Chapter 2
Mathematical Exposition of the Design Axioms

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Abstract Axiomatic design (AD) offers designers two fundamental principles to follow for a successful design: (1) identify and define the design objectives, i.e., functional requirements (FRs), in such a way that they are inherently independent; and (2) conceive solutions for the FRs that comply with two design axioms: the independent axiom and the information axiom. In the previous chapter, the rationale and origin for the axiomatic nature of the design axioms were provided. In this chapter, the two axioms are given a deeper mathematical understanding, thereby strengthening their value. Starting with the formal definition of functional independence, the criterion for functional independence of FRs in a design is derived as the Jacobian determinant $|J| \neq 0$. Since $| J | \neq 0$ implies independence of FRs and existence of design solutions, the $|J|$ criterion corroborates the declaration of independence axiom that a good design must "maintain the independence of the functional requirements." The $|J|$ criterion further reveals that AD criterion for functional independence—design with single input–single output—is only a sufficient condition. For rigor and completeness, the $|J|$ criterion is shown to be necessary and sufficient. In implementing information axiom, AD assessment of uncertainty in design should cover a larger extent than it currently does. AD has not and should begin to recognize and identify the sources of variability and the countermeasures to them. The chapter ends with a summary of implementation steps in AD expressed in mathematical terms.

2.1 Introduction

Axiomatic design (AD) offers designers two fundamental principles to follow for a successful design: (1) define the design objectives, i.e., functional requirements (FRs), in such a way that they are inherently independent; and (2) conceive solutions in terms of design parameters (DPs) that maintain the independence of FRs as

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originally intended and have the highest probability of achieving FRs' targets. AD uses independent axiom to check for functional independence and information axiom to assess probability of success. What follows are mathematical expositions of the two axioms. Chapter 1 discusses the dual application of the two axioms in design synthesis and design analysis. In this chapter, we will consider design analysis only.

2.2 Mathematical Exposition of Independence Axiom

Independent axiom in AD declares a criterion to check whether a conceived solution in terms of DPs maintains the functional independence of FRs. We develop another criterion derived from formal definition of functional independence. In the sections to follow, we discuss these two criteria. The discussion is confined to no more than 3 FRs and 3 DPs. However, the logic behind the discussion can be extended to n FRs and n DPs, $n > 3$.

2.2.1 AD Criterion for Functional Independence

Per independence axiom, "in an acceptable design, the DPs and the FRs are related in such a way that specific DP can be adjusted to satisfy its correspondent FR without affecting other functional requirements," p. 48 [1]. In other words, AD criterion for functional independence is that adjustment Δ, of one and only one DP should affect only correspondent FR but not other FRs. It implies a **s**ingle **i**nput–**s**ingle **o**utput (SISO) relationship of FRs to DPs. A mathematical representation of the criterion is as follows:

$$\begin{aligned} \Delta FR_1 &= \Delta FR_1(\Delta DP_1) \\ \Delta FR_2 &= \Delta FR_2(\Delta DP_2) \\ \Delta FR_3 &= \Delta FR_3(\Delta DP_3) \end{aligned}$$

An alternative representation is with a design matrix (DM). A DM is indexed row-wise by FR_j and column-wise by DP_k. If DP_k has an effect on FR_j, the cell DM (j, k) is marked "X". If it has no effect, the cell is marked "O".

In DM representation, AD's SISO criterion for independence is a diagonal DM. Such a design is called an uncoupled design.

$$\begin{bmatrix} \Delta FR_1 \\ \Delta FR_2 \\ \Delta FR_3 \end{bmatrix} = \begin{bmatrix} X & O & O \\ O & X & O \\ O & O & X \end{bmatrix} \begin{bmatrix} \Delta DP_1 \\ \Delta DP_2 \\ \Delta DP_3 \end{bmatrix}$$

Another representation that also satisfies the SISO criterion is as follows:

$$\begin{aligned}\Delta \mathrm{FR}_1 &= \Delta \mathrm{FR}_1(\Delta \mathrm{DP}_1);\\ \Delta \mathrm{FR}_2 &= \Delta \mathrm{FR}_2(\Delta \mathrm{DP}_1,\ \Delta \mathrm{DP}_2)\\ \Delta \mathrm{FR}_3 &= \Delta \mathrm{FR}_3(\Delta \mathrm{DP}_1,\ \Delta \mathrm{DP}_2,\ \Delta \mathrm{DP}_3).\end{aligned}$$

In the above representation, $\Delta \mathrm{FR}_k$ can be made a function solely of $\Delta \mathrm{DP}_k$ if the adjustment $\Delta \mathrm{DP}_k$ to satisfy the corresponding $\Delta \mathrm{FR}_k$ follows the sequence: $k = 1$ firstly—so that $\Delta \mathrm{FR}_1$ becomes a constant in subsequent equation for $\Delta \mathrm{FR}_2$—followed by $k = 2$ secondly, and so on:

$$\begin{aligned}\Delta \mathrm{FR}_1 &= \Delta \mathrm{FR}_1(\Delta \mathrm{DP}_1);\\ \Delta \mathrm{FR}_2 &= \Delta \mathrm{FR}_2(\Delta \mathrm{FR}_1,\ \Delta \mathrm{DP}_2);\\ \Delta \mathrm{FR}_3 &= \Delta \mathrm{FR}_3(\Delta \mathrm{FR}_1,\ \Delta \mathrm{FR}_2,\ \Delta \mathrm{DP}_3).\end{aligned}$$

This adjustment sequence is known as forward substitution; an algorithm used in solving lower triangular linear systems [2]. If the adjustment adheres to the sequence, the above shows $\Delta \mathrm{FR}_k$ is a function exclusively of $\Delta \mathrm{DP}_k$. The SISO rule is thereby fulfilled. The DM is triangular. The design is called a decoupled design.

$$\begin{Bmatrix}\Delta \mathrm{FR}_1\\ \Delta \mathrm{FR}_2\\ \Delta \mathrm{FR}_3\end{Bmatrix} = \begin{bmatrix} X & O & O\\ X & X & O\\ X & X & X\end{bmatrix}\begin{Bmatrix}\Delta \mathrm{DP}_1\\ \Delta \mathrm{DP}_2\\ \Delta \mathrm{DP}_3\end{Bmatrix}$$

In AD, only uncoupled and decoupled designs are acceptable. Since they are SISO, the FRs are obviously functionally independent of one another.

$$\mathrm{SISO} \ \Rightarrow \text{functional} \ \ \text{independence}$$

Any other design with DM that is neither diagonal nor triangular cannot satisfy SISO criterion. Such designs are called coupled designs. Per AD, they should be avoided since it is not obvious that the associated FRs are functionally independent of one another. We will show later that FRs in a design that does not satisfy SISO can still be functionally independent. In other words,

$$\text{Functional} \ \ \text{independence} \neg \Rightarrow \mathrm{SISO}$$

Accordingly, the independence axiom with SISO criterion is only a sufficient condition for functional independence.

To recap, per independent axiom, there are three categories of design: uncoupled, decoupled, and coupled. Within the coupled design, there are three subcategories.

One subcategory is designed with cyclic interaction: DP_1 affects FR_2, DP_2 affects FR_3, and DP_3 affects FR_1:

$$\begin{bmatrix} \Delta \mathrm{FR}_1 \\ \Delta \mathrm{FR}_2 \\ \Delta \mathrm{FR}_3 \end{bmatrix} = \begin{bmatrix} X & O & X \\ X & X & O \\ O & X & X \end{bmatrix} \begin{bmatrix} \Delta \mathrm{DP}_1 \\ \Delta \mathrm{DP}_2 \\ \Delta \mathrm{DP}_3 \end{bmatrix}.$$

Another subcategory is redundant design with more DPs than FRs:

$$\begin{bmatrix} \Delta \mathrm{FR}_1 \\ \\ \Delta \mathrm{FR}_2 \end{bmatrix} = \begin{bmatrix} X & X & X \\ \\ X & X & X \end{bmatrix} \begin{bmatrix} \Delta \mathrm{DP}_1 \\ \Delta \mathrm{DP}_2 \\ \Delta \mathrm{DP}_3 \end{bmatrix}.$$

A third subcategory is design with insufficient DPs:

$$\begin{bmatrix} \Delta \mathrm{FR}_1 \\ \Delta \mathrm{FR}_2 \\ \Delta \mathrm{FR}_3 \end{bmatrix} = \begin{bmatrix} X & X \\ X & X \\ X & X \end{bmatrix} \begin{bmatrix} \Delta \mathrm{DP}_1 \\ \\ \Delta \mathrm{DP}_2 \end{bmatrix}.$$

We next show examples of various categories and subcategories of design.

2.2.1.1 Examples of Various Categories of Design

Water Faucet Illustrating Coupled and Uncoupled Design

An example frequently used to illustrate the uncoupled and coupled design has the two alternative designs of a water faucet shown in Fig. 2.1. In the figure, Q subscripted h and c are respectively the flow rate of the hot and cold water. Both faucet designs **a** and **b** have the same functional requirements:

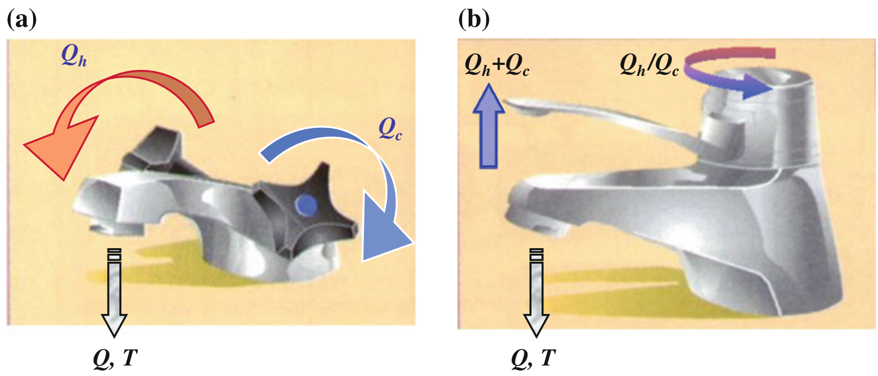

Fig. 2.1 Alternative designs **a** and **b** of a water faucet

$$\mathrm{FR}_1 = \text{control flow rate } Q;$$
$$\mathrm{FR}_2 = \text{control flow temperature } T.$$

For Faucet **a**, we choose as DP_1: the left knob controlling Q_h and as DP_2, the right knob controlling Q_c (see Fig. 2.1a). By assessing the effect of DPs on FRs, we arrive at the first category of coupled DM exhibiting cyclic interaction below.

