Chapter 8
Stabilization by Quantized Delayed State Feedback

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Abstract This chapter is devoted to the design of a static-state feedback controller for a linear system subject to saturated quantization and delay in the input. Due to quantization and saturation, we consider, for the closed-loop system, a weaker notion of stability, namely local ultimate boundedness. The closed-loop system is then modeled as a stable linear system subject to discontinuous perturbations. Then by coupling a certain Lyapunov–Krasovskii functional via S-procedure to adequate sector conditions, we derive sufficient conditions to ensure for the trajectories of the closed-loop system finite time convergence into a compact S_u surrounding the origin, from every initial condition belonging to a compact set S_0. Moreover, the size of the initial condition set S_0 and the ultimate set S_u are then optimized by solving a convex optimization problem over linear matrix inequality (LMI) constraints. Finally, an example extracted from the literature shows the effectiveness of the proposed methodology.

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8.1 Introduction

Networked control systems have been extensively studied due to their practical interest (see for instance [12]) and also because the introduction of communication networks in control loops poses new theoretical challenges. Indeed, the constraints induced by networks can seriously degrade the closed-loop system performances. Among all these constraints, two of them are particularly important: quantization and delay. For a complete overview of the issues arising in networked control systems see; [10, 12, 15].

Quantization is a phenomenon occurring in all data networks, where a real-valued signal is transformed into a piecewise constant signal. Whenever quantization is uniform, global asymptotic stability of the closed-loop cannot be established; see [6]. In particular, for large signals, the quantizer saturates and as a consequence, global stabilization cannot be obtained except for open-loop stable systems; see [17]. Furthermore, as uniform quantizers manifest a dead-zone around the zero, in general, local asymptotic stability for the origin cannot be guaranteed; see [2, 3]. Several methods have been proposed in the literature to deal with quantized systems. One first method to cope with quantized closed-loop systems consists of adopting a robust control point of view. Namely, the closed-loop system is modeled as a nominal system perturbed by a perturbation, i.e., the quantization error. Then, by using sector conditions and classical tools like small gain theorem or Lyapunov functions coupled with Input-to-State stability (ISS) properties [2, 3, 14] or S-procedure [1, 8], closed-loop stability can be assessed [10, 18].

Time delays naturally occur in networked control systems due to finite-time propagation of signals in communication media. Clearly, the presence of time delays can induce an additional degradation of the closed-loop performances; see [11]. In that case, the idea generally proposed is to extend classical results (without delay) to account time delay by considering either a Lyapunov–Razumikhin function [14] or a Lyapunov–Krasovskii functional [5, 9]. Using Lyapunov–Razumikhin function leads to very conservative results [11] but may deal with time-varying delays. Concerning Lyapunov–Krasovskii-based approach, the applicability of this methodology requires to properly define a Lyapunov–Krasovskii functional. Such a choice may entail some conservatism and some technical problems; see [16].

In this chapter, we focus on the stabilization problem for linear systems subject to constant time delay and saturated quantization in the input channel. The resulting closed-loop system is modeled as a linear time-delay system perturbed by two different nonlinearities allowing to describe more precisely the saturation and the quantization phenomena separately. Following the work of [19], the saturation function is embedded into a local sector condition, while the nonlinearity associated to the quantization error is encapsulated into a sector via the use of certain sector conditions. At this point, due to the presence of the uniform quantizer, (at least for unstable open-loop plants), it is impossible to prove asymptotic stability of the origin; see [18]. To overcome this problem, we focus on local ultimate boundedness for the closed-loop system. In particular, we design the controller gain K in a way such that

there exists a set (inner set) which is a finite time attractor for every initial condition belonging to a compact set (outer set) containing the previous one. Then, by relying on a Lyapunov–Krasovskii functional, we turn the solution to the considered control design problem into the solution to a quasi-LMI optimization problem. Finally, a procedure based on convex optimization is proposed to optimize the size of the outer and the inner sets.

