Chapter 14
Symmetrical Core and Shapley Value of an Information Transferal Game

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Abstract In this paper we study some properties and we characterize the Symmetrical Core. We analyze the relation of the Symmetrical Core with the Shapley value of a game modeling information transferal in a cooperative environment. This type of game was introduced by Galdeano et al. (Int Game Theor Rev 12(1):19–35, 2010) and it was also studied by Hou and Driessen (J Appl Math 2012:1–12, 2012). It consists of an information market game involving identical firms and an innovator having relevant information for the firms (e.g., a new technology).

We analyze how the symmetrical part of the Core varies according with the initial information level of the firms and the value of the information. We also present conditions in order that the Shapley value belongs to the Symmetrical Core. We compare the cooperative outcomes with the noncooperative equilibrium of the game studied by Quintas (Modell Meas Control D 11:11–28, 1995).

Keywords Cooperative game theory • Symmetrical Core • Shapley value • Information transferal

Mathematics Subject Classification: 91A80

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14.1 Introduction

Following the pioneer work by Arrow (1962), many studies on information markets appeared in the literature, analyzing economic effects of patent licensing or protection (Taylor and Silberston 1973; Kamien and Tauman 1986; Katz and Shapiro 1986; Gilbert and Shapiro 1990; Muto 1993; Quintas 1995; Nakayama et al. 1991; Wang 1998, 2002). More recently several studies have been done modeling the interaction of an innovator and n firms in an industry (Poddar and Sinha 2004; Sen and Tauman 2007; Tauman and Watanabe 2007; Schmidt 2008; Stamatopoulos and Tauman 2008; Galdeano et al. 2010; Hou and Driessen 2012).

We follow the approach presented by Galdeano et al. (2010). Besides n firms with identical characteristics, there exists an agent called the innovator, having relevant information for the firms. The innovator is not going to use the information for himself, but this information can be sold to the firms. Any firm that decides to acquire the new information (e.g., a new technology) is supposed to make use of the information. The n potential users of the information are the same before and after the innovator offers the new technology. The problem is modeled as a $(n+1)$-players cooperative game.

Galdeano et al. (2010) characterized the Shapley value of this game requiring 0-monotonicity. We now show that the game is also superadditive, we present a new formulation for the Shapley value, and we give conditions in order that the Shapley value belongs to the symmetrical part of the Core.

Hou and Driessen (2012) studied the nucleolus of this game. They also showed the equivalence between the nonemptiness of the Symmetrical Core and one of each conditions: supperadditivity, 0-monotonicity, or monotonicity.

In the present article, we present an explicit characterization of the Symmetrical Core, and we analyze how it varies depending on the initial information level of the firms and the value of the information. We also compare the cooperative outcomes with the noncooperative equilibrium studied by Quintas (1995).

The paper is organized as follows: in Sect. 14.2 we describe the information market and we define the corresponding game. In Sect. 14.3 we present some results on the symmetrical part of the Core, and we give conditions for the Shapley value to be in the symmetrical part of the Core. In Sect. 14.4 we study how the Symmetrical Core varies when firms have more or less prior information. We analyze some limit cases: when firms have full or no prior information and the variation of the value of the information. In Sect. 14.5 we present some conclusions and possible extensions.

14.2 The Information Market

We consider a market with n firms ($n \geq 2$) and an innovator who possess a patent or an information.

The set of agents will be denoted by $N = \{1, 2, \ldots, n+1\}$, where $I = \{1\}$ (the innovator) is the agent having a new information and $U = \{2, \ldots, n+1\}$ (the users) are firms who could be willing to obtain the new information.

The n users or firms interact in the same market, producing or performing the same activity with the same technology or the same information. Thus, all the users have the same incentives for the acquisition of the new information or technology. We will make the following assumptions about the problem we want to study:

S.1: The n information users are the same before and after the information holder offers the new technology. This indicates that there are no exits or incoming agents in the market.
S.2: All the players that acquire the new information make use of it.
S.3: In order to compute the utilities of the players, we use a conservative criteria, assuming that the uninformed agents make the right choice.

S.1 and S.2 are natural assumptions. S.3 avoids dealing with externalities (Macho-Stadler et al. 2006; De Clippel and Serrano 2008). As in several other papers (Amer et al. 2008; Belenky 2002; Sandholm et al. 1999), we adopt a principle of prudence: each coalition is assigned a utility corresponding to the worst possible scenario.

We will take into account these conditions in order to define an $(n+1)$- person cooperative game.

14.2.1 The Cooperative Game

Definition 1. An $(n+1)$-person game in characteristic function form is given by (N, v), where $N = \{1, 2, \ldots, n+1\}$ is the set of players, and $v : 2^N \to \mathbb{R}$ is the characteristic function.

Following Galdeano et al. (2010), we consider n firms with identical characteristics (the users) and an agent called the innovator, having relevant information for the firms. The innovator is not going to use the information for himself, but this information can be sold to the firms. Any firm that decides to acquire the new information (e.g., a new technology) is supposed to make use of the information. The n potential users of the information are the same before and after the innovator offers the new technology. The firms acquiring the information will be better than before obtaining it, while their utilities are computed under a conservator point of view, assuming that for any uninformed firm, the probability of making a right decision can be described by a *binomial probability distribution*, being $0 \leq c \leq 1$ the uniform probability of having success. We will first consider the case when $0 < c < 1$, and we will then consider the cases $c = 0$ and $c = 1$ in Sect. 14.4.

The probability that k among s firms take the right decision is given by $\binom{s}{k} c^k (1-c)^{s-k}$, and hence, the expected aggregated utility of k firms having success is $k\binom{s}{k} c^k (1-c)^{s-k} a_k$. Here $a_k \geq 0$ represents the utility if k firms make a right decision. Throughout the paper, the utility function is monotonic decreasing because when the number of firms taking a right decision increases, each firm receives a lower utility level, i.e., $a_{k+1} \leq a_k$ for all $k \geq 1$. We normalized it assuming that $a_1 = 1$.

