P-Simple Points and General-Simple Deletion Rules

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Abstract. Reductions transform binary pictures only by changing some black points to white ones. Parallel reductions can alter a set of black points simultaneously, while a sequential reduction traverses the black points of a picture, and changes the actually visited single point if the considered deletion rule is satisfied. Two reductions are called equivalent if they produce the same result for each input picture. General-simple deletion rules yield pairs of equivalent topology-preserving parallel and sequential reductions in arbitrary binary pictures. This paper bridges P-simple points and general-simple deletion rules: we show that some deletion rules that delete P-simple points are general-simple, and each point deleted by a general-simple deletion rule is P-simple.

Keywords: Topology preservation · Equivalent reductions · P-simple points · General-simple deletion rules

1 Introduction

Let $\mathcal{V}$ be a *digital space* (e.g., $\mathbb{Z}^2$ or $\mathbb{Z}^3$), where the elements of $\mathcal{V}$ are called *point*s. A *digital binary picture* [12] on $\mathcal{V}$ is a mapping that assigns only two possible values, *black* and *white* to each point. A *reduction* [7] transforms a binary picture only by changing some black points to white ones, which is referred to as *deletion*. Reductions play a key role in some topological algorithms, e.g., *thinning* [7] or *reductive shrinking* [8].

*Parallel reduction*s can delete a set of black points simultaneously, while a *sequential reduction* traverses the black points of a picture, and considers the actually visited point for possible deletion at a time. These two strategies are illustrated in Algorithms 1 and 2. Note that Algorithm 2 (i.e., the sequential approach) can specify a unique mapping of pictures to pictures with the help of the input parameter Π (i.e., the order in which the points are selected by the **foreach** loop). For practical purposes we assume that all input pictures are *finite* (i.e., they contain finitely many black points).

Algorithms 1 and 2 classify the set of black points B in the input picture into two (disjoint) subsets: points in the *constraint set* C are not taken into consideration (i.e., each element in C is absolutely protected), and the *deletion rule* R associated with these algorithms is evaluated only for the elements of

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Algorithm 1. Parallel reduction

Input: set of black points $B \subseteq \mathcal{V}$, constraint set $C \subseteq B$, and deletion rule R
Output: set of black points $PB \subseteq B$
// specifying the set points X to be investigated
$X = B \setminus C$
// determining the set of deletable points D
$D = \{\ p \mid p \in X \ \textit{and}\ R(p, B, C) = \textbf{true}\ \}$
// deletion of the set of points D
$PB\ =\ B \setminus D$

Algorithm 2. Sequential reduction

Input: set of black points $B \subseteq \mathcal{V}$, constraint set $C \subseteq B$, deletion rule R, and permutation Π of points in $B \setminus C$
Output: set of black points $SB \subseteq B$
// specifying the set points X to be investigated
$X = B \setminus C$
$SB = B$
// traversal of X according to permutation Π
foreach $p \in X$ **do**
  if $R(p, SB, C) = \textbf{true}$ **then**
    // deletion of the single point p
    $SB\ = SB \setminus \{p\}$

the set $X = B \backslash C$. Note that thinning algorithms can delete some *border point*s [12], hence their constraint sets contains all *interior point*s [12] and some types of *endpoint*s (i.e., points that provide relevant geometrical information with respect to the shape of the objects) [7,16] or accumulated *isthmus*es (i.e., generalization of curve/surface interior points) [3].

An investigated black point p is *deletable* by the deletion rule R, if $R\ (\ p,\ A,\ C\ )\ =\ \textbf{true}$, where $A = B$ in parallel reductions (see Algorithm 1), and $A = SB \subseteq B$ in the sequential case (see Algorithm 2). Thus parallel reductions consider the initial set of black points B when the deletion rule is evaluated. On the contrary, the set of black points is dynamically altered when a sequential reduction is performed. We lay stress upon the fact that both the constraint set C and the set of points $X = B \backslash C$ are not modified by Algorithms 1 and 2.

Sequential reductions (with the same deletion rule) suffer from the drawback that different visiting orders (raster scans) of points in $X = B \backslash C$ may yield various results. A deletion rule R is called *order-independent* [21] if the sequential reductions with R produce the same output picture for all the possible visiting orders. By extending this concept: a sequential reduction is said to be *order-independent* if its deletion rule is order-independent.

