Chapter 7
Marginal Pricing and Marginal Cost Pricing Equilibria in Economies with Externalities and Infinitely Many Commodities

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Abstract This paper considers a general equilibrium model of an economy in which some firms may exhibit various types of non-convexities in production, there are external effects among agents and the commodity space is infinite dimensional. The consumption sets, the preferences of the consumers and the production possibilities are represented by correspondences in order to take into account the external effects. The firms are instructed to follow the marginal pricing rule from which we obtain an existence theorem. Then, the existence of a marginal cost pricing equilibrium is proved by adding additional assumptions. The simultaneous presence of externalities and infinitely many commodities are sources of technical difficulties when attempting to generalize previous existence results in the literature.

Keywords General equilibrium • Marginal pricing rules • Externalities • Increasing returns • Infinitely many commodities • Correspondences

JEL Classification: D50.

7.1 Introduction

It is well known that the presence of increasing returns in production constitutes a particular case of market failure that leads us to use an alternative criteria for producer behaviour rather than profit maximization. From the outset, beginning with Hotelling (1938), it has been argued that when the firms exhibit increasing returns to scale, prices should be proportional to marginal costs. This is the so-called *marginal cost pricing rule*. Hotelling also paid attention to the fact that in some cases, a firm

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or even an industry which adopts marginal cost pricing will run at a loss if there are high fixed costs. The deficit must then be financed from income taxes. Indeed, Hotelling argued that general government revenues should *be applied to cover fixed costs of electric power plants, waterworks, railroad, and other industries in which the fixed costs are large, so as to reduce to the level of marginal cost the prices charged for the services and products of these industries*.

There also exists another notion, that of *marginal pricing rule*. When the production set is smooth, this mechanism means that prices should be proportional to the gradient of the transformation functional, i.e. the producer fulfils the first-order necessary condition for profit maximization. Both the marginal pricing and the marginal cost pricing rules are closely related in such a way that they are treated as equivalents in most papers in the literature. However, as pointed out by Guesnerie (1990), they are often not equivalents at all and can often be very misleading. Bonnisseau and Cornet (1990a,b) investigated the link between both notions of equilibrium. To do this, the authors needed to introduce both the cost function and the iso-output set which required them to distinguish a priori between inputs and outputs and to propose additional assumptions. Accordingly, robust results are obtained relating both notions of equilibria.

Despite many criticisms, the marginal (cost) pricing doctrine is in force today. A rigorous and general proof of such doctrine was first offered by Guesnerie (1975) but only for economies with certain kinds of non-convex technologies. Indeed, Guesnerie considered the polar of the cone of interior displacements of the mathematicians A. Dubovickii and A. Miljurin to formalize the notion of marginal cost pricing when the production sets are non-convex. The problem with this approach comes when the production set has "inward kinks" since in this case, the normal to such a cone is only the null vector. If there is only one firm in the economy, then this problem will never arise if we simply assume that the boundary of the production set is smooth as in Mantel (1979) and Beato (1982). However, in a model with many firms, even if we assume that each firm has a smooth technology, the aggregate production set may exhibit inward kinks as Beato and Mas-Colell (1985) have shown. To avoid this difficulty, Cornet (1990) introduced in the economic literature the use of the Clarke tangent and normal cones of the mathematician F. Clarke to represent (through Clarke normal cone) the marginal (cost) pricing rule. This cone is always convex and coincides with the profit maximization behaviour (and with the cone of interior displacements) when the technologies are convex.

For economies with finitely many commodities, there are quite robust results concerning existence of marginal pricing equilibria (for a survey, we refer to Brown 1991). In contrast, for economies with infinitely many commodities, although there is a large literature on competitive equilibria (for a survey, we refer to Mas-Colell and Zame 1991), there are few results concerning marginal pricing or marginal cost pricing equilibria. Shannon (1996) stated the first proof of marginal cost pricing equilibrium in an infinite dimensional setting. She considered a private ownership economy with a finite number of consumers and only one firm. The production possibility frontier was assumed to be smooth. In the existence proof, she used the Leray-Schauder degree theory. Later, Bonnisseau (2002) generalized the results of

Shannon to the case of many firms with non-smooth production sets. He had to introduce a new and larger normal cone since the Clarke´s cone does not have sufficient continuity properties in infinite-dimensional spaces. So far, there are no new results concerning marginal (cost) pricing equilibria with an infinite quantity of goods.

Furthermore, externalities constitute another basic market failure in the sense that when external effects are present, competitive equilibria are not Pareto optimal. Although it has been shown that if a competitive market exists for the externality, then optimality results (Villar 1997), this is not always the case. Take, for example, the case of an external effect produced by one individual on another. Here, price-taking behaviour is unrealistic. Moreover, by definition, the presence of external effects requires incorporating into the model the actions of other agents.

There is a large and growing literature on general equilibrium models with externalities. Laffont (1976, 1977), Laffont and Laroque (1972) and Bonnisseau (1997), among others, consider the very general case in which the action of any agent may affect the decisions on consumption and production, as well as the preferences, of the rest of the agents. In all cases, it is assumed that consumers have a non-cooperative behaviour in the sense that they maximize their preferences under their budget constraints taking the prices and the environment as given. More recently, this approach has been objected on the grounds that *price-taking assumptions inherent in the notion of competitive equilibrium are incompatible with the presence of agents who have market power-as all agents typically do when the total number of agents is finite* (Noguchi and Zame 2006). Consequently, there is also an important literature on competitive equilibria in exchange economies with externalities and a continuum of consumers (see also, Balder 2008 and Cornet and Topuzu 2005).

Another aspect of the external effects is that sometimes the presence of externalities leads to non-convexities in the underlying production processes (Mas-Colell et al. 1995, p. 375). Hence, models were proposed for combining both externalities and increasing returns. Given what was stated above on marginal pricing rule, we choose between these models, the one of Bonnisseau and Médecin (2001) where the authors develop a new marginal pricing rule with external factors. This is so because the pricing rule defined by means of Clarke´s normal cone to the production set for a fixed environment does not have sufficient continuity properties. As a consequence, the pricing rule thus obtained is less precise since the new cone is larger than the former.

The purpose of this article is to provide an existence theorem with an arbitrary number of non-convex producers and externalities in an infinite dimensional setting. Infinite-dimensional commodity spaces arise naturally when we consider economic activity over an infinite time horizon, or with uncertainty about the states of the world, or when there are an infinite variety of commodity differentiation. For the sake of technical simplicity, we assume that every production set has a smooth boundary. Consequently, apart from this assumption, our existence result encompasses all the other existence results of marginal pricing equilibria in the literature.

As in Bonnisseau and Cornet (1990a,b), we show the relation between marginal pricing and marginal cost pricing equilibria. The model is not a direct extension of that of Bonnisseau and Cornet (1990a) since the presence of externalities does not allow us to claim that if a production plan belongs to a production set, the one with positive outputs also belongs to this set. We can say the same about consumers: if a consumption stream belongs to a consumption set restricted by an externality, we cannot claim that the same consumption stream belongs to a consumption set when the externality has changed by including non-negative outputs. Another important difference is that in the proof of marginal cost pricing equilibrium, they construct an argument which relies on a property of the gradient of the cost function that does not work in functional gradients. These drawbacks lead us to consider production vectors with non-negative outputs. An additional assumption on prices (which is weaker than what can generally be seen in the literature) allows us to obtain equilibrium production vectors with this property. So it is shown that a marginal pricing equilibrium is a marginal cost pricing equilibrium.

In the proof of the theorems, we roughly follow the method developed by Bewley (1972). The majority of the papers on general equilibrium with infinitely many commodities rely crucially on the First Welfare Theorem, which fails for marginal pricing and marginal cost pricing equilibria (see Guesnerie 1975). In addition to the two major drawbacks cited above, there are other technical difficulties such as those in Fuentes (2011). We take care of these problems in Sects. 7.4.2 and 7.6.1 in the same way we did in that paper.

The pricing rule in Fuentes (2011) encompasses general pricing rules. Nevertheless, we remove both bounded losses and continuity on pricing rule assumptions together with strong lower hemi-continuity in the truncated production correspondence.

Since we are interested in the relationship between non-convexities, marginal pricing and externalities in an infinite-dimensional setting, we do not follow the "continuum agents approach". It is well known that when there is an atomless measure space of agents, there are convexifying effects on preferences and technologies (Aumann 1966; Rustichini and Yannelis 1991), so we do not consider this possibility.

The paper proceeds as follows. Section 7.2 presents the model and the notation to deal with externalities, increasing returns and marginal pricing equilibrium with infinitely many commodities. Section 7.3 is devoted to the basic assumptions. In Sect. 7.4, we first define the finite-dimensional auxiliary economies, and we posit additional assumptions in order to deal with problems arising in the model. In Sect. 7.5, we state the marginal pricing equilibrium theorem. Section 7.6 is devoted to the proof of the existence result. In Sect. 7.7, we state the marginal cost pricing equilibrium theorem and give additional assumptions and definitions. Lengthy or tedious proofs are contained in the appendix.

7.2 The Model

We consider an economy with m consumers labelled by subscript $i = 1, \ldots, m$ and n producers, labelled by subscript $j = 1, \ldots, n$. The (infinite-dimensional) commodity space is represented by the space of essentially bounded, real-valued, measurable functions on a σ-finite positive measure space $(M, \mathscr{M}, \mu)$. In the following, we denote by L the space $\mathscr{L}_\infty(M, \mathscr{M}, \mu)$.[1] Each element $z = \left((x_i)_{i=1}^m, (y_j)_{j=1}^n\right)$ is an environment or externality.

Each consumer i has a *consumption set* and a *preference relation* which depends upon the actions of the other economic agents. Formally, the consumption set is represented by a correspondence X_i from L^{m+n} to L_+. For the environment $z \in L^{m+n}$, $X_i(z) \subset L_+$ is the set of possible consumption plans of the i-th consumer. We denote by $\succsim_{i,z}$ the (complete, reflexive, transitive and binary) preference relation which is influenced by the actions of all economic agents.

The *production set* of the j-th producer is defined by a correspondence Y_j from L^{m+n} to L. $Y_j(z)$ is the set of all feasible production plans for the j-th firm when the actions of the economic agents are given by z.

A *price system* is a continuous linear mapping on L. If L is endowed with the norm topology, the set of prices is $L^* = ba(M, \mathscr{M}, \mu)$, the space of bounded additive set functions on $(M, \mathscr{M})$ absolutely continuous with respect to μ. Thus, the value of a *commodity bundle* $x \in L_\infty$ is $\int_M x d\pi$ (Dunford and Schwarz 1958). If some price vector p belongs to $\mathscr{L}_1(M, \mathscr{M}, \mu)$[2] $\subset ba(M, \mathscr{M}, \mu)$, then it is economically meaningful since for every $x \in L$, $p(x) = \int_{m\in M} p(m)\, x(m)\, d\mu(m)$ which is the natural generalization of the value of a commodity bundle concept in finite-dimensional spaces. The equilibrium prices can be chosen in the simplex $S = \{\pi \in ba_+(M, \mathscr{M}, \mu) : \pi(\chi_M) = 1\}$, where χ_M is the function equal to 1 for every m in M.