Throughout the paper, the size ($|S|$, or cardinality) of any coalition $S \subseteq N$ is denoted by s, $0 \leq s \leq n+1$. In case coalition S contains the innovator, then $v(S) = (s-1)a_n$ because any member of S, different from the innovator, made a right decision rewarding the expected utility a_n since the $n-s$ uninformed firms outside S are assumed to take a right decisions too.

Definition 2. The $(n+1)$-person information market game (N, v) in characteristic function form is given by,

$$\begin{aligned} &v(\emptyset) = 0 \\ &v(S) = (s-1)\, a_n && \text{if } 1 \in S. \\ &v(S) = w(s) = \sum_{j=0}^{s} j\binom{s}{j} c^j (1-c)^{s-j} a_{n-s+j} && \text{if } 1 \notin S \end{aligned} \tag{14.1}$$

$$\text{for all} \quad S \subseteq N,\ S \neq \emptyset \text{ and } s = |S|$$

14.2.1.1 Properties Fulfilled by the Characteristic Function *v*

A usual assumption is that the game is superadditive:

Definition 3. A game (N, v) is *superadditive* if for all sets $A \subseteq N$ and $B \subseteq N$ with $A \cap B = \phi$, we have that $v(A \cup B) \geq v(A) + v(B)$.

In superadditive games, the players have incentives to form coalitions.

Definition 4. By a superadditive $(n+1)$-person game in characteristic function form, we mean a real-valued function v, defined on the subsets of N, satisfying $v(\phi) = 0$ and *superadditivity*.

We will consider a weaker version of the superadditivity property, which will be fulfilled by the games we study here.

Definition 5. A game (N, v) is zero-monotonic If, for all sets $A \subseteq N$ and for all $i \notin A$, we have that $v(A \cup \{i\}) \geq v(A) + v(\{i\})$.

The following statements (Theorem 1, Proposition 1, Theorems 2 and 3) were given in Galdeano et al. (2010) in order to prove that the game was zero-monotonic. We will use them and we will now prove that the game is superadditive.

Theorem 1. *If the innovator is not in the coalition $S (1 \notin S)$ and he belongs to $T (1 \in T)$ such that $S \cap T = \phi$, then $v(S \cup T) \geq v(S) + v(T)$ if and only if*

$$v(S) \leq v(S \cup \{1\}) \tag{14.2}$$

Remark 1. The players in an uninformed coalition $S \subseteq U$ have incentives to join an informed coalition $T \subseteq N$, if the utility they obtain is less than they would obtain buying the information. We do not need a restriction on the set T because by assumption S.3, for the computation of the characteristic function $v(T)$, we assumed that the uninformed agents outside the coalition take the right decision. Thus it is always better for them to join the coalition.

It was analyzed the restrictions (14.2) depending on the number of agents in the market and it was obtained the following results:

Proposition 1. *If the innovator is not in the coalition S ($1 \notin S$) and he belongs to T ($1 \in T$) such that $S \cap T = \phi$, then $v(S \cup T) \geq v(S) + v(T)$ if and only if*

$$a_n \geq \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \tag{14.3}$$

Remark 2. Without assuming $a_1 = 1$, condition (14.3) takes the following form: $\frac{a_n}{a_1} \geq \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$.

The following theorem shows that the function v is *zero* $-$ *monotonic*

Theorem 2. *For any coalition S and any user $i \notin S$: $v(S \cup \{i\}) \geq v(S) + v(\{i\})$.*

Now we show under what conditions the game is *superadditive*. We will first prove the following lemma:

Lemma 1. *If $S \subseteq U$ and $T = \{i\} \in U \backslash S$ then*

$$\frac{w(s)}{s} \leq \frac{w(s \cup \{i\})}{s+1}$$

Proof. The proof easily follows by using Definition 2 and basic properties of combinatoric numbers. ∎

We will complete the proof of the superadditivity condition:

For disjoint, nonempty coalitions $S, T \subseteq N \backslash \{1\}$, by Definition 2 and by Lemma 1, we have that for $1 \leq s \leq n-1$ it holds $\frac{w(s)}{s} \leq \frac{w(s+1)}{s+1}$ then

$$v(\{i\}) = w(1) \leq \frac{w(2)}{2} \leq \ldots \leq \frac{w(n)}{n} = \frac{v(U)}{n}$$

We can assume without loss of generality $1 \leq s \leq t \leq s+t \leq n$ then

$$\frac{w(s)}{s} \leq \frac{w(s+1)}{s+1} \leq \frac{w(s+2)}{s+2} \leq \ldots \leq \frac{w(t)}{t} \leq \ldots \leq \frac{w(s+t)}{s+t} \tag{14.4}$$

Therefore

$$\frac{w(s+t)}{s+t} \geq \frac{w(s)}{s} \tag{14.5}$$

Theorem 3. *v is superadditive if and only if* $a_n \geq \dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$

Proof. By Proposition 1, if the innovator is not in the coalition S $(1 \notin S)$ and he belongs to T $(1 \in T)$ such that $S \cap T = \phi$, then $v(S \cup T) \geq v(S) + v(T)$ if and only if $a_n \geq \dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$.