The concept of *simple point* is fundamental in topological algorithms. A (black or white) point is called a simple point if its *alteration* (i.e., changing its 'color') *preserves topology* [12]. A reduction is said to be *topology-preserving* if the set of black points in the output picture can be obtained from the set of black points in the input picture by sequential deletions of simple points [12].

Unfortunately, simultaneous deletion of a (sub)set of simple points may alter the topology. That is why Bertrand has proposed the notion of *P-simple point* [1].

Two reductions are called *equivalent* if they produce the same result for each input picture [18]. With the help of the notion of *general-simple* deletion rule, the author showed that general-simple deletion rules yield pairs of equivalent and topology-preserving parallel and (order-independent) sequential reductions [18].

This paper bridges P-simple points and general-simple deletion rules. In Sect. 3, we prove that some deletion rules that delete P-simple points are general-simple, and Sect. 4 shows that each point deleted by a general-simple deletion rule is P-simple.

2 Basic Notions and Results

In this paper we consider the traditional paradigm of digital topology as reviewed by Kong and Rosenfeld [12].

Let $(k, \bar{k})$ be a pair of *adjacency* relations on *digital space* $\mathcal{V}$. A $(k, \bar{k})$ *digital picture* is a quadruple $(\mathcal{V}, k, \bar{k}, B)$, where each point in $B \subseteq \mathcal{V}$ is called a *black point*, each point in $\mathcal{V} \setminus B$ is said to be a *white point*, k-adjacency/connectivity is used for black points, and $\bar{k}$-adjacency/connectivity is assigned to white points.

A black point p is called an *interior point* if each point that is $\bar{k}$-adjacent to p is black. A black point is said to be a *border point* if it is not an interior point.

The concept of *simple* points is well established in digital topology: A point is called a *simple point* in a picture if its alteration *preserves topology* [12].

There are several useful characterizations of simple points in 2D [5,9–12], 3D [5,10,12,13,22], 4D [5,11], and higher dimensional [15] pictures. Since all examples presented in this paper assume 2D digital pictures sampled on $\mathbb{Z}^2$ (that is dual to the regular square grid), we recall the following characterization of simple points on $(8, 4)$ pictures.

Theorem 1. [9] *A (black or white) point $p \in \mathbb{Z}^2$ is simple in an $(8, 4)$ picture if and only if p is matched by at least one of the base templates depicted in Fig. 1 or their rotated and reflected versions.*

Fig. 1. Base matching templates for characterizing $(8, 4)$-simple (black or white) points. Notations: each black element matches a black point; each white element matches a white point; each element depicted in gray matches either a black or a white point; each central element marked a '$\star$' is coincident with the investigated point. Note that interior points are non-simple points.

In order to construct topology-preserving parallel reductions and provide a verification method, Bertrand has proposed the notion of *P-simple point* [1]. Let us consider a set of points $B \subseteq \mathcal{V}$ and a set of points $P \subseteq B$. Then a point $p \in P$ is P-simple in B if for each set of points $Q \subseteq P \backslash \{p\}$, point p is simple in $B \backslash Q$.

Bertrand showed that a parallel reduction that deletes only P-simple points is topology-preserving [1], and he gave a local characterization of P-simple points on $(26, 6)$ 3D pictures [2]. Note that Bertrand and Couprie linked *critical kernel*s to *minimal non-simple set*s and P-simple points [4].

Figure 2 classifies black points of an $(8, 4)$ picture into (non-simple) interior points, non-simple border points, P-simple points, and additional simple points (that are not P-simple points).

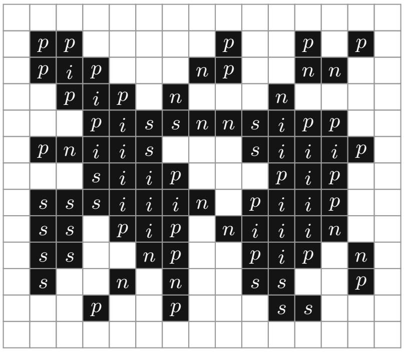

Fig. 2. Classifying black points in an $(8, 4)$ picture. Notations: (non-simple) interior points are marked 'i', non-simple border points are marked 'n', P-simple points are marked 'p', simple (border) points that are not P-simple points are marked 's'.