Notation: The notation $P > \mathbf{0}$, for $P \in \mathbb{R}^{n \times n}$, means that P is symmetric and positive definite. The sets $\mathbb{S}_n$ and $\mathbb{S}_n^+$ represent, the set of symmetric and symmetric positive definite matrices of $\mathbb{R}^{n \times n}$, respectively. Given a vector x (a matrix A), $x'(A')$ denotes the transpose of $x(A)$. The symmetric matrix $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ stands for $\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$. The matrix $diag(A, B)$ stands for the diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Moreover, for any square matrix $A \in \mathbb{R}^{n \times n}$, we define $\text{He}(A) = A + A'$. The matrix $\mathbf{1}$ represents the identity matrix of appropriate dimension. The notation $\mathbf{0}_{n,m}$ stands for the matrix in $\mathbb{R}^{n \times m}$ whose entries are zero and, when no confusion is possible, the subscript will be omitted. For any function $x : [-h, +\infty) \to \mathbb{R}^n$, the notation $x_t(\theta)$ stands for $x(t + \theta)$, for all $t \geq 0$ and all $\theta \in [-h, 0]$. $|x|$ refers to the classical Euclidean norm and $\|x_t\|$ refers to the induced norm defined by $\|x_t\| = \sup_{\theta \in [-h,0]} |x(t + \theta)|$. For $P > 0$, $|x|_P$ refers to the modified Euclidean norm $|x|_P = \sqrt{x'Px}$. Its induced norm is then denoted $\|x_t\|_P$. For a generic positive ρ, we define the functional sets $L_V(\rho) = \{x \in \mathscr{C}^1 : V(x_t, \dot{x}_t) \leq \rho\}$ and $\text{int } L_V(\rho) = \{x \in \mathscr{C}^1 : V(x_t, \dot{x}_t) < \rho\}$. For a generic positive ρ, we define the sets $S(\rho) = \{x \in \mathbb{R}^n : \eta(|x|, 0) \leq \rho\}$ and $\text{int } S(\rho) = \{x \in \mathbb{R}^n : \eta(|x|, 0) < \rho\}$, where η is a class $\mathscr{K}$ function.

8.2   Problem Statement

Consider the following continuous-time system with delayed saturated quantized input:

$$\dot{x}(t) = Ax(t) + B \,\text{sat}(\text{q}(u(t - h)))$$
$$x(0) \in \mathbb{R}^n \tag{8.1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $x(0)$ the initial state, u the control input and $h \geq 0$ the constant input delay. A, B are real constant matrices of appropriate dimensions. The saturation map, $\text{sat}(u) : \mathbb{R}^m \to \mathbb{R}^m$ is classically defined from the symmetric saturation function having as level the positive vector $\bar{u} : \text{sign}(\bar{u}_{(i)}) \min \{\bar{u}_{(i)}, |\bar{u}_{(i)}|\}$. The quantization function, depicted in Fig. 8.1, is defined as

$$\text{q}: \begin{cases} \mathbb{R}^p & \to \mathbb{R}^p \\ u & \mapsto \Delta \text{sign}(u) \lfloor \frac{|u|}{\Delta} \rfloor. \end{cases} \tag{8.2}$$

Fig. 8.1 Quantization function
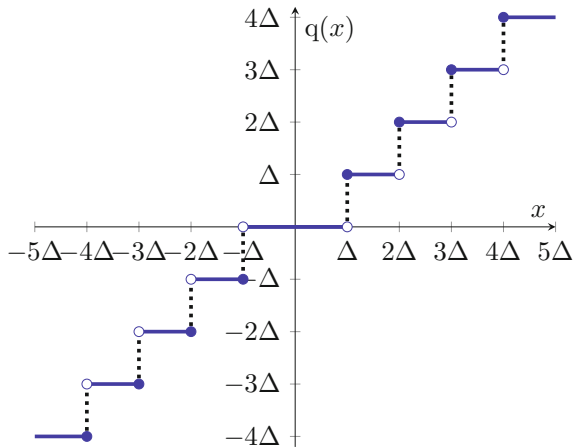
Assuming to fully measure the state x, we want to stabilize system (8.1) (in a sense to be specified) by a static-state feedback controller, that is

$$u = Kx.$$

Plugging the above controller yields the following dynamics for the closed-loop system

$$\dot{x}(t) = Ax(t) + B \operatorname{sat}(q(Kx(t-h))). \tag{8.3}$$

Now, due to the delayed input, the closed-loop system turns into a time-delay system, thus a suitable initial condition needs to be selected for the controller. Notice that, actually the infinite dimensionality is introduced in the closed-loop system by the controller. Thus, to make (8.3) an effective description of the closed-loop system, the initial condition of the delayed-system (8.3) needs to be chosen as

$$x(t) = x_0 = x(0) \ \forall t \in [-h, 0].$$

In particular this choice avoids jumps in the plant state at time $t = h$. Furthermore, note that x_0 is assumed to be a constant given vector of $\mathbb{R}^n$. Then, the closed-loop system can be described by the following functional differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B \operatorname{sat}(q(Kx(t-h))) \\ x(t) &= x_0 \ \forall t \in [-h, 0]. \end{aligned} \tag{8.4}$$