The weak-star topology $\sigma(\mathscr{L}_\infty, \mathscr{L}_1) = \sigma^\infty$ is the weakest topology for which the topological dual of L is $\mathscr{L}_1$. We denote by $\prod_{L^s} \sigma^\infty$ the product topology on the product space L^s. $\sigma(L, ba)$ and $\sigma(ba, L) = \sigma^{ba}$ are the weak and the weak-star topologies, respectively, on L and ba. Let $A : L^s \mapsto L$ be a correspondence. We say that A is $\left(\prod_{L^s} \sigma^\infty, \sigma^\infty\right)$-closed if it has a closed graph for the product of weak-star topologies. Let $\mathscr{S}$ be any topology on L^s. The net $(u^\alpha) \in L^s$ is said to $\mathscr{S}$-converge

[1] $\mathscr{L}_\infty(M, \mathscr{M}, \mu)$ is the set of equivalence classes of all μ-essentially bounded, $\mathscr{M}$-measurable functions on M. Let x be an element of $\mathscr{L}_\infty(M, \mathscr{M}, \mu)$, then $x \geq 0$ if $x(m) \geq 0$ μ-*a.e.* (almost everywhere); $x > 0$ if $x \geq 0$ and $x \neq 0$, and $x >> 0$ if $x(m) > 0$ μ-a.e. Hence, if x, x' $\in \mathscr{L}_\infty(M, \mathscr{M}, \mu)$, then $x \geq x'$ (respectively, $x > x'$, $x >> x'$) if $x - x' \geq 0$ (respectively, $x - x' > 0$, $x - x' >> 0$). $L_+ = \{x \in L : x \geq 0\}$ is the positive cone of L, and $L_{++} = \{x \in L : x > 0\}$ is the strict positive cone or the quasi-interior of L. Let A and B be subsets of L. The difference of A and B is defined by $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$. The open ball of centre x and radius ε is $B(x, \varepsilon) = \{x' \in L : \|x' - x\|_\infty < \varepsilon\}$, while the closed ball of centre x and radius ε is $\overline{B}(x, \varepsilon) = \{x' \in L : \|x' - x\|_\infty \leq \varepsilon\}$.

[2] $\mathscr{L}_1(M, \mathscr{M}, \mu)$ is classes of all $\mathscr{M}$-measurable functions f on M such that $\int_{m\in M} |f(m)|\, d\mu(m)$ $< \infty$.

to u if (u^{α}) converges for the topology $\mathscr{S}$. We denote by $\mathscr{T}$ the norm topology on L. The correspondence A is said to be $\left(\prod_{L^s}\sigma^{\infty},\mathscr{T}\right)$-lower hemi-continuous (for short l.h.c.) if for every net (z^{α}) in L^s which $\prod_{L^s}\sigma^{\infty}$-converges to z and $a\in A(z)$, there is a net (a^{α}) such that $a^{\alpha}\in A(z^{\alpha})$ for all α and a^{α} $\mathscr{T}$-converges to a. Let $\omega_i\in L_+$ be the *initial endowment* of the i-th agent and $\omega=\sum_{i=1}^{m}\omega_i$ the *total initial endowment* of the economy. Let $r_i : R^{1+n}\mapsto R$ be the *wealth function* of the i-th consumer. $r_i\left(\pi(\omega_i),\left(\pi\left(y_j\right)\right)_{j=1}^{n}\right)$ is his wealth whenever $\pi\in S$ and $\left(y_j\right)_{j=1}^{n}\in\prod_{j=1}^{n}Y_j(z)$. A special case of this structure is $r_i\left(\pi(\omega_i),\left(\pi\left(y_j\right)\right)_{j=1}^{n}\right)=\pi(\omega_i)+\sum_{j=1}^{n}\theta_{ij}y_j$ for $\theta_{ij}\geq 0$ and $\sum_{i=1}^{m}\theta_{ij}=1$, which holds for a private ownership economy.

We now assume that the graph of every production correspondence is smooth.

Assumption P (Smoothness). For all j

(i) For every $z\in L^{m+n}$, $Y_j(z)=\left\{y\in L: f_j(y,z)\leq 0\right\}$[3] and $\partial_{\infty}Y_j(z)=\left\{y\in L: f_j(y,z)=0\right\}$ where f_j is a transformation functional from $L\times L^{m+n}$ into R.
(ii) f_j is $\sigma^{\infty}\times\prod_{L^{m+n}}\sigma^{\infty}$-continuous on $L\times L^{m+n}$
(iii) For every $z\in L^{m+n}$, $f_j(\cdot,z)$ is Fréchet Differentiable, and if $f_j(y,z)\leq 0$ and $y'\leq y, f_j(y',z)\leq 0$ (free disposal)
(iv) $\nabla_1 f_j(y,z)$[4] $\in\mathscr{L}_1^{+}(M,\mathscr{M},\mu)\setminus\{0\}$ if $f_j(y,z)=0$ and $f_j(0,z)=0$
(v) $\nabla_1 f_j$ is $\left(\sigma^{\infty}\times\prod_{L^{m+n}}\sigma^{\infty}\right)$-continuous on $L\times L^{m+n}$, that is, for all $y\in\partial_{\infty}Y_j(z)$, for all $\varepsilon>0$, there exists a weak* open neighbourhood of (y,z), $U(y,z)$, in $L\times L^{m+n}$ such that $\nabla_1 f_j(y',z')\in B\left(\nabla_1 f_j(y,z),\varepsilon\right)$ for all $(y',z')\in U(y,z)$

Note that while non-convexities are allowed on the firms, they must be smooth ones (Assumptions P(i), P(ii) and P(iii)). However, no smoothness assumption is made in the aggregate production set $Y(z)=\sum_{j=1}^{n}Y_j(z)$, which would be far from being innocuous as Beato and Mas-Colell (1985) have shown. Assumption P(iii) also incorporates the free disposal condition. As for Assumption P(iv), we point out that $N_{Y_j(z)}\left(y_j\right)\subset ba_+(M,\mathscr{M},\mu)$ for all $y_j\in\partial_{\infty}Y_j(z)$. Indeed, let $x\in L_+$. For all $t\in(0,\varepsilon)$, $f_j(y+tx,z)\geq 0$ by Assumption P(i) and P(iii). Consequently, $\nabla_1 f_j\left(y_j,z\right)(x)=\lim_{t\downarrow 0}\frac{f_j\left(y_j+tx,z\right)}{t}\geq 0$. Thus, Assumption P(iv) only requires that prices be economically meaningful. Assumption P(v) says that f_j is continuously (Fréchet) differentiable on $L\times L^{m+n}$. This is a technical requirement for getting nice continuity properties in prices.

[3]We say that a production vector y is weakly efficient if and only if $y\in\partial_{\infty}Y(z)$. This is equivalent to say that $\left(\{y\}+\text{int}L_+\right)\cap Y(z)=\emptyset$. A stronger concept is that of efficiency. We say that a production vector y is efficient if and only if $\left(\{y\}+L_+\right)\cap Y(z)=\emptyset$.

[4]$\nabla_1 f_j(y,z)$ denotes the gradient vector of f_j with respect to y in the sense of Fréchet, that is, $\nabla_1 f_j\left(y_j,z\right)(x)=\lim_{t\to 0}\frac{f_j\left(y_j+tx,z\right)-f_j\left(y_j,z\right)}{t}$ for all $x\in L$, and the convergence is uniform with respect to x in bounded sets.

Remark 1. We point out that Assumptions P(i) and P(ii) imply that if $\left(y_j^\alpha\right) \in \partial_\infty Y_j(z^\alpha)$ for all α and $\left(y_j^\alpha, z^\alpha\right)$ $\sigma^\infty \times \prod_{L^s} \sigma^\infty$−converges to $\left(\bar{y}_j, \bar{z}\right)$, then $\bar{y}_j \in \partial_\infty Y_j(\bar{z})$.

Proposition 1. *Suppose that Assumption P holds. Then,* $Y_j : L^{m+n} \mapsto L$ *is a* $\left(\prod_{L^{m+n}} \sigma^\infty, \sigma^\infty\right)$*-closed and a* $\left(\prod_{L^{m+n}} \sigma^\infty, \mathcal{T}\right)$*-l.h.c correspondence.*

Proof. See Appendix

The smoothness assumptions allow us to introduce the marginal pricing rule for the j-th producer at $y \in \partial_\infty Y_j(z)$, as the closed half-line of outward normal vectors to $Y_j(z)$ at y_j, which also are in S, that is, $N_{Y_j(z)}\left(y_j\right) \cap S = \left\{\lambda \nabla_1 f_j(y, z) : \lambda \geq 0\right\} \cap S$. Indeed, for a given $z \in Z$, $N_{Y_j(z)}\left(y_j\right) = \left\{\lambda \nabla_1 f_j(y, z) : \lambda \geq 0\right\}$ since f is continuously differentiable on $L \times L^{m+n}$, $\nabla_1 f(y, z) \in \mathcal{L}_1^+ \setminus \{0\}$ and $f(y, z) = 0$ for all $y \in \partial_\infty Y_j(z)$ (Clarke 1983, Theorem 2.4.7, Corollary 2). Note that for all j and all $y_j \in \partial_\infty Y_j(z)$, $N_{Y_j(z)}\left(y_j\right) \cap S \neq \emptyset$, since $N_{Y_j(z)}\left(y_j\right) \subset \mathcal{L}_+^1 \setminus \{0\}$.

We characterize the economy by $\mathcal{E} = \left((X_i, \succsim_{i,z}, r_i)_{i=1}^m, \left(Y_j\right)_{j=1}^n, (\omega_i)_{i=1}^m\right)$. Before giving the definition of equilibrium, we need to introduce some useful definitions at first. The set of *weakly efficient allocations* is

$$Z = \left\{z \in L^{m+n} : \forall i \; x_i \in X_i(z), \forall j \; y_j \in \partial_\infty Y_j(z)\right\}.$$

We also define the set of *weakly efficient attainable allocations* corresponding to a given total initial endowment $\omega \in L$

$$A(\omega) = \{z \in Z : \textstyle\sum_{i=1}^m x_i \leq \sum_{j=1}^n y_j + \omega\}.$$

Finally, the set of *production equilibria* is

$$\text{PE} = \left\{(\pi, z) \in S \times Z : \pi \in \textstyle\bigcap_{j=1}^n N_{Y_j(z)}\left(y_j\right) \cap S\right\}.$$

We now formally define our notion of equilibrium.