The author established a new sufficient condition for arbitrary pictures with the help of *general-simple* deletion rules [18]. Let us recall and rephrase the corresponding concepts and results.

Definition 1. *Deletion rule R (with respect to the constraint set $C \subseteq B$) is* general *if $R(p, B, C) =$ **true** for a point $p \in B$, then $R(q, B, C) = R(q, B \backslash \{p\}, C)$ for each point $q \in B \backslash \{p\}$.*

Definition 2. *Deletion rule R is* general-simple *if R is general and it deletes only simple points.*

Theorem 2. *A deletion rule R yields an order-independent sequential reduction if and only if R is general.*

Theorem 3. *General deletion rules specify pairs of equivalent parallel and order-independent sequential reductions.*

Theorem 4. *A parallel reduction is topology-preserving if its deletion rule is general-simple.*

Let us summarize Theorems 2–4: if a deletion rule deletes only simple points, and the 'deletability' of any point does not depend on the 'color' of any 'deletable' point, then that deletion rule specifies a pair of topology-preserving equivalent parallel and (order-independent) sequential reductions.

3 From *P*-Simple Points to General-Simple Deletion Rules

In this section a special deletion rule is constructed that deletes P-simple points, and we verify that our deletion rule is general-simple. Throughout this section, we assume that set B contains all black points in the input picture (see Algorithms 1 and 2).

Let us define constraint set $\mathcal{C}$ as

$$\mathcal{C} = \{\ p \mid p \text{ is not a simple point in } B\ \}. \tag{1}$$

Let Y be a set of black points in a picture such that $Y \subseteq B$ and $Y \supseteq \mathcal{C}$. Then consider the following set of points

$$\mathcal{P}_{\mathcal{C},Y} = \{\ p \mid p \in Y \text{ and } p \text{ is a simple point in } Y\ \}. \tag{2}$$

The deletion rule $\mathcal{PS}$ (with respect to the constraint set $\mathcal{C}$) is defined as

$$\mathcal{PS}\,(\,p,\ Y,\ \mathcal{C}\,) = \begin{cases} \textbf{true} & \text{if } p \text{ is } P\text{-simple in } Y \\ \textbf{false} & \text{otherwise} \end{cases}, \tag{3}$$

where $P = \mathcal{P}_{\mathcal{C},Y}$.

Since a P-simple point is necessarily a simple point, $\mathcal{PS}$ deletes only simple points. We show that deletion rule $\mathcal{PS}$ is general-simple.

Let us introduce a relation marked '$\Bumpeq$' between two sets of black points. We say that $B_1 \Bumpeq B_2$ if B_2 can be derived from B_1 by sequential alteration of simple points. It can readily be seen that '$\Bumpeq$' is an equivalence relation. We can write that $B \Bumpeq B\backslash\{p\}$ if $p \in B$ is a simple black point, and $B \Bumpeq B \cup \{p\}$ if $p \notin B$ is a simple white point.

Lemma 1. *Assume that* $\mathcal{PS}(p, B, \mathcal{C}) =$ ***true****. Then* $\mathcal{P}_{\mathcal{C},B\backslash\{p\}} = \mathcal{P}_{\mathcal{C},B}\backslash\{p\}$.

Proof.

It is obvious that $\mathcal{P}_{\mathcal{C},B\backslash\{p\}} \subseteq \mathcal{P}_{\mathcal{C},B}\backslash\{p\}$. Then we prove that $\mathcal{P}_{\mathcal{C},B} \subseteq \mathcal{P}_{\mathcal{C},B\backslash\{p\}} \cup \{p\}$.

