students to "demonstrate a wide range of abilities" (p. 745). It stands to reason that they could also be used to reveal a teacher's abilities to experiment, investigate, make and test hypotheses, and create proofs (Sinclair and Robutti 2013). Also, there is a body of literature which states that when the virtual manipulative is designed so that the users can explore it (e.g., move it, change its features and parameters), its users may achieve more than when using physical manipulatives, although for the largest effect, it is advisable to combine virtual and physical manipulatives in mathematics teaching (Moyer-Packenham and Westenskow 2013). According to Sarama and Clements, careful design of virtual manipulatives, which brings forward specific features and allows for guided exploration, can make them superior to physical manipulatives. Virtual manipulatives can help learners to develop new conjectures about a mathematical topic. As they are created for a specific purpose, these virtual manipulatives provide an environment that is defined by the affordances and constraints of (a) the software, (b) the mathematical objects or artifacts implemented in them, and (c) the pedagogical ideas of their designers (Martinovic et al. 2015). By observing teachers when using these manipulatives and recording their reflections, we could identify aspects, such as (1) knowledge related to the area concept, units of measurement, areas of polygons, geometric transformations, congruence, congruent triangle conditions, similar triangles, parallel lines and angle relations, the concept of altitude in triangles and quadrilaterals; (2) knowledge of student challenges and understandings related to developing the formula for the area of a trapezoid; (3) ability to ask appropriate diagnostic questions; (4) pedagogical knowledge of appropriate instructional strategies and knowledge of proper use of manipulatives and technology; and (5) knowledge of geometric extensions designed to deepen students' understanding of developing the formula for the area of a trapezoid. Once we identify and measure the components of teachers' PSK related to this specific topic in geometry (using rubrics developed in the study) we map them as radar diagrams described later in this paper.

We selected the area of a trapezoid as a content focus of this study. It is important to have PSK about the area of a trapezoid because this is one of the commonly taught concepts at the middle and high school levels. Often, students do not get an opportunity to explore and discover the formula for the area of a trapezoid in their mathematics classrooms, however when given a chance, they can come up with multiple ways of justifying/discovering this formula. When teaching the concept of area, it is not sufficient to give students the pertinent formulas and have them merely compute the areas of various polygons. It is very important that students develop an understanding of the concept of area so they can reason about the relationships between shapes to determine area (Manizade and Mason 2014). We believe that teachers with a higher PSK are more likely to provide such opportunities to their students.

14.2 Methodology

The virtual manipulatives described in this chapter were created as part of the larger study in which we used multiple data collection methods to understand the status of the participants' PSK related to the area of a trapezoid.

14.2.1 Participants

The participants in the study were 39 volunteer Geometry teachers from 12 divisions in one of the eastern states of the United States. They had a wide range of teaching experience (i.e., 1–16 years, $M = 7$ years). Also, 37 of 39 (95 %) participants had at least 1 year of high school level Geometry teaching experience. All participants taught high school level mathematics classes at either high school or middle school. Based on self-reported data, they were familiar with the mathematical topic presented to them in the study, which was the area of the trapezoid. The volunteers were recruited from the M.S. program in mathematics education in a state university in the Southeastern United States. They were all taking an online graduate course on Euclidian and Non-Euclidian Geometry while the initial data collection was conducted.

14.2.2 Instrumentation

Nine virtual manipulatives (PCK instrument, adapted from Manizade 2006; Manizade and Mason 2011, 2014)—presented as dynamic GeoGebra files, were used as fictional samples of student work. The teachers were asked to interpret student work, rate the appropriateness, clarity, sophistication, and limitations of student strategies, and propose possible ways to address students' misconceptions and difficulties. Here we present four virtual manipulatives (presented as the work of Adam, Whitney, Donna, and Paul, see Appendix 1) together with guiding questions to help teachers evaluate the students' work, explain their thinking, and provide alternative approaches and proofs.

The selection of four examples is based on the three types of approaches one may use to derive the formula of an area of trapezoid (i.e., decomposing, using transformational geometry, and enclosing the trapezoid; Manizade and Mason 2014). For example, the student can decompose a trapezoid into simpler shapes (triangles, rectangles) and then find the area of a trapezoid as a sum (see Whitney's and Paul's approaches in Appendix 1). Another approach is used when the student uses transformational geometry (rotation, translation, reflection), as shown in Adam's approach. Finally, the student may attempt to construct a shape that encloses the original trapezoid. Then the student finds the area of a trapezoid by subtracting the areas of external pieces from that of the enclosing shape, as presented in Donna's approach.

14.2.3 Data Collection

The data used in this chapter was obtained in a larger study in which we collected data on different components of 39 mathematics teachers' PSK and triangulated information using multiple sources of qualitative and qualitative data. The data were collected using the following instruments: (1) A van Hiele test to determine each teacher's developmental level in Geometry; (2) a Trapezoid Questionnaire pre-test that we developed to measure teachers' knowledge of the area of the trapezoid concept; (3) a Trapezoid PCK instrument with nine fictional samples of student solutions (adapted from Manizade and Mason 2011, 2014) created in GeoGebra, where we asked teachers to interpret student work, rate the appropriateness, clarity, sophistication, and limitations of student strategies, and propose possible ways to address students' misconceptions and difficulties; (4) teachers' reflections on their perceived PCK learning in this process; (5) teachers' demographic survey, where we recorded their gender, school grades taught, years of teaching experience, and years of teaching geometry; and, 6) follow-up classroom observations using the Instructional Quality Assessment Classroom Observation Tool (IQA), where we observed the instructional practices of purposefully selected teachers and validated previous findings. All data, except during the classroom observations, were collected using online means (e.g., electronically created and submitted responses through Desire 2 Learn and Adobe Connect, the online course learning management systems).