Definition 1. A marginal pricing equilibrium of the economy $\mathcal{E}$ is an element $(\bar{z}, \bar{\pi}) = \left(\left((\bar{x}_i)_{i=1}^m, \left(\bar{y}_j\right)_{j=1}^n\right), \; \bar{\pi}\right)$ in $Z \times S$ such that:

a. For all i, $\bar{x}_i$ is $\succsim_{i,\bar{z}}$-maximal in $\left\{x_i \in X_i(\bar{z}) : \bar{\pi}(x_i) \leq r_i\left(\bar{\pi}(\omega_i), \left(\bar{\pi}\left(\bar{y}_j\right)\right)_{j=1}^n\right)\right\}$
b. For all j, $\bar{\pi} \in N_{Y_j(\bar{z})}\left(\bar{y}_j\right) \cap S$ and $\bar{y}_j \in \partial_\infty Y_j(\bar{z})$
c. $\sum_{i=1}^m \bar{x}_i = \sum_{j=1}^n \bar{y}_j + \omega$

Condition *a.* says that for a given price $\bar{\pi}$, and a given externality $\bar{z}$, each consumer maximizes his preference relation under his budget constraint. Condition b. says that for a given externality $\bar{z}$ and for the same price vector $\bar{\pi}$, every producer satisfies his first-order necessary condition for profit maximization. Condition *c.* says that the demand is equal to the supply.

If we replace in the above definition, condition c. by condition c': $\sum_{i=1}^{m} \bar{x}_i \leq \sum_{j=1}^{n} \bar{y}_j + \omega$ and $\bar{\pi}\left(\sum_{i=1}^{m} \bar{x}_i\right) = \bar{\pi}\left(\sum_{j=1}^{n} \bar{y}_j + \omega\right)$, then we have the definition of *WA-equilibrium.*[5]

Remark 2. If Y_j is a convex-valued correspondence which satisfies Assumption P, then $N_{Y_j(z)}\left(y_j\right) \cap S = \left\{\pi \in S : \pi\left(y_j\right) \geq \pi\left(y\right), \forall\, y \in Y_j\left(z\right)\right\}$. Consequently, for a private ownership economy with convex-valued correspondences, the marginal pricing equilibria are equivalent to the notion of walrasian equilibria (see Clarke 1983, Proposition 2.4.4).

We end this section with the following proposition:

Proposition 2. *Let* $(\Gamma, \leq)$ *be a directed set. Let* $(z^\alpha, \pi^\alpha)_{(\Gamma,\leq)}$ *be a net of* $Z \times S$*, such that*

$$\begin{cases} (z^\alpha, \pi^\alpha) \to (\bar{z}, \bar{\pi}) \text{ for the product topology } \prod_{L^{m+n}} \sigma^\infty \times \sigma^{ba} \\ \pi^\alpha \in N_{Y_j(z^\alpha)}\left(y_j^\alpha\right) \cap S \text{ for all } \alpha \in \Gamma \\ \left(\pi^\alpha\left(y_j^\alpha\right)\right)_{\alpha\in\Gamma} \text{ converges} \end{cases}$$

Then $\lim \pi^\alpha\left(y_j^\alpha\right) \geq \bar{\pi}\left(\bar{y}_j\right)$. If $\lim \pi^\alpha\left(y_j^\alpha\right) = \bar{\pi}\left(\bar{y}_j\right)$, then $\overline{\pi} \in N_{Y_j(\bar{z})}\left(\bar{y}_j\right) \cap S$.

The proof of this proposition is given in the Appendix. This result claims that the Clarke´s normal cone (with external factors) has sufficient continuity properties in the space L when the individual production set has a smooth boundary.

7.3 Other Basic Assumptions

We now posit the following assumptions:

Assumption (C). For every i

(i) X_i is a $\left(\prod_{L^{m+n}} \sigma^\infty, \sigma^\infty\right)$-closed correspondence with convex values and containing 0.

(ii) For every $z \in L^{m+n}$, for every x_i in $X_i\left(z\right)$, there exists x in $X_i\left(z\right)$ such that $x_i \prec_{i,z} x$, and for every $x_i, x_i' \in X_i\left(z\right)^2$, for every $t \in (0, 1)$, if $x_i \prec_{i,z} x_i'$, then $x_i \prec_{i,z} tx_i + (1-t)x_i'$.

(iii) The set $\Gamma_i = \left\{\left(z,\ x_i,\ x_i'\right) \in L^{m+n+2} : \left(x_i, x_i'\right) \in X_i\left(z\right)^2, x_i \precsim_{i,z} x_i'\right\}$ is a $\prod_{L^{m+n}} \sigma^\infty$-closed subset of L^{m+n+2}.

(iv) The wealth function r_i is continuous on R^{1+n} and strictly increasing in the second variable. Furthermore, $\sum_{i=1}^{m} r_i\left(\pi\left(\omega_i\right),\ \left(\pi\left(y_j\right)\right)_{j=1}^{n}\right) = \pi(\omega) + \sum_{j=1}^{n} \pi\left(y_j\right)$.

[5] See Guesnerie (1975).

Assumption (B). For every $\omega\prime \geq \omega$, the set
$A(\omega', z) = \{(y_j)_{j=1}^n \in \prod_{j=1}^n \partial_\infty Y_j(z) : \sum_{j=1}^n y_j + \omega' \in L_+\}$ is norm bounded.

Assumption (WSA). (Weak Survival) For all $(\pi, z, \lambda) \in \text{PE} \times R_+$, if $(y_j)_{j=1}^n \in A(\omega + \lambda\chi_M, z)$, then
$\pi\left(\sum_{j=1}^n y_j + \omega + \lambda\chi_M\right) > 0.$

Assumption (R). For all $(\pi, z) \in \text{PE}$, if $z \in A(\omega)$, then
$r_i\left(\pi(\omega_i), \ (\pi(y_j))_{j=1}^n\right) > 0.$

Assumption (C) is the natural generalization of the assumptions of Bonnisseau and Médecin (2001) to an infinite-dimensional context (see Fuentes 2011). Assumption (B) is essential for the existence of an equilibrium. It means that for every $\omega\prime \geq \omega$, the set of weakly efficient attainable production plans is relatively weakly compact, from which it follows that so is $A(\omega')$.

When the same price is offered by the producers, according to $N_{Y_j(z)}(y_j) \cap S$, Assumption WSA implies that the global wealth of the economy is strictly greater than the subsistence level. Assumption R states that the revenue functions are a way to redistribute the total wealth among the consumers and the individual revenues are above the survival level for each consumer when the global wealth is large enough to allow such redistribution. We point out that when $Y_j(z)$ is a convex subset of L for every j and every $z \in L^{m+n}$, $\omega \in \text{int}L_+$ and $0 \in Y_j(z)$, both assumptions (WSA) and (R) are satisfied.

Remark. Most papers in general equilibrium theory with infinite commodity spaces make use of a well-known assumption called *properness* since Mas-Colell (1986). This condition informally means that there is a commodity bundle v which is so desirable that the marginal rate of substitution of any other commodity for v is bounded away from zero. Properness was introduced to deal with the consequences of the emptiness of the (norm) interior of the positive cone, namely, the fact that price equilibrium functional $\overline{\pi}$ may be identically zero. We point out that the list of spaces for which the positive orthant has empty interior includes several Banach spaces with some few exceptions such as the space $\mathscr{L}_\infty(M, \mathscr{M}, \mu)$. That is why we do not need to impose any properness assumption.

7.4 Subeconomies

7.4.1 Construction of Finite-Dimensional Economies

Let F be a finite-dimensional subspace of L containing both χ_M and $(\omega_i)_{i=1}^m$. We denote by $\mathscr{F}$ the family of such subspaces F directed under set inclusion. For every $F \in \mathscr{F}$, we define its positive cone by $F_+ = F \cap L_+$ and its interior by $\text{int}F_+ = F \cap \text{int}L_+$ which is not empty since χ_M belongs to $\text{int}L_+$. Hence, it defines an order which allows us to endow each F with an euclidean structure such that $\| \chi_M \| = 1$ and $\left\{\chi_M^{\perp_F}\right\} \cap F_+ = \{0\}$, where $\chi_M^{\perp_F}$ denotes the orthogonal space to χ_M. Hence, the dual space of F is F itself,[6] and we denote by p^F the inner product $\langle p^F, \cdot \rangle_F$.

The truncated consumption correspondence for the commodity space F is given by $X_i^F : F^{m+n} \mapsto F_+$ and defined by $X_i^F\left(z^F\right) = X_i\left(z^F\right) \cap F_+$. Analogously, the truncated production correspondence $Y_j^F : F^{m+n} \mapsto F$ is defined by $Y_j^F\left(z^F\right) = Y_j\left(z^F\right) \cap F$, and, by the definition of Y_j, one easily checks that $Y_j^F\left(z^F\right) = \left\{y^F \in F : f_j\left(y^F, z^F\right) \leq 0\right\}$ and $\partial Y_j^F\left(z^F\right) = \left\{y \in F : f_j(y, z) = 0\right\} = \partial_\infty Y_j\left(z^F\right) \cap F$. Hence, $Z^F \subset Z$.

Let $S^F = \left\{p^F \in F_+^0 : \langle p^F, \chi_M \rangle_F = 1\right\}$, where F_+^0 denotes the positive polar cone of F_+. r_i^F is the revenue of the i-th consumer induced by r_i in the truncated economy. The relation $\succsim_{i,z^F}^F$ is the preorder induced on $X_i^F\left(z^F\right)$ by $\succsim$. We then denote the subeconomies by $\mathscr{E}^F = \left(\left(X_i^F, \succsim_{i,z^F}^F, r_i^F\right)_{i=1}^m, \left(Y_j^F\right)_{j=1}^n, (\omega_i)_{i=1}^m\right)$ for all $F \in \mathscr{F}$.

We point out that for all $F \in \mathscr{F}$, for all $z^F \in F^{m+n}$ and for all i and j, $X_i^F\left(z^F\right)$ and $Y_j^F\left(z^F\right)$ are non-empty subsets of F because of the Assumptions C(i) and P(iv) together with the fact that F is a subspace of L. We also remark that for all $F \in \mathscr{F}$ and all $\left(y_j, z\right) \in F^{m+n+1}$, $N_{Y_j^F(z)}^F\left(y_j\right) \cap S^F = \left\{\lambda \nabla_1 f_j(y, z)_{|F_+^0} : \lambda \geq 0\right\} \cap S^F$. The set of production equilibria and of weakly efficient attainable allocations in $\mathscr{E}^F$ are, respectively,

$$\mathrm{PE}^F = \left\{(p^F, z^F) \in S^F \times Z^F : p^F \in \bigcap_{j=1}^n N_{Y_j^F(z)}^F\left(y_j\right)\right\}$$

and

$$A^F(\omega) = \left\{z^F \in Z^F : \sum_{i=1}^m x_i^F \leq \sum_{j=1}^n y_j^F + \omega\right\} \subset A(\omega).$$

7.4.2 Bewley´s Limiting Technique and Additional Assumptions

In the paper of Bonnisseau and Médecin (2001), the consumption set is represented by a correspondence that is l.h.c. As noted in Fuentes (2011), if we assume that the correspondence X_i is l.h.c. for all i, the restriction to a finite-dimensional subspace may not be l.h.c. Hence, Bonnisseau and Médecin´s theorem (smooth case) does not apply, and, thus, we cannot follow the Bewley´s approach. One solution is to assume that for all i, the restriction of X_i to a finite-dimensional subspace is l.h.c.

