V.I. Ferronsky

Gravitation, Inertia and Weightlessness

Centrifugal and Gyroscopic Effects of the *n*-Body System's Interaction Energy

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V.I. Ferronsky
Water Problems Institute of the Russian
Academy of Sciences
Moscow
Russia

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To the Memory of
Sergey Ferronsky

and

Sergey Denisik
Colleagues and co-Authors on the Jacobi
Dynamics Study

Preface

Hundreds of works related to the physical meaning and nature of gravitation have been written and their number increases. It means that the problem remains unsolved. Our studies on Jacobi dynamics and, in particular, the recently discovered orbital velocity law of the Solar System's bodies, have allowed the secret of that mysterious phenomenon to be disclosed. It appears that Newtonian gravitation and Galileo's inertia are the centrifugal effects of interaction energy of a self-gravitating n-body system and its potential field. A self-gravitating celestial body appears to be an excellent natural centrifuge that is rotated by the energy of interacting elementary particles. Dynamical effects of such a centrifuge are the centrifugal and centripetal forces which are taken as the gravity and inertia forces. In every day practice centrifuges are used for separation of the components of matter in gaseous, liquid and solid states with respect to their density (force of weight). In nature, the same forces separate the shells and elementary particles of bodies and their systems. They also provide expansion and creation of bodies and their systems. Fundamentals of Jacobi dynamics completely correspond to the conditions of natural centrifuges. The centrifuge is an excellent experimental model for the study of dynamical effects in solving the many-body problem. In this book, we demonstrate some of those studies.

It was shown in our earlier publications (Ferronsky et al. 1978, 1979a, b, c, 1981a, b, 1982, 1984, 1987, 1996, 2011; Ferronsky and Ferronsky, 2010, 2013) that the needs of farther development of fundamentals in physics and mechanics have appeared for interpretation of new experimental facts obtained by artificial satellites in cosmic space. In fact, it was found by analysis of artificial satellite orbits that the Earth and the Moon do not stay in hydrostatic equilibrium, a conclusion that has been accepted as a basic postulate in the existing theories of their motion and the inner structure. That result means that the applied model of the hydrostatic equilibrium for celestial bodies in a uniform force field is not proved by the observed effects of gravitational mass interaction. The theory of the Earth and other planets' configuration is also based on hydrostatics. In this case, because the sum of the inner forces and moments are equal to zero, the bodies are considered as solid objects and their rotation is accepted as inertial, which also is not proved by the observation.

A serious discrepancy was found in the motion of Earth and other planets related to the ratio between potential and kinetic energy. It is well known that the potential energy of the Earth exceeds almost by 300 times the kinetic energy presented by inertial rotation of the planets. The same and even higher ratio is valid for the other planets, the Sun and the Moon. But according to the virial theorem the potential energy should be twice as much of kinetic energy. It means that the Earth and other planets exist without kinetic energy. An idea has appeared that there is some latent form of motion of the particles constituting the bodies, which has not been taken into account and is not considered in the existing theories.

Taking into account the relationship between gravitational moments and the gravitational field of the Earth observed by study of artificial satellites of the Earth, we come back to the derivation of the virial theorem in classic mechanics. Replacing the vector forces and moments by their volumetric values, we obtained for an n-particle system, an understanding of the condition of its dynamical equilibrium in its own force field. The new generalized form of the virial theorem remains in the framework of Newtonian laws of motion but with periodic components expressed by the second derivative from the polar moment of inertia. Thus, for study of a body dynamics in its own force field, the condition of hydrostatic equilibrium by dynamic (periodically oscillating) equilibrium is replaced. In this case the planet's kinetic energy is reanimated by oscillating motion of the interacting particles. And the ratio between the potential and kinetic energy to the classic virial theorem condition has reverted. In addition, a new phenomenon of the nature of gravitation as a dynamical effect of innate energy of the interacting elementary particles appears. On the basis of the obtained results we found that gravitation, inertia and weightlessness have a common innate nature in the form of elementary particles that provide interaction energy, which determines all the dynamical processes in creation and decay of natural systems.

Astrophysical science has proven that the forms of motion of material particles and objects observed in nature are determined by interaction of their constituting elementary particles. More than 300 sub-nuclear particles have been discovered until now. But a strict definition of the term "elementary particle of matter" until now does not exist, because such a particle has not been identified experimentally or theoretically. W. Heisenberg (1966) in his work "Introduction to the unified theory of elementary particles" notes that according to his and other researcher's experimental data, at collision of two particles of high energy, multiple other elementary particles appear. But they do not necessarily appear to be smaller than the colliding particles. Moreover, it appears that the new particles are always born of the same type independently of the nature of the collision. And also, the excess of kinetic energy of the colliding particles is converted into the matter of the created particles. It follows from the observation that the different elementary particles produced can be considered as various forms of existing matter or energy. The size of the new particles remains the same.

Taking into account the above experimental results, the elementary particles in this work are understood as the sub-nuclear material particles, which form the basis for all varieties of objects of the material world.

The most valuable result of our studies is discovery of the new law of planets and their satellites orbiting in the Solar System (Ferronsky and Ferronsky 2013). In this discovery, the astrophysics' postulate about the relationship of motion of the natural objects with interaction of their elementary particles has been proved. The law demonstrates that all planets and satellites have been orbited by the first cosmic velocity of their protoparents. Namely, the planets move in orbits with the first cosmic velocity of the protosun, the radius of which was equal to the semi-major axis of the modern orbit of each planet. The satellites of each planet move with mean orbital velocity equal to the first cosmic velocity of the corresponding planet having radius equal to the semi-major axis of the modern orbit of each satellite. This law holds for all the small planets of the asteroid belt and for all the comets. Theoretically the law follows from solution of Jacobi's virial equation and proved by astronomical observations. It follows from the discovered law that the postulate accepted until now on gravitational attraction of two interacting bodies appears to be a speculation. In fact, the orbital motion is initiated by the outer gravitational field of the central parental body. And the direction of the orbiting is determined by Lenz's rule. Thus, the gravitational field of a celestial body is the centrifugal effect of the body's interacting elementary particles energy and the matter and its energy are the innate natural discrete-wave phenomena. On this basis, we conclude that gravitation and inertia are centrifugal and equal to its centripetal effects of the elementary particles interaction energy leading to redistribution of the particles energy and changes in the body's mode motion. All other dynamical processes should follow from that effect. A self-gravitating body is an excellent example of a natural centrifuge.

The permanently acting process of the elementary particles interaction determines the evolution of a natural body. According to the Archimedes' law, continuous destruction of mass particles and their shell separation with respect to density takes place at their interaction. The upper lighter shell, after its density enriches the state of the weightlessness (relative to the whole body), separates and starts the formation of the secondary body. That is the process of body decay. Its elementary particles collision and scattering are the modes of interaction. The frequency of the particle interaction is the measure of their energy. In the Newtonian theory, that process the straight linear motion with acceleration under outer force action is proposed.

The body shell weightlessness is determined by its state of dynamical equilibrium with the other part of the body. In other words, the weightlessness determines the equilibrium state of the energy pressure between outer gravitational fields of two bodies or two shells. Weightlessness is a consequence of the centrifugal effect of elementary particles interaction that appears at differentiation of a body matter with respect to density. In natural conditions, weightlessness determines the effect of decay of a natural system by its constituting parts or elements at the system expansion. At the system contraction the process of creation of natural objects starts by creation of mass particles, their aggregates, bodies and galaxies. The equilibrium of larger aggregates here reaches out, gathering interacting particles that have the same frequency of oscillation. This can happen during simultaneous collision of *n* particles. Reality of such a process is proved by observation in the galaxy sleeves with almost the same orbital velocities of the stars having different distances from the common center. Those observations appear to be direct evidence of existing large masses of matter which are called "dark matter" and "dark energy".

In this work, the problem of physical meaning of gravitation, inertia and weightlessness is discussed. On the basis of effects of the new law of the Solar System, bodies orbiting the origin and nature of the above phenomenon are considered. The problem of creation of mass particles and elements from the elementary particles of "dark matter" is analyzed. The basic physics of the Jacobi dynamics from the viewpoint of quantum gravitation and general field theory based on the many body problem solution is discussed.

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Contents

Chapter 1
Introduction: Phenomenon of Gravitation, Inertia and Weightlessness

Abstract A brief story and the physical meaning of forces of gravitation, inertia, weightlessness and reference systems are discussed in this chapter. The theories of gravitation put forth by Newton and Einstein are considered in some detail. This is because, in spite of the criticism and enormous number of alternative versions, the above two theories have remained up to now to be the basis for construction of physical and mathematical models in celestial mechanics, astrophysics, geophysics and global dynamics as a whole. We draw attention to the fact that all the theories are based on the hydrostatic equilibrium of motion. In this connection the majority of researchers dealing with dynamics of the Earth and the planets (Munk and MacDonald, Jeffreys and others) come to the unanimous conclusion that the theories based on hydrostatics do not give satisfactory results in comparison with observations. Some of them straightly say that the theories are incorrect. In any case, the fact of j initiated this saying on the question about the nature of gravitation that "I *frame no hypotheses*". In our case, on the basis of the results obtained by studying celestial body motion in the framework of Jacobi dynamics, we come to the conclusion that the point of gravitation determines the integral dynamical effect of elementary particles' interaction energy which is the innate discrete-wave substance. The problem of inertia forces is most difficult in mechanics because there are too many different classifications depending on accepted reference systems and previous solutions. At some unknown time, a fiction force was introduced as a mathematical base for the D'Alambert principle. Polygamy of the forces is a weak place in mechanics and in different gravitation theories. Newton proposed three main forces that are inertial, impressed and centripetal. The centripetal force has three more varieties like absolute, accelerative and motive. Euler and D'Alambert also posited a number of forces. For such a large number of forces, use of the corresponding mathematical apparatus has to be developed. The vector, tensor, spinor and matrix calculus were developed and the work in that field became continuous. Each of them represents a special scientific direction in mathematically complicating solutions of practical physical, astrophysical and geophysical problems. In scientific literature, the physical meaning of the term "weightlessness" is defined as a complicated state. In relevant encyclopaedias one can find that weightlessness is the state of a material body moving in a gravity field by gravity

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forces that do not initiate mutual pressure of the body's particles on each other. The weightlessness effect in cosmic space is compared with man's feelings in the free fall of an elevator. Unfortunately, such a definition of weightlessness contains neither the nature of the unique phenomenon, nor real physical understanding. It is stated in physics that matter in the world, from the elementary particles to the Universe and their force fields, is continuously moving. Absolute rest is impossible. The philosophers say that the motion is the mode of existence of matter and this law is realized by energy. The forms of motion are different in quantity and in quality, and that difference is a subject of scientific and practical interest for human activity. Explanation of the relationship between different forms of motion appears to be the key for understanding a picture of the world development in the framework of the energy conservation law. Gravitation is the most mysterious natural phenomenon in the face of which even science shirks. Modern astrophysics states that the regularities of elementary particles' interaction may open a basic way for understanding laws of motion in the nature. Understanding of those laws is the subject of scientific research. In our case, on the basis of the results obtained by studying celestial body motion in the framework of Jacobi dynamics, we come to the conclusion that the point of gravitation determines the integral dynamical effect of elementary particles' interaction energy which is the innate discrete-wave substance. Let us start our analysis of the existing approaches in studying gravitation with Newtonian gravitation.

1.1 Newton's Law of Universal Gravitation

Newton's law of universal gravitation is accepted as one of the fundamental laws of nature ("gravity" is "weight" in Latin). "The world is governed by gravitation", said Newton. Physically this is a philosophical outlook which ancient Greek philosophers started to think about. Kepler has marked in this connection "gravity is a mutual tendency of all bodies". However, only Newton succeeded in formulation of the three laws of motion. On a physical–mechanical basis, he has shown that between any two bodies in the world, the forces of mutual attraction act in accordance with the equation

$$F = \frac{Gm_1 m_2}{R^2}, \tag{1.1}$$

where G is the gravitation constant determined experimentally; m_1, m_2 are the bodies' masses; R is the distance between the bodies; F is the attraction force.

Passing over from the mass points to the volumetric particles, Newton's law of gravitation leads to his theory of potential, which in the framework of non-relativistic classical physics describes the phenomenon of gravitation. It follows from (1.1) that the masses with density distribution $\rho(r)$ form the force field as described by Poisson's equation:

$$\Delta\varphi = 4\pi\rho, \tag{1.2}$$

where φ is the field potential; Δ is the Laplacian operator.

The Newtonian field of the potential assumes a long-ranged interaction with an infinite velocity. In this field the gravitating body acquires acceleration:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\mathrm{grad}\varphi. \tag{1.3}$$

It means that all the bodies in the force field move with the same acceleration.

The Poisson equation is not disclosed in the structure and mechanism of gravitation. That is why many tens of versions of gravitation theories were written. The mechanism and carrier of attraction forces up to now have not been discovered and the force F in Eq. (1.1) is the force of weight, but not attraction. Nevertheless, following Newton's definition of dynamical effects of interacted bodies is called "attraction". His famous work "Mathematical principles of natural philosophy", published in 1686, starts with a definition of matter, the quantity of motion and action, the innate, impressed and centripetal forces. Let us recall Newton's original formulations of the more important principles which we cite and discuss later on in the book. For that purpose we quote from the English translation of Newton's *Principia*, made by Andrew Mott in 1929 (Newton 1934).

Book I. The Motion of Bodies.

Definition I *The quantity of matter is the measure of the same, arising from its density and bulk conjointly.*

Definition II *The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.*

Definition III *The vis insita, or innate force of matter, is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be rest, or moving uniformly forwards in a right line.*

Definition IV *An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line.*

Definition V *A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.*

Of this sort is gravity, by which bodies tend to center of the earth; magnetism, by which iron tends to the load stone; and that force, whatever it is, by which the planets are continually drawn aside from the rectilinear motion, which otherwise they would pursue, and made to revolve in curvilinear orbits. A stone, whiled about in a sling, endeavors to recede from the hand that turns it; and by that endeavor, distends the sling, and that with so much the greater velocity, and as soon as it is let go, flier away. That force which opposes itself to this endeavor, and by which the *sling continually draws back the stone towards the hand, and retains in its orbit, because it is directed to the hand as the centre of the orbit, I call the centripetal force. And the same thing is to be understood of all bodies, revolved in any orbit. They all endeavor to recede from the centers of their orbits; and were it not for the opposition of a contrary force which restrains them to, and detains them in their orbits, which I therefore call centripetal, world fly off in right lines, with uniform motion…*

The quantity of any centripetal force may be considered as of three kinds: absolute, accelerative, and motive.

Definition VI *The absolute quantity of a centripetal force is the measure of the same, proportional to the efficiency of the cause that propagates from the centre, through the spaces round about.*

Definition VII *The accelerating quantity of a centripetal force is the measure of the same, proportional to the velocity which it generates in a given time.*

Definition VIII *The motive quantity of a centripetal force is the measure of the same, proportional to the motion which it generates in a given time.*

These quantities of forces, we may, for the sake of brevity, call by the names of motive, accelerative, and absolute forces; and for the sake of distinction, consider them with respect to the bodies that tend to the centre of forces towards which they tend; that is to say, I refer the motive force to the body as an endeavor and propensity of the whole towards a centre, arising from the propensities of the several parts taking together; the accelerative force to the place of the body, as a certain power diffused from the centre to all places around to move the bodies that are in them; and the absolute force to the centre, as endued with some cause, without which those motive forces would not be propagated through the space round about; whether that cause be some central body (such as is the magnet in the centre of the magnetic force, or the earth in the centre of the gravity force), or anything else that does not yet appear. For I here design only to give a mathematical notion of those forces, without considering their physical cause and seats…

I likewise call attractions and impulses, in the same sense, accelerative and motive; and use the words attraction, impulse, or propensity of any sort towards a centre, promiscuously, and indifferently, one for another; considering those forces not physically, but mathematically: wherefore the rider is not to imagine that by those words I anywhere take upon me to define the kind, or the manner of any action, the causes or the physical reason thereof, or that I attribute forces, in a true and physical sense, to certain centers (which are only mathematical points); when at any time I happen to speak as attracting, or as endued with attractive powers.

Law I *Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*

Law II *The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

Law III *To every action there is always opposite and equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

Section XI. Motion of Bodies Tending to each other with Centripetal Forces. Before discussing the problem, Newton essentially notes that "*...I approach to state a theory about the motion of bodies tending to each other with centripetal forces, although to express that physically it should be called more correct as pressure. But we are dealing now with mathematics and in order to be understandable for mathematicians let us leave aside physical discussion and apply the force as its usual name*".

Proposition LVII. Theorem X *Two bodies attracting each other mutually similar figures about their common centre of gravity, and about other mutually.*

For the distance of the bodies from their common centre of gravity are inversely as the bodies, and therefore in a given ratio to each other; and hence, by composition of ratios, in given ratio the whole distance between the bodies. Now these distances are carried round their common extremity with uniform angular motion, because lying in the same right line they never change their inclination to each other. But right line that are in a given ratio to each other, and carried round their extremities with an uniform angular motion, describe upon planes, which either rest together with them, or are moved with any motion not angular, figures entirely similar round those extremities. Therefore the figures described by the revolution of those distance are similar.

Proposition LVIII. Theorem XI *If two bodies attract each other with forces of any kind, and revolve about the common centre of gravity: I say, that, by the same forces, there may be described round either body unmoved a figure similar and equal to the figures which the bodies so moving describe round each other.*

Let the bodies S and P (Fig. 1.1a) *revolve about their common centre of gravity C proceeding from S to T, and from P to Q.*

From the given point s (Fig. 1.1b) *let there be continually drawn sp and sq equal and parallel to SP and TQ; and the curve pqv, which the point p described by point p at its revolution will be equal and similar to the curves which are described in its revolution round the fixed point S, will be similar and equal to the curve which the bodies S and P describes about each other; and therefore, by Theor. XX, similar to the curves in curves ST and PQV which the same bodies describe about their common centre of gravity C; and that because the proportions of the lines SC, CP, SP or sp, to each other given.*

Case 1 *The common centre of gravity C (by Cor. IV of The Laws of Motion) is either at rest, or moves uniformly in a right line. Let us first suppose it at rest, and in s and p let there be placed two bodies, one immovable in s, the other movable in p, similar and equal to the bodies S and P. Then let the right lines PR and pr touch the curves PQ and pq in P and p, and produce CQ and sq to R and r. And because the figures CPRQ, sprq are similar, RQ will be to tq as CP to sp, and therefore in a*

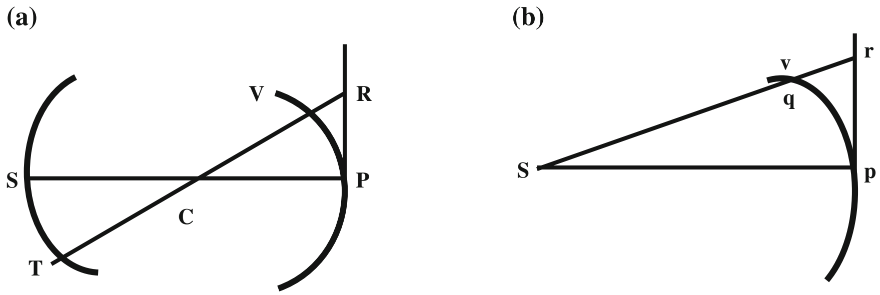

Fig. 1.1 The problem of two bodies mutually attracted

given ratio. Hence if the force with which the body P is attracted towards the body S, and by consequence towards the intermediate centre C, were to the force with which the body p is attracted towards the centre s, in the same given ratio, these forces would in equal times attract the bodies from the tangents PR, rq; and therefore this last force (tending to s) would make the body p revolve in the curve pqv, which would become similar to the curve PQV, in which the first force oblique the body P to revolve; and their revolutions would be completed in the same times. But because those forces are not to each other in the ratio of CP to sp, but (by reason of the similarity and equality of the distance SP, sp) mutually equal, the bodies in equal times will be equally drawn from the tangents; and therefore that the body p may be attracted through the grater interval rq, there is required a grater time, which will vary as the square root of the intervals; because, by Lem. X, the space described at the beginning of the motion are as the square of the times. Suppose, then, the velocity of the body p to be to the velocity of the body P as the square root of the ratio of the distance sp to distance cp, so that the arcs pq, PQ, which are in a similar proportion to each other, may be described in times that are as the square root to the distance; and the bodies P, p, always attracted by equal forces, will describe round the fixed centers C and s similar figures PQV, pqv, the latter of which pqv is similar and to be figure which the body P describes round the movable body S.

Case 2 *Suppose now that the common centre of gravity, together with the space in which the bodies are moved themselves proceeds uniformly in the right line; and (by Cor. VI of The Laws of Motion) all the motions in this space will be performed in the same manner as before; and therefore the bodies will describe about each other the same figures as before, which will be therefore similar and equal to the figure pqv.*

Corollary I *Hence two bodies attracting each other with forces proportional to the square of their distance, describe (by Prop. X), both round their common centre of gravity and round each other, conic sections having their focus in the centre about which the figures are described; and conversely, if such figures are described, the centripetal forces are inversely proportional to the square of the distance.*

Corollary II *And two bodies, whose focuses are inversely proportional to the square of their distance, describe (by Prop. XI, XII, XIII), both round their common centre of gravity, and round each other, conic sections having their focus in the centre about which the figures are described. And conversely, if such figures are described, the centripetal forces are inversely proportional to the square of distance.*

Corollary III *Any two bodies revolving round their common centre of gravity describe areas proportional to the time, by radii drawn both to the centre and to each other.*

Book II. The Motion of Bodies (*in resisting medium*).

Proposition XIX. Theorem XIV *All the parts of an homogeneous and uniform fluid in any unmoved vessel, and compressed on every side (setting aside the consideration of condensation, gravity, and all the centripetal forces), will be equally pressed on every side, and remain in their places without any motion arising from that pressure.*

Case 1 *Let a fluid be included in the spherical vessel ABC, and uniformly compressed on every side: I say, that no part of it will be moved by that pressure For it and part, other as D, be moved, all such parts at the same distance from the centre on every side must necessarily be moved at the same time by a like motion; because the pressure of them all in similar and equal; and all other motion is excluded that does not come all of them nearer to the centre, contrary to the supposition*

…

Proposition XXII. Theorem XVII *Let the density of any fluid be proportional to the compression, and its parts be attracted downwards by a gravitation inversely proportional to the square of the distances from the centre: I say, that if the distance be taken in harmonic progression, the densities of the fluid at those distances will be in a geometrical progression.*

Book Three. System of the World (*in mathematical treatment*).

Proposition II. Theorem II *That the forces by which the primary planets are continually drawn off from rectilinear motions, and retained in their orbits, tend to the sun; and are inversely as the squares of the distances of the places of those planets from the sun's centre.*

Proposition VII. Theorem VII *That there is a power of gravity pertaining to all bodies, proportional to the several quantities of matter which they contain.*

Proposition VIII. Theorem VIII *In two spheres gravitating each towards the other, if the matter in places an all sides round about and equidistant from the centers in similar, the weight of either sphere towards the other will be inversely as the square of the distance between their centers.*

Proposition IX. Theorem IX *That the force of gravity, considered downwards from the surface of the planets, decreases nearly in the proportion of the distances from the centre of the planets.*

If the matter of the planet were of an uniform density, this proportion would be accurate true. The error, therefore, can be no greater than what may arise from the inequality of the distance.

Proposition X. Theorem X *That the motions of the planets in the heavens may subsist an exceedingly long time.*

Hypothesis I *That the centre of the system of the world is immovable.*

Proposition XI. Theorem XI *That the common centre of gravity of the earth, the sun, and all the planets, is immovable.*

Proposition XII. Theorem XII *That the sun is agitated by a continual motion, but never recedes far from the common centre of gravity of all the planets.*

Based on the above proofs, Newton considers the other versions related to the two-body problem which have became basic principles for celestial and classic mechanics.

In Book III, Proposition XIX, Newton considers the problem of the Earth's oblateness as follows:

Proposition XIX. Theorem XIX *To find the proportion of the axis of a planet to the diameters perpendicular thereto.*

Our countryman, Mr. Norwood, measuring a distance of 905751 feet of London measure between London and York, in 1635, and observing the difference of latitudes to be 2°28′, determined the measure of one degree to be 367196 feet of London measure, that is, 57060 Paris toises. M.Picard, measuring an arc of one degree, and 22′55″ of the median between Amiens and Malvoisine, found an arc of one degree to be 57060 Paris toises. M.Cassini, the father, measured the distance upon the meridian from the town Collioure in Roussillon to the observatory of Paris; and his son added the distance from the Observatory to the Citadelo of Dunkirk. The whole distance was 486156½ toises and the difference of the latitudes of Collioure and Dunkirk was 8 degrees, and 31′11 $^5/_6$″. Hence an arc of one degree appears to be 57061 Paris toises. And from these measures arc conclude that the circumference of the earth is 123249600, and its semidiameter 19615800 Paris feet, upon the supposition that the earth is of a spherical figure.

Taking advantage of measurements that existed at that time, Newton calculated the ratio of the total gravitation force over the Paris latitude to the centrifugal force over the equator and found that the ratio is equal to 289:1. After that he imagined the Earth in the form of an ellipse of rotation with axis PQ and the channel *ACQqca* (Fig. 1.2).

If the channel is filled with water, then its weight in the branch *ACca* will be related to the water's weight in the branch *QCcq* as 289:288 because of the centrifugal force which decreases the water's weight in the last branch by the unit. He found by calculation that if the Earth has a uniform mass of the matter and has no

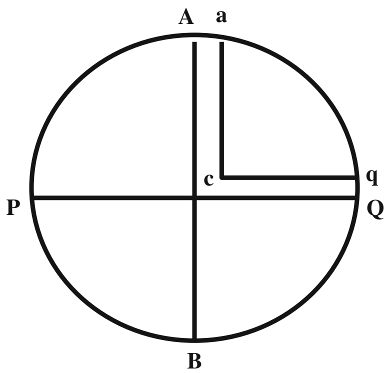

Fig. 1.2 Newton's problem of the Earth oblateness

motion, and the ratio of its axis *PQ* to the diameter AB is 100:101, then the gravity force of the Earth in point *Q* relates to the gravity force in the same point of the sphere with radius *CQ* or *CP* as 126:125. By the same argument the gravity in point *A* of a spheroid drawn by revolution around axis *AB* relates to the gravity in the same point of the sphere drawn from centre C with radius AC as 125:126. However, since there is one more perpendicular diameter, then this relation should be as $126{:}125^1/_2$. Having multiplied the above ratios, Newton found that the gravity force in point *Q* relates to the gravity force in point *A* as 501:500. Because of daily rotation, the liquid in the branches should be in equilibrium at a ratio of 505:501. So, the centrifugal force should be equal to 4/505 of the weight. In reality the centrifugal force composes 1/289. Thus, the excess in water height under the action of the centrifugal force in the branch *Acca* is equal to 1/289 of the height in branch *QCcq*.

After calculation by hydrostatic equilibrium in the channels, Newton obtained the ratio of the Earth's equatorial diameter to the polar diameter as 230:229, i.e. its oblateness is equal to (230–229)/230 = 1/230. This result, demonstrating that the Earth's equatorial area is higher than the polar region, was used by Newton for explanation of the observed slower swinging of pendulum clocks on the equator than on the higher latitudes.

At the end of Book III, after discussion of the Moon's motion, the tidal effects and the comets' motion, Newton concludes as follows.

Hitherto we have explained the phenomena of the heavens and our sea by the power of gravity, but have not yet assigned the cause of this power. This is certain, that it must proceed from a cause that penetrates to very centers of the sun, and planets, without suffering the least diminution of its force, that operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes used to do), but according to the quantity of the solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always as the inverse square of the distances. Gravitation towards the sun is made up out of the gravitations towards the several particles of which the body of the sun is composed; and in receding from the sun decreases accurately as *the inverse square of the distance as far as the orbit of Saturn, as evidently appears from the quiescence of the aphelion of the planets; nay even to the remotest aphelion of the comets, if those 4 aphelions are also quiescent.*

But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and of gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea.

And now we might add something concerning a certain most subtle spirit which pervades and lies in all gross bodies; by the force and action of which spirit the particles of bodies attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the solid filaments of the nerves, from the outward organs of sense to the brain, and from brain into the muscles. But these are things that cannot be explained in few wards, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic spirit operates.

Lagrange referred to Newton's work as "*the greatest creature of a human intellect*". It was published in England in Latin in 1686, 1713 and 1725 in his life-time and many times later on. We reiterate that the passages above are from the translation by Andrew Mott in 1729 that was printed in 1934.

As it follows from Newton's definition of the centripetal innate forces, his understanding of their meaning and action in the nature was very wide. The innate force of matter is the power of resistance. It can develop as the force of a body's resistance due to which it remains at rest or moves with constant velocity. It can develop as a body's resistance (reactive) force at outer effect and as a pressure when the body faces an obstacle. In modern mechanics this force is understood synonymously as the force of inertia. The resistance force or force of reaction has found its place in the theory of elasticity, and the pressure is used in hydrodynamics and aerodynamics.

The main meaning of the centripetal force which was introduced by Newton is that each body is attracted to a certain centre. He demonstrates this ability of bodies and objects on the Earth to attract to its geometric centre by action of the gravity force. Newton distinguishes three kinds of manifestation of the centripetal force, namely absolute, accelerating and moving. Absolute value of this force is a measure of the source power of its action from the centre to the outer space. The body's attraction to the centre and emission of attraction from the centre is demonstrated by

Newton in Book III "*The System of the World*", where in Theorem II he notes that gravity forces from the planets are directed to the Sun. In Theorem IX he says that attraction of the planets themselves goes from their surfaces to the centres. According to Newton's idea, the planet's surface is somewhat an area of formation of absolute value of the centripetal force from where it emits up and down.

The accelerating value of the centripetal force by Newton's definition is a measure proportional to velocity which it developed over a long time. The moving value of the centripetal force is a measure proportional to the moment, i.e. to the mass and velocity.

After such a wide spectrum of functions which Newton attributes to the centripetal force, it becomes clear why he was unable to understand its physical meaning and acknowledged: "*But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypotheses; and hypotheses, whether metaphysical or physical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea*".

It is worth noting that mathematicians, to whom Newton expounded the theory, because of complication in analytical operation with the forces, introduced to celestial mechanics and analytical dynamics the force function, i.e. energy with its ability to develop pressure. Doing so, they practically generalized the physical meaning of the force effects. As to the centripetal forces, later on in Sect. 2.2 of Chap. 2 we shall show that volumetric forces of mass–particle interaction in reality generate Newton's physical pressure which in formulation of practical problems is expressed by the energy. Once more note that Newton, as he expressed himself, instead of correct physical meaning of the concept "*pressure*" gave preference to the concept "*attraction*" to be more understandable to mathematicians.

Newton's problem about the mutual attraction of two bodies, which depict similar trajectories around their common centre of gravity and around each other, is based on the geometric solution of Kepler's problem formulated in his first two laws. Newton's solution is founded on his conception of the centripetal innate forces under which the bodies depict similar trajectories around their common centre of gravity and around each other. In celestial mechanics, developed on the basis of Newton's attraction law, the two-body problem is reduced to an analytical problem of one body, the motion of which takes place in the central field of the common mass. Both Newton's geometric theorem and analytical solution of celestial mechanics are based on the hydrostatic equilibrium state of averaged body motion being brought by Kepler's laws. That fact was well understood by Newton when presenting in detail the hydrostatics laws. However, in both cases the two-body problem was solved correctly in the framework of its formulation. The only difference is that, for Kepler, the planet motion occurs under the action of the

Sun forces, whereas Newton shows that this motion results from the mutual attraction of both the Sun and the planet (Ferronsky and Ferronsky 2010).

In Section V of Book II "*Density and Compression of Fluids: Hydrostatics*" Newton formulates the hydrostatics laws and on their basis in Book III "*The System of the World*" he considers the problem of the Earth's oblateness, applying real values of the measured distances between a number of points in Europe. Applying the found measurements and hydrostatic approach, he calculated the Earth's oblateness equal to 1/230, where in his consideration the centrifugal force plays the main contraction effect expanding the body along the equator. In fact the task is related to the creation of an ellipsoid of rotation from a sphere by action of the centrifugal force. Here Newton applied his idea that the attraction of the planet itself goes from the surface to its centre. In this case the total sum of the centripetal forces and the moments is equal to zero and rotation of the Earth should be inertial. It means that the planet's angular velocity has a constant value.

Inertial rotation of the Earth is accepted a priori. There is no evidence or other form of justification for this phenomenon. There are also no ideas relative to the mode of a planet's rotation, namely, whether it rotates as a rigid body or there is a differential rotation of separate shells. In modern courses of mechanics there is only analytical proof that, if a body occurs in the outer field of central forces, then the sum of its inner forces and torques is equal to zero. Thus, it follows that the Earth's rotation should have a mode of a rigid body and the velocity of rotation in time should be constant.

The proof of the conclusion, that if a body occurs in the field of the central forces then the sum of the inner forces and torques is equal to zero, and the moment of momentum has a constant value, is directly related to the Earth's dynamics (Kittel et al. 1965).

It is known from classical considerations that in the model of two interacting mass points reduced to the common mass centre, which Newton used for the solution of Kepler's problem having in mind the planets's motion around the Sun, the inner forces and torques being in the central force field are really equal to zero. The torque, which is a derivative with respect to time from the moment of momentum of material particles of the body, is determined here by the resultant of the outer forces and the planets' orbits in the central force field entering into the same plane. This conclusion follows from Kepler's laws of planetary motion.

Passing to the problem of Earth dynamics, Newton had no choice for the formulation of new conditions. The main conditions were determined already in the two-body problem where the planet appeared in the central force field of the reduced masses. The only difference that appears here is that the mass point has a finite dimension. The condition of zero equality of the inner forces and torques of the rotating planet should mean that the motion could result from the forces among which known were only the Galilean inertial forces. Such a choice followed from the inertial motion condition of two-body motion which he already applied. The second part of the problem related to reduction of the two bodies to their common centre of masses and to the accordingly appeared central force, has predetermined the choice of the equation of state. It became the hydrostatic equilibrium of the body's state being in the outer uniform central force field. The physical conception and mathematical expression of hydrostatic equilibrium of an object based on Archimedes' laws (3d century BP) and the Pascal law (1663) were well known in that time. This is the story of the sphere model with the equatorial and polar channels filled in by a uniform liquid mass in the state of hydrostatic equilibrium at inertial rotation.

In Newton's time the dynamics of the Earth in its direct meaning had not surfaced and it is absent up to now. The planet, rotating as an inertial body and deprived of its own inner forces and torques, has appeared as a dead-born creature. But up to now, the hydrostatic equilibrium condition, proposed by Newton, is the only theoretical concept of the planet's dynamics because it is based on the two-body problem solution which satisfies Kepler's laws and in practice plays the role of Hooke's law of elasticity.

In spite of the noted discrepancies, the problem of the Earth's oblateness was the first step towards the formulation and solution of the highly complicated planet's figure task on which theoretical and experimental study continues up to present time. As to the value of polar oblateness of the Earth, it appears to be much higher. Later observations and measurements show that relative flattening has a smaller value and Newton's solution was needed to have further development.

French mathematician and astronomer A. Clairaut continued Newton's solution of the figure problem of the Earth based on hydrostatics (Clairaut 1947). The degree measurements in the equatorial and northern regions done in the eighteenth century by French astronomers proved Newton's conclusion about the Earth's oblateness, which was at that time regarded with scepticism. However, the measured value of the relative flattening appeared to be different. In the equatorial zone it was equal to 1/314, and in the northern region—to 1/214 (Grushinsky 1976). Clairaut himself took part in the expeditions and found that Newton's results are not correct. It was also known to him that the Earth is not a uniform body. Therefore, he focused his strength on taking into account consideration of this effect. Clairaut's model represented an inertia rotating body filled in with liquid having a jumping density. By structure such a model was closer to the real Earth since it had a shell structure. But the hydrostatic equilibrium condition and the inertial rotation as the physical basis for the problem solution were out of hesitation and were taken as before. Clairaut introduced a number of assumptions to the problem formulation. In particular, since the velocity of inertial rotation and the value of the oblateness are small, the boundary areas of the shells and their equilibria were taken as ellipsoidal figures with a common axis of rotation. Clairaut's solution comprised obtaining a differential equation for the shell structured ellipsoid of rotation relative to geometric flattening of its main section.

Proposed by Newton and developed as a Clairaut model of the Earth in the form of a rotating by inertia spheroid filled in with a non-uniform liquid, the mass of which resides in hydrostatic equilibrium in the outer force field, it became generally accepted, commonly used and in principal has not changed up to now. Its purpose was to solve the problem of the planet's figure, i.e. the form of the planet's surface, and this goal in first approximation was reached. Moreover, having been obtained by a Clairaut equation on surface changes in the acceleration of the gravity force as a function of the Earth's latitude, it opened the way to experimental study of oblateness of the spheroid of rotation by means of measuring the outer gravity force field. Later on, in 1840 Stokes solved the direct and reversed task concerning surface gravity force for a rotating body and over its level applying the known parameters, namely, the mass, radius and angular velocity. The above parameters uniquely determined the gravity force at surface level, which is taken as the quiet ocean's surface, and in all outer space. By that task the relation between the Earth's figure and the gravity force was determined. In the middle of the last century, Molodensky (Molodensky and Kramer 1961) proposed the idea to consider the real surface of the Earth as a reduced surface and solved the corresponding boundary task. The doctrine of the spheroidal figure of the Earth has found common understanding and researchers, armed with theoretical knowledge, started to refine the dimensions and other details of the ellipsoid of rotation and to derive the corresponding corrections.

Earth dynamics was always of interest, not only to researchers of its configuration. Fundamentals of all the Earth, planetary and solar system sciences are defined first of all by the laws of motion of the Earth itself, where the confidence limit of the laws can be checked by observation. Moreover, all the sense of human life is connected with this planet. As far as the techniques and instruments for observation were developed, geodesists, astronomers and geophysicists have noticed that in the planet's inertial rotation, some irregularities and deviations relative to the accepted standard parameters and hydrostatic conditions have appeared. Those irregularities or inaccuracies, as they are often called, a number of which are counted by more than ten, were finally incorporated into two problems, namely, variation of the angular velocity in the daily, monthly, annually and secular time scale, and variation in the poles motion in the same time scales. Just after the problems became obvious and did not find resolution in the frame work of the accepted physical and theoretical conceptions of celestial mechanics, the latter has lost interest in the problems of Earth dynamics. In this connection the well-known German theoreticians in dynamics Klein and Sommerfeld stated that the Earth's mechanics appear to be more complicated than celestial mechanics and represent "*some confused labyrinths of geophysics*" (Klein and Sommerfeld 1903). The geophysicists themselves started to solve their own problems. They had no other way except to search for the causes of the observed inaccuracies. In order to study irregular velocity of the Earth's rotation and the pole motion, numerous projects of observation and regular monitoring were organized by the planetary network. As it was always in such cases, the cause of the observed effects was searched for in the effects of perturbations coming from the Moon and the Sun, and also in the influence of dynamical effects of their own shells like the atmosphere, the oceans and the liquid core, existence of which is justified by many researchers. In some works the absence of the hydrostatic equilibrium in distribution of the masses and strength in the planet's body is considered as the reason of irregular velocity of the Earth's rotation.

Many publications have been devoted to analysis of the observed inaccuracies in the Earth's rotation together with explanation of their possible causes, based on experimental data and theoretical solutions. The most popular review work in the twentieth century was the book of the known English geophysicist Harold Jeffreys "*The Earth: Its Origin, History and Physical Constitution*". The first publication of the book was in 1922 and later four more editions appeared, including the last one in 1970. Jeffreys was a great expert and direct participant of development of the most important geophysical activities. The originality of his methodological approach to describing the material lies in that, after formulation and theoretical consideration of the problem, he writes a chapter devoted to the experimental data and facts on the theme of the comparison with analytical solutions and discussion.

Remaining on the position of Newton's and Clairaut's models, Jeffreys considers the planet as an elastic body and describes the equation of the force equilibrium from hydrostatic pressure, which appears from the outer uniform central force field, and strengths in a given point in the form

$$\rho f_i = \rho X_i + \sum_{k=1,2,3} \frac{\partial p_{ik}}{\partial x_k}, \tag{1.4}$$

where ρ is the density; f_i is the acceleration component; $p_{ik} = p_{ki}$ is the stress component from the hydrostatic pressure; X_i is the gravity force on the unit mass from the outer force field.

Additionally, the equation of continuity (like the continuity equation in hydrodynamics) is written as the condition of equality of velocity of the mass inflow and outflow from elementary volume in the form

$$\frac{\partial \rho}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(\rho v_i), \tag{1.5}$$

where v_i is the velocity component in the direction of x_i.

Further, applying the laws of elasticity theory, he expresses elastic properties of the matter by Lame coefficients and writes the basic equations of the strength state of the body, which links the strengths and the deformations in the point as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x_i} + \Delta^2 u_i, \tag{1.6}$$

where u_i is the displacement component; λ and μ are the Lame coefficients; Δ is the component of the relative displacement; Δ is the Laplacian operator.

One may see that Jeffreys reduced Newton's effects of gravitation to the effects of Hooke's elasticity. The author introduces a number of supplementary physical ideas related to the properties of Earth's matter, assuming that it is not perfectly elastic. With development of stresses the matter reaches its limit of resistance and passes to the stage of plastic flow with a final effect of a break in the matter continuity. This break leads to a sharp local change in the strength state, which, in turn, leads to appearance of elastic waves in the planet's body causing earthquakes. For this case Eq. (1.6) after the same corresponding transformations is converted into the form of plane longitudinal and transversal waves, which propagate in all directions from the break place. Such is the physical basis of earthquakes which was a starting point of development of seismology as a branch of geophysics studying propagation of elastic longitudinal and transversal waves in the Earth's body. By means of seismic study, mainly by strong earthquakes and based on differences in velocity of propagation of the longitudinal and transversal waves through the shells having different elastic properties, the shell-structured body of the planet was identified.

Jeffreys analyzed the status of study in the theory of the Earth and the Moon figure following Newton's basic concepts. Namely, the planet has an inner and outer gravitational force field. The gravitational pressure is formed on the planet's surface and affects the outer space and the planet's centre. The Earth's figure is presented by an ellipsoid of rotation which is perturbed from the side of inaccuracies in the density distribution, as well as from the side of the Moon perturbations. The problem is to find the axes of the ellipsoid under action of both perturbations which occur because of a difference in the gravity field for the real Earth and the spherical body. It is accepted that the ocean's level is close to a spherical surface with deviation by a value of the first order of magnitude, and geometric oblateness of the ellipsoid is close to the value of $e \approx 1/297$. However, the value squares of deviation can not always be ignored because the value e^2 substantially differs from the value e. The observed data cannot be compared with theoretical solutions because the formulas depending on the latitudes give precise expressions neither for the radius vector from the Earth's centre to the sea level nor for the value of the gravity force. The problem of the planet's mass density distribution finds its resolution from the condition of the hydrostatic pressure at a known velocity of rotation. The value of oblateness of the outer spheroid can be found from the observed value of the precession constant with a higher accuracy than one can find from the theory of the outer force field. A weak side of such an approach is the condition of the hydrostatic stresses, which however are very small in comparison with the pressure in the centre of the Earth. The author also notes that deviation of the outer planet's gravitational field from spherical symmetry does not satisfy the condition of the inner hydrostatic stresses. Analysis of that discrepancy makes it possible to assess errors in the inner strengths related to hydrostatics. Because of the Earth' ellipticity, the attraction of the Sun and the Moon creates the force couple applied to the centre, which forces the instantaneous axis of rotation to depict a cone around the pole of the ecliptic and to cause the precession phenomenon. The same effect initiates an analogous conclusion that was made by the author relative to the Moon's oblateness, where the observed and calculated values show much more contrast.

These are the main physical fundamentals which Jeffreys used for the analysis and theoretical consideration of the planet's figure problem and for determination of its oblateness and of semi-major axis size. The author has found that the precession constant $H = 0.00327293 \pm 0.00000075$ and the oblateness $1/e = 297.299 \pm 0.071$. He assumes that the above figures could be accepted as a result which gives the hydrostatic theory. But in conclusion he says that the theory is not correct. If it is correct, then the solid Earth would be a bench mark of the planet's surface covered by oceans. There are some other data confirming that conclusion. However, this is the only and the most precise method for determining the spheroid flattening which needs non-hydrostatic corrections to be found. An analogous conclusion was made by the author relative to the Moon's oblateness, where the observed and calculated values exhibit much more contrast.

Other review works on the irregularity of rotation and the pole motion of the Earth are the monographs of Munk and MacDonald (1964), Melchior (1972–1973), Sabadini and Vermeersten (2004). The authors analyze the state of art and geophysical causes leading to an observed incorrectness in the planet's rotation and wobbling of the poles. They draw attention of readers to practical significance of the two main effects and designate about ten causes of their initiation. Among them are seasonal variations of air masses, moving of the continents, melting and growing of the glaciers, elastic properties of planets, convective motion in the liquid core. The authors stressed that solution of any of the above geophysical tasks should satisfy the dynamical equations of motion of the rotating body and the equations, which determine a relationship between the stresses and deformations inside the body. Theoretical formulation and solution of a task should be considered on a hydrostatic basis, where the forces, inducing stresses and deformations are formed by the outer uniform force field and the deformations occur in accordance with the theory of elasticity for the elastic body model, and in frame work of rheology laws for the elastic and viscous body model. The perturbation effects are the windy forcing, the ocean currents and convective flows in the core and in the shells.

The causes of the axis rotation wobbling and pole motion are considered in detail. The authors find that the problem of precession and nutation of the axis of rotation has been discussed since old times and it does not cause any extra questions. The cause of the phenomena is explained by the Moon and the Sun's perturbation of the Earth which has an equatorial swelling and obliquity of the axis to the ecliptic. The Euler equations for a rigid body form a theoretical basis for the problem's solution. In this case the free nutation of the rigid Earth according to Euler is equal to 10 months.

Summing up the above short excursion to the problem's history we found the situation as follows. The majority of researchers dealing with dynamics of the Earth and its figure come to the unanimous conclusion that the theories based on hydrostatics do not give satisfactory results in comparison with observations. For instance, Jeffreys straightly says that the theories are incorrect. Munk and MacDonald more delicately note that a dozen of the observed effects can be called which do not satisfy the hydrostatic model. It means that dynamics of the Earth as a theory is absent. The above state of art and the conclusion gave the idea to the authors to search for a novel physical basis for dynamics of the Earth.

Newton was the founder of classic mechanics where motion is considered on the basis of his three laws, his two-body problem and the Earth's oblateness solution. Those problems were the first step in formulating and solving a very difficult problem in searching for the nature of orbital motion forces and configuration of the planet.

As to the quantitative value of the Earth's oblateness, it appears that the Newtonian value was overestimated. The Newtonian gravitation here is some natural effect of the attraction.

1.2 Einstein's Gravitation

In 1632 G. Galiley in his book "Dialogue about two of the main world systems—Ptolemaic and Copernican" introduced the principle of relativity which asserted equivalency of different frames of reference. In 1864 on the basis of experimental data obtained by Oersted, Ampere and Faraday, James Clerk Maxwell wrote equations of the electromagnetic field. It followed from the equations that electromagnetic waves of the field are propagated in vacuum by light velocity. The progress in farther development of the gravitation theory in the twentieth century was connected with the name of Albert Einstein.

Einstein studied mathematics and physics in Zurich's Eidgenoessische Polytechnische Schule and finished it with a diploma degree in 1900 and started the work in Berne's patent bureau. Here he began his studies in theoretical physics. On the basis of the found idea that for a non-accelerating observer the light velocity in vacuum does not depend on velocity of its source motion, Einstein concluded that the light velocity should have constant value. That fact he used for development of the special theory of relativity. In 1905 Einstein submitted in Zurich University his doctoral thesis entitled "A new determination of molecular dimensions", which was soon accepted. After that in the article "On a heuristic point of view concerning the production and transformation of light" Einstein proposed that electromagnetic radiation must consist of photons and explained photoelectric effect. That paper was reviewed by Max Plank and rejected, but soon conformed and adopted. For this work Einstein was awarded the Nobel Prize in 1921.

Einstein continued this work on general theory of relativity for several years trying to construct it as a scalar theory, but he could not find such a model. Further, in 1912 he applied to his friend and classmate in Polytechnische Schule, Marcel Grossman, to help him in construction of a mathematical model for describing his physical theory. At this time Grossman was the head of the physical–mathematical department in the Zurich Politechnische Schule. He accepted the idea of his old friend and proposed that Einstein's theory be looked at via the tensor mathematical model. Initial ideas of the tensor calculus were put out by B. Riemann and E. Christoffel in 1864, and were completed by Italian physicists G. Ricci and T. Levi-Civita in 1901.

In 1913 Einstein and Grossman prepared their first common paper under the title "The Theory of Gravitation", which described general ideas of the theory. The paper included two chapters: physical 22 pages, prepared by Einstein, and mathematical 16 pages, prepared by Grossman. The paper was published in the journal Zeitschrift fur Matematik und Physik" в 1913 г. In 1916 Einstein published the paper "Foundation of the general theory of relativity". This is a short story of the appearance of Einstein's gravitation theory. Every 3 years since 1975, Marcel

Grossman is honoured at international meetings where farther results of development of mathematical and physical ideas in gravitational theory are discussed. The 14th Grossman meeting was held in Rome in 2015.

Einstein's theory of relativity is a world outlook theory. Here the Universe presents a visible part of the material world, in which the main attributes of existing matter are space and time. Space expresses the order of existing world objects, and time expresses the chronological order of events. These physics–philosophical categories serve as one of the most important instruments for construction of theoretical models in interpretation of experimental cosmological data. Philosophical ideas of Democritos related to space and time have been embodied in Newtonian physics in the form of absolute space and absolute time, which do not depend upon each other and do not have relationships with existing matter. Aristoteles, and later on Leibniz, developed the ideas of relativity of those philosophical categories. Einstein generalized those ideas in his relativistic theory in physics. He considered dependence of the space and time characteristics of objects from velocity of their motion relative to the definite reference system. This was the basis to join space and time in the special theory of relativity into a common four-dimensional continuum. Here, by the four-dimensional continuum relationship between energy and momentum conservation laws from one side and uniformity and isotropy of the space and time from the other side, the idea is generalized. In the general theory of relativity, dependency of metric parameters of space and time from distribution of gravitating masses, presence of which curves space, was revealed. Here from the character of mass distribution depend the ideas of universal finiteness and infinity, which are also relative. Generalization of the conservation laws has not yet been reached. Moreover, the difficulties related to use of a space and time continuum for description in micro-world events, in particular, for description of particle trajectories, have arisen. In this connection, in order to find an exit the principle of uncertainty was introduced.