By defining

$$\Psi(y) = q(y) - y$$
$$\phi(y) = \text{sat}(y) - y,$$

system (8.3) can be rewritten equivalently as

$$\dot{x}(t) = Ax(t) + BKx(t - h) + B\Psi(Kx(t - h)) + B\phi(\Psi(Kx(t - h)) + Kx(t - h))$$
$$x(t) = x_0 \quad \forall t \in [-h, 0] \tag{8.5}$$

Due to quantization, whenever the open-loop system (8.1) is unstable, asymptotic stabilization of the origin cannot be achieved via any gain K; see [15]. To overcome this drawback, we rest on ultimate boundedness for the closed-loop system trajectories, whose definition is recalled below (see [13] for more details):

Definition 1 Let S_0 and S_u be two compact sets containing the origin, the solutions to (8.4) with x_0 starting from S_0 are ultimately bounded in the set S_u if there exists a time $T = T(|x_0|) \geq 0$ such that for every $t \geq T$

$$x_0 \in S_0 \Rightarrow x_t \in S_u.$$

Hence, the problem we solve can be then summarized as follows:

Problem 1 Let A, B real matrices of adequate dimension, determine $K \in \mathbb{R}^{m \times n}$, and two compact sets S_0 and S_u, with $0 \in S_u \subset S_0$, such that for every $x_0 \in S_0$, the resulting trajectory to (8.5) is ultimately bounded in S_u.

8.3   Main Results

To solve Problem 1, we propose a Lyapunov-based result, which establishes local ultimate boundedness of a time-delay system, provided that certain inequalities involving a Lyapunov–Krasovskii functional hold.

Theorem 1 *Given a functional $V : \mathscr{C} \times \mathscr{C} \to \mathbb{R}^+$, assume that there exist κ, η and ω three class $\mathscr{K}$ functions such that for every positive t*

$$\kappa(|x(t)|) \leq V(x_t, \dot{x}_t) \leq \eta(\|x_t\|, \|\dot{x}_t\|) \tag{8.6}$$

and there exist two positive scalars γ and β, with $\beta < \gamma$ such that for every positive t and for every $x_t \in L_V(\gamma) \setminus \text{int } L_V(\beta)$ one gets:

$$\dot{V}(x_t, \dot{x}_t) \leq -\omega(|x_t(0)|). \tag{8.7}$$

Then,

- *for every initial condition $x(0) \in S(\gamma)$ the solutions to system (8.3) are bounded,*
- *for every $x(0) \in S(\gamma)$ there exists a positive time T such that for every $t \geq T$,*

$$x(t) \in S(\kappa^{-1}(\beta)).$$

Proof First, notice that

$$V(\varphi_0(t), \dot{\varphi}_0(t)) = V(x_0, 0) \leq \eta(|x_0|, 0) \leq \gamma.$$

Thus, thanks to (8.7), for every positive t one has

$$V(x_t, \dot{x}_t) < V(x_0, \dot{x}_0).$$

Furthermore due to (8.6) one has

$$\kappa(|x(t)|) \leq V(x_t, \dot{x}_t) < V(x_0, 0) \leq \eta(|x_0|, 0) \leq \gamma$$

which implies that
$$|x(t)| \leq \kappa^{-1}(\gamma), \quad \forall t \geq 0$$

and then boundedness is proven.

To prove finite time convergence, denote $T = \inf\{t > 0 \colon x_t \in L_V(\beta)\}$, for every $t \in [0, T]$ we have $x_t \in L_V(\gamma)$, then by integration of (8.7) along the trajectories of (8.5) yields

$$V(x_T, \dot{x}_T) \leq - \int_0^T \omega(|x_\tau(0)|)d\tau + V(x_0, 0).$$

Now, observe that whenever $x(\tau) \in L_V(\gamma)$ due to (8.6) we have

$$\kappa(|x(\tau)|) \leq \gamma.$$

Thus, one gets
$$\omega(|x(\tau)|) \leq \omega(\kappa^{-1}(\gamma)), \forall \tau \in [0, T]$$

and then
$$V(x_T, \dot{x}_T) \leq -T\omega(\kappa^{-1}(\gamma)) + \gamma.$$

From (8.6), to require that $\kappa(|x(T)|) \leq \beta$ if suffices to impose that

$$-T\omega(\kappa^{-1}(\gamma)) + \gamma \leq \beta$$

which leads to

$$T \geq \frac{\gamma - \beta}{\omega(\kappa^{-1}(\gamma))}.$$

Moreover since (8.7) holds also on the boundary of $L_V(\beta)$, trajectories cannot leave such a set and for all $t \geq T$