Now we should analyze the supperadditivity for the case of two disjoint nonempty coalitions $S, T \subseteq N \backslash \{1\}$. Using Lemma 1 and operating in (14.5), we get $w(s+t) \geq (s+t)\dfrac{w(t)}{t} = \dfrac{sw(t)}{t} + w(t)$ and by (14.4) $\frac{w(t)}{t} \geq \frac{w(s)}{s}$ then $w(s+t) \geq w(s) + w(t)$. Therefore $v(S \cup T) \geq v(S) + v(T)$. ∎

14.3 Cooperative Solutions of the Game

In this section we give the definition of the Core (Gillies 1953) and the Symmetrical Core of the game. We analyze how the Symmetrical Core varies for different values of c and a_i. We also study conditions for the Shapley value (Shapley 1953) to be in the Symmetrical Core.

14.3.1 The Symmetrical Core

Definition 6. An imputation or payoff distribution for the game (N, v) is a vector $x = (x_1, \ldots, x_{n+1})$ satisfying $\sum\limits_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for each $i \in N$.

The Core allocations are selected through efficiency and group rationality. Besides the appealing motivation for the definition of Core allocations, we might wonder if this set is nonempty. We will prove that for v given by (14.1), the Core is nonempty.

Definition 7. The Core is the set

$$C(v) = \left\{ (x_1, x_2, \ldots, x_{n+1}) : \sum_{i \in N} x_i = v(N) \ \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for each } \ S \subseteq N \right\}$$

The Core, however, is a set-valued solution concept which fails to satisfy the symmetry property in that users of the same type receive identical payoffs according to Core allocations. By symmetry (Condition S.1) we have $x_2 = x_3 = \cdots = x_{n+1}$, and then the Symmetrical Core allocations require equal payoffs to users, that is:

$$\mathrm{SymC}(v) = \{(x_1, x_{2,} \ldots, x_{n+1}) \in C(v) : x_2 = x_3 = \cdots = x_{n+1}\}$$

The following lemma gives a necessary condition in order that an imputation belongs to the Symmetrical Core.

Lemma 2. *Given a game* (N, v)*, with* v *defined by (14.1) and* a_n *fulfilling (14.3), if* $(x_1, x_{2,} \ldots, x_2) \in SymC(v)$*, then* $x_2 = a_n - \frac{1}{n}x_1$ *with* $ca_n \le x_2 \le a_n$*.*

Proof. If $(x_1, x_{2,} \ldots, x_2) \in \mathrm{SymC}(v)$ then $x_2 = x_3 = \cdots = x_{n+1}$ and $\sum_{i \in N} x_i = v\,(N)$. Now using (14.1) we obtain

$$na_n = x_1 + nx_2 = (x_1 + sx_2) + (n - s)x_2 \tag{14.6}$$

If $1 \in S$ we obtain

$$x_1 + \sum_{i \in S \setminus \{1\}} x_i = x_1 + sx_2 \ge sa_n \text{ with } |S| = s + 1 \tag{14.7}$$

Now using (14.6) in (14.7), we obtain $a_n \ge x_2$.

On the other hand, if $1 \notin S$ then we obtain $ca_n \le x_2$ therefore $ca_n \le x_2 \le a_n$. ■

Remark 3. This condition is not sufficient because, for example, if $(x_1, x_{2,} \ldots, x_2) \in \mathrm{SymC}(v)$, then $2x_2 \ge v(\{2, 3\})$, that is to say, $2x_2 \ge 2c(1 - c)a_{n-1} + 2c^2 a_n$. Now we assume it fulfills that $x_2 = ca_n$, then ($0 < c < 1$), we obtain $a_{n-1} < a_n$ which contradicts the general conditions of the game (a_j is decreasing).

Theorem 4. *Given a game* (N, v)*, then*
$$SymC(v) = \left\{(x_1, x_2, \ldots, x_2) \in C(v) : x_1 = na(n) - nx_2 \ \wedge\ \frac{v(U)}{n} \le x_2 \le \frac{v(N)}{n}\right\}.$$

Proof. On one hand, if $(x_1, x_{2,} \ldots, x_{n+1}) \in \mathrm{SymC}(v)$, then $x_2 = x_3 = \cdots = x_{n+1}$ and $(x_1, x_{2,} \ldots, x_{n+1}) \in C(v)$. Now using (14.1) we obtain a system with $2\binom{n}{s}$ inequalities:

$$\begin{cases} x_2 \ge a_n - \frac{1}{s}x_1 \\ x_2 \ge \sum_{j=0}^{s} \frac{j}{s}\binom{s}{j} c^j (1 - c)^{s-j} a_{n-s+j} \end{cases} \quad \text{with } s = 1, 2, \ldots, n \tag{14.8}$$

We obtain the following equivalent system:

$$a_n \ge \frac{\sum_{j=0}^{s-1} \frac{j}{s}\binom{s}{j} c^j (1 - c)^{s-j} a_{n-s+j}}{1 - c^s}, \qquad \text{with } s = 1, 2, \ldots, n$$

In Galdeano et al. (2010), it was proven that the solution of this system in $a_n, a_{n-1}, \ldots, a_2$ is $a_n \geq \dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$.

Moreover for each s in (14.8), we have a lower bound for x_2, $\sum_{j=0}^{s} \frac{j}{s}\binom{s}{j} c^j (1-c)^{s-j} a_{n-s+j} \leq x_2$.

To determine the largest of these lower bounds for x_2, using Lemmas 1 and 2, we have

$ca_n \leq \frac{v(S)}{s} \leq \frac{v(U)}{n} \leq x_2 \leq a_n = \frac{v(N)}{n}$ with $S \subseteq U$, $|S| = s$ and $|U| = n$. Then we obtain

$\frac{v(U)}{n} \leq x_2 \leq a_n = \frac{v(N)}{n}$ and $x_1 = na_n - nx_2$.