[6] F and F^*, the topological dual of F, are isomorphic (See MacLane and Garret 1999, Theorem 9, p. 357).

Assumption C(v). For all i

(v) There is a finite-dimensional subspace $\bar{F} \in \mathscr{F}$, such that for any finite-dimensional subspace $F \in \mathscr{F}$ such that $\bar{F} \subset F$, the correspondence X_i^F is l.h.c. on F^{m+n}.

Another problem in assuming that the correspondence X_i is l.h.c. for all i, relies in the fact that even if there is an equilibrium in each subeconomy $\mathscr{E}^F$, we cannot prove that a limit point $\left(\left((\bar{x}_i)_{i=1}^m, (\bar{y}_j)_{j=1}^n\right), \bar{\pi}\right)$ is an equilibrium vector in the original infinite-dimensional economy. Specifically, in the Claims 3 and 4 in the proof of Theorem 1 below, it can be seen that the lower hemi-continuity of X_i is not enough to prove that, for all i, if $x_i \succsim_{i,\bar{z}} \bar{x}_i$, then $\bar{\pi}(x_i) \geq r_i\left(\bar{\pi}(\omega_i), (\bar{\pi}(\bar{y}_j))_{j=1}^n\right)$. Consequently, we cannot use the limiting argument of the Bewley type. One solution is to establish the following assumption:

Assumption C(vi). For all i

The correspondence X_i is $\left(\prod_{L^{m+n}} \sigma^{\infty}, f\right)$-l.h.c. on L^{m+n}, that is, if z^{α} $\prod_{L^{m+n}} \sigma^{\infty}$−converges to z in L^{m+n} and $x \in X_i(z)$, there exists a finite-dimensional subspace $\dot{F}$ such that there is a net $(x^{\alpha}) \subset x + \dot{F}$ with $x^{\alpha} \in X_i(z^{\alpha})$ for all α and $x^{\alpha} \longrightarrow x$.

We point out that $\dot{F}$ may depend on $x \in X_i(z)$ and the net (z^{α}). We also note that the above Assumption implies that the correspondence X_i is $\left(\prod_{L^{m+n-1}} \sigma^{\infty}, \mathscr{T}\right)$-l.h.c. since the net (x^{α}) $\mathscr{T}$−converges to x due to the fact that it belongs to an affine finite-dimensional subspace.

When the boundary of the production set is smooth, such as in our case, if the production correspondence is l.h.c., then so is its restriction to a finite-dimensional subspace (see Remark 3 in the Appendix). Then, contrary to what is stated in Fuentes (2011), we do not need an additional assumption for the restricted production correspondences.

There are two remaining problems in the application of the Bewley technique. First, even if the original economy is supposed to satisfy the Weak Survival Assumption, this may not be true for the subeconomies. Secondly, even if the original economy is supposed to satisfy the Local Non-Satiation Assumption, we cannot say this is true in the subeconomies. Consequently, Theorem 3.1 of Bonnisseau and Médecin (2001) cannot be applied to $\mathscr{E}^F$. As we shall show later, if the commodity space F is large enough, then the economy satisfies weaker versions of Assumptions (WSA) and (LNS).

7.5 Existence of Marginal Pricing Equilibria

Now, we are ready to state the following result:

Theorem 1. *Under Assumptions (C), (P), (B), (WSA) and (R), the economy* $\mathscr{E} = \left((X_i, \succsim_{i,z}, r_i)_{i=1}^m, (Y_j)_{j=1}^n, (\omega_i)_{i=1}^m\right)$ *has a marginal pricing equilibrium.*

To compare this result with the literature, we first remark that it generalizes the one given in Shannon (1996) for the case without externalities and one producer and the one in Bonnisseau and Cornet (1990a) for the case with commodity space R^l. It also extends the main result of Bonnisseau (2002) under the particular circumstance of smooth production sets. In Fuentes (2011), the behaviour of the firms is defined through a general pricing rule. Nevertheless, the existence result uses a bounded losses assumption which is not necessary with the marginal pricing rule. Furthermore, we can suppress Assumption (PR) on the continuity of pricing rules (by Proposition 2 in this paper) and Assumption P(v) on the lower hemi-continuity of Y_j^F (See Remark 3 in the Appendix).

7.6 Proof of the Theorem

7.6.1 Equilibria in the Subeconomies

The results of this section follow the guidelines of Bonnisseau's proof of Proposition 2 (Bonnisseau 2002). The differences between our results and those of the author are due to the intrinsic differences between the finite-dimensional model without externalities (Bonnisseau and Cornet 1990a) and the one with external factors (Bonnisseau and Médecin 2001). We can observe that every subeconomy $\mathscr{E}^F$ satisfies Assumptions (P), (B), (R) and (C) (except LNS) of Theorem 3.1 of Bonnisseau and Médecin (2001). As we remarked at the end of Sect. 7.4, Assumptions (LNS) and (WSA) are not necessarily fulfilled by $\mathscr{E}^F$. The following lemma shows that each subeconomy satisfies weak versions of the survival and the local non-satiation of the preferences if F is large enough. Before stating the above result, we need to introduce the elements for its treatment. Let $\overline{\eta} > 0$ be a real number. Since $A(\omega + \overline{\eta}\chi_M, z)$ is norm bounded by Assumption (B), there exists (Schaefer and Wolf 1999, p. 25) $a > 0$ such that $a > 2\overline{\eta}$, $A(\omega + \overline{\eta}\chi_M, z) \subset B\left(0, \frac{a}{2}\right)^n$ and $A(\omega + \overline{\eta}\chi_M) \subset B\left(0, \frac{a}{2}\right)^{m+n}$. Let $\overline{r} > 2a$ such that $\{\omega + \overline{\eta}\chi_M\} + \overline{B}(0, na) \subset B(0, \overline{r})$. Let $\bar{\lambda}$ be a real number such that $\bar{\lambda} \geq 2n\overline{r} + \|\omega\|$. We point out that $\bar{\lambda}$ satisfies Lemma 4.2 of Bonnisseau and Médecin (2001) in our model.

Lemma 1. *Under Assumptions (C), (P), (B), (WSA) and (R), there exists a subspace $\hat{F} \in \mathscr{F}$ such that for all $F \in \mathscr{F}$, if $\hat{F} \subset F$, then the subeconomy $\mathscr{E}^F$ satisfies:*

(WSA^F): *For all $\left(p^F, z^F, \lambda^F\right) \in \mathrm{PE}^F \times [0, \bar{\lambda}]$, if $\left(y_j^F\right)_{j=1}^n \in A^F\left(\omega + \lambda^F \chi_M, z^F\right)$, then $\left\langle p^F, \sum_{j=1}^n y_j^F + \omega + \lambda^F \chi_M \right\rangle_F > 0$.*

(LNS^F): *For all $\left(\left(x_i^F\right)_{i=1}^m, \left(y_j^F\right)_{j=1}^n\right) \in A^F(\omega)$, and for all $\varepsilon > 0$, there exists $\left(x_i'^F\right)_{i=1}^m \in \prod_{i=1}^m \left(X_i^F\left(z^F\right) \cap B\left(x_i^F, \varepsilon\right)\right)$, such that $x_i'^F \succ_{i,z^F}^F x_i^F$ for all i.*

The proof of this lemma parallels the one given in Fuentes (2011). Just replace Assumption P by Remark 1 and Proposition 1 and Assumption PR by Proposition 2.

We recall that Bonnisseau and Médecin defined a new cone for the marginal pricing rule when there are external effects. Indeed, if we use the Clarke´s normal cone (with externalities), the equilibrium may not exist due to the presence of discontinuities. However, if the individual production set is smooth, their cone coincides with the Clarke´s cone.[7] The proposition below establishes that at least one equilibrium exists in the subeconomies.

Proposition 3. *Let $\bar{F}$ and $\hat{F}$ be the subspaces coming from Assumption C(v) and Lemma 1, respectively. Under Assumptions (C), (P), (B), (WSA) and (R), if we have $\bar{F} \subset F$, and $\hat{F} \subset F$, then the subeconomy $\mathscr{E}^F$ has an equilibrium $\left(z^F, p^F\right) \in Z^F \times S^F$.*

Proof. We remark that in the proof of Bonnisseau and Médecin (2001), the authors use Assumption (WSA) in Lemmas 4.2 (3) and 4.4 and in Claim 4.3. We also note that in the proof they fix belongs a parameter $\bar{t} > 0$ (p. 283). We replace it by $\overline{\eta}$ as given in paragraph before Lemma 1. For Lemma 4.2 (3) and Claim 4.3, Survival Assumption is applied only for production plans which satisfy that $\sum_{j=1}^{n} y_j + \omega + \eta\chi_M \geq 0$ with $\eta \leq \overline{\eta}$. Since $\overline{\eta} < \bar{\lambda}$ from the definition of $\bar{r}$, we have that condition (WSA^F) of Lemma 7 is enough to conclude. For Lemma 4.4, we shall prove that (WSA^F) is enough to use the deformation lemma. We now introduce the Bonnisseau and Médecin (2001)´s fundamental mathematical expressions we shall need. Let

$$\lambda_j^F : \chi_M^{\perp_F} \times F^{m+n} \longmapsto R$$
$$\left(s_j, z\right) \longmapsto \lambda_j^F\left(s_j, z\right)$$
$$\Lambda_j^F\left(s_j, z\right) = s_j - \lambda_j^F\left(s_j, z\right)\chi_M \in \partial Y_j^F(z)$$
$$X^F(z) = \sum_{i=1}^{m} X_i^F(z) + F_+ = F_+$$
$$Y_0^F(z) = -X^F(z)$$
$$\lambda_0^F : \chi_M^{\perp_F} \times F^{m+n} \longmapsto R$$
$$\left(s_j, z\right) \longmapsto \lambda_0^F\left(s_j, z\right)$$
$$\Lambda_0^F\left(s_j, z\right) = s_j - \lambda_0^F\left(s_j, z\right)\chi_M \in \partial\left(-F_+\right)$$
$$\theta^F\left(\left(s_j\right)_{j=1}^n, z\right) = \sum_{j=1}^{n} \lambda_j^F\left(s_j, z\right) + \lambda_0^F\left(-\sum_{j=1}^{n} s_j - proj_{\chi_M^{\perp_F}}\omega, z\right) - \langle \omega, \chi_M \rangle_F$$
$$\Delta^F\left(\left(s_j\right)_{j=1}^n, z\right) = \left\{ \left(p_j - p\right)_{j=1}^n \mid \begin{array}{c} p_j \in N_{Y_j(z)}\left(\Lambda_j^F\left(s_j, z\right), z\right),\ j = 1, \ldots, n \\ p \in N_{-F_+}\left(\Lambda_0^F\left(-\sum_{j=1}^{n} s_j - proj_{\chi_M^{\perp_F}}\omega, z\right)\right) \cap S^F \end{array} \right\}$$
$$M_{\overline{\eta}}^F(z) = \left\{\left(\left(s_j\right)_{j=1}^n\right) \in \left(\chi_M^{\perp_F}\right)^n : \sum_{j=1}^{n} \Lambda_j^F\left(s_j, z\right) + \omega + \overline{\eta}\chi_M \in F_+\right\} \text{ for every } z \in Z_D^F$$
$$GM_{\overline{\eta}}^F = \left\{\left(\left(s_j\right)_{j=1}^n, z\right) \in \left(\chi_M^{\perp_F}\right)^n \times Z_D^F : \sum_{j=1}^{n} \Lambda_j^F\left(s_j, z\right) + \omega + \overline{\eta}\chi_M \in F_+\right\}$$
$$GM_{\overline{\eta},\alpha}^F = \left\{\left(\left(s_j\right)_{j=1}^n, z\right) \in \left(\chi_M^{\perp_F}\right)^n \times Z_D^F : \overline{\eta} \leq \theta^F\left(\left(s_j\right)_{j=1}^n, z\right) \leq \alpha\right\}$$
$$\alpha = \max\left\{\theta^F\left(\left(s_j\right)_{j=1}^n, z\right) : \left(\left(s_j\right)_{j=1}^n, z\right) \in \left(\overline{B}^F(0, 2a) \cap \left\{\chi_M^{\perp_F}\right\}\right)^n \times Z_D^F\right\}$$

[7] Bonnisseau and Médecin 2001, p. 277

where, $\overline{B}^F(0,a) = \overline{B}(0,a) \cap F$, $D^F := \overline{B}^F\left(0,\overline{\lambda}\right)^m \times \overline{B}^F(0,\overline{r})^n$ and $Z_D^F := Z^F \cap D^F$.