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \end{bmatrix} = \begin{bmatrix} X & X \\ X & X \end{bmatrix} \begin{bmatrix} \mathrm{DP}_1 \\ \mathrm{DP}_2 \end{bmatrix}$$

Similarly for Faucet **b**, we choose as DP_1, the up/down of the lever to control total flow rate (Q_h + Q_c); and as DP_2, the clockwise/counterclockwise of the lever to control the ratio of the hot/cold water flow rate ($Q_\mathrm{h}/Q_\mathrm{c}$), see Fig. 2.1b. Again, by considering the effect of DPs on FRs, we obtain the uncoupled DM below that is acceptable per AD's SISO criterion for functional independence.

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \end{bmatrix} = \begin{bmatrix} X & O \\ O & X \end{bmatrix} \begin{bmatrix} \mathrm{DP}_1 \\ \mathrm{DP}_2 \end{bmatrix}$$

The examples above illustrate a practical aspect of AD. That is, by simply considering the effects of DPs on FRs with (X, O), we can differentiate an acceptable design from an unacceptable design without going into the details of the physics. This practicality is very useful at the concept selection stage of design.

Projector Illustrating Redundant Design

Projector has two FRs:

$$\mathrm{FR}_1 = \text{magnify the image};$$
$$\mathrm{FR}_2 = \text{focus the image on the projection plane}.$$

Figure 2.2a shows a projector; and Fig. 2.2b, the associated ray tracing of the light beam from the object plane, through the lens, and to the projection plane.

From the similar triangles shown in Fig. 2.2b, we have

$$\mathrm{FR}_1 = \frac{\text{image height}}{\text{object height}} = \frac{D}{d};$$

Also per camera equation, the image is focused whenever

$$\mathrm{FR}_2 = \frac{1}{D} + \frac{1}{d} + \frac{1}{f} = 0.$$

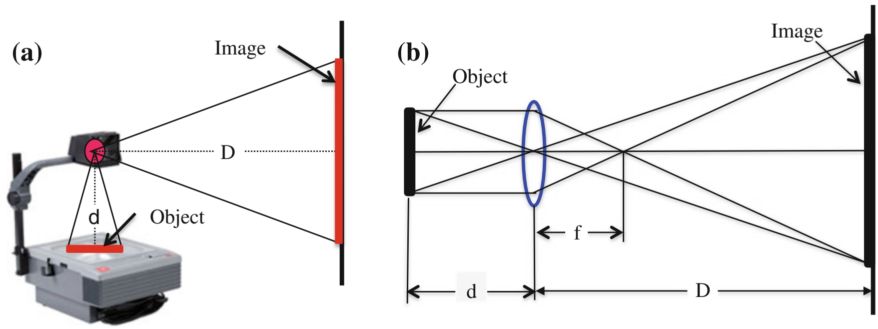

Fig. 2.2 Schematic **a** of a projector and its ray-tracing **b**

Thus, we have a redundant design with 2 FRs and 3 DPs as shown below.

$$\begin{bmatrix} FR_1 \\ \\ FR_2 \end{bmatrix} = \begin{bmatrix} X & X & O \\ \\ X & X & X \end{bmatrix} \begin{bmatrix} DP_1 \\ DP_2 \\ DP_3 \end{bmatrix}.$$

In the above, DP_1 is D, the distance of the lens from the screen aka the throw of the projector; DP_2 is d, the distance of the lens from the object, and DP_3 is f, the focal length of the lens.

A redundant design with more DPs than FRs cannot satisfy SISO criterion unless we fix the extra DPs. For example in the type of overhead projector shown in Fig. 2.2a, the focal length of the lens DP_3 is fixed. Hence, we have a design with equal number of FRs and DPs shown below. In this case, the design is coupled. With this type of projector, it would take several trial and errors to get the right magnification of the image focused at a given throw.

$$\begin{bmatrix} FR_1 \\ FR_2 \end{bmatrix} = \begin{bmatrix} X & X \\ X & X \end{bmatrix} \begin{bmatrix} DP_1 \\ DP_2 \end{bmatrix}$$

By contrast, in a portable projector used in a variety of room sizes that requires various throws, the distance of the object from the lens DP_2 is fixed. A zoom lens with varying focal length DP_3 is used. Thus we have a decoupled design:

$$\begin{bmatrix} FR_1 \\ FR_2 \end{bmatrix} = \begin{bmatrix} X & O \\ X & X \end{bmatrix} \begin{bmatrix} DP_1 \\ DP_3 \end{bmatrix}.$$

With a portable projector, to attain a certain magnification focused at a certain throw is easy: set the throw for magnification then adjust the zoom lens for focus.

Disbursement Algorithm Illustrating a Redundant Design

Let us say we have an ATM that has in it the following bank notes denomination: \$20, \$10, \$5, and \$1. Three demands are made of the ATM as follows:

FR_1: disburse bills that sum up to \$Total.
FR_2: disburse number of bills that totals to N_{Total}.
FR_3: ensure the number of small bills is twice that of large bills.

There are four design parameters (DPs):

DP_1: number of \$20 bills, $N_{\$20}$.
DP_2: number of \$10 bills, $N_{\$10}$.
DP_3: number of \$5 bills, $N_{\$5}$.
DP_4: number of \$1 bills, $N_{\$1}$.

The 4 DPs would satisfy the three FRs as follows:

$$\begin{aligned}
\mathrm{FR}_1 &= \$20N_{\$20} + \$10N_{\$10} + \$5N_{\$5} + \$1N_{\$1} = \$\text{Total} \\
\mathrm{FR}_2 &= N_{\$20} + N_{\$10} + N_{\$5} + N_{\$1} = N_{\text{Total}} \\
\mathrm{FR}_3 &= 2N_{\$20} - N_{\$1} = 0.
\end{aligned}$$

Above is a redundant design with 3 FRs and 4 DPs with a DM as shown below:

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \\ \mathrm{FR}_3 \end{bmatrix} = \begin{bmatrix} X & X & X & X \\ X & X & X & X \\ X & O & O & X \end{bmatrix} \begin{bmatrix} N_{\$20} \\ N_{\$10} \\ N_{\$5} \\ N_{\$1} \end{bmatrix}$$

If we were to fix an extra DP to get equal number of FRs and DPs, we would have 4 (=${}_4C_3$) possible DM solutions as follows:

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \\ \mathrm{FR}_3 \end{bmatrix} = \begin{bmatrix} \$20 & \$10 & \$5 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} N_{\$20} \\ N_{\$10} \\ N_{\$5} \end{bmatrix}; \tag{2.1a}$$

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \\ \mathrm{FR}_3 \end{bmatrix} = \begin{bmatrix} \$20 & \$10 & \$1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} N_{\$20} \\ N_{\$10} \\ N_{\$1} \end{bmatrix}; \tag{2.1b}$$

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \\ \mathrm{FR}_3 \end{bmatrix} = \begin{bmatrix} \$20 & \$5 & \$1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} N_{\$20} \\ N_{\$5} \\ N_{\$1} \end{bmatrix}; \tag{2.1c}$$

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \\ \mathrm{FR}_3 \end{bmatrix} = \begin{bmatrix} \$10 & \$5 & \$1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} N_{\$10} \\ N_{\$5} \\ N_{\$1} \end{bmatrix}. \tag{2.1d}$$

Note in solution (2.1a), $\mathrm{FR}_3 = (\mathrm{FR}_1 - \$10 \times \mathrm{FR}_2)/\$5$. So that $\mathrm{FR}_3 = f(\mathrm{FR}_1, \mathrm{FR}_2)$ is no longer independent as originally planned. In fact, design Eq. (2.1a) is itself a redundant design with 2 FRs and 3 DPs. In short, fixing extra DPs in a redundant design to obtain a square DM does not guarantee a solution. It may induce coupling and destroy the functional independence originally planned for.

Hubcap Illustrating Insufficient DPs

Figure 2.3a shows the front of GM 1986–88 Pontiac 6000 hubcap. Figure 2.3b shows the wheel rim with a circumferential ledge, shown white, onto which the hubcap snapped on. The diameter of the ledge is D_{rim}. Figure 2.3c shows three pairs of clips at the back of the hubcap. The pair at the 4 o'clock position is shown enlarged in Fig. 2.3d. The clips are cantilevers fixed on a post. As seen in Fig. 2.3c, the three pairs of clips are spaced 120° apart such that the 6 clips form a circle of diameter D_{clip}, larger than D_{rim}. As the hubcap is snapped on to the rim, the rim ledge catches the cantilever clips. Wheel retention is developed through interference fit = $k\delta$; where k is the spring rate of the cantilever clips and δ is the interference = $(D_{\mathrm{clip}} - D_{\mathrm{rim}})/2$.

The are two FRs for the hubcap design:

FR_1 = retain the hubcap over road bumps and on cornering;
FR_2 = ease the removal of hubcap during a flat tire repair.

There is only one design parameter:

DP_1 = interference, the larger the better for retention; the smaller the better for removal.