We now prove the reverse implication , i.e.,

on the other hand, if $(x_1, x_2, \ldots, x_{n+1})$ is such that $x_1 = na(n) - nx_2 \wedge \frac{v(U)}{n} \leq x_2 \leq a_n$, then $x_2 = x_3 = \cdots = x_{n+1}$, and using (14.1), we obtain

$$\sum_{i \in N} x_i = na_n = v\,(N)$$

If $S \subseteq U$ then by Lemma 1, $\frac{v(S)}{s} \leq \frac{v(U)}{n}$, and $x_2 = x_3 = \cdots = x_{n+1}$, we have $v(S) \leq sx_2$.

If $S \cup \{1\} \subseteq N$ then

$$\sum_{i \in S \cup \{1\}} x_i = x_1 + sx_2 = (na_n - nx_2) + sx_2 = na_n + (s-n)\,x_2\,. \tag{14.9}$$

with $s \leq n$ and $x_2 \leq a_n$, then $na_n + (s-n)\,x_2 \geq na_n + (s-n)\,a_n = sa_n$. Using (14.1) we obtain

$$\sum_{i \in S \cup \{1\}} x_i \geq v(S \cup \{1\})$$

We have for all $S \subseteq N :\ \sum_{i \in S} x_i \geq v(S)$.

Therefore we obtain that

$\mathrm{SymC}(v) = \left\{(x_1, x_2, \ldots, x_2) \in C(v) : x_1 = na(n) - nx_2 \wedge \frac{v(U)}{n} \leq x_2 \leq \frac{v(N)}{n} = a(n)\right\}$

■

Corollary 5. *The Symmetrical Core and by extension the Core is nonempty.*

Proof. It is easy to show that $(x_1, x_{2,..}, x_2) = (0, a_n, .., a_n) \in \mathrm{SymC}(v)$ and $\left(na_n - v(U), \frac{v(U)}{n}, .., \frac{v(U)}{n}\right) \in \mathrm{SymC}(v)$. ■

Remark 4. Thus, the imputation $(0, a_n, .., a_n) = (0, \frac{v(N)}{n}, .., \frac{v(N)}{n}) \in \mathrm{SymC}(v)$ is the most appealing for the users because they get $\frac{v(N)}{n}$ and the less appealing for the innovator that obtains 0.

The imputation $\left(na_n - v(U), \frac{v(U)}{n}, .., \frac{v(U)}{n}\right) \in \text{SymC}(v)$ is the less appealing for the users because they get $\frac{v(U)}{n}$ and the most appealing to the innovator that obtains $na_n - v(U)$.

14.3.2 Shapley Value and Symmetrical Core

Definition 8. Given a game (N, v), the Shapley value (Shapley 1953) is defined by the following vector $\varphi(v) = \left(\varphi_1(v), \ldots, \varphi_{n+1}(v)\right)$ where

$$\varphi_i(v) = \sum_{S \subseteq N \text{-}\{i\}} \frac{s!\,(n-s)!}{(n+1)!}\left[v\left(S \cup \{i\}\right) - v(S)\right]$$

with $|S| = s$ and $|N| = n + 1$.

In Galdeano et al. (2010), it was given a formulation of the Shapley value. Now we present a more concise expression:

Theorem 6. *Given a game* (N, v)*, with* v *defined by (14.1) fulfilling (14.3) and* $0 < c < 1$*, then the Shapley value* $\varphi(v) = \left(\varphi_1(v), \ldots, \varphi_{n+1}(v)\right)$ *is an imputation for the game* (N, v) *with*

$\varphi_i(v) = \varphi_2(v) = \varphi_3(v) = \cdots = \varphi_{n+1}(v) = \frac{a_n}{2} + \frac{1}{n(n+1)} \sum_{s=0}^{n} w(s),$

$\varphi_1(v) = \frac{1}{2} v(N) - \frac{1}{n+1} \sum_{s=0}^{n} v(s) = \frac{1}{2} na_n - \frac{1}{n+1} \sum_{s=0}^{n} w(s)$ *with* $s = |S|$, $1 \notin S$ *and* $i \neq 1$.

The proof follows by splitting the sums considering informed and uninformed coalitions, respectively.

The following lemma shows that the Shapley value fulfills the necessary condition of the Symmetrical Core.

Lemma 3. *If* $(\varphi_1(v), \varphi_2(v), \ldots, \varphi_2(v))$ *is as in Theorem 6, then* $0 \leq \varphi_1(v) \leq n(1-c)\,a_n$ *and* $c\,a_n \leq \varphi_2(v) \leq a_n$.

The proof is similar to Lemma 2.

The following example shows that the Shapley value could be outside the Symmetrical Core.

Example 1. For $n = 2$ the Symmetrical Core is in a line in $\mathbb{R}^3$ given by

$$\text{SymC}(v) = \begin{cases} x_1 = 2a_2 - 2x_2 \\ x_2 = x_2 \\ x_3 = x_2 \end{cases} \quad \text{with } c^2 a_2 + c\,(1-c) \leq x_2 \leq a_2,$$

The Shapley value is $(\varphi_1(v), \varphi_2(v), \varphi_2(v)) \in \mathbb{R}^3$, with

$$\varphi_1(v) = a_2 - \frac{1}{3}ca_2 - \frac{2}{3}c + \frac{2}{3}c^2 - \frac{2}{3}c^2a_2.$$

$$\varphi_2(v) = \frac{1}{2}a_2 + \frac{1}{6}ca_2 + \frac{1}{3}c - \frac{1}{3}c^2 + \frac{1}{3}c^2a_2$$

If $c = \frac{1}{2}$ and $a_2 = 0.667$, then $(0.277, 0.528, 0.528) \in \text{SymC}(v)$.

If $c = \frac{1}{2}$ and $a_2 = 0.369$, then $(0.078, 0.328, 0.328) \notin \text{SymC}(v)$.

The following theorem gives necessary and sufficient conditions for the Shapley value to be in the Symmetrical Core.