For λ_j^F and λ_0^F, $\sum_{j=1}^n \Lambda_j^F\left(s_j,z\right)+\omega+\eta\chi_M \geq 0$ if and only if $\theta^F\left(\left(s_j\right)_{j=1}^n, z\right) \leq \eta$ (Lemma 4.3). The authors apply a deformation lemma for which it must prove that the conditions of the lemma are satisfied. One of these conditions (the one which uses Survival Assumption) requires that $0 \notin \Delta^F\left(\left(s_j\right)_{j=1}^n, z\right)$ for all $\left(\left(s_j\right)_{j=1}^n, z\right) \in GM_{\overline{\eta},\alpha}^F$. If it is not, then (see the proof of Lemma 4.4) there exists $\left(\left(s_j\right)_{j=1}^n, z\right) \in \left(\chi_M^{\perp_F}\right)^n \times Z_D^F$ such that $\overline{\eta} \leq \theta^F\left(\left(s_j\right)_{j=1}^n, z\right) \leq \alpha$ and $p \in N_{-F_+}\left(\Lambda_0^F\left(-\sum_{j=1}^n s_j - proj_{\chi_M^{\perp_F}}\omega, z\right)\right) \cap S$ such that $p \in \cap_{j=1}^n N_{Y_j^F(z)}(y) \cap S^F$. By the above result, $\sum_{j=1}^n \Lambda_j^F\left(s_j,z\right) + \omega + \alpha\chi_M \geq 0$, and it can be proved that $p\left(\sum_{j=1}^n \Lambda_j^F\left(s_j,z\right) + \omega + \alpha\chi_M\right) = 0$ contradicting Survival Assumption since $\left(\Lambda_j^F\left(s_j,z\right)\right)_{j=1}^n \in A^F(\omega + \alpha\chi_M, z)$. Therefore, Assumption (WSA^F) is enough to conclude if one proves that $\alpha \leq 2n\overline{r} + \|\omega\|$.

Since $\Lambda_j^F\left(s_j,z\right) \in \partial Y_j^F(z)$, $\Lambda_j^F\left(s_j,z\right) \notin \text{int}F_+$ (otherwise, $0 \notin \partial Y_j^F(z)$). Consequently, for $\varepsilon > 0$, there exists $\xi \in B\left(\Lambda_j^F\left(s_j,z\right), \varepsilon\right) \cap (F \diagdown F_+)$ and $M' \subset M$ such that $\mu(M') \neq 0$ and $\Lambda_j^F\left(s_j,z\right)(m) = s_j(m) - \lambda_j^F\left(s_j,z\right) - \varepsilon < \xi(m) \leq 0$ for all $m \in M'$. Hence, one deduces that $\lambda_j^F\left(s_j,z\right) > -\left\|s_j\right\| - \varepsilon$. In the same way, $\Lambda_j^F\left(s_j,z\right) \notin \text{int}(-F_+)$ (otherwise, $\Lambda_j^F\left(s_j,z\right) \notin \partial Y_j^F(z)$). Hence, for $\varepsilon > 0$, $B\left(\Lambda_j^F\left(s_j,z\right), \varepsilon\right) \cap (F \diagdown (-F_+)) \neq \emptyset$, from which one deduces that $\lambda_j^F\left(s_j,z\right) < \left\|s_j\right\| + \varepsilon$. Consequently, $-\left\|s_j\right\| - \varepsilon < \lambda_j^F\left(s_j,z\right) < \left\|s_j\right\| + \varepsilon$. Since the inequality is true for all $\varepsilon > 0$, one has $\left|\lambda_j^F\left(s_j,z\right)\right| \leq \left\|s_j\right\|$ for all j. On the other hand, for $\Lambda_0^F(u,z) \in \partial(-F_+)$, one easily checks that $\left|\lambda_0^F(u,z)\right| \leq \|u\|$ since $-F_+$ is convex.

Let $\left(\left(s_j\right)_{j=1}^n, z\right) \in \left(\overline{B}^F(0,2a) \cap \left\{\chi_M^{\perp_F}\right\}\right)^n \times Z_D^F$. From the above remarks and the fact that $\left|proj_{\chi_M^{\perp_F}}\omega\right| \leq \|\omega\|$, it follows that $\theta^F\left(\left(s_j\right)_{j=1}^n, z\right) \leq 4na + \|\omega\| < 2n\overline{r} + \|\omega\| \leq \overline{\lambda}$, which in turn implies that $\alpha \leq 2n\overline{r} + \|\omega\|$.

For the Local Non-Satiation Assumption, we remark that it is used in Bonnisseau and Médecin (2001) only in Claim 4.6 where $z^F \in A^F(\omega)$. Consequently, condition (LNS^F) of Lemma 1 is enough to conclude, and the proof of the Proposition 3 is complete.

7.6.2 The Limit Point

Let $\left(\left(\left(x_i^F\right)_{i=1}^m, \left(y_j^F\right)_{j=1}^n\right), p^F\right)_{F\in\mathscr{F}}$ be the net of equilibria of the subeconomies $\left(\mathscr{E}^F\right)_{F\in\mathscr{F}}$ given by Proposition 3. From the definition of $N_{Y_j^F(z)}^F\left(y_j\right) \cap S^F$, there exist price vectors $\left(\pi_j^F\right)_{j=1}^n \in \prod_{j=1}^n N_{Y_j(z^F)}\left(y_j^F\right) \cap S$ such that $p^F = \pi_{j|F}^F$ for all j. Hence, we obtain the net $\left(\left(x_i^F\right)_{i=1}^m, \left(y_j^F\right)_{j=1}^n, \left(\pi_j^F\right)_{j=1}^n\right)_{F\in\mathscr{F}}$. Proposition 3 implies that $\left(\left(x_i^F\right)_{i=1}^m, \left(y_j^F\right)_{j=1}^n\right)_{F\in\mathscr{F}} \in A(\omega)$, which is norm bounded by Assumption (B). Hence, from the Banach-Alaoglu theorem, it remains in a $\prod_{L^{m+n}} \sigma^\infty$−compact subset of L^{m+n}. Furthermore, the net $\left(\pi_j^F\right)_{F\in\mathscr{F}}$ belongs to S which is σ^{ba}-compact. Consequently, there exists a subnet

$\left(\left(x_i^{F(t)}\right), \left(y_j^{F(t)}\right), \left(\pi_j^{F(t)}\right)\right)_{t\in(T,\geq)}$ which $\prod_{L^{m+n}} \sigma^\infty \times \sigma^{ba}$−converges to $\left((\bar{x}_i), (\bar{y}_j), (\bar{\pi}_j)\right)$. This also implies that the subnets of real numbers $\left(\left\langle p^{F(t)}, y_j^{F(t)}\right\rangle_{F(t)}\right) = \left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)$ and $\left(\left\langle p^{F(t)}, x_i^{F(t)}\right\rangle_{F(t)}\right) = \left(\pi_j^{F(t)}\left(x_i^{F(t)}\right)\right)$ are bounded so that they can be supposed to converge.

We now prove that at least one limit point exists which in turn is a marginal pricing equilibrium of the economy $\mathscr{E}$.

Claim 1. $\bar{\pi}_1 = \bar{\pi}_2 = \ldots = \bar{\pi}_n > 0$

Proof. We first prove that $\bar{\pi}_1 = \bar{\pi}_2 = \ldots = \bar{\pi}_n \geq 0$. Let $x \in L$. There exists $F \in \mathscr{F}$ such that $x \in F$. There exists $t_0 \in T$ such that $F \subset F(t)$ for all $t > t_0$. As $p^{F(t)} = \pi_{j|F(t)}^{F(t)}$ for all j, we have that, for all $t > t_0$, $\langle p^{F(t)}, x\rangle_{F(t)} = \pi_j^{F(t)}(x)$ for all j. Without loss of generality, we denote the limit of $\langle p^{F(t)}, x\rangle_{F(t)}$ by $\bar{\pi}(x)$. Hence, $\lim \pi_j^{F(t)}(x) = \bar{\pi}(x)$ for all j. Since $ba^+(M, \mathscr{M}, \mu)$ is closed, we have the first part of the Claim. Since $\pi_j^{F(t)}(\chi_M) = 1$ for all j and $t \in T$, we have that $\bar{\pi}(\chi_M) = 1$. Therefore, the proof is complete.

Claim 2. $\left((\bar{x}_i)_{i=1}^m, (\bar{y}_j)_{j=1}^n\right) \in \prod_{i=1}^m X_i(\bar{z}) \times \prod_{j=1}^n \partial_\infty Y_j(\bar{z})$ and $\sum_{i=1}^m \bar{x}_i = \sum_{j=1}^n \bar{y}_j + \omega$

Proof. $\left(\left(x_i^{F(t)}\right)_{i=1}^m, \left(y_j^{F(t)}\right)_{j=1}^n\right) \in Z^{F(t)}$. Since $\left(z^{F(t)}\right)_{t\in(T,\geq)}$ $\prod_{L^{m+n}} \sigma^\infty$−converges to $\bar{z}$, we get $\bar{z} = \left((\bar{x}_i)_{i=1}^m, (\bar{y}_j)_{j=1}^n\right) \in \prod_{i=1}^m X_i(\bar{z}) \times \prod_{j=1}^n \partial_\infty Y_j(\bar{z})$ by Assumption C(i) and Proposition 1. Since $\sum_{i=1}^m x_i^{F(t)} = \sum_{j=1}^n y_j^{F(t)} + \omega$ for all $t \in T$, one obtains $\sum_{i=1}^m \bar{x}_i = \sum_{j=1}^n \bar{y}_j + \omega$.