Thus, the conception of space and time in philosophy and physics as a long evolutionary way of development from ancient Greek philosophers and physicists to modern naturalists has passed.

Before Einstein Euclid's geometry with its brilliant axioms had began completion by new principles, Russian mathematician N. Lobachevsky and German mathematicians B. Riemann and K. Gauss developed new ideas and postulates of space geometry. The postulates are properties of the triangle on the curved surface for which the sum of angles is not equal to 2 π, the velocity of light is equal to the constant value, the needs in motion velocity measuring relative to some reference system, the difference in clock running in different inertial reference systems with respect to which they are stay in rest.

The physical basis of Einstein's general theory of relativity is the principle of equivalency between the forces of inertia and gravity. Galileo had found that principle in the seventeenth century by observation. They say that Galileo studied the free fall of kernels with different densities which were thrown from Pisa's tower. From that experience Galileo derived the laws of bodies in free fall either on a sloping plain or thrown upward at an angle to the horizon and also their application to isochronism of a pendulum's oscillation. After Galileo the problem of equality of the inertial and gravitational masses was checked by R. Dickke, L. Etvesh, V. Braginsky and others, which proved the Galileo principle. Those results gave to Einstein an idea to accept that fact as the postulate in his theory. It was used also as an example for explanation of weightlessness by interpretation of a passenger behaviour in lift free fall.

But the equivalency principle is not an absolute postulate. Because Newton's gravitational field is both vector-valued and centre-valued, it does not extend on an extended object and the inertial acceleration has a divergent effect. Contrary to gravitation, the inertial force does not work in a vacuum.

In addition to equivalency, Einstein's theory is based on the principle of absence of an absolute reference system and absolute light velocity.

On the basis of those principles Einstein made a number of predictions. The first is connected with probability of the light beam curve in the field of the Sun's gravity. The second was prediction of the Mercury perihelion motion. Finally, existence of the gravitational emission, waves, and finiteness and infinity of the Universe was predicted.

1.3 Other Theories on Gravitation

Gravitation creates physical and mathematical bases for the development of theories of matter and motion in nature. Newton's and Einstein's gravitational theories appear to be the only foundation now for solution of practical problems and for farther progress in nuclear physics, astrophysics, celestial mechanics and space sciences. After appearance of new theories, the process of improvement, farther development and presentation of new versions started in earnest. There is an enormous number of alternative theories now published. We do not here take on the task of reviewing them. We only note that classifications of alternative gravitation theories include mechanical, electrical, Lorentz-invariant, scalar, metrical, quasi-linear, scalar-tensor, non-metrical and so on models and theories.

The fact of existence of so many new ideas and proposals for improvement of the gravitational theory evidences that the problem of farther developing of the theory is topical. Newton himself initiated this by saying on the question about the nature of gravitation "Hypothesis I: I am not a frame". Let us see as examples a number of alternative proposals.

N. Fatio (1680) and G.-L. Le Sage (1748), Swiss mathematicians, proposed a kinetic theory of beam ether which, on the basis of mechanical collision, the effects of gravitation are explained. Fatio's work was for a long time unpublished and was not known to readers. Le Sage's work was published at the time when the kinetic theory of gases had just appeared andit was discussed in this connection. The point of the theory is as follows. Gravitation is an effect of motion with high velocity and collision of "extraterrestrial particles". They move in all directions with uniform intensity. In this connection one object receives impacts and pressure from all sides.

Another neighbouring object is also exposed to pressure from all sides but with a little bit less intensity from the side of the first object due to the screen effect. As a result, both objects because of decreased pressure between them started to bring together. Thus, the gravitational attraction here was developed by the collision pressure decrease between neighbouring interacting particles. Sometimes this effect is called “shadow” gravitation. The Fatio–Le Sage theory was supported by Newton, Galley and Bernoulli. However, negative reaction from many others was stronger and the theory has not obtained recognition.

Some years earlier, M. Lomonosov interpreted gravitation as did Le Sage on the basis of corpuscles approaching each other.

However, Lorenz had noted that particles in the Le Sage model are incompatible with the electron theory. He proposed that a particle be changed by more penetrating radiation which is absorbed by matter. But Hilbert had noted that the force $1/r^2$ does not appear if the distance between the atoms is much more than wavelength. Lorenz’s wave idea was rejected by Thomson, Poincare and other scientists.

Fedosin (1999) published a theory of similarity between nuclear and star systems. On the basis of mathematical derivation of the reverse square law and parameters of the graviton, a gravitational mechanism was proposed. His gravitons were represented by different relativist and metagalaxy particles like photons, neutrinos, cosmic rays and others. The presence of gravitons induces gravitational field and matter condensation in the form of planets and stars. Here the inertia forces act on a body from the graviton field and that force prevents changes in the state of motion that had been reached. The weak and strong interactions after matter condensation in the stars and planets were assumed to be secondary events. Conception of space and time in cosmology and relativistic physics has joined space and time into one abstract Universe.

1.4 Inertia Forces and Reference Systems

Force is the quantitative measure of a body’s interactions. The force conception as a cause of motion and change of velocity was introduced by Galileo. He was the founder of the dynamics that studies the influence of a body’s interaction with its motion. According to Descartes the only impacting force at a body’s contact is its motion.

In 1632 Galileo had formulated the inertia law. It is the body’s property to keep its velocity with respect to module and direction. Inertia can be progressive and rotary. The measure of the first is the body mass, and of the second is the moment of inertia. The nature of inertia hides in the body’s force of weight matter. The force of weight matter for a mass point is a linear vector and for a volumetric body it is a scalar value. The term “force of inertia” is a multi-meaning conception. Kepler was the first to use it in celestial mechanics. Force of inertia follows from the first and second Newtonian motion laws in classical mechanics. That force is indissolubly connected with the inertial frame of reference, and Newton’s laws of formation study are authorized only in inertial reference systems. For solution of problems in dynamics in non-inertial reference systems, in addition to the interaction of mass forces, a friction force of inertia is introduced. That force of inertia determines a body's acceleration. The fiction force of inertia also is used in equations of motion, where it is a vector value. Fiction force is applied because the sum of inner body forces in a Newtonial central force field is equal to zero. Otherwise, equilibrium of the body is absent.

According to Newton's definition, inertia force is an innate property of matter. Force works continually and the body also moves continually. The body should continuously move rectilinear and uniform up to meeting with another force, which brakes or turns or stops motion. The force is proportional to mass by module. In classical mechanics the force of inertia determines the cause of acceleration of a body motion. Newton's third law develops and supplements the sense of the inertia force. In Euler's dynamics and in d'Alambert's equations the inertia force is used for description of a body motion in a non-inertial reference system. They used the inertial force there as a figurative, imaginary, fiction force, which in meaning is not inertial. An isolated body not interacting with other bodies is moving rectilinear and uniformly. In other reference systems it will move with acceleration. Mach's principle states that inertia of a material body is a resistance to the body motion outside from the Universe. This is an inalienable property of the matter.

The problem of the inertia forces is most difficult in mechanics because of too many different classifications depending on the accepted reference system and solving task. At some time the fiction force as mathematical admission based on the d'Alambert principle was introduced.

Polygamy of forces is a weak place in mechanics and in different gravitation theories. Newton has three main forces: inertial, impressed and centripetal. The centripetal force has three more varieties like absolute, accelerative, and motive. Euler and d'Alambert have also a number of forces. In order to use such number of forces, corresponding mathematical apparatuses have been developed. The vector, tensor, spinor and matrix calculus were developed and the work in that field continues unabated. Each of the calculi represents a special scientific direction in mathematics which complicates solutions of practical physical, astrophysical and geophysical problems.

In order to solve a task on a body's motion, a reference system has to be introduced. From a kinematic point of view, all reference systems are equivalent, but the parameters of motion such as velocity and trajectory in different reference systems appear different. The reference system connected with the Earth is accepted as the inertial one. But the Foucault pendulum demonstrates that the Earth's rotation affects the motion of a system. The Galileo law separates the class of inertial reference systems. In that system the inertial force affects not the accelerating body but the connections. The inertial force in the non-inertial reference system is equal in value but has opposite direction. For quantitative description of a body's motion under action of another body, its mass and force should be known. Mass characterizes the measure of body inertia. At the interaction of two bodies the acceleration acquires both. The ratio of acceleration of two interacting bodies is equal to their mass ratio. Body mass is a scalar value.

In accordance with the second law in mechanics (F = ma), which was formulated by Descartes in 1644 in his “Principia phylosophy”, acceleration appears at the body’s interaction. By the third law of mechanics ($F_1 = -F_2$), which was formulated by Huygens in 1669, the bodies affect one another with forces equal in modules and opposite direction. The acceleration transmitted to bodies is reversed proportionally to their masses. The developed forces at the interaction have a common nature. But the fact that force as a vector value applies to a mass point remains convenient but far from a reality model.

It is worth noting that a natural body and any type of elementary particle motion occur under the action of volumetric force, i.e. energy. The forces as vector values are absent in the nature as well as the non-volumetric mass points are absent. Energy is the universal measure of all different forms of matter. Energy as a measure of motion has a privilege compared to forces. First, there is the law of energy conservation which is conserved during energy redistribution between interacting particles and bodies. Energy does not disappear and does not spend its power for a motion. It only transforms from one form to another and is redistributed between interacting objects. That circumstance allows the uncertainty to appear in identification of the force function to be excluded. Secondly, energy is a scalar value that has solved the problem for gravitation to be volumetric force for interacting bodies and particles. And third, as it will be shown in this work, the use of energy as a measure of motion, the existing theories of motion of Newton, Euler, Hamilton, Shrodinger and Einstein to be reduced to a unified theory of motion based on Jacobi dynamics, where the energy and moment of inertia (i.e. mass) occur in functional relationship.

1.5 Effect of Weightlessness

In scientific literature the physical meaning of the term “weightlessness” is defined as a complicated state. In relevant encyclopaedias one can find that weightlessness is the state of a material body moving in a gravity field by gravity forces which do not initiate mutual pressure of the body’s particles on each other. The weightlessness effect in cosmic space is compared with a man’s feelings in free fall in an elevator. Unfortunately, such a definition of weightlessness contains neither the nature of the unique phenomenon nor any real physical understanding. Later on it will be shown that the complexities related to understanding of nature of many dynamical effects are hidden in hydrostatics, which is the basis for solving of problems of a celestial body’s dynamics. Here we just note that the gravitational forces in hydrostatics act as outer forces. On the contrary, at dynamical equilibrium these forces, including gravitation, are inner. In the free fall of an elevator, a man having less density of his mass is moving with less velocity relative to the elevator. It means that our understanding of weightlessness by free fall of the man in the elevator, in the plain and so on, is incorrect because of their different density.

In this book it will be shown that weightlessness physics is determined by equilibrium of the energy pressure between interacting bodies or particles within their outer force fields. In other words, weightlessness is an effect of weight force (gravity) appearing at different interacting particles, bodies or their shells having different mass density. In that case Archimedes' forces, which represent the mass of an object's radial component of a body's weight force, are developed. We try to show here that all the dynamical effects of motion in nature have been developed by the energy of interaction of elementary particles constituting the body.

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Chapter 2
Gravitation, Inertia and Weightlessness as the Centrifugal Effects of Interaction Energy of the *n*–Body System

Abstract Newton's problem on the mutual attraction of two bodies, which depicts similar trajectories around their common centre of gravity and around each other, is based on the geometric solution of Kepler's problem formulated in his first two laws. Applying Jacobi dynamics, we analyzed orbital motion of the Earth, the Moon and other planets and satellites and discovered one more law of orbital velocity common to all Solar System bodies. The new law allows a new look at the nature of gravitation and the processes of creation and decay in celestial bodies.

For separation of light fraction in different products, people produce centrifuges. Centrifugation is a method of separation of dispersed solid, liquid and gaseous products into components by means of centrifugal and centripetal forces. For realizing the centrifugation method, there are a wide range of centrifuges from a human hand to electro-mechanical construction. The main part of a centrifuge is its rotating drum on which the separating material is placed. The energy of the drum rotation is distributed between its material components and by centrifugal forces their light and heavier parts, according to force of weight, are divided. The centrifugal and equal in value centripetal forces are developed in the matter by interaction of the components during drum rotation.

The same effects are characteristic for a self-gravitating celestial body which represents a natural centrifuge. Separation of shells of an initial common solar cloud with respect to force of weight of components and formation of the Solar System by orbiting of planets and satellites with the first cosmic velocity and a corresponding period of virial oscillation of the protoparents prove the centrifugal nature of that process.

Interpretation of artificial satellite data proving the absence of the Earth's and the Moon's hydrostatic equilibrium states are discussed in detail. It is shown that the Earth and the Moon are triaxial bodies and their axial rotation is not an inertial effect. The earthquake's observational data demonstrate the planet's oscillating dynamics with periods from 8.4 up to 57 min. Two general modes of the Earth's oscillation were found, namely, spectral with a vector of radial direction and torsion with a vector perpendicular to the radius.

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The main problem of a celestial body's equilibrium state, which is the ratio of kinetic and potential energies, is discussed thoroughly. It is shown that the ratio of Earth's kinetic and potential energy is equal to $\sim 1/300$. The other planets, the Sun and the Moon, the hydrostatic equilibrium for which is also accepted as a fundamental condition, remain in an analogous situation. This is because the hydrostatic approach does not take into account the kinetic energy of the interacting elementary mass particles, which is, in fact, Newton's gravity energy (force). As a result, the celestial body dynamics has been left without kinetic energy.

In order to correct the above situation, the generalized virial theorem was derived by introduction of volumetric forces and moments into the classic one. As a result, the oscillating mode of the body motion has appeared in the form of Jacobi's virial equation in the form $\ddot{\Phi} = 2E - U$ (where Φ is the Jacobi's function; E and U are the total and potential energy). In addition, the inner and outer force fields and energy as the measure of interacting mass particles of a celestial body were revealed. The reduced inner gravitational (weighting) field was obtained.

2.1 The Centrifuge as a Model of Dynamical Effects of the *n*–Body's Interaction Energy and Its Potential Field

The great Newton, author of the attraction force of interacting bodies, confessed that he did not know the nature of gravitation. Now after more than 300 years from Newton's time, scientists are still unsuccessfully trying to solve the gravitation problem. At the same time, ordinary people have long past given up on thoughts alone and have discovered practical ways to find and fabricate apparatuses based on gravity action for solving their domestic problems. For separation of light fraction in different products, they produce centrifuges. Centrifugation is a method of separation of dispersed solid, liquid and gaseous products into components by means of centrifugal and centripetal forces.

For realizing the centrifugation method, there is a wide range of centrifuges from hand- to electro-mechanical construction. The main part of a centrifuge is its rotating drum on which the separating material is placed. The energy of the drum rotation is distributed between the material components and by centrifugal forces their light and heavier, according to weight force, parts are divided. The centrifugal and equal-in-value centripetal forces are developed in the matter by interaction of the components during drum rotation. Its velocity reaches up to 10,000 rot/min and in ultracentrifuges it can be more of 40,000 rot/min.

The most primitive method of fluid separation is leaving them stand. For example, in order to obtain cream from milk we leave the milk standing for a short time. By natural centrifugal force the cream merges itself and can be skimmed off. We observe the same effect in different reservoirs and seas during separation of sediments. For separation of spirits and brew, is not enough that there be natural energy and that energy is introduced by heating the brew.

At separation of suspensions and dispersed matter by centrifuges, the drum rotation energy transmits to the components where it redistributed in proportions of weight forces (densities) of each component.

Centrifuges are used in medicine for blood component analysis. A high-velocity centrifuge is applied for separation of fine dispersion suspensions. In agriculture, they are used for grain separation. In mining they are applied for enrichment of ores. High-velocity centrifuges here are used for uranium isotope separation from hexafluorine UF_6.

Centrifugal modelling is used for scientific purposes in studying properties of materials and engineering constructions, most of which work under weight forces like slopes, embankments, dams, foundations, and underground constructions. The purpose of such studies is to determine at what levels of stresses and deformations in construction material will cause destruction.

Recently, in connection with development of high-speed motion transport like aviation and cosmic flights, centrifuges have been used to train pilots and to adapt them to overloading. The developed apparatus models the gravitation and inertial action effects and is based on the use of centrifugal and centripetal forces.

Any action of the centrifugal force of weight is always accompanied by an equal-in-value centripetal force. Separation of matter in nature and in a centrifuge always happens by means of the integral energy of the matter components interaction. And their motion appears to be reactive. This is because a prop is always needed for any motion. The role of the prop is played by the matter component separation of the force of inertia.

It is worth knowing that, in the case of drum rotation velocity, one more dynamical effect of centrifugation change appears. The effect occurs in the form of apparatus vibration which soon disappears. This is because during velocity change, the equilibrium in distribution of the matter component in the drum is broken. But as soon as the matter distribution reaches self-balance, the vibration breaks off. In natural centrifuges, the matter components imbalance by the same cause is also periodically developed. In this case, the vibration is caused by a different cataclysm. The other centrifugal effects on the Earth, like variation in a day's duration and the observed Chandler's wobbling of a pole with a period of 14 months in comparison with 10 months, given by the Euler rigid model, precession and nutation, are developed.

Coming to analysis of the centrifugal effects of energy interaction of a many-body system, we note that a self-gravitating body is an excellent natural centrifuge. In that centrifuge, the mass of the interacting elementary particles, possessing innate energy, performs the role of a multilayer drum. In case of non-elastic interaction of the particles, their interaction energy is redistributed proportionally to their force of weight. The process of particle separation with respect to the force of weight accompanies the collision. And the tangential component of the particle motion in addition to the radial oscillation motion appears. That component provides celestial bodies rotation (see Chap. 5). Thus, the centrifugal effects of a self-gravitating body are responsible for formation of centrifugal and centripetal fields in the form of active and reactive pressure which we have accepted up to now as a mysterious gravitation and inertia. The confirmation of that conclusion was only recently discovered to be the law of orbital motion of Solar system bodies with the first cosmic velocity of their protoparents (Ferronsky and Ferronsky 2013).

Let us start our analysis by consideration of the law's details and its effects.

2.2 Illusion of Two-Body Mutual Attraction at Their Gravitational Interaction

In celestial mechanics, developed on the basis of Newton's attraction law, the two-body problem is reduced to an analytical problem of one body, the motion of which takes place in the central field of the common mass. Both Newton's geometric theorem and analytical solution of celestial mechanics are based on the hydrostatic equilibrium state of a body motion due to Kepler's laws. We analyzed orbital motion of the Earth, the Moon and other planets and satellites applying the Jacobi dynamics and discovered one more law of orbital velocity common for all the solar system bodies (Ferronsky and Ferronsky 2013). The new law allows us to take a new look on the nature of gravitation and the processes of creation and decay of celestial bodies. Let us discuss some details of the new law and its effects.

It appears that the mean orbital velocity and the period of revolution of every planet are equal to the first cosmic velocity and corresponding period of virial oscillation of the protosun, having its radius equal to the semi-major axes of the planet's orbit. And also, the mean orbital velocity and periods of revolution of every satellite is equal to the first cosmic velocity and corresponding period of oscillation of the protoplanet, having its radius equal to the semi-major axes of the satellite's orbit. The same effect is valid for the asteroids, comets and other small bodies. The subsequent body evolution has not broken the above regularity.

The conception of "cosmic velocity" became especially popular at the time of development of the artificial satellite techniques. It is known that in order to overcome its own force weight of the launching satellite, its minimal velocity should be ~ 7.9 км/с, which follows from Kepler's third law. After reaching that velocity, the reactive engine is switched off and the the apparatus is moved on equilibrium orbit by the energy of the Earth's potential field with the same velocity, which is called 'the first cosmic velocity'. For the other planets, their satellites and the Sun, their first cosmic velocities have other different values.

Applying Jacobi dynamics in a framework of dynamical equilibrium state, the first cosmic velocity has the same (~ 7.9 km/s) value. The first cosmic velocity v_1 of the Protosun's and protoplanetary bodies and the period of oscillation of the corresponding outer shell T_1 of the created bodies were calculated by the formulae, from which, in fact, the third Kepler's law follows (see Chap. 4):

$$v_1 = \omega R = R\sqrt{\frac{GM}{R^3}} = \sqrt{\frac{GM}{R}}, \tag{2.1}$$

$$T_1 = \frac{2\pi}{\omega} = \frac{2\pi R}{v_1}, \tag{2.2}$$

$$\frac{(2\pi)^2}{T_1^2} = \frac{Gm}{R^3} \tag{2.3}$$

where M is the body's mass; G is the gravity constant; R is the semi-major axis; $\omega = \sqrt{GM/R^3} = v_1/R$ is the frequency of virial oscillation of the outer shell, which appears to be equal to the angular velocity of orbital motion of the created body. Note, the frequency of virial oscillation of the outer weighty shell does not equal to its angular velocity because the frequency is the parameter of the energy (force) field.

For example, when the protosun's radius R extended up to the present-day Earth's orbit ($M = 1.99 \times 10^{30}$ kg, $R = 1.496 \times 10^{11}$ m), then its first cosmic velocity was equal to

$$v_1 = \omega R = \sqrt{\frac{GM_s}{R}} = \sqrt{\frac{6.67 \times 10^{11} \cdot 1.99 \times 10^{30}}{1.496 \times 10^{11}}} = 29{,}786.786\,\text{m/s}$$
$$= 29.786786\,\text{km/s}$$

This value corresponds to the observed mean orbital velocity of the Earth.

The period of oscillation of the interacting mass particles of the protosun's outer shell ($R = 1.496 \times 10^{11}$ m, $v_1 = 29786.786$ m/s) was equal to

$$T_1 = \frac{2\pi R}{v_1} = \frac{6.28 \cdot 1.496 \times 10^{11}}{29786.786} = 3.1540428 \times 10^7\,\text{s} = 1\,\text{year},$$

which is equal to the observed period of the planet's orbital revolution.

When the Protoearth's radius R extended up to the present-day Moon's orbit ($m_e = 5.976 \times 10^{24}$ kg, $R = 3.844 \times 10^8$ m), then its first cosmic velocity was equal to

$$v_1 = \sqrt{\frac{GM_e}{R}} = \sqrt{\frac{6.67 \times 10^{11} \cdot 5.976 \times 10^{24}}{3.844 \times 10^8}} = 1018.3018\,\text{m} = 1.0183918\,\text{km/s},$$

which is the present-day Moon's mean orbital velocity.

The period of oscillation of the interacting mass particles of the Protoearth's outer shell ($R = 3.844\times 10^8$ m, $v_1 = 1018.3018$ m/s) was equal to

Table 2.1 Observed orbital periods of revolution of the planets around the Sun and calculated periods of oscillation of its corresponding outer shell

Planets	Mass $\times 10^{24}$ (kg)	Orbital radius, $R \times 10^{11}$ (m)	Observed period of revolution T_1 (year)	Calculated period of oscillation T (year)
Mercury	0.327	0.579	0.2408	0.24
Venus	4.87	1.082	0.6153	0.62
Earth	5.976	1.496	1.0	1.0
Mars	0.639	2.28	1.8823	1.88
Vesta	0.000275	3.53	3.7594	3.63
Juno	0.0000282	3.997	4.3733	4.37
Themis	0.0000113	4.68	5.5397	5.539
Jupiter	1899	7.784	11.8781	11.86
Saturn	569.4	14.271	29.4802	29.48
Uranus	87.01	28.708	84.1951	84.01
Neptune	103.3	44.969	164.9185	164.8
Pluto	0.01305	59.466	250.8882	248.09

$$T_1 = \frac{2\pi R}{v_1} = \frac{2 \cdot 3.14 \cdot 3.844 \times 10^8}{1018.3018} = 23.706449 \times 10^5\,\mathrm{s} = 27.438019\,\mathrm{days},$$

which corresponds to the present-day Moon's period of orbital revolution.

Tables 2.1 and 2.2 demonstrate the observed and calculated values of the orbital periods of revolution of the planets, asteroids (small planets) and satellites obtained by applying first cosmic velocities of the protosun and the protoplanets which prove the above claim.

The obtained results mean that all the planets and satellites were launched by the first cosmic velocity of the self-gravitating protosun and protoplanets after their outer shells have acquired weightlessness. As it will be shown below, the process of evolutionary loss of energy by emission led to redistribution and differentiation of the body's mass density: it increases in the inner shells and decreases in the outer one by the light components dilution. In general, due to this process of accumulation of the less-dense matter in the outer shell its density decrease up to the state of weightlessness and creation of the secondary self-gravitating body by the eddy currents results. The process of the outer shell separation appears to be the mechanism of contraction (volume decrease by increase of the density) of a body during evolution (Fig. 2.1).

The discovered regularity of creation of the Solar system's planets and satellites seems to be valid for the process of separation of the proto-Sun itself and other protostars from the proto-galaxy Milky Way. If we accept the known Galaxy's astrometrical data (mass $m_g = 2.5 \times 10^{41}$ kg, and distance of the Sun from the Galaxy centre $R_s = 2.5 \times 10^{20}$ m), then it is not difficult to calculate that the first cosmic velocity of the proto-Galaxy, the size of which was limited by the Sun's semi-major orbital axes, is equal to 230 km/s, and the orbital period of revolution is

Table 2.2 Observed orbital periods of revolution of the satellites around the planets and calculated periods of oscillation of their corresponding outer shells

Planets	Satellites	Orbital radius $R \times 10^3$ (m)	Calculated period of revolution T_1 (day)	Observed period of revolution T (day)
Earth	Moon	384.4	27.438	27.32
Mars	Phobos	9.4	0.3208	0.319
	Deimos	23.5	1.2604	1.262
Jupiter	V	181	0.4973	0.498
	Io	422	1.7706	1.769
	Europa	671	3.5508	3.551
	Ganimede	1.070	7.1541	7.155
	Callisto	1.880	16.6709	16.69
	XIII	11.100	239.0960	240.92
	VII	11.750	259.5899	259.14
	XII	21.000	660.7744	620.77
	1X	23.700	745.1833	758.90
Saturn	Janos	151.5	0.6956	0.7
	Mimas	185.6	0.9431	0.94
	Encelados	238.1	1.3704	1.37
	Tethys	294.7	1.8869	1.89
	Dione	377.4	2.7366	2.74
	Titan	1212.9	15.7548	15.95
	Iapetus	3560.8	79.2494	79.33
	Phoebe	12.944	549.2722	548.2
Uranus	Cordelia	49.751	0.3348	0.3350
	Cupid	74.8	0.6172	0.618
	Miranda	129.39	1.4043	1.4135
	Ariel	191.02	2.5189	2.5204
	Umbriel	266.3	4.1463	4.1442
	Titania	435.91	8.6840	8.7058
	Oberon	583.52	13.4503	13.4632
Neptune	Triton	354.8	5.8523	5.877
	Nereid	5513.4	359.8227	360.14
Pluto	Charon	19.571	6.7453	6.387
	Nix	48.675	26.4568	24.856
	Hydra	64.780	40.6198	38.206

220×10^6 year. The values are close to those found by observation, namely: mean orbital velocity of the Sun is defined as (230–250) km/s, and the orbital period of revolution T_s = (220–250) $\times$ 10^6 year.

The today-observed picture of the Milky Way, consisting of a bar-shaped core surrounded by a disc of gaseous matter and stars, which creates two major and four smaller logarithmic spiral arms with a spherical halo of old stars and globular

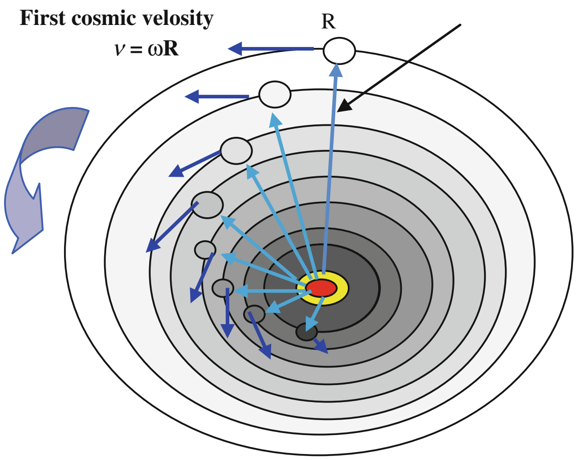

Fig. 2.1 Scheme of successive creation, separation and orbiting of planets from the upper weightlessness shells of the protosun with its first cosmic velocity

clusters, prove the common mechanism of creation of the galactic system. From the viewpoint of Jacobi's dynamics the observed picture evidences the generally common vortex mechanism of creation of a hierarchic system from the initial heterogeneous baryonic and non-baryonic (dark) matter of the compressing Universe. During its present-day expansion stage, due to redistribution of mass density and after reaching a state of weightlessness, the proto-stars creation and separation process will be continued in the spiral arms.

The analogous unified process was repeated for all the planets and their satellites.

The creation of the other small bodies like comets, meteors and meteorites are also found their explanation within the considered mechanism and physics. In fact, the only condition for separation of outer body's shell is its weightlessness (its corresponding mean density relative to the body's mean density), but not a limit of some amount of mass. In this connection, any volume and amount of mass could probability be separated at any time. For example, we found by calculation that the short-periodic Encke's Comet (1970 I, $T = 3.302$ year) has semi-major orbital axis $R \approx 1.5 \times 10^{11}$ m and has separated from the protosun after small planet Vesta and before the Mars. The short-periodic Halley's Comet (1910 II, $T = 76.1$ year) has a semi-major orbital axis $R = 2.7 \times 10^{12}$ m and has separated from the protosun after Saturn and before Jupiter. The long-periodic Ikeya-Seki's Comet (1965 III, $T = 874$ year) has semi-major orbital axis $R = 1.35 \times 10^{14}$ m and has separated from the protosun before the Pluto. Like the asteroid belt between Jupiter and Mars, the comet belts should definitely exist between the orbits of all the Jupiter group planets. As to meteors and meteorites, they all should be separated from planets in the same way. From the point of view of dynamical equilibrium of their orbital motion, the orbits of all the small bodies (comets, meteors and meteorites) should have large eccentricities and steep angles of inclination to the equator of their central bodies. This is because of probable oblateness of the Protosun body, where its polar regions should have higher values of the first cosmic velocity. Those small bodies and meteorites, which have not reached or have later on lost dynamical equilibrium, fell down on the planet's or satellite's surface.

As shown in Table 2.2, the small planets of the asteroid belt are separated from the protosun by the same mechanism. From the point of view of orbital motion and first cosmic velocities, there are no features of their separation from a broken planet.

Thus, the bullet point of creation and orbiting of the Solar System bodies is the inner energy released by the elementary particles interaction of their protoparents. The conditions of creation and orbiting of the planets and their satellites look like the conditions of launching of an artificial satellite, which are orbiting upon reaching weightlessness. The indicator of the body's weightlessness is its first cosmic velocity in orbital motion, which represents the energy of the outer force field of the parental body at a given height. So, all the planets appear to be weightless relative to the Sun and move on their orbits by the solar outer gravitational field. All the satellites are also weightless relative to their planets' gravitational fields and move along the orbits by first cosmic velocities of the inner energy of the planets' interacting masses. Dynamical equilibrium of their orbits' motion is guaranteed by their own outer force fields, which are generated by interaction of their own masses.

It is worth noting that in the scientific literature, the physical meaning of the term "weightlessness" is the state of a material body moving in a gravity field by gravity forces, which do not initiate mutual pressure of each other's body's particles. The weightlessness effect in cosmic space is compared with man's feelings in the free fall of the elevator. Unfortunately, such a definition of weightlessness does not contain both the nature of the unique phenomenon and the real physical understanding. In Chaps. 3 and 4, we show that the complexities related to understanding the nature of many dynamical events are placed in hydrostatics, which is the basis for solving of problems of a celestial body's dynamics. Here we just note that the gravitational forces in hydrostatics act as outer forces. To the contrary, at dynamical equilibrium these forces, including the gravitation, are inner. By Tables 2.1 and 2.2, we can say that the orbital moment of momentum of each planet is not its parameter but a parameter of the kinetic energy of the protosun. So, the existing discussion related to the mass and orbital momentum of the planets and the Sun are meaningless.

We have to study and explain the nature, mechanism and conditions which lead to the creation and decay of the Solar System's bodies in the Galaxy. These proofs can be found by experimental data of artificial satellites and with the help of dynamical equilibrium introduction.

It was shown earlier (Ferronsky and Ferronsky 2010) that by the satellite orbit study the Earth and the Moon do not stay in hydrostatic equilibrium. Therefore, the existing results and conclusions based on hydrostatics need to be corrected.

Moreover, we discovered the main discrepancy related to the hydrostatic equilibrium of the planets, satellites and the Sun. Namely, the potential energy of Earth, Mars, Jupiter, Saturn, Uranus and Neptune by about 300 times exceed their kinetic energy. And, for Mercury, Venus, Moon and the Sun this ratio equal to about 10^4.

In fact, all the celestial bodies with their inertial rotation are without kinetic energy. This is dynamics based on hydrostatics.

The consideration takes off an old misunderstanding about the difference in the orbital planet's and the Sun's moment of momentum. The planet conserves creation energy of the Sun in accordance with the third Kepler's law and its orbital moment of momentum is the parameter of the Sun's outer force field as well as the first cosmic velocity. As to the direction of a body's axial rotation and orbital revolution, then these parameters enter by the inner and outer force field, like in electrodynamics, in accordance with Lenz's law.

Now, after the conditions of the solar system bodies creation have been found, we have to continue related studies that reveal the dynamical processes of the problem, including the nature of gravitation, inertia and weightlessness based on Jacobi dynamics.

As it follows from the discussed dynamical effects related to the solar system bodies creation, oscillation, rotation and shell separation by means of the matter differentiation with respect to its density according to Archimedes' and Coriolis' laws, those processes take place by the inner energy of the matter itself without interference of any outer forces. In addition, the inner forces of a body occur as the dynamical effect of the elementary particles of their matter interaction energy which is the innate natural phenomenon. The matter here is also not detachable from the energy, as the energy is not detachable from the matter. Any infinite small value of the matter has a volume that is more than zero. That is why interaction of the infinitely small values of matter leads to volumetric deformation. It is the main cause of the way the model of the mass points of matter resulting in dynamical effect of interaction in the form of vector force with rectilinear motion cannot lead to the correct result in any acceptable form for practice approximation. In this connection, the use in celestial mechanics of Newton's mass point model for solution of dynamical problems on celestial bodies interaction appears incorrect. Because of the absence of experimental data on planet and satellite dynamics, Newton's innate force was attributed to Galileo's inertia. But it appears now that the innate volumetric force is the energy of the elementary particle interaction or energy of the gravitation and particles.

2.3 Relationship Between a Body's Moment of Inertia and Outer Gravitational Field by Satellite Data

The effects of the Earth's oblateness and the related problems of irregularity in the rotation and the planet's pole motion and also the continuous changes in the gravity and electromagnetic field have a direct relation to solution of a wide range of scientific and practical problems in Earth dynamics, geophysics, geology, geodesy, oceanography, physics of the atmosphere, hydrology and climatology. In order to understand the physical meaning and regularities of these phenomena regular observations are carried out. Newton's first attempts to find the quantitative value of

Table 2.3 Parameters of the Earth's oblateness by degree measurement data

Author	Year	a (m)	e	e_e	λ
D'Alambert	1800	6,375,553	1/334.00		
Valbe	1819	376,895	1/302.78		
Everest	1830	377,276	1/300.81		
Eri	1830	376,542	1/299.33		
Bessel	1841	377,397	1/299.15		
Tenner	1844	377,096	1/302.5		
Shubert	1861	378,547	1/283.0		
Clark	1866	378,206	1/294.98		
Clark	1880	378,249	1/293.47		
Zhdanov	1893	377,717	1/299.7		
Helmert	1906	378,200	1/298.3		
Heiford	1909	378,388	1/297.0		
Heiford	1909	378,246	1/298.8	1/38,000	38°E
Krasovsky	1936	378,210	1/298.6	1/30,000	10°E
Krasovsky	1940	378,245	1/298.3		
International	1967	378,160	1/298.247		

Here e is the oblateness of the polar axis; a is the semi-major axis; e_e is the equatorial oblateness; λ is the longitude of the maximal equatorial radius

the Earth's oblateness were based on degree measurements done by Norwood, Pikar, Kassini. As mentioned above, by his calculation of the Paris latitude, the oblateness value appears to be 1/230. Very soon after, some analogous measurements were taken in the equatorial zone in Peru and in the northern zone in Lapland Clairaut, Mopertui, Buge, and other known astronomers also took part in these works. They confirmed the fact of Earth's oblateness as calculated by Newton. The degree of the arc in the northern latitudes appeared to be maximal and the oblateness was equal to 1/214. In the equatorial zone, the arc length was minimal and the oblateness was equal to 1/314. So the Earth pole axis from these measurements was found to be shorter of the equatorial approximately by 20 km.

In the twentieth century, more than 20 large degree measurements were done from which the values of the oblateness and dimension of the semi-major axis were found. In Tables 2.3 and 2.4 the parameters of the triaxial ellipsoid are shown (Grushinsky 1976).

It is worth noting that in geodesy, a practical application of the triaxial ellipsoid has not been found, because it needs more complicated theoretical calculations and more reliable experimental data. In the theory this important fact is ignored, because it is not inscribed into the hydrostatic theory of the body.

In addition to the local degree measurements, which allow determination of the Earth's geometric oblateness, more precise integral data can be obtained by observation of the precession and nutation of the planet's axis of rotation. It is assumed that the obateness depends on deflection of the body's mass density

Table 2.4 Parameters or the Earth's equatorial ellipsoid

Author	Year	$a_1 - a_2$ (m)	λ
Helmert	1915	230 ± 51	17°W
Berrot	1916	150 ± 58	10°W
Heyskanen	1924	345 ± 38	18°E
Heyskanen	1929	165 ± 57	38°E
Hirvonen	1933	139 ± 16	19°W
Krasovsky	1936	213	10°E
Isotov	1948	213	15°E

Here a_1 and a_2 are the semi-major and semi-minor axes of the equatorial ellipsoid

distribution from spherical symmetry and is initiated by a force couple that appeared to be an interaction of the Earth with the Moon and the Sun. The precession of the Earth axis is proportional to the ratio of the spheroid's moments of inertia relative to the body's axis of rotation in the form of the dynamical oblateness ε:

$$\varepsilon = \frac{C - A}{C}.$$

At the same time, the retrograde motion of the Moon's nodes (points of the ecliptic intersection by the Moon orbit) is proportional to the second spherical harmonics coefficient J_2 of the Earth's outer gravitational potential in the form

$$J_2 = \frac{C - A}{\mathrm{Ma}^2}$$

It is difficult to obtain a rigorous value of geometric oblateness from its dynamic expression because we do not know the radial density distribution. Moreover, the Moon's mass is known up to a fraction of a percent but it is inconvenient to calculate analytically the joint action of the Moon and the Sun on the precession. In spite of that, some researchers succeeded to make such calculations, assuming that the Earth's density is increasing proportionally to the depth. Their data are the following:

$$\begin{array}{lll}
\text{by Newcomb} & \varepsilon = 1/305.32 = 0.0032753; & e = 1/297.6; \\
\text{by de Sitter} & \varepsilon = 1/304.94 = 0.0032794; & e = 1/297.6; \\
\text{by Bullard} & \varepsilon = 1/305.59 = 0.00327236; & e = 1/297.34;
\end{array}$$

After appearance of the Earth artificial satellites and some special geodetic satellites the situation with observation procedures has in principle changed. The satellites made it possible to determine directly, by measuring of the even zonal moments, the coefficient J_n in expansion of the Earth gravitational potential by spherical functions. In this case at hydrostatic equilibrium the odd and all the tesseral moments should be equal to zero. It was assumed before the satellite era that the correction coefficients of a higher degree of J_2 will decrease and the main expectations to improve the calculation results were focused on the coefficient J_4. But it has appeared, that all the gravitational moments of higher degrees are the values proportional to square of oblateness, i.e. $\sim(1/300)^2$ (Zharkov 1978).

On the basis of the calculated harmonic, the coefficients of the expanded gravitational potential of the Earth published by Smithsonian Astrophysical observatory and the Goddard cosmic centre of the USA, the fundamental parameters of the gravitational field and the shape of the so-called standard Earth were determined. Among them are coefficient of the second zonal harmonic $J_2 = 0.0010827$, equatorial radius of the Earth ellipsoid $a_e = 6{,}378{,}160$ m, angular velocity of the Earth's rotation $\omega_e = 7.292 \times 10^{-5}$ rad/s, equatorial acceleration of the gravity force $\gamma_e = 978031.8$ mgl, and oblateness $1/e = 1/298.25$ (Grushinsky 1976; Melchior 1972). At the same time, if the Earth stays in hydrostatic equilibrium, then, applying the solutions of Clairaut and his followers, the planet's geometric oblateness should be equal to $e' = 1/299.25$. On the basis of that contradiction Melchior (1972) concluded, that the Earth does not stay in hydrostatic equilibrium. It represents either a simple equilibrium of the rigid body, or there is equilibrium of a liquid and not static but dynamic with an extra hydrostatic pressure. Coming to interpretation of the density distribution inside the Earth by means of the Wilyamson-Adams equation, Melchior (1972) adds, that in order to eliminate there the hydrostatic equilibrium, one needs a supplementary equation. Since such equation is absent, we are obliged to accept the previous conditions of hydrostatics.

The situation with absence of hydrostatic equilibrium of the Moon is much more striking. The polar oblateness e_p of the body is (Grushinsky 1976)

$$e_\mathrm{p} = \frac{b - c}{r_0} = 0.94 \times 10^{-5},$$

and the equatorial oblateness e_e is equal

$$e_\mathrm{e} = \frac{a - c}{r_0} = 0.375 \times 10^{-4},$$

where a, b and c are the equatorial and polar semi-axes; r_0 is the body mean radius.

It was found by observation of the Moon libration, that

$$e_\mathrm{p} = 4 \times 10^{-4} \text{ and } e_\mathrm{e} = 6.3 \times 10^{-4}.$$

The calculation of the ratio of theoretical values of the dynamic oblateness $e_\mathrm{d} = e_\mathrm{p}/e_\mathrm{e} = 0.25$ substantially differs from the observation, which is $0.5 \le e_\mathrm{d} \le 0.75$. At the same time, the difference of the semi-axes is $a_1 - a_3 = 1.03$ km and $a_2 - a_3 = 0.83$ km, where a_1 and a_2 are the Moon's equatorial semi-axes.

After the works of Clairaut, Stokes and Molodensky, on the basis of which the relationship between the gravity force changes at sea level and on the real Earth surface with its angular velocity of rotation was established, one more problem has appeared. During measurements of the gravity force at any point of the Earth's surface two effects are revealed. The first is an anomaly of the gravity force, and second is a declination of the plumb line from the normal in a given point.

Analysis of the gravity force anomalies and the geoid heights (a conventional surface of a quiet ocean) based on the existent schematic maps compiled from the calculated coefficients of expansion of the Earth's gravity potential and ground-level gravimetric measurements, allows us to derive some specific features related to the parameter forming the planet. As Grushinsky (1976) notes, elevation of the geoid over the ellipsoid of rotation with the observed oblateness reaches 50–70 m only in particular points of the planet, namely, in the Bay of Biscay, North Atlantic, near the Indonesian Archipelago. In the case of triaxial ellipsoid, the equatorial axis passes near those regions with some asymmetry. The maximum of the geoid heights in the western part is shifted towards the northern latitudes, and the maximum in the eastern part remains in the equatorial zone. The western end of the major radius reaches the latitudes of 0–10° to the west of Greenwich and the western end falls on latitudes of about 30–40° to the west of a meridian of 180°. This also indicates asymmetry in distribution of the gravity forces and the forming masses. And the main feature is that the tendency to asymmetry of the northern and the southern hemispheres as a whole is observed. The region of the geoid's northern pole rises above the ellipsoid up to 20 m, and the Antarctic region is situated lower by the same value. The asymmetry in planetary scale is traced from the north-west of Greenland to the south-east through Africa to Antarctic with positive anomalies, and from Scandinavia to Australia through the Indian Ocean with negative anomalies up to 50 mgl. Positive anomalies up to 30 mgl are fixed within the belt from Panama to Fiery Land and to the peninsula Grechem in Antarctic. The negative anomalies are located on both sides, which extend from the Aleut bank to the south-east of the Pacific Ocean and from Labrador to the south of the Atlantics. The structure of the positive and negative anomalies is such that their nature can be interpreted as an effect of spiral curling of the northern hemisphere relative to the southern one.

As to the plumb line declination, this effect is considered only in geodesy from point of view of practical application in the corresponding geodetic problems. Physical aspects of the problem are not touched. Later on, we will discuss this problem.

The problem of the Earth rotation has been discussed at the NATO workshop (Cazenave 1986). It was stated, that both aspects of the problem still remain unsolved. The problems are variations in the day's duration and the observed Chandler's wobbling of the pole with the period of 14 months in comparison with 10 months, given by the Euler rigid body model. The Chandler's results are based on the analysis of 200-year observational data of motion of the Earth's axis of rotation, done in the USA in the 1930s. He found that there is an effect of free wobbling of the planet's axis with the period of about 420 days. Since that time the discovered effect remains the main obstacle in explanation of the nature and theoretical justification of the pole's motion.

Summing up the above short excursion to the problem's history we found the situation as follows. The majority of researchers dealing with the dynamics of the Earth and its shape come to the unanimous conclusion that the theories based on hydrostatics do not give satisfactory results in comparison with the observations. For instance, Jeffreys straightforwardly says that the theories are incorrect. Munk and Macdonald more delicately note that dozen of the observed effects can be called which do not satisfy the hydrostatic model. It means that dynamics of the Earth as a theory is absent. The above state of art and the conclusion motivated the authors to search for a novel physical basis for dynamics and creation of the Earth, planets and satellites.

Let us come back to the fact of absence of the Earth's hydrostatic equilibrium found by the satellite data. The initial factual material for the problem study is presented by the observed orbit elements of the geodetic satellites, which move on perturbed Kepler's orbits. The satellite motion is fixed by means of observational stations located within zones of a visual height range of 1000–2500 km, which is optimal for the planet's gravity field study. It was found, that the satellite's perturbed motion at such a close distance from the Earth surface is connected with the non-uniform distribution of mass density, the consequences of which are the non-spherical shape of the figure and the corresponding non-uniform distribution of the outer gravity field around the planet. These non-uniformities cause corresponding changes in trajectories of the satellite's motion, which are fixed by the tracking stations. Thus, distribution of the Earth's mass density determines the adequate equipotential trajectory in the planet's gravity field, which follows the satellite. The main goal of the geodetic satellites, launched under different angles relative to the equatorial plane, is the measurement of all deviations in the trajectory from the unperturbed Kepler's orbit.

The satellite orbits data for solving the nature of the Earth oblateness problem are interpreted on the basis of the known (in celestial mechanics) theory of expansion of the gravity potential of a body, the structure and the shape of which do not much differ from the uniform sphere. The expression of the expansion, by spherical functions, recommended by the International Union of Astronomy, is the following equation (Grushinsky 1976):

$$U(r,\varphi,\lambda) = \frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P_n(\sin\varphi) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n P_{nm}(\sin\varphi)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right] \tag{2.4}$$

where r, φ and λ are the heliocentric polar coordinates of an observation point; G is the gravity constant; M and R_e are the mass and the mean equatorial radius of the Earth; P_n is the Legendre polynomial of n order; $P_{nm}(\sin\varphi)$ is the associated spherical functions; J_n, C_{nm}, S_{nm} are the dimensionless constants characterizing the Earth's shape and gravity field.

The first terms of Eq. (2.4) determine the zero approximation of Newton's potential for a uniform sphere. The constants J_n, C_{nm}, S_{nm} represent the dimensionless gravitational moments, which are determined through analyzing the satellite orbits. The values J_n express the zonal moments, and C_{nm} and S_{nm} are the tesseral moments. In the case of hydrostatic equilibrium of the Earth as a body of rotation, in the expression of the gravitational potential (2.4) only the even n-zonal moments J_n are rapidly decreased with growth, and the odd zonal and all tesseral moments turn into zero, i.e.,

$$U = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\theta) - \sum_{n=3}^{\infty} J_n\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\theta)\right], \tag{2.5}$$

where θ is the angle of the polar distance from the Earth's pole.

Here the constant J_2 represents the zonal gravitational moment, which characterizes the axial planet's oblateness and makes the main contribution to correction of the unperturbed potential. That constant determines the dimensionless coefficient of the moment of inertia relative to the polar axis and equal to

$$J_2 = \frac{C - A}{MR_e^2}, \tag{2.6}$$

where C and A are the Earth moments of inertia with respect to the polar and equatorial axes accordingly, and R_e is the equatorial radius.

For expansion by spherical functions of the Earth's gravity forces potential, the rotation of which is taken to be by action of the outer inertial forces, but not by its own force field, the centrifugal force potential is introduced into Eq. (2.5). Then for the hydrostatic condition with the even zonal moments J_n one has

$$W = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\theta) - \sum_{n=3}^{\infty} J_n\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\theta)\right] + \frac{\omega^2 r^2}{3}[1 - P_2(\cos\theta)]. \tag{2.7}$$

where W is the potential of the body rotation; $\omega^2 r$ is the centrifugal force.