In this chapter, we focus on the aspect of the study related to the use of virtual manipulatives in order to gather information about the teachers' professionally situated knowledge (a Trapezoid PCK instrument) and to analyze the role of incorporated affordances when designing teachers' PD. We used virtual manipulatives in the instrument items by asking teachers to assess students' approaches and knowledge (Nührenbörger and Steinbring 2008). All teachers were given GeoGebra files to explore and to describe student thinking and come up with possible misconceptions the student might have (to see the GeoGebra files go to the GeoGebra book at the link: http://ggbtu.be/buLXjTUHI).

We also asked participants to develop a critical understanding of the epistemological character of each manipulative (Nührenbörger and Steinbring 2008). The teachers needed to come up with a strategy to use the current manipulative or to design a new one, which would address the student's misconception or misunderstanding and help the student to further his/her understanding of the area of a trapezoid. All of the qualitative data were collected from the teachers as written responses to the open ended questions. We allowed teachers to take as much time as needed to answer the questions. We consistently used the aforementioned instruments to collect qualitative data from every participant.

14.2.4 Data Analysis

The validity, reliability, trustworthiness, and rigor of the PSK-related questions included in the instruments used in the Delphi study (Manizade and Mason 2011) have been established and reported in the literature. The Delphi methodology included three rounds of surveys of a diverse panel of 20 experts. The data analysis and data collection were done reflectively. Initial items were developed based on the research literature in the field of teacher knowledge, and geometry teaching and learning. New categories for the analysis emerged from the data and were used to complete the analysis. The experts reached consensus after the third round of surveys, producing the assessment instrument (Manizade and Mason 2011). The adapted versions of instrument Items W-Z we developed for this study went through multiple rounds of peer review (five mathematics education researchers reviewed the instruments at least three times each).

Grounded Theory (Charmaz 2014; Glaser and Strauss 1967) was used to finalize the instrument and to develop rubrics to evaluate teachers' responses. This methodology provided us with a way to identify how different levels of teachers' content knowledge about the area of a trapezoid interact with teachers' PCK, what the role of affordances in teachers' PD was, and how to develop PSK profiles for each teacher. We achieved credibility by collecting a multitude of different data to merit our claims. One of the authors was the instructor in the course and had first-hand familiarity with the topic and the participants.

Qualitative data collected using the aforementioned items were coded using an open coding technique and analyzed for emerging themes related to teachers' PCK, according to our theoretical framework. We created rubrics to evaluate teachers' responses on the PCK items. This was done through a reflexive process of constant comparison between data and emerging rubrics and consisted of open-coding the responses, so that the core categories and the main themes became apparent. Memos were written throughout the entire process. We conducted selective coding and theoretical sampling. Additional sampling was conducted to saturate the core category and related categories. Once the categories became saturated, the memos were sorted out to find the theoretical code(s) which best organized the substantive codes.

Based on the themes that emerged from the data, we modified our initial definition of teachers' PSK used in the measures incorporating virtual manipulatives. The new working definition of the PSK included five subcomponents: Geometric knowledge, knowledge of student challenges and conceptions, ability to ask diagnostic questions, knowledge of applicable instructional strategies and tools, and ability to provide geometric extensions. These became rays in a visual representation of teachers' profiles as depicted in Fig. 14.1. Based on grounded theory techniques, once the working definition of PSK was generated, the literature related to the use of virtual manipulatives in education research, as well as the role of affordances (incorporated in the virtual manipulatives) in a PD for the mathematics teachers, were integrated to show how virtual manipulatives could be used to to gather information on teachers' PSK.

The rubrics were designed to discriminate between five levels of teachers' PSK (i.e., 0–4) and its sub-components. The initial versions of the rubrics were developed using the mathematics education literature and our professional experiences.

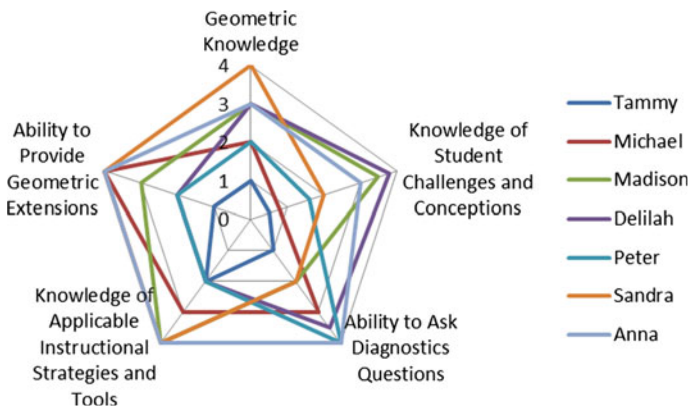


Fig. 14.1 Sample of individual profiles of marker teachers

Initial coding led us to develop new ideas and strategies for further data collection. We modified the rubrics after we coded teachers' responses as qualitative data for the additional emerging themes, and the new themes were identified and included in the corresponding PSK subcomponents of the developed rubrics. Then the new rubrics were checked against the qualitative data collected through the PCK Trapezoid instrument to look for any additional categories and themes, thus pulling us into an "interactive space" (Charmaz 2014, p. 115), where we critically inspected and challenged our preconceived ideas. We conducted coding with gerunds, "to define implicit meanings and actions" (p. 121), and to realize directions for exploration and comparison of data. The rubrics were then modified three to four times and refined to differentiate between levels of teacher competencies through a reflexive process of linking rubrics to the collected sets of raw data from 39 teachers. The developed rubrics allowed us to identify levels 0–4 (4 being the highest) of teacher responses for each of the aforementioned dimensions of PSK in the instrument using virtual manipulatives designed with GeoGebra.