Theorem 7. *The Shapley value is in the Symmetrical Core if* $v(U) \leq \frac{1}{2}(n+1)a_n + \frac{1}{n}\sum_{s=1}^{n-1} v(s)$ *with* $s = |S|$ *and* $1 \notin S$.

Proof. Let $(\varphi_1(v), \varphi_2(v), \ldots, \varphi_2(v))$ be a solution for the game (N, v) given by the Shapley value, and then by Theorem 4, the Shapley value will be in the Symmetrical Core if

$$\varphi_1(v) + n\varphi_2(v) = v(N) = na_n \text{ with } \frac{v(U)}{n} \leq \varphi_2(v) \leq a_n$$

The Shapley value verifies: $\sum_{i=1}^{n+1} \varphi_i(v) = \varphi_1(v) + n\varphi_2(v) = v(N)$. As $\varphi_1(v) \geq 0$, then $n\varphi_2(v) \leq v(N)$. Thus $\varphi_2(v) \leq \frac{v(N)}{n} = a_n$.

Now we analyze when the following condition holds: $\frac{v(U)}{n} \leq \varphi_2(v)$.

By Theorem 6, $\varphi_2(v) = \frac{a_n}{2} + \frac{1}{n(n+1)}\sum_{s=0}^{n} v(s)$. Then $\frac{v(U)}{n} \leq \varphi_2(v)$ if and only if

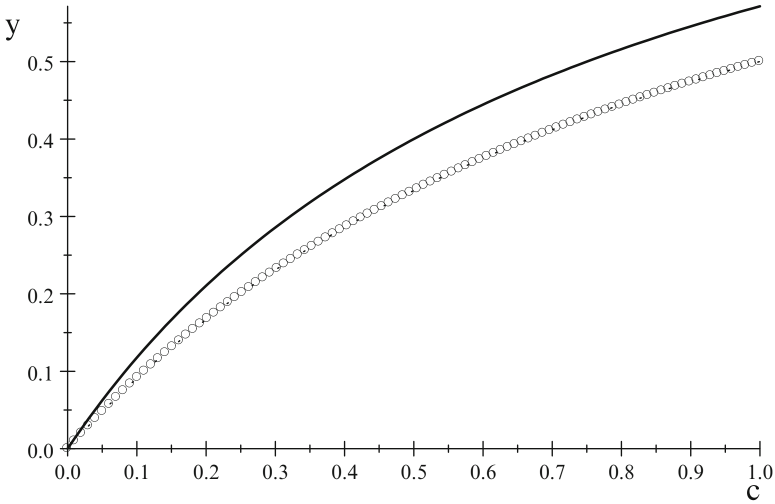
$$\frac{v(U)}{n} \leq \frac{a_n}{2} + \frac{1}{n(n+1)}\sum_{s=0}^{n} v(s) \tag{14.10}$$

Operating in (14.10) we obtain $v(U) \leq \frac{1}{2}(n+1)a_n + \frac{1}{n}\sum_{s=1}^{n-1} v(s)$. ∎

Remark 5. If $n = 2$ then the Shapley value is in the Symmetrical Core if $a_2 \geq \frac{4c}{(4c+3)}$.

As $a_2 \geq \frac{c}{1+c}$ and $\frac{4c}{(4c+3)} \geq \frac{c}{1+c}$, we have a critical zone $\frac{c}{1+c} \leq a_2 < \frac{4c}{(4c+3)}$ where the Shapley value is not in the Symmetrical Core.

It is shown in the following graphic, where $\frac{4c}{(4c+3)}$ (solid) and $\frac{c}{1+c}$ (dots).



If $n \geq 3$ the Shapley value is in the Symmetrical Core if

$$a_n \geq \frac{n\sum_{j=1}^{n-1} j\binom{n}{j} c^j (1-c)^{n-j-1} a(j) - \sum_{s=2}^{n-1}\sum_{j=1}^{s-1} j\binom{s}{j} c^j (1-c)^{s-j-1} a(n-s+j)}{\sum_{s=0}^{n-1}\left[\left(n^2 - sn + \frac{s(s+1)}{2}\right)\right] c^{n-1-s}}$$

Increasing n, the critical zone where the Shapley value is not in the Symmetrical Core shrinks.

14.3.3 Cooperative and Noncooperative Model

The cooperative game studied in this paper was analyzed by Quintas (1995) from a noncooperative point of view. It was observed that the innovator obtained a neat profile by selling the information to the n firms. However the situation was not so appealing for the buyers. The expected utility each one finally obtained after buying the information was that one he would have obtained if he was the only uninformed agent. Nevertheless they couldn't ignore the existence of the information and they should buy it.

The main result of the noncooperative study mentioned above states as follows:

Theorem 8. *The price P that the innovator can ask to the n users such that all of them acquire the information is determined by the unique Nash equilibrium of the noncooperative game.*

This price is $P = (1-c)a_n - \varepsilon$, with $\varepsilon \geq 0$ arbitrarily small, and the payoff n-tupla is

$$((1-c)na_n + n\varepsilon, ca_n + \varepsilon, \ldots, ca_n + \varepsilon)$$

and $\varepsilon \to 0$, *then the payoff n-tupla is*

$$((1-c)na_n, ca_n, \ldots, ca_n)$$

Let's compare the expected utility given in Theorem 8 with the results presented in our article.

Theorem 9. *The Nash equilibrium payoff of the noncooperative game,* $P = ((1-c)na_n, ca_n, \ldots, ca_n)$ *verifies:*

1) It is an imputation of the cooperative game (N, v).
2) It does not belong to the Symmetrical Core.