The first two terms and the term of the centrifugal force in Eq. (2.7) express the normal potential of the gravity force

$$W = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\theta) + \frac{\omega^2 r^2}{3}[1 - P_2(\cos\theta)].\right. \tag{2.8}$$

The potential (2.8) corresponds to the spheroid's surface which coincides with the ellipsoid of rotation with accuracy up to its oblateness. Rewriting term $P_2(\cos\theta)$ in this equation through sinus of the heliocentric latitude and the angular velocity—through the geodynamic parameter q, one can find the relationship of the Earth's oblateness ε with the dynamic constant J_2. Then the equation of the dynamic oblateness ε is obtained in the form (Grushinsky 1976; Melchior 1972)

$$\varepsilon = \frac{3}{2}J_2 + \frac{q}{2}, \tag{2.9}$$

where the geodynamic parameter q is the ratio of the centrifugal force to the gravity force at the equator

$$q = \frac{\omega^2 R}{GM/R^2}. \tag{2.10}$$

Geodynamic parameter J_2, found by satellite observation in addition to the oblateness calculation, is used for determination of a mean value of the Earth's moment of inertia. For this purpose the constant of the planet's free precession is also used, which represents one more observed parameter expressing the ratio of the moments of inertia in the form:

$$H = \frac{C - A}{C}. \tag{2.11}$$

This is the theoretical base for interpretation of satellite observations. But its practical application gave very contradictory results (Grushinsky 1976; Melchior 1976; Zharkov 1978). In particular, the zonal gravitation moment calculated by means of observation was found to be J_2 = 0.0010827, from where the polar oblateness ε = 1/298.25 appeared to be shorter of the expected value and equal to 1/297.3. All zonal moments J_n, starting from J_3, which relate to the secular perturbation of the orbit, were close to constant value and equal, by the order of magnitude, to square of the oblateness i.e. $\sim (1/300)^2$ and slowly decreasing with an increase of n. The tesseral moments C_{nm} and S_{nm} appeared to be not equal to zero, expressing the short-term nutational perturbations of the orbit. In the case of hydrostatic equilibrium of the Earth at the found value of J_2 the polar oblateness ε should be equal to 1/299.25. On this basis, the conclusion was made that the Earth does not stay in hydrostatic equilibrium. The planet's deviation from the hydrostatic equilibrium evidenced that there is a bulge in the planet's equatorial region with amplitude of about 70 m. It means that the Earth's body is forced by normal and tangential forces which develop corresponding stresses and deformations. Finally, by the measured tesseral and sectorial harmonics, it was directly confirmed, that the Earth has an asymmetric shape with reference to the axis of rotation and to the equatorial plane.

Because the Earth does not stay in hydrostatic equilibrium, then the above-described initial physical fundamentals for interpretation of the satellite observations should be recognized as incorrect and the related physical concepts cannot explain the real picture of the planet's dynamics.

The question is raised how to interpret the obtained actual data and where the truth should be sought. First of all, we should verify the correctness of the oblateness interpretation and the conclusion about the Earth's equatorial bulge. It is known from observation that the Earth is a triaxial body (see Table 2.3). Theoretical application of the triaxial Earth model was not considered because it contradicts the hydrostatic equilibrium hypothesis. But after it was found that the hydrostatic equilibrium is absent, the triaxial Earth alternative should be considered first.

Let us analyze Eq. (2.10). It is known from the observation data that the constant of the centrifugal oblateness q is equal to

$$q = \frac{\omega^2 R}{GM/R^2} = \left(\frac{1}{17.01}\right)^2 = \frac{1}{289.37}. \tag{2.12}$$

Determine a difference between the centrifugal oblateness constant q and the polar oblateness ε′ found by the satellite orbits, assuming that the desired value has a relationship with the perturbation caused by the equatorial ellipsoid

$$\begin{aligned}\varepsilon' &= \frac{a-b}{a} = \frac{a-c}{a} - \frac{b-c}{a} = \frac{1}{289.37} - \frac{1}{298.25} = \frac{1}{9720} = \left(\frac{1}{98.59}\right)^2 \\ &= 1.713\left(\frac{1}{289.37}\right)^2,\end{aligned} \tag{2.13}$$

where a, b and c are the semi-axes of the triaxial Earth.

The differences between the major and minor equatorial semi-axes can be found from Eq. (2.13). If the major semi-axis is taken in accordance with recommendation of the International Union of Geodesy and Geophysics as a = 6378160 m, then the minor equatorial semi-axis b can be equal to:

$$a - b = 6{,}378{,}160/9720 = 656\,\text{m}; \quad b = 6{,}377{,}504\,\text{m}.$$

One can see that the second semi-axes shorter of the first one by 656 m. There is a reason now to assume, that the value of equatorial oblateness ε′ = 1/9720 is a component in all the zonal gravitation moments J_n, related to the secular perturbations of the satellite orbits including J_2. They are perturbed both by the polar and the equatorial oblateness of the Earth. This effect ought to be expected because it was known long ago from observation that the Earth is a triaxial body. If our conclusion is true, then there is no ground for discussion about the equatorial bulge. And also the problem of the hydrostatic equilibrium is closed automatically because in this case the Earth is not a figure of rotation. And the nature of the observing fact of rotation of the Earth should be looked for rather in the action of the own inner force field but not in the effects of the inertial forces. As to the nature of the Earth oblateness, then for its explanation later on the effects of perturbation arising during separation of the Earth's shells by mass density differentiation and separation of the Earth itself from the Protosun will be considered. In particular, the effect of the heredity in creation of the body's oblateness is evidenced by the ratio of kinetic energy of the Sun and the Moon expressed through the ratio of square frequencies of oscillation ε'' of their polar moments of inertia which is close to the planet's equatorial oblateness:

$$\varepsilon'' = \frac{\omega_c^2}{\omega_л^2} = \frac{\left(10^{-4}\right)^2}{\left(0.96576 \cdot 10^{-2}\right)^2} = \left(\frac{1}{96.576}\right)^2 = 1{,}713\left(\frac{1}{289.37}\right)^2,$$

where $\omega_c = 10^{-4}\ \mathrm{s}^{-1}$ and $\omega_л = 0.96576 \times 10^{-2}\ \mathrm{s}^{-1}$ are the frequencies of oscillation of the Sun's and the Moon's polar moment of inertia correspondingly.

The most prominent effect, which was discovered by investigation of the geodetic satellite orbits is the fact of a physical relationship between the Earth mean (polar) moment of inertia and the outer gravity field. That fact without exaggeration can be called a fundamental contribution to understanding the nature of the planet's self-gravity. The planet's moment of inertia is an integral characteristic of the mass density distribution. Calculation of the gravitational moments based on measurement of elements of the satellite orbits is the main content of the satellite geodesy and geophysics. Short-1 s of the gravity field fixed at revolution of a satellite around the Earth, the period of which is small compared to the planet's period, provides evidence about oscillation of the moment of inertia or, to be more correct, about oscillating motion of the interacted mass particles. It will be shown, that oscillating motion of the interacting particles forms the main part of a body's kinetic energy and the moment of inertia itself is the periodically changing value.

Oscillation of the Earth's moment of inertia and also the gravitational field is fixed not only during the study by artificial satellites. Both parameters have also been registered by surface seismic investigations. Consider briefly the main points of those observations.

2.4 Observation of Moment of Inertia and Inner Gravitational Field Oscillation During Earthquakes

The study of the Earth eigenoscillation started with Poisson's work on oscillation of an elastic sphere, which was considered in the framework of the theory of elasticity. At the beginning of the twentieth century, Poisson's solution was generalized by Love in the framework of the problem solution of a gravitating uniform sphere of the Earth's mass and size. The calculated values of periods of oscillation were found to be within the limit of some minutes to one hour.

In the middle of the twentieth century during the powerful earthquakes in 1952 and 1960 in Chile and Kamchatka, an American team of geophysicists headed by Beneoff, using advanced seismographs and gravimeters, reliably succeeded in recording an entire series of oscillations with periods from 8.4 up to 57 min. Those oscillations in the form of seismograms have represented the dynamical effects of the interior of the planet as an elastic body, and the gravimetric records have shown the "tremor" of the inner gravitational field (Zharkov 1978). In fact, the effect of the simultaneous action of the potential and kinetic energy in the Earth's interior was fixed by these experiments.

About 1.000 harmonics of different frequencies were derived by expansion of the line spectrum of the Earth's oscillation. These harmonics appear to be integral characteristics of the density, elastic properties and effects of the gravity field, that is, of the potential and kinetic energy of separate volumetric parts of the non-uniform planet. As a result two general modes of the Earth's oscillations were found by the above spectral analysis, namely, spherical with a vector of radial direction and torsion with a vector perpendicular to the radius.

From the point of view of the existing conception about the planet's hydrostatic equilibrium, the nature of the observed oscillations was considered to be a property of the gravitating non-uniform (regarding density) body in which the pulsed load of the earthquake excites elementary integral effects in the form of elastic gravity quanta (Zharkov 1978). Considering the observed dynamical effects of earthquakes, geophysicists came close to a conclusion about the nature of the oscillating processes in the Earth's interior. But the conclusion itself still has not been expressed because it continues to relate to the position of the planet's hydrostatic equilibrium.

Now we move to one of the main problems related to the Earth's equilibrium or, more correctly, to the absence of the Earth equilibrium if it is considered on the basis of hydrostatics.

2.5 Kinetic Energy of a Body at Gravitational Interaction of Its Elementary Particles

It seems, we discovered the cause, for which even formulation of the problem of the body's dynamics based on the hydrostatic equilibrium is incorrect. The point is that the ratio of the kinetic energy to the potential one of a celestial body is too small. For example, this ratio for the Earth is equal to $\sim 1/300$, i.e. the same as its oblateness. Such a ratio does not satisfy the fundamental condition of the virial theorem, the equation of which expressed the hydrostatic equilibrium state. According to that condition the considered energies' ratio should be equal to 1/2. Taking into account that the kinetic energy of the Earth is presented by the planet's inertial rotation, then assuming it to be a rigid body rotating with the observed angular velocity $\omega_r = 7.29 \times 10^{-5}\ \mathrm{s}^{-1}$, the mass $M = 6 \times 10^{24}$ kg, and the radius $R = 6.37 \times 10^6$ m, the energy is equal to:

$$T_e = I \cdot \omega_r^2 = 0.6 \cdot MR^2\omega_r^2 = 0.6 \cdot 6 \times 10^{24} \cdot (6.37 \times 10^6)^2 \cdot (7.29 \times 10^{-5})^2$$
$$= 7.76 \times 10^{29}\ \mathrm{J} = 7.76 \times 10^{36}\ \mathrm{erg}.$$

The potential energy of the Earth at the same parameters is

$$U_e = 0.6 \cdot GM^2/R = 0.6 \cdot 6.67 \times 10^{-11} \cdot (6 \times 10^{24})^2/6.37 \times 10^6 = 2.26 \times 10^{32}\,\text{J} = 2.26 \times 10^{39}\,\text{erg}.$$

The ratio of the kinetic and potential energy comprises

$$J_2 = \frac{T_e}{U_e} = \frac{7.76 \times 10^{29}}{2.26 \times 10^{32}} = \frac{1}{292}.$$

One can see that the ratio is close to the planet's oblateness. It does not satisfy the virial theorem and does not correspond to any condition of equilibrium of a really existing natural system because in accordance with the third Newton's law equality between the acting and the reacting forces should be satisfied. The other planets, such like Mars, Jupiter, Saturn, Uranus and Neptune, exhibit the same behaviour. But for Mercury, Venus, the Moon and the Sun, the equilibrium states of which are also accepted as hydrostatic, their potential energy exceeds their kinetic energy by 10^4 times. Since the bodies in reality exist in equilibrium and their orbital motion strictly satisfies the ratio of the energies, then the question arises where the kinetic energy of the body's own motion has disappeared. Otherwise, the virial theorem for the celestial bodies is not valid. Moreover, if one takes into account that the energy of inertial rotation does not belong to the body, then the celestial body equilibrium problem appears to be out of discussion. And the celestial body dynamics is left without kinetic energy.

Thus, we came to the problem of a body equilibrium from two positions. From one side, it does not stay in hydrostatic equilibrium, and from the other side, it does not stay in general mechanical equilibrium because there is no reaction forces to counteract to the acting potential forces. The answer to both questions is given below while deriving an equation of the dynamical equilibrium of a body by means of generalization of the classical virial theorem.

2.6 Generalized Classical Virial Theorem as Equation of Dynamical Equilibrium of a *n*–Body's Mass Point System

The main methodological question arises: in which state of equilibrium the Earth exists? The answer to the question results from the generalized virial theorem for a self-gravitating body, which is the body that generates the energy for motion by interaction of the constituent particles having the innate moments. The guiding effect which we use here is the motion observed by an artificial satellite, which is the functional relationship between changes in the gravity field of the Earth and its mean (polar) moment of inertia. The deep physical meaning of this relationship is as follows: the planet's mean (polar) moment of inertia is an integral (volumetric) parameter, which does not represent location of the interacting mass particles, but expresses changes in their motion under the constrained energy. The virial theorem of Clausius for a perfect gaseous cloud or a uniform body is presented in its averaged form. In order to generalize the theorem for a nonuniform body, we introduce there the volumetric moments of interacted particles, taking into account their volumetric nature. Moreover, the interacted mass particles of a continuous medium generate volumetric forces (pressure or capacity of energy) and volumetric momentums, which, in fact, generate the motion in the form of oscillation and rotation of masses. The oscillating form of motion of the Earth and other celestial bodies is the dominating part of their kinetic energy which up to now has not been taken into account. We wish to fill in this gap in dynamics of celestial bodies by applying the volumetric forces of their gravitational interaction.

The virial theorem is an analytical expression of the hydrostatic equilibrium condition and follows from the Newton's and the Euler's equations of motion. Let us recall its derivation in accordance with the classical mechanics (Goldstein 1980).

Consider a system of mass points, the location of which is determined by the radius vector $\mathbf{r}_i$ and the force F_i including the constraints. Then equations of motion of the mass points through their moments p_i can be written in the form

$$\dot{p}_i = F_i, \tag{2.14}$$

The value of the moment of momentum is

$$Q = \sum_i p_i \cdot r_i,$$

where summation is done over all masses of the system. The derivative with respect to time from that value is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \sum_i \dot{r}_i \cdot p_i + \sum_i \dot{p}_i \cdot r_i. \tag{2.15}$$

The first term in the right-hand side of (2.15) is reduced to the form

$$\sum_i \dot{r}_i \cdot p_i = \sum_i m_i \cdot \dot{r}_i \cdot \dot{r}_i = \sum_i m_i v_i^2 = 2T$$

where T is the kinetic energy of particle motion under action of forces $\boldsymbol{F_i}$.

The second term in the Eq. (2.15) is

$$\sum_i \dot{p}_i \cdot r_i = \sum_i F_i \cdot r_i.$$

Now Eq. (2.15) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i p_i \cdot r_i = 2T + \sum_i F_i \cdot r_i. \tag{2.16}$$

The mean values in (2.16) within the time interval τ are found by their integration from 0 to t and division by τ:

$$\frac{1}{\tau}\int_0^t \frac{d\mathrm{Q}}{\mathrm{d}t}\mathrm{d}t = \overline{\frac{dQ}{\mathrm{d}t}} = \overline{2T} + \overline{\sum_i F_i \cdot r_i}$$

or

$$\overline{2T} + \overline{\sum_i F_i \cdot r_i} = \frac{1}{\tau}[Q(\tau) - Q(0)]. \tag{2.17}$$

For the system in which the coordinates of mass point motion are repeated through the period $\boldsymbol{\tau}$, the right-hand side of Eq. (2.17) after its averaging is equal to zero. If the period is too large, then the right-hand side becomes a very small quantity. Then, the expression (2.17) in the averaged form gives the relation

$$-\overline{\sum_i F_i \cdot r_i} = 2T, \tag{2.18}$$

or in mechanics it is written in the form

$$-U = 2T.$$

Equation (2.18) is known as the virial theorem, and its left-hand side is called the virial of Clausius (German *virial* is from the Latin *vires* which means forces). The virial theorem is a fundamental relation between the potential and kinetic energy and is valid for a wide range of natural systems, the motion of which is provided by action of different physical interactions of their constituent particles. Clausius proved the theorem in 1870 when he solved the problem of work of the Carnot thermal machine, where the final effect of the water vapour pressure (the potential energy) was connected with the kinetic energy of the piston motion. The water vapour was considered as a perfect gas. And the mechanism of the potential energy (the pressure) generation at the coal burning in the firebox was not considered and was not taken into account.

The starting point in the above-presented derivation of virial theorem in mechanics is the moment of the mass point system, the nature of which is not considered either in mechanics or by Clausius. By Newton's definition the moment "*is the measure of that determined proportionally to the velocity and the mass*". The nature of the moment by his definition is "*the innate force of the matter*". By his understanding that force is an inertial force, that is, the motion of a mass continues with a constant velocity.

The observed (by satellites) relationship between the potential and the kinetic energy of the gravitation field and the Earth's moment of inertia provides evidence that the kinetic energy of the interacting mass particle motion, which is expressed as a volumetric effect of the planet's moment of inertia, is not taken into account. The evidence of that was given in the previous Section in the quantitative calculation of a ratio between the kinetic and potential energies, equal to $\sim 1/300$.

In order to remove that contradiction, the kinetic energy of motion of the interacting particles should be taken into account in the derived virial theorem. Because if any mass has volume, the moment p should be written in volumetric form:

$$p_i = m_i \dot{r}_i. \tag{2.19}$$

Now the volumetric moment of momentum acquires the wave nature and is presented as

$$Q = \sum_i p_i \cdot r_i = \sum_i m_i \cdot \dot{r}_i \cdot r_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{m_i r_i^2}{2}\right) = \frac{1}{2}\dot{I}_\mathrm{p}, \tag{2.20}$$

where I_p is the polar moment of inertia of the system (for the sphere it is equal to 3/2 of the axial moment).

The derivative from that value with respect to time is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{2}\ddot{I}_\mathrm{p} = \sum_i \dot{r}_i \cdot p_i + \sum_i \dot{p}_i \cdot r_i. \tag{2.21}$$

The first term in the right-hand part of (2.21) remains without change

$$\sum_i \dot{r}_i \cdot p_i = \sum_i m_i \cdot \dot{r}_i \cdot \dot{r}_i = \sum_i m_i v_i^2 = 2T \tag{2.22}$$

The second term represents the potential energy of the system

$$\sum_i \dot{p}_i \cdot r_i = \sum_i F_i \cdot r_i = U. \tag{2.23}$$

Equation (2.21) is written now in the form

$$\frac{1}{2}\ddot{I}_\mathrm{p} = 2T + U. \tag{2.24}$$

Expression (2.24) represents a generalized equation of the virial theorem for a mass point system interacting by Newton's law. Here in the left-hand side of (2.24), the previously ignored inner kinetic energy of interaction of the mass particles appears. Solution of Eq. (2.24) gives a variation of the polar moment of inertia within the period τ. For a conservative system, averaged expression (2.21) by integration from 0 to t within time interval τ gives

$$\frac{1}{\tau}\int_0^t \frac{\mathrm{d}Q}{\mathrm{d}t}\mathrm{d}t = \overline{\frac{\mathrm{d}Q}{\mathrm{d}t}} = \overline{2T} + \overline{U} = \ddot{I}_\mathrm{p} = 0. \tag{2.25}$$

Equation (2.25) at $\ddot{I}_\mathrm{p} = 0$ gives $\dot{I}_\mathrm{p} = E = \text{const.}$, where E is the total system's energy. It means that the interacting mass particles of the system move with constant velocity. In the case of dissipative system, Eq. (2.25) is not equal to zero and the interacted mass particles move with acceleration. Now the ratio between the potential and kinetic energy has a value in strict accordance with the Eq. (2.24). The kinetic energy of the interacted mass particles in the form of oscillation of the polar moment of inertia in that equation is taken into account. And now in the frame of the law of energy conservation, the ratio of the potential to kinetic energy of a celestial body has a correct value.

Expression (2.24) appears to be an equation of dynamical equilibrium of the self-gravitating planets with the force field of the Sun and the self-gravitating moons with the force fields of their planets. Here, static equilibrium is absent because interacting particles continuously move and generate energy due to their inner potential. The integral effect of the moving particles is fixed by the satellite orbits in the form of changing zonal, sectorial and tesseral gravitational moments. We used the resulting energy of the initial moment (2.19) for derivation of the generalized virial theorem. The initial moments form the inner or "*innate*" by Newton's definition, energy of the body which has an inherited origin.

Thus, we obtained a differential equation of the second order (2.24), which describes a celestial body dynamics by its innate gravitation and its dynamical equilibrium.

The virial Eq. (2.24) was obtained by Jacobi already one and a half centuries ago from Newton's equations of motion in the form (Jacobi 1884)

$$\ddot{\Phi} = U + 2T, \tag{2.26}$$

where Φ is the Jacobi's function (the polar moment of inertia).

At that time Jacobi was not able to consider the physical meaning of the equation. For that reason he assumed that as there are two independent variables Φ and U in the equation, then it cannot be resolved. We succeeded to find an empirical relationship between the two variables and at first obtained an approximate and later on rigorous solution of the equation (Ferronsky et al. 1978, 1987, 2011; Ferronsky 2005). The relationship is proved now by means of the satellite observation.

We can explain now the cause of discrepancy between the geometric (static) and dynamic oblateness of the Earth. The reason is as follows: the planet's moments of inertia with respect to the main axes and their integral form of the polar moment of inertia do not stay in time as constant values. The polar moment of inertia of a self-gravitating body has a functional relationship with the potential energy, the generation of which results by interaction of the mass particles in regime of periodic oscillations. The hydrostatic equilibrium of a body does not express the picture of the dynamic processes from which, as it follows from the averaged virial theorem, the energy of the oscillating effects was lost from consideration. Because of that it was not possible to understand the nature of the energy. The main part of the body's kinetic energy of the body's oscillation was also lost. As to the rotational motion of the body shells, it appears only in the case of the non-uniform radial distribution of the mass density. The contribution of rotation to the total body's kinetic energy comprises a very small part.

The cause of the accepted incorrect ratio between the Earth potential and kinetic energy is the following. Clairaut's equation (Ferronsky and Ferronsky 2010) derived for the planet's hydrostatic equilibrium state and applied to determine the geometric oblateness, because of the above reason, has no functional relationship between the force function and the moment of inertia. Therefore for the Earth's dynamical problem the equation gives only the first approximation. In formulation of the Earth's oblateness problem, Clairaut accepted Newton's model of action of the centripetal forces from the surface of the planet to its geometric centre. In such a physical conception, the total effect of the inner force field becomes equal to zero. Section 2.7 it will be shown, that the force field of the continuous body's interacting masses represents volumetric pressure, but not a vector force field. That is why the accepted postulate relating to the planet's inertial rotation appears to be physically incorrect.

The question was raised about how it happened, that geodynamic problems and first of all the problem of stability of the Earth's motion up to now were solved without knowing the planet's kinetic energy. The probable explanation of that seems to lie in the history of the development of science. In Kepler's problem and in Newton's two-body problem solution, the transition from the averaged parameters of motion to the real conditions is implemented through the mean and the eccentric anomalies, which by geometric procedures indirectly take into account the above energy of motion. In the Earth's shape problem this procedure of Kepler is not applied. Therefore, the so-called "inaccuracies" in the Earth motion appear to be the regular dynamic effects of a self-gravitating body, and the hydrostatic model in the problem is irrelevant. The hydrostatic model was accepted by Newton for the other problem, where just this model allowed discovery and formulation of the general laws of the planets' motion around the Sun. Newton's centripetal forces in principle satisfy Kepler's condition when the distance between bodies is much more than their size accepted as mass points. Such a model gives a rough approximation in the problem solution.

Kepler's laws express a picture of the planets and satellites motion around their parent bodies in averaged parameters. All the deviations of those averaged values related to the outer perturbations are not considered as it was done in the Clausius' virial theorem for the perfect gas.

Newton solved the two-body problem, which had been already formulated by Kepler. The solution was based on the heliocentric world system of Copernicus, on the Galilean laws of inertia and free fall in the outer force field and on Kepler's laws of the planet's motion in the central force field considered as a geometric plane task. The goal of Newton's problem was to find the force in which the planet's motion resulted. His centripetal attraction and the inertial forces in the two-body problem satisfy Kepler's laws.

As it was mentioned, Newton understood the physical meaning of his centripetal or attractive forces as a pressure, which is accepted now like a force field. But by his opinion, for mathematical solutions the force is a more convenient instrument. And in the two-body problem, the force-pressure is acting from the centre (of point) to the outer space.

It is worth discussing briefly Newton's preference given to the force but not to the pressure. In mechanics, the term "*mass point*" is understood as a geometric point of space, which has no dimension but possesses a finite mass. In physics, a small amount of mass is known by the term "*particle*", which has a finite value of size and mass. But very often physicists use models of particles, which have neither size nor mass. A body model like mass point has been known since ancient times. It is simple and convenient for mathematical operations. The point is an irreplaceable geometric symbol of a reference point. The physical point, which defines inert mass of a volumetric body, is also suitable for operations. But interacting and physically active mass point creates a problem. For instance, in the field theory, the point value is taken to denote the charge, the meaning of which is not better understood than is the gravity force. But it is considered often there, that the point model for mathematical presentation of charges is not suitable because operations with it lead to zero and infinite values. Then for resolution of the situation the concept of charge density is introduced. The charge is presented as an integral of density for the designated volume and in this way the solution of the problem is resolved.

The point model in the two-body problem allowed reduction of it to the one-body problem and for a spherical body of uniform density to write the main seven integrals of motion. In the case when a body has a finite size, then not the forces but the pressure becomes an effect of the body particle interaction. The interacting body's mass particles form a volumetric gravitational field of pressure, the strength of which is proportional to the density of each elementary volume of the mass. In the case of a uniform body, the gravitational pressure should be also uniform within the whole volume. The outer gravitational pressure of the uniform body should be also uniform at the given radius. The non-uniform body has a non-uniform gravitational pressure of both inner and outer field, which has been observed in studying the real body field. Interaction of mass particles results appear in their collision, which leads to oscillation of the whole body system. In general, if the mass density is higher, then the frequency of body oscillation is also higher.

It was known from the theory of elasticity, that in order to calculate the stress and the deformation of a beam from a continuous load, the latter can be replaced by the equivalent lumped force. In that case the found solution will be approximate because the beam's stress and deformation will be different. The question is what degree of approximation of the solution and what kind of the error is expected. Volumetric forces are not summed up by means of the parallelogram rule. Volumetric forces by their nature are not reduced for application either to a point, or to a resultant vector value. Their action is directed to the $1/4\pi r^2$ space and they form inner and outer force fields. The force field by its action is proportional to action of the energy. This is because the force is the derivative of the energy.

The centrifugal and Coriolis forces are also proved to be inertial forces as a consequence of inertial rotation of the body. And the Archimedes force has not found its physical explanation, but it became an observational fact of hydrostatic equilibrium of a body mass immersed into a liquid.

Such is the short story of appearance and development of the hydrostatic equilibrium of the celestial bodies in the outer uniform gravity field. The force of gravity of a body mass is an integral value. In this connection, Newton's postulate about the gravity centre as a geometric point should be considered as a model for presentation of two interacting particles. It is shown in the next section, that the reduced physical, but not geometrical, gravity centre of a volumetric body is represented by an envelope of the figure, which draws an averaged value of radial density distribution of the body.

Due to the above discussion, the theorem of the classical mechanics stated that if a body is found in the central force field, then the sum of inner forces and torques are equal to zero, from the mathematical point of view is correct in the frame of the given initial conditions. As in case of the derived virial theorem, the moment of momentum can be presented by the first derivative from the polar moment of inertia. And then the torque equal to zero in the central field will be presented by oscillation of the polar moment of inertia not equal to zero.

The problem of dynamics of a self-gravitating body, including its shape problem, in its formulation and solution needs for a higher degree of approximation. A generalized virial theorem (2.24) satisfies the condition of a body's dynamical equilibrium state and creates a physical and theoretical basis for farther development of theory. It follows from the theorem that in a hydrostatic equilibrium state there is the particular case of dynamics. Solution of the problem of the body's dynamics based on the equation of dynamical equilibrium appears to be the next natural and logistic step from the hydrostatic equilibrium model to a more perfect method without loss of the previous preference.

Below we consider the problem of "decentralization" of the own force field for a self-gravitating body.

2.7 Jacobi's *n*–Mass Point Problem and Centrifugal Effects of the Reduced Inner Gravitational Field of a Body

Between 1842–1843, when Jacobi was a Professor at Königsberg University, he delivered a special series of lectures on dynamics. The lectures were devoted to the dynamics of a system of n–mass points the motion of which depends only on mutual distance between them and is independent of velocities. In this connection, by deriving the law of conservation of energy, where the force function is a homogeneous function of space coordinates, Jacobi gave an unusual form and a new content to this law. In transforming the equations of motion, he introduced an expression for the system's centre of mass. Then, following Lagrange, he separated the motion of the centre of mass from the relative motion of the mass points. Making the centre of mass coincident with the origin of the coordinate system, he obtained the following equation (Jacobi 1884):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum m_i r_i^2\right) = -(2k+4)U + 4E,$$

where m_i is the mass point i; $r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$ is the distance between the points and the centre of mass; k is the degree of homogeneity of the force function; U is the system's potential energy; and E is its total energy.

When $k = -1$, which corresponds to the interaction of mass points according to Newton's law, and denoting it by

$$\frac{1}{2}\sum m_i r_i^2 = \Phi$$

Jacobi obtained

$$\ddot{\Phi} = U + 2T = 2E - U,$$

where Φ is the Jacobi function (the polar moment of inertia.

This is Jacobi's generalized (non-averaged) virial equation. In the Russian scientific literature it is known as the Lagrange-Jacobi equation since Jacobi derived it by applying Lagrange's method of separation of the motion of the mass centre from the relative motion of mass points.

On the right-hand side of the virial equation there is a classical expression of the virial theorem, i.e. the relation between the potential and kinetic energy. In the case of constancy of its left-hand side, when motion of the system happens with a constant velocity, the equation acquires conditions of hydrostatic equilibrium of a system in the outer force field. The left-hand side of the equation, i.e. the second derivative with respect to the Jacobi function, expresses oscillation of the polar moment of inertia of the system, which, in fact, is kinetic energy of the inner volumetric torques of the interacted mass points moving in accordance with Kepler's laws.

Jacobi has not paid attention to physics of his equation, which expresses kinetic energy of the interacting volumetric particles in the form of their oscillation. He used the equation for a quantitative analysis of stability of the Solar System and noted that the system's potential and kinetic energies should always oscillate within certain limits. In the contemporary literature of celestial mechanics and analytical dynamics, Jacobi's virial equation is used for the same purposes (Whittaker 1937; Duboshin 1975). Since this equation contains two independent variables, it found no any other practical applications. The functional relationship between the potential (kinetic) energy and the polar moment of inertia was disclosed in our works. On that basis, the rigorous solution of the equation will be found and applied to study dynamics of a self-gravitating body (Ferronsky and Ferronsky 2010; Ferronsky et al. 2011).

Consider a planet as a self-gravitating sphere with uniform and one-dimensional interacting media. The motion of the body proceeds both in its own and in the Sun's force fields. It is known from theoretical mechanics that any motion of a body can be represented by a translation motion of its mass centre, rotation around that centre and motion of the body mass related to its changes in the shape and structure (Duboshin 1975). In the two-body problem, the last two effects are neglected due to their smallness.

In order to study the planet's motion in its own force field, the translational (orbital) motion relative to the fixed point (the Sun) should be separated from the two other components of motion. After that one can consider the rotation around the geometric centre of the planet's masses under action of its own force field and changes in the shape and structure (oscillation). Such separation is required only for the moment of inertia, which depends on what frame of reference is selected. The force function depends on a distance between the interacting masses and does not depend on selection of a frame of reference (Duboshin 1975). The moment of inertia of the planet relative to the solar reference frame should be split into two parts. The first is the moment of the body mass centre relative to the same frame of reference and the second is moment of inertia of the planet's mass relative to the own mass centre.

So, set up the absolute Cartesian coordinates $O_c\xi\eta\zeta$ with the origin in the centre of the Sun and transfer it to the system Oxyz with the origin in the geometrical centre of the planet's mass (Fig. 2.2).

Then, the moment of inertia of the Earth in the solar frame of reference is

$$I_c = \sum m_i R_i^2, \tag{2.27}$$

where m_i is the planet mass of particles; R_i is its distance from the origin in the same frame.

The Lagrange method is applied to separate the moment of inertia (2.27). The method is based on his algebraic identity

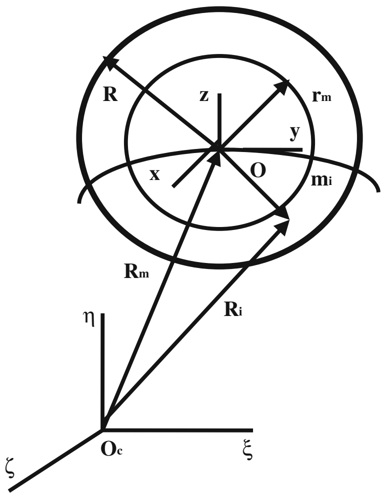

Fig. 2.2 Body motion in its own gravitation field

$$\left(\sum_{1\le i\le n} a_i^2\right)\left(\sum_{1\le i\le n} b_i^2\right) = \left(\sum_{1\le i\le n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n}\left(a_i b_j - b_i a_j\right)^2, \tag{2.28}$$

where a_i and b_i are whichever values; n is any positive number.

Jacobi in his "Vorlesungen über Dynamik" was the first who performed the mathematical transformation for separation of the moment of inertia of n interacting mass points into two algebraic sums (Jacobi 1884; Duboshin 1975; Ferronsky et al. 1987, 2011). It was shown that if we posit that (Fig. 2.2)

$$\xi_i = x_i + A; \quad \eta_i = y + B; \quad \zeta_i = z + C;$$

$$\sum m_i = M; \ \sum m_i\xi_i = MA; \ \sum m_i\eta_i = MB; \ \sum m_i\zeta_i = MC; \tag{2.29}$$

where A, B and C are the coordinates of the mass centre in the solar frame of reference.

Then, using identity (2.28), one has

$$\sum m_i r_i^2 = \sum m_i\xi_i^2 + \sum m_i\eta_i^2 + \sum m_i\zeta_i^2 = \sum m_i x_i^2 + 2A\sum m_i x_i A^2 \sum m_i + \\ + \sum m_i y_i^2 + 2B\sum m_i y_i + B^2\sum m_i + \sum m_i z_i^2 + 2C\sum m_i z_i + C^2\sum m_i.$$

Since

$$MA = \sum m_i\xi_i = \sum m_i x_i + \sum m_i A = \sum m_i x_i + MA,$$

then

$$\sum m_i x_i = 0, \text{ and also } \sum m_i y_i = 0, \sum m_i z_i = 0.$$

Now, the moment of inertia (2.27) acquires the form

$$\sum m_i R_i^2 = M\left(A^2 + B^2 + C^2\right) + \sum m_i\left(x_i^2 + y_i^2 + z_i^2\right), \tag{2.30}$$

where

$$M\left(A^2 + B^2 + C^2\right) = MR_m^2, \tag{2.31}$$

$$\left\{\sum m_i(x_i^2 + y_i^2 + z_i^2) = Mr_m^2,\right\} \tag{2.32}$$

M is the planet's mass; R_m and r_m are the radii of inertia of the planet in the Sun's and the planet's frame of reference.

Thus, we have separated the moment of inertia of the planet, rotating around the Sun in the inertial frame of reference, into two algebraic terms. The first one (2.31) is the planets' moment of inertia in the solar reference system $O_c\xi\eta\zeta$. The second term (2.32) presents the moment of inertia of the planet in its own frame of reference O_{xyz}. The planet mass here is distributed over the spherical surface with the reduced radius of inertia r_m. In literature, the geometrical centre of mass O in the plant reference system is erroneously identified with the centre of inertia and centre of gravity of the planet.

For farther consideration of the problem of the Earth's dynamics, we accept the polar frame of reference with its origin at centre O. Then expression (2.32) for the planet's polar moment of inertia I_p acquires the form

$$I_\mathrm{p} = \sum m_i(x_i^2 + y_i^2 + z_i^2) = \sum m_i r_i^2 = Mr_m^2. \tag{2.33}$$

Now the reduced radius of inertia r_m, which draws a spherical surface, is

$$r_m^2 = \frac{\sum m_i r_i^2}{M}. \tag{2.34}$$

Here $M = \sum m_i$ is the planet's mass relative to its own frame of reference.

Taking into account the spherical symmetry of the uniform and one-dimensional planet, we consider the sphere as a concentric spherical shell with the mass $\mathrm{d}m(r) = 4\pi r^2 \rho(r)\mathrm{d}r$. Then the expression (2.32) in the polar reference system can be rewritten in the form

$$r_m^2 = \frac{1}{M}\int_0^R r^2 4\pi r^2 \rho(r)\mathrm{d}r = \frac{4\pi R^2}{MR^2}\int_0^R r^4(r)\mathrm{d}r, \tag{2.35}$$

or

$$\frac{r_m^2}{R^2} = \frac{4\pi \int_0^R r^4 \rho(r) \mathrm{d}r}{MR^2} = \frac{\beta^2 MR^3}{MR^2} = \beta^2 \tag{2.36}$$

from where

$$r_m = \beta R,$$

where $\rho(r)$ is the law of radial density distribution; R is the radius of the sphere; β is the dimensionless coefficient of the reduced spheroid (ellipsoid) of inertia $\beta^2 MR^2$.

The value of β depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \beta > 0$. Earlier (Ferronsky et al. 1987, 2011) it was defined as a structural form-factor of the polar moment of inertia.

Analogously, the reduced radius of gravity r_g, expressed as a ratio of the potential energy of interaction of the spherical shells with density $\rho(r)$ to the potential energy of interaction of the body mass distributed over the shell with radius R. The potential energy of the sphere is written as

$$U = 4G\pi \int\limits_0^R r\rho(r)m(r)\mathrm{d}r = \alpha \frac{GM^2}{R},$$

from where

$$\alpha = \frac{4G\pi \int_0^R r\rho(r)m(r)\mathrm{d}r}{\frac{GM^2}{R}} = \frac{r_g}{R}. \tag{2.37}$$

The form-factor α of the inner force field, which controls its reduced radius, can be written as

$$\alpha = \frac{r_g}{R} = \frac{4G\pi \int_0^R r\rho(r)m(r)\mathrm{d}r}{\frac{GM^2}{R}}, \tag{2.38}$$

where in expressions (2.37) и (2.38) $m(r) = 4\pi \int_0^r r^2 \rho(r)\mathrm{d}r$, and $r_g = \alpha R$.

The value of β depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \alpha > 0$. Earlier (Ferronsky et al. 1987, 2011) it was defined as a structural form-factor of the force function.

Numerical values of the dimensionless form-factors α and β for a number of density distribution laws $\rho(r)$, obtained by integration of the numerators in Eqs. (2.36) and (2.37) for the polar moment of inertia and the force function, are presented in Table 2.5 (Ferronsky et al. 1987, 2011). Note, that the value of the polar I_p and axial I_a moments of inertia of a one-dimensional sphere are related as $I_p = 3/2 I_a$.

Table 2.5 Numerical values of form-factors α and β^2 for radial distribution of mass density and for sphere politropic models

Distribution law index of Politrope	α	$\beta^2_\perp$	β^2
Radial distribution of mass density			
$\rho(r) = \rho_0$	0.6	0.4	0.6
$\rho(r) = \rho_0(1 - r/R)$	0.7428	0.27	0.4
$\rho(r) = \rho_0(1 - r^2/R^2)$,	0.7142	0.29	0.42
$\rho(r) = \rho_0\exp(1 - kr/R)$	0.16k	8/k^2	12/k^2
$\rho(r) = \rho_0\exp(1 - kr^2/R^2)$	$\sqrt{k/2\pi}$	1/k	1.5/k
$\rho(r) = \rho_0\delta(1 - r/R)$	0.5	0.67	1.0
Politrope model			
0	0.6	0.4	0.6
1	0.75	0.26	0.38
1.5	0.87	0.20	0.30
2	1.0	0.15	0.23
3	1.5	0.08	0.12
3.5	2.0	0.045	0.07

It follows from Table 2.5 that for a uniform sphere with ρ(r) = const its reduced radius of inertia coincides with the radius of gravity. Here, both dimensionless structural coefficients α and β^2 are equal to 3/5, and the moments of gravitational and inertial forces are equilibrated and because of that the rotation of the mass is absent (Fig. 2.3a). Thus

$$\frac{r_m^2}{R^2} = \frac{r_g}{R} = \frac{3}{5}, \tag{2.39}$$

from where

$$r_m = r_g = \sqrt{3/5R^2} = 0.7745966R. \tag{2.40}$$

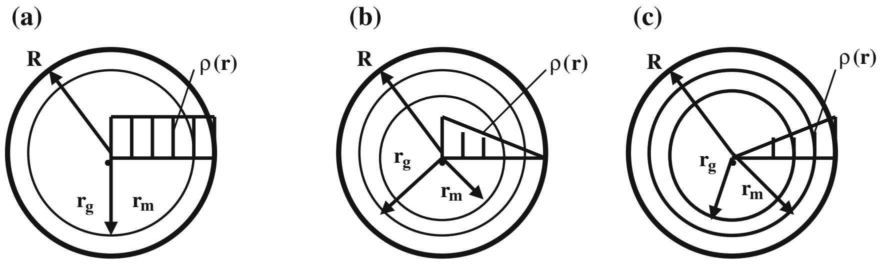

Fig. 2.3 Radius of inertia r_m (**a**) and radius of gravity r_g (**b**) as a function of radial density distribution ρ = $f(r)$

For a non-uniform sphere at $\rho(r) \neq$ const from Eqs. (2.36) to (2.38) one has

$$0 < \frac{r_m^2}{R^2} < \frac{3}{5} < \frac{r_g}{R} < 1 \tag{2.41}$$

It follows from the inequality (2.41) and Table 2.5 that in comparison with the uniform sphere, the reduced radius of inertia of the non-uniform body decreases and the reduced gravity radius increases (Fig. 2.3b). Because of $r_m \neq r_g$ and $r_m < 0.77\mathrm{R} < r_g$ the torque appears as a result of an imbalance between gravitational and inertial volumetric forces of the shells. Then from Eq. (2.31) it follows that

$$r_m = r_{mo} - \delta\, \mathrm{r_{mt}} \quad \text{и} \quad r_g = r_{go} + \delta\, r_{gt}\,, \tag{2.42}$$

where subscripts 0 and t relate to the uniform and non-uniform sphere.

In accordance with (2.41) and (2.42) rotation of shells of a one-dimensional body should be hinged-like and asynchronous. In the case of increasing mass density towards the body surface, then the signs in (2.41) and (2.42) are reversed (Fig. 2.3c). This remark is important because the direction of rotation of a self-gravitating body is function of its mass density distribution.

The main conclusion from the above consideration is that the inner force field of a self-gravitating body is reduced to a closed envelope (spheroid, ellipsoid or more complicated shape) of gravitational pressure, but not to a resulting force passing through the geometric centre of the masses. In the case of a uniform body, the envelopes have a spherical shape and both gravitational and inertial radii coincide. For a non-uniform body, the radius of inertia does not coincide with the radius of gravity, the reduced envelope is closed but has non-spherical (ellipsoidal or any other) shape. Analytical solutions done below justify the above.

So, we accept the force pressure as an effect of mass particle interaction which is the property producing work in the form of motion. In the other words, the pressure of interacted masses appears to be the force function or a flux of the potential energy.

The scheme of forces defining conditions of dynamical equilibrium of a body based on inner energy of the mass particles interaction is shown in Fig. 2.4.

Now we pass to derivation of the equation of dynamical equilibrium (Jacobi virial equation) for the well-known physical interaction models of natural systems. The only restriction here is the requirement of uniformity of the potential energy function of the system relative to the frame of reference. But that requirement appears to be not always obligatory. A specific physical model which is used for description of the system's dynamics in classical mechanics, hydrodynamics, statistic mechanics, quantum mechanics, theory of relativity in that case will not be an important factor.

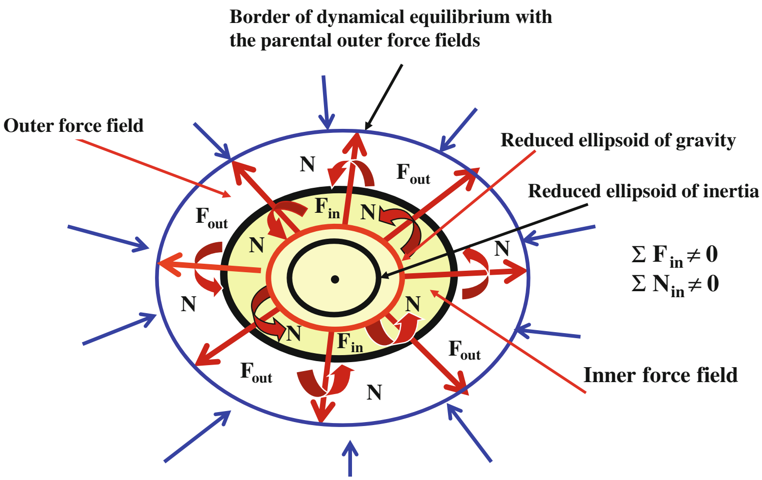

Fig. 2.4 Scheme of dynamic (oscillating) equilibrium of a body based on its inner energy of the mass particles interaction for uniform (**a**) and non-uniform density distribution growing to centre (**b**) and from centre (**c**)

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Chapter 3
Derivation of Unified Jacobi's Equation for Different Types of Physical Interactions

Abstract In this chapter, we derive a universal Jacobi's virial equation for description of the gravitation and dynamics of natural systems. It is derived from the main existing equations, describing a wide range of physical models of the systems. In particular, Jacobi's virial equation is derived from equations of motion of Newton, Euler, Hamilton, Einstein and quantum mechanics. The obtained equation appears to be a scalar-quantum mathematical model of matter's motion and gravitation. The derived equation represents not only a formal mathematical transformation of the initial equations of motion. The physical quintessence of mathematical transformation of the equations of motion involves changes of the vector forces and moment of momentums by the volumetric forces or pressure and oscillation of interacting mass particles (inner energy) expressed through the energy of oscillation of the polar moment of inertia of a body. Here, the potential and kinetic energy and the polar moment of inertia of a body have a functional relationship and within periods of oscillation are inversely changed by the same law. Moreover, the virial oscillations of a body represent the main part of a body's kinetic energy, which is lost in the hydrostatic equilibrium model. The change of vector forces and moment of momentums by force pressure and oscillation of the interacting mass particles disclose the physical meaning of the gravitation and mechanism of generation of the gravitational and electromagnetic energy and their common nature. The most important advantage given by Jacobi's virial equation is its independence from the choice of a coordinate system, transformation of which, as a rule creates many mathematical difficulties.

It was shown in Sect. 2.4 that the bullet point of the Solar System cosmogony and cosmology as a whole is the inner energy of interaction of elementary particles, which leads to weightlessness and self-gravitation of the system's upper shell of the matter. It means that the body's matter and its force fields (inner and outer) are responsible for the origin and evolutionary processes. Jacobi's virial equation represents the n-body problem and, in fact, is the generalized virial theorem. Its solution involves fundamentals of the theory of dynamics of a self-gravitating system considered by the main classical authors.

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Let us begin by deriving Jacobi's virial equation from the equations of Newton, Euler, Hamilton, Einstein and also from equations of quantum mechanics. By doing so we can show that Jacobi's virial equation appears to be a unified instrument for a description of the gravitational dynamics of natural systems in the framework of various physical models of the matter interaction employed. The Jacobi virial equation for a system motion in its own force field and establishing a relationship between the potential and kinetic energy of the oscillating polar moment of inertia is defined as the generalized (non-averaged) virial theorem or the equation of dynamical equilibrium of a body.

The theory presented in this book can be applied to study the body which, by its structure, presents a system that includes gaseous, liquid and solid shells. For this purpose derivation of Jacobi's virial equation from the equations of Newton, Euler, Hamilton, Einstein, and also from the equations of quantum mechanics is presented. In this part of the work, we justify physical applicability of the above fundamental equation for study of the gravitational dynamics and structure of stars, planets, satellites and their shells. For this purpose, the volumetric forces and moments into the transformed equations are introduced. In this case the gravitational energy becomes the measure of the matter interaction as it is observed in nature.

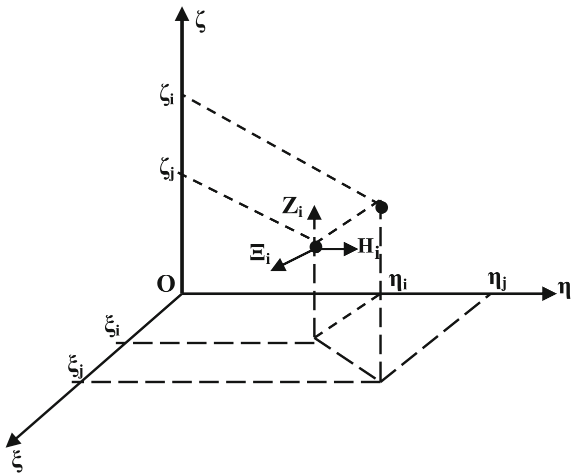
3.1 Derivation of Jacobi's Virial Equation from Newtonian Equations of Motion

Throughout this section, the term 'system' is defined as an ensemble of material mass points m_i ($i = 1, 2, 3, \ldots, n$) which interact by Newton's law of universal attraction. This physical model of a natural system forms the basis for a number of branches of physics, such as classical mechanics, celestial mechanics and stellar dynamics.

We shall not present the traditional introduction in which the main postulates are formulated; we shall simply state the problem (see, for example Landau and Lifshitz 1973a). We start by writing the equations of motion of the system in some absolute Cartesian coordinates ξ, η, ζ. In accordance with the conditions imposed, the mass point m_i is not affected by any force from the other $n-1$ points except that of gravitational interaction. The projections of this force on the axes of the selected coordinates ξ, η, ζ can be written (Fig. 3.1):

$$\begin{aligned}
\Xi_i &= -Gm_i \sum_{1\le i<j\le n} \frac{m_j(\xi_j - \xi_i)}{\Delta_{ij}^3},\\
H_i &= -Gm_i \sum_{1\le i<j\le n} \frac{m_j(\eta_j - \eta_i)}{\Delta_{ij}^3},\\
Z_i &= -Gm_i \sum_{1\le i<j\le n} \frac{m_j(\zeta_j - \zeta_i)}{\Delta_{ij}^3},
\end{aligned} \tag{3.1}$$

Fig. 3.1 Absolute descartes frame of reference



where G is the gravitational constant and

$$\Delta_{ji} = \sqrt{\left(\xi_j - \xi_i\right)^2 \left(\eta_j - \eta_i\right)^2 \left(\zeta_j - \zeta_j\right)^2}$$

is the reciprocal distance between points i and j of the system.