14.3 Results

As a result of the study, we developed teacher profiles represented by radar diagrams as shown in Fig. 14.1. To create each ray of the radar diagram, we developed and implemented rubrics to discriminate between levels 0–4 for each of the subcomponents of PSK—geometric knowledge, knowledge of student challenges and conceptions, ability to ask diagnostic questions, knowledge of applicable instructional strategies and tools, and ability to provide geometric extensions. Then we plotted the numerical score for each teacher on the axes and connected the vertices to create a visual representation of individual teachers' PSK in geometry.

Figure 14.1 contains a visual depiction of the PSK of the seven teachers purposefully selected to cover the range of the teachers' developmental levels in Geometry, as determined through a van Hiele test (see the Geometric Knowledge axis) and the parameters of theoretical sampling (Charmaz 2014). Consequently, we selected one case each at Levels 1 and 4, two cases at Level 2 and three cases at Level 3.

In the further text, we present some examples of different levels of development of teachers' PSK identified using the manipulatives. In these examples, we focus on one of the five identified subcomponents of PSK titled 'Knowledge of Student Challenges and Conceptions'. For the comparison purposes we present examples of teachers' work rated as 1 versus 4 using the rubrics developed in the study. Table 14.1 (see Appendix 2) presents components of the rubric used to identify teachers' knowledge of student challenges and conceptions at Level 4. As a contrast, Table 14.2 (see Appendix 2) includes the rubric related to the teachers' knowledge of student challenges and conceptions at Level 1.

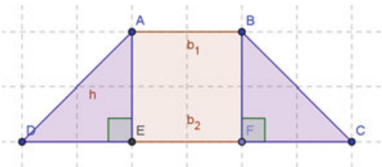
Table 14.1 Rubric used to evaluate teacher's knowledge of student challenges and conceptions, level 4

Teacher is able to identify A and (B or C) and (D or E) and F
(**A**) A student's limited conception of a trapezoid (e.g., isosceles, right). Or generalizability of the student's approach
(**B**) A student's limited strategy/method (e.g., using only decomposition; composition is basic; strategy that may not always work—decomposing trapezoid into a rectangle and two triangles, transformation may not always work, while enclosing and subtracting excess will always work) **OR**
(**C**) A special case potentially resulting in a limited or wrong formula. Or a generalizable case applicable for any trapezoid resulting in a proper formula
(**D**) A student's developmental level in geometry using the van Hiele theory of a trapezoid concept **OR**
(**E**) A student's developmental level in geometry using the van Hiele theory with respect to area concept (0-not understanding area; 1-basic understanding of adding units; 2-if the shapes match then their areas are equal; 3-if you re-arrange them they will still be the same; 4-using transformational geometry or simple Euclidian proof to claim equal areas)
(**F**) A student potentially developing these challenges due to the limited experiences with different types of trapezoids or tools used or lack of motivation

Table 14.2 Rubric used to evaluate teacher's knowledge of student challenges and conceptions, level 1

Teacher's response covers G and (H or I)
(**G**) Teacher recognizes that there is a misconception (if any) in student thinking but does not provide sufficient explanation of the actual misconception **OR** his/her explanation is mathematically incorrect
(**H**) The main focus is on the formula, algebra and counting the area units. **OR**
(**I**) The mathematical terminology is incorrect/poor. **OR** there is evidence of limitations in the teacher's understanding of the concepts of trapezoids and/or area (e.g. considers a set of special cases of trapezoids)

14.3.1 Whitney's Case



The following answer was evaluated as Level 4, strong:

Whitney has the automatic response of using the "traditional trapezoid" typically used in the classroom, which in turn is not inherently a negative viewpoint. This simply limits the ability to test her derivation on a more "interesting" example. Based on this figure, it seems that by cutting a right triangle and then pairing it with a mate on the other side would form a square when adding the areas together. This is noted by having both triangles outlined in blue. They complete each other. Then there is a square in the middle which is depicted by the altitudes of the two triangles adjacent it. Thus, creating two very simplistic shapes that can be easily analyzed. The grid in the background details exactly how many blocks (units) this trapezoid has in total.

I don't believe this would work on all trapezoids simply because not all trapezoids have a pair of triangles that can be dissected from it. Some consist of merely one triangle and a square or there may be two triangles that are not identical; therefore, complicating the task of "counting" square units.