Proof. 1) By the definition of v, we have $v(\{i\}) = c\,a_n$, $v(\{1\}) = 0$, and $v(N) = na_n$.
Then $\sum_{i \in N} x_i = (1-c)na_n + nca_n = na_n$ and we obtain

$$\sum_{i \in N} x_i = v(N) \tag{14.11}$$

If $i = 1$ then $v(\{1\}) = 0 \leq (1-c)na_n = x_1$, and if $i \neq 1$ then $v(\{i\}) = ca_n = x_2$, and we have

$$x_i \geq v(\{i\}) \quad \text{for all } i \in N \tag{14.12}$$

From (14.12) and (14.11) we conclude that P is an imputation for the game (N, v).
Now we prove 2).
If $P = ((1-c)na_n, ca_n, \ldots, ca_n) \in \text{SymC}(v)$, then $2x_2 \geq v(\{2,3\})$, and we obtain
$2x_2 \geq 2c(1-c)a_{n-1} + 2c^2 a_n$. As $0 < c < 1$ and $x_2 = ca_n$, then we obtain $a_{n-1} < a_n$. It is impossible, because a_j increases. Then $P \notin \text{SymC}(v)$. ■

Remark 6. In Galdeano et al. (2010), it was proved that $\varphi_i(v) \geq ca_n$ with i being a user. By Lemma 2, the Symmetrical Core imputations for the users verify $ca_n \leq x_i \leq a_n$. By Theorem 8, ca_n is the equilibrium outcome for the users. Thus, we conclude that the users are better off in the cooperative environment and an opposite situation results for the innovator.

14.4 Limit Cases

In Theorem 6 it was given a characterization of the Shapley value and in Theorem 4, it was presented a characterization of the Symmetrical Core, being $0 < c < 1$ and $\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \leq a_n \leq a_{n-1} \leq \ldots\ a_2 \leq 1$.

Now we analyze properties of some limit cases. These limit cases correspond to completely uninformed users ($c = 0$) or completely informed users ($c = 1$). The other extreme cases result when $a_n = a_{n-1} = \cdots = a_2 = \dfrac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$ and $a_n = a_{n-1} = \cdots = a_2 = 1$

14.4.1 *Users with No Prior Information* ($c = 0$): *Big Boss Games*

We will show that in this case the game (N, v) is a Big Boss Game (Muto et al. 1988).

Definition 9. A monotonic game (N, v) is called a Big Boss Game if there is one player, denoted by i^*, satisfying the following two conditions:

B1) $v(S) = 0$ if $i^* \notin S$ and **B2**) $v(N) - v(S) \geq \sum_{i \in N \setminus S} (v(N) - v(N - \{i\}))$ if $i^* \in S$.

The Big Boss Games are denoted by BBG^N.

***B*1** implies that one player i^* is very powerful. Coalitions not containing i^* cannot get anything.

***B*2** implies that for every coalition not containing i^*, its contribution to the grand coalition is not less than the sum of the contributions of its players to the grand coalitions. Hence, weak players may increase their influence by forming coalitions. We also notice that a Big Boss Game v is *superadditive* (Definition 3), because of the monotonicity of v and $B1$.

When $c = 0$, the characteristic function $v : 2^N \to \mathbb{R}$ results

$$v(S) = \begin{cases} (s-1)\, a_n & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S \end{cases} \quad \text{for all } S \subseteq N \text{ and } |S| = s \tag{14.13}$$

Here player i^* is the innovator $i^* = 1$. As a consequence of (14.13), we have:

Theorem 10. *Let* (N, v) *be a game with* v *given by (14.13), then* $(N, v) \in BBG^N$.

Proof. We must prove that (N, v) is monotonic, that is, $v(S) \leq v(T)$ for all $S \subseteq T$ and $|S| = s$, $|T| = t$.

If $1 \in S$ then $1 \in T$, and by (14.13) $v(S) = (s-1)\, a_n \leq (t-1)\, a_n = v(T)$.

If $1 \notin S$ then we have two cases: $1 \in T$ or $1 \notin T$, then:

i) $1 \notin S$ and $1 \in T$, by (14.13) $v(S) = 0 \leq (t-1)\, a_n = v(T)$.
ii) $1 \notin S$ and $1 \notin T$, $v(S) = 0 = v(T)$. Then (N, v) is monotonic.

Now we must also prove that $B1$ and $B2$ hold.

Condition $B1$ immediately follows from (14.13).

Let be $1 \in S$ and $i \neq 1$; then by (14.13), we have

$$\sum_{i \in N \setminus S} (v(N) - v(N - \{i\})) = \sum_{i \in N \setminus S} (na_n - (n-1)\,a_n) = \sum_{i \in N \setminus S} a_n = (n+1-s)\,a_n$$

at the same time $v(N) - v(S) = n\,a_n - (s-1)\,a_n = (n+1-s)\,a_n$ thus

$$v(N) - v(S) = \sum_{i \in N \setminus S} (v(N) - v(N - \{i\})).$$

Therefore $B2$ holds with equality. ■

Now we compute the Shapley value and the Symmetrical Core.

Theorem 11. *1. The Shapley value for the users is given by* $\varphi_i(v) = \frac{1}{2}a_n$ *for* $i \neq 1$ *and for the innovator is given by* $\varphi_1(v) = \frac{1}{2}v(N) = \dfrac{na_n}{2}$.

2. The Symmetrical Core is given by $SymC(v) = \{(x_1, x_2, \ldots, x_2) : 0 \leq x_2 \leq a_n$ *with* $x_1 + nx_2 = na_n\}$.