It is easy to check that the forces affect the ith material point of the system and are determined by the scalar function U, which is called the potential energy function of the system, and is given by

$$U = -G \sum_{1 \le i < j \le n} \frac{m_i m_j}{\Delta_{ij}}. \tag{3.2}$$

Now Eq. (3.1) can be rewritten in the form

$$\Xi_i = -\frac{\partial U}{\partial \xi_i},$$
$$\mathrm{H}_i = -\frac{\partial U}{\partial \eta_i},$$
$$\mathrm{Z}_i = -\frac{\partial U}{\partial \zeta_i}.$$

Then Newton's equations of motion for the ith point of the system take the form

$$\begin{aligned} m_i\ddot{\xi}_i &= \Xi_i, \\ m_i\ddot{\eta}_i &= \mathrm{H}_i, \\ m_i\ddot{\zeta}_i &= \mathrm{Z}_i, \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} m_i\ddot{\xi}_i &= -\frac{\partial U}{\partial \xi_i}, \\ m_i\ddot{\eta}_i &= -\frac{\partial U}{\partial \eta_i}, \\ m_i\ddot{\zeta}_i &= -\frac{\partial U}{\partial \zeta_i}, \end{aligned} \tag{3.4}$$

where dots over coordinate symbols mean derivatives with respect to time.

The motion of a system is described by Eqs. (3.3) and (3.4) and is completely determined by the initial data. In classical mechanics, the values of projections ξ_{i0}, η_{i0}, ζ_{i0} and velocities $\dot{\xi}_{i0}$, $\dot{\eta}_{i0}$, $\dot{\zeta}_{i0}$ at the initial moment of time $t = t_0$ may be known from the initial data.

The study of motion of a system of n material points affected by self-forces of attraction forms the essence of the classical many-body problem. In the general case, ten classical integrals of motion are known for such a system, and they are obtained directly from the equations of motion.

Summing all the Eq. (3.4) for each coordinate separately, it is easy to be convinced of the correctness of the expressions:

$$\begin{aligned} \sum_{1\le i\le n} \Xi_i &= 0, \\ \sum_{1\le i\le n} H_i &= 0, \\ \sum_{1\le i\le n} Z_i &= 0. \end{aligned}$$

From those equations it follows that

$$\begin{aligned} \sum_{1\le i\le n} m_i\ddot{\xi}_i &= 0, \\ \sum_{1\le i\le n} m_i\ddot{\eta}_i &= 0, \\ \sum_{1\le i\le n} m_i\ddot{\zeta}_i &= 0. \end{aligned} \tag{3.5}$$

Equations (3.5), appearing as a sequence of equations of motion, can be successively integrated twice. As a result, the first six integrals of motion are obtained:

$$\begin{aligned}
&\sum_{1\le i\le n} m_i\dot{\xi}_i = a_1,\\
&\sum_{1\le i\le n} m_i\dot{\eta}_i = a_2,\\
&\sum_{1\le i\le n} m_i\dot{\zeta}_i = a_3.\\
&\sum_{1\le i\le n} m_i(\xi_i - \dot{\xi}_i t) = b_1,\\
&\sum_{1\le i\le n} m_i(\eta_i - \dot{\eta}_i t) = b_2,\\
&\sum_{1\le i\le n} m_i(\zeta_i - \dot{\zeta}_i t) = b_3,
\end{aligned} \tag{3.6}$$

where $a_{1,}$ $a_{2,}$ $a_{3,}$ $b_{1,}$ $b_{2,}$ b_3 are integration constants.

These integrals are called integrals of motion of the centre of mass. The integration constants $a_{1,}$ $a_{2,}$ $a_{3,}$ $b_{1,}$ $b_{2,}$ b_3 can be determined from the initial data by substituting their values at the initial moment of time for the values of all the coordinates and velocities.

Let us obtain one more group of first integrals. To do this, the second of Eq. (3.3) can be multiplied by $-\zeta_{i,}$ and the third третье by η_i . Then all expressions obtained should be added and summed over the index *i*. In the same way, the first of Eq. (3.3) should be multiplied by $\zeta_{i,}$ and the third by $-\xi_i$ added and summed over index *i*. Finally, the second of Eq. (3.3) should be multiplied by $\xi_{i,}$ and the first by $-\eta_i$ added and summed over index *i*. It is easy to show directly that the right-hand sides of the expressions obtained are equal to zero:

$$\sum_{1\le i\le n} (Z\eta_i - H\zeta_i) = 0,$$

$$\sum_{1\le i\le n} (\Xi\zeta_i - Z\xi_i) = 0,$$

$$\sum_{1\le i\le n} (H\xi_i - \Xi\eta_i) = 0.$$

Consequently their left-hand sides are also equal to zero:

$$\begin{aligned}
&\sum_{1\le i\le n} m_i(\ddot{\zeta}_i\eta_i - \ddot{\eta}_i\zeta_i) = 0,\\
&\sum_{1\le i\le n} m_i(\ddot{\xi}_i\zeta_i - \ddot{\zeta}_i\xi_i) = 0,\\
&\sum_{1\le i\le n} m_i(\ddot{\eta}_i\xi_i - \ddot{\xi}_i\eta_i) = 0.
\end{aligned} \tag{3.7}$$

Integrating Eq. (3.7) over time, three more first integrals can be obtained

$$\begin{aligned}
&\sum_{1\le i\le n} m_i\left(\dot{\xi}_i\eta_i - \dot{\eta}_i\xi_i\right) = c_1,\\
&\sum_{1\le i\le n} m_i\left(\dot{\zeta}_i\xi_i - \dot{\xi}_i\zeta_i\right) = c_2,\\
&\sum_{1\le i\le n} m_i\left(\dot{\eta}_i\zeta_i - \dot{\zeta}_i\eta_i\right) = c_{3.}
\end{aligned} \tag{3.8}$$

The integrals (3.8) are called area:of area integrals or integrals of moments of momentum. Three integration constants c_1, c_2, c_3 are also determined from the initial data by changing over from the values of all the coordinates and velocities to their values at the initial moment of time.

The last of the classical integrals can be obtained by multiplying the three Eq. (3.4) by $\dot{\xi}_i$, $\dot{\eta}_i$ and $\dot{\zeta}_i$ respectively, and adding and summing all the expressions obtained. As a result, the following equation is obtained:

$$\sum_{1\le i\le n} m_i\left(\ddot{\xi}_i\dot{\xi}_i + \ddot{\eta}_i\dot{\eta}_i + \ddot{\zeta}_i\dot{\zeta}_i\right) = -\sum_{1\le i\le n}\left(\frac{\partial U}{\partial \xi_i}\dot{\xi}_i + \frac{\partial U}{\partial \eta_i}\dot{\eta}_i + \frac{\partial U}{\partial \zeta_i}\dot{\zeta}_i\right). \tag{3.9}$$

It is not difficult to see that the right-hand side of Eq. (3.9) is the complete differential over time of the potential energy function U of the system as a whole. The left-hand side of the same equation is also the complete differential of some function T called the kinetic energy function of the system, and equal to

$$T = \frac{1}{2}\sum_{1\le i\le n} m_i\left(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2\right). \tag{3.10}$$

Equation (3.10) can be written finally in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(T) = -\frac{\mathrm{d}}{\mathrm{d}t}(U),$$

from which, after integration, one finds that

$$\mathrm{E} = T + U, \tag{3.11}$$

where E is the integration constant, determined from the initial conditions.

Equation (3.11) is called the integral of motion or the integral of living (kinetic) forces.

To derive the equation of dynamic equilibrium, or Jacobi's virial equation, each of the Eq. (3.4) should be multiplied by $\xi_{i,}$ η_i and ζ_i respectively; then, after summing all the expressions, one can obtain

$$\sum_{1 \le i \le n} m_i(\xi_i\ddot{\xi}_i + \eta_i\ddot{\eta}_i + \zeta_i\ddot{\zeta}_i) = -\sum_{1 \le i \le n}\left(\xi_i\frac{\partial U}{\partial \xi_i} + \eta_i\frac{\partial U}{\partial \eta_i} + \zeta_i\frac{\partial U}{\partial \zeta_i}\right). \tag{3.12}$$

We can take farther advantage of the obvious identities:

$$m_i\xi_i\ddot{\xi}_i = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i\xi_i^2\right) - m_i\dot{\xi}_i^2,$$

$$m_i\eta_i\ddot{\eta}_i = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i\eta_i^2\right) - m_i\dot{\eta}_i^2,$$

$$m_i\zeta_i\ddot{\zeta}_i = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i\zeta_i^2\right) - m_i\dot{\zeta}_i^2$$

from the Eulerian theorem concerning the homogenous functions. For the interaction of the system points, according to Newton's law of universal attraction, the degree of homogeneity of the potential energy function of the system is equal to –1, and hence

$$-\sum_{1 \le i \le n}\left(\xi_i\frac{\partial U}{\partial \xi_i} + \eta_i\frac{\partial U}{\partial \eta_i} + \zeta_i\frac{\partial U}{\partial \zeta_i}\right) = U.$$

Substituting the above expressions into the right- and left-hand side of Eq. (3.12), one obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{1}{2}\sum_{1 \le i \le n} m_i\left(\xi_i^2 + \eta_i^2 + \zeta_i^2\right)\right] - 2\sum_{1 \le i \le n}\frac{1}{2}m_i\left(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2\right) = U.$$

For a system of material points we now introduce the Jacobi function expressed through the moment of inertia of the system and presented in the form

$$\Phi = \frac{1}{2}\sum_{1 \le i \le n} m_i\left(\xi_i^2 + \eta_i^2 + \zeta_i^2\right).$$

Then taking into account (3.11), the previous equation can be rewritten in a very simple form as follows:

$$\ddot{\Phi} = 2E - U. \tag{3.13}$$

This is the equation of dynamic equilibrium of a self-gravitating body or Jacobi's virial equation describing both the dynamics of a system and its dynamic equilibrium using integral (volumetric) characteristics Φ and U or T.

Let us derive now another form of Jacobi's virial equation where the translational moment of the centre of mass of the system is separated and all the characteristics depend only on the relative distance between the mass points of the system. For this purpose, the Lagrangian identity can be used:

$$\left(\sum_{1\le i\le n} a_i^2\right)\left(\sum_{1\le i\le n} b_i^2\right) = \left(\sum_{1\le i\le n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n}\left(a_i b_j - b_i a_j\right)^2, \tag{3.14}$$

where a_i and b_i may acquire any values and n is any positive number.

Let us now put $a_i = \sqrt{m_i}$, and b_i equal to $\sqrt{m_i}\xi_i$, $\sqrt{m_i}\eta_i$ and $\sqrt{m_i}\zeta_i$, respectively. Then three identities can be obtained from (3.14):

$$\left(\sum_{1\le i\le n} m_i\right)\left(\sum_{1\le i\le n} m_i\xi_i^2\right) = \left(\sum_{1\le i\le n} m_i\xi_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\left(\xi_j - \xi_i\right)^2,$$
$$\left(\sum_{1\le i\le n} m_i\right)\left(\sum_{1\le i\le n} m_i\eta_i^2\right) = \left(\sum_{1\le i\le n} m_i\eta_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\left(\eta_j - \eta_i\right)^2,$$
$$\left(\sum_{1\le i\le n} m_i\right)\left(\sum_{1\le i\le n} m_i\zeta_i^2\right) = \left(\sum_{1\le i\le n} m_i\zeta_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\left(\zeta_j - \zeta_i\right)^2.$$

In summing up one finds

$$2m\Phi = \left(\sum_{1\le i\le n} m_i\xi_i\right)^2 + \left(\sum_{1\le i\le n} m_i\eta_i\right)^2 + \left(\sum_{1\le i\le n} m_i\zeta_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\Delta_{ij}^2.$$

Using now Eq. (3.6), the last equality can be rewritten in the form

$$2m\Phi = \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\Delta_{ij}^2 + (a_1 t + b_1)^2 + (a_2 t + b_2)^2 + (a_3 t + b_3)^2, \tag{3.15}$$

where $m = \sum\limits_{1\le i\le n} m_i$ is the total mass of the system.

Let us put

$$\Phi_0 = \frac{1}{4m} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \Delta_{ij}^2.$$

The value Φ_0 does not depend on the choice of the coordinate system and coincides with the value of the Jacobi function in the barycentric coordinate system. Moreover, from Eq. (3.15) it follows that

$$\ddot{\Phi} = \ddot{\Phi}_0 + \frac{a_1^2 + a_2^2 + a_3^2}{m}.$$

Excluding the value Φ from Jacobi's Eq. (3.13) with the help of the last equality, the same equation can be obtained in the barycentric coordinate system:

$$\ddot{\Phi}_0 = 2E_0 - U, \tag{3.16}$$

where $E_0 = T_0 + U_0$ is the total energy of the system in the barycentric coordinate system equal to

$$E_0 = E - \frac{a_1^2 + a_2^2 + a_3^2}{2m}.$$

We can now show that the value of E_0 does not depend on the choice of the coordinate system. For this purpose, we can again use the Lagrangian identity (3.14). In this case $a_i = \sqrt{m_i}$, and $b_i = \sqrt{m_i}\dot{\xi}_i$, $\sqrt{m_i}\dot{\eta}_i$ and $\sqrt{m_i}\dot{\zeta}_i$. Then, the following three identities can be justified:

$$\left(\sum_{1 \le i \le n} m_i\right)\left(\sum_{1 \le i \le n} m_i \dot{\xi}_i^2\right) = \left(\sum_{1 \le i \le n} m_i \dot{\xi}_i\right)^2 + \frac{1}{2} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \left(\dot{\xi}_j - \dot{\xi}_i\right)^2,$$

$$\left(\sum_{1 \le i \le n} m_i\right)\left(\sum_{1 \le i \le n} m_i \dot{\eta}_i^2\right) = \left(\sum_{1 \le i \le n} m_i \dot{\eta}_i\right)^2 + \frac{1}{2} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \left(\dot{\eta}_j - \dot{\eta}_i\right)^2,$$

$$\left(\sum_{1 \le i \le n} m_i\right)\left(\sum_{1 \le i \le n} m_i \dot{\zeta}_i^2\right) = \left(\sum_{1 \le i \le n} m_i \dot{\zeta}_i\right)^2 + \frac{1}{2} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \left(\dot{\zeta}_j - \dot{\zeta}_i\right)^2.$$

After summing and using (3.16), one obtains

$$2mT = \left(a_1^2 + a_2^2 + a_3^2\right) + \frac{1}{2}\sum_{1 \le i \le n}\sum_{1 \le j \le n} m_i m_j\left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right],$$

or

$$T = \frac{\left(a_1^2 + a_2^2 + a_3^2\right)}{2m} + \frac{1}{2m}\left\{\frac{1}{2}\sum_{1 \le i \le n}\sum_{1 \le j \le n} m_i m_j\left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right]\right\}. \tag{3.17}$$

Here, the second term on the right-hand side of Eq. (3.17) coincides with the expression for the kinetic energy T_0 of a system.

Substituting (3.17) into an expression for E_0, one obtains

$$\begin{aligned} E_0 = T_0 + U = \frac{1}{2m}\sum_{1 \le i \le j \le n} m_i m_j\left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right] \\ - G\sum_{1 \le i \le j \le n} \frac{m_i m_j}{\Delta_{\mathrm{ij}}}. \end{aligned} \tag{3.18}$$

Thus, the total energy of the system E_0 depends only on the distance between the points of the system and on the velocity changes of these distances. But Jacobi's Eq. (3.16) appears to be invariant with respect to the choice of the coordinate system.

We can show now that the requirement of homogeneity of the potential energy function for deriving Jacobi's virial equation is not always obligatory. For this purpose, we consider two examples.

3.2 Derivation of Jacobi's Virial Equation for Dissipative Systems

Let us derive Jacobi's virial equation for a non-conservative system. We consider a system of n material points, the motion of which is determined by the force of their mutual gravitation interaction and the friction force. It is well known that the friction force always appears in the course of evolution of any natural system. It is also known that there is no universal law describing the friction force (Bogolubov and Mitropolsky 1974). The only general statement is that the friction force acts in the direction opposite to the vector of velocity of a considered mass point.

Consider as an example the simplest law of Newtonian friction when its force is proportional to the velocity of motion of the mass:

$$\begin{aligned} \Xi_f &= -km_i\dot{\xi}_i, \\ H &= -km_1\dot{\eta}_i, \\ Z &= -km_i\dot{\zeta}_i, \end{aligned} \tag{3.19}$$

where $\dot{\xi}_i$, $\dot{\eta}_i$, $\dot{\zeta}_i$ are the components of the radius vector of the velocity of the ith mass point in the barycentric coordinate system; k is a constant independent of i; $k > 0$.

Sometimes the friction force is independent of the velocity of the mass point. There are also some other laws describing the friction force.

We derive the equation of dynamical equilibrium for a system of n material points using the equations of motion (3.4) and taking into account the friction force expressed by Eq. (3.19):

$$\begin{aligned} m_i\ddot{\xi}_i &= -\frac{\partial U}{\partial \xi_i} - km_i\dot{\xi}_i, \\ m_i\ddot{\eta}_i &= -\frac{\partial U}{\partial \eta_i} - km_i\dot{\eta}_i, \\ m_i\ddot{\zeta}_i &= -\frac{\partial U}{\partial \zeta_i} - km_i\dot{\zeta}_i, \end{aligned} \tag{3.20}$$

where the value of the system's potential energy is determined by Eq. (3.2).

Multiplying each of Eq. (3.20) by ξ_i, η_i and ζ_i, respectively, and summing through all i, one obtains

$$\begin{aligned} \sum_{1\le i\le n} m_i\left(\xi_i\ddot{\xi}_i + \eta_i\ddot{\eta}_i + \zeta_i\ddot{\zeta}_i\right) = &- \sum_{1\le i\le n}\left(\frac{\partial U}{\partial \xi_i}\xi_i + \frac{\partial U}{\partial \eta_i}\eta_i + \frac{\partial U}{\partial \zeta_i}\zeta_i\right) \\ &- k\sum_{1\le i\le n} m_i\left(\xi_i\dot{\xi}_i + \eta_i\dot{\eta}_i + \zeta_i\dot{\zeta}_i\right). \end{aligned} \tag{3.21}$$

Transforming the right- and left-hand sides of Eq. (3.21) in the same way as in deriving Eq. (3.13), one obtains

$$\ddot{\Phi} = 2E - U - k\dot{\Phi}. \tag{3.22}$$

Let us show that the total energy E of the system is a monotonically decreasing function of time. For this purpose, we multiply each of the Eq. (3.20) by the vectors $\dot{\xi}_i$ $\dot{\eta}_i$ $\dot{\zeta}_i$, respectively, and sum over all from 1 to n, which results in

$$\sum_{1\le i\le n} m_i\left(\xi_i\ddot{\xi}_i+\eta_i\ddot{\eta}_i+\zeta_i\ddot{\zeta}_i\right) = -\sum_{1\le i\le n}\left(\frac{\partial U}{\partial \xi_i}\xi_i+\frac{\partial U}{\partial \eta_i}\eta_i+\frac{\partial U}{\partial \zeta_i}\zeta_i\right) - k\sum_{1\le i\le n} m_i\left(\dot{\xi}_i^2+\dot{\eta}_i^2+\dot{\zeta}_i^2\right).$$

The last expression can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(T) = -\frac{\mathrm{d}}{\mathrm{d}t}(U) - 2kT$$

or

$$\mathrm{d}E = -2kT\mathrm{d}t. \tag{3.23}$$

Since the kinetic energy T of the system is always greater than zero, $\mathrm{d}E \leq 0$, that is the total energy of a gravitating system is a monotonically decreasing function of time. Thus, the expression for the total energy $E(t)$ of the system can be written as

$$E(t) = E_0 - 2k\int_{t_0}^{t} T(t)\mathrm{d}t = E_0[1+q(t)],$$

where $q(t)$ is a monotonically increasing function of time.

Finally, the equation of dynamical equilibrium for a non-conservative system takes the form

$$\ddot{\Phi} = 2E_0[1+q(t)] - U - k\dot{\Phi}. \tag{3.24}$$

The second example where the requirement of homogeneity of the potential energy function for deriving Jacobi's virial equation is not obligatory is as follows. We derive Jacobi's virial equation for a system whose mass points interact mutually in accordance with Newton's law and move without friction in a spherical homogenous cloud whose density ρ_o is constant in time. Let, the geometric centre of the cloud coincide with the centre of mass of the considered system. The equations of motion for such a system can be written in the form

$$\begin{aligned} m_i\frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2} &= -\frac{4}{3}\pi G\rho_0 m_i\xi_i - \frac{\partial U}{\partial \xi_i},\\ m_i\frac{\mathrm{d}^2\eta_i}{\mathrm{d}t^2} &= -\frac{4}{3}\pi G\rho_0 m_i\eta_i - \frac{\partial U}{\partial \eta_i},\\ m_i\frac{\mathrm{d}^2\zeta_i}{\mathrm{d}t^2} &= -\frac{4}{3}\pi G\rho_0 m_i\zeta_i - \frac{\partial U}{\partial \zeta_i}, \end{aligned} \tag{3.25}$$

where $i = 1, 2, \ldots, \underline{n}$.

It is obvious that the above system of equations possesses the ten first integrals of motion and that Jacobi's virial equation, written in the form

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} = 2E - U - \frac{8}{3}\pi G\rho_0\Phi. \tag{3.26}$$

is valid for it.

The equation in the form (3.26) was first obtained by Duboshin et al. (1971). Equations (3.24) and (3.26) can be written in a more general form:

$$\ddot{\Phi} = 2E - U + X\left(t, \Phi, \dot{\Phi}\right), \tag{3.27}$$

where $X\left(t, \Phi, \dot{\Phi}\right)$ is a given function of time t, the Jacobi function Φ and first derivative $\dot{\Phi}$. Moreover, we can call Eq. (3.27) a generalized equation of dynamical equilibrium.

The above examples prove the statement that for conditions of homogeneity of the potential energy function, required for the derivation of Jacobi's virial equation, is not always necessary. This condition is required for description of dynamics of conservative systems but not for dissipative systems or for systems in which motion is restricted by some other conditions.

3.3 Derivation of Jacobi's Virial Equation from Eulerian Equations

We now derive Jacobi's virial equation by transforming of the hydrodynamic or continuum model of a physical system. As is well known, the hydrodynamic approach to solving problems of dynamics is based on the system of differential equations of motion supplement, in the simplest case, by the equations of state and continuity, and by the appropriate assumptions concerning boundary conditions and perturbations affecting the system.

In this section, we understand by the term 'system' some given mass M of ideal gas localized in space by a finite volume V and restricted by a closed surface S. Let the gas in the system move by the forces of mutual gravitational interaction and of baric gradient. In addition, we accept the pressure within the volume to be isotropic and equal to zero on the surface S bordering the volume V. Then for a system in some Cartesian inertial coordinate system ξ, η, ζ, the Eulerian equations can be written in the form

$$\begin{aligned}
&\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial}{\partial\xi}u+\rho\nu\frac{\partial}{\partial\eta}u+\rho w\frac{\partial}{\partial\zeta}u=-\frac{\partial p}{\partial\xi}+\rho\frac{\partial U_{\mathrm{G}}}{\partial\xi},\\
&\rho\frac{\partial \nu}{\partial t}+\rho u\frac{\partial}{\partial\xi}\nu+\rho\nu\frac{\partial}{\partial\eta}\nu+\rho w\frac{\partial}{\partial\zeta}\nu=-\frac{\partial p}{\partial\eta}+\rho\frac{\partial U_{\mathrm{G}}}{\partial\eta},\\
&\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial}{\partial\xi}w+\rho\nu\frac{\partial}{\partial\eta}w+\rho w\frac{\partial}{\partial\zeta}w=-\frac{\partial p}{\partial\zeta}+\rho\frac{\partial U_{\mathrm{G}}}{\partial\zeta},
\end{aligned}\tag{3.28}$$

where $\rho(\xi, \eta, \zeta, t)$ is the gas density; u, v, w are components of the velocity vector $\bar{v}(\xi, \eta, \zeta, t)$ in a given point of space; $p(\xi, \eta, \zeta, t)$ is the gas pressure; U_{G} is Newton's potential in a given point of space.

The value U_{G} is given by

$$U_{\mathrm{G}}=G\int\limits_{(V)}\frac{\rho(x,y,z,t)}{\Delta}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,\tag{3.29}$$

where G is the gravity constant; $\Delta=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$ is the distance between system points.

The potential energy of the gravitational interaction of material points of the system is linked to the Newtonian potential (3.29) by

$$U=-\frac{1}{2}\int\limits_{(V)}U_{\mathrm{G}}\rho(\xi,\eta,\zeta,t)\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta.$$

To supplement the system of equations of motion we write the equation of continuity

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial\xi}(\rho u)+\frac{\partial}{\partial\eta}(\rho v)+\frac{\partial}{\partial\zeta}(\rho w)=0,\tag{3.30}$$

and the equation of state

$$p=f(\rho),\tag{3.31}$$

assuming at the same time that the processes occurring in the system are barotropic.

Let us obtain the ten classical integrals for the system whose motion is described by Eq. (3.28).

We derive the integrals of the motion of the centre of mass by integrating each of the Eq. (3.28) with respect to all the volume filled by the system. Integrating the first equation, we obtain

$$c\int\limits_{(V)}\rho\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta+\int\limits_{(V)}\rho\left(u\frac{\mathrm{d}u}{\mathrm{d}\xi}+v\frac{\mathrm{d}u}{\mathrm{d}\eta}+w\frac{\mathrm{d}u}{\mathrm{d}\zeta}\right)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$
$$=-\int\limits_{(V)}\frac{\mathrm{d}p}{\mathrm{d}\xi}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta+G\int\limits_{(V)}\rho(\xi,\eta,\zeta,t)\left[\int\limits_{(V)}\rho(x,y,z,t)\frac{x-\xi}{\Delta^3}dxdydz\right]\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta. \tag{3.32}$$

The second term in the right-hand side of Eq. (3.32) disappears because of the symmetry of the integral expression with respect to x and ξ. In accordance with the Gauss–Ostrogradsky theorem, the first term in the right-hand side of Eq. (3.32) turns to zero. In fact

$$\int\limits_{(V)}\frac{\mathrm{d}p}{d\xi}\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta=\int\limits_{(S)}p\mathrm{d}\eta\mathrm{d}\zeta=0, \tag{3.33}$$

as pressure p on the border of the considered system is equal to zero owing to the absence of outer effects

Bearing in mind the possibility of passing to a Lagrangian coordinate system, and taking into account the law of the conservation of mass $\mathrm{d}m=\rho\mathrm{d}V=\rho_0\mathrm{d}V_0=\mathrm{d}m_0$, we get

$$\int\limits_{(V)}\rho\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}\xi\;\mathrm{d}\eta\mathrm{d}\zeta+\int\limits_{(V)}\rho\left(u\frac{\mathrm{d}u}{\mathrm{d}\xi}+v\frac{\mathrm{d}u}{\mathrm{d}\eta}+w\frac{\mathrm{d}u}{\mathrm{d}\zeta}\right)\mathrm{d}\xi\;\mathrm{d}\eta\;\mathrm{d}\zeta$$
$$=\int\limits_{(V)}\rho\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}V=\int\limits_{(V_0)}\rho_0\frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{d}V_0=\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V_0)}u\rho_0\mathrm{d}V_0=\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)}\rho u\mathrm{d}V,$$

where V_0 and ρ_0 are the volume and the density in the initial moment of time t_0.

Finally, Eq. (3.32) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)}\rho u\mathrm{d}V=0. \tag{3.34}$$

Integrating (3.34) with respect to time and writing analogous expressions for two other equations of the system (3.28), we obtain the first three integrals of motion:

$$\int_{(V)} \rho u \mathrm{d}V = a_1,$$
$$\int_{(V)} \rho v \mathrm{d}V = a_2, \tag{3.35}$$
$$\int_{(V)} \rho w \mathrm{d}V = a_3.$$

Equations (3.35) represent the law of conservation of the system moments. Integration constants $a_{1,}\ a_{2,}\ a_3$ can be obtained from the initial conditions.

We consider, the first equation of the system (3.35) using again the law of conservation of mass. Then it is obvious that

$$\int_{(V)} \rho u \mathrm{d}V = \int_{(V)} \frac{\mathrm{d}\xi}{\mathrm{d}t}\rho \mathrm{d}V = \int_{(V)} \frac{\mathrm{d}\xi}{\mathrm{d}t}\rho_0 \mathrm{d}V_0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{(V)} \xi\rho_0 \mathrm{d}V_0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{(V)} \xi\rho \mathrm{d}V = a_1. \tag{3.36}$$

Analogous expressions can be written for the two other Eq. (3.35). Integrating them with respect to time, we obtain integrals of motion of the centre of mass of the system in the form

$$\int_{(\vec{V})} \xi\rho \mathrm{d}V = a_1 t + b_1,$$
$$\int_{(V)} \eta\rho \mathrm{d}V = a_2 t + b_2, \tag{3.37}$$
$$\int_{(V)} \zeta\rho \mathrm{d}V = a_3 t + b_3.$$

We now derive three integrals of the moment of momentum of motion. For this purpose we multiply the second of Eq. (3.28) by $-\zeta$, the third by η, and then sum and integrate the resulting expressions with respect to volume V occupied by the system. We obtain

$$\int_{(V)} \rho\left(\eta\frac{\mathrm{d}w}{\mathrm{d}t} - \zeta\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}V = -\int_{(V)} \left(\eta\frac{\partial p}{\partial\zeta} - \zeta\frac{\partial p}{\partial\eta}\right)\mathrm{d}V + \int_{(V)} \rho\left(\eta\frac{\partial U_{\mathrm{G}}}{\partial\zeta} - \zeta\frac{\partial U_{\mathrm{G}}}{\partial\eta}\right)\mathrm{d}V. \tag{3.38}$$

Analogously, multiplying the first of Eq. (3.28) by ζ, the third by −ξ, then summing and integrating with respect to volume V, we obtain

$$\int\limits_{(V)} \rho\left(\zeta\frac{\mathrm{d}u}{\mathrm{d}t} - \xi\frac{\mathrm{d}w}{\mathrm{d}t}\right)\mathrm{d}V = -\int\limits_{(V)} \left(\zeta\frac{\partial p}{\partial \xi} - \xi\frac{\partial p}{\partial \zeta}\right)\mathrm{d}V + \int\limits_{(V)} \rho\left(\zeta\frac{\partial U_{\mathrm{G}}}{\partial \xi} - \xi\frac{\partial U_{\mathrm{G}}}{\partial \zeta}\right)\mathrm{d}V. \tag{3.39}$$

Multiplying the second of Eq. (3.28) by ξ, the first by –η, and summing and integrating as above, the third equality can be written as

$$\int\limits_{(\mathrm{V})} \rho\left(\xi\frac{\mathrm{d}v}{\mathrm{d}t} - \eta\frac{\mathrm{d}u}{\mathrm{d}t}\right)\mathrm{d}V = -\int\limits_{(\mathrm{V})} \left(\xi\frac{\partial p}{\partial \eta} - \eta\frac{\partial p}{\xi}\right)\mathrm{d}V + \int\limits_{(\mathrm{V})} \rho\left(\xi\frac{\partial U_{\mathrm{G}}}{\partial \eta} - \eta\frac{\partial U_{\mathrm{G}}}{\partial \xi}\right)\mathrm{d}V. \tag{3.40}$$

We write the second integral in the right-hand side of Eq. (3.38) in the form

$$\begin{aligned}\int\limits_{(V)} \rho\left(\eta\frac{\mathrm{d}w}{\mathrm{d}t} - \zeta\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}V =& G\int\limits_{(V)} \rho(\xi,\eta,\zeta,t)\eta\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \int\limits_{(V)} \rho(x,y,z,t)\frac{z-\zeta}{\Delta^3}\mathrm{d}x,\mathrm{d}y,\mathrm{d}z \\ &- G\int\limits_{(V)} \rho(\xi,\eta,\zeta,t)\zeta\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \int\limits_{(V)} \rho(x,y,z,t)\frac{y-\zeta}{\Delta^3}\mathrm{d}x,\mathrm{d}y,\mathrm{d}z.\end{aligned}$$

The integral is equal to zero owing to the asymmetry expressed by the integral expressions with respect to z, ζ and y, η. Because the pressure at the border of the domain S is equal to zero, the first term in the right-hand side of Eq. (3.38) is also equal to zero. Actually,

$$\int\limits_{(V)} \left(\eta\frac{\partial p}{\partial \zeta} - \zeta\frac{\partial p}{\partial \eta}\right)\mathrm{d}V = \int\limits_{(V)} \left[\frac{\mathrm{d}}{\mathrm{d}\eta}(\xi p) - \frac{\mathrm{d}}{\mathrm{d}\xi}(\eta p)\right]\mathrm{d}V = \int\limits_{(V)} [\xi p\mathrm{d}\xi\mathrm{d}\zeta - \eta p\mathrm{d}\eta\mathrm{d}\zeta] = 0$$

Taking into account the law of mass conservation, the left-hand side of Eq. (3.38) in the Lagrange coordinate system can be rewritten as

$$\int\limits_{(V)} \rho\left(\eta\frac{\mathrm{d}w}{\mathrm{d}t} - \zeta\frac{\mathrm{d}\nu}{\mathrm{d}t}\right)\mathrm{d}V = \int\limits_{(V)} p\frac{\mathrm{d}}{\mathrm{d}t}(\eta w - \zeta\nu)\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} p(\eta w - \zeta\nu)\mathrm{d}V = 0. \tag{3.41}$$

Integrating this equation with respect to time, the first of the three integrals is obtained:

$$\int\limits_{(V)} p(\eta w - \zeta v)\mathrm{d}V = C_1.$$

The other two integrals can be obtained analogously. Thus, the system of integrals of the moment of momentum has the form

$$\begin{aligned}
&\int\limits_{(V)} p(\eta w - \zeta v)\mathrm{d}V = C_1,\\
&\int\limits_{(V)} p(\zeta u - \xi w)\mathrm{d}V = C_2,\\
&\int\limits_{(V)} p(\xi v - \eta u)\mathrm{d}V = C_3.
\end{aligned} \tag{3.42}$$

To derive the tenth integral of motion representing the law of energy conservation, we multiply each of the system of Eq. (3.28) by u, v and w accordingly, and then sum and integrate the equality obtained with respect to the system volume

$$\begin{aligned}
\int\limits_{(V)} \rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}u + \frac{\mathrm{d}v}{\mathrm{d}t}v + \frac{\mathrm{d}w}{\mathrm{d}t}w\right)\mathrm{d}V = &- \int\limits_{(V)} \left(\frac{\partial p}{\partial \xi}u + \frac{\partial p}{\partial \eta}v + \frac{dp}{d\zeta}w\right)\mathrm{d}V\\
&+ \int\limits_{(V)} \rho(\xi,\eta,\zeta,t)\left(\frac{\partial U_{\mathrm{G}}}{\partial \xi}u + -\frac{\partial U_{\mathrm{G}}}{\partial \eta}v + \frac{\partial U_{\mathrm{G}}}{\partial \zeta}w\right)\mathrm{d}V.
\end{aligned} \tag{3.43}$$

Applying the law of mass conservation for an elementary volume, it can easily be seen that the left-hand side of Eq. (3.43) expresses the change of the velocity of kinetic energy of the system

$$\int\limits_{(V)} \rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}u + \frac{\mathrm{d}v}{\mathrm{d}t}v + \frac{\mathrm{d}w}{\mathrm{d}t}w\right)\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int\limits_{(V)} \left(u^2 + v + w^2\right)\mathrm{d}V\right] = \frac{\mathrm{d}}{\mathrm{d}t}(T).$$

The first integral in the right-hand side of Eq. (3.43) can be transferred into

$$-\int\limits_{(V)} \left(\frac{\partial p}{\partial \xi}u + \frac{\partial p}{\partial \eta}v + \frac{dp}{d\zeta}w\right)\mathrm{d}V = 3\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} p\mathrm{d}V$$

and gives the change of velocity of the internal energy of the system.

The second integral in the right-hand side of the same equation expresses the velocity of the potential energy change:

$$\int_{(V)} \rho(\xi,\eta,\zeta,t)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\left(\frac{\partial U_{\mathrm{G}}}{\partial\xi}\frac{\mathrm{d}\xi}{\mathrm{d}t} + -\frac{\partial U_{\mathrm{G}}}{\partial\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\partial U_{\mathrm{G}}}{\partial\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{1}{2}\int_{(V)} \rho(\xi,\eta,\zeta,t)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta U_{\mathrm{G}}\right] = -\frac{\mathrm{d}}{\mathrm{d}t}(U).$$

Finally, the law of energy conservation can be written in the form

$$T + U = W = E = \mathrm{const}, \tag{3.44}$$

where W is the internal energy of the system.

We now derive Jacobi's virial equation for a system described by Eqs. (3.28)–(3.31). For this purpose, we multiply each of Eq. (3.28) by ξ, η and ζ, respectively, summing and integrating the resulting expressions with respect to the volume of the system:

$$\int_{(V)} \rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}\xi + \frac{\mathrm{d}v}{\mathrm{d}t}\eta + \frac{\mathrm{d}w}{\mathrm{d}t}\zeta\right)\mathrm{d}V = -\int_{(V)} \left(\frac{\partial p}{\partial\xi}\xi + \frac{\partial p}{\partial\eta}\eta + \frac{\mathrm{d}p}{d\zeta}\zeta\right)\mathrm{d}V$$
$$+ \int_{(V)} \rho\left(\frac{\partial U_{\mathrm{G}}}{\partial\xi}\xi + -\frac{\partial U_{\mathrm{G}}}{\partial\eta}\eta + \frac{\partial U_{\mathrm{G}}}{\partial\zeta}\zeta\right)\mathrm{d}V. \tag{3.45}$$

Using the obtained identities considered in the previous section, we have

$$\frac{\mathrm{d}u}{\mathrm{d}t}\xi = \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2) - u^2\right),$$

$$\frac{dv}{\mathrm{d}t}\eta = \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\eta^2) - v^2\right),$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}\zeta = \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\zeta^2) - w^2\right).$$

Taking into account the law of conservation of mass for elementary volume, we transform the left-hand side of Eq. (3.45) as follows:

$$\int\limits_{(V)} \rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}\xi + \frac{\mathrm{d}v}{\mathrm{d}t}\eta + \frac{\mathrm{d}w}{\mathrm{d}t}\zeta\right)\mathrm{d}V = \frac{1}{2}\int\limits_{(V)} \rho\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\xi^2+\eta^2+\zeta^2\right)\mathrm{d}V - \int\limits_{(V)} \rho\left(u^2+v^2+w^2\right)\mathrm{d}V = \ddot{\varphi} - 2T, \tag{3.46}$$

where

$$\ddot{\varphi} = \frac{1}{2}\int\limits_{(V)} \rho\left(\xi^2+\eta^2+\zeta^2\right)\mathrm{d}V$$

is the Jacobi function and

$$T = \frac{1}{2}\int\limits_{(V)} \rho\left(u^2+v^2+w^2\right)\mathrm{d}V$$

is the kinetic energy of the system.

We now transform the first integral in the right-hand side of Eq. (3.45). Using the Gauss–Ostrogradsky theorem and the equality with zero pressure at the border of the system, we can write

$$-\int\limits_{(\mathrm{V})} \left(\frac{\partial p}{\partial\xi}\xi + \frac{\partial p}{\partial\eta}\eta + \frac{\partial p}{\partial\zeta}\zeta\right)\mathrm{d}V = -\int\limits_{(\mathrm{V})} \left[\frac{\partial}{\partial\xi}(p\xi) + \frac{\partial}{\partial\eta}(p\eta) + \frac{\partial}{\partial\zeta}(p\zeta)\right]\mathrm{d}V + 3\int\limits_{(V)} p\mathrm{dV} = 3\int\limits_{(V)} p\mathrm{d}V. \tag{3.47}$$

The obtained equation expresses the doubled internal energy of the system.

The second integral in the right-hand side of Eq. (3.45) is equal to the potential energy of the gravitational interaction of mass particles within the system

$$\int\limits_{(\mathrm{V})} \rho\left(\frac{\partial U_{\mathrm{G}}}{\partial\xi}\xi + -\frac{\partial U_{\mathrm{G}}}{\partial\eta}\eta + \frac{\partial U_{\mathrm{G}}}{\partial\zeta}\zeta\right)\mathrm{d}V = U. \tag{3.48}$$

Substituting Eqs. (3.46)– (3.48) into (3.45), Jacobi's virial equation is obtained in the form

$$\ddot{\Phi} - 2T = 3\int\limits_{(\mathrm{V})} p\mathrm{d}V + U. \tag{3.49}$$

Taking into account the law of conservation of energy (3.44), we rewrite Eq. (3.49) in a form which will be used farther:

$$\ddot{\Phi} = 2E - U, \tag{3.50}$$

where $E = T + U + W$ is the total energy of the system

3.4 Derivation of Jacobi's Virial Equation from Hamiltonian Equations

Let the system of material points be described by Hamiltonian equations of motion. Let also the considered systems consist of n material points with masses m_i. Its generalized coordinates and moments are q_i and $p_i = m_i(\mathrm{d}q_i/\mathrm{d}t)$. Hamiltonian equations for such a system can be written as

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\mathrm{d}q_i}, \\ \dot{q}_i &= \frac{\partial H}{\mathrm{d}p_i}, \end{aligned} \tag{3.51}$$

where $H\,(p, q)$ is the Hamiltonian; $i = 1, 2, \ldots, n$.

Using values q_i and p_i, we can construct the moment of momentum

$$\sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} m_i q_i \dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} \frac{m_i q_i^2}{2}\right).$$

Now the Jacobi function may be introduced:

$$\sum_{i=1}^{n} p_i q_i = \dot{\Phi}. \tag{3.52}$$

Differentiating Eq. (3.52) with respect to time, Jacobi's virial equation is obtained in the

$$\ddot{\Phi} = \sum_{i=1}^{n} \dot{p}_i q_i + \sum_{i=1}^{n} p_i \dot{q}_i. \tag{3.53}$$

Substituting expressions for $\dot{p}_i$ and $\dot{q}_i$ taken from the Hamiltonian Eq. (3.51) into the right-hand side of (3.52), we obtain Jacobi's virial equation written in Hamiltonian form:

$$\ddot{\Phi} = \sum_{i=1}^{n} \left(-\frac{\partial H}{\partial q_i} q_i + \frac{\partial H}{\partial p_i} p_i \right). \tag{3.54}$$

The Hamiltonian of the system of material points interacting according to the law of the inverse squares of distance is a homogeneous function in terms of moments p_i with a degree of homogeneity of the function equal to 2, and in terms of coordinates q_{i} with a degree of homogeneity equal to −1. It follows from this that

$$H(p,q) = T(p) + U(q)$$

and hence

$$\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} = 2T,$$

$$\sum_{i=1}^{n} q_i \frac{\partial H}{\partial q_i} = -U.$$

Taking these relationships into account, Eq. (3.54) acquires the usual form of Jacobi's virial Eq. (3.50) for the system of mass points interacting according to the law of inverse squares of distance.

Equation (4.54) is more general than Eq. (3.50). The use of generalized coordinates and moments as independent variables permits us to obtain the solution of Jacobi's virial equation, taking into account gravitational and electromagnetic perturbations as well as quantum effects, both in the framework of classical physics and in terms of the Hamiltonian written in an operator form. In the general case, Eq. (3.54) can be reduced to (3.50) as the potential energy of interaction of the system's points is a homogenous function of its coordinates.

3.5 Derivation of Jacobi's Virial Equation in Quantum Mechanics

It is known that in quantum mechanics some physical value L by definition takes the linear Hermitian operator $\hat{L}$. Any physical state of the system takes the normalized wave function ψ . The physical value of L can take the only eigenvalues of the operator $\hat{L}$. The mathematical expectation $\hat{L}$ of the value L at state ψ is determined by the diagonal matrix element

$$\overline{L} = <\psi|\hat{L}|\psi> . \tag{3.55}$$

The matrix element of the operators of the Cartesian coordinates $\hat{x}_i$ and the Cartesian components of the conjugated moments $\hat{p}_k$ calculated within wave functions f and g of the system satisfy Hamilton's equations of classical mechanics:

$$\frac{\mathrm{d}}{\mathrm{d}t} <f|\hat{p}_i|g> \; = - <f|\frac{\partial \hat{H}}{\mathrm{d}\hat{x}_i}|g> , \tag{3.56}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} <f|\hat{x}_i|g> \; = - <f|\frac{\partial \hat{H}}{\partial \hat{p}_i}|g> , \tag{3.57}$$

Where $\hat{H}$ is the operator which corresponds to the classical Hamiltonian.

Operators $\hat{p}_i$ and $\hat{x}_k$ satisfy the commutation relations

$$\begin{aligned} [\hat{p}_i, \hat{x}_k] &= i\hbar\delta_{ik}, \\ [\hat{p}_i, \hat{p}_k] &= 0, \\ [\hat{x}_i, \hat{x}_k] &= 0, \end{aligned} \tag{3.58}$$

where $\hbar$ is Planck's constant; δ_{ik} is the Kronecker's symbol; $\delta_{ik} = 1$ at $i = k$ and $\delta_{ik} = 0$ at $i \neq k$.

Operator components of momentum $\hat{p}_i$ for the functions whose arguments are Cartesian coordinates $\hat{x}_i$ have the form

$$\hat{p}_i = i\hbar \frac{\partial}{\partial x_i} \tag{3.59}$$

and reverse vector

$$\hat{p} = -i\hbar\nabla.$$

The derivative taken from the operator with respect to time does not depend explicitly on time; it is defined by the relation

$$\hat{L} = -\frac{i}{\hbar}\left[\hat{L}, \hat{H}\right], \tag{3.60}$$

where $\hat{H}$ is the Hamiltonian operator that can be obtained from the Hamiltonian of classical mechanics in accordance with the correspondence principle.

We have already noted that in the classical many-body problem the translational motion of the centre of mass can be separated from the relative motion of the mass points if only the inertial forces affect the system. We can show that in quantum mechanics the same separation is possible.

The Hamiltonian operator of a system of n particles which is not affected by external forces in coordinates is

$$\hat{\mathrm{H}} = -\frac{\hbar^2}{2}\sum_{i=1}^{n}\frac{\nabla_i^2}{m_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{i=1}^{n} U_{ik}(x_i - x_k, y_i - y_k, z_i - z_k). \tag{3.61}$$

Let us replace in (3.61) the three n coordinates x_i, y_i, z_i by coordinates X, Y, Z of the centre of mass and by coordinates $\xi_\lambda, \eta_\lambda, \zeta_\lambda$, which determine the position of a particle λ (λ = 1, 2, …, $n-1$) relative to particle n. We obtain

$$\begin{aligned} X &= \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \\ M &= \sum_{i=1}^{n} m_i, \\ \xi_\lambda &= x_\lambda - x_n, \end{aligned} \tag{3.62}$$

where λ = 1, 2, …, $n-1$.

Analogously the corresponding relations for Y, Z, η_λ, ζ_λ are obtained.

It is easy to obtain from (3.62) the following operator relations:

$$\frac{\mathrm{d}}{\mathrm{d}x_p} = \frac{m_p}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial \xi_p},\ \ p = 1, 2, \ldots, n-1,$$

$$\frac{\partial}{\partial X_n} = \frac{m_n}{M}\frac{\partial}{\partial X} - \sum_{\lambda=1}^{n-1}\frac{\partial}{\partial \xi_\lambda},$$

$$\begin{aligned} \sum_{\lambda=1}^{n-1}\frac{1}{\partial x_i}\frac{\partial^2}{\partial x_i^2} = \sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\left(\frac{m_\lambda^2}{M^2}\frac{\partial^2}{\partial X^2} + 2\frac{m_\lambda}{M}\frac{\partial^2}{\partial X \partial \xi_\lambda} + \frac{\partial^2}{\partial \xi_\lambda^2}\right) \\ + \frac{1}{m_n}\left(\frac{m_n^2}{M^2}\frac{\partial^2}{\partial X^2} - 2\frac{m_n}{M}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial X \partial \xi_\lambda} + \sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial \xi_\mu \partial \xi_\lambda}\right) \\ = \frac{1}{m_n}\frac{\partial^2}{\partial X^2} + \left(\sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\frac{\partial^2}{\partial \xi_\lambda^2} + \frac{1}{m_n}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial \xi_\mu \partial \xi_\lambda}\right), \end{aligned}$$

where summing on the Greek index is provided from 1 до n−1. It is seen that all the combined derivatives $\partial^2/\partial x \partial \xi_\lambda$ were mutually reduced and do not enter into the final expression. This allows the Hamiltonian to be separated into two parts:

$$H = H_o + H_r,$$

where, in the right-hand side, the first term

$$H_o = \frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial Z^2}\right)$$

describes the motion of the centre of mass, and the second term

$$H_r = -\frac{\hbar^2}{2}\left(\sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\nabla_\lambda^2 + \frac{1}{m_n}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\nabla_\lambda\nabla\mu\right) + U \tag{3.63}$$

describes the relative motion of the particles.

The potential energy in (3.63), which is

$$U = \frac{1}{2}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}U_{\lambda\mu}(\xi_\lambda - \xi_\mu, \eta_\lambda - \eta_\mu, \zeta_\lambda - \zeta_\mu) + \sum_{\lambda-1}^{n-1}U_{\lambda\mu}(\xi_\lambda, \eta_\lambda, \zeta_\lambda), \tag{3.64}$$

also certainly does not depend on the coordinates of the centre of mass.

Now the Schrödinger's equation

$$(H_o + H_r)\psi = E\psi \tag{3.65}$$

permits the separation of variables.

Assuming $\psi = \varphi(X, Y, Z)$ and $(\xi_\lambda, \eta_\lambda, \zeta_\lambda,)$, we obtain

$$-\frac{\hbar^2}{2V}\nabla^2\varphi = E_0\varphi, \tag{3.66}$$

$$H_r u = E_r u, \tag{3.67}$$

$$E_0 + E_r = E. \tag{3.68}$$

The solution of Eq. (3.66) has the form of a plane wave:

$$\begin{aligned}\varphi &= e^{i\overrightarrow{\mathrm{k}}\,\overrightarrow{\mathrm{R}}},\\ E_0 &= h^2 k_2/2M,\end{aligned} \tag{3.69}$$

where R is a vector with coordinates X, Y and Z.

The result obtained is in full accordance with the classical law of the conservation of motion of the centre of mass. This means that the centre of mass of the system moves like a material point with mass m and momentum $\hbar\bar{\mathrm{k}}$. The mode of relative motion of the particles is determined by Eq. (3.67), which does not depend on the motion of the centre of mass.