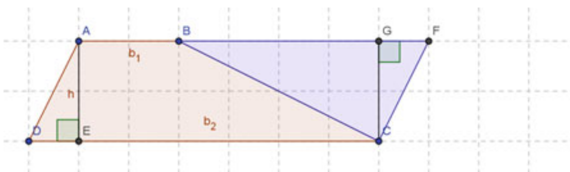
We concluded that the teacher understands the student's challenges related to the area of the trapezoid. She recognized the student's limited conception of a trapezoid (e.g., isosceles). In addition, she acknowledged a limited strategy used by the student (e.g., using only decomposition; composition is basic; strategy that may not always work—decomposing trapezoid into a square and two triangles). The teacher identified this as a special case potentially resulting in a limited or wrong formula. She also referenced the student's low developmental level in geometry with respect to the area concept (grid in the background for counting the units of the area). Finally, this teacher referenced the potential reason for the student's challenges as being due to the student's limited experiences with different types of trapezoids.

On the other hand, the following answer was evaluated as Level 1, weak:

The given trapezoid easily decomposes to a square and two triangles. This is a very common visual, however, it will not work for all trapezoids. There are some irregular trapezoids on which this would not work. Whitney assumes that a segment can be drawn from one end of the top base to be perpendicular to the bottom base. This is not true in all trapezoids.

In this example, the teacher recognized that there is a misconception in the student's thinking but did not provide a sufficient explanation of the actual misconception. She did not mention that this is a special case. Her use of mathematical terminology was poor. Terms such as "congruent", "height" and "segment" were not properly used. She also mentioned that for "some irregular trapezoids" this method would not work, while "some" should be "the most."

14.3.2 Donna's Case



This response was evaluated as Level 1, weak and not concrete:

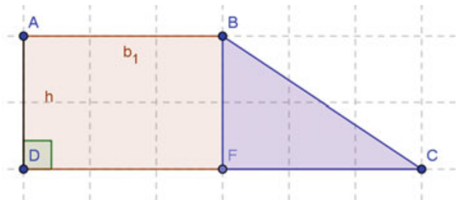
Based on Donna's approach, I believe that she has tried to simplify finding the area of the original trapezoid by decomposing it into simpler polygons so that the area would be easier to find.

In this case, the explanation was mathematically incorrect and terminology used was poor. The teacher in this sample data does not recognize that the trapezoid in this sample is not being decomposed, but instead it is being enclosed into a parallelogram.

Compare it to the response below, rated as Level 4, that had details and used proper mathematics language, referenced a higher level of geometric development presented in the manipulative, and recognized the generalizability of the proposed method:

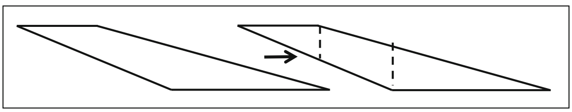
> Donna constructed another shape to contain the trapezoid. Donna recognized that since a trapezoid only has one pair of parallel sides, it is always possible to construct a line through one point of the trapezoid that is parallel to one of the non-parallel sides. By extending one of the bases, a parallelogram is constructed, since the bases are already parallel. The triangle formed by the gap between the parallelogram and the trapezoid will always have a base congruent to the opposite base in the trapezoid and the heights are also congruent.

14.3.3 Paul's Case



The following response was evaluated as Level 4, strong. It had enough detail and included a counterexample developed as a virtual manipulative. The teacher recognized the student's limited conception of a trapezoid (i.e., right angle trapezoid), limited strategy in deriving the formula, and lower level of geometric development related to the concept of a trapezoid. Finally, this teacher used mathematical terms correctly:

> Paul's method is similar to Whitney's: he can see that this particular trapezoid, with its right angles, lends itself well to being decomposed into a right triangle and a rectangle. However, this method will not work for any trapezoid since not all trapezoids have right angles. When presented with a trapezoid like the one Whitney had, Paul would have to adapt his method to incorporate a second triangle. If given a trapezoid like the one below, Paul would have no way to use his method since no rectangle can be easily derived. Even with the two right angles, a rectangle does not exist, since there are not two pairs or parallel sides or right angles for the quadrilateral:



The following answer was evaluated as Level 1, weak, due to the lack of the teacher's understanding of the mathematics involved in this sample of a student's work:

> Paul has broken the trapezoid into a right triangle and a rectangle. This would not work for any trapezoid because two right triangles will be formed in many trapezoids.

The teacher recognized that there was a misconception in the student's thinking but did not provide a sufficient explanation of the actual misconception. In addition, there is evidence that the teacher has a limited understanding of trapezoids. The participant's response indicates that he or she considered the case of a trapezoid in which the height creates two right triangles, but did not consider the case of a trapezoid presented in the image above.

Based on our data analysis, when teachers described how they would use manipulatives or technology to address student understanding of the concept, they mostly gave examples of using rubber bands and geoboards, tangrams, grid paper, Ang-legs, protractors and rulers to explore various trapezoids and to find their area. One teacher had the idea of giving the students "scissors, a pencil, and a ruler and ask [them] to try partitioning the trapezoid into the shapes that [they] used." The teachers also mentioned using technology such as Geometer's Sketchpad and GeoGebra, but in some cases called it a "paint program," presenting a very limited understanding of its affordances.