Let q be a point, such that $q \in \mathcal{P}_{\mathcal{C},B}\backslash\{p\}$. In consequence, q is a simple point in B. Since $\mathcal{PS}(p, B, \mathcal{C}) = \textbf{true}$ for a point $p \in B\backslash\mathcal{C}$, p is a simple point in B. As point p is $\mathcal{P}_{\mathcal{C},B}$-simple in B, p is a simple point in $B\backslash\{q\}$. Hence we can write

$$\begin{aligned} B &\Bumpeq B\backslash\{q\}, \\ B &\Bumpeq B\backslash\{p\}, \\ B\backslash\{q\} &\Bumpeq (B\backslash\{q\}) \setminus \{p\}. \end{aligned} \tag{4}$$

Since '$\simeq$' is an equivalence relation,

$$B\backslash\{p\} \;\simeq\; B \;\simeq\; B\backslash\{q\} \;\simeq\; (B\backslash\{q\}) \setminus \{p\} = (B\backslash\{p\})\backslash\{q\}. \tag{5}$$

Consequently, $B\backslash\{p\} \;\simeq\; (B\backslash\{p\})\backslash\{q\}$, thus q is a simple point in $B\backslash\{p\}$. Hence $q \in \mathcal{P}_{\mathcal{C},B\backslash\{p\}} \cup \{p\}$. ☐

Lemma 1 states that deletion of a point by $\mathcal{PS}$ does not alter the simpleness of the (black) simple points that are not in the constraint set $\mathcal{C}$. Thus we can state that

$$\mathcal{P}_{\mathcal{C},B} = B\backslash\mathcal{C}, \tag{6}$$

and since $\mathcal{PS}(p, B, \mathcal{C}) = \mathbf{true}$, then

$$\mathcal{P}_{\mathcal{C},B\backslash\{p\}} = (B\backslash\{p\}) \setminus \mathcal{C}. \tag{7}$$

We are now ready to state the key theorem.

Theorem 5. *Deletion rule $\mathcal{PS}$ (with respect to constraint set $\mathcal{C}$) is general.*

Proof.
It is to be proved that if $\mathcal{PS}(p, B, \mathcal{C}) = \mathbf{true}$, then $\mathcal{PS}(q, B, \mathcal{C}) = \mathcal{PS}(q, B\backslash\{p\}, \mathcal{C})$ for any two points $p \in B$ and $q \in B\backslash\{p\}$.

Since $\mathcal{PS}(p, B, \mathcal{C}) = \mathbf{true}$, $p \in \mathcal{P}_{\mathcal{C},B}$ (see Eqs. 1–3, 6). If $q \in \mathcal{C}$, then $\mathcal{PS}(q, B, \mathcal{C}) = \mathcal{PS}(q, B\backslash\{p\}, \mathcal{C}) = \mathbf{false}$ (see Eq. 3). Hence, by Eqs. 6 and 7, it is sufficient to consider a point q such that $q \in \mathcal{P}_{\mathcal{C},B\backslash\{p\}}$.

– First we prove that if $\mathcal{PS}(q, B, \mathcal{C}) = \mathbf{true}$, then $\mathcal{PS}(q, B\backslash\{p\}, \mathcal{C}) = \mathbf{true}$.
  We give an indirect proof. Let us assume that $\mathcal{PS}(q, B, \mathcal{C}) = \mathbf{true}$ and $\mathcal{PS}(q, B\backslash\{p\}, \mathcal{C}) = \mathbf{false}$ for a point $q \in \mathcal{P}_{\mathcal{C},B}$. Thus there is a set $Q \subseteq \mathcal{P}_{\mathcal{C},B\backslash\{p\}} \setminus \{q\}$ such that q is not a simple point in $(B\backslash\{p\})\backslash Q = B\backslash(Q \cup \{p\})$. Hence we can write

$$B\backslash(Q \cup \{p\}) \;\not\simeq\; (B\backslash(Q \cup \{p\}))\backslash\{q\}. \tag{8}$$

  Since, by Lemma 1, $Q \cup \{p\} \subseteq \mathcal{P}_{\mathcal{C},B}\backslash\{q\}$ holds, and $\mathcal{PS}(q, B, \mathcal{C}) = \mathbf{true}$, q is simple in $B\backslash(Q \cup \{p\})$. Hence

$$B\backslash(Q \cup \{p\}) \;\simeq\; (B\backslash(Q \cup \{p\}))\backslash\{q\}. \tag{9}$$