Claim 3. For all i, if $x_i \succsim_{i,\bar{z}} \bar{x}_i$, then $\bar{\pi}(x_i) \geq r_i\left(\bar{\pi}(\omega_i), \lim\left(\pi_j^{F(t)}\left(y_j^{F(t)}\right)\right)_{j=1}^n\right)$.

Proof. See Fuentes (2011).

Claim 4. For all i, $\bar{\pi}(\bar{x}_i) = r_i\left(\bar{\pi}(\omega_i), \left(\bar{\pi}\left(\bar{y}_j\right)\right)_{j=1}^n\right)$ and for all j, $\bar{\pi}\left(\bar{y}_j\right) = \lim \pi_j^{F(t)}\left(y_j^{F(t)}\right)$.

Proof. By Proposition 2, we have $\lim \pi_j^{F(t)}\left(y_j^{F(t)}\right) \geq \bar{\pi}_j\left(\bar{y}_j\right)$ for all j. The rest of the proof is identical to the proof of Step 6 of Fuentes (2011).

From Claims 2 and 4 together with Proposition 2, one obtains $\bar{z} \in Z$, $\bar{\pi} \in \bigcap_{j=1}^n N_{Y_j(\bar{z})}\left(\bar{y}_j\right) \cap S$ and $\sum_{i=1}^m \bar{x}_i = \sum_{j=1}^n \bar{y}_j + \omega$. It only remains to show that condition a. of Definition 1. is satisfied.

Claim 5. For all i, $\bar{x}_i$ is a greater element for $\succsim_{i,\bar{z}}$ in the budget set $\left\{x_i \in X_i(\bar{z}) : \bar{\pi}(x_i) \leq r_i\left(\bar{\pi}(\omega_i), \left(\bar{\pi}\left(\bar{y}_j\right)\right)_{j=1}^n\right)\right\}$.

Proof. We have to show that for every agent i, if $x_i \succ_{i,\bar{z}} \bar{x}_i$, then $\bar{\pi}(x_i) > \bar{\pi}(\bar{x}_i)$. From Claims 3 and 4, one has $\bar{\pi}(x_i) \geq \bar{\pi}(\bar{x}_i)$. Suppose $\bar{\pi}(x_i) = \bar{\pi}(\bar{x}_i)$. From Claims 3, 4 and Assumptions (WSA) and (R), $\bar{\pi}(\bar{x}_i) = r_i\left(\bar{\pi}(\omega_i), \left(\bar{\pi}\left(\bar{y}_j\right)\right)_{j=1}^n\right) > 0$. For all $t \in (0,1)$, we have $\bar{\pi}(tx_i) < \bar{\pi}(x_i) = \bar{\pi}(\bar{x}_i)$. For t close enough to 1, $tx_i \in X_i(\bar{z})$ and, since preferences are continuous, $tx_i \succ_{i,\bar{z}} \bar{x}_i$. From Claim 4, we get $\bar{\pi}(tx_i) \geq \bar{\pi}(\bar{x}_i)$, a contradiction with the above inequality.

7.7 Existence of Marginal Cost Pricing Equilibria

An equilibrium as defined in Definition 1 is called marginal cost pricing equilibrium in Shannon (1996) and many other papers. The terminology has been adopted because it is suggestive even though it is not always correct as indicated earlier by Guesnerie (1990). Indeed, $\bar{\pi} \in N_{Y_j(\bar{z})}\left(\bar{y}_j\right)$ implies that $\bar{\pi}$ is proportional to the marginal cost only if the set of input combinations for producing a given level of output is convex. Marginal cost pricing equilibrium also means that every producer minimizes its costs. Bonnisseau and Cornet (1990a,b) investigated and established a formal link between the marginal pricing rule and the one of marginal cost pricing in the finite-dimensional case. We are now interested in having a marginal cost pricing equilibrium for an economy with externalities and infinitely many commodities. We must introduce both the notions of *iso-output set* and *cost functional*, for which we have to distinguish a priori between inputs and outputs. Although we follow the approach of Bonnisseau and Cornet (1990a), there appear significant drawbacks in using their technique in our economy as we shall see later.

Let I^j and O^j be partitions of the set M for the j-th producer, such that $M = I^j \cup O^j$ and $I^j \cap O^j = \emptyset$. We define the following subspaces of L.

$L^{I^j} = \{u \in L : u(m) = 0 \ \mu - a.e. \text{ if } m \notin I^j\}$

$L^{O^j} = \{u \in L : u(m) = 0 \ \mu - a.e. \text{ if } m \notin O^j\}$

For every $y_j \in L$, we denote $proj_{L^{I^j}}\left(y_j\right)$ as y_{I^j}. Note that $y_{I^j} \in L^{I^j}$ since $proj_{L^{I^j}}\left(y_j\right)$ is measurable. The same applies for $y_{O^j} = proj_{L^{O^j}}\left(y_j\right)$.

We now define the *iso-output* set: for all $(r, b, z) \in \left(L^{I^j}\right)^*_+ \times L^{O^j} \times L^{m+n}$, we let

$$Y_j(b, z) = \left\{-y_{I^j} \in L : \text{ there exists } y_j \in Y_j(z), y_j = y_{O^j} + y_{I^j} \text{ and } y_{O^j} = b\right\}.$$

For all $(r, b, z) \in \left(L^{I^j}\right)^*_+ \times L^{O^j} \times L^{m+n}$, we define the cost functional c_j as follows:

$$c_j(r, b, z) = \inf\left\{r(a) : a \in Y_j(b, z)\right\}$$

if $Y_j(b, z) \neq \emptyset$.

For every $(r, b, z) \in \left(L^{I^j}\right)^*_+ \times L^{O^j} \times L^{m+n}$, we denote by $\nabla_O c_j(r, b, z)$ the (Fréchet) gradient vector of c_j with respect to b. Thus, for every x in L^{O^j}, $\nabla_O c_j(r, b, z)(x) = \lim_{t \to 0} \frac{c_j(r, b+tx, z) - c_j(r, b, z)}{t}$, and hence, $\nabla_O c_j(r, b, z) \in \left(L^{O^j}\right)^*$.

As in Bonnisseau and Cornet (1990a), we separate between the first $n-1$ producers and the n−th one which maximizes his profit. For the $n-1$ first ones, we posit the following assumption:

Assumption C(vi) (P'). For $z \in L^{m+n}$

(i) There exists a partition of the set M into two non-empty subsets I^j and O^j such that $\mu\left(I^j\right) \neq \emptyset$ and $\mu\left(O_j\right) \neq \emptyset$. For every $y_j \in Y_j(z)$, $y_j(m) = y_{I^j}(m) \leq 0$ if $m \in I^j$. Furthermore, there exists $\tilde{y}_j \in Y_j(z)$ such that $y_{O^j} \leq \tilde{y}_{O^j}$ and $\tilde{y}_j(m) = \tilde{y}_{Oj}(m) \geq 0$ if $m \in O^j$.

(ii) The set $Y_j(b, z)$ is convex

(iii) The set $\Omega_j = \left\{b \in L^{O^j} : Y_j(b, z) \neq \emptyset\right\}$ is $\sigma^{\infty}_{L^{O^j}}$−open.

(iv) For every $r \in \left(L^{I^j}\right)^*_+$, the cost functional $c_j(r, \cdot, z)$ is $\mathcal{T}_{L^{O^j}}$−differentiable on Ω_j.

For the nth producer, we let

Assumption C(vi) (P"). The correspondence $Y_n : L^{m+n} \longmapsto L$ is convex valued.

We remark that in an economy without externalities and with R^l as commodity space, the above assumptions are the same as those in Bonnisseau and Cornet (1990a). We refer to that paper for an economic interpretation. We note that every $y_j \in Y_j(z)$ has a unique representation $y_j = y_{I^j} + y_{O^j}$ since $L^{I^j} \cap L^{O^j} = \{0\}$.

For every $\pi \in L^*_+$, we denote by π_{I^j} (π_{O^j}) the restriction of π to L^{I^j} (L^{O^j}). We now can give a precise definition of marginal cost pricing equilibrium.

Definition 2. A marginal cost pricing equilibrium of the economy $\mathscr{E}$ is an element $(\hat{z}, \hat{\pi}) = \left(\left((\hat{x}_i)^m_{i=1}, (\hat{y}_j)^n_{j=1}\right), \hat{\pi}\right)$ in $Z \times S$ such that

a. For all i, $\hat{x}_i$ is $\succsim_{i,\hat{z}}$-maximal in $\left\{x_i \in X_i(\hat{z}) : \hat{\pi}(x_i) \leq r_i\left(\hat{\pi}(\omega_i), \left(\hat{\pi}\left(\hat{y}_j\right)\right)_{j=1}^n\right)\right\}$

b$'$. For all $j = 1, \ldots n-1$, $\hat{\pi}_{I^j}(-\hat{y}_{I^j}) = c_j(\hat{\pi}_{I^j}, \hat{y}_{O^j}, \hat{z})$ (cost minimization), $\hat{y}_{O^j} \geq 0$ (output condition) and $\hat{\pi}_{O^j} = \nabla_O c_j(\hat{\pi}_{I^j}, \hat{y}_{O^j}, \hat{z})$ (marginal cost pricing). For $j = n$, $\hat{\pi}(\hat{y}_n) \geq \hat{\pi}(y)$ for all $y \in Y_n(\hat{z})$ (profit maximization)

c$'$. $\sum_{i=1}^m \hat{x}_i \leq \sum_{j=1}^n \hat{y}_j + \omega$ and $\hat{\pi}\left(\sum_{i=1}^m \hat{x}_i - \sum_{j=1}^n \hat{y}_j - \omega\right) = 0$

One easily checks that Condition c. of Definition 1 implies Condition c'. above. Condition b' says that at equilibrium every producer minimizes his cost, prices equal marginal cost and resultant production vectors are non-negative.

Lemma 2. *Let us assume that P and P' hold. Let* $p \in L_+ \setminus \{0\}$*, let* $y_j = y_{I^j} + y_{O^j} \in \partial_\infty Y_j(z)$ *such that* $p(-y_{I^j}) = c_j(p_{I^j}, y_{O^j}, z)$ *and* $p_{O^j} \leq \nabla_{O^j} c_j(p_{I^j}, y_{O^j}, z)$*. Then* $p \in N_{Y_j(z)}\left(y_j\right)$.

Proof. The proof is a direct transcription of the proof of Lemma 2 (a) in Bonnisseau and Cornet (1990a) since, in this point, there are not relevant differences when considering externalities and infinitely many commodities.

The next proposition is the key argument of the proof of Theorem 2.