We identified affordances for each of the virtual manipulatives developed in this study using GeoGebra. These affordances may enhance or hinder the quality of discussions in the PD for geometry and measurement. They could also provide additional data points related to teachers' PSK. As teachers moved, stretched, and dragged trapezoids in the aforementioned cases of the manipulatives, they were able to gather information on how students created these tools and obtain insight on students' geometric reasoning related to trapezoids and the area. Teachers' pedagogical analysis of students' thinking provided supplementary data for the following two rays of the radar diagram presented in Fig. 14.1: geometric knowledge, and knowledge of student challenges and conceptions. Additional research is needed to identify the role of affordances in relation to the remaining rays of teachers' profiles. In future studies, it would be useful to further explore the relationship between affordances of virtual manipulatives and their potential to enhance or hinder PD in geometry and measurement.

14.4 Discussion

In this chapter, we discussed some ways in which we could incorporate virtual manipulatives into designing measures of mathematics teachers' PSK (i.e., our first research question). Our experience in doing so was positive. The virtual manipulatives we used (see Items W-Z at http://ggbtu.be/buLXjTUHI) have helped us develop the PSK profiles of mathematics teachers presented in Fig. 14.1 with respect to the area of a trapezoid. This process allowed us to identify the range of the in-service teachers' knowledge of geometry concepts and pedagogy, and also the ways in which their teaching philosophies and attitudes differ.

For example, some participants claimed that their main responsibility is to have an ordered, teacher-centered classroom, as exploration "often leads to chaos and a lot of wasted time." Other participants were inspired to use more exploration in their classes, followed by a whole-group discussion, so students can compare and review their ideas. Some teachers thought that their students do not have the skills and mindset conducive to exploration (e.g., "I anticipate that students will not want to come up with a formula-they would rather be told"; "Students are so used to being given a formula and a set of algorithms to follow I think at first they will find it difficult to come up with their own ideas"). Teachers with lower geometric knowledge retained this perspective even after working with virtual manipulatives, while the teachers with higher geometric knowledge expressed more openness to use virtual manipulatives with their students as exploration tools. As we continue to collect data in the future years of this longitudinal study, it would be interesting to know in what way, if any, the difference in teachers' perspectives affects their PSK profiles.

We also looked into affordances of virtual manipulatives developed in GeoGebra to see if they may enhance or hinder their use in a PD for geometry and measurement (i.e., the second research question). Using virtual manipulatives purposefully created as samples of students' varied understanding of a concept provided opportunities for laboratory sessions (Bartolini Bussi and Maschietto 2008), in which teachers moved, stretched, and dragged manipulatives on the screen in an attempt to understand how they were created and why. When teachers answer questions that target this understanding and relate it to their pedagogy, they reveal aspects of their PSK (that we presented as radar diagrams). The virtual manipulatives used in this project also present opportunities for the construction of mathematical knowledge (Moyer et al. 2002) and for this reason could be implemented in PD situations.

Our data also showed a difference in teachers' attitudes towards using technology. For example, one self-confident teacher claimed that every activity in a geometry classroom has to be a technology-based exploratory activity. While other teachers acknowledged that technology may be useful for developing conjectures, most of them thought that it would take substantial time to learn to use it effectively. Teacher attitudes towards technological tools and other manipulatives influenced the 4th subcomponent of PSK—knowledge of applicable instructional strategies and tools—as referenced in Fig. 14.1.

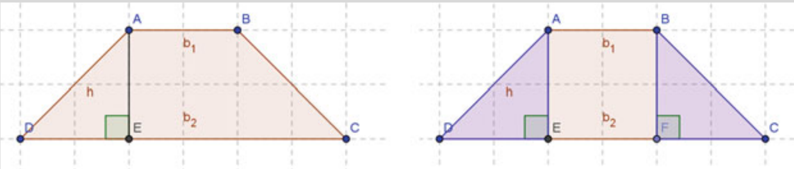
While here we presented one approach to developing profiles of mathematics teachers' PSK in geometry and measurement (i.e., the third research question), further studies are needed to explore connections between different subcomponents of the PSK and the ways they influence each other. The profiles themselves can be used to create a differentiated PD experience for geometry teachers. This study can also be replicated and used for other branches of mathematics.

Appendix 1

Virtual manipulatives (presented as the work of Whitney, Paul, Adam, and Donna) together with guiding questions to help teachers evaluate the students' work, explain their thinking, and provide alternative approaches and proofs.

Item W: Whitney's Approach

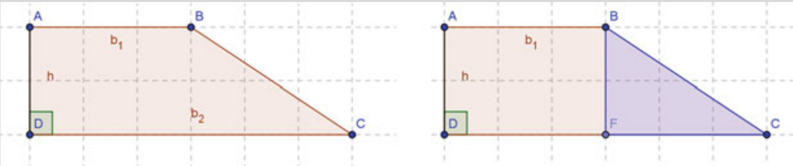
When presented with the task of *developing a formula for the area of any trapezoid* in her high school geometry class, Whitney developed the diagrams as a strategy for deriving the formula for the area of a trapezoid described by the sketches below. She decomposed a trapezoid into a rectangle and two congruent triangles. Then, she added the areas of all three shapes to calculate the area of the trapezoid.