Obviously the design is flawed since it has insufficient DPs: one DP, the interference, cannot satisfy two conflicting FRs, retention, and removal. It violates the AD's SISO criterion of one DP affecting only one FR.

$$\begin{bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \end{bmatrix} = \begin{bmatrix} X \\ X \end{bmatrix} [\mathrm{DP}_1]$$

The consequence was 25 % of hubcaps fell off as the car corners or hits bumps or potholes. And some customers have difficulty removing the hubcap for a flat repair. The solution [3] back then was to implement robust design optimization to find a clip spring rate k that reduces the design sensitivity to the variation in interference. The solution had limited success, as performance of an ill-conceived design cannot be improved through subsequent optimization.

Fig. 2.3 Attachment of hubcap to wheel rim, **a** Hubcap front, **b** Wheel rim, **c** Hubcap back, **d** Cantilever clips

2.2.1.2 Car Door-to-Body Integration—Coupling in Large System

Thus far, we have illustrated the use of AD's SISO criterion to accept or reject various categories of design. These illustrations involve designs with small number of FRs. We now apply SISO to designs with large number of FRs; specifically to a car door-to-body integration. Figure 2.4 shows the car door integrated to the body opening.

Problems in car door-to-body integration, e.g., poor fit of door with neighboring panels; noisy ride and water leak; high opening and closing effort, are typical system problems. They appear only after the system is assembled since only then are couplings triggered. Fixing them is like playing a whack-a-mole game. As one solves a problem in one area, new problems pop up in other areas. This is because

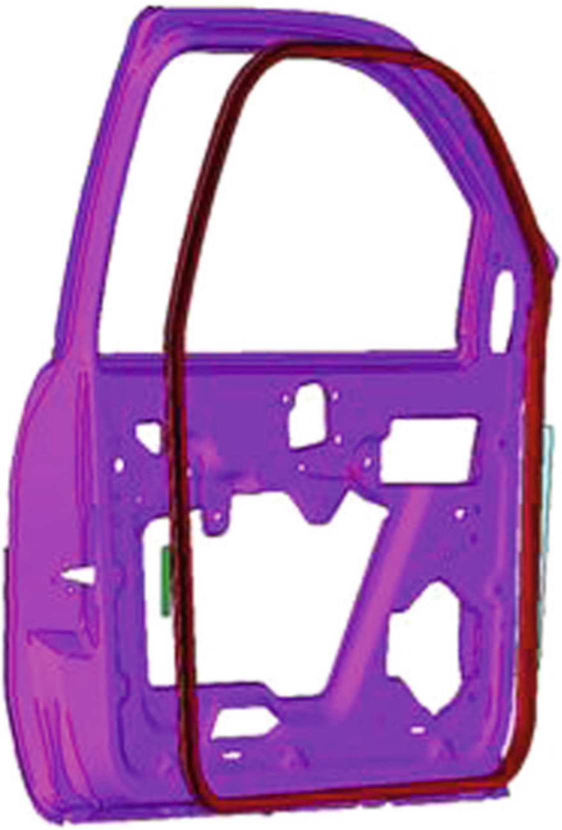

Fig. 2.4 Integration of car door to body

attempt to fix one FR failure inadvertently triggers other FR failures due to coupling. These types of system failures are not detectable by the traditional recursive design/build/test of components since they cannot capture the inter-dependence of FRs among subsystems and components.

AD takes a top-down approach. It starts with system-level FRs and decomposes them down through the subsystems until they reach the levels where known, implementable solutions exist. These levels reached are called leaf-level FRs, and the corresponding solutions, leaf-level DPs. Along the way, AD examines the interrelationships across the subsystems and components. In this way, functional couplings are understood and captured.

The AD top-down approach is as follows. First we decompose the system-level FRs down the design hierarchy to the leaf-level design parameters. From the decomposition, we construct the DM that shows the relationship among the leaf-level FR–DP pairs. We then successively remove those FR–DP pairs that are not part of the coupling. What remains is a reduced DM that contains all the couplings of the original DM. Retracing the leaf-level couplings up the design hierarchy reveals the roots of the system-level functional interactions.

Capturing Functional Couplings with the Design Matrix

Appendix A1 shows the FR decomposition and Appendix A2 the corresponding DP decomposition of the car door–to–body system. The decomposition starts with three subsystems: FR_1: fit door to neighboring panels; FR_2: keep interior quite and intrusion free; and FR_3: ensure door opens and closes properly. The subsystem FRs are then decomposed down the hierarchy, zigzagging between FRs and DPs, i.e., between Appendices A1 and A2, until the leaf-level DPs marked by "+" are reached. These leaf-level DPs, together with the corresponding FRs they satisfy, are listed in Table 2.1. For succinct presentation, Table 2.1 uses serial notations 1–28 in place of hierarchical notations in Appendix A. For example, FR_1 in Table 2.1 refers to $FR_{1.1.1}$ in Appendix A1. Note that FR_1 thru FR_7 are leaf-level FRs that flow from the first subsystem; FR_8 thru FR_{18} are leaf-level FRs, from the second subsystem and FR_{19} thru FR_{28}, from the third subsystem.

Table 2.1 Leaf-level FR–DP of car door-to-body system

FR_1 achieve uniform gap on both edges	DP_1 hinge tip in x–z plane
FR_2 balance leading and trailing edge gaps	DP_2 fore/aft position of hinge axis
FR_3 align feature lines	DP_3 vertical position of hinge datum
FR_4 achieve flushness at leading edge	DP_4 in/out position of hinge axis
FR_5 achieve flushness along both edges	DP_5 hinge tip in y–z plane
FR_6 achieve flushness at trailing edge	DP_6 in/out position of striker
FR_7 achieve flushness above beltline	DP_7 header over bent
FR_8 maintain adequate seal margin	DP_8 position of door interior surface
FR_9 maintain adequate seal height	DP_9 a system to maintain uniform seal height
FR_{10} maintain seal footprint	DP_{10} contour of contacting surfaces
FR_{11} divert away water	DP_{11} channel slope
FR_{12} detune seal from noise transmission.	DP_{12} modal property of seal section
FR_{13} dissipate noise energy	DP_{13} seal damping characteristic
FR_{14} eliminate seal itch	DP_{14} lubricant, substrate loss modulus
FR_{15} prevent gap-induced turbulence	DP_{15} gap filler
FR_{16} stop flushness-induced turbulence	DP_{16} header stiffness
FR_{17} control leakage across seals	DP_{17} sealing energy as barrier to intrusion
FR_{18} maintain mass flow rate of inlet air	DP_{18} fan
FR_{19} ensure reaction force > gravity	DP_{19} stiffness and preloads of check link spring
FR_{20} bar opening door swing thru stops	DP_{20} site, depth and climb of check link valleys
FR_{21} let closing door swing thru stops	DP_{21} site, depth and descent of check link valleys
FR_{22} eliminate resistance to swing	DP_{22} hinge axes aligned with axis of rotation
FR_{23} lower KE to surmount latch misalign	DP_{23} up-down adjustable striker
FR_{24} lower KE to compress seal	DP_{24} area under weather strip CLD
FR_{25} lower KE to deflect header	DP_{25} area under header load-deflection curve
FR_{26} lower KE to overcome air bind	DP_{26} pressure relief valve
FR_{27} store spring energy from opening	DP_{27} preloaded check link torsional spring
FR_{28} reduce effort to unlatch	DP_{28} mechanism to relieve reaction at latch

From Table 2.1, we construct the DM that captures the functional interdependencies among the 28 leaf-level FR–DP pairs, see Fig. 2.5. Row-wise are leaf-level FRs; column-wise are associated leaf-level DPs. Leaf-level FR_1 thru FR_7 and their associated DPs, which flow from the first subsystem, are labeled to the left and top of the DM, not shaded.

The leaf-level FR_8 thru FR_{18} and their associated DPs, which flow from the second subsystem, are shown lightly shaded green. And FR_{19} thru FR_{28}, which flow from the third subsystem are heavily shaded blue. For each cell DM (j, k) of the 28 × 28 cells, assessment is made whether the DP_k has an effect on FR_j. If it has an effect (no effect), the cell is marked with an “X” (blank). While cell-by-cell evaluation is tedious, it is crucial because functional inter-dependencies so obtained among the leaf-level FR–DP decide how the design functions at the system level.

Reducing the Design Matrix to Uncover Functional Coupling

When a DM is sparse or small, we can check for coupling by inspection. When it is large, the task becomes difficult. For DM with $n \times n = 28 \times 28$ and a total off-diagonal elements of $m = 88$ as in Fig. 2.5, the number of possible couplings equals $2^{m-n+1} - 1 = 2.306E18$ [4]. It is prohibitive to check for couplings among this

	DP1	DP2	DP3	DP4	DP5	DP6	DP7	DP8	DP9	DP10	DP11	DP12	DP13	DP14	DP15	DP16	DP17	DP18	DP19	DP20	DP21	DP22	DP23	DP24	DP25	DP26	DP27	DP28
FR1	X																											
FR2	X	X																										
FR3	X		X																				X					
FR4				X																								
FR5				X	X																							
FR6					X	X																						
FR7				X	X	X	X	X	X							X								X	X			
FR8				X	X	X	X	X															X	X				
FR9									X																			
FR10				X	X	X	X	X		X													X					
FR11											X																	
FR12												X																
FR13													X															
FR14				X	X	X	X	X	X	X			X	X														
FR15	X	X	X												X								X					
FR16				X		X			X							X									X			
FR17				X	X	X	X	X	X	X						X	X							X	X			
FR18																		X										
FR19	X				X														X								X	
FR20	X				X														X	X							X	
FR21	X				X														X		X						X	
FR22	X				X															X		X					X	
FR23	X																						X					
FR24				X	X	X	X	X	X							X							X	X				
FR25				X	X	X	X	X								X							X	X	X			
FR26																										X		
FR27																											X	
FR28				X	X	X		X															X					X

Fig. 2.5 Design matrix relating the leaf-level FR–DP

large number of candidates. Thus, we reduce the dimension of DM by isolating the submatrix that contains the couplings from the rest. We do this by successively relocate rows (columns) whose row-wise (column-wise) entries are all zeros but the diagonal element. For example in Fig. 2.5, we find five zero-rows: FR_1, FR_4, FR_9, FR_{13}, and FR_{27}; six zero-columns: DP_{14}, DP_{15}, DP_{17}, DP_{21}, DP_{22}, and DP_{28} and four combined (zero-rows, zero-columns): (FR_{11}, DP_{11}), (FR_{12}, DP_{12}), (FR_{18}, DP_{18}), and (FR_{26}, DP_{26}). A zero-row corresponds to an FR that is not affected by other DPs; a zero-column corresponds to a DP that does not affect any other FRs; and a combined (zero-row, zero-column) corresponds to an FR–DP pair that does not affect nor be affected by other FRs and DPs. All the three categories do not belong to any coupling loop. They are thus moved from their original locations. Namely through row and column interchange, all combined (zero-row, zero-column) are relocated to the upper left corner of the DM; all zero-columns (zero-rows) and their associated rows (columns) are relocated to the lower right (upper left) corner of the DM. What remains is a 13 × 13 submatrix outlined in thick border as shown in Fig. 2.6.