The existence in the right-hand side of Eq. (3.63) of the third term restricts further factorization of the function $u(\xi_\lambda, \eta_\lambda, \zeta_\lambda)$. Only in the two-body problem, where $n = 2$ and at $\lambda = \mu = 1$, a part of the Hamiltonian connected with the relative motion simplified and takes the form

$$H_r = -\frac{\hbar^2}{2}\left(\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2\right) + U_{12}(\xi_1, \eta_1, \zeta_1). \tag{3.70}$$

It seems that choosing the corresponding system of coordinates can lead us to an approach for separating the motion of the centre of mass to the many-body problem.

Introducing into Eq. (3.70) the reduced mass m^*, which is determined as in classical mechanics by the relation

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{m^*}, \tag{3.71}$$

and omitting indices in the notation for relative coordinates and potential energy U_{12}, we come to

$$-\frac{\hbar^2}{2m^*}\nabla^2 u + U(\xi, \eta, \zeta)u = E_r u, \tag{3.72}$$

This is Schrödinger's equation for the equivalent one-particle problem.

Considering the hydrogen atom in the framework of the one-particle problem, it is assumed that the nucleus is in ground state. In accordance with Eq. (3.72), the normalized mass of the nucleus and electron m^* should be introduced. No changes which account for the effect of the nucleus on the relative motion should be introduced. Because of the nucleus, mass m is much heavier than electron mass m_e; instead of Eq. (3.71) we can use its approximation

$$m^* = m\left(1 - \frac{m}{M}\right).$$

Comparing, for example the frequency of the red line H_α ($n = 3 - n' = 2$) in the spectrum of a hydrogen atom:

$$\omega(H_\alpha) = \frac{5}{36}\frac{m_H^* e^4}{2\hbar^2 h}$$

with the frequency of the corresponding line in the spectrum of a deuterium atom:

$$\omega(D_\alpha) = \frac{5}{36}\frac{m_D^* e^4}{2\hbar^2 h},$$

and taking into account that $m_D \approx 2m_H$, for the difference of frequencies, we obtain

$$\omega(D_\alpha) - \omega(H_\alpha) = \frac{m_D^* - m_H^*}{m_H^*}\omega(H_\alpha) \approx \frac{m}{2M_H}\omega(H_\alpha).$$

This difference is not difficult to observe experimentally. At wavelength 6563 Å it is equal to 4.12 cm^{-1}. Heavy hydrogen was discovered by Urey, Brickwedde and Murphy (1932), who observed a weak satellite D_α in the line H_α of the spectrum of natural hydrogen. This proves the practical significance of even the first integrals of motion.

We now show that the virial theorem is valid for any quantum mechanical system of particles retained by Coulomb (outer) forces:

$$2\overline{T} + U = 0.$$

We prove this by means of scale transformation of the coordinates keeping unchanged normalization of wave functions of a system.

The wave function of a many-particle system with masses m$_i$ and electron charge e_i satisfies Schrödinger's equation:

$$-\frac{\hbar^2}{2}\sum_{i=1}^{n-1}\frac{1}{m_i}\nabla_i^2\psi + \frac{1}{2}\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\frac{e_i e_k}{r_{\mathrm{ik}}}\psi = E\psi \tag{3.73}$$

and the normalization condition

$$\int \mathrm{d}\tau_1 \ldots \int \psi^*\psi \mathrm{d}\tau_n = 1. \tag{3.74}$$

The mean values of the kinetic and potential energies of a system at stage ψ are determined by the expressions

$$T = -\frac{\hbar^2}{2}\sum_{i=1}^{n-1}\frac{1}{m_i}\int \mathrm{d}\tau_1 \ldots . \int \psi^*\nabla_i^2\psi \mathrm{d}\tau_n \tag{3.75}$$

$$U = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{i=1}^{n-1} e_i e_k \int \mathrm{d}\tau_1 \ldots \int \mathrm{d}\tau_n \frac{\psi^*\psi}{r_{ik}}\mathrm{d}\tau_n. \tag{3.76}$$

The scale transformation

$$\bar{r}_i' = \lambda \bar{r}_i, \tag{3.77}$$

keeps in force the condition (3.74) and means that the wave function

$$\psi(\bar{r}_i, \ldots, \bar{r}_n) \tag{3.78}$$

is replaced by the function

$$\psi_\lambda = \lambda^{3n/2}\psi(\lambda\bar{r}_i, \ldots, \bar{r}_n). \tag{3.79}$$

Substituting (3.79) into Eqs. (3.76) and (3.75) and passing to new variables of integration (3.77), and taking into account that

$$\nabla_i^2 = \lambda^2 \nabla_i'^2, \quad \frac{1}{r_{ik}} = \lambda \frac{1}{r_{ik}'},$$

instead of the true value of the energy, $\overline{E} = \overline{T} + \overline{U}$, we obtain

$$\overline{E}(\lambda) = \lambda^2 \overline{T} + \lambda \overline{U}. \tag{3.80}$$

Equation (3.80) should have a minimum value in the case when the function which is the solution of the Schrödinger's equation is taken from the family of functions (3.79), i.e. when λ = 1. So, at λ = 1 the expression

$$\frac{\partial \overline{E}(\lambda)}{\partial \lambda} = 2\lambda^2 \overline{T} + \overline{U}$$

should turn into zero, and thus

$$2\overline{T} + \overline{U} = 0,$$

which is what we want to prove.

We now derive Jacobi's virial equation for a particle in the inner force field with the potential $U(q)$ and fulfilling the condition

$$-q\nabla U(q) = U. \tag{3.81}$$

using the quantum mechanical principle of correspondence. We shall also show that in quantum mechanics Jacobi's virial equation has the same form and content as in classical mechanics. The only difference being that its terms are corresponding operators.

In the simplest case, the Hamiltonian of a particle is written as

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \hat{U}, \tag{3.82}$$

and its Jacobi function is

$$\hat{\Phi} = \frac{1}{2} m\hat{q}^2. \tag{3.83}$$

It is clear that the following relations are valid:

$$\nabla\hat{\Phi} = m\hat{q},$$

$$\nabla^2\hat{\Phi} = m,$$

Following the definition of the derivative with respect to time from the operator of the Jacobi function of a particle (3.60), we can write

$$\ddot{\Phi} = -\frac{1}{h}\left[\hat{\Phi}, \hat{H}\right],$$

where, after corresponding simplification, quantum mechanical Poisson brackets can be reduced to the form

$$\left[\hat{\Phi}, \hat{H}\right] = \frac{h^2}{2m}\left\{\nabla^2\hat{\Phi} + 2\left(\nabla\hat{\Phi}\right)\nabla\right\} = \frac{h^2}{2m}(m + 2mq\nabla). \tag{3.84}$$

The second derivative with respect to time from the operator of the Jacobi function is

$$\ddot{\Phi} = -\frac{1}{h^2}\left\{\left[\hat{\Phi}, \hat{H}\right], \hat{H}\right\} \tag{3.85}$$

Substituting the corresponding value of $\left[\hat{\Phi}, \hat{H}\right]$ and $\hat{H}$ from (3.84) and (3.82) into the right-hand side of (3.85), we obtain

$$\ddot{\Phi} = -\frac{h^2}{2m}\frac{1}{h^2}\left[(m + 2m\hat{q}\nabla), \left(-\frac{h^2}{2m}\nabla^2 + \hat{U}\right)\right]. \tag{3.86}$$

After simple transformation, the right-hand side of (3.86) will be

$$\ddot{\Phi} - \frac{1}{2m}\left\{2h^2\nabla^2 + 2m\hat{q}\left(\nabla\hat{U}\right)\right\} = -\frac{2h^2}{2m\nabla^2} + \hat{U}, \tag{3.87}$$

where in writing this expression in the right-hand side, we used condition (3.81).

Add and subtract the operator $\hat{\mathrm{U}}$ from the right-hand side of Eq. (3.87) and, following the definition of the Hamiltonian of the system (3.82), we obtain the quantum mechanical Jacobi virial equation (equation of dynamical equilibrium of the system), which has the form

$$\ddot{\Phi} = 2\hat{H} - \hat{U}. \tag{3.88}$$

From Eq. (3.88), by averaging with respect to time, we obtain the quantum mechanical analogue of the classical virial theorem (equation of hydrostatic equilibrium of the system). In accordance with this theorem, the following relation is kept for a particle performing finite motion in space

$$2\hat{H} = \hat{U}. \tag{3.89}$$

Analogously, one can derive Jacobi's virial equation and the classical virial theorem for a many-particle system, the interaction potential for which depends on distance between any particle pair and is a homogeneous function of the coordinates. In particular, Jacobi's virial equation for Coulomb interactions will have the form of Eq. (3.88).

3.6 General Covariant Form of Jacobi's Virial Equation

Jacobi's initial equation,

$$\ddot{\Phi} = 2E - U,$$

which was derived in the framework of Newtonian mechanics and is correct for the system of material points interacting according to Newton and Coulomb laws, includes two scalar functions Φ and U relates to each other by a differential relation. We draw attention to the fact that neither function, in its structure, depends explicitly on the motion of the particles constituting the body. The Jacobi function Φ is defined by integrating the integrand $\rho(r)r^2$ over the volume (where $\rho(r)$ is the mass density and r is the radius vector of the material point) and is independent in the explicit form of the particle velocities. The potential energy U also represents the integral of $m(r)\mathrm{d}m(r)/r$ over the volume (where $m(r)$ is the mass of the sphere's shell of radius r; $dm(r)$ is the shell's mass) independent of the motion of the particles for the same reason.

Let us derive Jacobi's equation from Einstein's equation written in the form

$$\Delta G = 2\pi T, \tag{3.90}$$

where ΔG and T are Einstein tensor energy–momentum tensor accordingly.

In fact, since the covariant divergence of Einstein's tensor is equal to zero, we consider the covariant divergence of the energy–momentum tensor T only of substance and fields (not gravitational). Moreover, the ordinary divergence of the sum of the tensor T and pseudotensor t of the energy–momentum of the gravitational field can be substituted for the covariant divergence of the tensor T. This ordinary divergence leads to the existence of the considered quantities.

Let us define the sum of the tensor T and pseudotensor t through T_{ij} and derive Jacobi's equation in this notation.

The equation for ordinary divergence of the sum $T_{ij} = (T + t)_{ij}$ can be written as

$$T_{0,k,k} - T_{00,0} \tag{3.91}$$

$$T_{jk,k} - T_{j0,0} = 0. \tag{3.92}$$

We multiply Eq. (3.92) by x^j and integrate over the whole space (assuming the existence of a synchronous coordinate system). Integrating by parts, neglecting the surface integrals (they vanish at infinity), and transforming to symmetrical form with respect to indices, we obtain

$$\int T_{ij}\mathrm{d}V = \frac{1}{2}\left[\int \left(T_{i0}x^j + T_{j0}x^i\right)\mathrm{d}V\right] = 0, \tag{3.93}$$

where i and j are spatial indices.

Similarly, multiplying (3.91) by x^i x^j and integrating over the whole space, it follows that

$$\left[\int T_{00}x^i x^j \mathrm{d}V\right]_0 = -\int \left(T_{i0}x^j + T_{j0}x^i\right)\mathrm{d}V. \tag{3.94}$$

From (3.93) and (3.94), we finally get

$$\int T_{ij}\mathrm{d}V = \frac{1}{2}\left[\int T_{00}x^i x^j\right]_{0,0}. \tag{3.95}$$

It is worth recalling that T_{00} also includes the gravitational defect of the mass due to the pseudotensor t by definition.

The integral $\int T_{00}x^i x^j \mathrm{d}V$ represents the generalization of the Jacobi function $\Phi = \frac{1}{2}\int \rho r^2 \mathrm{d}V$ introduced earlier, if we take the spur (also commonly known as the trace) of Eq. (3.95). Let us clarify this operation.

In Eq. (3.95) the spur is taken by the spatial coordinates. It is therefore necessary either to represent the total zero spur by four indices, as happens in the case of a transverse electromagnetic field, or to represent the relationship between the reduced spur with three indices and the total spur, as happens in the case of the energy–momentum tensor of matter.

Special care should be taken while representing the spur of the pseudotensor of the energy–momentum t. Consider the post-Newtonian approximation. In this approximation, assuming the value of $2u$ to be $-g_{oo}$ -1, the components of the pseudotensor t are written in the form

$$t^{00} = -\frac{7}{8\pi} u_{j,i},$$

$$t^{ij} = -\frac{1}{4\pi}\left(u_{j,i} - \frac{1}{2}\delta_{ij} u_k u_k\right),$$

so that

$$S_p t = t^{00} + S_p\left(t^{ij}\right) = -\frac{1}{\pi} u_i u_j = \frac{1}{7} t^{00},$$

$$S_p\left(t^{ij}\right) = \frac{6}{7} t^{00}.$$

The spur in the left-hand side of Eq. (3.95) can therefore be reduced to the energy of the Coulomb field, the total energy of the transverse electromagnetic field and the gravitational energy (when it can be separated, i.e. post-Newtonian approximation).

Finally, it follows in this case that the scalar form of Jacobi's equation holds:

$$\Phi_{0,0} = mc^2, \tag{3.96}$$

where m is the mass, accounting for the baryon defect of the mass and the total energy of the electromagnetic radiation. We do not take into account the radiation of the gravitational waves.

The result obtained by Tolman for the spherical mass distribution (Tolman 1969) is of interest:

$$m = 4\pi \int \hat{\varepsilon} r^2 \mathrm{d}r, \tag{3.97}$$

where r is the radius and $\hat{\varepsilon}$ is the energy density.

The integral (3.97) acquires a form which is also valid in the case of flat space-time. This result can be explained as follows. The curvature of space-time is exactly compensated by the mass defect. This probably explains the fact that Jacobi's virial equation, derived from Newton's equations of motion which are valid in the case of non-relativistic approximation for a weak gravitational field, becomes more universal than the equations from which it was derived.

We shall not study the general tensor of Jacobi's virial equation, since in the framework of the assumed symmetry for the considered problems we are interested only in the scalar form of the equation as applied to electromagnetic interactions. As follows from the above remarks, in this case Jacobi's equation remains unchanged and the energy of the free electromagnetic field is accounted for in the term defining the total energy of the system. Total energy enters into Jacobi's equation without the electromagnetic energy irradiated up to the considered moment of time. Moreover, for the initial moment of time we take the moment of system formation. This irradiated energy appears also to be responsible for the growth of the gravitational mass defect in the system, as was mentioned above.

3.7 Relativistic Analogue of Jacobi's Virial Equation

Let us derive Jacobi's virial equation for asymptotically flat space-time. We write the expression of a 4-moment of momentum of a particle:

$$p^i x_i, \tag{3.98}$$

where $p^i = mcu^i$ is the 4-momentum of the particle; c is the velocity of light; $u^i = \mathrm{d}x^i/\mathrm{d}s$ is the 4- velocity; x^i is the 4-coordinate of the particle; s is the interval of events, and i is the running index with values 0, 1, 2 and 3.

In asymptotically flat space-time, we write

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc\frac{\mathrm{d}}{\mathrm{d}s}\left(u^i x_i\right) = mc\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{x^i x_i}{2}\right). \tag{3.99}$$

Since

$$x^i x_i = c^2 t^2 - r^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s} = \frac{\gamma}{c}\frac{\mathrm{d}}{\mathrm{d}t},$$

where $\gamma = 1/\sqrt{1-(\mathrm{v}^2/c^2)}$, and r is the radius of mass particle.

Then we continue the transformation of Eq. (3.99):

$$mc\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{x^i x_i}{2}\right) = mc\frac{\gamma^2}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{c^2t^2 - r^2}{2}\right) = mc\gamma^2 - \frac{\gamma^2}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{mr^2}{2}\right),$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc\gamma^2 - \frac{\gamma^2}{c}\ddot{\Phi}, \tag{3.100}$$

where

$$\ddot{\Phi} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{mr^2}{2}\right)$$

is the Jacobi function.

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc\frac{\mathrm{d}}{\mathrm{d}s}\left(u^i x_i\right) = mcu^i u_i + mc\frac{\mathrm{d}u^i}{\mathrm{d}s}x_i. \tag{3.101}$$

Using the identity $u_i u^i \equiv 1$ and the geodetic equation

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} = -\Gamma^i_{k\ell}\mathrm{u}^k\mathrm{u}^i,$$

where

$$\Gamma^i_{k\ell} = \frac{1}{2}g^{\mathrm{im}}\left(\frac{\partial g_{km}}{\partial x^\ell} + \frac{\partial g_{\ell m}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^m}\right)$$

are Christoffel's symbols, the Eq. (3.101) will be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc - mcx_i\Gamma^i_{k\ell}u^k\mathrm{u}^\ell. \tag{3.102}$$

The metric tensor g_{ik} for a weak stationary gravitational field is

$$g_{ik} = \eta_{ik} + \xi_{ik}, \tag{3.103}$$

where in our notation η_{ik} is the Lorentz tensor with signature (+, -, -, -).
For the Schwarzschild metric tensor ξ_{ik}, we write

$$\xi_{00} = -\frac{r_g}{r};\ \xi_{11} = -\frac{1}{1 - r_g/r} + 1 \approx -\frac{r_g}{r};$$

$$\xi_{ik} = 0 \text{ if } i \neq k \text{ and } \underset{-}{/} \neq 0.1. \tag{3.104}$$

Here $r_g = 2GV/c^2$ is the Schwarzschild gravitational radius of the mass m'.
Now we can rewrite the second term in the right-hand side of Eq. (3.102), using Eqs. (3.103) and (3.104)

$$\begin{aligned} mcx_i\Gamma^i_{k\ell}u^k\mathrm{u}^\ell = {} & mcx^m u^k \mathrm{u}^\ell\left(\frac{\partial\xi_{km}}{\partial x^m} - \frac{1}{2}\frac{\partial\xi_{k\ell}}{\partial x^m}\right) \\ & = mc\left(x^0u^0u^1\frac{\partial\xi_{00}}{\partial x^1} + x^1u^1u^1\frac{\partial\xi_{11}}{\partial x^1} - \frac{1}{2}x^1u^0u^0\frac{\partial\xi_{00}}{\partial x^1} - x^1u^1u^1\frac{\partial\xi_{11}}{\partial x^1}\right). \end{aligned} \tag{3.105}$$

But $u^1 << u^0 = \gamma$ and $x^1 = r$.

We therefore obtain for Eq. (3.105)

$$
\begin{aligned}
mcx_i\Gamma^i_{k\ell}u^k\mathrm{u}^\ell &= -\frac{mc}{2}x^1u^0u^0\frac{\partial\xi_{00}}{\partial x^1} \\
&= -\frac{mc}{2}r\gamma^2\frac{r_g}{r^2} = \frac{mc}{2}\gamma^2\frac{2Gm'}{c^2r} = -\frac{\gamma^2}{c}\frac{Gm'm}{r}.
\end{aligned}
\tag{3.106}
$$

Finally, we see that

$$
\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc - \frac{\gamma^2}{c}U, \tag{3.107}
$$

where U is the potential energy of the mass in the gravitational field of the mass m'.

Identification of the expression $(\mathrm{d}/\mathrm{d}s)(p^i x_i)$ obtained from Eqs. (3.100) and (3.107) gives

$$
mc\gamma^2 - \frac{\gamma^2}{c}\ddot{\Phi} = mc - \frac{\gamma^2}{c}U. \tag{3.108}
$$

It is easy to see that

$$
mc\left(\gamma^2 - 1\right) = mc\left(\frac{1}{1 - v^2/c^2} - 1\right) = mc\frac{v^2}{c^2}\frac{1}{1 - v^2/c^2} = \frac{\gamma^2}{c}mv^2 = \frac{\gamma^2}{c}2T.
$$

We then obtain

$$
\frac{\gamma^2}{c}\ddot{\Phi} = \frac{\gamma^2}{c}U + \frac{\gamma^2}{c}2T.
$$

which gives

$$
\ddot{\Phi} = U + 2T,
$$

or

$$
\ddot{\Phi} = 2E + U, \tag{3.109}
$$

where T is the kinetic energy of the particle m and $E = U + T$ is its total energy.

Equations (3.109) are known as classical Jacobi's virial equations, and the expression (3.102) represents its relativistic analogue for asymptotically flat space-time.

3.8 Direct Derivation of the Equation of Virial Oscillation from Einstein's Equations

Weinberg (1972) reduced Einstein's equation for homogeneous isotropic space with the help of the Robertson–Walker metric, to the following scalar form:

$$3\ddot{R} = -4G(\rho + 3p)R, \tag{3.110}$$

$$\ddot{R}R + 2\left(\dot{R}\right)^2 + 2k = 4\pi G(\rho - p)R^2, \tag{3.111}$$

where R is the radius of the Universe; p is the radiation pressure (mass defect) and ρ is the density of matter without mass defect.

Multiplying Eq. (3.110) by $R/3$ and summing it with (3.111), we obtain:

$$\left(\ddot{R^2}\right) + 2k = 8\pi G R^2\left(\frac{1}{3}\rho - p\right). \tag{3.112}$$

When $\rho << p$ and ρR^3 = const (dust cloud), and taking into account that for curved space (Landau and Lifshitz 1973b)

$$\rho R^3 = \frac{m}{2\pi^2}, \tag{3.113}$$

where m is the total mass of the particles constituting the cloud, expression (3.113) is transformed into

$$\left(\ddot{R^2}\right) + 2k = \frac{8\pi}{3} G \frac{m}{2\pi^2}\frac{1}{R}. \tag{3.114}$$

Since from the Jacobi function we have $\Phi = mR^2/2$, Eq. (3.114) can be rewritten as

$$\ddot{\Phi} + km = \frac{2}{3\pi} G m^2 \sqrt{\frac{m}{2}} \frac{1}{\sqrt{\Phi}} \tag{3.115}$$

or

$$\ddot{\Phi} + km = \frac{\sqrt{2}}{3\pi}\sqrt{G^2 m^5}\frac{1}{\sqrt{\Phi}}. \tag{3.116}$$

Finally, the equation of virial oscillations can be easily obtained in the known form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \tag{3.117}$$

where $A = km = E$ is the total energy, and B is a constant equal to $Gm^{5/2}$ multiplied by a factor which depends on the realization of the mass defect and on the period of the $\alpha^2\beta$ form factors (equal to $1/\sqrt{2}$).

When $p = \rho/3$, the equation of virial oscillations for radiation can be obtained from Eq. (3.112):

$$\ddot{\Phi} = -A.$$

Equations (3.110) and (3.111) are valid for all natural systems which exhibit a central symmetry of mass distribution. For celestial bodies Eq. (3.112) is written as

$$\rho R^3 = \frac{3m}{4\pi}.$$

Then, from (3.112) it follows that

$$\left(\ddot{R}\right)^2 + 2k = \frac{8\pi}{3} G R^2 \left(1 - \frac{3p}{\rho}\right) = \frac{2Gm}{R}\left(1 - \frac{3p}{\rho}\right).$$

Now, Eq. (3.117) becomes

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}\left(1 - \frac{3p}{\rho}\right). \tag{3.118}$$

As Weinberg (1972)pointed out, the inequality $0 < 3p \leq \rho$ holds for celestial bodies, and in the most general case we can write

$$p = (\gamma - 1)(\rho - n\mu),$$

where n is the density of particles and μ is the mass of a particle.

Therefore, $(\rho - n\mu)$ is the mass defect and γ is the polytropic index, which for stable system ranges from 0 to 5/3 for non-relativistic objects, and $\gamma \geq 4/3$ for ultra-relativistic objects. For $\gamma > 5/3$, the body expands indefinitely, and at $\gamma \leq 4/3$ collapse of the body occurs.

For actually existing celestial bodies, where the absence of heat equilibrium is taken into account (in the case of a discrete system), pressure is defined as (Weinberg 1972)

$$p = \frac{1}{3}[\rho + f(\rho, n)],$$

where $f(\rho,n) = T^0_\alpha$ is a function of the energy density ρ and the density n (number of particles per unit volume). This function is equal to zero in the ultra-relativistic limit and in the non-relativistic limit it is equal to

$$[-n\mu + (\rho - n\mu)] = -2n\mu + \rho.$$

In both limiting cases, pressure p is

$$p = \frac{1}{3}\rho \quad \text{and} \quad p = \frac{2}{3}(\rho - n\mu).$$

Hence, in Eq. (3.118) the undetermined factor in B is equal to zero and $[(2\mu n/\rho) -1)$ or $(1-(2\Delta/\rho)]$, where $\Delta = \mu n - \rho$ is the mass defect.

Finally, taking into account the mass defect in Eq. (3.118) shows that the constant $B = B_o D$, where B_o is of Newtonian nature ($aGm^{5/2}$) and D, a relativistic correction, is smaller than 1.

Now let us estimate this correction D in the case of the white dwarf and the neutron star models according to Weinberg.

The equation determining the density of particles of particles when Fermi–Dirac statistics hold can be written as

$$n = \frac{k_F^3}{3\pi\hbar^2},$$

where n is the number of particles in the volume; k_F the radius of the Fermi sphere and $\hbar$ is Planck's constant.

The density of matter of a star is written as

$$\rho = n\mu_p n_p,$$

where μ_p is the mass of a proton and n_p the average number of protons in a nuclei.

The critical density of matter in a star is

$$\rho_{cr} = \frac{\mu_p n_p \mu_e^3}{3\pi\hbar^3},$$

where μ_e is the electron mass.

Introducing the new variables $Z_1 = \rho/\rho_{cr}$ and $Z_2 = \rho/\rho_{cr}$, the equation of state for white dwarfs can be rewritten as follows:

$$Z_1 = \frac{3\mu_e}{\mu_p} F_1(Z_1),$$

$$Z_2 = \frac{3\mu_e}{\mu_p} F_2(Z_2),$$

where F_1 and F_2 are some transcendental functions.

For neutron stars, the critical density is

$$\rho_{cr} = \frac{\mu_p^4}{3\pi\hbar^3},$$

and the equations of state are written

$$Z_1 = 3F_1(Z_1),$$

$$Z_2 = 3F_2(Z_2).$$

Solving the equations of state for the two limiting cases when $\rho << \rho_{cr}$ (i.e. when the polytropic indexes are 5/3 and 4/3 respectively), we obtain for white dwarfs, respectively

$$\rho_e = \frac{3}{2}p \quad \text{и} \quad \rho_e = 3p.$$

For neutron stars, in the limiting cases ($\rho << \rho_{cr}$) and ($\rho >> \rho_{cr}$), we have the same form of relations:

$$\rho = \frac{3}{2}p \quad \text{и} \quad \rho = 3p,$$

where ρ is the total density of matter.

In the ultra-relativistic limit, the relativistic correction will have very large values ($D = 0$), which means that the total collapse of the star (Oppenheimer–Volkoff limit) is leading to the formation of a black hole.

Thus, we have obtained the equation of virial oscillations (8.152) directly in the most general case and without having to assume the constancy of the form factor product $\alpha^2\beta$. Since the same equation follows from Jacobi's equation with the use of the hypothesis, we conclude that the relation $\alpha\beta$ = const was proven.

We should also note that modern astrophysical studies of the oscillation of celestial bodies in the non-relativistic approximation are based on the supposition that these movements have a homologous structure (Misner et al. 1973; Weinberg 1972; Frank-Kamenetsky 1959; Zeldovich and Novikov 1967). It can easily be verified that the supposition of homology is a sufficient condition to prove the constancy of the form factor product $\alpha^2\beta$ which is the main point in the derivation of the equation of virial oscillations from Jacobi's equation.

The mathematical formulation of the homologous motion of matter in the course of oscillation of a celestial body is written as follows:

$$r(t) = t(0)\cdot f(t),$$

where $r(t)$ is the radius of a given layer-shell of the body and $f(t)$ is an arbitrary function of time.

Let us introduce the Lagrange coordinates, where m is the mass inside the sphere of radius r, and dm is the mass of shell of radius r and thickness dr. According to the property of Lagrange coordinates, they are independent of time.

Then, the Jacobi function and the potential energy are written as:

$$\Phi = \frac{1}{2}\int_{0}^{m} r^2 \mathrm{d}m,$$

$$|U| = G\int_{0}^{m} \frac{m\mathrm{d}m}{r}.$$

Using the assumption that the motion is homologous, these expressions can be rewritten as:

$$\Phi = \frac{1}{2}f^2(t)\int_{0}^{m} r^2(0)\mathrm{d}m,$$

$$U = \frac{G}{f(t)}G\int_{0}^{m} \frac{m\mathrm{d}m}{r(0)}.$$

Integrals on the right-hand side of these expressions do not depend on time and are therefore constants. Thus, the product $U^2\Phi$ does not depend on time and is also a constant.

Note that in the works of the authors mentioned above, the formula for the pulsation frequency of celestial bodies has been obtained assuming small amplitudes and the validity of the harmonic law of pulsations. Our approach allows the same frequency of pulsations to be obtained without the above restricting assumptions. Moreover, by comparing the two expressions which give equivalent results, it is possible to obtain the politropic index which enters into the astrophysical formula for the frequency of pulsations.

3.9 Derivation of Jacobi's Virial Equation in Statistical Mechanics

Statistical mechanics accepts the considered system in equilibrium state a priori at the stage of the problem formulation. Let us derive the virial theorem also for this branch of mechanics.

Denote by r_i generalized moment p_i, …, p_f or generalized coordinate q_i, …, q_f of the system points. Assume also that the value r_i of a physical system is changing from a to b and there is equality $H(a) = \infty$, or $a = 0$, or there are both effects, or also $H(b) = \infty$, or $b = 0$, or both effects. Let symbol $< \ldots >$ denotes mean value of the classical canonic distribution. Then it is possible to show the correctness of the following statement:

$$< r_i \frac{\partial H}{\partial r_i} > \ = kT, \tag{3.119}$$

where k is the Boltzmann's constant; T is the temperature and H is the system's Hamiltonian.

In fact, the normalization integral for the canonical distribution is

$$1 = A \int \ldots \int e^{-\frac{H}{kT}} \mathrm{d}q_1 \ldots \mathrm{d}p_f. \tag{3.120}$$

Integrating expression (3.120) by parts on q_1, one has

$$1 = A \int \ldots \int \left(q_1 e^{-\frac{H}{kT}}\right)\Big|_a^b . \mathrm{d}q_2 \ldots \mathrm{d}p_f + \frac{A}{kT} \int \ldots \int q_1 \ldots \mathrm{d}p_f. \tag{3.121}$$

According to the above limitation, first integral contributes nothing, and from this expression follows correctness of the Eq. (3.119), which is called the theorem of the uniform distribution.

We can derive now the virial theorem in classical statistical mechanics. For this, we assume that a particle i occurs in the point $r_i = (q_{ix}, q_{iy}, q_{iz})$ and it is acted on by the force $\bar{F}_i = \frac{\mathrm{d}\bar{p}}{\mathrm{d}t}$, where $\bar{p}_i = (p_{ix}, p_{iy}, p_{iz})$. By definition the system's virial of n particles is the expression $C = -\frac{1}{2}\sum_{i=1}^{n} \bar{F}_i \cdot \bar{r}_i$ which is averaged in time. Assuming that the motion of particles is described by the Hamiltonian equations of motion ($\frac{\mathrm{d}q_{ij}}{\mathrm{d}t} = \frac{\mathrm{d}H}{\mathrm{d}p_{ij}}, \frac{\mathrm{d}p_{ij}}{\mathrm{d}t} = -\frac{\mathrm{d}H}{\mathrm{d}q_{ij}}$; $i = 1, 2, \ldots, n$; $j = x, y, z$) and for the system the ergodic hypothesis system is correct, according to which the averaging over the ensemble and on time leads to the same results, we can show that the system virial C is equal to

$$C = \frac{3}{2} nkT. \tag{3.122}$$

According to expression (3.119)

$$< r_i \frac{\partial H}{\partial r_i} > \ = \ < -q_{ij} F_{ij} > \ = kT. \tag{3.123}$$

Now, the correctness of expression (3.122) follows from Eq. (3.123) and from the definition of the virial system.

Farther, it is easy to show that in the case when the force is defined by the potential W, that is,

$$F_{ij} = -\frac{\partial W}{\partial q_{ij}},$$

and the moment enters only to the kinetic energy $k = \sum_{i=1}^{n} \frac{p_i^2}{2m}$, then the following equality is correct:

$$\bar{k} = \frac{1}{2}\sum_{i=1}^{n} \bar{\nabla} \bar{W} \bar{r}_i = \frac{3}{2} knT = C. \tag{3.124}$$

If the forces interacting between the gas particles are $f\left(\left|\bar{r}_j - \bar{r}_k\right|\right) = f\left(r_{jk}\right)$ and depend only on distance between the particles, then these forces contribute to virial as

$$-\frac{1}{2}\sum_{1\leq j<k\leq n} r_{jk} f_{jk}\left(K_{jk}\right), \tag{3.125}$$

where the summation is over all particle pairs. In fact, taking the force $f(r_{jk})$, like in the case of repulsion, as positive value, the force affecting on j-particle in the form

$$\bar{F}_j = \frac{\bar{r}_j - \bar{r}_k}{r_{jk}} f\left(r_{jk}\right),$$

and the force affecting on k-particle in the form

$$\bar{F}_k = \frac{\bar{r}_k - \bar{r}_j}{r_{\mathrm{jk}}} f\left(r_{\mathrm{jk}}\right),$$

then the contribution to virial from the pair (j, k) is equal to

$$-\frac{1}{2}\left(\bar{r}_j \bar{F}_j + \bar{r}_k \bar{F}_k\right) = -\frac{1}{2}\left(\bar{r}_j - \bar{r}_k\right)\frac{\bar{r}_j - \bar{r}_k}{\bar{r}_{jk}} = -\frac{1}{2}\bar{r}_{jk} f\left(\bar{r}_{jk}\right),$$

from where the correctness of (3.125) follows.

If the gas occurs in a vessel of v volume, then the force affecting from the side on the gas of p pressure contributes to virial by $3/2pv$. In fact, the force acting from the vessel side on an element da of the surface to – $p\bar{n}$da, where n is the unit vector of the outer normal. The contribution to virial here is

$$\frac{1}{2}p\int_{(S)} \bar{n}\cdot\bar{r}\mathrm{d}a = \frac{1}{2}\mathrm{d}iv\bar{r}\mathrm{d}v = \frac{3}{2}pv, \tag{3.126}$$

where the Gauss theorem and equality $\mathrm{d}iv\bar{r} = 3$ were used.

Let us show now that for classical non-ideal gas of n particle volume at temperature T, the following expression is correct:

$$pv = nkT + \frac{1}{3}\sum_{1 \le j < k \le n} r_{jk} f\left(r_{jk}\right). \tag{3.127}$$

In fact, applying expressions (3.124) and (3.126), we can write

$$C = \bar{k} = \frac{3}{2}nkT = \frac{3}{2}pv - \frac{1}{2}\sum_{1 \le j < k \le n} r_{jk} f\left(r_{jk}\right),$$

from where the required result follows.

For the gas, where the energy and its potential results by interaction of gas particles, the virial theorem follows in the form

$$(u+2)\bar{k} = u\bar{E} + 3pv. \tag{3.128}$$

Really, in this case

$$\sum_{1 \le j < k \le n} r_{jk} f\left(r_{jk}\right) = -r_{jk}\left(\frac{\partial W}{\partial r_{jk}}\right) = -uU,$$

from where one has that

$$\bar{k} = \frac{3}{2}pv + \frac{1}{2}u\bar{U}$$

Multiplying both parts of the above expression by 2 and adding nk to both sides, the expression (3.128) appears, where E is the total energy of the system equal to

$$E = T + U.$$

The generalized virial theorem we derived in Sect. 2.5 of Chapter 2, or Jacobi's virial equation, is valid for the considered physical system in the framework of statistical mechanics.

3.10 Universality of Jacobi's Virial Equation for Description of Dynamics of Natural Systems

It follows from the derivation of Jacobi's virial equation, where the linear forces and momentums were substituted by their volumetric values, that it appears to be a universal mathematical model for description of gravitational interaction energy and corresponding dynamics of natural bodies in the framework of the existing physical models, their total, potential and kinetic energy and the polar moment of inertia. As it is seen, the body's energy and moment of inertia are in a functional relationship and change by oscillating motion. Moreover, the second derivative of the moment of inertia $\ddot{\Phi}$ expresses the potential and kinetic energy of the body's interacting particles, which in fact, is the sought for the Newton force function. This is a unique property of the virial Eq. (3.1).

At averaging of virial equation $\ddot{\Phi} = 2E + U$, when the first derivative from the system's moment of inertia Φ has a constant value ($\dot{\Phi} = 2E + U = \text{const.}$) it can represent the classical virial theorem like $2E = U$ or $-U = 2T$, which determines the condition of the hydrostatic equilibrium state.

The starting point for derivation of the virial theorem is particle momentum. By Newton's definition this value "is a certain measure determined proportionally to the velocity and the mass". This value is defined or is found experimentally. All other force parameters are obtained by transformation of the initial momentum and those actions are explained by physical interaction of the mass particles, which are the carrier of momentum. In fact, we recognize the momentum to be "innate", according to Newton's terminology, value, i.e. the hereditary value. Under the "innate" value Newton understood "both the resistance and the pressure of the mass" and finally the effect acquires its status of the inertial force. But the essence does not change, because the momentum appears together with the mass. Thus, the circle of philosophical speculations is locked up by the momentum, i.e. by the mass with energy and its oscillation. All other attributes of the motion are formed by particle interaction which is shown by mathematical transformations.

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Chapter 4
Solution of Jacobi's Virial Equation for Self-gravitating Systems

Abstract It is shown in this chapter that Jacobi's virial equation provides, first of all, a solution for the models of natural systems, which have explicit solutions in the framework of the classical many-body problem. The particular example of this is the unperturbed problem of Keplerian motion, when the system consists of only two material points interacting by Newtonian law. The parallel solutions for both the classical and dynamical approaches are given, and in doing so we show that, with the dynamical approach, the solution acquires a new physical meaning, namely, oscillating motion. That solution appeared to be possible because of an existing relationship between the gravitational energy and the polar moment of inertia in the form $|U|\sqrt{\Phi} = B = \text{const}$. It is also done for the solution of Jacobi's virial equation in hydrodynamics, in quantum mechanics for dissipative systems, for systems with friction and in the framework of the theory of relativity. The above solutions acquire a new physical meaning because the dynamics of a system is considered with respect to new parameters, i.e. its Jacobi function (polar moment of inertia) and potential (kinetic) energy. The solution, with respect to the Jacobi function and the potential energy, identifies the evolutionary processes of the structure or redistribution of the mass density of the system. Moreover, the main difference of the two approaches is that the classical problem considers motion of a body in the outer central force field. The virial approach considers motion of a body both in the outer and in its own force field applying, instead of linear forces and moments, the volumetric forces (pressure) and moments (oscillations). Finally, analytical solution of the generalized equation of perturbed virial oscillations in the form $\ddot{\Phi} = -A + B/\sqrt{\Phi} + X(t, \Phi, \dot{\Phi})$ was done. Derivation of the equation of dynamical equilibrium and its solution for conservative and dissipative systems shows that dynamics of celestial bodies in their own force field puts forward a wide class of geophysical, astrophysical and geodetic problems which can be solved by the methods of celestial mechanics and with new physical concepts.

In the previous chapter, we derived Jacobi's virial equation of dynamics (gravitation) and dynamical equilibrium in the framework of various physical models which are used for describing the motion of natural systems. It was shown that, instead of the traditional description of such systems, like the Sun, planets and

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satellites, based on hydrostatics, the problem of dynamics can be studied on a more correct physical basis, which appears to be dynamical equilibrium.

By transformation of the linear forces and momentums into their volumetric values, we obtain equations of dynamics of a celestial body applying the fundamental integral characteristics, namely, the energy and moment of inertia. Moreover, such a form of equation description does not depend on the choice of the reference system and becomes universal for solving dynamical problems in the framework of any physical models. In addition, we begin to understand the nature of the force field source, which is the effect of interaction of the body's elementary particles expressed through the moment of inertia. In this case, we succeeded in restoring the kinetic energy lost at the hydrostatic approach.

The problem now is to find the general solution of Jacobi's virial equation relative to oscillation and rotation of a body and to apply the solution to study its dynamics, origin and evolution. This application is valid for studying the Sun, the Earth, the Moon and other celestial bodies.

In this chapter, we show that Jacobi's virial equation provides, first of all, the solution for the models of natural systems, which have explicit solutions in the framework of the classical many-body problem. We shall give parallel solutions for both the classical and dynamical approaches, and in doing so we show that, with the dynamical approach, the solution acquires a new physical meaning. We shall also consider a general case of the solution of Jacobi's virial equation for conservative and dissipative systems.

4.1 Solution of Kepler's Problem in Classical and Virial Approach

The many-body problem is known to be the key problem in classical mechanics and especially in celestial mechanics. A particular example of this is the unperturbed problem of Keplerian motion, when the system consists of only two material points interacting by Newtonian law. The explicit solution of the problem of unperturbed Keplerian motion permits the many-body problem to be solved with some approximation by varying arbitrary constants. In this case, the problem of dynamics, for example that of the Solar System, is transferred into the solution of the problem of dynamics of nine pairs of bodies in each of which one body is always the Sun and the second is each of the nine planets forming the system. Considering each planet–sun sub-system, the influence of the other eight planets of the system is taken into account by introducing the perturbation function. By the virial approach, we can obtain for the Sun, one characteristic period of circulation with respect to the centre of mass of the system which will not coincide with any period of the planets. The dynamical approach evidences that the planet's orbital motion is performed by the central body, i.e. by the Sun, by the energy of its outer force field or by the field of the energy pressure. Each planet interacts with the solar force field by the energy of its own outer force field. The planet's orbit is the certain curve of its equilibrium motion which results from the two interacting fields of pressure. The planet's own oscillation and rotation perform by action of its inner gravitational fields of pressure.

Following these brief physical comments on the dynamical equilibrium motion of a planet, we now present two approaches of solving the Keplerian problem: the classical and the integral.

4.1.1 The Classical Approach

The traditional way of solving the unperturbed Keplerian problem is excellently described in the university courses for celestial mechanics found in Duboshin (1978). Here we present only the principle ideas. The method consists in transforming the two-body problem described by the system of Eq. (3.3) into the one-body problem using six integrals of motion of the centre of mass (3.6). The system of equations obtained is sixth order and expresses the change of barycentric co-ordinates of one point with respect to the centre of mass of the system as a whole. Let us write it in the form

$$\begin{aligned} \ddot{x} &= -\frac{\mu x}{r^3}, \\ \ddot{y} &= -\frac{\mu y}{r^3}, \\ \ddot{z} &= -\frac{\mu z}{r^3}, \end{aligned} \tag{4.1}$$

where μ is the constant depending on the number of the point and for which the second point is equal to

$$\mu = \frac{Gm_1^3}{(m_1+m_2)^2}.$$

We then pass on from that Cartesian system of co-ordinates $OXYZ$ to orbital $\xi\,\eta\,\zeta$, using first integrals of the system of Eq. (4.1). Those are three integrals of the area,

$$\begin{aligned} y\dot{z} - z\dot{y} &= c_1, \\ z\dot{x} - x\dot{z} &= c_2, \\ x\dot{y} - y\dot{x} &= c_3, \end{aligned} \tag{4.2}$$

the energy integral,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{2\mu}{r} + h, \tag{4.3}$$

and the Laplacian integrals,

$$
\begin{aligned}
-\frac{\mu x}{r} + c_3\dot{y} - c_2\dot{z} &= f_1,\\
-\frac{\mu y}{r} + c_1\dot{z} - c_3\dot{x} &= f_2,\\
-\frac{\mu z}{r} + c_2\dot{x} - c_1\dot{y} &= f_3.
\end{aligned} \tag{4.4}
$$

As these seven integrals are not independent, we conclude that they cannot form a general solution of the system (4.1). In fact there are two relations for these integrals:

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0,$$

$$f_1^2 + f_2^2 + f_3^2 = \mu^2 + h\left(c_1^2 + c_2^2 + c_3^2\right),$$

showing that only five of them are independent. But the last integral needed can be found by simple quadrature. Using these integrals, we can pass on to the system of orbital co-ordinates Oξηζ using the transformation relations (see Fig. 4.1):

$$
\begin{aligned}
\xi &= \frac{f_1}{f}x + \frac{f_2}{f}y + \frac{f_3}{f}z,\\
\eta &= \frac{C_2 f_3 - C_3 f_2}{Cf}x + \frac{C_3 f_1 - C_1 f_2}{Cf}y + \frac{C_1 f_2 - C_3 f_1}{Cf}z,\\
\zeta &= \frac{C_1}{C}x + \frac{C_2}{C}y + \frac{C_3}{C}z.
\end{aligned} \tag{4.5}
$$

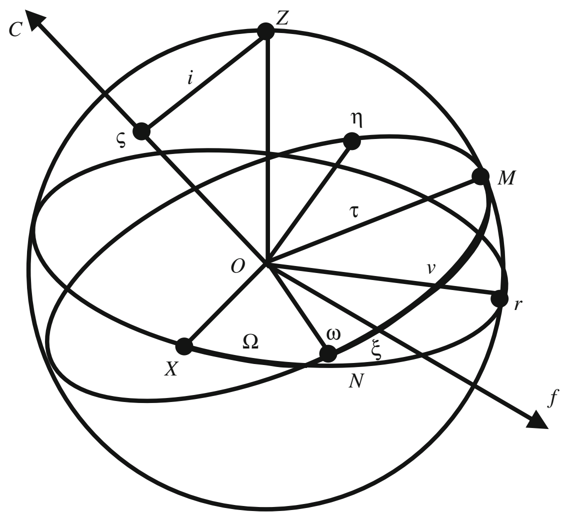

Fig. 4.1 Transition from Cartesian co-ordinate system *OXYZ* to orbital Oξηζ

The equation of the curve along which the point moves in accordance with (4.1) has the simplest form in the system of initial co-ordinates. The equation is

$$\begin{aligned} \zeta &= 0, \\ \mu r &= C^2 - f\xi. \end{aligned} \tag{4.6}$$

Finally, introducing the polar orbital co-ordinates r and v, which are related to the rectangular orbital co-ordinates ξ and η by the expressions (see Fig. 4.2)

$$\xi = r \cos \nu \, \text{and} \, \eta = r \sin \nu$$

and using the integral of areas $r^2 \nu = C$, we come to the equation

$$C(t - r) = \left(\frac{C^2}{\mu}\right)^2 \int_0^v \frac{\mathrm{d}v}{\left(1 + \frac{f}{\mu}\cos \nu\right)^2}. \tag{4.7}$$

The solution of Eq. (4.7) gives the change of function v with respect to time. Repetition of the transformation in the reverse order leads to solution of the problem. In doing this, we obtain the expression for the change of co-ordinates of the material point with respect to the initial data ξ_{10}, η_{10}, ζ_{10}, ξ_{20}, η_{20}, ζ_{20}, $\dot{\xi}_{10}$, $\dot{\eta}_{10}$, $\dot{\zeta}_{10}$, $\dot{\xi}_{20}$, $\dot{\eta}_{20}$, $\dot{\zeta}_{20}$. It is remarkable that if the total energy (4.3) has negative value, then the solution of Eq. (4.7) leads to the Keplerian equation

$$E - e \sin E = n(t - \tau), \tag{4.8}$$

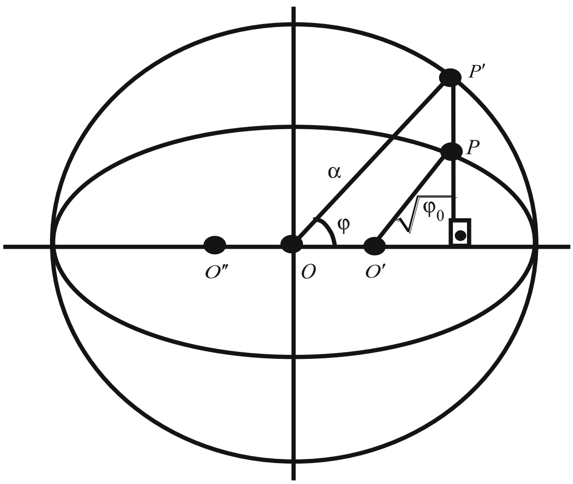

Fig. 4.2 Relationship between the polar and the rectangular coordinates

where the function ν is related to the variable E' by the expression

$$tg\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}tg\frac{E}{2},$$

$$e = \frac{f}{\mu}, \quad n = \frac{\sqrt{\mu}}{a^{3/2}}, \quad p = \frac{C^2}{\mu} = a\left(1-e^2\right).$$

Because energy by definition is the property to do work (motion) and can be only a positive value, the physical meaning of negative total energy, which defines the elliptic orbit of a body moving in the central field of the two-body problem should be revealed. In the presented solution of the two-body problem, the left-hand side of the energy integral (4.3) expresses the kinetic energy and the right-hand side means the potential energy of the mass interaction. The integral of energy (4.3) as a whole, in the co-ordinates and in the velocities represents the averaged virial theorem, where the potential energy has formally a negative value. Here, the physical meaning of the total energy determination consists in comparison of magnitude of the potential and kinetic energy. A negative value of the total energy means that the potential energy exceeds the kinetic one by that value. As it follows from analysis of the inner force field of a self-gravitating body presented in Chap. 2, the potential energy exceeds the kinetic one only in the case of non-uniform distribution of the mass density and cannot be less than it. In the case of equality of both energies, the total potential energy is realized into oscillating motion. The excess of the potential energy is used for rotation of the masses and in the dissipation. The last case is discussed later.

4.1.2 The Dynamic Approach

Let us consider the solution of the problem of unperturbed motion of two material points on the basis of Jacobi's virial equation which in accordance with Eq. (3.16) is written in the form

$$\ddot{\Phi}_0 = 2E_0 - U,$$

where $E_0 = T_0 + U = \text{const}$ is the total energy of the system in a barycentric co-ordinate system;

The Jacobi function Φ_0 is expressed by (3.15):

$$\ddot{\Phi}_0 = \frac{m_1 m_2}{2(m_1+m_2)}\left[(\xi_1-\xi_2)^2 + (\eta_1-\eta_2)^2 + (\zeta_1-\zeta_2)^2\right]$$

and the potential energy U in accordance with (3.2) is

$$U = \frac{Gm_1 m_2}{\sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}}.$$

It is easy to see that between the Jacobi function Φ_0 and the potential energy U, the relationship exists in the form

$$|U|\sqrt{\Phi} = \frac{G(m_1 m_2)^{3/2}}{\sqrt{2(m_1 m_2)}} = \frac{G}{\sqrt{2}} m\mu^{3/2} = B = \text{const}, \tag{4.9}$$

where μ is the generalized mass of the two bodies; m is the total mass of the system and B is a constant value.