(a) Based on the diagram above, describe Whitney's thinking. If she were to complete the formal derivation of the area formula using her diagrams, would her method work for any trapezoid? Why, or why not?
(b) If Whitney's approach presents a challenge or misunderstanding, what underlying geometric conception(s) or understanding(s) might lead her to the error presented in this item?
(c) If Whitney's approach presents a challenge or misunderstanding, how might she have developed them?
(d) What further question(s) might you ask Whitney to understand her thinking?
(e) What instructional strategies and/or tasks would you use during the next instructional period to address Whitney's challenge(s) (if any presented)? Why?
(f) If applicable, how would you use technology or manipulatives to address Whitney's challenge or misunderstanding?
(g) How would you extend this problem to help Whitney further develop her understanding of the area of a trapezoid?

Item X: Paul's Approach

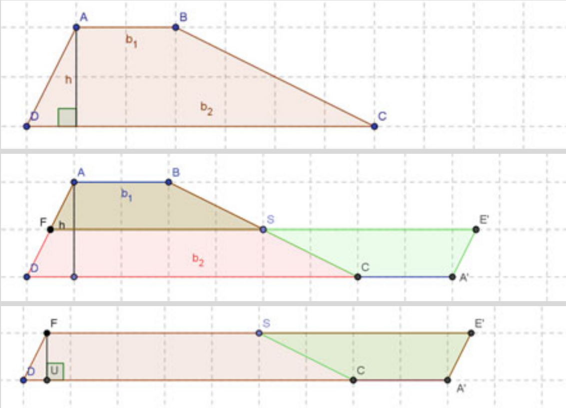
When presented with the task of *developing a formula for the area of any trapezoid* in his high school geometry class, Paul developed the diagrams as a strategy for deriving the formula for the area of a trapezoid described by the sketches below. He decomposed a trapezoid into a rectangle and a right triangle. Then he added the areas of these shapes to calculate the area of the trapezoid.



(a) Based on the diagram above, describe Paul's thinking. If he were to complete the formal derivation of the area formula in his diagrams, would his method work for any trapezoid? Why, or why not?
(b) If Paul's approach presents a challenge or misunderstanding, what underlying geometric conception(s) or understanding(s) might lead him to the error presented in this item?
(c) If Paul's approach presents a challenge or misunderstanding, how might he have developed them?
(d) What further question(s) might you ask Paul to understand his thinking?
(e) What instructional strategies and/or tasks would you use during the next instructional period to address Paul's challenge(s) (if any presented)? Why?
(f) If applicable, how would you use technology or manipulatives to address Paul's challenge or misunderstanding?
(g) How would you extend this problem to help Paul further develop his understanding of the area of a trapezoid?

Item Y: Adam's Approach

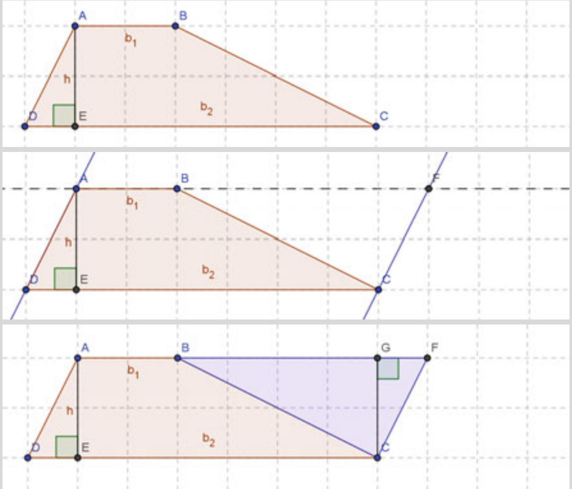
When presented with the task of *developing a formula for the area of any trapezoid* in his high school geometry class, Adam developed the diagrams as a strategy for deriving the formula for the area of a trapezoid described by the sketches below. He created a midsegment FS of trapezoid ABCD. This allowed him to create a new trapezoid, FABS. He rotated this trapezoid around point S to create a parallelogram, FE'A'D. He calculated the area of parallelogram FE'A'D to find the area of the original trapezoid, ABCD.



(a) Based on the diagram above, describe Adam's thinking. If he were to complete the formal derivation of the area formula in his diagrams, would his method work for any trapezoid? Why, or why not?
(b) If Adam's approach presents a challenge or misunderstanding, what underlying geometric conception(s) or understanding(s) might lead him to the error presented in this item?
(c) If Adam's approach presents a challenge or misunderstanding, how might he have developed them?
(d) What further question(s) might you ask Adam to understand his thinking?
(e) What instructional strategies and/or tasks would you use during the next instructional period to address Adam's challenge(s) (if any presented)? Why?
(f) If applicable, how would you use technology or manipulatives to address Adam's challenge or misunderstanding?
(g) How would you extend this problem to help Adam further develop his understanding of the area of a trapezoid?

Item Z: Donna's Approach

When presented with the task of *developing a formula for the area of any trapezoid* in her high school geometry class, Donna developed the diagrams as a strategy for deriving the formula for the area of a trapezoid described by the sketches below. She created a line CF parallel to the side AD of the trapezoid ABCD. She then extended side AB. This allowed her to create a parallelogram, AFCD. Then she subtracted the area of triangle BFC from the area of parallelogram AFCD to calculate the area of the original trapezoid ABCD.