	DP11	DP12	DP18	DP26	DP1	DP4	DP9	DP13	DP27	DP2	DP3	DP5	DP6	DP7	DP8	DP10	DP16	DP19	DP20	DP23	DP24	DP25	DP14	DP15	DP17	DP21	DP22	FR28
FR11	X																											
FR12		X																										
FR18			X																									
FR26				X																								
FR1					X																							
FR4						X																						
FR9							X																					
FR13								X																				
FR27									X																			
FR2					X					X																		
FR3					X						X									X								
FR5						X						X																
FR6												X	X															
FR7						X	X					X	X	X	X		X				X	X						
FR8						X						X	X	X	X					X	X							
FR10						X						X	X	X	X	X				X								
FR16						X	X						X				X					X						
FR19					X				X			X						X										
FR20					X				X			X						X	X									
FR23					X															X								
FR24						X	X					X	X	X	X		X			X	X							
FR25						X						X	X	X	X		X			X	X	X						
FR14						X	X	X				X	X	X	X	X							X					
FR15					X					X	X									X				X				
FR17						X	X					X	X	X	X	X	X				X	X			X			
FR21					X				X			X						X								X		
FR22					X				X			X							X								X	
FR28						X						X	X		X					X								X

Fig. 2.6 Relocating zero-rows and zero-columns

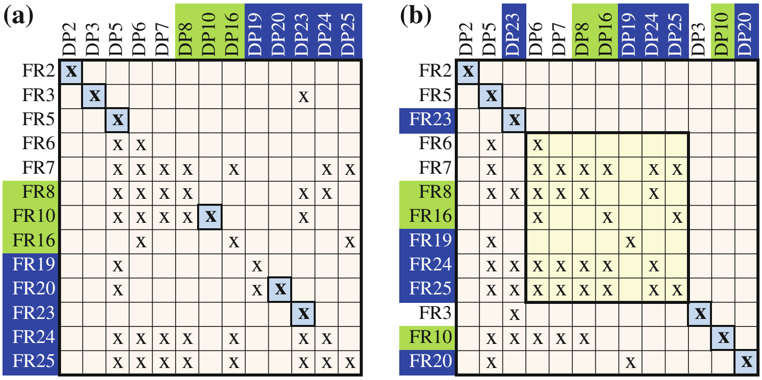

Fig. 2.7 Further detection **a** and relocation **b** of zero-row and zero-column

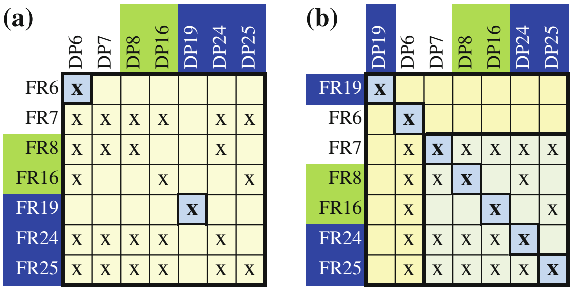

Fig. 2.8 Final detection **a** and relocation **b** of zero-row and zero-column

Further examination of the 13 × 13 submatrix reveals 3 more zero-rows: FR_2, FR_5 and FR_{23}; and 3 more zero-columns: DP_3, DP_{10} and DP_{20}, as shown in Fig. 2.7a. We repeat the successive relocation of these zero-rows and zero-columns and arrive at a further reduced 7 × 7 submatrix outlined in thick border in Fig. 2.7b.

Continuing the search for zero-rows and zero-columns in the 7 × 7 reduced matrix, we found 1 zero-row FR_6 and 1 zero-column DP_{19} (see Fig. 2.8a). Upon relocation of these two, we finally obtain a 5 × 5 matrix that containing neither zero-row nor zero-column as outlined in thick border, Fig. 2.8b. All the couplings in the original decomposed DM are now isolated and condensed into this 5 × 5 DM.

Implications from the Reduction of DM

Figure 2.9a shows DM as decomposed juxtaposed with Fig. 2.9b DM as condensed. The as-condensed DM shows three submatrices: a (4 × 4) uncoupled submatrix in the upper left; a (24 × 24) decoupled submatrix in the lower right; and a protruding (5 × 5) coupled submatrix within the decoupled submatrix. These

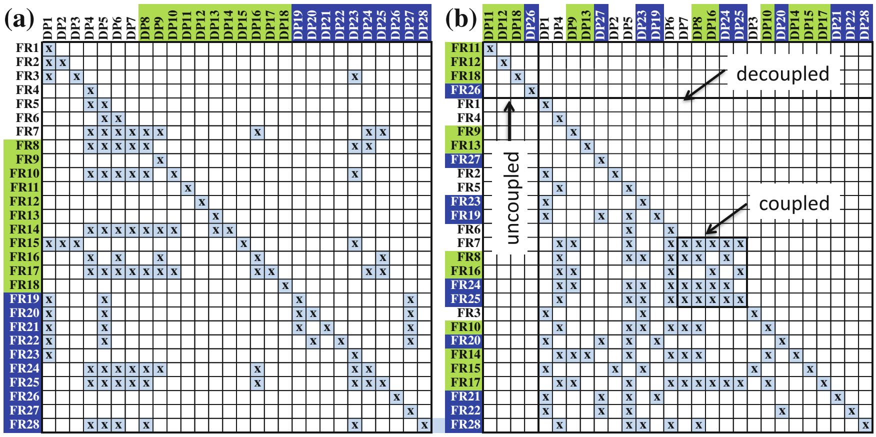

Fig. 2.9 Design matrix, **a** as decomposed and **b** as condensed

results reflect the reduction algorithm: relocating the combined (zero-rows, zero-columns) to DM upper left produces the uncoupled submatrix; relocating the zero-rows to DM upper left plus the zero-columns to DM lower right produce the decoupled submatrix. What remains is a protruding coupled submatrix within the decoupled submatrix. As indicated on top of DM Fig. 2.9b, the algorithm also produces a mingling of leaf-level DPs from the three subsystems. The implications of these results are as follows.

The protruding (5 × 5) coupled submatrix is the source of the whack-a-mole type of failures. The couplings need to be resolved first and foremost by effectively identify and eliminate functional couplings following for example a graph theory-based method described in [4].

Once couplings in the (5 × 5) coupled matrix are resolved, what remains is a (24 × 24) lower triangular DM. The lower triangular DM serves as a road map and provides a sequence to follow in satisfying ΔFRs: adjust ΔDP_j, $j = 1$ *to* i, to satisfy/fix ΔFR_i. Without the road map, we will still be fighting the whack-a-mole type of failure.

We must recognize which FR–DP pair falls unto the uncoupled submatrix and take advantage of the information, as they are the easiest to fix and satisfy.

Engineers in a door group are typically tasked with specific leaf-level functions of the door. As indicated in Fig. 2.9b, there is a mingling of the leaf-level DP that forms the triangular DM. The engineers must be made aware of this interdependency, i.e., mingling, of functions since their tasks must conform to the sequence dictated by the triangular DM.

The implications described above hold for any assembly of subsystems and components that form a large system.

2.2.2 |J| Criterion for Functional Independence

2.2.2.1 Derivation of $|J|$ as a Criterion for Functional Dependence

Unlike AD's SISO criterion for functional independence that is derived from empirical observations, in this section we will derive the criterion based on the formal mathematical definition of functional independence.

To illustrate, consider the car door-to-body integration example in Sect. 2.2.1.2. We start with the system-level FRs and decompose them down the hierarchy through the subsystems level to the leaf-level FRs. The FRs, which are conceptual at the system level, get more specific and detailed as they are decomposed down the hierarchy. When the leaf levels are reached, the FRs are realized by DPs that are known, implementable physical solutions. Thus, the FRs can be expressed in terms of the DPs through physics. For example, Table 2.1 relates 28-leaf-level FRs to corresponding leaf-level DPs. We denote these relationships as follows:

$$\begin{gathered}\mathrm{FR}_1 = f_1(\mathrm{DP}_1, \ldots, \mathrm{DP}_\mathrm{m}) \\ \vdots \\ \mathrm{FR}_\mathrm{n} = f_\mathrm{n}(\mathrm{DP}_1, \ldots, \mathrm{DP}_\mathrm{m})\end{gathered}$$

Or in vector notation,

$$\mathbf{FR} = \mathbf{f}(\mathbf{DP});$$

In the above and hereafter, a bolded quantity denotes a vector, a bracketed quantity denotes a matrix, and $\mathbf{f}(\bullet)$ denotes vector valued functions.