The relationship (4.9) is remarkable because it is independent of the initial data, that is, of its co-ordinates and velocities. Being an integral characteristic of the system and dependent only on the total mass and the generalized mass of the two points, the relationship permits Jacobi's virial equation to be transformed to an equation with one variable as follows:

$$\ddot{\Phi}_0 = 2E_0 + \frac{B}{\sqrt{\Phi_0}}. \tag{4.10}$$

We consider the solution of Eq. (4.10) for the case when total energy E_0 has negative value. Introducing $A = -2E_0 > 0$, Eq. (4.10) can be rewritten:

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}. \tag{4.11}$$

We apply the method of variable for solution of Eq. (4.11) and show that partial solution of two linear equations (Ferronsky et al. 1984):

$$\left(\sqrt{\Phi_0}\right)'' + \sqrt{\Phi_0} = \frac{B}{A}, \tag{4.12}$$

$$t'' + t = \frac{4B\lambda}{(\sqrt{2A})}, \tag{4.13}$$

which include only two integration constants is also the solution of Eq. (4.11).

We now introduce the independent variable λ into Eqs. (4.12) and (4.13), where primes denote differentiation with respect to λ. Note that time here is not an independent variable. This allows us to search for the solution of two linear equations instead of solving one non-linear equation. The solution of Eqs. (4.12) and (4.13) can be written in the form

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)], \tag{4.14}$$

$$t = \frac{4B}{(2A)^{3/2}}[1 - \varepsilon\sin(\lambda - \psi)]. \tag{4.15}$$

Let us prove that the partial solution (4.14) and (4.15) of differential equations (4.12) and (4.13) is the solution of Eq. (4.10) that is sought. For this purpose, we express the first and second derivatives of the function $\sqrt{\Phi_0}$ with respect to λ through corresponding derivatives with respect to time using Eq. (4.15). From (4.15) it follows that

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{4B}{(2A)^{3/2}}[1 - \varepsilon\sin(\lambda - \psi)] \tag{4.16}$$

We can replace the right-hand side of the obtained relationship by $\sqrt{\Phi_0}$ from (4.14),

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \sqrt{\Phi_0}\sqrt{\frac{2}{A}}. \tag{4.17}$$

Transforming the derivative from $\sqrt{\Phi_0}$ with respect to λ into the form

$$\frac{\mathrm{d}\sqrt{\Phi_0}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\sqrt{\Phi_0}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\dot{\Phi}_0}{2\sqrt{\Phi_0}}\frac{\mathrm{d}t}{\mathrm{d}\lambda}$$

and taking into account (4.17), we can write

$$\left(\sqrt{\Phi_0}\right)' = \frac{\dot{\Phi}_0}{\sqrt{2A}}.$$

The second derivative can be written analogously:

$$\left(\sqrt{\Phi_0}\right)'' = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\Phi}_0}{\sqrt{2A}}\right) = \frac{\ddot{\Phi}_0}{\sqrt{2A}}\sqrt{\Phi_0}\sqrt{\frac{2}{A}} = \frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A}. \tag{4.18}$$

Putting Eq. (4.18) into (4.12), we obtain

$$\frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A} + \sqrt{\Phi_0} = \frac{B}{A}.$$

Dividing the above expression by $\sqrt{\Phi_0}/\text{A}$, we can finally write

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}.$$

This shows that the partial solution of the two linear differential equations (4.12) and (4.13) appears to be the solution of the non-linear equation (4.11).

4.2 Solution of n-Body Problem in the Framework of Conservative System

After solving Jacobi's virial equation for the unperturbed two-body problem, we come to dynamics of a system of n material particles where $n \to \infty$.

Let us assume that an external observer studying the dynamics of a system of n particles in the framework of classical mechanics has the following information. He has the mass of the system, its total and potential energy and can determine the Jacobi function and its first derivative with respect to time in any arbitrary moment. Then we can use Jacobi's virial Eq. (3.13) and making only the assumption needed for its solution that $|U|\sqrt{\Phi} = B = \text{const}$, may predict the dynamics of the system, i.e. the dynamics of its integral characteristics at any moment of time. The assumption $|U|\sqrt{\Phi_0} = \text{const}$ will be considered separately in Chap. 8.

If the total energy E_0 of the system has negative value, the external observer can immediately write the solution of the problem of the Jacobi function change with respect to time in the form of (4.14) and (4.15):

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)],$$

$$t = \frac{4B}{(2A)^{3/2}}[1 - \varepsilon\sin(\lambda - \psi)]$$

where $A = -2E_0$; ε and ψ are constants depending on the initial values of the Jacobi function Φ_0 and its first derivative $\dot{\Phi}_0$ at the moment of time t_0.

Let us obtain the values of constants ε and ψ, in explicit form expressed through the values Φ_0 and $\dot{\Phi}_0$ at the initial moment of time t_0. For convenience, we introduce a new independent variable φ, connected to λ by the relationship $\lambda - \psi = \varphi$. Then Eqs. (4.14) and (4.15) can be rewritten:

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos], \tag{4.19}$$

$$t - \frac{4B}{(2A)^{3/2}} \quad \psi = \frac{4B}{(2A)^{3/2}}[\varphi - \varepsilon \sin \varphi]. \tag{4.20}$$

Using Eq. (4.19) we write the expression for φ:

$$\varphi = \arccos \frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}, \tag{4.21}$$

and taking into account the equality

$$\frac{\mathrm{d}\sqrt{\Phi_0}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\sqrt{\Phi_0}}{\mathrm{d}\varphi}.$$

If we substitute Eq. (4.21) into the expression

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\varepsilon \sin \varphi,$$

the Eq. (4.19) can be rewritten finally in the form

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\varepsilon\sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}. \tag{4.22}$$

Equation (4.22) allows us to determine the first constant of integration ε as a function of the initial data Φ_0 and $\dot{\Phi}_0$ at $t = t_0$. Solving Eq. (4.22) with respect to ε after simple algebraic transformation, we obtain

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}|_{t=t_0} = \text{const}. \tag{4.23}$$

The second constant of integration ψ can be expressed through the initial data after solving Eq. (4.20) with respect to ψ and change of value φ by its expression from Eq. (4.21). Defining

$$t - \frac{4B}{(2A)^{3/2}}\psi = \tau,$$

we obtain

$$-\tau\left\{\frac{4B}{(2A)^{2/3}}\left[\arccos\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} - \varepsilon\varepsilon\sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}\right] - t\right\}|_{t=t_0} = \text{const}. \tag{4.24}$$

The physical meaning of the integration constants ε, τ, and the parameter $T_v = 8\pi B/(2A)^{3/2}$ can be understood after the definitions

$$T_v = \frac{8\pi B}{(2A)^{3/2}}, \quad n = \frac{2\pi}{T_v} = \frac{(2A)^{3/2}}{4B}, \quad a = \frac{B}{A}$$

and rewriting Eqs. (4.19) and (4.20) in the form

$$\sqrt{\Phi_0} = a[\varphi - \varepsilon \sin\varphi], \tag{4.25}$$

$$M = \varphi - \varepsilon \sin\varphi, \tag{4.26}$$

where $M = n(t - \tau)$.

The value $\sqrt{\Phi_0}$ draws an ellipse during the period of time $T_v = 8\pi B/(2A)^{3/2}$ (see Fig. 4.3). The ellipse is characterized by a semi-major axis a equal to B/A and by the eccentricity ε which is defined by expression (4.23). In the case considered where $E_0 < 0$, the value ε is changed in time from 0 to 1. The value τ characterizes the moment of time when the ellipse passes the pericentre.

Let us obtain explicit expressions with respect to time for the functions $\sqrt{\Phi_0}$, Φ_0 and $\dot{\Phi}_0$. For this purpose we write Eq. (4.26) in the form of a Lagrangian:

$$F(\varphi) = \varphi - \varepsilon \sin\varphi - M = 0. \tag{4.27}$$

It is known (Duboshin 1978) that by application of Lagrangian formulas, we can write in the form of a series the expressions for the root of the Lagrange Eq. (4.27) and for the arbitrary function f which is dependent on φ:

$$\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[\sin^k M\right] = M + \varepsilon \sin M + \frac{\varepsilon^2}{1 \cdot 2} \frac{d}{\mathrm{d}M} \left[\sin^2 M\right] + \cdots, \tag{4.28}$$

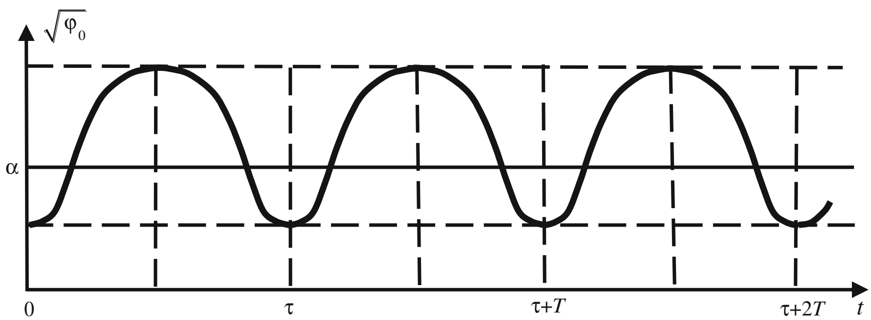

Fig. 4.3 Changes of the Jacobi function over time

$$f(\varphi) = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[f'(M) \sin^k M \right] = f(M) + \varepsilon f'(M) \sin M \\ + \frac{\varepsilon^2}{1 \cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[f(M) \sin^2 M \right] + \cdots \tag{4.29}$$

Using Eq. (4.29), we write expressions for cosφ, $\cos^2\varphi$ and sinφ in the form of a Lagrangian series of parameter ε power:

$$\cos \varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[(-1) \sin M \sin^k M \right] = \cos M + \varepsilon(-1) \sin M \sin(M) \\ + \frac{\varepsilon^2}{1 \cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[(-1) \sin(M) \sin^2 M \right] + \cdots = \cos M - \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cos 2M \\ - \frac{3}{4} \varepsilon^3 \cos M + \frac{3}{8} \varepsilon^2 \cos 3M + \cdots \tag{4.30}$$

$$\cos^2 \varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[(-2) \sin M \cos M \sin^k M \right] = \cos^2 M \\ + \varepsilon(-2) \sin M \cos M \sin M + \frac{\varepsilon^2}{1 \cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[(-2) \sin M \cos M \sin^2 M \right] + \cdots \\ = \cos^2 M - 2\varepsilon \sin^2 M \cos M + \frac{\varepsilon^2}{2} (-2) \left(3 \sin^2 M \cos^2 M - \sin^4 M \right) + \cdots \tag{4.31}$$

$$\sin \varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} [\cos M \sin M] = \sin M + \varepsilon \cos M \sin M \\ + \frac{\varepsilon^2}{1 \cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[\cos M \sin^2 M \right] + \cdots = \sin M + \varepsilon \cos M \sin M \\ + \frac{\varepsilon^2}{1 \cdot 2} \left[2 \sin M \cos^2 M - \sin^3 M \right] + \cdots \tag{4.32}$$

We write the expressions for $\sqrt{\Phi_0}$, Φ_0, $\dot{\Phi}_0$ using Eqs. (4.25) and (4.26) in the form

$$\sqrt{\Phi_0} = a(1 - \varepsilon \cos \varphi), \tag{4.33}$$

$$\Phi_0 = a^2 \left(1 - 2\varepsilon \cos \varphi + \varepsilon^2 \cos^2 \varphi \right), \tag{4.34}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\varepsilon B \sin\varphi. \tag{4.35}$$

Substituting into (4.33) to (4.35) the expressions for cosφ, $\cos^2\varphi$ and sinφ in the form of the Lagrangian series (4.30) to (4.32) we obtain

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\varepsilon + \frac{3}{8}\varepsilon^3\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{3}{8}\varepsilon^3\cos 3M + \cdots\right], \tag{4.36}$$

$$\Phi_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \cdots\right], \tag{4.37}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\varepsilon B\left[\sin M + \frac{1}{2}\varepsilon\sin 2M + \frac{\varepsilon^2}{2}\sin M\left(2\cos^2 M - \sin^2 M\right) + \cdots\right] \tag{4.38}$$

The series of Eqs. (4.36), (4.37) and (4.38) obtained are put in order of increased power of parameter ε and are absolutely convergent at any value of M in the case when the parameter ε satisfies the condition

$$\varepsilon < \bar{\varepsilon} = 0.6627\ldots, \tag{4.39}$$

where $\bar{\varepsilon}$ is the Laplace limit.

In some cases, it is convenient to expand the values $\sqrt{\Phi_0}$, Φ_0, $\dot{\Phi}_0$ in the form of a Fourier series, using conventional methods (see, e.g., Duboshin 1978). Figure 4.4 demonstrates the changes of $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$.

It is also possible to consider the case solution of Jacobi's virial equation for $E_0 = 0$ and $E_0 > 0$. Readers can find here, without difficulty, a full analogy of these results as well as the solution of the two-body.

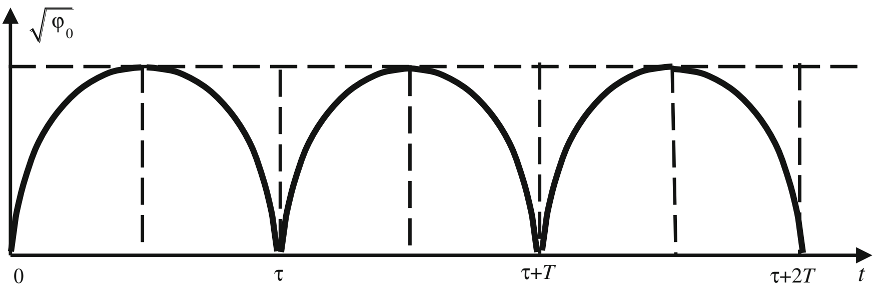

Fig. 4.4 Changes of the value $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$

4.3 Solution of Jacobi's Virial Equation in Hydrodynamics

Let us consider the solution of the problem of the dynamics of a homogeneous isotropic gravitating sphere in the framework of traditional hydrodynamics and the virial approach we have developed.

4.3.1 The Hydrodynamic Approach

The sphere is assumed to have radius R_0 and be filled by an ideal gas with a density ρ_0. We assume that at the initial time the field of velocities which has the only component is described by equation

$$u = H_0 r, \tag{4.40}$$

where u is the radial component of the velocity of the sphere's matter at the distance r from the centre of mass; H is independent of the quantity r and equal to H_0 at time t_0.

We also assume that the motion of the matter of the sphere goes on only under action of the forces of mutual gravitational interaction between the sphere particles. In this case, the influence of the pressure gradient is not taken into account assuming that the matter of the sphere is sufficiently diffused. Then, the symmetric spherical shells will move only under forces of gravitational attraction and will not coincide. In this case, the mass of the matter of any sphere shell will keep its constant value and the condition (4.40) will be satisfied at any moment of time, and constant H should be dependent on time.

Under those conditions the Eulerian system of equations (3.28) can be written in the form

$$\rho \frac{\partial u}{\partial t} + \rho(u\nabla)u = \rho \frac{\partial U_G}{\partial r}, \tag{4.41}$$

where $\rho(t)$ is the density of the matter of the sphere at the moment of time t; u is the radial component of the velocity of matter at distance r from the sphere's centre; U_G is the Newtonian potential for the considered point of the sphere.

The expression for the Newtonian potential U_G (3.29) can be written as follows:

$$U_G = G\frac{4}{3}\pi\rho r^2, \tag{4.42}$$

and the continuity equation will be

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial U}{\partial r} = 0. \tag{4.43}$$

Within the framework of the traditional approach, the problem is to define the sphere radius R and the value of the constant H at any moment of time, if the radius R_0, density ρ_0 and the value of the constant H_0 at the initial moment of time t_0 are given. If we know the values $H(t)$ and $R(t)$, we can then obtain the field of velocities of the matter within the sphere which is defined by Eq. (4.40), and also the density ρ of matter at any moment of time, using the relationship

$$\frac{4}{3}\pi R_0^3 \rho_0 = \frac{4}{3}\pi R^3 \rho = m.$$

Hence the formulated problem is reduced to identification of the law of motion of a particle which is on the surface of the sphere and within the field of attraction of the entire sphere mass $m = 4/3\pi\ \rho_0 R_0^3$.

The equation of motion for a particle on the surface of the sphere, which follows from Eq. (4.41) after transforming the Eulerian co-ordinates into a Lagrangian, has the form

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -G\frac{m}{R^2}. \tag{4.44}$$

It is necessary to determine the law of change of $R(t)$, resolving Eq. (4.44) at the initial data:

$$R(t_0) = R_0,$$

$$\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{t=t_0} = H_0 R_0. \tag{4.45}$$

We reduce the order of Eq. (4.44). To do so we multiply it by $\mathrm{d}R/\mathrm{d}t$,

$$\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{\mathrm{d}R}{\mathrm{d}t}\frac{Gm}{R^2}$$

and integrate with respect to time:

$$\int_{t_0}^{t} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{R}\right)^2 = \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Gm}{R}\right)\mathrm{d}t.$$

After integration, we obtain

$$\frac{1}{2}\dot{R}^2 - \frac{1}{2}\dot{R}_0^2 = \frac{Gm}{R} - \frac{Gm}{R_0}$$

or

$$\frac{1}{2}\dot{R}^2 = \frac{Gm}{R} + k, \tag{4.46}$$

where the constant k is determined as

$$\begin{aligned} k &= \frac{1}{2}\dot{R}_0^2 - \frac{Gm}{R_0} = \frac{1}{2}H_0^2R_0^2 - G\frac{4\pi}{3}\rho_0\frac{R_0^3}{R_0} \\ &= \frac{1}{2}H_0^2R_0^2\left[1 - \frac{8\pi}{3}\frac{G\rho_0}{H_0^2}\right] = \frac{1}{2}H_0^2R_0^2[1-\Omega] = \text{const} \end{aligned} \tag{4.47}$$

Here the quantity $\Omega = \rho_o / \rho_{cr}$, where $\rho_{cr} = 3H_0^2/8\pi G$.

Note that, Eq. (4.46) obtained after reduction of the order of the initial Eq. (4.44) is in its substance the energy conservation law. Equation (4.46) permits the variables to be divided and can be rewritten in the form

$$\int_{R_0}^{R} \frac{\mathrm{d}R}{\sqrt{\frac{2Gm}{R} + 2k}} = \int_{t_0}^{t} \mathrm{d}t. \tag{4.48}$$

The plus sign before the root is chosen assuming that the sphere at the initial time is expanding, that is, $H_0 > 0$.

The differential Eq. (4.46) has three different solutions at $k = 0$, $k > 0$ and $k < 0$ depending on the sign of the constant k, which is in its turn defined by the value of the parameter Ω at the initial moment of time. First we consider the case when $k = 0$ which relates, by analogy with the Keplerian problem, to the parabolic model at $k = 0$. Equation (4.46) is easily integrated and for the expression case, i.e. $\dot{R} > 0$, we obtain

$$\dot{R}^2 = \frac{2Gm}{R},$$

$$\dot{R} = \frac{(2Gm)^{1/2}}{R^{1/2}},$$

from which it follows that

$$R^{1/2}\mathrm{d}R = (2Gm)^{1/2}\mathrm{d}t$$

or

$$\frac{2}{3}R^{3/2} = (2Gm)^{1/2}t + \text{const.} \tag{4.49}$$

We choose as initial counting time $t = 0$, the moment when $R = 0$. In this case, the integration constant disappears

$$R = \left(\frac{9}{2}Gm\right)^{1/3} t^{2/3}. \tag{4.50}$$

The density of the matter changes in accordance with the law

$$\rho(t) = \frac{m}{\frac{4}{3}\pi R^3} = \frac{1}{6\pi G t^2}, \tag{4.51}$$

and the quantity $H(t)$, as a consequence of (4.50) has the form

$$H(t) = \frac{\dot{R}}{R} = \frac{2}{3}\frac{1}{t}. \tag{4.52}$$

For the case when $k > 0$, which corresponds to so-called hyperbolic motion, the solution of Eq. (4.46) can be written in parametric form (Zeldovich and Novikov 1967):

$$R = \frac{Gm}{2k}(\text{ch}\eta - 1), \tag{4.53}$$

$$t = \frac{Gm}{(2k)^{3/2}}(\text{sh}\eta - \eta),$$

where the constants of integration in (4.53) have been chosen so that $t = 0$, $\eta = 0$ at $R = 0$.

Finally we consider the case when $k < 0$, which corresponds to elliptic motion. At $k < 0$, the expansion of the sphere cannot continue for unlimited time and the expansion phase should be changed by attraction of the sphere.

The explicit solution of Eq. (4.46) at $k < 0$ can be written in parametric form (Zeldovich and Novikov 1967)

$$R = \frac{Gm}{2|k|}(1 - \text{ch}\eta), \tag{4.54}$$

$$t = \frac{Gm}{(2|k|)^{3/2}}(\eta - \text{sh}\eta).$$

The maximum radius of the sphere is determined from Eq (4.46) on the condition $dR/dt = 0$ and equals

$$R_{\max} = \frac{Gm}{|E|}. \tag{4.55}$$

The time needed for expansion of the sphere from $R_0 = 0$ at $t_0 = 0$ to $R_{\max}$ is

$$t_{\max} = \frac{\pi Gm}{(2|k|)^{3/2}}. \tag{4.56}$$

So the sphere should make periodic pulsations with period T_p equal to

$$T_p = \frac{2\pi Gm}{(2|k|)^{3/2}}. \tag{4.57}$$

The considered solution has important cosmologic application.

4.3.2 The Virial Approach

We shall limit ourselves by formal consideration of the same problem in the framework of the condition of the dynamical equilibrium of a self-gravitating body based on the solution of Jacobi's virial equation, which we discussed earlier.

As shown in Chap. 3, Jacobi's virial equation (3.50), derived from Eulerian equations (3.28), is valid for the considered gravitating sphere. It was written in the form

$$\ddot{\Phi} = 2E - U, \tag{4.58}$$

where Φ is the Jacobi function for a homogeneous isotropic sphere and is defined by

$$\Phi = \frac{1}{2}\int_0^R 4\pi r^2 \rho r^2 \mathrm{d}r = \frac{2\pi\rho R^5}{5} = \frac{3}{10} mR^2. \tag{4.59}$$

The potential gravitational energy of the matter of the sphere is expressed as

$$U = -4\pi G \int_0^R r\rho(r)m(r)\mathrm{d}r = -\frac{16\pi^2}{15} G\rho^2 R^2 = -\frac{3}{5} G\frac{m^2}{R}. \tag{4.60}$$

The total energy of the sphere E will be equal to the sum of the potential U and kinetic T energies.

The kinetic energy T is expressed as

$$T = \frac{1}{2}\int_0^R 4\pi u^2 \rho r^2 \mathrm{d}r = \frac{1}{2}\int_0^R 4\pi H^2 r^2 \rho r^2 \mathrm{d}r = \frac{4\pi\rho H^2 R^5}{10} = \frac{3}{10} m H^2 R^2. \quad (4.61)$$

For a homogeneous isotropic gravitating sphere, the constancy of the relationship between the Jacobi function (4.59) and the potential energy (4.60) can be written as

$$|U|\sqrt{\Phi} = B = \frac{3}{5} G \frac{m^2}{R}\sqrt{\frac{3}{10} m R^2} = \frac{1}{\sqrt{2}}\left(\frac{3}{5}\right)^{3/2} G m^{3/2}, \quad (4.62)$$

where B has constant value because of the conservation law of mass m of the considered sphere.

The total energy E of the sphere also has a constant value:

$$E = T + U = \frac{A}{2}. \quad (4.63)$$

Then, if the total energy of the sphere has a negative value, Jacobi's virial equation can be written in the form:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}. \quad (4.64)$$

Let us consider the conditions under which the total energy of the system will have a negative value. For this purpose we write it explicitly:

$$E = T + U = -\frac{16}{15}\pi^2 G \rho^2 R^5 + \frac{2\pi\rho H^2 R^5}{5} = \frac{2}{5}\pi\rho H^2 R^5\left[1 - \frac{8\pi G\rho}{3H^2}\right] \quad (4.65)$$

It is clear from Eq. (4.65) that the total energy E has a negative value, when $\rho > \rho_c$, where $\rho_c = 3H^2/8\pi G$.

The general solution of Eq. (4.64) has the form of Eqs. (4.14) and (4.15):

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)], \quad (4.66)$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)], \quad (4.67)$$

where ε and ψ are constants dependent on the initial values of the Jacobi function Φ_0 and its first derivative $\dot{\Phi}_0$ at the moment of time t_o. The constants ε and ψ are determined by Eqs. (4.23) and (4.24) accordingly.

If we express all the constants in Eq. (4.23),

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}|_{t=t_0} = \text{const} \tag{4.68}$$

through mass m of the system, it is not difficult to see that

$$-\dot{\Phi}_0^2 + 4B\sqrt{\Phi_0} - 2A\Phi_0 = 0$$

Then the constant ε will be equal to zero. Hence the solutions (4.28) and (4.29) coincide with the solution (4.54), which was obtained in the framework of the traditional hydrodynamic approach. In this case the period of eigenpulsations of the Jacobi function (the polar moment of inertia) of the sphere $T_p = 8\pi R/(2A)^{3/2}$ will be equal to the period of change of its radius $T_r = 2\pi Gm/(2|k|)^{3/2}$ obtained from Eq. (4.54).

4.4 The Hydrogen Atom as a Quantum Mechanical Analogue of the Two-Body Problem

Let us consider the problem concerning the energy spectrum of the hydrogen atom, which is a unique example of the complete conformity of the analytical solution with experimental results. The problem consists of a study of all the forms of motion using the postulates of quantum mechanics and based on the solution of Jacobi's virial equation.

The classical Hamiltonian in the two-body problem is written as

$$H = \frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} + U(|\bar{r}_1 - \bar{r}_2|), \tag{4.69}$$

where

$$\bar{p}_1 = \frac{\partial H}{\partial \dot{\vec{r}}_1} = m_1\dot{\bar{r}}_1$$

$$\bar{p}_2 = \frac{\partial H}{\partial \dot{\vec{r}}_2} = m_2\dot{\bar{r}}_2$$

which, after separation of the centre of mass can be transformed into the form

$$H = \frac{\bar{P}^2}{2M} + \frac{\bar{p}^2}{2m} + U(r), \tag{4.70}$$

where $r = |\bar{r}_1 - \bar{r}_2|$ is the distance between two particles and

$$\bar{P} = M\dot{\bar{R}}; \quad \bar{p} = m\dot{\bar{r}}; \quad M = m_1 + m_2;$$

$$\bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}; \quad m = \frac{m_1 m_2}{m_1 + m_2}.$$

We obtain the Hamiltonian operator for the quantum mechanical two-body problem by changing the pulses and radii by the corresponding operators with the communication relations

$$[\hat{p}_i, \hat{p}_k] = -i\hbar\delta_{ik},$$

$$[\hat{p}_i, \hat{r}_k] = -i\hbar\delta_{ik}.$$

Then

$$\hat{H} = -\frac{\hbar^2}{2M}\Delta_R - \frac{\hbar^2}{2m}\Delta_r + \hat{U}(r).$$

The wave function $u(\bar{r}_1, \bar{r}_2) = \varphi(\bar{R})\psi(\bar{r})$, which satisfies the Schrödinger equation,

$$\hat{H}u = \varepsilon u,$$

describes the motion of the inertia centre (free motion of the particle of mass m_c is described by the function $\varphi(R)$ and the motion of the particle of mass m in the $U(r)$ is described by the wave function $\Psi(\bar{r})$). Subsequently we consider only the wave function of the motion of particle m.

The Schrödinger equation

$$\Delta\Psi + \frac{2m}{\hbar^2}[E - U(r)]\Psi = 0$$

written here for the stationary state in a central symmetrical field in spherical co-ordinates has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Psi}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2\Psi}{\partial\varphi^2}\right] + \frac{2m}{\hbar^2}[E - U(r)]\psi = 0 \tag{4.71}$$

Using the Laplacian operator $\hat{\ell}^2$,

$$\hat{\ell}^2 = \left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

we obtain

$$\frac{\hbar^2}{2m}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\hat{\ell}^2}{r^2}\Psi\right] + U(r)\Psi = E\Psi.$$

The operators $\hat{\ell}^2$ and $\hat{\ell}_z$ ($\hat{\ell}_z = -i\partial/\partial\varphi$) commutate with the Hamiltonian $\hat{H}(r)$ and therefore there are common eigenfunctions of the operators $\hat{H}$, $\hat{\ell}^2$ и $\hat{\ell}_z$. We consider only such solutions of Schrödinger equations. This condition determines the dependence of the function Ψ on the angles

$$\Psi(r,\Theta,\varphi) = R(r)Y_{\ell k}(\Theta,\phi),$$

where the quantity $Y_{\ell k}(\Theta,\varphi)$ is determined by the expression

$$Y_{\ell k}(\Theta,\varphi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{ik\varphi}(-1)^k i^\ell\sqrt{\frac{(2\ell+1)(\ell-k)!}{2(\ell+k)!}}P_\ell^k(\cos\Theta),$$

and $P_\ell^k(\cos\Theta)$ is the associated Legendre polynomial, which is

$$P_\ell^k(\cos\Theta) = \frac{1}{2^\ell\ell!}\sin^k\Theta\frac{\mathrm{d}^{r+\ell}}{\mathrm{d}\cos\Theta^{r+\ell}}\left(\cos^2\Theta - 1\right)^\ell.$$

Since

$$\hat{\ell}^2 Y_{\ell k} = \ell(\ell+1)Y_{\ell k},$$

we obtain for the radial part of the wave function $R(r)$,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}R + \frac{2m}{\hbar^2}[E - U(r)]R = 0. \tag{4.72}$$

Equation (4.72) does not contain the value $\ell_z = m$, i.e. at the given ℓ the energy level E corresponds to $2\ell+1$ states differing by the value ℓ_z.

The operator

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right)$$

is equivalent to the expression

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR)$$

and thus it is convenient to make the change of variables assuming that

$$X(r) = rR(r).$$

So that Eq. (4.71) can be rewritten in the form

$$\frac{\mathrm{d}^2X}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2}X + \frac{2m}{\hbar^2}[E - U(r)]X = 0. \tag{4.73}$$

We now consider the demand following from the boundary conditions and related to the behaviour of the wave function $X(r)$. At $r \to 0$ and the potentials satisfying the condition

$$\lim_{r\to 0} U(r)r^2 = 0, \tag{4.74}$$

only the first two terms play an important role in Eq. (4.73). $X(r) \sim r^{\nu}$ and we obtain

$$\nu(\nu-1) = \ell(\ell+1).$$

This equation has roots $\nu_1 = \ell + 1$ and $\nu_2 = -\ell$.

The requirement of normalization of the wave function is incompatible with the values $\nu = -\ell$ at $\ell \neq 0$ because the normalization integral

$$\int_0^{\infty} |X_r^2(r)\mathrm{d}r|$$

will be divergent for the discrete spectrum, and the condition

$$\int \Psi(\lambda, \xi)\ \Psi(\lambda, X)\ \mathrm{d}\lambda = \delta(X - \xi)$$

does not hold for the continuous spectrum.

At $\ell = 0$ the boundary conditions are determined by the demand for the finiteness of the mean value of the kinetic energy which is satisfied only at $\nu = 1$. So, when the condition (4.74) is satisfied, the wave function of a particle is everywhere finite and at any ℓ

$$\mathrm{X}(0) = 0.$$

Let us consider the energy spectrum and the wave function of the bounded states of a system of two charges. The bounded states exist only in the case of the attracted particles. Such a system defines the properties of the hydrogen atom and hydrogen-like ions.

The equation for the radial wave function is

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R + \frac{2m}{\hbar^2}\left(E + \frac{\alpha}{r}\right)R = 0, \tag{4.75}$$

where $\alpha = Ze^2$ is constant, characterizing the potential; e is the electron charge; Z is the whole number equal to the nucleus charge in the charge units.

The constants e^2, m and $\hbar$ allow us to construct the value with the dimension of length

$$a_0 = \frac{\hbar^2}{me^2} = 0.529 \cdot 10^{-8}\mathrm{cm},$$

known as the Bohr radius, and the time

$$t_0 = \frac{\hbar^3}{me^4} = 0.242 \cdot 10^{-11}\mathrm{cm}.$$

These quantities define the typical space and time scale for describing a system, and it is therefore convenient to use these units as the basic system of atomic units.

Equation (4.75) in atomic units (at $Z = 1$) takes the form

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R + 2\left(E + \frac{1}{r}\right)R = 0. \tag{4.76}$$

At $E < 0$ the motion is finite and the energy spectrum is discrete. We need the solutions (4.76) quadratically integrable with r^2. Let us introduce the specification

$$n = \frac{1}{\sqrt{-2E}}, \quad \rho = \frac{2r}{n}.$$

Equation (4.76) can be written as

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{2}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} + \left[\frac{n}{\rho} - \frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2}\right]R = 0. \tag{4.77}$$

We find the asymptotic forms of the radial function $R(r)$. At $\rho \to \infty$ and omitting the terms $\sim\rho^{-1}$ and $\sim\rho^{-2}$ in (4.77), we obtain

$$\frac{d^2R}{d\rho^2} = \frac{R}{4}.$$

Therefore, at high values of ρ, $R \propto e^{\pm\rho/2}$. The normalization demand is satisfied only by $R(\rho) \propto e^{-\rho/2}$. The asymptotic forms at $r \to 0$ have already been determined.

Substituting

$$R(\rho) = \rho^{\ell} e^{-\rho/2}\omega(\rho),$$

Eq. (4.77) is reduced to the form

$$\rho\frac{d^2\omega}{d\rho^2} + (2\ell + 2 - \rho)\frac{d\omega}{d\rho} + (n - \ell - 1)\omega = 0. \tag{4.78}$$

To solve this equation in the limit of $\rho = 0$, we substitute $\omega(\rho)$ in the form of a power series

$$\omega(\rho) = 1 + \frac{(0-\nu)}{(0+\lambda)}\rho + \frac{(0-\nu)(1-\nu)}{(0+\lambda)(1+\lambda)}\frac{\rho^2}{2!} + \frac{(0-\nu)(1-\nu)(2-\nu)}{(0+\lambda)(1+\lambda)(2+\lambda)}\frac{\rho^3}{3!} + \cdots \tag{4.79}$$

where $\lambda = 2\ell + 2$ and $-v = -n + \ell + 1$.

At $\rho \to \infty$, the function $\omega(\rho)$ should increase, but not faster then the limiting power ρ. Then $\omega(\rho)$ has to be a polynomial of v power. So, $-\mathrm{n} + \ell + 1 = -\mathrm{k}$, and $\mathrm{n} = \ell + 1 + \mathrm{k}$ ($k = 0, 1, 2, \ldots$) at a given value of ℓ. Hence, using the definition for n, we can find the expression for the energy spectrum

$$E_n = -\frac{1}{2n^2}. \tag{4.80}$$

The number n is called the principal quantum number. In general units it has the form

$$E = -Z^2\frac{me^4}{2\hbar^2 n^2}. \tag{4.81}$$

This formula was obtained by Bohr in 1913 on the basis of the old quantum theory, by Pauli in 1926 from matrix mechanics and by Schrödinger in 1926 by solving the differential equations.

Let us solve the problem of the spectrum of the hydrogen atom using the equation of dynamical equilibrium of the system. In Chap. 3 we obtained Jacobi's virial equation for a quantum mechanical system of particles whose interaction is defined by the potential being a homogeneous function of the co-ordinates. This equation in the operator form is

$$\ddot{\hat{\Phi}} = 2\hat{H} - \hat{U}, \tag{4.82}$$

where $\hat{\Phi}$ is the operator of the Jacobi function, which, for the hydrogen atom is written

$$\hat{\Phi} = \frac{1}{2} m\hat{r}^2. \tag{4.83}$$

The Hamiltonian operator is

$$\hat{\mathrm{H}} = -\frac{\hbar^2}{2\mathrm{m}} \Delta_{\mathrm{r}} + \hat{\mathrm{U}}, \tag{4.84}$$

and the operator of the function of the potential energy for the hydrogen atom is

$$\hat{U} = -\frac{e^2}{r}. \tag{4.85}$$

We solve the problem with respect to the eigenvalues of Eq. (4.82), using the main idea of quantum mechanics. For this we use the Schrödinger equation

$$\hat{H}\Psi = E\Psi$$

and rewrite Eq. (4.82) in the form

$$\ddot{\hat{\Phi}} = 2E - \hat{U}. \tag{4.86}$$

This equation includes two (unknown in the general case) operator functions $\hat{\Phi}$ and $\hat{U}$. In the case of the interaction, the potential is determined by the relation (4.85), and we can use a combination of the operators $\hat{\Phi}$ and $\hat{U}$ in the form

$$|\hat{U}|\sqrt{\hat{\Phi}} = \frac{e^2 m^{1/2}}{\sqrt{2}} = B. \tag{4.87}$$

We now transform (4.86) into the form which was considered in classical mechanics:

$$\ddot{\hat{\Phi}} = 2E + \frac{B}{\sqrt{\hat{\Phi}}}. \tag{4.88}$$

Equation (4.88) is a consequence of Eq. (4.86) when the Schrödinger equation and the relationship (4.87) are satisfied. Its solution for the bounded state, that is, when total energy E is determined in parametric form, can be written

$$\sqrt{\Phi} = \frac{B}{2|E|}(1 - \varepsilon \cos \varphi), \tag{4.89}$$

$$\varphi - \varepsilon \sin \varphi = M, \tag{4.90}$$

where the parameter M is defined by the relation

$$M = \frac{(4|E|)^{3/2}}{4B}(t - \tau), \tag{4.91}$$

where ε and τ are integration constants and where

$$\varepsilon = \sqrt{1 - \frac{AC}{2B^2}},$$

$$C = -\dot{\hat{\Phi}}_0^2 + 4B\sqrt{\hat{\Phi}_0} - 2A\hat{\Phi}_0.$$

Moreover, the solution can be written in the form of Fourier and Lagrange series. Thus, the expression (4.37) describes the expansion of the operator $\hat{\Phi}$ into a Lagrange series including the accuracy of ε^3, and has the form

$$\hat{\Phi}_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \cdots\right]. \tag{4.92}$$

Using the general expression for the mean values of the observed quantities in quantum mechanics

$$\langle \Psi|\hat{\Phi}|\Psi\rangle = \bar{\Phi},$$

and taking into account that the mean value of the Jacobi function of the hydrogen atom should be different from zero, we find that our system has multiple eigen-frequencies $\nu_n = n\nu_0$ with respect to the basic v_0 which corresponds to the period

$$T_0 = \frac{8\pi B}{(4|E|)^{3/2}}. \tag{4.93}$$

In accordance with the expression

$$E_n = \hbar\omega_n = \frac{\hbar 2\pi n}{T_0}, \tag{4.94}$$

each of these frequencies corresponds to the energy level E_n of the hydrogen atom. We substitute the expression (4.93) for T_v into Eq. (4.94) and resolve it in relation to E_n:

$$|E_n| = \frac{\hbar 2\pi n(4|E_n|)^{3/2}}{8\pi B} = \frac{\hbar n(4|E_n|)^{3/2}}{\frac{4e^2m^{1/2}}{\sqrt{2}}} = \frac{\hbar n 2\sqrt{2|E_n|}}{e^2m^{1/2}}. \tag{4.95}$$

The expression obtained by Bohr follows from (4.95):

$$E_n = \frac{e^4 m}{2\hbar^2 n^2}. \tag{4.96}$$

This equation solves the problem.

4.5 Solution of a Virial Equation in the Theory of Relativity (Static Approach)

We consider now the solution of Jacobi's virial equation in the framework of the theory of relativity showing its equivalence to Schwarzschild's solution.

Let us write down the known expression for the radius of curvature of space-time as a function of mass density:

$$\frac{1}{R^2} = \frac{8\pi}{3}\frac{G\rho}{c^2}, \tag{4.97}$$

where R is the curvature radius; ρ is the mass density; G is the gravitational constant and c is the velocity of light.

Equation (4.97) can also be rewritten in the form

$$\rho R^2 = \frac{3}{8\pi}\frac{c^2}{G}. \tag{4.98}$$

If the product ρR^2 in Eq. (4.98) is the Jacobi function ($\Phi = \rho R^2$) is the density of the Jacobi function) then, from (4.98)

$$\Phi = \frac{3}{8\pi}\frac{c^2}{G} \tag{4.99}$$

and it follows that the Jacobi function is a fundamental constant for the Universe. (In general relativity, the spatial distance does not remain invariant. Therefore, instead of this the Gaussian curvature is used, which has the dimension of the universe distance and is the invariant or, more precisely, the covariant.)

The constancy of the Jacobi function in this case reflects the smoothness of the description of motion in general relativity. The oscillations relative to this smooth motion described by Jacobi's equation are the gravitational waves and horizons, in particular the collapse and all types of singularity up to the process of condensation of matter in galaxies, stars and so on.

Now we can show that Schwarzschild's solution in general relativity is equivalent to the solution of Jacobi's equation when $\ddot{\Phi} = 0$. Let us write the expression for the energy-momentum tensor

$$T_i^k = (\rho + p)u_i u^k + p\delta_i^k. \tag{4.100}$$

In the corresponding co-ordinate system, we obtain

$$u^i = \left(0, 0, 0, \frac{1}{\sqrt{-g_{00}}}\right), \tag{4.101}$$

where $\rho = \rho(r)$ and $p = p(r)$.

The independent field equations are written as

$$G_1^1 = T_1^1, \quad G_0^0 = T_0^0, \tag{4.102}$$

$$R^{-2} = \frac{1}{3} G\rho c^2.$$

The expression for the metric is written in the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{r^2}{R^2}} + r^2(\mathrm{d}\Omega)^2 - \left\{A - B\sqrt{1 - \frac{r^2}{R^2}}\right\}^2 c^2 r^2, \tag{4.103}$$

where

$$\frac{\mathrm{d}r^2}{1 - \frac{r^2}{R^2}} + r^2(\mathrm{d}\Omega)^2,$$

is the spatial element.

In this case the expression for the volume occupied by the system is written

$$V = \int_0^r \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\Theta}{\sqrt{1 - \frac{r^2}{R^2}}} \mathrm{d}r\mathrm{d}\Theta r d = \frac{4\pi\pi^3}{3}\left[\arcsin\frac{r}{R} - \frac{r}{R}\sqrt{1 - \frac{r^2}{R^2}}\right] \tag{4.104}$$

It can be easily verified that the right-hand side of Eq. (4.104) coincides with solution (4.14) and (4.15) of the equation of virial oscillations (4.11) at $\ddot{\Phi} = 0$, that is,

$$\arcsin x - x\sqrt{1-x^2} = \arccos\left(\frac{\frac{A}{B}\sqrt{\Phi}-1}{\sqrt{1-\frac{AC}{2B^2}}}\right) - \sqrt{1-\frac{AC}{2B^2}}\sqrt{1-\left(\frac{\frac{A}{B}\sqrt{\Phi}-1}{\sqrt{1-\frac{AC}{2B^2}}}\right)^2}. \tag{4.105}$$

In fact, Eq. (4.105) is satisfied for

$$x = \frac{\frac{A}{B}\sqrt{\Phi}-1}{\sqrt{1-\frac{AC}{2B^2}}} \quad \text{и} \quad x = \sqrt{1-\frac{AC}{2B^2}},$$

that is,

$$\frac{A}{B}\sqrt{\Phi} - 1 = 1 - \frac{AC}{2B^2}, \quad \text{or} \quad \frac{AC}{2B^2} + \frac{A\sqrt{\Phi}}{B} = 2.$$

At $\ddot{\Phi} = 0$, the parameter of virial oscillations

$$e = \sqrt{1-\frac{AC}{2B^2}} \quad \text{and} \quad \sqrt{\Phi} = \frac{B}{A},$$

so the last condition is satisfied.

Schwarzschild's solution is rigorous and unique for Einstein's equation for a static model of a system with spherical symmetry.

Since this solution coincides with the solution of virial oscillations at the same conditions, the solutions (4.14) and (4.15) of Eq. (4.11) obtained in this chapter should be considered rigorous. Thus, we can conclude that the constancy of the product $U\sqrt{\Phi}$ in the framework of the static system model is proven. In Chap. 8 we will come back to this condition and will obtain another proof of the same very important relationship.

4.6 General Approach to Solution of Virial Equation for a Dissipative System

In the previous sections, we considered a number of cases of explicitly solved problems in mechanics and physics for the dynamics of n-body system and have shown that all those classical problems have also explicit solutions in the framework of the virial approach. But in the latter case, the solutions acquires a new physical meaning because the dynamics of a system is considered with respect to new parameters, i.e. its Jacobi function (polar moment of inertia) and potential (kinetic) energy. In fact, the solution of the problem in terms of co-ordinates and velocities specifies the changes in location of a system or its constituents in space. The solution, with respect to the Jacobi function and the potential energy, identifies the evolutionary processes of the structure or redistribution of the mass density of the system. Moreover, the main difference of the two approaches is that the classical problem considers motion of a body in the outer central force field. The virial approach considers motion of a body both in the outer and in the own force field applying, instead of linear forces and moments, the volumetric forces (pressure) and moments (oscillations).

It appears from the cases considered that the existence of the relationship between the potential energy of a system and its Jacobi function written in the form

$$U\sqrt{\Phi} = B = \text{const}. \tag{4.106}$$

is the necessary condition for the resolution of Jacobi's equation.

This is the only case when the scalar equation

$$\ddot{\Phi} = 2\mathrm{E} - \mathrm{U}$$

is transferred into a non-linear differential equation with one variable in the form

$$\ddot{\Phi} = 2\mathrm{E} + \frac{\mathrm{B}}{\sqrt{\Phi}}. \tag{4.107}$$

It was shown earlier that if the total energy of a system $E_o = -A/2 < 0$, then the general solution for Eq. (4.107) can be written as

$$\sqrt{\Phi_0} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)], \tag{4.108}$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)]$$

where ε and ψ are integration constants, the values of which are determined from initial data using Eqs. (4.23) and (4.24).

Equation (4.107) is called the equation of virial oscillation because its solution discovers a new physical effect—periodical non-linear change of the Jacobi function and hence the potential energy of a system around their mean values determined by the virial theorem. Thus, in addition to the static effects determined by the hydrostatic equilibrium, in the study of dynamics of a system, the effects determined by condition of dynamical equilibrium expressed by the Jacobi function are introduced.

The equation of virial oscillations (4.107) reflects physics of motion of the interacted mass particles of a body or masses of bodies themselves by the inverse square law. Its application opens the way to study the nature and the mechanism of generation of the body's energy, which performs its motion, and search the law of change the system's configuration, that is, mutual change location of particles or the law of redistribution of the mass density for the system matter during its oscillations. This problem was considered earlier in our works (Ferronsky et al. 1984, 1987, 2011). We continue its study in Chap. 7.

As described above, cases of solution of Eq. (4.107) relate to unperturbed conservative systems. But in reality, in nature all systems are affected by internal and external perturbations which, from a physical point of view, are developed in the form of dissipation or absorption of energy. In this connection, as was shown, the right-hand side of the equation of virial oscillations (4.107), an additional term appears which is proportional to the Jacobi function Φ (indicating the presence of gravitational background or the existence of interaction between the system particles in accordance with Hook's law) and its first derivative $\dot{\Phi}$ depending on time t (indicating the existence of energy dissipation). All these and other possible cases can be formally described by a generalized equation of virial oscillations (3.27):

$$\ddot{\Phi} = 2E + \frac{B}{\sqrt{\Phi}} + X\left(t, \Phi, \dot{\Phi}\right), \tag{4.109}$$

where $X\left(t, \Phi, \dot{\Phi}\right)$ is the perturbation function, the value of which is small in comparison with the term $B/\sqrt{\Phi} \neq \text{const}$.

In this chapter, we consider general as well as some specific approaches to the solution of Eq. (4.109) in the framework of different physical models of a system.

4.7 Analytical Solution of Generalized Equation of Virial Oscillations

The equation of perturbed virial oscillations is generalized in the form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + X\left(t, \Phi, \dot{\Phi}\right), \tag{4.110}$$

where $A = -2E$; B is constant; $X\left(t, \Phi, \dot{\Phi}\right)$ is the perturbation function which we assume is given and dependent in general cases on time t, the Jacobi function Φ and its first derivative $\dot{\Phi}$.

We consider two ways for analytical construction of the solution of Eq. (4.110). In addition, let the function $X\left(\mathrm{t}, \Phi, \dot{\Phi}\right)$ in Eq. (4.110) depend on some small parameter e in relation to which the function can be expanded into absolute convergent power series of the form

$$X(t, \Phi, \dot{\Phi}) = \sum_{r=1}^{\infty} e^k X^k(t, \Phi, \dot{\Phi}) \tag{4.111}$$

Let the series be convergent in some time interval t absolute for all values of e, which are satisfied to condition $|e| < \bar{e}$. Then Eq. (4.110) can be rewritten in the form

$$\mathrm{X}(t, \Phi, \dot{\Phi}) = -A + \frac{B}{\sqrt{\Phi}} \sum_{r=1}^{\infty} e^k X^k(t, \Phi, \dot{\Phi}) \tag{4.112}$$

We look for the solution of Eq. (4.112) also in the form of the power series of parameter e. For this purpose, we write the function Φ(t) in the form of a power series, the coefficients of which are unknown:

$$\Phi(t) = \sum_{k=0}^{\infty} e^k \Phi^{(k)}(t). \tag{4.113}$$

Putting (4.113) into (4.112), the task can be reduced to the determination of such function $\Phi^{(k)}(t)$ which identically satisfy Eq. (4.112). In this case, the coefficient $\Phi^{(o)}(t)$ becomes the solution of the unperturbed oscillation Eq. (4.107), which can be obtained from (4.112) by putting $e = 0$.

One can consider the series (4.113) as a Taylor series expansion in order to determine all the other coefficients $\Phi^{(k)}(t)$, that is,

$$\begin{aligned} \Phi^{(k)} &= \frac{1}{k!}\left(\frac{d^k\Phi}{\mathrm{d}e^k}\right)|_{e=0}, \\ \dot{\Phi}^{(k)} &= \frac{1}{k!}\left(\frac{\mathrm{d}^k\dot{\Phi}}{\mathrm{d}e^k}\right)|_{e=0}. \end{aligned} \tag{4.114}$$

Accepting the series (4.113) for introduction into Eq. (4.112), it becomes identical with respect to the parameter e. Thus, we have justified the differentiation of the identity with respect to the parameter e several times assuming that the identity remains after repeated differentiation.