  Thus Eq. 9 contradicts Eq. 8.
– Then we prove that if $\mathcal{PS}(q, B\backslash\{p\}, \mathcal{C}) = \mathbf{true}$, then $\mathcal{PS}(q, B, \mathcal{C}) = \mathbf{true}$.
  Proving indirectly, let us assume that $\mathcal{PS}(q, B\backslash\{p\}, \mathcal{C}) = \mathbf{true}$ and $\mathcal{PS}(q, B, \mathcal{C}) = \mathbf{false}$. Consequently, there is a set $Q \subseteq \mathcal{P}_{\mathcal{C},B}\backslash\{q\}$ such that q is not a simple point in $B\backslash Q$. Thus

$$B\backslash Q \;\not\simeq\; (B\backslash Q)\backslash\{q\}. \tag{10}$$

There are two cases to be investigated:

- $p \in Q$:

  Consider the set $Q' = Q \backslash \{p\}$. Then by Lemma 1, $Q' \subseteq \mathcal{P}_{\mathcal{C},B\backslash\{p\}}$ holds. Since $\mathcal{PS}(q, B\backslash\{p\}, \mathcal{C}) = \textbf{true}$, point q is simple in $(B\backslash\{p\})\backslash Q'$. Thus

  $$(B\backslash\{p\})\backslash Q' \;\; \risingdotseq \;\; ((B\backslash\{p\})\backslash Q')\backslash\{q\}. \tag{11}$$

  Since $Q = Q' \cup \{p\}$, Eq. 11 can be rewritten as

  $$B\backslash Q \;\; \risingdotseq \;\; (B\backslash Q)\backslash\{q\}. \tag{12}$$

  Thus Eq. 12 contradicts Eq. 10.

- $p \notin Q$:

  In this case $Q \subseteq \mathcal{P}_{\mathcal{C},B}$ and $(Q \cup \{q\}) \subseteq \mathcal{P}_{\mathcal{C},B}$.

  Since we assumed that $\mathcal{PS}(p, B, \mathcal{C}) = \textbf{true}$, point p is simple in sets $B\backslash Q$ and $B\backslash(Q \cup \{q\})$. Hence the following two equations hold

  $$B\backslash Q \;\; \risingdotseq \;\; (B\backslash Q)\backslash\{p\} \;\; = \;\; (B\backslash\{p\})\backslash Q, \tag{13}$$

  $$(B\backslash Q)\backslash\{q\} \;\; = \;\; B\backslash(Q \cup \{q\}) \;\; \risingdotseq \;\; B\backslash(Q \cup \{q\})\backslash\{p\} \;\; = \;\; B\backslash Q\backslash\{p, q\}. \tag{14}$$

  Since, by Lemma 1, $Q \subseteq \mathcal{P}_{\mathcal{C},B\backslash\{p\}}\backslash\{q\}$ holds, and $\mathcal{PS}(q, B\backslash\{p\}, \mathcal{C}) = \textbf{true}$, point q is simple in $(B\backslash\{p\})\backslash Q$. Thus

  $$(B\backslash\{p\})\backslash Q \;\; \risingdotseq \;\; ((B\backslash\{p\})\backslash Q)\backslash\{q\} \;\; = \;\; B\backslash Q\backslash\{p, q\}. \tag{15}$$

  Since '$\risingdotseq$' is an equivalence relation, by Eqs. 13–15, we can write

  $$B\backslash Q \;\; \risingdotseq \;\; (B\backslash\{p\})\backslash Q \;\; \risingdotseq \;\; B\backslash Q\backslash\{p, q\} \;\; \risingdotseq \;\; (B\backslash Q)\backslash\{q\}. \tag{16}$$

  Thus Eq. 16 contradicts Eq. 10.

We proved that if $\mathcal{PS}(p, B, \mathcal{C}) = \textbf{true}$, then $\mathcal{PS}(q, B, \mathcal{C}) = \mathcal{PS}(q, B\backslash\{p\}, \mathcal{C})$ for any two points $p \in B$ and $q \in B\backslash\{p\}$. $\square$

As deletion rule $\mathcal{PS}$ is general by Theorem 5, and it deletes only simple points, $\mathcal{PS}$ is general-simple (see Definition 2). Thus the following theorem is an easy consequence of Theorems 2–5.