$$\kappa(|x(t)|) \leq V(\varphi_t, \dot{\varphi}_t) \leq \beta$$

and this concludes the proof. ∎

Remark 1 In the previous result, any assumption on the considered norm was given. In particular, the norms involved in the above result may be chosen differently from each other. For example, one can assume relation (8.6) and (8.7) hold as follows

$$\kappa(|x(t)|_a) \leq V(x_t, \dot{x}_t) \leq \eta(\|x_t\|_b, \|\dot{x}_t\|_c) \tag{8.8}$$

$$\dot{V}(x_t, \dot{x}_t) \leq -\omega(|x_t(0)|_d) \tag{8.9}$$

where $|\cdot|_a$, $|\cdot|_b$, $|\cdot|_c$ and $|\cdot|_d$ are any vector norms and $\|\cdot\|_b$ and $\|\cdot\|_c$ the corresponding induced function norms. The proof of Theorem 1 is almost the same, except that combining the left-hand side of (8.8) and (8.9) one should deal with different norms, i.e. $|\cdot|_a$ and $|\cdot|_d$. However, since $|\cdot|_a$ and $|\cdot|_d$ are equivalent, there exists a positive scalar δ such that

$$|x_t(0)|_d \leq \delta|x_t(0)|_a,$$

allowing to obtain a similar result to Theorem 1.

Now, we present two lemmas that will be useful in the sequel:

Lemma 1 ([7]) *For every $u \in \mathbb{R}^m$ the following conditions hold:*

$$\Psi(u)'T_1\Psi(u) - \Delta^2 \text{trace}(T_1) \leq 0, \tag{8.10}$$

$$\Psi(u)' T_2 (\Psi(u) + u) \leq 0, \tag{8.11}$$

for any diagonal positive definite matrices $T_1, T_2 \in \mathbb{R}^{m \times m}$.

Lemma 2 ([19]) *Consider a matrix $G \in \mathbb{R}^{m \times m}$, the nonlinearity $\phi(u) = \text{sat}(u) - u$ satisfies*

$$\phi(u)'T_3(\text{sat}(u) + Gx) \leq 0, \tag{8.12}$$

for every diagonal positive definite matrix $T_3 \in \mathbb{R}^{m \times m}$, if $x \in \mathscr{S}(\bar{u})$ defined by

$$\mathscr{S}(\bar{u}) = \left\{ x \in \mathbb{R}^n : |G_{(i)}x| \leq \bar{u}, \forall i \in \{1, \ldots, m\} \right\}.$$

Based on the use of a Lyapunov–Krasovskii functional, the following theorem provides sufficient conditions to solve Problem 1.

Theorem 2 *If there exist a matrix $W \in \mathbb{R}^{n \times n}$, three symmetric positive define matrices $J, L, U \in \mathbb{R}^{n \times n}$, two matrices $Y, Z \in \mathbb{R}^{m \times n}$, three positive definite diagonal matrices $T_1, T_2, T_3 \in \mathbb{R}^{m \times m}$, and two positive scalars σ, σ_2 such that $\sigma_2 < \sigma$ and*

$$\begin{bmatrix} h^2 J - \text{He}(W) & L + AW' - W & BY - W & \mathbf{0} & B & B \\ \star & \sigma L + U - e^{-\sigma h} J + \text{He}(WA') & e^{-\sigma h} J + BY + WA' & \mathbf{0} & B & B - Z'T_3 \\ \star & \star & -e^{-\sigma h}(J + U) + \text{He}(BW) & \mathbf{0} & B - Y'T_2 & B - Y'T_3 \\ \star & \star & \star & (\Delta^2 \, \text{trace}(T_1) - \sigma_2)\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -T_1 - 2T_2 & -T_3 \\ \star & \star & \star & \star & \star & -2T_3 \end{bmatrix} < 0$$

(8.13)

$$\begin{bmatrix} L & Z_{(i)} \\ \star & \bar{u}_{(i)}^2 \end{bmatrix} \geq \mathbf{0} \quad i \in \{1, \ldots, m\} \tag{8.14}$$

then the gain $K = YW'^{-1}$, *the sets* $S_0 = L_v(1)$ *and* $S_u = L_v(\beta)$ *with* $\frac{\sigma_2}{\sigma} < \beta < 1$
are a solution to Problem 1, where $P = W^{-1}LW'^{-1}$ *and* $Q = W^{-1}UW'^{-1}$.