We next obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}e}\right) = -\frac{1}{2}\frac{B}{\Phi^{3/2}}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}e}\right) + \sum_{k=1}^{\infty} k e^{k-1} X^{(k)} + \sum_{k=1}^{\infty} e^k \left(\frac{\mathrm{d}X^{(k)}}{\mathrm{d}e}\right), \tag{4.115}$$

where $\mathrm{d}X^{(k)}/\mathrm{d}e$ is the total derivative of the function $\mathrm{X}^{(\mathrm{k})}$ with respect to parameter e, expressed by

$$\frac{dX^{(k)}}{de} = \frac{\partial X^{(k)}}{\partial \Phi}\left(\frac{d\Phi}{de}\right) + \frac{\partial X^{(k)}}{\partial \dot{\Phi}}\left(\frac{d\dot{\Phi}}{de}\right)$$

Now let $e = 0$ in (4.115). Then by taking into account (4.113) and (4.114), we obtain

$$\frac{d^2\Phi^{(1)}}{dt^2} + p_1\Phi^{(1)} = X_1 \tag{4.116}$$

where

$$p_1 = \frac{1}{2}\frac{B}{\Phi^{3/2}}|_{e=0} = \frac{1}{2}\frac{B}{\Phi^{(0)3/2}} \quad X_1 = X^1\left(t, \Phi^{(0)}, \dot{\Phi}^{(0)}\right)$$

are known functions of time, since the solution of the equation in the zero approximation [unperturbed oscillation Eq. (4.108)] is known

Carrying out differentiation of Eq. (4.112) with respect to parameter e for the second, third and so on $(k - 1)$ times, and assuming after each differentiation that $e = 0$, we will step by step obtain equations determining second, third and so on approximations. It is possible to show that in each succeeding approximation the equation will have the same form and the same coefficient p_1 as in Eq. (4.116). If so, the equation determining the functions $\Phi^{(k)}$ and $\dot{\Phi}^{(k)}$ has the form

$$\frac{d^2\Phi^{(k)}}{dt^2} + p_1\Phi^{(k)} = X_k\left(t, \Phi^{(0)}, \dot{\Phi}^{(0)}, \ldots, \Phi^{(k-1)}, \Phi^{(k-1)}\right) \tag{4.117}$$

where the function X_k depends on $\Phi^{(0)}$, $\dot{\Phi}^{(0)}$, …, $\Phi^{(k-1)}$, $\dot{\Phi}^{(k-1)}$, which were determined earlier and are the functions of t and unknown functions $\Phi^{(0)}$ and $\dot{\Phi}^{(0)}$.

It is known that there is no general way of obtaining a solution for any linear differential equation with variable coefficients, but in our case we can use the following theorem of Poincare (Duboshin 1975). Let the general solution of the unperturbed virial oscillation equation be determined by the function $\Phi^{(0)} = f(t, C_1, C_2)$, where C_1 and C_2 are, for instance, arbitrary constants ε and Ψ in the solution (4.108) of Eq. (4.107). Then, Poincare's theorem confirms that the function determined by the equalities

$$\Phi_1 = \frac{\partial f}{\partial C_1},$$

$$\Phi_2 = \frac{\partial f}{\partial C_2}$$

satisfies the linear homogeneous differential equation reduced by omission of the right-hand side of Eq. (4.117).

Thus, the general solution of the linear homogeneous equation

$$\frac{\mathrm{d}^2\Phi^{(k)}}{\mathrm{d}t^2} + p_1\Phi^{(k)} = 0$$

has the form

$$\Phi_1 C_1^{(k)} + \Phi_2 C_2^{(k)} = \Phi^{(k)} \tag{4.118}$$

and the general solution of Eq. (4.117) can be obtained by the method of variation of arbitrary constants, i.e. assuming that $C_2^{(k)}$ are functions of time. Then, using the key idea of the method of variation of arbitrary constants we obtain a system of two equations

$$\begin{aligned} \dot{C}_1^{(k)}\Phi_1 + \dot{C}_2^{(k)}\Phi_2 &= 0,_k\,. \\ \dot{C}_1^{(k)}\dot{\Phi}_1 + \dot{C}_2^{(k)}\dot{\Phi}_2 &= X_k. \end{aligned} \tag{4.119}$$

Solving this system with respect to $\dot{C}_1^{(k)}$ and $\dot{C}_2^{(k)}$ and integrating the expression obtained, we write the general solution of Eq. (4.117) as follows:

$$\Phi^{(k)}(t) = \Phi_2 \int_{t_0}^{t} \frac{\Phi_1 X_k \mathrm{d}t}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1} - \Phi_1 \int_{t_0}^{t} \frac{\Phi_2 X_k \mathrm{d}t}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1}$$

where

$$\Phi_1 = \frac{\partial f(t, C_1, C_2)}{\partial C_1},$$

and

$$\Phi_2 = \frac{\partial f(t, C_1, C_2)}{\partial C_2}.$$

Thus, we can determine any coefficient of the series (4.113), reducing Eq. (4.112) into an identity, and therefore write the general solution of Eq. (4.110) in the form

$$\Phi = \sum_{k=0}^{\infty} e^k\Phi^{(k)} = \sum_{k=0}^{\infty} e^k \left[\Phi_2 \int_{t_0}^{t} \frac{\Phi_1 X_k \mathrm{d}t}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1} - \Phi_1 \int_{t_0}^{t} \frac{\Phi_2 X_k \mathrm{d}t}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1}\right] \tag{4.120}$$

Let us consider the second way of approximate integration of the perturbed virial Eq. (4.110), based on Picard's method (Duboshin 1975). It is convenient to apply this method of integrating the equations which were obtained using the Lagrange method of variation of arbitrary constants.

Assuming that the first integrals (4.23) and (4.24)

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)} \tag{4.121}$$

$$-\tau = \left\{\frac{4B}{(2A)^{3/2}}\left[\arccos\frac{1-\frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} - \varepsilon\sqrt{1-\left(\frac{1-\frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}\right] - t\right\} \tag{4.122}$$

of the unperturbed virial oscillation Eq. (4.107) are also the first integrals of the perturbed oscillation Eq. (4.110). But constants ε and τ are now unknown functions of time. Let us derive differential equations which are satisfied by these functions using the first integrals (4.121) and (4.122). For convenience, we replace the integration constant ε by C, using the expression

$$\varepsilon = \sqrt{1 - \frac{AC}{2B^2}}.$$

Now we rewrite Eq. (4.121) in the form

$$C = -\dot{\Phi}_0^2 + 4B\sqrt{\Phi_0} - 2A\Phi_0 \tag{4.123}$$

Then using the main idea of the Lagrange method, after variation of the first integrals (4.122) and (4.123) and replacement of $\ddot{\Phi}$ by

$$\left(-A + \frac{B}{\sqrt{\Phi}} + X\left(t, \Phi, \dot{\Phi}\right)\right),$$

we write

$$\dot{C} = -2\dot{\Phi}X\left(t, \Phi, \dot{\Phi}\right) \tag{4.124}$$

$$\dot{\tau} = \Psi(\Phi, C)\dot{C} = -2\dot{\Phi}X\left(t, \Phi, \dot{\Phi}\right)\Psi(\Phi, C) \tag{4.125}$$

where

$$\Psi(\Phi, C) = -\frac{4B}{(2A)^{3/2}}\frac{\mathrm{d}}{\mathrm{d}C}\left[\arccos\frac{1-\frac{A}{B}\sqrt{\Phi}}{\sqrt{1-\frac{AC}{2B^2}}} - \sqrt{1-\frac{AC}{2B^2}}\sqrt{1-\left(\frac{1-\frac{A}{B}\sqrt{\Phi}}{\sqrt{1-\frac{AC}{2B^2}}}\right)^2}\right].$$

We now express Φ and $\dot{\Phi}$ in explicit form through C, τ and t, using, for example, the Lagrangian series:

$$\Phi(t) = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \cdots\right], \tag{4.126}$$

$$\dot{\Phi}(t) = \sqrt{\frac{2}{A}}\varepsilon B\left[\sin M + \frac{1}{2}\varepsilon\sin 2M + \frac{\varepsilon^2}{2}\sin M\left(2\cos^2 M - \sin^2 M\right) + \cdots\right] \tag{4.127}$$

Thus, taking into account Eqs. (4.126) and (4.127) for the functions Φ and $\dot{\Phi}$, Eqs. (4.124) and (4.25) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t} &= F_1(t, C, \tau), \\ \frac{\mathrm{d}\tau}{\mathrm{d}t} &= F_2(t, C, \tau). \end{aligned} \tag{4.128}$$

To solve the system of differential Eqs. (4.128), we use Picard's successive approximation method, obtained in the k-th approximation expressions for $C^{(k)}$ and $\tau^{(k)}$ in the form

$$C^{(k)} = C^{(0)} + \int_{t_0}^{t} F_1\left(t, C^{(k-1)}, \tau^{(k-1)}\right)\mathrm{d}t, \tag{4.129}$$

$$\tau^{(k)} = \tau^{(0)} + \int_{t_0}^{t} F_2\left(t, C^{(k-1)}, \tau^{(k-1)}\right)\mathrm{d}t, \tag{4.130}$$

where $C^{(0)}$ and $\tau^{(0)}$ are the values of arbitrary constants C and τ at initial time t_0, and $k = 1, 2, \ldots$

Then, in the limit of $k \to \infty$, we obtain the solution of the system (4.128):

$$\begin{aligned} C &= \lim_{k\to\infty} C^{(k)}, \\ \tau &= \lim_{k\to\infty} \tau^{(k)}. \end{aligned} \tag{4.131}$$

Consider now, two possible cases of the perturbation function behaviour. First, assume that the perturbation function X does not depend explicitly on time. Then, since it is possible to expand functions Φ and $\dot{\Phi}$ into a Fourier series in terms of sine and cosine of argument M, the right-hand sides of the system (4.128) can also be expanded into a Fourier series in terms of sine and cosine of M.

Finally, we obtain

$$\frac{dC}{dt} = \left[A_0 + \sum_{k=1}^{\infty}(A_k \cos kM + B_k \sin kM)\right], \tag{4.132}$$

$$\frac{d\tau}{dt} = \left[a_0 + \sum_{k=1}^{\infty}(a_k \cos kM + b_k \sin kM)\right], \tag{4.133}$$

where A_0, A_k, B_k, a_0, a_k, b_k are the corresponding coefficients of the Fourier series which are

$$A_0 = \frac{2}{\pi}\int_0^{2\pi} F_1(M, C)dM,$$

$$A_k = \frac{2}{\pi}\int_0^{2\pi} F_1(M, C)\cos kM dM,$$

$$B_k = \frac{2}{\pi}\int_0^{2\pi} F_1(M, C)\sin kM dM,$$

$$a_0 = \frac{2}{\pi}\int_0^{2\pi} F_2(M, C)dM,$$

$$a_k = \frac{2}{\pi}\int_0^{2\pi} F_2(M, C)\cos kM dM,$$

$$b_k = \frac{2}{\pi}\int_0^{2\pi} F_2(M, C)\sin kM dM.$$

Following Picard's method, in order to solve Eqs. (4.132) and (4.133) in the first approximation, we introduce into the right-hand side of the equations the values of arbitrary constants C and τ corresponding to the initial time t_0.

Then we obtain

$$C^{(1)}(t) = C^{(0)} + A_0^{(0)}(t - t_0) + \sum_{k=1}^{\infty} \frac{1}{kn} \left\{ A_k^{(0)}[\sin kM - \sin kM_0] + B_k^{(0)}[\cos kM - \cos kM_0] \right\} \tag{4.134}$$

$$\tau^{(1)}(t) = \tau^{(0)} + a_0^{(0)}(t - t_0) + \sum_{k=1}^{\infty} \frac{1}{kn} \left\{ a_k^{(0)}[\sin kM - \sin kM_0] + b_k^{(0)}[\cos kM - \cos kM_0] \right\} \tag{4.135}$$

Thus, when the function X does not depend explicitly on time t, solutions (4.134) and (4.135) of Eq. (4.110) have three analytically different parts. The first is a constant term, depending on the initial values of the arbitrary constants. It is usually called the constant term of perturbation of the first order. The second part is a function monotonically increasing in time. It is called the secular term of the perturbation of the first order. The third part consists of an infinite set of trigonometric terms. All of them are periodic functions of M and consequently of time t. This is called periodic perturbation.

Similarly, we can obtain solutions in the second, third and so on orders. Here we limit our consideration only within the first order of perturbation theory. In practice, few terms of the periodic perturbation can be taken into account and the solution obtained becomes effective only for a short period of time.

When the perturbation function X is a periodic function of some argument M',

$$M' = n'(t - \tau'),$$

the right-hand side of the system of Eqs. (4.128) are periodic functions of the two independent arguments M and M'. Therefore, they can be expanded into a double Fourier series in terms of sine and cosine of the linear combination of arguments M and M'. Then in the first approximation of perturbation theory we obtain the following system of equations:

$$\frac{dC^{(1)}}{dt} = A_{00}^{(0)} + \sum_{k',k=-\infty}^{\infty} \left[A_{k,k'}^{(0)} \cos(kM + k'M') + B_{k,k'}^{(0)} \sin(kM + k'M') \right], \tag{4.136}$$

$$\frac{d\tau^{(1)}}{dt} = a_{00}^{(0)} + \sum_{k',k=-\infty}^{\infty} \left[a_{k,k'}^{(0)} \cos(kM + k'M') + b_{k,k'}^{(0)} \sin(kM + k'M') \right]. \tag{4.137}$$

Integrating Eqs. (4.136) and (4.137) with respect to time, we obtain a solution of the system:

$$C^{(1)}(t) = C^{(0)} + A_{00}^{(0)}(t - t_0) + \sum_{k',k=-\infty}^{\infty} \frac{1}{kn + k'n'} \Big\{ A_{k,k'}^{(0)} [\cos(kM + k'M') - \cos\big(kM_0 + k'M'_0\big)] + B_{k,k'}^{(0)} \Big[\sin(kM + k'M') - \sin\big(kM_0 + k'M'_0\big)\Big] \Big\} \tag{4.138}$$

$$\tau^{(1)}(t) = \tau^{(0)} + a_{00}^{(0)}(t - t_0) + \sum_{k',k=-\infty}^{\infty} \frac{1}{kn + k'n'} \Big\{ a_{k,k'}^{(0)} [\cos(kM + k'M') - \cos\big(kM_0 + k'M'_0\big)] + b_{k,k'}^{(0)} \Big[\sin(kM + k'M') - \sin\big(kM_0 + k'M'_0\big)\Big] \Big\} \tag{4.139}$$

Equations (4.138) and (4.139) have the same analytical structure as (4.134) and (4.135). At the same time, in this case, the periodic part of the perturbation can be divided into two groups depending on the value of the divisor $kn + k'n'$. If the values of k and k' are such that the divisor is sufficiently large, then period $T_{k,k'} = 2\pi/(kn + k'n')$ of the corresponding inequality will be rather small. Such inequalities are called short-periodic. Their amplitudes are also rather small, and they can play a role only within short periods of time.

If the values of k and k' are such that the divisor $kn + k'n'$ is sufficiently small but unequal to zero, then the period of the corresponding inequality will become large. The amplitude of such terms could also be large and play a role within large periods of time. Such terms form series of long-periodic inequalities. In the case of such k and k', when $kn + k'n' = 0$, the corresponding terms are independent of t and change the value of the secular term in the solutions (4.138) and (4.139).

4.8 Solution of the Virial Equation for a Dissipative System

In Chap. 3 we derived Jacobi's virial equation for a non-conservative system in the form

$$\ddot{\Phi} = 2E_0[1 + q(t)] - U - k\dot{\Phi} \tag{4.140}$$

At $k \ll 1$, $t \gg t_o$, $|U|\sqrt{\Phi} = B = \text{const}$, $2E_o = -A_o$, and when the magnitude of the term $k\dot{\Phi}$ is sufficiently small, Eq. (5.140) can be rewritten in a parametric form

$$\ddot{\Phi} = -A_0[1 + q(t)] + \frac{B}{\sqrt{\Phi}} \tag{4.141}$$

where $q(t)$ is a monotonically increasing function of time due to dissipation of energy during 'smooth' evolution of a system within a time interval $t \in [0, \tau]$.

Using the theorem of continuous solution depending on the parameter, we write the solution of Eq. (4.141) as follows:

$$\begin{aligned} &- \arccos W + \arccos W_0 - \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}}\sqrt{1 - W^2} \\ &+ \sqrt{1 - \frac{A_0 C}{2B^2}}\sqrt{1 - W_0^2} = \sqrt{\frac{(2 - -A_0[1 + q(t)])^{3/2}}{4B}}(t - t_0) \end{aligned} \tag{4.142}$$

$$\begin{aligned} &\arccos W - \arccos W_0 + \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}}\sqrt{1 - W^2} \\ &- \sqrt{1 - \frac{A_0 C}{2B^2}}\sqrt{1 - W_0^2} = \sqrt{\frac{(2 - -A_0[1 + q(t)])^{3/2}}{4B}}(t - t_0) \end{aligned} \tag{4.143}$$

where

$$W = \frac{\frac{A_0[1 + q(t)]}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}}}; \quad W_0 = \frac{\frac{A_0}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{A_0 C}{2B^2}}};$$

$$A_0[1 + q(t)] > 0; \quad C < \frac{2B^2}{A_0[1 + q(t)]};$$

$$|-A_0[1 + q(t)]\sqrt{\Phi} + B| < B\sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}};$$

$$C = -2A_0\Phi_0 + 4B\sqrt{\Phi_0} - \dot{\Phi}_0^2$$

Equations of discriminant curves which bound to oscillations of the Jacobi function Φ by analogy with the case of the conservative system can be written as

$$\sqrt{\Phi_1} = \frac{B}{A_0[1 + q(t)]}\left[1 + \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}}\right], \quad t \in [0, \tau], \tag{4.144}$$

$$\sqrt{\Phi_2} = \frac{B}{A_0[1 + q(t)]}\left[1 - \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}}\right], \quad t \in [0, \tau]. \tag{4.145}$$

It is obvious that the solution of Jacobi's virial equation for a non-conservative system is quasi-periodic with period

$$T_v(q) = \frac{8\pi B}{(2A_0[1+q(t)])^{3/2}}, \tag{4.146}$$

and an amplitude of Jacobi function oscillations

$$\Delta\sqrt{\Phi} = \frac{B}{A_0[1+q(t)]}\left(1 - \frac{A_0[1+q(t)]C}{2B^2}\right)^{1/2}. \tag{4.147}$$

As $q(t)$ is monotonically and continuously increasing parameter confined in time, the period and the amplitude of the oscillations will gradually decrease and tend to zero in the time limit.

In Fig. 4.5a the integral curves (4.142) and (4.143) and the discriminant curves (4.144) and (4.145) are shown in a general case when $0 < C < 2B^2/A_0$. At the point O_b, the integral and discriminant curves tend to coincide and the value of the amplitude of the Jacobi function (polar moment of inertia) oscillations of the system goes to zero.

When $C = 0$ (Fig. 4.5b) the discriminant line (4.144) coincides with the axis of abscissa, $\Phi_2 = 0$. In the accepted case of constancy of the system mass, the point O_b, where the integral and discriminant curves coincides will be reached in the time limit $t \rightarrow \infty$.

When $2B^2/A_o \rightarrow C$ and $C < 0$ the solutions (4.142), (4.143) and (4.144), (4.145) could be complex so the processes considered are not physical.

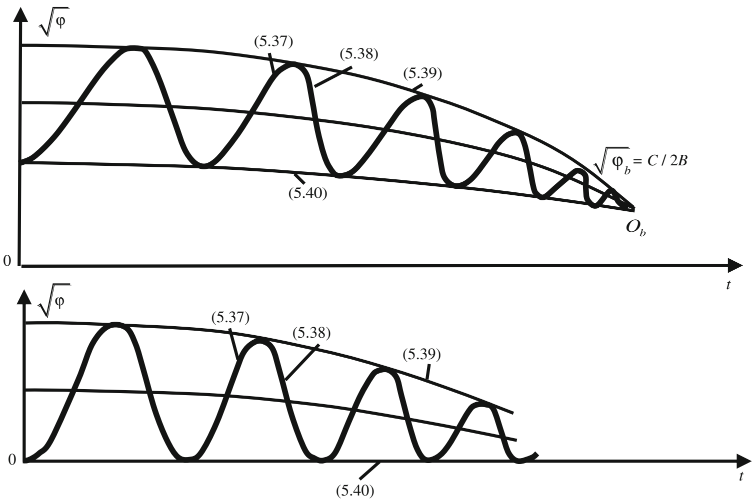

Fig. 4.5 Virial oscillations of Jacobi function in time for non-conservative system (**a**) and for general (Wintner's) case (**b**)

We note that, by analogy with the case for conservative system, considered in Chap. 3, we can show here that the asymptotic relations (4.136)–(4.137) for the solutions (4.142) and (4.143) of Jacobi's equation (4.141) in the points of contact of the discriminant line $\Phi_2 = 0$, are justified. In the points of contact for the integral curves (4.142) and (4.143) and the discriminant curves (4.144) and (4.145) for which Φ_1 and Φ_2 are not equal to zero, the following asymptotic relations are also justified:

$$\left(\sqrt{\Phi_1} - \sqrt{\Phi}\right) \propto (t' - t)^2, \tag{4.148}$$

$$\left(\sqrt{\Phi} - \sqrt{\Phi_2}\right) \propto (t - t')^2, \tag{4.149}$$

where t' is time of a tangency point for the corresponding integral curve of the discriminant lines $\Phi_{1,2}$ when $\Phi_{1,2} \neq 0$.

4.9 Solution of the Virial Equation for a System with Friction

Let us consider the solution of Jabobi's virial equation for conservative systems, but let the relationship between its potential energy and the Jacobi function be as follows:

$$U\sqrt{\Phi} = B + k\dot{\Phi} \tag{4.150}$$

In this case, the equation of virial oscillations (4.107) can be written

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} - k\frac{\dot{\Phi}}{\sqrt{\Phi}}. \tag{4.151}$$

The term $-k\dot{\Phi}/\sqrt{\Phi}$ in (4.151) plays the role of perturbation function reflecting the effect of internal friction of the matter while the system is oscillating.

In principle, Eq. (4.151) can be solved using the above perturbation theory methods. However, we can show that a particular solution exists for the system of two differential equations of the second order, which satisfies Eq. (4.151). These differential equations are as follows:

$$\left(\sqrt{\Phi}\right)'' + \sqrt{\frac{2}{A}}k\left(\sqrt{\Phi}\right)' + \sqrt{\Phi} = \frac{B}{A}, \tag{4.152}$$

$$t'' + \sqrt{\frac{2}{A}}kt' + t = \frac{4B}{(2A)^{3/2}}\lambda. \tag{4.153}$$

In Eqs. (4.152) and (4.153) we introduced a new variable λ, so the primes at Φ and t mean differentiation with respect to λ. Note also that time t here is not an independent variable. This allowed us to transfer the non-linear equation into two linear equations. The partial solution of Eqs. (4.152) and (4.153) containing two integration constants is

$$\sqrt{\Phi} = \frac{B}{A}\left[1 - \varepsilon e^{-r/2\sqrt{2/A}\lambda} \cos\left(\sqrt{\frac{4A - 2k^2}{4A}}\lambda + \psi + \tau\right)\right] \tag{4.154}$$

$$t = \frac{4B}{(2A)}\left[\lambda - \varepsilon e^{-r/2\sqrt{2/A}\lambda} \sin\left(\sqrt{\frac{4A - 2k^2}{4A}}\lambda + \psi\right)\right] - \frac{4B}{(2A^{3/2})}\sqrt{\frac{2}{A}}k \tag{4.155}$$

where ε and ψ are arbitrary constants

$$\tau = \operatorname{arctg}\sqrt{\frac{2}{A}}k\left(\frac{4A - 2k^2}{4A}\right)^{-1/2}.$$

To show that Eqs. (4.154) and (4.155) of the two linear differential equations (4.152) and (4.153) are also general solutions of (4.46), let us do as follows.

Differentiating (4.151) with respect to λ, we obtain

$$t' = \sqrt{\frac{2}{A}}\sqrt{\Phi}. \tag{4.156}$$

We write the derivative from function $\sqrt{\Phi}$ with respect to λ using Eq. (4.156) in the form

$$\left(\sqrt{\Phi}\right)' = \frac{\dot{\Phi}}{\sqrt{2A}} \tag{4.157}$$

We write the derivative from function $\sqrt{\Phi}$ with respect to λ using Eq. (4.156) in the form

$$\left(\sqrt{\Phi}\right)'' = \frac{\ddot{\Phi}}{\sqrt{2A}}t' = \frac{\ddot{\Phi}\sqrt{\Phi}}{A} \tag{4.158}$$

Substituting Eqs. (4.157) and (4.158) for $\left(\sqrt{\Phi}\right)'$ and $\left(\sqrt{\Phi}\right)''$ into Eq. (4.152), we obtain

$$\frac{\ddot{\Phi}\sqrt{\Phi}}{A} + \sqrt{\frac{2}{A}}k\frac{\dot{\Phi}}{\sqrt{\Phi}} + \sqrt{\Phi} = \frac{B}{A} \tag{4.159}$$

Dividing Eq. (4.159) by $\sqrt{\Phi}/A$ we have

$$\ddot{\Phi} + k\frac{\dot{\Phi}}{\sqrt{\Phi}} + A = \frac{B}{\sqrt{\Phi}}$$

which is in fact our Eq. (4.151). This means that Eqs. (4.154) and (4.155) are the general solution of Eq. (4.151).

Note that Eq. (4.155) differs in general from Kepler's equation both by the exponential factor before the sine function and by the constant term in the right-hand side of Eq. (4.155). In addition, it follows from Eq. (4.154) that the period of virial oscillations of the Jacobi function depends on the parameter k. Therefore, when λ changes its value by $2\pi/\left[\sqrt{(4A-2k^2)/4A}\right]$ the value of $\sqrt{\Phi}$ remains unchanged (we neglect the changes of the amplitude of virial oscillations due to existence of the exponential factor) assuming that

$$\frac{k}{2}\sqrt{\frac{2}{A}}2\pi/\sqrt{\frac{4A-2k^2}{4A}} \ll 1.$$

It follows from Eq. (4.155) that time t changes by the relationship of $T = 8\pi B/(2A)^{3/2}\sqrt{(4A-2k^2)/4A}$ defining the period of the damping virial oscillations. Therefore, from solutions (4.154) and (4.155) of Eq. (4.151) it follows that if during the evolution of the system the value $U\sqrt{\Phi}$ varies only slightly around the constant; this leads to damping of the virial oscillations of the integral characteristics of the system around their averaged virial theorem value.

In conclusion, we have to note that derivation of the equation of dynamical equilibrium and its solution for conservative and dissipative systems shows that dynamics of celestial bodies in their own force field puts forward a wide class of geophysical, astrophysical and geodetic problems which can be solved by the methods of celestial mechanics and with new physical concepts of dynamics, gravitation and inertia that we considered here.

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Chapter 5
Centrifugal Effects as the Mechanism of the Solar System Creation from a Common Gaseous Cloud

Abstract The irresistible difficulty in cosmogony is the observed fact that the planets of the Solar System, having only ~ 0.015 % of the system mass, possesses 98 % of their orbital angular momentum. At the same time, ~ 99.85 % of the Sun's mass produces no more than 2 % of the angular momentum, which is accepted to be the conservative parameter. This fact was found in the framework of the hydrostatic equilibrium of the relevant bodies. It is shown in this chapter that, in the framework of Jacobi dynamics, based on new understanding of physics in the matter of gravitational interaction, creation of a new body occurs within the parental cloud as a result of its separation in density on shells, where the outer shell reaches the state of weightlessness. Here the orbital moment of momentum of a created secondary body represents the total kinetic moment of the parental body's cloud owing to the energy conservation law. It means that the orbital moment of momentum of each planet represents the kinetic momentum of the protosun at the time of planet separation and orbiting. The planet's orbital moment of momentum is formed by the total potential energy of the protosun. But the planet's angular moment of axial rotation is formed by the tangential component of the planet's own potential energy. So, the above fact appears to be a misunderstanding. Appearance of weightlessness of the upper shell during a body's matter differentiation, conditions of a body separation and orbiting, the structure of the potential end kinetic energies of a non-uniform body, conditions of dynamical equilibrium of oscillation and rotation of a body, equations of oscillation and rotation of a body and their solution, the nature and mechanism of body shells differentiation and physical meaning of the Archimedes and Coriolis forces are considered as particular tasks, which found here a mathematical solution. The problem of initial values of mean density and radius of a created body also has its own solution. The discussed physics and kinematics of creation and separation of Solar System bodies prove Huygens' law of motion on the semi-cubic parabola of his watch pendulum, which synchronously follows the Earth's motion. Relationship between the evolute and evolvent represents the relationship between function and its derivative or between function and its integral. In the case of the Huygens' oscillating pendulum, the suspension filament starts unrolling in a fixed point. In the case of a celestial body, creation of a satellite starts

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in a fixed point of its parental body where the initial conditions are transferred by the third Kepler's law, which is the consequence of a body creation.

5.1 The Conditions for a Body Separation and Orbiting

It is known that all Solar System bodies (the planets, their satellites, comets and meteoric bodies) are identical in their substantial and chemical content, and in this respect they are of common origin. But the search for a unified mechanism of body creation has encountered an irresistible difficulty in their dynamics. The point is that the planets, having only ~0.015 % of the system mass, possess 98 % of the orbital angular momentum. At the same time, ~99.85 % of the Sun's mass produce no more than 2 % of the angular momentum, which is accepted to be a conservative parameter. Also, the specific (for unit of the mass) angular momentum of the planets is increased together with the distance from the Sun. As it was discussed in previous chapters, the above results follow from a calculation model based on the hydrostatic equilibrium state of the system, where the body motion results from the outer forces. It was shown that the hydrostatic equilibrium for celestial body dynamics appeared not to be a correct physical conception.

We analyze the evolutionary problem of the Solar System based on fundamentals of Jacobi dynamics, where the body motion initiates by the inner forces' action. Here, the energy loss in the form of radiation is accepted as the physical basis of the body evolution. And the centrifugal effect of elementary particles collision and scattering appears to be the mechanism of the energy generation and its redistribution. It is clear from observation that all celestial bodies are self-gravitating systems.

It is shown next that creation of a new body occurs within the parental cloud because of its separation in density on shells by the Archimedes law, when the outer shell reaches the state of weightlessness. Here, the orbital moment of momentum of a created secondary body represents the total kinetic moment of the parental body's cloud owing to the energy conservation law, as it was shown in (2.20)–(2.21):

$$Q = \sum_i p_i r_i = \sum_i m_i v_i r_i = \sum_i m_i \dot{r}_i r_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{m_i r_i^2}{2}\right) = \frac{1}{2}\dot{I}_p, \tag{5.1}$$

where Q is the moment of momentum of the parental cloud; p is the moment of a particle; r is the radius; I_p is the polar moment of inertia of the cloud.

It means that the orbital moment of momentum of each planet represents the kinetic momentum of the protosun at the time of planet separation and orbiting. Kinetic moment of a body is equal to the sum of the rotational and oscillating moments, and the kinetic energy is equal to the sum of the rotational and oscillating energy, which follows from the energy conservation law. At the same time, the planet's orbital moment of momentum is formed by the total potential energy of the protosun. But the planet's angular moment of the axial rotation is formed by the tangential component of the own planet's potential energy (see below Eqs. (5.8) and (5.9)). Here the energy of axial rotation compiles a small portion of the oscillating energy. As it was noted in Sect. 2.4, kinetic energy of the planets Earth, Mars, Jupiter, Saturn, Uranus and Neptune compiles 10^{-3}–10^{-2}, and of the Mercury, Venus, the Moon and the Sun is about 10^{-4} from the total kinetic energy of each body. For bodies with uniform mass density distribution the kinetic energy of rotation is equal to zero.

The interaction (collision and scattering) of mass particles is accompanied by continuous redistribution of the body's mass density. According to Roche's tidal dynamics, redistribution of mass density leads to shell separation. It will be shown later that, when the density of the upper shell reaches less then two thirds, with respect to the underlying shells, then the upper shell becomes weightless (i.e. it loses weight). From a physical point of view, it means that its own force field of the upper shell is in dynamical equilibrium with the parental force field. In this case, if the density of the upper shell has non-uniform density distribution, then by the difference in the potentials of the force field, the shell is converted into a secondary body. If the upper shell has uniform mass density distribution, then the shell forms a ring around the equatorial plane of the parental body. In the general case, the upper weightless shell decays into fragments with different amounts of mass. The comets were formed from the solar shell, the satellites and meteorites were created from the planet's shells. During evolution of a non-uniform gaseous body, it undergoes axial and equatorial oblateness by an outer force field of the central parental body. This can be observed by inclination of the planet's and satellite's orbital plane slope relative to the parental equatorial plane. The polar outer force field pressure appears to be higher than the equatorial. As a result, the outer polar force field values appeared to be higher than the equatorial. Because of this, the polar matter of the upper shell is continuously removed to the equatorial plane. This is why the created bodies are formed mainly in the equatorial plane and form into an equatorial disk.

So the orbital motion of a separated secondary body is defined by the outer force field at the surface of the parental body. The value of this field at the body's surface is a fundamental parameter, which is determined by the body's law of energy conservation. That is why the orbital velocity of a newly created body is equal to its parental first cosmic velocity. The direction of the orbital motion is determined by Lenz's law (see Fig. 2.3). In this connection, it is worth noting that from the point of view of the Solar System creation problem, attempts to find an explanation of the observed distribution of the moment of momentum between the axial rotation of the Sun and the planets' orbital motion are not fruitful. This is because the planets' orbital velocity demonstrates parental relationship between the planets and the Sun by proving its identity with the first cosmic velocity and the law of energy conservation.

Thus, it follows from the above scenario that, induced by matter, the interaction of the outer force field of the Sun is responsible for orbital motion of the planets. Analogously, the planets' force field is orbiting their satellites. Doing so, each body with high accuracy records the value of the parent's potential energy at the moment of orbiting. As to the shell's axial rotation, then its potential energy is determined by the value of its tangential component. The normal and tangential components of body's potential energy comprise the total potential energy, which is a conservative parameter.

Justifications of the above dynamical effects, which take part in creation of the Solar System bodies in the framework of Jacobi's dynamics, are presented below. The main dynamical effect, related to the nature of the Solar System body creation, is proved by observational data seen in Tables 2.1 and 2.2.

It was shown earlier in Sect. 4.1 that, in the framework of Jacobi dynamics, solution of the Kepler's problem is given by equations:

$$\sqrt{\Phi} = \frac{B}{A}[1 - \varepsilon\cos(\lambda - \psi)], \tag{5.2}$$

$$t = \frac{4B}{(2A)^{3/2}}[\lambda - \varepsilon\sin(\lambda - \psi)] \tag{5.3}$$

$$\omega = \frac{2\pi}{T} = \frac{(2A)^{3/2}}{4B} = \sqrt{\frac{GM}{R^3}} = \sqrt{\frac{4}{3}\pi G\rho_0} \tag{5.4}$$

where ε and ψ are the integration constants depending on the initial values of Jacobi's function Φ and its first derivative $\dot{\Phi}$ at the time moment t_0 (the time here is an independent variable); T is the period of virial oscillations; ω is the oscillation; λ is the auxiliary an independent variable; $A = A_0 = 1/2E > 0,\ B = B_0 = U\sqrt{\Phi_0}$ for radial oscillations; $A = A_r = 1/2E > 0,\ B = B_r = U\sqrt{\Phi_r}$ for rotation of the body.

The product of the oscillation frequency ω of the outer force field and R of the body gives the value of the first cosmic velocity of an artificial satellite, that is, the velocity with which the satellite undertakes gravity attraction (the pressure induced by the outer force field). In order to undertake the attraction, a satellite uses its own inner energy of the reactive engine. In this way, the satellite reaches the first cosmic velocity and becomes weightless, i.e. its own outer force field reaches equilibrium with the planet's outer force field. After that, the engine is switched off and its motion continues by the energy of an outer force field. In order to be separated from the parental body, its outer shell must reach a state of weightlessness, that is, its own force field reaches dynamical equilibrium with the parental force field. The secondary body, created from the outer shell, being completely in a non-weighty state and in dynamical equilibrium with the parental outer force field, moves farther by that force field with the first cosmic velocity.

The data of Tables 2.1 and 2.2 show that the existing discrepancies in the moment of momentum distribution between the Sun and the planets and also the problems of capture or separation of the planets' cloud are taking off. The secondary body at its creation conserves the parental potential energy through the first cosmic velocity. As to the direction of orbital motion, the Lenz law works, which evidences about common nature of the gravity and electromagnetic fields. The specific value (per mass unit) of the planets' and satellites' orbital moment of momentum, which increases with the distance from a central body, has found an explanation by the same reasoning.

Now we come to the problem of appearance of the weightlessness for the body's outer shell at the evolution by radiation of energy. First, we discuss the structure of the potential and kinetic energy of a celestial body.

5.2 The Structure of Potential and Kinetic Energies of a Non-uniform Body

In fact, all the celestial bodies of the Solar System, including the Sun, are non-uniform creatures. They have a shell structure and the shells themselves are also non-uniform components of a body. It was shown in Sect. 2.2 that, according to the artificial satellite data, all the measured gravitational moments of the Earth, including tesseral ones, have significant values. In geophysics, this fact is interpreted as a deviation of the Earth from the hydrostatic equilibrium and attendance of the tangential forces which are continuously developed inside the body. From the viewpoint of the planet's dynamical equilibrium, the fact of the measured zonal and tesseral gravitational moments is a direct evidence of permanent development of the normal and tangential volumetric forces which are the components of the inner gravitational force field. In order to identify the above effects the inner force field of the body should be accordingly separated.

The expressions (2.39)–(2.42) in Chap. 2 indicate that the force function and the polar moment of a non-uniform self-gravitating sphere can be expanded with respect to their components related to the uniform mean density mass and its non-uniformities. In accordance with the superposition principle, these components are responsible for the normal and tangential dynamical effects of a non-uniform body. Such a separation of potential energy and polar moment of inertia through their dimensionless form-factors α and β was done by Garcia et al. (1985) with our interpretation (Ferronsky et al. 1996). Taking into account that the observed satellite irregularities are caused by a non-uniform distribution of the mass density, an auxiliary function relative to the radial density distribution was introduced for the separation:

$$\Psi(s) = \int_0^s \frac{(\rho_r - \rho_0)}{\rho_0} x^2 \mathrm{d}x \tag{5.5}$$

where $s = r/R$ is the ratio of the running radius to the radius of the sphere R; ρ_0 is the mean density of the sphere of radius r; ρ_r is the radial density; x is the running coordinate; the value $(\rho_r - \rho_0)$ satisfies $\int_0^R (\rho_r - \rho_0) r^2 \mathrm{d}r = 0$ and the function $\Psi(1) = 0$.

The function Ψ(s) expresses a radial change in the mass density of the non-uniform sphere relative to its mean value at the distance *r*/*R*. Now we can write expressions for the force function and the moment of inertia by using the structural form-factors α and β which were introduced in Sect. 2.7:

$$U = \alpha \frac{GM^2}{R} = 4\pi G \int_0^R r\rho(r)m(r)\mathrm{d}r, \tag{5.6}$$

$$I = \beta^2 MR^2 = 4\pi \int_0^R r^4 \rho(r)\mathrm{d}r. \tag{5.7}$$

By (5.5) we can do a corresponding change of variables. As a result, the expressions for the potential energy U and polar moment of inertia I are found in the form of their components composed of their uniform and non-uniform constituents (Garcia et al. 1985; Ferronsky et al. 1996):

$$\begin{aligned} U &= 4\pi G \int_0^R r\rho(r)m(r)\mathrm{d}r = \alpha \frac{GM^2}{R} \\ &= \left[\frac{3}{5} + 3\int_0^1 \psi\, x\mathrm{d}x + \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x\right] \frac{GM^2}{R}, \end{aligned} \tag{5.8}$$

$$I = \beta^2 MR^2 = \left[\frac{3}{5} - 6\int_0^1 \psi\, x\mathrm{d}x\right] MR^2. \tag{5.9}$$

It is known that the moment of inertia multiplied by the square of the frequency ω of the oscillation-rotational motion of the mass is the kinetic energy of the body. Then Eq. (5.9) can be rewritten as

$$K = I\omega^2 = \beta^2 MR^2\omega^2 = \left[\frac{3}{5} - 6\int_0^1 \psi x\mathrm{d}x\right] MR^2\omega^2. \tag{5.10}$$

Let us clarify the physical meaning of the terms in expressions (5.8) and (5.10) of the potential and kinetic energies.

As it follows from (2.36) and Table 2.5, the first terms in (5.8) and (5.10), numerically equal to 3/5, represent α_0 and β_0^2 being the structural coefficients of the uniform sphere with radius *r*, the density of which is equal to its mean value. The ratio of the potential and kinetic energies of such a sphere corresponds to the condition of the body's dynamical equilibrium when its kinetic energy is realized in the form of oscillations.

The second terms of the expressions can be rewritten in the form

$$3\int_0^1 \psi\, x\mathrm{d}x \equiv 3\int_0^1 \left(\frac{\psi}{x}\right) x^2\mathrm{d}x, \tag{5.11}$$

$$-6\int_0^1 \psi\, x\mathrm{d}x \equiv -6\int_0^1 \left(\frac{\psi}{x}\right) x^2\mathrm{d}x. \tag{5.12}$$

One can see here that the additive parts of the potential and kinetic energies of the interacting masses of the non-uniformities of each sphere shell, with the uniform sphere having a radius r of the sphere shell that are written there. Note that the structural coefficient β of the kinetic energy is twice as high as the potential energy and has the minus sign. It is known from physics that interaction of mass particles, uniform and non-uniform with respect to density is accompanied by their elastic and inelastic scattering of energy and appearance of a tangential component in their trajectories of motion. In this particular case, the second terms in Eqs. (5.8) and (5.10) express the tangential (torque) component of the potential and kinetic energy of the body. Moreover, the rotational component of the kinetic energy is twice as much as the potential one.

The third term of Eq. (5.8) can be rewritten as

$$\frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \equiv \frac{9}{2}\int_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2\mathrm{d}x. \tag{5.13}$$

Here, there is another additive part of the potential energy of the interacting non-uniformities. It is the non-equilibrated part of the potential energy which does not have an appropriate part of the reactive kinetic energy and represents a dissipative component. Dissipative energy represents the electromagnetic energy that is emitted by the body and it determines the body's evolutionary effects. This energy forms the electromagnetic field of the body (see Chap. 7).

Non-uniformity of the density in this case and later is determined as the difference between the density of the given spherical layer and mean value of density of the sphere with radius of the spherical layer.

Thus, by expansion of the expression of the potential energy and the polar moment of inertia, we obtained the components of both forms of energy which are responsible for oscillation and rotation of the non-uniform body. Applying the above results, we can write separate conditions of the dynamical equilibrium for each form of the motion and separate virial equations of the dynamical equilibrium of their motion.

5.3 Equations of Oscillation and Rotation of a Body and Their Solution

Equations (5.8) and (5.10) can be written in the form

$$U = \left(\alpha_0 + \alpha_t + \alpha_\gamma\right)\frac{GM^2}{R}, \tag{5.14}$$

$$K = \left(\beta_0^2 - 2\beta_t^2\right)MR^2\omega^2, \tag{5.15}$$

where $\alpha_0 = \beta_0^2$ and $\alpha_t = -2\beta_t^2$, the subscripts 0, t, γ define the radial, tangential and dissipative components of the considered values.

Because the potential and kinetic energies of the uniform body are equal $\left(\alpha_0 = \beta_0^2 = 3/5\right)$ then from (5.8) and (5.10) one has

$$U_0 = K_0, \tag{5.16}$$

$$E_0 = U_0 + K_0 = 2U_0. \tag{5.17}$$

In order to express dynamical equilibrium between the potential and kinetic energies of the non-uniform interacting masses we can write, from (5.8) and (5.10),

$$U_t = 2K_t, \tag{5.18}$$

$$E_t = U_t + K_t = 3U_t, \tag{5.19}$$

where E_t, U_0, K_0, U_t, K_t are the total, potential and kinetic energies of oscillation and rotation accordingly. Note, that the energy is always a positive value.

Equations (5.15)–(5.19) present expressions for uniform and non-uniform components of an oscillating system which serves as the conditions of their dynamical equilibrium. Evidently, the potential energy U_γ of interaction between the non-uniformities, being irradiated from the body's outer shell, is irretrievably lost and provides a mechanism of body's evolution.

In accordance with classical mechanics, for the above-considered non-uniform gravitating body, being a dissipative system, the torque N is not equal to zero, the angular momentum L of the sphere is not a conservative parameter, and its energy is continuously spent during the motion, that is,

$$N = \frac{\mathrm{d}L}{\mathrm{d}t} \neq 0, \quad L \neq \text{const.}, \quad E \neq \text{const.} > 0.$$

A system physically cannot be conservative if friction or other dissipation forces are present, because F·ds due to friction is always positive and an integral cannot vanish (Goldstein 1980),

$$\oint F \cdot \mathrm{d}s > 0.$$

After we have found that the resultant of the body's gravitational field is not equal to zero and the system's dynamical equilibrium is maintained by the virial relationship between the potential and kinetic energies, the equations of a self-gravitating body motion can be written.

Earlier we (Ferronsky et al. 1987) used the obtained virial equation for describing and studying the motion of both uniform and non-uniform self-gravitating spheres. Jacobi (1884) derived it from Newton's equations of motion of n mass points and reduced the n-body problem to the particular case of the one-body task with two independent variables, namely, the force function U and the polar moment of inertia Φ, in the form

$$\ddot{\Phi} = 2E - U, \tag{5.20}$$

Equation (5.20) represents the energy conservation law and describes the system in scalar U and Φ volumetric characteristics. In Chap. 3, it was shown that Eq. (5.20) is also derived from Euler's equations for a continuous medium, and from the equations of Hamilton, Einstein, and quantum mechanics. Its time-averaged form gives the Clausius virial theorem for a system with outer source of forces. It was earlier mentioned that Clausius was deducing the theorem for application in thermodynamics and, in particular, as applied to assessment and designing of Carnot's machines. As the machines operate in the Earth's outer force field, Clausius introduced the coefficient 1/2 to the term of "living force" or kinetic energy, that is,

$$K = \frac{1}{2}\sum_i m_i v_i^2.$$

As Jacobi has noted, the meaning of the introduced coefficient was to taken into account only the kinetic energy generated by the machine, but not by the Earth's gravitational force. That was demonstrated, for instance, by the work of a steam hammer for driving piles. The machine raises the hammer, but it falls down under the action of the force of the Earth's gravity. That is why the coefficient 1/2 of the kinetic energy of a uniform self-gravitating body in Eqs. (5.8)–(5.10) has disappeared. In its own force field, the body moves due to release of its own energy.

Earlier, by means of relation $U\sqrt{\Phi} \approx \text{const}$, an approximate solution of Eq. (5.20) for a non-uniform body was obtained (Ferronsky et al. 1987, 2011). Now, after expansion of the force function and polar moment of inertia, at $U_\gamma = 0$ and taking into account the conditions of the dynamical equilibrium (5.16) and (5.18), Eq. (5.20) can be written separately for the radial and tangential components in the form

$$\ddot{\Phi}_0 = \frac{1}{2}E_0 - U_0, \tag{5.21}$$

$$\ddot{\Phi}_t = \frac{1}{3}E_t - U_t. \tag{5.22}$$

Taking into account the functional relationship between the potential energy and the polar moment of inertia

$$|U|\sqrt{\Phi} = B = \text{const}$$

and also taking into account that the structural coefficients $\alpha_0 = \beta_0^2$ and $2\alpha_0 = \beta_t^2$, both Eqs. (5.21) and (5.22) are reduced to an equation with one variable and have a rigorous solution

$$\Phi_n = -A + \frac{B_n}{\sqrt{\Phi_n}}, \tag{5.23}$$

where A_n and B_n are the constant values and subscript n defines the non-uniform body.

The general solution of Eq. (5.23) is (5.13) and (5.14):

$$\sqrt{\Phi_n} = \frac{B_n}{A_n}[1 - \varepsilon\cos(\xi - \varphi)], \tag{5.24}$$

$$\omega = \frac{2\pi}{T_v} = \frac{(2A_n)^{3/2}}{4B_n}, \tag{5.25}$$

where ε and φ are, as previously, the integration constants depending on the initial values of Jacobi's function Φ_n and its first derivative $\dot{\Phi}_n$ at the time moment t_0 (the time here is an independent variable); T_v is the period of virial oscillations; ω is the oscillation frequency; ξ is the auxiliary independent variable; $A_n = A_0 - 1/2E_0 > 0$; $B_n = B_0 = U_0\sqrt{\Phi_0}$ for radial oscillations; $A_n = A_t = -1/3E_t, > 0$; $B_n = B_t = U_t\sqrt{\Phi_t}$ for rotation of the body.