Proposition 4. *Let* (z, π) *be a MPE of* $\mathscr{E}$ *such that* $y_{O^j} = y_{O^j}^+$ *for all j. Then* (z, π) *is a MCPE of* $\mathscr{E}$ *if Assumptions P,* P' *and* P'' *hold.*

The proof of this proposition is given in the Appendix. This shows the relationship between the two notions of marginal pricing equilibrium and marginal cost pricing equilibrium under the particular circumstance that, at marginal pricing equilibrium, all outputs are non-negative. We remark that in the paper of Bonnisseau and Cornet (1990a), they show the relationship between the two notions of equilibrium also in the case $y_{Oj} \neq y_{Oj}^+$. We refer to the Appendix for more details on this subject.

A sufficient condition for $y_{Oj} = y_{Oj}^+$ is that the price system is (punctually) strictly positive.

Lemma 3. *Let* $z \in L^{m+n}$*, let* $Y_j : L^{m+n} \mapsto L$ *be a correspondence satisfying Assumption P and P'(i). Let* $y_j \in \partial_\infty Y_j(z)$ *such that* $\nabla_1 f_j\left(y_j, z\right) \in \mathscr{L}_1^{++}$*. Then,* $y_{Oj} = y_{Oj}^+$.

Proof. Suppose that $y_{Oj} \neq y_{Oj}^+$. Hence, $y_j' = y_{I^j} + y_{O^j}^+ > y_j = y_{I^j} + y_{O^j}$ and $y_j' \in \partial_\infty Y_j(z)$ by Assumptions P'(i) and free disposal. Consequently, $\nabla_1 f_j\left(y_j, z\right)\left(y_j - y_j'\right) < 0$ since $\nabla_1 f_j\left(y_j, z\right)$ is in the quasi-interior of $\mathscr{L}_1^+$. Since $\nabla_1 f_j$ is $\left(\sigma^\infty \times \prod_{L^{m+n}} \sigma^\infty\right)$-continuous by Assumption P, there exists an σ^∞–open neighbourhood $U\left(y_j\right)$ of y_j, such that $\nabla_1 f_j\left(y_j'', z\right)\left(y_j - y_j'\right) < 0$ for all $y'' \in U\left(y_j\right)$. Let $y_j^\kappa = \kappa y_j' + (1-\kappa) y_j$ such that $\kappa > 0$. For all $\kappa \in (0,1)$, $y_j' > y_j^\kappa > y_j$, so that $y_j^\kappa \in \partial_\infty Y_j(z)$; and for κ close enough to 0, $y_j^\kappa \in U\left(y_j\right)$. Consequently, $\nabla_1 f_j\left(y_j^\kappa, z\right)\left(y_j - y_j'\right) < 0$ which implies that $\left(y_j - y_j'\right) \in \operatorname{int}\left[\nabla_1 f_j\left(y_j^\kappa, z\right)\right]^o =$

$\mathrm{int}T_{Y_j(z)}\left(y_j^\kappa\right)$ (Clarke 1983, Theorem 2.4.7). Since the set of vectors hypertangent to $Y_j(z)$ at y_j is non-empty, $\left(y_j - y_j'\right)$ is hypertangent to $Y_j(z)$ at y_j (Clarke 1983, Theorem 2.4.8). Consequently, $y_j^\kappa + \varepsilon\left(y_j - y_j'\right) + \varepsilon a\chi_M \in Y_j(z)$ for all $\varepsilon > 0$ small enough and a suitably chosen $a > 0$. Let us take $\varepsilon < \kappa$, then $y_j^\kappa + \varepsilon\left(y_j - y_j'\right) \in \mathrm{int}Y_j(z)$. Since $y_j^\kappa + \varepsilon\left(y_j - y_j'\right) = (\kappa - \varepsilon)\, y_j' + (1 - \kappa + \varepsilon)\, y_j$, one has that $y_j^\kappa + \varepsilon\left(y_j - y_j'\right) > y_j$ a contradiction.

Actually, the above proof shows a stronger result than the statement of the lemma: y_j is efficient. We remark that the proof parallels that of Proposition 2 in Bonnisseau and Créttez (2007) for the finite-dimensional case. The only difference is that we use Theorem 2.4.8 of Clarke (1983) instead of Theorem 2.5.8 of Clarke´s book.

We posit an additional assumption before stating the main result of this section.

Assumption SPP (Strictly Positive Prices). For all j, if $(z, \pi) \in A(\omega) \times \cap_{j=1}^{n} N_{Y_j(z)}\left(y_j\right) \cap S$, then $\nabla_1 f_j\left(y_j, z\right) \in \mathscr{L}_1^{++}$.

From Assumption P(iv), $\nabla_1 f_j\left(y_j, z\right) \in \mathscr{L}_1^{+}$ for all $y_j \in \partial_\infty Y_j(z)$. Hence, Assumption SPP only requires that the common price vector π, which is given by the marginal pricing rule of each producer, be strictly positive when the allocation is feasible and weakly efficient. Assumption SPP is weaker than Assumption P(4) in Shannon (1996) where it is required that $\nabla_1 f_j\left(y_j, z\right) \in \mathscr{L}_1^{++}$ for all $y_j \in Y_j(z)$.

Theorem 2. *Under Assumptions (C), (P), (P'), (P"), (WSA), (R) and (SPP), the economy* $\mathscr{E} = \left(\left(X_i, \succsim_{i,z}, r_i\right)_{i=1}^{m}, \left(Y_j\right)_{j=1}^{n}, \left(\omega_i\right)_{i=1}^{m}\right)$ *has a marginal cost pricing equilibrium.*

Proof. The proof follows immediately from Theorem 1, Assumption SPP and Proposition 4.

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Appendix

Proof of Proposition 1

Before proving Proposition 1, we show that, given Assumption P, for every $z \in L^{m+n}$ and every j, $\overline{\mathrm{int}Y_j(z)}$[8] $= Y_j(z)$. Let y belong to $Y_j(z)$. If y belongs to $\mathrm{int}Y_j(z)$, then y belongs to $\overline{\mathrm{int}Y_j(z)}$. If y belongs to $\partial_\infty Y_j(z)$, then for all $\varepsilon > 0$, $y - \frac{\varepsilon}{2}\chi_M$ belongs to $\mathrm{int}Y_j(z)$ by free disposal. Consequently, $B(y,\varepsilon) \cap \mathrm{int}Y_j(z) \neq \emptyset$ for all $\varepsilon > 0$, and thus, y belongs to $\overline{\mathrm{int}Y_j(z)}$.

We now prove that the correspondence $Y_j : L^{m+n} \mapsto L$ is $\left(\prod_{L^{m+n}}\sigma^\infty, \mathscr{T}\right)$-l.h.c. From the above result and Lemma 14.21 in Aliprantis and Border (1994), it is enough to prove that $\mathrm{int}Y_j : L^{m+n} \mapsto L$ is $\left(\prod_{L^{m+n}}\sigma^\infty, \mathscr{T}\right)$-l.h.c. Let $y \in \mathrm{int}Y_j(z)$ and let z^α be a net which $\prod_{L^{m+n}}\sigma^\infty$−converges to z. Since f_j is $\sigma^\infty \times \prod_{L^{m+n}}\sigma^\infty$-continuous, there exists $\alpha_0 \in \Gamma$ such that $\alpha > \alpha_0$ implies $f_j(y, z^\alpha) < 0$. Hence, there exists a net $y^\alpha\ (= y) \in \mathrm{int}Y_j(z^\alpha)$ for all α and $y^\alpha \to y$.

The weak* closeness of Y_j is immediate from Assumption P(ii).

Remark 3. Given Assumption P, if the correspondence Y_j is convex valued, then $\overline{\mathrm{int}Y_j(z)} = Y_j(z)$ without free disposal requirement (Schaefer and Wolf 1999, p. 38, 1.3). On the other hand, we can repeat the argument made above to show that the correspondence $Y_j^F : F^{m+n} \mapsto F$ is l.h.c.

Proof of Proposition 2

We omit the index j in order to simplify the notation. We first state the following Lemma:

Lemma 4. *For a given $\bar{z} \in L^{m+n}$, let $T_{Y(\bar{z})}(\bar{y})$ be the Clarke tangent cone of $Y(\bar{z})$ at $\bar{y}$. Let $v \in T_{Y(\bar{z})}(\bar{y})$ and $\delta > 0$. There exist weak* open neighbourhoods of $\bar{z}$ and $\bar{y}$, $W^{\bar{z}}$ and $W^{\bar{y}}$, respectively, such that for all $\varepsilon > 0$, for all $z \in W^{\bar{z}}$ and for all $y \in W^{\bar{y}} \cap B(\bar{y},\varepsilon)$, $v + \bar{y} - y - \delta\chi_M \in T_{Y(z)}(y)$.*

Proof. Given $\bar{z} \in Z$, we have to prove that $\nabla_1 f(\bar{y},\bar{z})(v + \bar{y} - y - \delta\chi_M) \leq 0$. Let $0 < \alpha < \frac{\delta\nabla_1 f(\bar{y},\bar{z})(\chi_M)}{2(\|v\|+\varepsilon+\delta)}$. From Assumption P(v), there exists a $\prod_{L^{m+n+1}}\sigma^\infty$−open neighbourhood of $(\bar{z},\bar{y})$, $U^{\bar{z}} \times U^{\bar{y}}$, such that for all $(z,y) \in U^{\bar{z}} \times U^{\bar{y}}$, $|\nabla_1 f(y,z) - \nabla_1 f(\bar{y},\bar{z})| < \alpha$. Let us consider the following σ^∞−open neighbourhood of $\bar{y}$,

$$V^{\bar{y}} = \left\{ y \in L : |\nabla_1 f(\bar{y},\bar{z})(\bar{y} - y)| < \frac{\delta\nabla_1 f(\bar{y},\bar{z})(\chi_M)}{2} \right\}$$

[8] For $z \in L$, $\mathrm{int}Y_j(z) = \{y \in L : f_j(y,z) < 0\}$.