Proof Consider the following Krasovskii–Lyapunov functional

$$V(x_t, \dot{x}_t) = x(t)' P x(t) + \int_{t-h}^{t} e^{\sigma(s-t)} x(s)' Q x(s) ds + h \int_{-h}^{0} \int_{t+\theta}^{t} e^{\sigma(s-t)} \dot{x}(s)' R \dot{x}(s) ds d\theta \tag{8.15}$$

where P, Q, R are symmetric positive definite matrices. Notice that for $t > 0$ the functional satisfies

$$x(t)' P x(t) \leq V(x_t, \dot{x}_t) \leq x(t)' P x(t) + h \sup_{\tau \in [t-h,t]} x(\tau)' Q x(\tau) + \frac{h^3}{2} \sup_{\vartheta \in [t-h,t]} \dot{x}(\vartheta)' R \dot{x}(\vartheta). \tag{8.16}$$

and then

$$|x(t)|_P^2 \leq V(x_t, \dot{x}_t) \leq \|x_t\|_P^2 + \|x_t\|_{hQ}^2 + \frac{h^3}{2} \|\dot{x}_t\|_R^2. \tag{8.17}$$

Now, by computing the time-derivative along the solutions to system (8.5) of the above functional, one gets

$$\dot{V}(x_t, \dot{x}_t) = 2\dot{x}(t)' P x(t) + x(t)' Q x(t) - e^{-\sigma h} x(t-h)' Q x(t-h)$$
$$- \sigma \int_{t-h}^{t} e^{\sigma(s-t)} x(s)' Q x(s) ds - h\sigma \int_{-h}^{0} \int_{t+\theta}^{t} e^{\sigma(s-t)} \dot{x}(s)' R \dot{x}(s) ds d\theta$$
$$+ h^2 \dot{x}(t)' R \dot{x}(t) - h \int_{-h}^{0} e^{\sigma\theta} \dot{x}(t+\theta)' R \dot{x}(t+\theta) d\theta, \tag{8.18}$$

and

$$\dot{V}(x_t, \dot{x}_t) = 2\dot{x}(t)' P x(t) + x(t)' Q x(t) - e^{-\sigma h} x(t-h)' Q x(t-h)$$
$$- \sigma \int_{t-h}^{t} e^{\sigma(s-t)} x(s)' Q x(s) ds - h\sigma \int_{-h}^{0} \int_{t+\theta}^{t} e^{\sigma(s-t)} \dot{x}(s)' R \dot{x}(s) ds d\theta$$
$$+ h^2 \dot{x}(t)' R \dot{x}(t) - h e^{\sigma h} \int_{-h}^{0} \dot{x}(t+\theta)' R \dot{x}(t+\theta) d\theta. \tag{8.19}$$

Moreover, by Jensen's inequality, it follows

$$
\dot{V}(x_t, \dot{x}_t) \leq 2\dot{x}(t)' P x(t) + x(t)' Q x(t) - e^{-\sigma h} x(t-h)' Q x(t-h)
$$
$$
- \sigma \int_{t-h}^{t} e^{\sigma(s-t)} x(s)' Q x(s) ds + h \left(-\sigma \int_{-h}^{0} \int_{t+\theta}^{t} e^{\sigma(s-t)} \dot{x}(s)' R \dot{x}(s) ds d\theta \right)
$$
$$
+ h^2 \dot{x}(t)' R \dot{x}(t) - e^{-\sigma h} \left(x(t)' - x(t-h)' \right) R \left((x(t) - x(t-h)) \right).
$$
(8.20)

We want to show that if (8.13) and (8.14) hold, then there exists a positive small enough constant ε such that, along the solutions to system (8.5),

$$
\dot{V}(x_t, \dot{x}_t) \leq -\varepsilon x(t)' x(t),
$$
(8.21)

whenever $x_t \in L_V(1) \setminus \operatorname{int} L_V(\beta)$. To prove (8.21), by following S-procedure arguments, it suffices to prove that there exists a positive scalar θ, such that

$$
\mathscr{L}_1 = \dot{V}(x_t, \dot{x}_t) - \theta(\beta - V(x_t, \dot{x}_t)) \leq -\varepsilon x(t)' x(t).
$$

From (8.20), one can write

$$
\mathscr{L}_1 \leq \mathscr{L}_0 - \theta(\beta - V(x_t, \dot{x}_t)),
$$
(8.22)

where $\mathscr{L}_0$ denotes the right-hand side of (8.20). Define

$$
\zeta = \left[\dot{x}(t)' \ x(t)' \ x(t-h)' \ \mathbf{1}'_m \right]'.
$$
(8.23)