(a) Based on the diagram above, describe Donna's thinking. If she were to complete the formal derivation of the area formula in her diagrams, would her method work for any trapezoid? Why, or why not?
(b) If Donna's approach presents a challenge or misunderstanding, what underlying geometric conception(s) or understanding(s) might lead her to the error presented in this item?
(c) If Donna's approach presents a challenge or misunderstanding, how might she have developed them?
(d) What further question(s) might you ask Donna to understand her thinking?
(e) What instructional strategies and/or tasks would you use during the next instructional period to address Donna's challenge(s) (if any presented)? Why?
(f) If applicable, how would you use technology or manipulatives to address Donna's challenge or misunderstanding?
(g) How would you extend this problem to help Donna further develop her understanding of the area of a trapezoid?

Appendix 2

Components of the rubric used to identify teachers' knowledge of student challenges and conceptions at Level 4 (Table 14.1) and Level 1 (Table 14.2).

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Index

A
Activity, 26–29, 31, 86, 127, 128, 130, 133, 151, 164, 165, 265, 312, 314, 325, 336
Activity Theory, 26, 27, 31, 38, 60
Adding (addition), 17, 27
Affordance, 30, 37, 38, 47, 49–51, 53, 84, 96, 105, 129, 130, 133, 140, 141, 143, 151, 166, 197, 214, 266, 303, 313, 314, 326, 335
Algebra, 149, 157, 190
Ambitious teaching, 301–303, 312, 319, 320
Apps, 4, 10, 16, 41, 44–46, 50, 51, 53, 54, 59, 63, 96, 101, 248, 250–254, 257, 259, 260, 262–266, 270, 275–279, 281–284, 286, 287, 289, 290, 294, 296
Arithmetic(s), 76, 78, 80, 83, 125, 132, 133, 148, 152, 174, 290
Artifact, 27–29, 31–33, 37, 38, 151, 326
Artifact-Centric Activity Theory (ACAT), 25, 26, 30–32, 35, 38
Attribute, 27, 45, 46, 49, 50, 52, 61, 63, 64

C
Coding, 221, 331
Cognitive involvement, 284, 285, 296
Cognitive technology tools, 11
Computer, 4, 18, 84–87, 151, 174, 177, 196, 210, 214, 218, 233, 238, 240, 250, 276, 310
Computer-based, 12
Computer-based mathematical cognitive tools, 11, 12
Computer manipulative, 6
Concrete, 16, 72–78, 105, 109, 164, 196, 249, 250, 255, 333
Concrete manipulative(s), 74, 96, 97, 99, 105, 108, 109, 112, 113, 156, 190, 248, 250
Cultural-Historical Activity Theory (CHAT), 127–130
Curricula, 275, 283, 284, 296

D
Definition, 3, 4, 6, 7, 9, 12, 13, 21, 75, 129, 249, 259, 263
Design, 15, 19, 20, 26, 30, 31, 38, 48, 61, 84, 86, 96–98, 124, 128, 131, 134, 136, 143, 148, 149, 162, 164, 208, 220, 260, 276, 277, 280, 301, 324
Design-based research (DBR), 127, 131, 143, 304
Developmental progression(s), 76
Developmental trajectory, 141
Device, 4, 10, 266, 282, 286
Digital, 96, 102, 295, 301
Digital manipulative(s), 6
Distance, 19, 41, 45, 48–53, 59, 61, 225, 228, 231, 232, 234
Dynamic, 3, 7–9, 13, 128, 129, 214, 281, 302, 324
Dynamic mathematical object, 5, 7, 13–15, 17–19
Dynamic object, 3, 5, 7, 9, 10, 13, 42, 148, 190, 196, 214, 249, 276, 303, 326

E
Ecological psychology, 127, 129
Emergent, 275, 283–286, 289, 296, 297
Environment, 4, 9, 10, 13–16, 18, 43, 84, 96, 105, 129, 130, 166, 224, 231, 239, 297, 317, 326
Equation, 19, 43, 50, 57, 216, 312
Evaluation criteria, 277, 283
Experimentation, 282

F
Feature(s), 4, 7–9, 21, 34, 97, 191, 216, 238, 279, 326
Fidelity, 42, 44, 251, 253, 255, 257, 263, 264, 270, 279, 314

P.S. Moyer-Packenham (ed.), *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives*, Mathematics Education in the Digital Era 7,
DOI 10.1007/978-3-319-32718-1

Finger patterns, 127, 131, 132, 134, 137, 142, 143
Fingu, 124, 127–141, 143
Fraction, 14, 15, 17, 19, 48, 96, 190, 193–198, 200, 202, 208, 210, 314, 315
Framework, 26, 27, 37, 38, 42, 45, 52–55, 57, 59–64, 72, 97, 99, 112, 114, 128, 302, 304

G
Game(s), 4, 13, 19, 127, 129, 131, 132, 134, 136, 139, 141, 143, 177, 185, 262, 278, 296, 310, 311
Gamification, 9
Geometry, 20, 72, 86, 157, 190, 218, 231, 247, 248, 252, 255, 269, 287
Graphics, 275, 282, 284, 294

H
High achievement, 18
High school, 214, 217, 220, 222, 231, 235, 237, 239, 240, 287, 327