Let $W^{\bar{y}} = U^{\bar{y}} \cap V^{\bar{y}}$, $W^{\bar{z}} = U^{\bar{z}}$ and $\varepsilon > 0$. For all $(y, z) \in W^{\bar{y}} \cap B(\bar{y}, \varepsilon) \times W^{\bar{z}}$,

$$\begin{aligned}
\nabla_1 f(y,z)(v + \bar{y} - y - \delta\chi_M) &= (\nabla_1 f(y,z) - \nabla_1 f(\bar{y},\bar{z}))(v + \bar{y} - y - \delta\chi_M) \\
&\quad + \nabla_1 f(\bar{y},\bar{z})(v + \bar{y} - y - \delta\chi_M) \\
&\quad < \alpha(\|v\| + \varepsilon + \delta) \\
&+ \nabla_1 f(\bar{y},\bar{z})(v) + \nabla_1 f(\bar{y},\bar{z})(\bar{y} - y) - \delta\nabla_1 f(\bar{y},\bar{z})(\chi_M) \\
&\quad < \frac{\delta\nabla_1 f(\bar{y},\bar{z})(\chi_M)}{2} + \nabla_1 f(\bar{y},\bar{z})(v) \\
&\quad + \frac{\delta\nabla_1 f(\bar{y},\bar{z})(\chi_M)}{2} - \delta\nabla_1 f(\bar{y},\bar{z})(\chi_M) \\
&= \nabla_1 f(\bar{y},\bar{z})(v) \leq 0
\end{aligned}$$

We now proceed to the proof of Proposition 2. Let $(z^\alpha, \pi^\alpha)_{(\Gamma,\leq)}$ be a net of $Z \times S$, $\prod_{L^{m+n}} \sigma^\infty \times \sigma^{ba}$—converging to $(\bar{z}, \bar{\pi})$. Let $v \in T_{Y(\bar{z})}(\bar{y})$ and $\delta > 0$. There exist $\varepsilon > 0$ and $\alpha_0 \in \Gamma$ such that for all $\alpha > \alpha_0$, $y^\alpha \in B(0, \varepsilon)$. We note that $\|y^\alpha - \bar{y}\| < \varepsilon + \|\bar{y}\| = \varepsilon'$. Hence, for all $\alpha > \alpha_0$, $y^\alpha \in B(\bar{y}, \varepsilon')$. From the above lemma, there exist weak*-open neighbourhoods of $\bar{z}$ and $\bar{y}$, $W^{\bar{z}}$ and $W^{\bar{y}}$, respectively, such that for $\varepsilon' > 0$ and all $\alpha > \alpha_0$, $(y^\alpha, z^\alpha) \in W^{\bar{y}} \cap B(\bar{y}, \varepsilon') \times W^{\bar{z}}$ and $v + \bar{y} - y^\alpha - \delta\chi_M \in T_{Y(z^\alpha)}(y^\alpha)$.

Since $\pi^\alpha \in N_{Y(z^\alpha)}(y^\alpha)$, $\pi^\alpha(v + \bar{y} - y^\alpha - \delta\chi_M) \leq 0$ for all $\alpha > \alpha_0$. Passing to the limit, we obtain $\bar{\pi}(v) + \bar{\pi}(\bar{y}) - \lim_\alpha \pi^\alpha(y^\alpha) - \delta \leq 0$. Since $0 \in T_{Y(\bar{z})}(\bar{y})$, $\bar{\pi}(\bar{y}) \leq \lim_\alpha \pi^\alpha(y^\alpha) + \delta$, and since this inequality holds true for all $\delta > 0$, we have $\bar{\pi}(\bar{y}) \leq \lim_\alpha \pi^\alpha(y^\alpha)$.

Let $v \in T_{Y(\bar{z})}(\bar{y})$. If $\lim_\alpha \pi^\alpha(y^\alpha) = \bar{\pi}(\bar{y})$, then $\bar{\pi}(v) \leq 0$. Consequently, $\bar{\pi} \in N_{Y(\bar{z})}(\bar{y}) \cap S$ since $\pi^\alpha \in S$ for all α.

Proof of Proposition 4

We first state and prove the following lemma, which is used in the proof of Proposition 4. To simplify, we suppress index *j*.

Lemma 5. *Let* $p_I \in (L^I)^*_+$, *then there exists* $\hat{p}_I \in L^*_+$ *such that* $\hat{p}_I(x) = p_I(x^I)$ *if* $x \notin L^O$ *and* $\hat{p}_I(x) = 0$ *if* $x \in L^O$.

Proof. Let $p_I \in (L^I)^*_+$. By a classical extension theorem, there exists a functional $\tilde{p}_I \in L^*_+$, and hence, a measure $\tilde{v}_I \in ba^+(M, \mathscr{M}, \mu)$ such that $\tilde{p}_I(x) = \int_{m \in M} x(m)\, d\tilde{v}_I(m)$ and $p_I(x) = \tilde{p}_I(x)$ for all $x \in L^I$, since L^* and $ba(M, \mathscr{M}, \mu)$ are isometrically isomorphic (Dunford and Schwarz 1958). We now define the measure $\hat{v}_I$ as:

$$\hat{v}_I(A) = \begin{cases} \tilde{v}_I(A^I) & \text{if } A \subsetneq O \\ 0 & \text{otherwise} \end{cases}.$$

One easily checks that $\hat{v}_I \in ba^+(M, \mathscr{M}, \mu)$ which is identified with a functional $\hat{p}_I \in L_+^*$. Take $x \notin L^O$. There exists $M' \subset I$ such that $\mu(M') \neq 0$ and $x^I(m) \neq 0$ for all $m \in M'$. Consequently, $\hat{p}_I(x) = \hat{p}_I(x^I) + \hat{p}_I(x^O) = \int_{m\in M} x^I(m)\, d\hat{v}_I(m) + \int_{m\in M} x^O(m)\, d\hat{v}_I(m) = \int_{m\in I} x^I(m)\, d\hat{v}_I(m) = \int_{m\in I} x^I(m)\, d\tilde{v}_I(m) = \tilde{p}_I(x^I) = p_I(x^I)$. If $x \in L^O$, $\hat{p}_I(x) = \int_{m\in M} x^O(m)\, d\hat{v}_I(m) = 0$.

Remark 4. The above lemma can be rewritten in terms of the subspace $(L^O)^*$ as follows: for every $p_O \in (L^O)^*_+$, there exists a functional $\hat{p}_O \in L_+^*$ such that $\hat{p}_O(x) = p_O(x^O)$ if $x \notin L^I$ and $\hat{p}_O(x) = 0$ if $x \in L^I$.

First, we claim that for all $t > 0$, $-y_{I^j}$ does not belong to the relative interior of $Y_j(y_{O^j} + t\chi_{O^j}, z)$. Otherwise, $y_j \in \text{int}Y_j(z)$. We also note that for all $t > 0$, the relative interior of $Y_j(y_{O^j} + t\chi_{O^j}, z)$ is non-empty. Finally, since for all $t > 0$, $Y_j(y_{O^j} + t\chi_{O^j}, z)$ is convex, $\cup_{t>0}\text{int}Y_j(y_{O^j} + t\chi_{O^j}, z)$ is open, non-empty and convex (Schaefer and Wolf 1999, p. 38, 1.2).

Since $-y_{I^j} \notin \cup_{t>0}\text{int}Y_j(y_{O^j} + t\chi_{O^j}, z)$, there exists a continuous linear functional $p_{I^j} \in \left(L^{I^j}\right)^*_+$ such that $p_{I^j}(-y_{I^j}) \leq p_{I^j}(a)\ \forall a \in \cup_{t>0}\text{int}Y_j(y_{O^j} + t\chi_{O^j}, z)$,[9] whence $p_{I^j}(-y_{I^j}) \leq p_{I^j}(a')$ for all $a' \in \cup_{t>0} Y_j(y_{O^j} + t\chi_{O^j}, z)$. Consequently, $p_{I^j}(-y_{I^j}) = c_j(p_{I^j}, y_{O^j} + t\chi_{O^j}, z)$ since $-y_{I^j} \in Y_j(y_{O^j} + t\chi_{O^j}, z)$ for all $t > 0$. By the above lemma, we can extend the functional p_{I^j} to an element of L_+^*—denoted by p_{I^j} as well—such that $p_{I^j}(\xi) = 0$ for all $\xi \in L^{O^j}$. Let $p_{O^j} = \nabla_{O^j} c_j(p_{I^j}, y_{O^j}, z) \in \left(L_+^{O^j}\right)^*$. We also extend p_{O^j} to L_+^*—denoted by p_{O^j} as well—such that $p_{O^j}(\xi) = 0$ for all $\xi \in L^{I^j}$. Consequently, by Lemma 2, $p_j = p_{I^j} + p_{O^j} \in N_{Y_j}(z)(y_j)$. Since, (z, π) is a marginal pricing equilibrium, $\pi = \lambda p_j$ for some $\lambda > 0$. Hence, $\pi_{I^j} = \lambda p_{I^j}$ and $\pi_{O^j} = \lambda p_{O^j} = \lambda \nabla_{O^j} c_j(p_{I^j}, y_{O^j}, z) = \nabla_{O^j} c_j(\lambda p_{I^j}, y_{O^j}, z) = \nabla_{O^j} c_j(\pi_{I^j}, y_{O^j}, z)$. Consequently, $\pi_{I^j}(-y_{I^j}) = c_j(\pi_{I^j}, y_{O^j}, z)$ and $\pi = \pi_{I^j} + \nabla_{O^j} c_j(\pi_{I^j}, y_{O^j}, z) \in N_{Y_j(z)}(y_j)$. Hence, conditions a., b'. and c'. of Definition 2 are satisfied.

Remark. We point out that Bonnisseau and Cornet show that if (z, π) is a marginal pricing equilibrium, then there exists a vector $(w_j)_{j=1}^n \in L^n$ (our notation) defined as $w_j = y_{I^j} + y_{O^j}^+$, such that $\left((x_i)_{i=1}^m, (w_j)_{j=1}^n, \pi\right)$ is a marginal cost pricing equilibrium. A significant difference between our approach and theirs is that in their case, $\left((x_i)_{i=1}^m, (w_j)_{j=1}^n\right) \in \Pi_{i=1}^m X_i \times \Pi_{j=1}^n Y_j$, while in ours, if $z \in Z$, $\left((x_i)_{i=1}^m, (w_j)_{j=1}^n\right)$ may not be in $\Pi_{i=1}^m X_i\left(\left((x_i)_{i=1}^m, (w_j)_{j=1}^n\right)\right) \times \Pi_{j=1}^n Y_j\left(\left((x_i)_{i=1}^m, (w_j)_{j=1}^n\right)\right)$ since the sets are not comparable. This justifies Assumption SPP.

[9]Let us suppose that $p_{I^j}(-y_{I^j}) \geq p_{I^j}(a)$ for all $a \in \cup_{t>0}\text{int}Y_j(y_{O^j} + t\chi_{O^j}, z)$. For any $t > 0$ and a sufficiently large $\alpha > 0$, we have $-y_{I^j} + \alpha\chi_{I^j} \in \text{int}Y_j(y_{O^j} + t\chi_{O^j}, z)$ by free disposal condition. Hence, $p_{I^j}(-y_{I^j}) \geq p_{I^j}(-y_{I^j} + \alpha\chi_{I^j})$, a contradiction.

Another important difference with the above paper is that, even if $\left((x_i)_{i=1}^{m}, (w_j)_{j=1}^{n}\right)$ belongs to $\Pi_{i=1}^{m} X_i\left((x_i)_{i=1}^{m}, (z_j)_{j=1}^{n}\right) \times \Pi_{j=1}^{n} Y_j\left(\left((x_i)_{i=1}^{m}, (z_j)_{j=1}^{n}\right)\right)$, we cannot prove that $\nabla_{O^j} c_j(\pi_{I^j}, y_{O^j}, z) = \nabla_{O^j} c_j\left(\pi_{I^j}, y_{O^j}^{+}, \quad z\right)$ as they did, since the argument they constructed does not work in Fréchet derivatives in infinite-dimensional spaces. Consequently, in the present context, $\nabla_{O^j} c_j(\pi_{I^j}, y_{O^j}, z) = \nabla_{O^j} c_j\left(\pi_{I^j}, y_{O^j}^{+}, z\right)$ whenever $y_{O^j} = y_{O^j}^{+}$ which also justifies the Assumption SPP.

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