Then, inequality (8.22) reads:

$$
\mathscr{L}_1 \leq \zeta' M \zeta + (-\sigma + \theta) V(x_t, \dot{x}_t) + \sigma_2 - \theta\beta.
$$
(8.24)

where

$$
M = \begin{bmatrix} h^2 R & P & \mathbf{0} & \mathbf{0} \\ \star & \sigma P + Q - e^{-\sigma h} R & e^{-\sigma h} R & \mathbf{0} \\ \star & \star & -e^{-\sigma h}(R+Q) & \mathbf{0} \\ \star & \star & \star & -\frac{\sigma_2}{m}\mathbf{1} \end{bmatrix}.
$$
(8.25)

Moreover, by selecting $\theta = \sigma$ and $\beta = \frac{\sigma_2}{\sigma}$, from the above expression we obtain

$$
\mathscr{L}_1 \leq \zeta' M \zeta.
$$
(8.26)

Now define

$$
\begin{aligned}
\Omega = {}& \zeta' M \zeta - \Psi(Kx(t-h))' T_1 \Psi(Kx(t-h)) + \Delta^2 \operatorname{trace}(T_1) \\
& - 2\Psi(Kx(t-h))' T_2 \Psi(Kx(t-h)) - 2\Psi(Kx(t-h))' T_2 Kx(t-h) \\
& - 2\phi\big(\Psi(Kx(t-h)) + Kx(t-h)\big)' T_3 \Big(\phi\big(\Psi(Kx(t-h)) + Kx(t-h)\big) \\
& + \Psi(Kx(t-h)) + Kx(t-h) + Gx \Big).
\end{aligned}
\tag{8.27}
$$

The satisfaction of (8.14), with the change of variables $P = W^{-1} L W'^{-1}$ and $Z = Y W'^{-1}$ guarantees that the set $\mathscr{E}(P,1) = \{x \in \mathbb{R}^n : x'Px \le 1\}$ is contained in the polyhedral set $\mathscr{S}(\bar{u})$ defined in Lemma 2. Moreover, due to (8.16), it follows that if $x_t \in L_v(1)$ then $x(t) \in \mathscr{E}(P,1)$ and therefore also $x(t) \in \mathscr{S}(\bar{u})$, that guarantees the satisfaction of relation (8.12) in Lemma 2. Furthermore, note that from (8.15), one can write:

$$
V(x_0, \dot{x}_0) = x_0' P x_0 + \int_{-h}^0 e^{\sigma s} x_0' Q x_0 ds \le x_0'(P + hQ)x_0,
$$

since x_0 is supposed to be constant on $[-h, 0]$. Then, from (8.16), for any $x_0 \in \mathscr{E}(P + hQ, 1)$ then it follows that $x_0 \in Lv(1)$. Thus, by using Lemmas 1 and 2, for every $x_t \in Lv(1)$, one has

$$
\zeta' M \zeta \le \Omega = \xi' \mathscr{N} \xi
\tag{8.28}
$$

where

$$
\mathscr{N} =
\begin{bmatrix}
h^2 R & P & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\star & \sigma P + Q - e^{-\sigma h} R & e^{-\sigma h} R & \mathbf{0} & \mathbf{0} & -G' T_3 \\
\star & \star & -e^{-\sigma h}(R+Q) & \mathbf{0} & -K' T_2 & -K' T_3 \\
\star & \star & \star & \mathscr{N}_4 & \mathbf{0} & \mathbf{0} \\
\star & \star & \star & \star & -T_1 - 2T_2 & -T_3 \\
\star & \star & \star & \star & \star & -2T_3
\end{bmatrix},
$$

with $\mathscr{N}_4 = \frac{1}{m}(-\sigma + \Delta^2 \operatorname{trace}(T_1))\mathbf{1}$ and

$$
\xi' = \big[\zeta' \ \Psi(Kx(t-h))' \ \phi(\Psi(Kx(t-h)) + Kx(t-h))'\big].
$$

To prove our claim, we show that having (8.13) satisfied implies that (8.21) holds. To this end, in light of (8.24), (8.26), and (8.28), it suffices to show that along the trajectories of (8.5) one has

$$
\xi' \mathscr{N} \xi < -\varepsilon x' x.
$$

By Finsler lemma [18], the above inequality is equivalent to find a matrix X of appropriate dimensions, such that

$$
\mathscr{Q} = \mathscr{N} + X \bar{B} + \bar{B}' X' < \mathbf{0},
$$

where $\bar{B} = \begin{bmatrix} -1 & A & BK & 0 & B & B \end{bmatrix}$. Now, select X as follows