The vector equation above is Eq. (2.6) of Chap. 1, with $\mathbf{f}(\mathbf{DP}) \equiv \mathbf{f}_a(\mathbf{DP})$. The vector **DP** represents the physical quantities of the design, and the vector valued function $\mathbf{f}(\mathbf{DP})$ represents the laws of physics relating **FR** to **DP**. Since $\mathbf{f}(\mathbf{DP})$ is drawn from laws of physics, it may be assumed as continuous. So that we may expand $\mathbf{f}(\mathbf{DP})$ in a Taylor series about a design point $\mathbf{DP}^*$:

$$\begin{aligned}\mathbf{f}(\mathbf{DP}) &= \mathbf{f}(\mathbf{DP}^*) + [J](\mathbf{DP} - \mathbf{DP}^*) + \mathbf{o}(||\mathbf{DP} - \mathbf{DP}^*||) \\ &\approx \mathbf{f}(\mathbf{DP}^*) + [J](\mathbf{DP} - \mathbf{DP}^*)\end{aligned}$$

Thus: $\quad \mathbf{FR} - \mathbf{FR}^* \approx [J](\mathbf{DP} - \mathbf{DP}^*)$

where $[J]$ is the Jacobian matrix whose element $J_{ij} = \partial \mathrm{FR}_i / \partial \mathrm{DP}_j$ evaluated at the design point $\mathbf{DP}^*$ is a constant.

We recognize the Jacobian $[J]$ above is in fact the design matrix $[A]$ in AD, Eq. (3.3) in [1]. Thus, we may rewrite the vector equation as follows:

$$\Delta\mathbf{FR} = [A]\Delta\mathbf{DP}. \tag{2.2}$$

If **FRs** are linear functions of **DP**, the above differential vector equation reduces:

$$\mathbf{FR} = [A]\mathbf{DP}. \tag{2.2a}$$

Equation (2.2a) is known as a "design equation" and appears on page 55 of the first axiomatic design text [1]. It has since been used extensively for conceptual applications in the AD literature. However, it is important to recognize that in most cases Eq. (2.2a) is only a notation to convey a relation between **FR** and **DP**. In actuality, the equation to solve for is the differential form in Eq. (2.2).

Expanding the differential vector equation for the special case of $i, j = 1, 2$:

$$\Delta \mathrm{FR}_1 = A_{11}\Delta \mathrm{DP}_1 + A_{12}\Delta \mathrm{DP}_2; \tag{2.3}$$

$$\Delta \mathrm{FR}_2 = A_{21}\Delta \mathrm{DP}_1 + A_{22}\Delta \mathrm{DP}_2. \tag{2.4}$$

Equation (2.3) × A_{22} minus Eq. (2.4) × A_{12} to eliminate $\Delta\mathrm{DP}_2$ gives:

$$A_{22}\Delta \mathrm{FR}_1 - A_{12}\Delta \mathrm{FR}_2 = (A_{22}A_{11} - A_{12}A_{21})\Delta \mathrm{DP}_1 \tag{2.5}$$

Note that $(A_{22}\ A_{11} - A_{12}\ A_{21}) = \left(\frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_2}\right)\left(\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_1}\right) - \left(\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_2}\right)\left(\frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_1}\right)$ is the determinant $|\mathrm{A}|$ of the DM. It is known as $|J|$, the Jacobian determinant in vector calculus.

$$|J| = \begin{vmatrix} \frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_1} & \frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_2} \\ \frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_1} & \frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_2} \end{vmatrix}.$$

Thus, if $(A_{22}\ A_{11} - A_{12}\ A_{21}) = |J| = 0$ in Eq. (2.5), then $\Delta\mathrm{FR}_2 = A_{22}\Delta\mathrm{FR}_1/A_{12}$. Or $\mathrm{FR}_2 = \mathrm{FR}_2^* + A_{22}\,(\mathrm{FR}_1 - \mathrm{FR}_1^*)/A_{12}$. Namely, FR_2 is dependent on FR_1. Hence, $|J| = 0$ implies functional dependence.

$$\text{Proof A:} \quad (|J| = 0) \Rightarrow \text{functional dependence}$$

We next prove the converse is true. Namely if FR_2 is functionally dependent on FR_1, then $|J| = 0$. We start with the formal definition of functional dependency. Namely, FR_2 is dependent on FR_1 if it is a function of FR_1:

$$\mathrm{FR}_2 = \mathrm{FR}_2(\mathrm{FR}_1)$$

Applying the chain rule for differentiation on above equation, we have

$$\frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_1} = \frac{\partial \mathrm{FR}_2}{\partial \mathrm{FR}_1}\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_1}$$
$$\frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_2} = \frac{\partial \mathrm{FR}_2}{\partial \mathrm{FR}_1}\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_2}.$$

In the above, multiply 1st equation by $\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_2}$ and the 2nd by $\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_1}$. Subtract one resulting equation from the other to eliminate $\left(\frac{\partial \mathrm{FR}_2}{\partial \mathrm{FR}_1}\right)\left(\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_2}\right)\left(\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_1}\right)$, we have:

$$\left(\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_2}\right)\left(\frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_1}\right) - \left(\frac{\partial \mathrm{FR}_1}{\partial \mathrm{DP}_1}\right)\left(\frac{\partial \mathrm{FR}_2}{\partial \mathrm{DP}_2}\right) \equiv |J| = 0$$

Hence, functional dependence implies $|J| = 0$. This is the proof of the converse:

$$\text{Proof B:} \quad \text{Functional dependence} \Rightarrow (|J| = 0).$$

Combining both proofs A and B, we have

$$\text{Functional dependence} \Leftrightarrow (|J| = 0).$$

Namely, $|J| = 0$ is a necessary and sufficient condition for FR_2 to be functionally dependent on FR_1. Likewise, FR_2 is functionally independent of FR_1 if and only if (iff) $|J| \neq 0$. Thus by formal definition of functional dependency, we have derived the criterion: FRs are functionally independent iff $|J| \neq 0$; dependent iff $|J| = 0$.

2.2.2.2 Implications of |J| as a Criterion for Functional Independence

The differential form of Eq. (2.2) may be rewritten as follows.

$$\Delta\mathbf{FR} = [J]\Delta\mathbf{DP}$$

So that the adjustments $\Delta\mathbf{DP}$ necessary to bring **FR** to its target **FR*** is,

$$\Delta\mathbf{DP} = [J]^{-1}\Delta\mathbf{FR}$$

Note that if **FRs** of the design are functionally dependent, then $|J| = 0$ and its inverse $|J|^{-1}$ does not exist. In mathematical lingua, the design has a "singularity" in its first derivative and is non-differentiable. Consequently, no adjustments in $\Delta\mathbf{DP}$ can bring the design to its target value **FR***.

AD's independence axiom declares that a good design must "maintain the independence of the functional requirements (**FR**)." The $|J|$ criterion corroborates this declaration since $|J| \neq 0$ implies independence of **FR** and it guarantees a design solution. Therefore, the $|J|$ criterion provides formidable theoretical evidence that a violation of the independence axiom will impede the design from finding a final value **DP*** that fulfills the design equation and meets all functional requirements **FR***. While the AD independence axiom was established through extensive empirical study to yield "good" designs, the $|J|$ criterion shows that these "good" designs not only fulfill all their functional requirements but also can be found through well established analytical and numerical methods.

AD further proposes SISO as the criterion for independence. Since determinant of a diagonal or triangular DM—a SISO design—is the product of all the diagonal elements none of which is zero, their $|J| \neq 0$. This validates SISO as a criterion for functional independence. However, SISO criterion is only a sufficient condition. Namely, SISO implies functional independence but functional independence does not imply SISO:

$$\begin{aligned} &\text{SISO } \Rightarrow \text{Functional independence;} \\ &\text{Functional independence} \neg \Rightarrow \text{SISO.} \end{aligned}$$

While SISO criterion is more conservative, it does not detract from its utility. It remains as a sufficiency condition. Furthermore, during design synthesis it is relatively easy to mentally keep track of a diagonal or lower triangular design matrix. In comparison, calculating a Jacobian is significantly harder; especially at high-level conceptual design synthesis whereat the mathematical form of the design equations is not well known.

Nevertheless, there are cases where the SISO criterion is inadequate as there are coupled, non-SISO designs with $|J| \neq 0$. Such cases are functionally independent thus admit design solutions but will be rejected per AD's SISO criterion. For example in robotics, the robot Jacobian that relates joint velocities to end-effector velocities is used routinely to plan and execute robot paths and transform forces and torques from the end effector to the manipulator. The robot Jacobian is in fact a design matrix that relates output (end-effector velocities) to input (joint velocities). In most cases, it is not SISO so that most robot designs would have been rejected per the SISO criterion.

To recap, the Jacobian matrix $[J]$ relates $\Delta\mathbf{FR}$ to $\Delta\mathbf{DP}$ of a conceived solution. Its determinant $|J|$ is a test for functional independence of **FR**: yes iff $|J| \neq 0$; no iff $|J| = 0$. Furthermore, iff $|J| \neq 0$, then the conceived solution in term of **DP** can satisfy the **FR**. Otherwise, it cannot. In short, $|J|$ acts as a qualifier: accept a design solution iff $|J| \neq 0$.

We end this section with Table 2.2 which provides a contrast between SISO criterion and $|J|$ criterion derived per formal definition of functional dependency.

Table 2.2 Contrasting AD SISO criterion with $|J|$ criterion

	$\begin{bmatrix} x & o & x \\ x & x & o \\ o & x & x \end{bmatrix}$	$\begin{bmatrix} x & o & x \\ x & x & o \\ o & x & x \end{bmatrix}$	$\begin{bmatrix} x & o & o \\ x & x & o \\ x & x & x \end{bmatrix}$	$\begin{bmatrix} x & o & o \\ o & x & o \\ o & o & x \end{bmatrix}$
SISO criterion:	Bad	Bad	Good	Better
–	Reject	Reject	Accept	Accept
$\|J\|$ criterion:	Bad	Good	Better	Best
–	Reject	Accept	Accept	Accept
–	if $\|J\| = 0$	if $\|J\| \neq 0$	since $\|J\| \neq 0$	since $\|J\| \neq 0$

2.2.2.3 |*J*| for Various Categories of Design

Water Faucet

Both designs **a** and **b** have the same FRs

FR_1 = control flow rate Q;
FR_2 = control water temperature T.