Theorem 6. *The followings hold for deletion rule $\mathcal{PS}$ (that deletes P-simple points, and considers constraint set $\mathcal{C}$):*

1. *The sequential reduction (see Algorithm 2) with deletion rule $\mathcal{PS}$ is order-independent and topology-preserving.*
2. *The sequential and parallel reductions specified by $\mathcal{PS}$ are equivalent (i.e., they produce the same result for each input picture).*

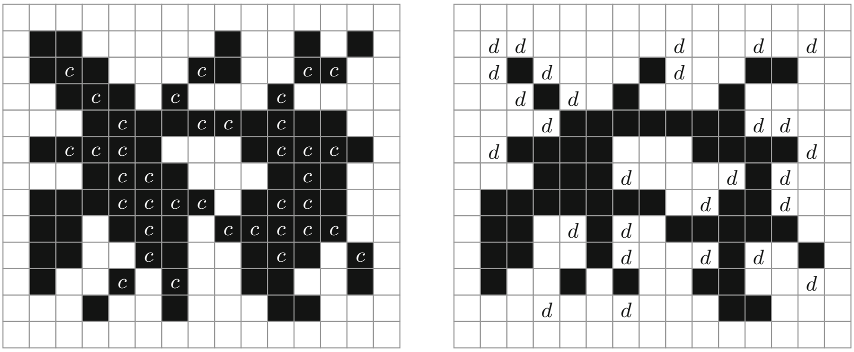

Fig. 3. Example of the pair of equivalent topology-preserving parallel and (order-independent) sequential reductions that use deletion rule $\mathcal{PS}$ with respect to constraint set $\mathcal{C}$. Elements in the constraint set (i.e., non-simple points) are marked 'c' in the input $(8,4)$ picture (left), and deleted points are marked 'd' in the output picture (right).

3. *The parallel reduction (see Algorithm 1) with deletion rule $\mathcal{PS}$ is topology-preserving.*

It can readily be seen that if we combine deletion rule $\mathcal{PS}$ with an arbitrary constraint set $C \supset \mathcal{C}$, then Lemma 1, Theorems 5 and 6 hold. Hence we can get various pairs of equivalent and topology-preserving parallel and (order-independent) sequential reductions by considering different geometrical constraints (e.g., characterizations of endpoints [7,16] and types of isthmuses [3]).

Figure 3 gives an example of a pair of equivalent topology-preserving parallel and order-independent sequential reductions with deletion rule $\mathcal{PS}$.

4 From General-Simple Deletion Rules to *P*-Simple Points

In this section we show that each general-simple deletion rule deletes P-simple points.

Theorem 7. *If a deletion rule R (with respect to a constraint set $C \subseteq B$) is general-simple, then each point that can be deleted by R is P-simple in B, where $P = \{ \; p \mid p \in B \backslash C$ and $R(p, B, C) = \boldsymbol{true} \; \}$.*

Proof. Let $p \in P$ be a point such that $R(p, B, C) = \textbf{true}$, and consider a set $Q \subseteq P \backslash \{p\}$. It is to be proved that point p is simple in $B \backslash Q$.

There are two cases to be investigated:

- $Q = \emptyset$: Since deletion rule R is general-simple, it deletes only simple points. Hence deletable point p is simple in $B = B \backslash \emptyset = B \backslash Q$.

– $Q \neq \emptyset$: The general-simple deletion rule R is general as well (see Definitions 1 and 2), hence it is order-independent by Theorem 2. It means that 'deletable' point p remains 'deletable' after deletion of some previously visited 'deletable' points. Thus $R(p, B\backslash Q, C) = \textbf{true}$.
  Since the general-simple deletion rule R deletes only simple points, point p is simple in $B \setminus Q$. □

Let us recall the 2D parallel thinning algorithm proposed by Manzanera et al. [14]. That algorithm falls into the category of *fully parallel thinning* [7] since it uses the same reduction in each thinning phase (i.e., iteration step). The deletion rule $\mathcal{M}$ of that algorithm was given by three classes of matching templates. The base templates α_1, α_2, and β are depicted in Fig. 4. All their rotated versions are templates as well, where the rotation angles are 90°, 180°, and 270°. All elements of α_1-type and α_2-type are *removing templates*, while β and its rotated versions are *preserving templates*. A black point is designated to be deleted if at least one removing template matches it, but it is not matched by any preserving template. The constraint set $\mathcal{I}$ comprises all interior points in the input picture of the actual iteration step.