Proof. 1. Let be $i \neq 1$. By Definition 6, splitting the sum between informed and uninformed coalitions, and by (14.13) we have $\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!\,(n-s)!}{(n+1)!}\,[v\,(S \cup \{i\}) - v\,(S)] =$
$\sum_{\substack{S \subseteq N \setminus \{i\} \\ 1 \in S}} \frac{s!\,(n-s)!}{(n+1)!}\,[sa_n - (s-1)\,a_n] + \sum_{\substack{S \subseteq N \setminus \{i\} \\ 1 \notin S}} \frac{s!\,(n-s)!}{(n+1)!} 0$ then

$$\varphi_i(v) = \sum_{\substack{S \subseteq N \setminus \{i\} \\ 1 \in S}} \frac{s!\;(n-s)!}{(n+1)!} a_n \tag{14.14}$$

Now we analyze how many subsets S are in each sum of (14.14).

If $S \subseteq N \setminus \{i\}$, with $1 \in S$, we count how many subsets of the type $S \setminus \{1\} \subseteq N \setminus \{1, i\}$ we have (the innovator is a fixed player in all the coalitions S we could form), they are $\binom{n-1}{s-1}$.

Then $\binom{n-1}{s-1}\frac{s!\,(n-s)!}{(n+1)!} = \frac{s}{n(n+1)}$, with $|S| = s = 1, \ldots, n$.

As the function $v\,(S)$ depends only on the cardinality s of the set S, we have for $i \neq 1$

$$\varphi_i(v) = a_n \sum_{\substack{s=1 \\ 1 \in S}}^{n} \frac{s}{n(n+1)} \tag{14.15}$$

and $\sum_{s=1}^{n} \dfrac{s}{n(n+1)} = 1/2$; thus, we have

$$\varphi_i(v) = \frac{1}{2}a_n \text{ with } i \neq 1 \tag{14.16}$$

Fix $i = 1$. As $\sum_{i=1}^{n} \varphi_i(v) = v(N)$. Then by (14.16) and by (14.13), we have $\varphi_1(v) = na_n - n\frac{1}{2}a_n = \frac{n}{2}a_n$.

Thus the Shapley value is

$$(\varphi_1(v), \varphi_2(v), \ldots, \varphi_2(v)) = \left(\frac{n}{2}a_n, \frac{1}{2}a_n, \ldots, \frac{1}{2}a_n\right)$$

2. The proof of this part is similar to Theorem 4. ■

Corollary 12. *1. The Shapley value is in the Symmetrical Core and it is the midpoint of the segment.*
2. The payoff $(na_n, 0, \ldots, 0)$ *found in Quintas (1995) in the noncooperative game is an extreme point in the Symmetrical Core.*

14.4.2 Completely Informed Users ($c = 1$)

In the case $c = 1$, the characteristic function $v : 2^N \to \mathbb{R}$ becomes

$$v(S) = \begin{cases} (s-1)\,a_n & if\ \ 1 \in S \\ sa_n & if\ \ 1 \notin S \end{cases} \quad \text{for all } S \subseteq N \text{ and } |S| = s \tag{14.17}$$

As an immediate consequence of (14.17), we have:

Theorem 13. *1. The Shapley value for the users is given by* $\varphi_2(v) = \frac{1}{n}v(N) = a_n$ *and* $\varphi_1(v) = 0$
2. The Symmetrical Core is given by $SymC(v) = \{(x_1, x_2, \ldots, x_2) : x_1 = \varphi_1(v) = 0 \wedge\ x_2 = \varphi_i(v) = a_n\}$.

The proof is similar to the case $c = 0$.

Remark 7. 1. In this case the Symmetrical Core is a single point set and it coincides with the Shapley value.
2. From (14.17) the innovator is a dummy player. Thus the users have no incentives to form coalitions with the innovator, but they do have incentives to form coalitions among themselves because if $i \neq 1$ then $v(S) = v(S \cup \{1\})$ and $v(S) < v(S \cup \{i\})$.
3. The users in this case have complete information and the payoff outcome $(0, a_n, \ldots, a_n)$ coincides with the result obtained by Quintas (1995) in the noncooperative game.

On the other hand, if we denote by SymC(v, c) the Symmetrical Core corresponds to value of c. It is easy to verify that:

Theorem 14. *If* $0 \leq c_1 \leq c_2 \leq \ldots \leq c_n \leq 1$ *then*

$$SymC(v, 0) \supseteq SymC(v, c_1) \supseteq SymC(v, c_2) \supseteq \ldots \supseteq SymC(v, c_n) \supseteq SymC(v, 1)$$

Proof. It follows from the characterization of the Symmetrical Core in the general case $0 < c < 1$ and the cases $c = 0$ and $c = 1$. ■

14.4.3 Extreme Values of a_j

Now for each c fixed, we analyze the Symmetrical Core and the Shapley value for extreme values of a_j. Namely, we analyze the cases $a_n = 1$ and $a_n = a_{n-1} = \cdots = a_2 = \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$.

In the case $a_n = 1$, the characteristic function $v : 2^N \to \mathbb{R}$ becomes

$$v(S) = \begin{cases} (s-1) & \textit{if } \ 1 \in S \\ sc & \textit{if } \ 1 \notin S \end{cases} \text{ for all } S \subseteq N \text{ and } |S| = s \tag{14.18}$$

As $a_n = 1$ then $a_j = 1$, for all $j = 1, ..n$.

Lemma 4. *For v given by (14.18):*

1. *The Symmetrical Core is the segment given by: $SymC(v) = \{(n(1-x_2), x_2, \ldots, x_2) : c \leq x_2 \leq 1\}$.*
2. *The Shapley value $\left(\frac{1}{2}n\,(1-c)\,, \frac{c+1}{2}, \ldots \frac{c+1}{2}\right)$ is in the Symmetrical Core and it is the midpoint of the segment.*
3. *The equilibrium payoff $((1-c)n, c, \ldots, c)$ found by Quintas (1995) for the noncooperative game is an extreme point of the Symmetrical Core.*

Proof. Let's prove 1.