Both designs have the same governing physics:

$$\text{Mass conservation:} \quad Q = Q_h + Q_c; \tag{2.6a}$$

$$\text{Energy conservation:} \quad QT = Q_h T_h + Q_c T_c.$$

$$T = \frac{Q_h T_h + Q_c T_c}{Q_h + Q_c} \tag{2.6b}$$

$$= \frac{(Q_h/Q_c)T_h + T_c}{(Q_h/Q_c) + 1} \tag{2.6c}$$

For Faucet **a**, we choose the left knob controlling Q_h as DP_1; and the right knob controlling Q_c as DP_2 (see Fig. 2.1a). Substituting into Eqs. (2.6a) and (2.6b):

$$\begin{aligned}
&FR_1 = DP_1 + DP_2; && FR_2 = \frac{DP_1 T_h + DP_2 T_c}{DP_1 + DP_2}.\\
&\frac{\partial FR_1}{\partial DP_1} = 1; && \frac{\partial FR_1}{\partial DP_2} = 1.\\
&\frac{\partial FR_2}{\partial DP_1} = \frac{(T_h - T_c)DP_2}{(DP_1 + DP_2)^2}; && \frac{\partial FR_2}{\partial DP_2} = -\frac{(T_h - T_c)DP_1}{(DP_1 + DP_2)^2}
\end{aligned}$$

$$|J| = \begin{vmatrix} 1 & 1 \\ \frac{(T_h - T_c)DP_2}{(DP_1 + DP_2)^2} & -\frac{(T_h - T_c)DP_1}{(DP_1 + DP_2)^2} \end{vmatrix} = -\frac{T_h - T_c}{DP_1 + DP_2} \tag{2.7}$$

Per AD's SISO criterion, FRs in Faucet **a** are coupled and the design should be rejected. However, according to formal definition of functional dependence, FRs of the design are functionally independent since $|J| \neq 0$. It is therefore acceptable.

For Faucet **b**, we choose as DP_1, the up/down lever controlling total flow rate ($Q_h + Q_c$), and as DP_2, the left/right lever controlling ratio of the hot/cold water flow rate (Q_h/Q_c) (see Fig. 2.1b). Substituting into Eqs. (2.6a) and (2.6c), we have an uncoupled design as indicated by the design equation:

$$\begin{aligned}
&FR_1 = DP_1; && FR_2 = \frac{DP_2 T_h + T_c}{DP_2 + 1}.\\
&\frac{\partial FR_1}{\partial DP_1} = 1; && \frac{\partial FR_1}{\partial DP_2} = 0;\\
&\frac{\partial FR_2}{\partial DP_1} = 0; && \frac{\partial FR_2}{\partial DP_2} = \frac{(T_h - T_c)}{(DP_2 + 1)^2}.
\end{aligned}$$

$$|J| = \begin{vmatrix} 1 & 0 \\ 0 & \frac{(T_h - T_c)}{(DP_2+1)^2} \end{vmatrix} = \frac{(T_h - T_c)}{(DP_2+1)^2} \tag{2.8}$$

Note that as the water heater temperature T_h is set closer to outside water temperature T_c, the $|J|$ value gets closer to zero so that the faucet becomes less capable of providing the two independent functions. In short, the $|J|$ criterion provides a quantitative measure of independence which the AD SISO criterion cannot.

Note further that the physics governing both faucets are the same. Yet their [J] matrices in Eqs. (2.7) and (2.8) are different: one is coupled and the other is uncoupled. This conveys a fundamental message in AD. Namely, it is the choice of design solutions DPs, not the physics that determine the goodness of a design.

Projector Design

The two FRs of a projector are as follows:

$$\begin{aligned} \mathrm{FR}_1 &= \text{magnify the image} = \frac{D}{d}; \\ \mathrm{FR}_2 &= \text{focus the image} = \frac{1}{D} + \frac{1}{d} + \frac{1}{f} = 0. \end{aligned} \tag{2.9}$$

In the above, D = DP_1 is the distance of the lens from the screen aka the throw of the projector; d = DP_2 is the distance of the lens from the object and f = DP_3 is the focal length of the lens. Thus, we have a redundant design: 2 FRs and 3 DPs. If we were to fix the extra DP to get an equal number of FRs and DPs, we would have three (= ${}_3C_2$) possible $|J|$ solutions as follows:

$$\begin{aligned} &\frac{\partial \mathrm{FR}_1}{\partial \mathrm{D}} = \frac{1}{d}; \qquad \frac{\partial \mathrm{FR}_1}{\partial d} = -\frac{D}{d^2}; \qquad \frac{\partial \mathrm{FR}_1}{\partial f} = 0. \\ &\frac{\partial \mathrm{FR}_2}{\partial \mathrm{D}} = -\frac{1}{D^2}; \qquad \frac{\partial \mathrm{FR}_2}{\partial d} = -\frac{1}{d^2}; \qquad \frac{\partial \mathrm{FR}_2}{\partial f} = -\frac{1}{f^2}. \end{aligned}$$

$$(D, d) \text{ as DPs}: \ |J| = \begin{vmatrix} \frac{1}{d} & -\frac{D}{d^2} \\ -\frac{1}{D^2} & -\frac{1}{d^2} \end{vmatrix} = -\frac{1}{d^2}\left(\frac{1}{D} + \frac{1}{d}\right); \tag{2.10a}$$

$$(D,f) \text{ as DPs}: \quad |J| = \begin{vmatrix} \frac{1}{d} & 0 \\ -\frac{1}{D^2} & -\frac{1}{f^2} \end{vmatrix} = -\frac{1}{df^2}. \tag{2.10b}$$

$$(d,f) \text{ as DPs}: \quad |J| = \begin{vmatrix} -\frac{D}{d^2} & 0 \\ -\frac{1}{d^2} & -\frac{1}{f^2} \end{vmatrix} = \frac{D}{d^2 f^2}. \tag{2.10c}$$

All three candidate sets of DPs for the FRs do ensure functional independence of FRs because their $|J| \neq 0$. However, AD SISO criterion will reject solution (2.10a). Nonetheless solution (2.10b), which is acceptable to both AD SISO and $|J|$ criteria, would be the preferred choice since it permits wider latitude of throw D for the projector to accommodate various room sizes.

Disbursement Algorithm

The $|J|$ values for the four possible DM solutions, Eqs. (2.1a) thru (2.1d), in Sect. 2.2.1.1 are as follows:

$$|J| = \begin{vmatrix} \$20 & \$10 & \$5 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \$0; \tag{2.11a}$$

$$|J| = \begin{vmatrix} \$20 & \$10 & \$1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \$8; \tag{2.11b}$$

$$|J| = \begin{vmatrix} \$20 & \$5 & \$1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = -\$7; \tag{2.11c}$$

$$|J| = \begin{vmatrix} \$10 & \$5 & \$1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = \$3. \tag{2.11d}$$

We showed in Sect. 2.2.1.1 that for design solution (2.1a), FR_3 is dependent on FR_1 and FR_2. This is confirmed in Eq. (2.11a) above which shows $|J| = 0$. This example demonstrates that an improper choice of DPs can destroy functional independency as originally intended. This is why we need to continually check for it.

Design with Insufficient DPs

To show that $|J|$ of a design with insufficient DPs is zero, i.e., the design FRs are functionally dependent, consider a design with two (FR_1, FR_2) and one DP_1 whose effect on (FR_1, FR_2) are (A_{11}, A_{21}):

$$\begin{bmatrix} \Delta FR_1 \\ \Delta FR_2 \end{bmatrix} = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} [\Delta DP_1].$$

We conjure up a second DP_2 identical to DP_1 to make up for the insufficiency in DP. This second DP_2 has identical effects of (A_{11}, A_{21}) on (FR_1, FR_2):

$$\begin{bmatrix} \Delta FR_1 \\ \Delta FR_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11} \\ A_{21} & A_{21} \end{bmatrix} \begin{bmatrix} \Delta DP_1 \\ \Delta DP_2 \end{bmatrix}$$

The $|J|$ of the above design equals $A_{11}A_{21} - A_{11}A_{21} = 0$. This confirms that FR_1 and FR_2 are functionally dependent when there are insufficient DPs.

2.3 Mathematical Exposition of Information Axiom

The information axiom states that a good design solution must minimize its information contents I, or equivalently maximize its probability of success, P_s. Figure 2.10 illustrates the evaluation of P_s. In the presence of variability, an FR_i will exhibit a range of values called the system range or the spread. Cognizant of its variability, a designer would accept the FR_i if it falls within a specified range called the design range. The overlap of the two ranges shown shaded in Fig. 2.10 is P_s, the probability of success of the design. To consider P_s, we need to identify sources of variation that are generating the variability in FRs; how the variability is magnified by the design; and what are the countermeasures for them. We consider these in the next several sections.