$$X = \begin{bmatrix} H' & H' & H' & 0 & 0 & 0 \end{bmatrix}'.$$

Then, by pre- and post-multiplying $\mathcal{Q}$, respectively, by $\mathrm{diag}\{H^{-1}, H^{-1}, H^{-1}, \sqrt{m}\mathbf{1}, \mathbf{1}, \mathbf{1}\}$ and $\mathrm{diag}\{H'^{-1}, H'^{-1}, H'^{-1}, \sqrt{m}\mathbf{1}, \mathbf{1}, \mathbf{1}\}$ with

$$\begin{array}{lll} H^{-1} = W, & WRW' = J, & WQW' = U, \\ WPW' = V, & KW' = Y, & GW' = Z, \end{array}$$

we obtain the left-hand side of (8.13). Then the satisfaction of (8.13) implies that the inequality

$$\xi' \mathcal{N} \xi < -\varepsilon x' x$$

holds along the trajectories of system (8.5). Thus, by invoking Theorem 1, thanks to Remark 1, setting

$$|\cdot|_a = |\cdot|_P, \qquad |\cdot|_b = \sqrt{|\cdot|_P^2 + |\cdot|_{hQ}^2}, \quad |\cdot|_c = |\cdot|_R, \qquad |\cdot|_d = |\cdot|_2,$$

and

$$\kappa(s) = s^2, \qquad \eta(s, y) = s^2 + \frac{h^3}{2}y^2, \quad \gamma = 1, \qquad \beta = \frac{\sigma_2}{\sigma},$$

establishes the result. ∎

Remark 2 The above result is based on the use of a classical Lyapunov functional. As pointed by [9], this result can be extended to the more realistic case of time-varying delay by considering a more complex functional and by following analogous arguments as the ones shown in the proof of Theorem 2.

8.4 Optimization Issues

In solving Problem 1, the implicit objective is to obtain a set S_0 as large as possible and a set S_u as small as possible. The problem of maximizing the size of S_0 and minimizing the size of S_u relies on the choice of a good measure of such sets. At this stage, it is interesting to remark that as mentioned in the proof of Theorem 2, if $x(0) \in \mathcal{E}(P + hQ, 1)$, then $x_0 \in S_0 = L_v(1)$. This implies that we can use the ellipsoid $\mathcal{E}(P + hQ, 1)$ to implicitly maximize S_0. Hence, when considering ellipsoidal sets, several measures and therefore associated criteria can be considered. Volume, minor axis, directions of interest, inclusion of a given shape set defined

through the extreme points v_r , $r = 1, \ldots, n_r$ (see for example [1]) are some of the tools usually adopted to indirectly optimize the size of such sets. The natural choice to maximize the size of S_0 in an homogeneous way, consists in minimizing the trace of the matrix $P + hQ$.

By the same way, if $x_t \in Lv(\beta)$ then from (8.16), $x(t) \in \mathcal{E}(P, \beta)$. Then, to minimize the size of S_u, one can simultaneously minimize β and maximize the trace of P. Therefore the minimization of $\mathrm{trace}(P + hQ)$ clashes with the minimization of the set S_u. Thus a trade-off between the two objectives needs to be considered. Moreover, the matrices P and Q do not directly appear in (8.13) and (8.14), thus the trace minimization problem needs to be rewritten in the decision variables. To this end, it is worthwhile to notice that the minimization of $\mathrm{trace}(P + hQ)$, due to positive definiteness of $P + hQ$, can be implicitly performed by maximizing the $\mathrm{trace}(P + hQ)^{-1}$. In particular, since

$$Q = W^{-1}UW'^{-1}, \quad P = W^{-1}LW'^{-1},$$

it turns out that $(P + hQ)^{-1} = W'(hU + L)^{-1}W$. Then, thanks to [4], we get

$$W'(hU + L)^{-1}W \geq W + W' - (hU + L).$$

Thus, let δ_1 and δ_2 be two tuning parameters, we can consider the following criterion

$$\text{minimize } \delta_1\sigma_2 - \delta_2\,\mathrm{trace}(2W - hU - L).$$

At this point, it is important to note that conditions provided by Theorem 2 are nonlinear in the decision variables, which prevents from solving directly a convex optimization problem. This is more specifically a problem in case of products of decision matrices. On the other hand, products of a decision matrix with a scalar are numerically tractable if the scalar is considered either as a tuning parameter (but there is no guarantee that the problem remains feasible) or fixed via an iterative search. Next, we consider the following additional constraints in the decision variables:

- $T_2 = \tau_2\mathbf{1}$,
- $T_3 = \tau_3\mathbf{1}$.