I
Image, 3, 7, 15, 16, 83, 84, 95, 99, 113–115, 151, 215, 221, 223, 226, 228, 230–232, 234, 238, 239, 301, 315, 326
Individual, 7, 18, 26, 29, 35, 96, 110, 128, 129, 135, 137, 139, 141, 165, 166, 175, 177, 255, 258, 315, 324, 330
Instrument, 12, 26, 27, 29, 31, 37, 179, 275, 279, 284, 286, 297, 324, 328, 329, 331
Instrumental genesis, 28, 29, 31, 37
Instrumental orchestration, 29, 37
Integer(s), 11, 109, 277, 284, 287–290, 294, 296
Interact, 5, 53, 96, 114
Interactive, 3, 5, 7–9, 11–13, 15, 18, 42, 47, 149
Interactivity, 7–10, 163, 165, 249, 280, 284, 293, 296
Interface design, 279, 281
iPad, 10, 19, 32, 42, 44–48, 50, 52, 54, 101, 142, 247, 248, 250, 286, 288, 294, 310
Iterative design, 301

L
Learning objects, 11, 42, 214, 283, 297
Learning tool(s), 136, 275, 276, 279, 280, 282, 284, 294, 295
Learning trajectory (trajectories), 83, 282, 320
Low achieving, 18

M
Manipulate, 5, 9, 14, 15, 84, 131, 256, 290, 309, 326
Manipulative(s), 4, 6, 7, 9–13, 15, 20, 21, 38, 52, 72–75, 77, 80, 82, 84–86, 96–99, 101, 104, 106, 110, 112, 114, 149, 151, 152, 155, 156, 158, 160, 162–165, 167, 174, 183, 184, 191, 197, 210, 214, 222, 224, 229, 231, 233, 235, 239, 240, 248, 263, 277, 295, 309, 325, 336
Mathematical understanding, 44, 106, 214–217, 220–223, 225–227, 229, 231–240, 249, 251, 253, 297
Mathematics, 4, 5, 7, 12, 14, 21, 35, 46, 48, 50, 56, 79, 98, 101, 111–113, 128, 152, 163, 164, 182, 184, 190, 198, 203, 206, 209, 210, 219, 248, 249, 252, 264, 266, 270, 277, 279, 280, 282, 289, 295, 301, 304, 308, 310, 313, 315, 317, 319, 325, 330
Math object, 12
Meta-analysis (meta-analyses), 16, 74, 151, 165, 197, 214, 250
Mouse, 4, 5, 9, 49, 84, 214
Multiplication, 15, 72, 277, 294, 296, 311
Multi-representation, 14–17

N
National Library of Virtual Manipulatives (NLVM), 4, 6, 14, 17, 20, 148, 149, 219, 316
Non-routine, 191, 200, 208, 209

O
Object(s), 3, 7, 8, 37, 197, 326

P
Pairs, 15, 16, 334
Part-whole relations, 124–126, 131–134, 137
Pattern blocks, 14, 15, 49, 82, 96
Pedagogical, 44, 99, 106, 113, 248, 255, 279, 282, 307, 335
Pedagogical collective subject, 29
Physical manipulative(s), 12, 14, 72–74, 81–85, 87, 148, 165, 190, 196–198, 208, 210, 217–220, 222, 223, 225–227, 229, 232, 235, 236, 239, 265, 276, 309, 326
Pictorial, 5, 15, 52, 190, 314
Pictorial Image, 3, 14, 15, 18
Place value, 26, 28, 32, 33, 35, 96, 114
Practice-based, 304, 305, 313, 319

Primary school, 148, 156, 158, 162–164, 167, 174, 178, 253, 260
Problem solving, 11, 15, 16, 72, 96, 111, 113, 137, 164, 192, 320
Programmable feature, 7–9, 13, 14, 16

R
Representation(s), 7, 9, 12, 14–16, 34, 43, 44, 54, 73, 81, 99, 100, 102, 103, 105, 111, 114, 166, 190, 217, 222, 236, 282, 314

S
SAMAP, 149–153, 155–157, 159, 163–167
Simulation, 14, 113
Single-representation, 14, 15
Situated knowledge, 324, 329
Smart phone, 276
Socio-cultural theory, 128
Subitize (subitizing), 126, 133, 140, 143
Subject, 26, 27, 31, 154, 325

T
Tablet, 4, 286, 310
Task(s), 10, 27, 47, 57, 101, 110, 130, 132, 134, 137, 139, 141, 178, 228, 303, 316
Technology-enabled, 6, 13
Tool, 3, 27, 35, 42, 44–46, 48, 53, 64, 84, 106, 157, 250, 281, 297, 318, 324, 336
Touch-screen (or touch screen), 4, 10, 248, 276, 282
Transformation, 30, 37, 39, 217, 220, 222, 240, 324
Tutorial, 13, 18

V
Virtual manipulative(s), 3–16, 19–21, 26, 28, 30–32, 34, 35, 37–39, 42–50, 52–54, 56, 59, 61–64, 72–74, 81, 83–87, 96, 97, 99, 101, 105, 108, 112–115, 124, 129, 132, 143, 148, 149, 151, 156, 164–167, 174, 175, 177, 181, 183, 184, 190, 191, 196–198, 200, 206, 208–210, 214, 216, 217, 219–240, 249, 251, 266, 276, 280, 295, 301, 303, 309, 310, 314–316, 318, 323–331, 334–336
Virtual manipulative environments, 4, 13, 15, 17–20, 303
Virtual materials, 12, 13
Virtual math objects, 11, 12
Visual representation, 3, 5, 7–9, 13–15, 17–20, 31, 42, 109, 148, 214, 218, 226, 249, 302, 309, 326, 330, 331

W
Worked examples, 190–192, 197
Working in pairs, 15, 18