2.3.1 Recognition of Noise Variables

Since variability in FRs is a consequence of variation, we need to recognize and identify the sources that are generating the variation. We denote these sources as the noise variables, NVs. For example in the faucet design, the temperature T_c of the cold water in Eqs. (2.6b) and (2.6c), Sect. 2.2.2.3, entering the faucet from the outside is a NV since it fluctuates with the uncontrollable temperature outside. It is a NV induced by the environment. If the water heater in a building does not have sufficient capacity to meet the demand of multiple faucets, hot water pressure will fluctuate with the number of faucets turned on or off at a given time. This will affect hot water flow Q_h in Eqs. (2.6a), (2.6b), and (2.6c). This is a NV induced by customer usage. If a projector is used for a variety of room size, it will need a variety of throws to magnify the image. Hence, the throw D in Eq. (2.9), Sect. 2.2.2.3 is a NV induced by customer usage. In the

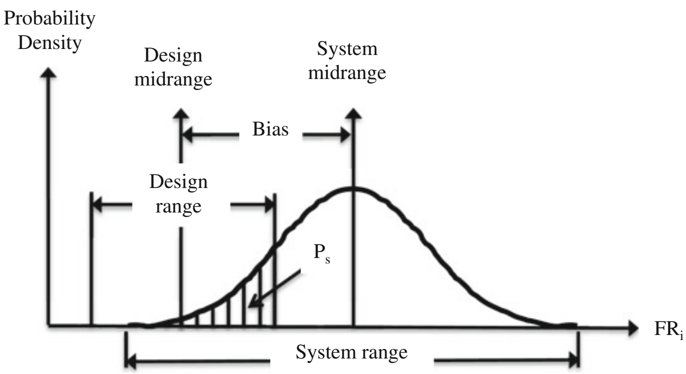

Fig. 2.10 Evaluating the probability of success

hubcap design, the NV is the interference caused by the manufacturing variation in D_{rim} and D_{clip}.

Let **NV** denotes the noise variables that cause variation in **FR**. A **NV** triggers a random deviation in **FR** from its current value **FR*** given by the amount:

$$\mathbf{FR} - \mathbf{FR}^* = \left[J^{\mathrm{NV}}\right](\mathbf{NV} - \mathbf{NV}^*). \tag{2.12}$$

In Eq. (2.12), **NV*** is a reference value, e.g., the midrange of **NV**; $[J^{\mathrm{NV}}]$ is the Jacobian matrix of $\partial \mathrm{FR}_i / \partial \mathrm{NV}_j$ given by

$$\left[J^{\mathrm{NV}}\right] = \begin{bmatrix} \frac{\partial \mathrm{FR}_1}{\partial \mathrm{NV}_1} & \cdots & \frac{\partial \mathrm{FR}_1}{\partial \mathrm{NV}_\mathrm{m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{FR}_\mathrm{n}}{\partial \mathrm{NV}_1} & \cdots & \frac{\partial \mathrm{FR}_\mathrm{n}}{\partial \mathrm{NV}_\mathrm{m}} \end{bmatrix}_{n \times m}$$

The superscript NV is used to distinguish $[J^{\mathrm{NV}}]$ from $[J]$, the Jacobian matrix of $\partial \mathrm{FR}_i / \partial \mathrm{DP}_j$ which hereafter will be superscripted with "DP." While $[J^{\mathrm{DP}}]$ relates to the functional independence of **FR** in a design, $[J^{\mathrm{NV}}]$ relates to the sensitivity of **FR** to noise **NV**. To illustrate, the $[J^{\mathrm{NV}}]$ for some earlier designs are as follows.

For the faucet designs **a** and **b**:

$$\mathrm{FR}_1 = Q_\mathrm{h} + Q_\mathrm{c}; \qquad \mathrm{FR}_2 = \frac{Q_\mathrm{h} T_\mathrm{h} + Q_\mathrm{c} T_\mathrm{c}}{Q_\mathrm{h} + Q_\mathrm{c}}.$$

$$\mathrm{NV}_1 = Q_\mathrm{h}; \qquad \mathrm{NV}_2 = T_\mathrm{c}.$$

$$\frac{\partial \mathrm{FR}_1}{\partial \mathrm{NV}_1} = 1; \qquad \frac{\partial \mathrm{FR}_1}{\partial \mathrm{NV}_2} = 0.$$

$$\frac{\partial \mathrm{FR}_2}{\partial \mathrm{NV}_1} = \frac{Q_\mathrm{c}(T_\mathrm{h} - T_\mathrm{c})}{(Q_\mathrm{h} + Q_\mathrm{c})^2} \qquad \frac{\partial \mathrm{FR}_2}{\partial \mathrm{NV}_2} = \frac{Q_\mathrm{c}}{Q_\mathrm{h} + Q_\mathrm{c}}.$$

$$\left[J^{\mathrm{NV}}\right] = \begin{bmatrix} 1 & 0 \\ \frac{Q_\mathrm{c}(T_\mathrm{h} - T_\mathrm{c})}{(Q_\mathrm{h} + Q_\mathrm{c})^2} & \frac{Q_\mathrm{c}}{Q_\mathrm{h} + Q_\mathrm{c}} \end{bmatrix}$$

For the projector with fixed d equals to a constant, Eq. (2.9) gives:

$$\mathrm{FR}_1 = \frac{D}{\mathrm{d}}; \qquad \mathrm{FR}_2 = \frac{1}{D} + \frac{1}{d} + \frac{1}{f}.$$

$$\mathrm{NV} = D.$$

$$\frac{\partial \mathrm{FR}_1}{\partial \mathrm{NV}} = \frac{1}{d}; \qquad \frac{\partial \mathrm{FR}_2}{\partial \mathrm{NV}} = -\frac{1}{D^2}.$$

$$\left[J^{\mathrm{NV}}\right] = \begin{bmatrix} \frac{1}{d} \\ -\frac{1}{D^2} \end{bmatrix}$$

For the hubcap design,

$$\begin{aligned}
&\mathrm{FR} = k\delta;\\
&\mathrm{NV} = \delta;\\
&\frac{\partial \mathrm{FR}}{\partial \mathrm{NV}} = k.\\
&\left[J^{\mathrm{NV}}\right] = \left[\frac{\partial \mathrm{FR}}{\partial \mathrm{NV}}\right] = k
\end{aligned}$$

2.3.2 *Countermeasures to Noise Variables*

The countermeasures to noise sources are as follows: (1) to reduce if not eliminate them, (2) to compensate for them, and (3) to desensitize the design against them. Action (1) refers to the reduction if not elimination of ($\mathbf{NV} - \mathbf{NV}^*$) in Eq. (2.12). This involves tightening the design tolerances, identifying and eliminating process variables that cause variation, and a host of other activities associated with fighting **NV** head-on.

2.3.2.1 Compensation as a Countermeasure

Compensation avoids fighting noise head-on. Instead, it provides a mechanism that further adjust **DP** to nullify $[J^{\mathrm{NV}}](\mathbf{NV} - \mathbf{NV}^*)$:

$$\mathbf{FR} - \mathbf{FR}^* = \left[J^{\mathrm{NV}}\right](\mathbf{NV} - \mathbf{NV}^*) - \left[J^{\mathrm{DP}}\right](\mathbf{DP} - \mathbf{DP}^*) = 0. \tag{2.13}$$

An example people most familiar with is tire balancing in which correction weights ($\mathbf{DP} - \mathbf{DP}^*$) are added to counteract the combined effect of the tire and wheel unbalance. Other examples are water faucet and projector designs discussed earlier. Per Eq. (2.13), the amount of compensation needed is as follows:

$$\mathbf{DP} - \mathbf{DP}^* = \left[J^{\mathrm{DP}}\right]^{-1}\left[J^{\mathrm{NV}}\right](\mathbf{NV} - \mathbf{NV}^*).$$

As revealed in above equation, a prerequisite to compensation is that the design satisfies independent axiom, i.e., $| \mathrm{J}^{\mathrm{DP}} | \neq 0$. Otherwise, $[J^{\mathrm{DP}}]^{-1}$ does not exist, and compensation is not possible. The uncoupled design, e.g., the single-handle faucet, is most easy to compensate since its $[J^{\mathrm{DP}}]$ provides a one-to-one relationship between **DP** and **NV**. The decoupled design, e.g., projector of Eq. (2.10b), is equally easy to compensate if we follow the forward substitution scheme dictated by $[J^{\mathrm{DP}}]$ described in Sect. 2.2.1. The coupled design while possible is difficult to compensate. In short, AD criterion for independence is most applicable in designing for compensation. Per information axiom, information content in a compensated design is zero since variation is completely nullified.

2.3.2.2 Robust Design as a Countermeasure

Another countermeasure action (3) is to move activities to the design stage. Parameters are designed into the design that reduce the sensitivity $[J^{NV}]$ in Eq. (2.13), thereby reducing if not eliminating activities in countermeasures (1) and (2) altogether. For example in hubcap design, in which retention force FR equals $k\delta$, instead of fighting variability in δ head-on, we use a less stiff cantilever spring $k = [J^{NV}] \rightarrow$ small. So that variability in δ is not amplified and transmitted to the retention force. The strategy not to fight **NV** head-on but to reduce the sensitivity to **NV** is known as robust design. Robust design has been the centerpiece of Design for Six Sigma (DFFS).

2.3.3 Implementing Countermeasures to Minimize Information Content

Referring to Fig. 2.10, we minimize information content or equivalently maximize P_s in two steps:

1. reduce bias (= system mid-range − design mid-range) to zero by compensation;
2. minimize the system range to within the design range through robust design.

To begin with, we take the expected value of the random variables **FR** and **NV** on both sides of on Eq. (2.13) to arrive at the expression for bias:

$$\text{Bias} = \mathrm{E}(\mathbf{FR}) - \mathbf{FR}^* = \left[J^{\mathrm{NV}}\right]\{\mathrm{E}(\mathbf{NV}) - \mathbf{NV}^*\} - \left[J^{\mathrm{DP}}\right](\mathbf{DP} - \mathbf{DP}^*) \qquad (2.14)$$

It follows that adjustment in **DP** needed to reduce bias to zero by compensation is as follows:

$$(\mathbf{DP} - \mathbf{DP}^*) = \left[J^{\mathrm{DP}}\right]^{-1}\left[J^{\mathrm{NV}}\right]\{\mathrm{E}(\mathbf{NV}) - \mathbf{NV}^*\} \qquad (2.15)$$

Note again that bias cannot be reduced to zero if FRs are functionally dependent since $| J^{DP} | = 0$ implies $[J^{DP}]^{-1}$ does not exist; thus, no solution is possible.