In [17] the author proved that the deletion rule of the thinning algorithm proposed by Manzanera et al. [14] is general-simple. Thus that algorithm is topology-preserving, and it is equivalent to an order-independent sequential algorithm that uses the same deletion rule (see Theorems 2–4). By Theorem 7, we can state that the existing thinning algorithm proposed by Manzanera et al. [14] deletes P-simple points from $(8, 4)$ input pictures.

Figure 5 is to illustrate one iteration step of the thinning algorithm in question. It is easy to check that each deletable point by $\mathcal{M}$ is P-simple, where P is the set of deletable points.

Note that Palágyi, Németh, and Kardos proposed a pair of equivalent 4-subiteration 2D sequential and parallel thinning algorithms [19], and four pairs of equivalent 6-subiteration 3D sequential and parallel surface-thinning algorithms [20]. In addition, they showed in [20] that the deletion rule of the 3D parallel surface-thinning algorithm of Gong and Bertrand [6] is general-simple. Since the deletion rules of the 2D and 3D thinning algorithm mentioned above are all general simple, all of these algorithms delete P-simple points by Theorem 7.

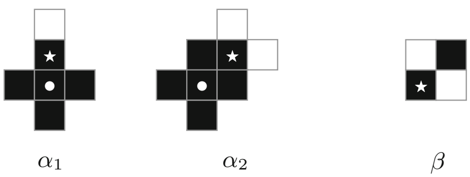

Fig. 4. The three base templates associated with the deletion rule $\mathcal{M}$. Notations: each black element matches a black point; each white element matches a white point; black elements marked '★' are the central positions of the templates; black elements with white bullets match interior points (i.e., elements in the constraint set $\mathcal{I}$).

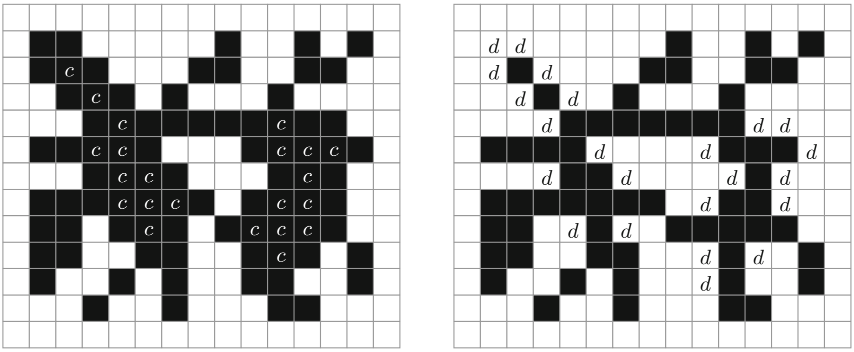

Fig. 5. Example of the pair of equivalent topology-preserving parallel and (order-independent) sequential reductions that use deletion rule $\mathcal{M}$ with respect to constraint set $\mathcal{I}$. These reductions are associated to one iteration step of the fully parallel thinning algorithm proposed by Manzanera et al. [14]. Elements in the constraint set $\mathcal{I}$ (i.e., interior points) are marked 'c' in the input $(8, 4)$ picture (left), and deleted points by $\mathcal{M}$ are marked 'd' in the output picture (right).

5 Conclusions

This work bridges P-simple points and general-simple deletion rules that specify pairs of equivalent topology-preserving parallel and (order-independent) sequential reductions. On the one hand, we showed that deletion rules that delete P-simple points (with respect to constraint sets containing all non-simple points) are general-simple. Hence parallel reductions with these deletion rules are equivalent to topology-preserving and order-independent sequential reductions. On the other hand, we proved that each point deleted by a general-simple deletion rule is P-simple. Note that the results presented in this work are all valid for digital pictures in arbitrary dimensions.

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