By (14.18) we have

$$v(N) = n \ \text{ and } \ v(U) = nc \tag{14.19}$$

Using (14.19), and Theorem 4, we have

$$\text{SymC}(v) = \{(n(1-x_2), x_2, \ldots, x_2) : c \leq x_2 \leq 1\}$$

Now we prove 2.

The Shapley value for the users is given by

$$\varphi_i\,(v) = \frac{v\,(n+1)}{2n} + \frac{1}{n(n+1)} \sum_{s=0}^{n} v(s) \tag{14.20}$$

By (14.18) we have

$$v\,(S) = sc \text{ with } 1 \notin S \text{ and } |S| = s \tag{14.21}$$

Using (14.21) , (14.19), and Theorem 6, we have $\varphi_i\,(v) = \frac{v(n+1)}{2n} + \frac{1}{n(n+1)} \sum\limits_{s=0}^{n} sc$.

As $\sum\limits_{s=0}^{n} s = \frac{n(n+1)}{2}$, it results

$$\varphi_i\,(v) = \frac{n}{2n} + \frac{1}{n(n+1)} \frac{n\,(n+1)\,c}{2} = \frac{1+c}{2} \tag{14.22}$$

As $\varphi_1\,(v) + n\varphi_i\,(v) = v(N)$, then $\varphi_1\,(v) = na_n - n\varphi_i\,(v)$.

Therefore by (14.19) and (14.22), we obtain $\varphi_1(v) = n - n\left(\frac{1+c}{2}\right) = n\left(\frac{1+c}{2}\right)$.

Then the Shapley value $\left(\frac{1}{2}n(1-c), \frac{c+1}{2}, \ldots \frac{c+1}{2}\right)$ is in the Symmetrical Core, and it is the midpoint of the segment.

In order to prove 3, let's consider the payoff of the noncooperative game $((1-c)na_n, ca_n, \ldots, ca_n)$ and $a_n = 1$, then it becomes: $((1-c)n, c, \ldots, c)$. It's an extreme point of the Symmetrical Core. ■

Now let's analyze the case $\boldsymbol{a}_n = \boldsymbol{a}_{n-1} = \cdots = \boldsymbol{a}_2 = \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$

Lemma 5. *If* $\boldsymbol{a}_n = \boldsymbol{a}_{n-1} = \cdots = \boldsymbol{a}_2 = \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$, *then* $v(N) = v(U)$.

Proof. By (14.1) we have

$$v(U) = \sum_{j=1}^{n} j \binom{n}{j} c^j (1-c)^{n-j} a_j \tag{14.23}$$

Using that $a_n = a_{n-1} = \cdots = a_2 = \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$, $a_1 = 1$ and by (14.1), we have

$$v(U) = \sum_{j=1}^{n} j \binom{n}{j} c^j (1-c)^{n-j} a_j = nc(1-c)^{n-1} + \left(\sum_{j=2}^{n} j \binom{n}{j} c^j (1-c)^{n-j}\right) \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$$

As we have $\sum_{j=2}^{n} j\binom{n}{j} c^j (1-c)^{n-j} = nc\left(1-(1-c)^{n-1}\right)$ then

$$v(U) = nc(1-c)^{n-1} + \left(nc - nc(1-c)^{n-1}\right) \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}} \tag{14.24}$$

Operating in (14.24) we obtain $v(U) = n\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}$, and by (14.1) $v(N) = na_n$, then $v(U) = v(N)$. ■

Corollary 15. *The Symmetrical Core is a single point set*

$$SymC(v) = \left\{\left(0, \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}, \ldots, \frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right)\right\}$$

Remark 8. 1. The Shapley value is not in the Symmetrical Core.

2. The equilibrium payoff $\left((1-c)n\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}, c\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}, \ldots, c\frac{c(1-c)^{n-2}}{1+c(1-c)^{n-2}}\right)$ found by Quintas (1995) for the noncooperative game is not in the Symmetrical Core.

14.5 Conclusions

We studied several properties of the game modeling information transferal. We computed the Symmetrical Core. We showed the Symmetrical Core (and also the Core) is nonempty . We characterized the Symmetrical Core and we analyzed the relation with the Shapley value. We showed examples of cases when the Shapley value is not in the Symmetrical Core. We presented conditions for the Shapley value to be in the Symmetrical Core.

We compared the cooperative outcomes with the noncooperative outcomes. The Nash equilibrium found in Quintas (1995) in the noncooperative game is an imputation for the game but in the general cases is not in the Symmetrical Core (Theorem 9) .The Nash equilibrium gives the users a worse payoff than the Shapley value and the Symmetrical Core allocations (Remark 6).

We also analyzed some limit cases.

In the case of users with no prior information, the game was a Big Boss Game (Muto et al. 1988). The innovator had a huge power, and the payoff corresponding to the noncooperative equilibrium found in Quintas (1995) in the noncooperative game was an extreme point in the Symmetrical Core, giving the best outcome to the innovator and no utilities to the users. The Shapley value was in the Symmetrical Core. It is a segment of a line, being the Shapley value the midpoint of the segment.

In the case of fully informed users, the role of the innovator was irrelevant (it is a "dummy" player), and the Symmetrical Core is a single point.

In these cases both the Shapley value and the noncooperative outcome were in the Symmetrical Core.

Our approach was different from that introduced in the Bi-Form Games (Brandemburger and Stuart 2007), where it is considered a hybrid noncooperative-cooperative model. Instead of that, we made a comparison of the outcomes in noncooperative and cooperative scenarios because both models have a role to play in understanding business strategy, and many times it is not known beforehand if the game is going to be played with or without cooperation among the agents. As it was expected, the users were better off in a cooperative environment, and we explicitly compared the cooperative solution outcomes with the noncooperative equilibrium.

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