We next subtract Eq. (2.14) from Eq. (2.13) to obtain,

$$\mathbf{FR} - E(\mathbf{FR}) = \left[J^{\mathrm{NV}}\right]\{\mathbf{NV} - E(\mathbf{NV})\}.$$

The variance–covariance of **FR** is then given as follows:

$$\underset{n\times n}{\left[V^{\mathrm{FR}}\right]} = \underset{n\times m}{\left[J^{\mathrm{NV}}\right]}\;\underset{m\times m}{\left[V^{\mathrm{NV}}\right]}\;\underset{m\times n}{\left[J^{\mathrm{NV}}\right]^T}$$

where $[V^{FR}]$ and $[V^{NV}]$ are the variance–covariance of **FR** and **NV** shown below.

$$[V^{\mathrm{FR}}] = E\{\{\mathbf{FR} - E(\mathbf{FR})\}\{\mathbf{FR} - E(\mathbf{FR})\}^T\}$$
$$[V^{\mathrm{NV}}] = E\{\{\mathbf{NV} - E(\mathbf{NV})\}\{\mathbf{NV} - E(\mathbf{NV})\}^T\}.$$

Assuming the **NVs** are probabilistically independent, the matrix $[V^{\mathrm{NV}}]$ would be diagonal. The variance of $\mathrm{FR_i}$ is then the ith diagonal element of $[V^{\mathrm{FR}}]$ given by

$$v_{ii}^{\mathrm{FR}} = \sum_{k=1}^{m} j_{ik}^{\mathrm{NV}} v_{kk}^{\mathrm{NV}} j_{ik}^{\mathrm{NV}}$$

The total variance of **FR** is the trace of $[V^{\mathrm{FR}}]$:

$$\text{Variance of } \mathbf{FR} = \sum_{i=1}^{n} \mathrm{v}_{\mathrm{ii}}^{\mathrm{FR}} = \sum_{i=1}^{n} \sum_{k=1}^{m} j_{ik}^{\mathrm{NV}} v_{kk}^{\mathrm{NV}} j_{ik}^{\mathrm{NV}} \tag{2.16}$$

To maximize P_s, we reduce bias to zero by compensation per Eq. (2.15) and minimize system range, equals to squared root of variance of **FR,** to within the design range of **FR** by robust design per Eq. (2.16).

Summarizing, the steps in AD in mathematical terms are as follows.

1. Define **FR** in a solution neutral environment, free of functional inter-dependence among them.

2. Conceive solution **DP** that maintains the functional independence in **FR** so that $\mathbf{FR}^*$ can be achieved:

$$|J^{\mathrm{DP}}| \neq 0$$

3. Minimize the spread of **FR** with robust design. Namely, reduce $[J^{\mathrm{NV}}]$:

$$\sum_{i=1}^{\mathrm{n}} \sum_{k=1}^{\mathrm{m}} \mathrm{j}_{ik}^{\mathrm{NV}} v_{kk}^{\mathrm{NV}} j_{ik}^{\mathrm{NV}} \rightarrow \text{minimum}$$

4. Subject to constraint that the bias is zero:

$$(\mathbf{DP} - \mathbf{DP}^*) = [J^{\mathbf{DP}}]^{-1} [J^{\mathbf{NV}}] \{E(\mathbf{NV}) - \mathbf{NV}^*\}$$

Step 2 and 3 express, respectively, the independence axiom and information axiom in mathematical terms. Step 4 states in mathematical term that independent axiom takes precedence over information axiom. Namely, if **FRs** are not functionally independent, then $|\mathrm{J}^{\mathrm{DP}}| = 0$; $[J^{\mathrm{DP}}]^{-1}$ does not exist; and the constraint that bias = 0 cannot be satisfied. This point is missed in DFSS courses that do not include AD. Common sense tells us that Robust Design optimization has to be subsequent to requirement definition **FR**, and solution conception **DP** because performance of a poorly defined and ill-conceived design cannot be improved via subsequent optimization.

Appendix A1: FR Decomposition of Door-to-Body System

FR_1: fit door exterior to adjacent panels (showroom)
- $FR_{1.1}$: achieve uniform gap around perimeter
  - $FR_{1.1.1}$: achieve uniform gap on both edges
  - $FR_{1.1.2}$: balance leading and trailing edge gaps
  - $FR_{1.1.3}$: align feature lines
- $FR_{1.2}$: achieve flushness to adjacent panels
  - $FR_{1.2.1}$: achieve flushness below beltline
    - $FR_{1.2.1.1}$: achieve flushness at leading edge
    - $FR_{1.2.1.2}$: achieve flushness along both edges
    - $FR_{1.2.1.3}$: achieve flushness at trailing edge
  - $FR_{1.2.2}$: achieve flushness above beltline

FR_2: keep interior quiet and intrusion free
- $FR_{2.1}$: block water and airborne noise
  - $FR_{2.1.1}$: maintain contact pressure
    - $FR_{2.1.1.1}$: maintain adequate seal margin
    - $FR_{2.1.1.2}$: maintain adequate seal height
  - $FR_{2.1.2}$: maintain seal footprint
- $FR_{2.2}$: divert away water
- $FR_{2.3}$: reduce noise transmission through seal
  - $FR_{2.3.1}$: detune seal from noise transmission
  - $FR_{2.3.2}$: dissipate noise energy
- $FR_{2.4}$: eliminate noise sources from mating surfaces
  - $FR_{2.4.1}$: eliminate seal itch
  - $FR_{2.4.2}$: prevent gap-induced turbulence
  - $FR_{2.4.3}$: stop flushness-induced turbulence
- $FR_{2.5}$: prevent CO emission intrusion
  - $FR_{2.5.1}$: control leakage across seals
  - $FR_{2.5.2}$: maintain mass flow rate of inlet air

FR_3: ensure proper opening/closing of the door
- $FR_{3.1}$: hold open at distinct positions on incline
  - $FR_{3.1.1}$: ensure reaction force > gravity
  - $FR_{3.1.2}$: bar opening door swing thru stops
- $FR_{3.2}$: reduce effort to swing door
  - $FR_{3.2.1}$: let closing door swing thru stops
  - $FR_{3.2.2}$: eliminate resistance to swing
- $FR_{3.3}$: latch / unlatch easily
  - $FR_{3.3.1}$: reduce effort to latch
    - $FR_{3.3.1.1}$: decrease energy resisting latch
      - $FR_{3.3.1.1.1}$: lower KE to surmount latch misalign
      - $FR_{3.3.1.1.2}$: lower KE to compress seal
      - $FR_{3.3.1.1.3}$: lower KE to deflect header
      - $FR_{3.3.1.1.4}$: lower KE to overcome airbind
    - $FR_{3.3.1.2}$: store spring energy from opening
  - $FR_{3.3.2}$: reduce effort to unlatch

Appendix A2: DP Decomposition of Door-to-Body System

DP_1: door fitting scheme
  $DP_{1.1}$: hinge / latch system
  + $DP_{1.1.1}$: hinge tip in x-z plane
  + $DP_{1.1.2}$: fore/aft position of hinge axis
  + $DP_{1.1.3}$: vertical position of hinge datum
  $DP_{1.2}$: system for achieving flushness
    $DP_{1.2.1}$: hinge / latch system
    + $DP_{1.2.1.1}$: in/out position of hinge axis
    + $DP_{1.2.1.2}$: hinge tip in y-z plane
    + $DP_{1.2.1.3}$: in/out position of striker
  + $DP_{1.2.2}$: header over bent
DP_2: system for a quiet & intrusion-free interior
  $DP_{2.1}$: sealing energy as barrier
    $DP_{2.1.1}$: seal indentation by header
    + $DP_{2.1.1.1}$: position of door interior surface
    + $DP_{2.1.1.2}$: a system to maintain uniform seal height
  + $DP_{2.1.2}$: contour of contacting surfaces
+ $DP_{2.2}$: channel slope
  $DP_{2.3}$: noise transmission management
  + $DP_{2.3.1}$: modal property of seal section
  + $DP_{2.3.2}$: seal damping characteristic
  $DP_{2.4}$: noise elimination system
  + $DP_{2.4.1}$: lubricant, substrate loss modulus
  + $DP_{2.4.2}$: gap filler
  + $DP_{2.4.3}$: header stiffness
  $DP_{2.5}$: positive cabin pressure
  + $DP_{2.5.1}$: sealing energy as barrier to intrusion
  + $DP_{2.5.2}$: fan
DP_3: system for opening/closing door
  $DP_{3.1}$: detent mechanism
  + $DP_{3.1.1}$: stiffness & preloads of check link spring
  + $DP_{3.1.2}$: site, depth & climb of check link valleys
  $DP_{3.2}$: system to reduce occupant effort
  + $DP_{3.2.1}$: site, depth & descent of check link valleys
  + $DP_{3.2.2}$: hinge axes aligned with axis of rotation
  $DP_{3.3}$: system for counterbalancing opening/closing door
    $DP_{3.3.1}$: system to reduce effort to latch
      $DP_{3.3.1.1}$: KE threshold reducer
        + $DP_{3.3.1.1.1}$: up-down adjustable striker
        + $DP_{3.3.1.1.2}$: areas under weather-strip CLD
        + $DP_{3.3.1.1.3}$: area under header load-deflection curve
        + $DP_{3.3.1.1.4}$: pressure relief valve
    + $DP_{3.3.1.2}$: pre-loaded check link torsional spring
  + $DP_{3.3.2}$: mechanism to relieve reaction at latch

References

1. N.P. Suh, *The Principles of Design* (Oxford University Press, New York, NY, 1990)
2. G. Strang, *Linear Algebra and Its Applications*, 2nd edn. (Academic Press Inc., New York, NY, 1980)
3. H.L. Oh, Modeling variation to enhance quality in manufacturing. Presented at the Conference on *Uncertainty in Engineering Design*, NBS, 10–11 May 1988
4. H.L. Oh, T. Lee, R. Lipowski, A graph theory based method for functional decoupling of a design with complex interaction structure, ASME 2010 International Design Engineering Technical Conferences, DETC2010-28609