Considering σ, τ_2, τ_3 as tuning parameters. Problem 1 can be solved by solving the following LMI optimization problem:

$$\underset{L,J,U,W,\sigma_2,T_1}{\text{minimize}} \quad \delta_1\sigma_2 - \delta_2\,\mathrm{trace}(2W - hU - L)$$

subject to

$$(8.13), (8.14), \sigma_2 < \sigma, L > \mathbf{0}, J > \mathbf{0}, U > \mathbf{0}, T_1 > \mathbf{0}.$$

8.5   Example

Consider the system described by [9]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{sat}(u(t-h)).$$

The time delay is fixed to $h = 0.2$ and the level of saturation is $\bar{u} = 5$. We choose the quantization error bound as $\Delta = 1$. Notice that, since the open-loop system is unstable ($spec(A) = \{-0.5, \ 1\}$), the closed-loop trajectories cannot converge to the origin due to the quantizer. Using iterative research for scalars σ, τ_2, τ_3, we obtain the gain

$$K = \begin{bmatrix} -1.2773 & -2.5541 \end{bmatrix}.$$

For different initial conditions, several trajectories are depicted in Fig. 8.2. The ellipsoid $\mathscr{E}(P + hQ, 1)$ and $\mathscr{E}(P, \beta)$ are depicted in blue and dashed blue, respectively. The two red crosses are the two unstable equilibrium points due to the saturation. Furthermore, notice that a trajectory starting from an initial point outside the ellipsoid $\mathscr{E}(P, 1)$ and necessary such that x_0 is outside $L_v(1)$ still converges into $\mathscr{E}(P, \beta)$ which indicates clearly the conservatism of our technique. For $x_0 = [-7.7074 \ 6.5731]'$, Fig. 8.3 shows the evolution of the control u. Its value saturates for less than half a second and does not converge to zero.
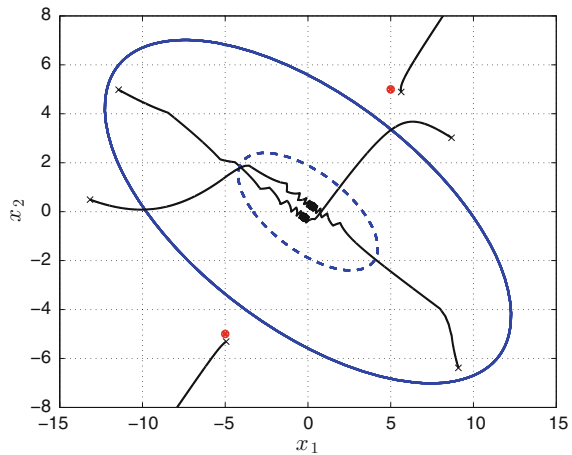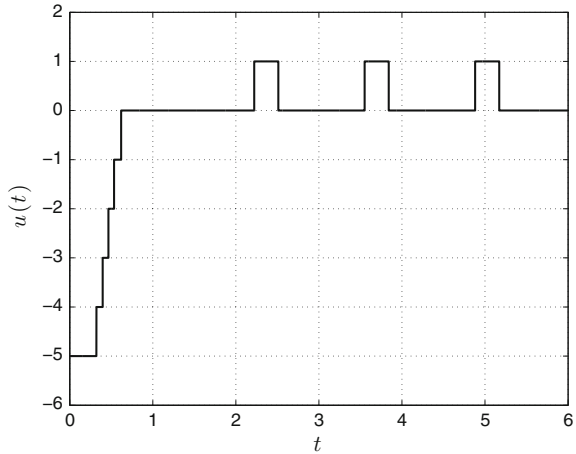

Fig. 8.2 Trajectories of the closed-loop system

Fig. 8.3 Evolution of u for $x_0 = [-7.7074, 6.5731]'$

8.6 Conclusion

In this chapter, we tackled the stabilization problem of linear systems with saturated, quantized, and delayed input. Specifically, the proposed methodology allows to design the controller to achieve local ultimate boundedness for the closed-loop system. Moreover, the control design problem is turned into an optimization problem aiming at the optimization of the size of the outer set S_0 and of the inner set S_u. The solution to such an optimization problem can be performed in a convex optimization setup, providing a computer-oriented solution. The effectiveness of the proposed strategy is supported by a numerical example. Future works should be devoted to the use of more complex Lyapunov functionals depending on extra states and slack variables, as well as the extension to these results to the case of time-varying time delay.

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