The expressions for the Jacobi function and its first derivative in an explicit form can be obtained after transforming them into the Lagrange series:

$$\begin{aligned}
\sqrt{\Phi_n} &= \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\varepsilon + \frac{3}{8}\varepsilon^3\right)\cos M_c - \frac{\varepsilon^2}{2}\cos 2M_c - \frac{3}{8}\varepsilon^3\cos 3M_c + \ldots\right],\\
\Phi_n &= \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M_c - \frac{\varepsilon^2}{2}\cos 2M_c - \frac{\varepsilon^3}{4}\cos 3M_c + \ldots\right],\\
\dot{\Phi}_n &= \sqrt{\frac{2}{A}}\varepsilon B\left[\sin M_c + \frac{1}{2}\varepsilon\sin 2M_c + \frac{\varepsilon^2}{2}\sin M_c\left(2\cos^2 M_c - \sin^2 M_c\right) + \ldots\right].
\end{aligned} \tag{5.26}$$

Radial frequency of oscillation ω_{or} and angular velocity of rotation ω_{tr} of the shells of radius r can be rewritten from (5.25) as

$$\omega_{0r} = \frac{(2A_0)^{3/2}}{4B_0} = \sqrt{\frac{U_{0r}}{J_{0r}}} = \sqrt{\frac{\alpha_{0r}^2 G m_r}{\beta_{0r}^2 r^3}} = \sqrt{\frac{4}{3}\pi G \rho_{0r}}, \tag{5.27}$$

$$\omega_{tr} = \frac{(2A_t)^{3/2}}{4B_t} = \sqrt{\frac{2U_{tr}}{J_{tr}}} = \sqrt{\frac{2\alpha_{tr}^2 G m_r}{\beta_{tr}^2 r^3}} = \sqrt{\frac{4}{3}\pi G \rho_{0r} k_{er}}, \tag{5.28}$$

where U_{0r} and U_{tr} are the radial and tangential components of the force function (potential energy); J_{0r} and $J_{tr} = 2/3 J_{or}$ are the polar and axial moment of inertia; $\rho_{0r} = \frac{1}{V_r}\int\limits_{V_r} \rho(r)\mathrm{d}V_r$; $\rho(r)$ is the law of radial density distribution; ρ_{0r} is the mean density value of the sphere with a radius r; V_r is the sphere volume with a radius r; $2\alpha_{tr} = \beta_{tr}^2$; k_{er} is the dimensionless coefficient of the energy dissipation or tidal friction of the shells equal to the shell's oblateness.

The relations (5.24)–(5.25) represent Kepler's laws of body rotation in dynamical equilibrium. In the case of uniform mass density distribution, the frequency of oscillation of the sphere's shells with radius r is $\omega_{0r} = \omega_0 = \text{const}$. It means that here all the shells are oscillating with the same frequency. Thus, it appears that only non-uniform bodies are rotating systems.

Rotation of each body's shell depends on the effect of the potential energy scattering at the interaction of masses of different density. As a result, a tangential component of energy appears which is defined by the coefficient k_{er}. In geodynamics, the coefficient is known as the geodynamical parameter. Its value is equal to the ratio of the radial oscillation frequency and the angular velocity of a shell and can be obtained from Eqs. (5.27)–(5.28), that is,

$$k_e = \frac{\omega_t^2}{\omega_0^2} = \frac{\omega_t^2}{\frac{4}{3}\pi G \rho_0}. \tag{5.29}$$

It was found in the general case of a three-axial (a, b, c) ellipsoid with the ellipsoidal law of density distribution, the dimensionless coefficient $k_e \in [0, 1]$ is equal (Ferronsky et al. 1987, 2011)

$$k_e = \frac{F(\varphi, f)}{\sin\varphi} \Bigg/ \frac{a^2 + b^2 + c^2}{3a^2},$$

where $\varphi = \arcsin\sqrt{\frac{a^2 - c^2}{a^2}}$, $f = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$, and $F(\varphi, f)$ is an incomplete elliptic integral of the first degree in the normal Legendre form.

Thus, in addition to the earlier obtained solution of radial oscillations (Ferronsky et al. 1987, 2011), now we have a solution of its rotation. It is seen from expression

(5.27) that the shell oscillations do not depend on the phase state of the body's mass and are determined by its density.

It follows from Eqs. (5.24) and (5.28) that in order to obtain the frequency of oscillation and angular velocity of rotation of a non-uniform body, the law of radial density distribution should be revealed. This problem will be considered later on. But before that the problem of the nature of a body shells separation with respect to their density needs to be solved.

5.4 The Nature and Mechanism of Body's Shell Differentiation by Action of the Centrifugal Roche's Dynamics

It is well known that celestial bodies have a quasi-spherical shell structure. This phenomenon has been confirmed by recording and interpretation of seismic longitudinal and transversal wave propagation during earthquakes. The phenomenon is explained by the centrifugal effect of the energy interaction of a body's elementary particles. In order to demonstrate physics and mechanism of a body mass differentiation with respect to its density, we apply Roche's tidal (centrifugal) dynamics.

Newton's theorem of gravitational interaction between a material point and a spherical layer states that the layer does not affect a point located inside the layer. On the contrary, the outside-located material point is affected by the spherical layer. Roche's tidal dynamics is based on the above theorem. His approach is as follows (Ferronsky et al. 1996).

There are two bodies of masses M and m interacting in accordance with Newton's law (Fig. 5.1a)

Let $M \gg m$ and $R \gg r$, where r is the radius of the body m, and R is the distance between the bodies M and m. Assuming that the mass of the body M is uniformly distributed within a sphere of radius R, we can write the accelerations of the points A and B of the body m as

$$q_{\mathrm{A}} = \frac{GM}{(R-r)^2} - \frac{Gm}{r^2}, \quad q_{\mathrm{B}} = \frac{GM}{(R+r)^2} + \frac{Gm}{r^2}.$$

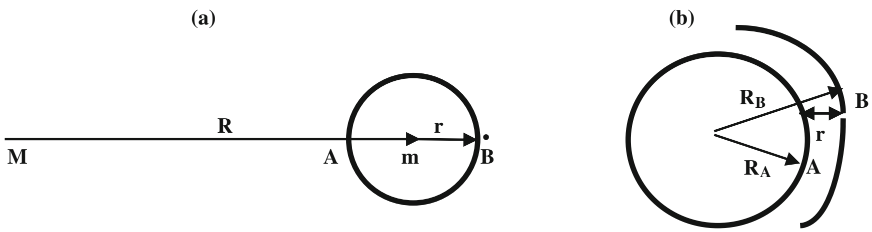

Fig. 5.1 The tidal gravitational stability of a sphere (**a**) and the sphere layer (**b**)

The relative tidal acceleration of the points A and B is

$$
\begin{aligned}
q_{\mathrm{AB}} &= G\left[\frac{M}{(R-r)^2} - \frac{M}{(R+r)^2} - \frac{2m}{r^2}\right] \\
&= \frac{4\pi}{3} G\left[\rho_{\mathrm{M}} R^3 \frac{Rr}{(R^2-r^2)^2} - 2\rho_{\mathrm{m}} r\right] \approx \frac{8\pi}{3} Gr(2\rho_{\mathrm{M}} - \rho_{\mathrm{m}})
\end{aligned}
\tag{5.30}
$$

Here $\rho_{\mathrm{M}} = M/\frac{4}{3}\pi R^3$ and $\rho_{\mathrm{m}} = m/\frac{4}{3}\pi r^3$ are the mean density distributions for the spheres of radius R and r. Roche's criterion states that the body with mass m is stable against the tidal force disruption of the body M, if the mean density of the body m is at least twice as high as that of the body M in the sphere with radius R. Roche considered the problem of the interaction between two spherical bodies without any interest in their creation history and in how the forces appeared. From the point of view of the origin of celestial bodies and of the interpretation of dynamical effects, we are interested in the tidal stability of separate envelopes of the same body. For this purpose, we can apply Roche's tidal dynamics to study the stability of a non-uniform spherical envelope.

Let us assess the tidal stability of a spherical layer of radius R and thickness $r = R_{\mathrm{B}} - R_{\mathrm{A}}$ (Fig. 5.1b). The layer of mass m and mean density $\rho_m = m/4\pi R_A^2 r$ is affected at point A by the tidal force of the sphere of radius R_{A}. The mass of the sphere is M and mean density $\rho_{\mathrm{M}} = M / \frac{4}{3}\pi R_{\mathrm{A}}^3$. The tidal force in point B is generated by the sphere of radius $R + r$ and mass $M + m$. Then the accelerations of the points A and B are

$$
q_{\mathrm{A}} = \frac{GM}{R_{\mathrm{A}}^2} \text{ and } q_{\mathrm{B}} = \frac{G(M+m)}{(R_{\mathrm{A}}+r)^2}.
$$

Then the accelerations of the points A and B are

$$
q_{\mathrm{A}} = \frac{GM}{R_{\mathrm{A}}^2}, \quad q_{\mathrm{B}} = \frac{G(M+m)}{(R_{\mathrm{A}}+r)^2}.
$$

The relative tidal acceleration of the points A and B\` is

$$
\begin{aligned}
q_{\mathrm{AB}} &= GM\left[\frac{1}{R_{\mathrm{A}}^2} - \frac{1}{(R_{\mathrm{A}}+r)^2}\right] - \frac{Gm}{(R_{\mathrm{A}}+r)^2} \\
&= \left(\frac{8}{3}\pi G\rho_{\mathrm{M}} - 4\pi G\rho_{\mathrm{m}}\right) r = 4\pi\pi G\left(\frac{2}{3}\rho_{\mathrm{M}} - \rho_{\mathrm{m}}\right), (R \gg r.).
\end{aligned}
\tag{5.31}
$$

Equations (5.30)–(5.31) give the possibility to understand the centrifugal nature of a body shell separation including some other dynamical effects.

5.5 Self-similarity Principle and Radial Component of Non-uniform Sphere

It follows from Eq. (5.31) that in the case of the uniform density distribution ($\rho_m = \rho_M$), all spherical layers of the gravitating sphere move to the centre with accelerations and velocities which are proportional to the distance from the centre. It means that such a sphere contracts without loss of its uniformity. This property of self-similarity of a dynamical system without any discrete scale is unique for a uniform body (Ferronsky et al. 1996).

A continuous system with a uniform density distribution is also ideal from the point of view of Roche's criterion of stability with respect to the tidal effect. That is why there is a deep physical meaning in separation of the first term of potential energy in expression (5.8). A uniform sphere is always similar in its structure in spite of the fact that it is a continuously contracting system. Here, we do not consider the Coulomb forces effect. In this case, we have considered the specific proton and electron branches of the evolution of the body.

Note that in Newton's interpretation the potential energy has a non-additive category. It cannot be localized even in the simplest case of the interaction between two mass points. In our case of a gravitating sphere as a continuous body, for the interpretation of the additive component of the potential energy we can apply Hooke's concept. According to Hooke, there is a linear relationship between the force and the caused displacement. Therefore, the displacement is in square dependence on the potential energy. Hooke's energy belongs to the additive parameters. In the considered case of a gravitating sphere, the Newton force acting on each spherical layer is proportional to its distance from the centre. Thus, here from the physical point of view, the interpretations of Newton and Hooke are identical.

At the same time in the two approaches there is a principal difference even in the case of uniform distribution of the body density. According to Hooke, the cause of displacement, relative to the system, is the action of the outer force. And if the total energy is equal to the potential energy, then equilibrium of the body is achieved. The potential energy here plays the role of elastic energy. The same uniform sphere with Newton's forces will be contracted. All the body's elementary shells will move without change of uniformity in the density distribution. But the first terms of Eqs. (5.8)–(5.10) show that the tidal effects of a uniform body restrict motion of the interacting shells towards the centre. In accordance with Newton's third law and the d'Alembert principle the attraction forces, under the action of which the shells move, should have equally and oppositely direct forces of Hooke's elastic counteraction. In the framework of the elastic gravitational interaction of shells, the dynamical equilibrium of a uniform sphere is achieved in the form of its elastic oscillations with equality between the potential and kinetic energy. The uniform sphere is dynamically stable relative to the tidal forces in all of its shells during the time of the system contraction. Because the potential and kinetic energies of a sphere are equal, then its total energy in the framework of the averaged virial theorem within one period of oscillation is accepted formally as equal to zero. Equality of the potential and kinetic energy of each shell means the equality of the centripetal (gravitational) and centrifugal (elastic constraint) accelerations. This guarantees the system remaining in dynamical equilibrium. On the contrary, all the spherical shells will be contracted towards the gravity centre which, in the case of the sphere, coincides with the inertia centre but does not coincide with the geometric centre of the masses. Because the gravitational forces are acting continuously, the elastic constraint forces of the body's shells are reacting also continuously. The physical meaning of the self-gravitation of a continuous body consists in the permanent work which applies the energy of the interacting shell masses on one side and the energy of the elastic reaction of the same masses in the form of oscillating motion on the other side. At dynamical equilibrium the body's equality of potential and kinetic energy means that the shell motion should be restricted by the elastic oscillation amplitude of the system. Such an oscillation is similar to the standing wave which appears without transfer of energy into outer space. In this case the radial forces of the shell's elastic interactions along the outer boundary sphere should have a dynamical equilibrium with the forces of the outer gravitational field. This is the condition of the system to be held in the outer force field of a mother's body. Because of this, while studying the dynamics of a conservative system, its rejected outer force field should be replaced by the corresponding equilibrated forces as they do, for instance, in Hooke's theory of elasticity.

Thus, from the point of view of dynamical equilibrium, the first terms in Eqs. (5.8) and (5.10) represent the energy which provides the field of the radial forces in a non-uniform sphere. Here, the potential energy of the uniform component plays the role of the active force function, and the kinetic energy is the function of the elastic constraint forces.

5.6 Charges-like Motion of Non-uniformities and Tangential Component of the Force Function

Let us now discuss the tidal motion of non-uniformities due to their interactions with the uniform body. The potential and kinetic energies of these interactions are given by the second terms in Eqs. (5.8) and (5.10). In accordance with (5.31), the non-uniformity motion looks like the motion of electrical charges interacting on the background of a uniform sphere contraction. Spherical layers with densities exceeding those of the uniform body (positive anomalies) come together and move to the centre in elliptic trajectories. The layers with deficit of the density (negative anomalies) come together, but move from the centre on the parabolic path. Similar anomalies come together, but those with the opposite sign are dispersed with forces proportional to the layer radius. In general, the system tends to reach a uniform and equilibrium state by means of redistribution of its density up to the uniform limit.

Both motions happen not relative to the empty space, but relative to the oscillating motion of the uniform sphere with a mean density. Separate consideration of motion of the uniform and non-uniform components of a heterogeneous sphere is justified by the superposition principle of the forces action which we keep here in mind. The considered motion of the non-uniformities looks like the motion of the positive and negative charges interacting on the background of the field of the uniformly dense sphere (Ferronsky et al. 1996). One can see here that in the case of gravitational interaction of mass particles of a continuous body, their motion is the consequence not only of mutual attraction, but also mutual repulsion by the same law $1/r^2$. In fact, in the case of a real natural non-uniform body it appears that the Newton and Coulomb laws are identical in details. Later on, while considering a body's by-density differentiated masses, the same picture of motion of the positive and negative anomalies will be seen.

If the sphere shells, in turn, include density non-uniformities, then by means of Roche's dynamics it is possible to show that the picture of the non-uniformity motion does not differ from that considered above.

In physics, the process of interaction of particles with different masses without redistribution of their moments is called elastic scattering. The interaction process resulting in redistribution of their moments and change in the inner state or structure is called inelastic scattering. In classical mechanics, while solving the problems of motion of the uniform conservative systems (like motion of the material point in the central field or motion of the rigid body), the effects of the energy scattering do not appear. In the problem of dynamics of the self-gravitating body, where interaction of the shells with different masses and densities are considered, the elastic and inelastic scattering of the energy becomes an evident fact following from consideration of the physical meaning of the expansion of the energy expressions in the form of (5.8) and (5.10). In particular, their second terms represent the potential and kinetic energies of gravitational interaction of masses having a non-uniform density with the uniform mass and express the effect of elastic scattering of density-different shells. Both terms differ only in the numeric coefficient and sign. The difference in the numerical coefficient evidences that the potential energy here is equal to half of the kinetic one ($U_t = 1/2K_t$). This part of the active and reactive force function characterizes the degree of the non-coincidence of the volumetric centre of inertia and that of the gravity centre of the system. This effect is realized in the form of the angular momentum relative to the inertia centre.

Thus, we find that inelastic interaction of the non-uniformities with the uniform component of the system generates the tangential force field which is responsible for the system rotation. In other words, in the scalar force field of the by-density uniform body the vector component appears. In such a case, we can say that, by analogy with an electromagnetic field, in the gravitational scalar potential field of the non-uniform sphere $U(R, t)$ the vector potential $A(R, t)$ appears for which $U = \mathrm{rot}\ \mathrm{A}$ and the field $U(R, t)$ will be solenoidal. In this field, the conditions for vortex motion of the masses are born, where $\mathrm{div}\ A = 0$. This vector field, which in electrodynamics is called solenoidal, can be represented by the sum of the potential and vector fields. The fields, in addition to the energy, acquire moments and have a discrete-wave structure. In our case, the source of the wave effects appears to be the interaction between the elementary shells of the masses by means of which we can construct a continuous body with a high symmetry of forms and properties. The source of the discrete effects can be represented by the interacting structural components of the shells, namely, atoms, molecules and their aggregates. We shall continue discussion about the nature of the gravitational and electromagnetic energy in Chap. 7.

5.7 Centrifugal Nature of the Archimedes and Coriolis Forces

The Archimedes principle states: *The apparent loss in weight of a body totally or partially immersed in a liquid is equal to the weight of the liquid displaced.* We saw in Sect. 5.4 that the principle is described by Eq. (5.31) and the forces that sink down or push out the body or the shell are of a gravitational nature. In fact, in the case of $\rho_m = \rho_M$ the body immersed in a liquid (or in any other medium) is kept in place due to equilibrium between the forces of the body's weight and the forces of the liquid reaction. In the case of $\rho_m > \rho_M$ or $\rho_m < \rho_M$, the body is sinking or floating up depending on the resultant of the above forces. Thus, the Archimedes forces seem to have a gravity nature and are the radial component of the body's inner force field.

It is assumed that the Coriolis forces appeared as an effect of the body motion in the rotational system of co-ordinates relative to the inertial reference system. In this case, rotation of the body is accepted as the inertial motion and the Coriolis forces appear to be the inertial ones. It follows from the solution of Eq. (5.22) that the Coriolis' forces appear to be the tangential component of body's inner force field, and the body rotation is caused by the moment of those forces that are relative to the three-dimensional centre of inertia which also does not coincide with the three-dimensional gravity centre.

In accordance with Eq. (5.31) of the tidal acceleration of an outer non-uniform spherical layer at $\rho_M \neq \rho_m$, the mechanism of the centrifugal density differentiation of masses is revealed. If $\rho_M < \rho_m$, then the shell immerses (is attracted) up to the level where $\rho_M = \rho_m$. At $\rho_M > \rho_m$ the shell floats up to the level where $\rho_M = \rho_m$ and at $\rho_M > 2/3\rho_m$ the shell becomes a self-gravitating one. Thus, in the case when the density increases towards the sphere's centre, which is the Earth's case, then each overlying stratum appears to be in a suspended state due to repulsion by the Archimedes forces which, in fact, are a radial component of the gravitational interaction forces.

The effect of the gravitational differentiation of masses explains the nature of creation of shell-structured celestial bodies and corresponding processes (for instance, the Earth's crust and its oceans, geotectonic, orogenic and seismic processes, including earthquakes). All these phenomena appear to be a consequence of the continuous gravitational differentiation in density of the planet's masses. We assume that creation of the Earth and the Solar System as a whole resulted from this effect. For instance, the mean value of the Moon's density is less than 2/3 of the Earth's, i.e., $\rho_M < 2/3\rho_m$. If one assumes that this relation was maintained during the Moon's formation, then, in accordance with Eq. (5.31), this body separated at the earliest stage of the Earth's mass differentiation. Creation of the body from the separated shell should occur by means of the cyclonic eddy mechanism, which was proposed in due time by Descartes and which was unjustly rejected. If we take into account existence of the tangential forces in the non-uniform mass, then the above mechanism seems to be realistic.

Thus, we learned the nature and mechanism of an initially heavy outer shell of a self-gravitating body into a weightlessness state. Such a weightlessness shell, by its own tangential component of the potential energy is transferred into vortex cloud and after reaching dynamical equilibrium (self-gravitating state) becomes a planet, satellite or any other body. In the case of uniform density of the weightless shell, it transfers into a nebula, equatorial ring or diffuse matter. The orbital motion of a newly created planet, or a satellite of another body is determined by the first cosmic velocity of the parental body. And the axial rotation depends on the value of non-uniformity in density.

5.8 Initial Values of Mean Density and Radius of a Secondary Body

Thus, it follows from Eq. 5.31 that the outer shell of a gaseous body, after reaching its density equal to 2/3 from the mean value of the total body, becomes weightless. If the shell's own density is non-uniform, by its tangential component of the energy the shell is transferred into a secondary body in the form of a vortex creature. As seen from observation, new bodies are formed in different regions of a protoparent body's surface. The large-in-mass bodies like stars, planets and satellites are firmed mainly in the equatorial zone due to difference in value for the polar and equatorial outer force field. Because of this a new body inherits the polar and equatorial obliquity, the value of which reflects degree of the non-uniformity of its density. The comets, asteroids and smaller bodies are formed in the other regions of the parental bodies. The high eccentric orbits of such bodies prove this fact. The inclination of the new body's orbital plane relative to the parental equatorial plane can be up to close to 180°.

The following initial values of density ρ_i and radius R_i of the protosun and protoplanets can be obtained on the basis of their dynamic equilibrium state.

The protosolar gaseous cloud has separated from the protogalaxy body when its outer shell in the equatorial domain has reached the state of weightlessness. In fact, the gaseous cloud should represent a chemically non-homogeneous rotating body. As it follows from Roche's dynamics (Eq. 5.31), the mean density of the gaseous protogalaxy outer shell should be $\rho_s = 2/3\rho_g$. The condition $\rho_s = 2/3\rho_g$ is the starting point of separation and creation of the protosun from the outer protogalaxy shell.

Accepting the above described mechanism of formation of the secondary body, we can find the mean density of the protogalaxy at the moment of the protosun separation as

$$\rho_g = \frac{m_\Gamma}{\frac{4}{3}\pi R^3} = \frac{2.5 \times 10^{41}}{\frac{4}{3} \times 3.14 \times (2.5 \times 10^{20})^3} = 1.67 \times 10^{-21}\,\text{kg/M}^3$$
$$= 1.67 \times 10^{-24}\,\text{g/cm}^3.$$

Here the protogalaxy radius is equal to the semi-major orbital axis of the protosun, i.e. $R_u = 2.5 \times 10^{20}$ m.

The mean density of the separated protogalaxy shell is

$$\rho_c = 2/3\rho_g = 2/3 \times 1.67 \times 10^{-24} = 1.11 \times 10^{-24}\,\text{g/cm}^3.$$

In accordance with Eq, (5.30), the mean density and radius of the initially created protosun body should be

$$\rho_s = 2\rho_g = 2 \times 1.67 \times 10^{-24} = 3.34 \times 10^{-24}\,\text{g/cm}^3;$$

$$R_c = \sqrt[3]{\frac{2 \times 10^{33}}{\frac{4}{3}3.34 \times 10^{-24}}} = 7.5 \times 10^{18}\,\text{cm} = 7.5 \times 10^{16}\,\text{m}$$

The mean density and the radius of the initially created protojupiter, protoearth and Protomoon are as follows:

the protojupiter: $\rho_j = 2 \times 10^{-9}\,\text{g/cm}^3,\ R_j = 6.2 \times 10^{13}\,\text{cm} = 6.2 \times 10^{11}\,\text{m};$
the protoearth: $\rho_e = 2.85 \times 10^{-7}\,\text{g/cm}^3,\ R_e = 1.9 \times 10^{11}\,\text{cm} = 1.9 \times 10^{9}\,\text{m};$
the Protomoon: $\rho_m = 5 \times 10^{-4}\,\text{g/cm}^3,\ R_m = 1.1 \times 10^{9}\,\text{cm} = 1.1 \times 10^{7}\,\text{m};$

An analogous unified process was repeated for all the planets and their satellites.

From the analysis of the above observational and calculated data, the following conclusions are made:

1. The planets of the Solar System were created from a common non-uniform in density self-gravitating protosolar cloud, which has separated during evolution on shells with different densities. In accordance with Roche's tidal dynamics, after the outer shell has reached a density equal to 2/3 from the cloud's mean value (the condition of the weightlessness relative to the total body), by the inner force field and the tangential component of the potential energy, the protoplanets after became self-gravitating bodies, were formed and separated. Analogous processes have taken place at creation of satellites from planets. In addition, accumulation of "light" matter in the outer shells took place gradually and accompanied by separation of small portions in the form of comets and other bodies and dust matter being weightlessness relative to the surrounding weighted bodies.

2. The orbital velocities of the planets and satellites, which corresponds to the first cosmic velocity of the parental bodies, appears to be an effect of the outer force field, which is realized at the moment when the shell reaches its weightlessness state. The orbital motion of the planets, satellites and other bodies in the outer force field results by the laws of electrodynamics.
3. The small planets of the asteroid belt have created from the protosolad cloud by the common law. Appraising by orbital velocities, there are no features of their creation because of a body destruction.
4. The axial rotation of the Sun, planets and satellites has taken and takes place by tangential component of the inner force field. The axial rotation has never been inertial like a rigid body. The body's angular moment depends on the friction (weight) of the rotating masses and, to the contrary of the orbital moment of momentum, it does not remain a conservative value. The orbital angular momentum is the fundamental and conservative parameter because it expresses the law of the body's energy conservation law. The angular momentum of the Sun itself expresses only the tangential component of its potential energy which is a small part of the total potential energy of the body (see minas sign in Eq. 5.10). The direction of revolution and rotation of all the planets and satellites is governed by the force field of the parental body and determined, as in electrodynamics, by Lenz's law.

The discussed physics and kinematics of creation and separation of Solar System bodies prove the Huygens' law of motion on semi-cubic parabola of his watch pendulum, which synchronously follows the Earth's motion. Relationship between the evolute and Huygens pendulum clockevolvent represents the relationship between a function and its derivative or between function and its integral. For the Huygens' oscillating pendulum the suspension filament starts unrolling in a fixed point. In the case of a celestial body, creation of a satellite starts in a fixed point of its parental body where the initial conditions are transferred by the third Kepler's law, which is the consequence of a body creation.

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Chapter 6
The Body's Evolutionary Processes as Centrifugal and Gyroscopic Effects of Interaction Energy Emission

Abstract We consider problems of the gravitational evolution of a gaseous sphere based on Jacobi's virial equation and the relationship between potential energy and the moment of inertia of the sphere in the form $-U\sqrt{\Phi} = \alpha\beta G m^{5/2}$. The problems to be solved are as follows:

- Equilibrium boundary conditions for a self-gravitating gaseous sphere;
- Velocity of gravitational differentiation of a gaseous sphere;
- The luminosity–mass relationship;
- Bifurcation of a dissipative system;
- Cosmochemical effects;
- Radial distribution of mass density and the body's inner force field;
- Oscillation frequency and angular velocity of shell rotation;
- The nature of precession, nutation and body's equatorial plane obliquity;
- The nature of Chandler's effect of the Earth pole wobbling.

All the above tasks have physical formulations and mathematical solutions.

We consider here several problems of the gravitational evolution of a gaseous sphere based on Jacobi's virial equation and the relationship between the potential energy and the moment of inertia of the sphere in the form

$$-U\sqrt{I} = \alpha \frac{Gm^2}{R}\sqrt{m(\beta R)^2} = \alpha\beta G m^{5/2}, \tag{6.1}$$

where U is the gravitational potential energy of the sphere; I is the polar moment of inertia; G is the gravitational constant; m is the body mass; R is the sphere radius; and α and β are dimensionless structural parameters depending on the radial mass density distribution of the spherical body.

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From (6.1), and taking into account Eqs. (2.36), (2.38) and (5.9), (5.11), we have the following relations between the structural form factors:

$$\alpha = \frac{r_g}{R} \quad \text{и} \quad \beta = \frac{r_m}{R} \tag{6.2}$$

$$\alpha\beta = \text{const.}, \tag{6.3}$$

where $\alpha = (\alpha_0 + \alpha_t + \alpha_\gamma)$; $\beta = (\beta_0 - \beta_t)$; $\alpha_0 = \beta_0^2 = 0.6$; $\alpha_t = 2\beta_t^2$; $\alpha_0\beta_0 = a_0 = \text{const}$; r_g and r_m are the reduced gravity radius and radius of inertia; α_0; β_0; α_t; α_γ; β_t are form factors of the normal, tangential and dissipative components of the energy for non-uniform mass density distribution of a system.

In Chap. 5 we found that the constancy of the form factors product (6.3) is independent of the body mass, radius and radial mass density distribution for spherical and elliptic bodies. Equation (6.1) is therefore a key expression in our further consideration.

6.1 Equilibrium Boundary Conditions for a Self-gravitating Gaseous Sphere

It is well known that polytropic models require the boundary mass density of a gravitating body to be rigorously equal to zero. Hence this condition gives us no opportunity to consider any physical processes during evolution.

If Eq. (6.1) for the spherical and elliptical gravitating systems is valid, it allows us to consider convenient boundary conditions which can be used in the study of evolutionary problems.

In deriving the physical boundary conditions for a self-gravitating and rotating gaseous sphere, we consider its rotation as an effect of the tangential component of energy generated by the interacting non-uniform particles. As it was shown in Chap. 5, the ellipticity of the body is formed not as a result of its rotation but because of its self-gravitation. The key relationship (6.1) used here as the basis of our consideration prevents any possible errors. When we have to introduce the moment of inertia, the rotating sphere boundary at the equator will be defined by Kepler's law.

The fact that gaseous sphere boundary equilibrium conditions differ from those of the interior explains the difference between a free molecular boundary particle movement and an internal chaotic one. It is a consequence of the discrete matter structure dominant at the boundary (Jeans 1919).

Let us now consider the thermodynamic boundary conditions. Surely, we can define the boundary temperature only in the case of its real existence which, in turn, depends on the existence of the thermodynamic equilibrium between matter and radiation. Otherwise, it cannot be considered as black body radiation, and the Stefan–Boltzmann equation is inapplicable.

Thermodynamic equilibrium at the boundary can be reached only when the energy and momentum carried away by the radiation flow are greater than that carried away by the flow of particles from the sphere surface per unit time. Such a surface cannot increase further without disturbing the thermodynamic equilibrium.

We shall consider the evolutionary process of the gaseous sphere to be a successive series of hydrodynamic states in equilibrium. We shall also assume that the radiation energy loss causes the sphere to contract during the time periods between the equilibrium states.

Taking these ideas into account, we can express the hydrodynamic equilibrium at the boundary either by an expression representing particle flow 'locking' by the gravitational (centrifugal) force, or, equivalently, by an equation showing the absence of particle dissipation from the boundary surface, which can be written in the form

$$\frac{Gm\mu}{R^2} = \frac{\mu\bar{v}^2}{R}, \tag{6.4}$$

where μ is the mass of the particle, and $\bar{v}$ is the velocity of the particle heat movement at the sphere boundary of the pole (more precisely, it is the velocity of a particle running from the gravitational field).

For gravitational contraction between any two equilibrium states, Eq. (6.4) must be written as

$$\frac{Gm\mu}{R^2} > \frac{\mu\bar{v}^2}{R}. \tag{6.5}$$

Let us prove that the expression (6.4) for the gaseous spherical body boundary satisfies the virial relations.

First, we consider one particle at the sphere boundary surface with mass μ and moving in the volumetric central field of the body with mass m and radius R. Then it is easy to see that

$$\left(\frac{\mu\ddot{R}^2}{2}\right) = \mu\left[\overline{\ddot{R}R} + \left(\dot{\bar{R}}\right)^2\right], \tag{6.6}$$

where the kinetic energy K_p of the particle is

$$\mu\left(\dot{\bar{R}}\right)^2 = \frac{2\mu v^2}{2} = 2K_p \tag{6.7}$$

From Newton's law, we have

$$\ddot{\bar{R}} = -\frac{Gm}{R^3}R. \tag{6.8}$$

The potential energy U_p of the particle in the gravitational field of the body is

$$\mu\ddot{\bar{R}}\bar{R} = -\frac{Gm\mu}{R^3}\left(\overline{RR}\right) = -\frac{Gm\mu}{R} = U_\mathrm{p}. \tag{6.9}$$

Therefore,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\mu R^2}{2}\right) = U_\mathrm{p} + 2K_\mathrm{p}. \tag{6.10}$$

Summing over all particles at the boundary layer and neglecting their interaction energy, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{m_\mathrm{s} R^2}{2}\right) = U_\mathrm{s} + 2K_\mathrm{s}, \tag{6.11}$$

where m_s is the mass of the boundary spherical layer.

Or finally

$$\frac{3}{4}\ddot{I}_\mathrm{s} = U_\mathrm{s} + 2K_\mathrm{s},$$

$$\ddot{\Phi}_\mathrm{s} = U_\mathrm{s} + 2K_\mathrm{s} \tag{6.12}$$

which represents Jacobi's virial equation for a spherical gaseous layer.

The exchange of particles between the gaseous body and its boundary layer takes place at the same radius R and lasts for a short time while the total mass of the layer remains constant. So Eq. (6.12) is rigorous.

The solution of Eq. (6.12) will be exactly the same as that obtained in Chap. 4 for a gravitating sphere, except that corresponding parameters of the sphere must be replaced by those of the boundary layer.

If one time averages over time intervals that are longer than the period of boundary-layer oscillations, then the left-hand side of Eq. (6.12) tends to zero (i.e. the layer enters into the outer force field) and a quasi-equilibrium boundary state is obtained determined by the generalized classical virial relation between the potential and kinetic energies:

$$\dot{\Phi} = U_\mathrm{s} + 2K_\mathrm{s}. \tag{6.13}$$

Thus, we have proved that Eq. (6.4) written for the gaseous sphere boundary is a virial relation. We shall use this expression further in solving the problem of contraction velocity for a gravitating gaseous sphere.

6.2 Velocity of Centrifugal Differentiation of a Gaseous Sphere

In considering the evolution of a gaseous sphere, one does not usually take into account its rotation because the total kinetic energy exceeds the rotational energy. Other authors who accepted the rotation of the gaseous sphere could not manage with the angular momentum accepted as conservative value during contraction (Zeldovich and Novikov 1967; Spitzer 1968; Alfvén and Arrhenius 1970).

It was shown in Chaps. 2 and 5 that the main part of kinetic energy of a celestial body is represented by oscillatory energy of the interacting elementary particles. The rotational part has much smaller oscillatory energy and appears to be an indication of degree of the body matter non-homogeneity. Slowly rotating bodies like the Sun, Mercury, Venus, and Moon have more homogeneous density distribution. Their part of rotational energy from the total kinetic one is $\sim 1/10^4$. For the other planets of the Solar System this figure is $\sim 1/300$. It follows from (5.10) of Chap. 5 that the value of oscillatory energy for a body as a whole is a conservative parameter. The value of rotary energy is a changeable parameter.

The solution of the virial equation obtained earlier enables us to propose the following mechanism for centrifugal contraction of a gaseous sphere. During each period of the sphere's oscillation, a certain amount of energy is lost through radiation. Hence, the contraction amplitude is larger than the expansion amplitude. The difference between the two amplitudes is the value of the gaseous sphere contraction averaged over one period of oscillation. Taking into account the adiabatic invariant relation (Landau and Lifshitz 1973), we may consider the problem of the gravitational contraction of a gaseous sphere using the virial relations and the key relationships (6.1) and (6.3). Note that we consider here the process of evolution without loss of body equilibrium.

Since we consider the evolution process of a gaseous sphere as a successive moment from one equilibrium state to another, it is natural that the minimum time interval for averaging varying parameters should be a little larger than that required for establishing the hydrodynamic equilibrium. So it is not difficult to control the variations of parameters during evolution which are not in contradiction with the equilibrium. (Later, we shall consider these restrictions to be nonexistent).

It is convenient for our purpose to write the generalized virial theorem in the form

$$-U = -2(E - E_\gamma) \equiv 2(E_\gamma - E), \tag{6.14}$$

where $E = U + K$ is the total energy of the gaseous sphere which is a constant over time; E_γ is the electromagnetic energy radiated up to the considered moment of time; K is kinetic energy which includes the energy of rotation and oscillation of the interacted mass particles; E and U are negative parameters.

The time derivative of E_γ is the gaseous sphere luminosity L which is a function of the sphere radius R and the boundary surface temperature T_0:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_\gamma\right) = L = 4\pi\sigma R^2 T_0^4, \tag{6.15}$$

where σ is the Stefan–Boltzmann constant.

From Eq. (6.14) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_\gamma\right) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(E_\gamma - E\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-U).$$

The potential energy is in turn a function of the radius R:

$$-U = \alpha\frac{Gm^2}{R}.$$

The time derivative of $(-U)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}(-U) = v_c\frac{\mathrm{d}}{\mathrm{d}R}(-U),$$

where $v_c = \mathrm{d}R/\mathrm{d}t$ is the gaseous sphere contraction velocity. To find this velocity we write

$$\frac{1}{2}v_c\frac{\mathrm{d}}{\mathrm{d}R}\left(\alpha\frac{Gm^2}{R}\right) = \frac{\mathrm{d}E_\gamma}{\mathrm{d}t} = L$$

and finally, with the help of Eq. (6.3), we obtain

$$v_c = \frac{8\pi\sigma}{Gm^2}\frac{R^2T_0^4}{(\mathrm{d}/\mathrm{d}R)(\alpha^2/R)}. \tag{6.16}$$

From Eq. (6.16) it is easy to see that v_c contains two unknown functions: $\alpha = \alpha(R)$ and $T_0 = T_0(R)$.

As was found in Chap. 2, the structural form factor α, as well as β, is the function of radial mass density distribution of the sphere. In Chap. 5 we considered this function presented by (6.2) and (6.3). It was found that the contraction velocity of the gaseous sphere depends on the mass density redistribution which determines kinetic energy of the body and its shells. So, the function $\beta = \beta(R)$ can be found from the condition of kinetic energy conservation of the body's upper shell after its separation.

It follows from (6.3) that during the centrifugal (gravitational) contraction of the gaseous sphere its radius $R \to R_1$ and $\beta \to 1$ (where R_1 is the orbital radius of separation). If $R \to R_1$ then velocity of rotation $v \to v_1$ (v_1 is the first cosmic velocity).

The kinetic energy of the body's upper shell before K_b and after K_a shell is written as

$$K_b = I\omega^2 = \beta^2 m\omega^2 R^2, \tag{6.17}$$

$$K_a = m\text{v}_1^2 = m_s\omega^2 R_1^2 \tag{6.18}$$

where I is the polar moment of inertia of the body; ω is the frequency of the radial oscillations; m and m_s are the body and its upper shell mass; $R - R_1$ is the thickness of the upper shell or the contraction value.

From Eqs. (6.17)–(6.18) we can write

$$\begin{aligned} \beta^2 &= \frac{m_s\omega^2 R_1^2}{m\omega^2 R^2} = \kappa\frac{R_1^2}{R^2}, \\ \beta &= \sqrt{\kappa}\frac{R_1}{R}, \\ \alpha &= \frac{a}{\beta} = \frac{a}{\sqrt{\kappa}}\frac{R}{R_1}, \end{aligned} \tag{6.19}$$

where κ is the ratio of the protosun's mass to the mass of a separated body.

Thus, we obtained an expression for α as a function of R, which is valid when the kinetic energy of the upper body's shell conserves in the orbital motion of the separated creature.

Let us now try to obtain the relationship between the gaseous sphere boundary temperature T_0 and the radius R. We introduced the virial equilibrium boundary conditions by Eq. (6.4). This equilibrium was defined as particle flow 'locking' by the gravitational force, or, equivalently, by an equation showing the absence of particle dissipation from the boundary surface. Let us now rewrite it:

$$\frac{Gm\mu}{R^2} = \frac{\mu\bar{\text{v}}^2}{R}. \tag{6.20}$$

The heat velocity $\bar{\text{v}}^2$ depends on the boundary temperature T_0 as

$$\mu\bar{v}^2 = 3kT_0, \tag{6.21}$$

where k is the Boltzmann constant.

Therefore, we can rewrite the condition for particle flow 'locking' (6.20) with the help of Eq. (6.21) as

$$\frac{Gm\mu}{3k} = T_0 R. \tag{6.22}$$

From the law of equal energy distribution over the degrees of freedom for the case of a gas particle mixture in equilibrium, it follows that

$$\mu_1 \bar{v}_1^2 = \mu_2 \bar{v}_2^2. \tag{6.23}$$

It is easy to see from (6.22) that the equilibrium radius of a gaseous sphere depends on the chemical composition of the gas. This conclusion follows from Eqs. (6.22) and (5.71) of Chap. 5, where the mechanical equilibrium condition of a body's upper shell is considered. Those results explain the effect of the particle flow 'locking' on the pole by the gravitational force which is based on the concept of mass and radiation equilibrium. Care must therefore be taken when the gas mixture is analyzed, that is, if there are a small number of particles with light masses, the mixture will dissipate easily and the particles flow 'locking' will take place in the case of the heavier particles of the gas mixture. The results explain also the observing orbital motion of planets and satellites mainly in the equatorial plane of the parental body. The conditions here for body separation from the viewpoint of dynamical equilibrium appear to be preferential.

When the quantities of the various mass particles are approximately equal, the particle flow 'locking' condition can be found only by a numerical solution. The gaseous sphere radius can be determined only after the equilibrium equation is solved, and to solve it we must consider all the given types and concentrations of particles in the flow. Formally, we can apply the effective particle mass μ which depends on a value averaged over all the particle masses. The problem can also be solved by numerical methods for a gas mixture consisting of many particles, and especially when the processes of ionization and recombination and chemical reactions occur.

Another interesting phenomenon, which we shall discuss, arises from the fact that electromagnetic forces are much stronger then gravitational forces. When some electrons escape the gravitating body it becomes positive by charges that create huge forces which tend to stop the process of electron dissipation. That is why it is necessary to use the proton mass μ_p when the gaseous cloud consists of neutral hydrogen partly ionized at the gaseous sphere boundary surface (the position of the boundary shell is specified by the radius R). The flow of electrons will be 'locked' by the extra forces appearing as a result of their primary dissipation. In addition, this uncompensated positive charge should have a drift at the boundary surface and small flow of cold plasma should be observed.

In the course of contraction of the gaseous sphere and the increase of its average temperature, the process of gas ionization should also increase. When the flow of electrons is large enough, and the limiting equilibrium between the gravitational forces and the charged protons is achieved, the protons should also start to run off the body's gravitational field. In this case, the increasing electron flux has to be 'locked' by electrostatic forces. The boundary equilibrium change from the proton 'locking' to electron 'locking' should start at this moment.

Thus, we come to the conclusion that at least two phases of gaseous sphere evolutions should exist: that of the proton and that of the electron, with a transitional domain between them which can be calculated by numerical methods in each specific case.

Figure 6.1 illustrates all that we have said. The process of centrifugal contraction of the gaseous sphere is represented by the curve ABCD. Within the AB range, the body equilibrium is kept by the gravitational field 'locking' of the proton flow (the proton phase). Within the same range of sphere contraction, the radius R decreases while the temperature T_0 increases. Point B is the critical one; here the transformation of equilibrium boundary conditions from proton 'locking' to electron 'locking' begins. The process spreads up to point C. While the sphere radius decreases in the range BC, the boundary temperature remains constant.

In the electron equilibrium phase in the range CD, we can see that during the contraction process the boundary temperature increases again.

Let us check the derived expression (6.22) and the conclusion concerning the existence of two boundary equilibrium phases on the observed Sun data.

First, we calculate the numerical value of T_0R in the CGS system with the help of Eq. (6.22). Assuming numerical values for proton and electron masses, we obtain

$$T_pR_p = A_p = \frac{Gm\mu_p}{3k} = \frac{6.67 \times 10^{-8} \times 2 \times 10^{33} \times 1.67 \times 10^{-24}}{3 \times 1.38 \times 10^{-16}} = 5 \times 10^{17}\,\text{cm K},$$

$$\begin{aligned} T_eR_e = A_e = \frac{Gm\mu_e}{3k} &= \frac{6.67 \times 10^{-8} \times 2 \times 10^{33} \times 9.1 \times 10^{-28}}{3 \times 1.38 \times 10^{-16}} \\ &= 2.73 \times 10^{14}\,\text{cm K}. \end{aligned}$$

For the contemporary Sun we know that $R = 7 \times 10^{10}$ cm, $T_0 = 5000$ K and $T_0R = 3.5 \times 10^{14}$ cm K.

As at $T_0 = 5000$ K, where gas ionization must be fairly complete, we have a very good coincidence of the calculated and the observed data for the products T_0R and T_eR_e. For this temperature the proton radius of the Sun R_p is equal to 10^{14} T_0R cm, which corresponds to the orbit radius of Jupiter.

Thus, we have found α, β and T_0 as functions of the radius R. We can now obtain the gaseous sphere contraction velocity. We rewrite Eq. (6.16):

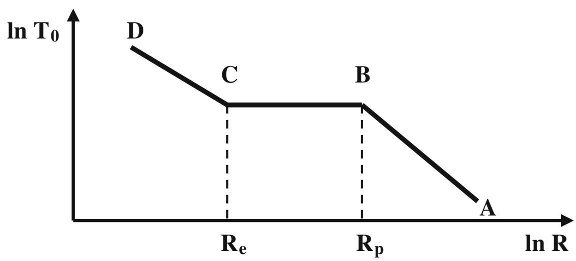


Fig. 6.1 Proton AB and electron CD equilibrium phases of the boundary shell of contracting gaseous sphere

$$\nu_c = \frac{8\pi\sigma}{Gm^2}\frac{(RT_0)^4}{R^2(\mathrm{d}/\mathrm{d}R)(\alpha/R)} \tag{6.24}$$

and, using (6.19), we can evaluate the denominator as

$$R^2\left|\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{\alpha}{R}\right)\right| = R^2\left|\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{a}{\sqrt{\kappa}}\frac{1}{R}\frac{R}{R_1}\right]\right| = \frac{a}{2\sqrt{\kappa}}\frac{R}{R_1}.$$

Finally, we write contraction velocity v_c as

$$\nu_c = \frac{16\pi\sigma}{Gm^2}\frac{A^4}{a}\sqrt{\kappa}\frac{R_1}{R}, \tag{6.25}$$

where $A = A_e = R_eT_e$ and $A = A_p = R_pT_p$ are for the electron and the proton phases of the gaseous sphere evolution, respectively.

Let us use Eq. (6.25) to obtain the contraction velocity and the time of contraction of the protosun during the proton and the electron phases of the gaseous sphere evolution using its corresponding parameters.

If we take for the proton phase of the protosun, after its separation from the protogalaxy, $A_p = 5 \times 10^{17}$ cm K, initial radius $R = 7.5 \times 10^{18}$ cm, final radius of the proton phase evolution (at the asteroid belt, after separation of the protojupiter), $R_1 = 4.2 \times 10^{13}$ cm, $a = 0.46$ and $\sigma = 5.76 \times 10^{-5}$ erg cm^{-2} s (K)^4 as initial, we obtain

$$\bar{v}_{\mathrm{orb}} = \frac{16 \times 3.14 \times 5.76 \times 10^{-5}(5 \times 10^{17})^4}{6.67 \times 10^{-8}(2 \times 10^{33})^2 \times 0.46} \times \sqrt{\frac{2 \times 10^{33}}{2.65 \times 10^{30}}} \times \frac{4.2 \times 10^{13}}{7.5 \times 10^{18}} = 2.23 \times 10^5\,\mathrm{cm\,s^{-1}},$$

$$t_{\text{ыз}} = \frac{7.5\mathrm{x}10^{18}}{2.23\mathrm{x}10^5} = 3.36\mathrm{x}10^{13}\,\mathrm{s} = 1.06\mathrm{x}10^6\,\text{years}.$$

We can find now the contraction velocity and the time of contraction of the protosun during the electron phase of the gaseous sphere evolution. We take now for the electron phase $A_e = 2.73 \times 10^{14}$ cm K, initial radius of the protosun, after separation of the protojupiter, $R = 4.2 \times 10^{13}$ m, let final radius of the electron phase be present-day value $R_1 = 7 \times 10^{10}$ m, $a = 0.46$, and $\sigma = 5.76 \times 10^{-5}$ erg cm^{-2} s K^4. Then we obtain

$$\nu_{\mathrm{cse}} = \frac{16 \times 3.14 \times 5.76 \times 10^{-5}(2.73 \times 10^{14})^4}{6.67 \times 10^{-8}(2 \times 10^{33})^2 \times 0.46} \times \sqrt{\frac{2 \times 10^{33}}{1.18 \times 10^{28}}} \times \frac{7 \times 10^{10}}{4.2 \times 10^{13}} = 8.96 \times 10^{-5}\,\mathrm{cm\,s^{-1}},$$

$$t_{se} = \frac{4.2 \times 10^{13}}{8.96 \times 10^{-5}} = 4.7 \times 10^{17}\,\mathrm{s} = 14.9 \times 10^9\,\text{years}.$$

The found values show that the contemporary solar system formed during the proton phase (the Jupiter's group of planets) within 1 million years and during the electron phase (the Earth group of planets) within the next 15 billion years. Here we have not taken into account the effects of chemistry of the gaseous sphere on the equilibrium boundary conditions of the evolutionary process. But the obtained figures of evolution time show that our calculations give good approximation to the reality.

6.3 The Luminosity–Mass Relationship

To obtain the luminosity–mass relationship we again consider the gaseous sphere evolution plot given in Fig. 6.1 It follows from (6.15) that in proton (AB) and electron (CD) evolutionary phases, the gaseous sphere luminosity is proportional to $1/R^2$. The boundary surface temperature T_0 remains practically constant during the transition period (BC) when the equilibrium transformation from the proton to the electron phase takes place. But the gaseous sphere luminosity will decrease sharply. One can see that the luminosity decrease here is proportional to

$$L \propto \frac{\mu_p^2}{\mu_e^2}, \tag{6.26}$$

that is, it is proportional to the ratio of the proton and electron mass squared as the gaseous sphere surface decreases proportionally to R^2. Thus, while going from point B to point C of the plot, the luminosity of the contracting body decreases by six orders of magnitude. We can suppose that the observed variations of variable star brightness are related to their virial energy pulsations, when stars are at the stage of evolution being considered.

As shown in the previous section, the most continuous period of proton or electron phase evolution is on the right-hand end of the plot intercept (AB) and (CD). For these principal evolution time intervals we can write

$$L = 4\pi\sigma R^2 T_0^4 = \frac{(RT_0)^4}{R^2} \propto m^4. \tag{6.27}$$

This expression, derived from our theoretical considerations, is in good agreement with the well-known luminosity–mass relation which follows from observations. That is why Eq. (6.22) can be considered as an additional relation between the luminosity, the radius and the boundary surface temperature.

Let us take one more example. In Campbell's work (1962), 13 elliptical galaxies from the Virgo Cluster are considered and an analysis of the radius–mass relation for the observed data is given. To interpret these data, Jeans' relation (1919) is used:

$$Gm\mu = \frac{3}{2}kT_0 R \quad \text{или} \quad \frac{m}{R} = \frac{3kT_0}{2G\mu}, \tag{6.28}$$

where μ is the proton mass.

On the plot presented in this work reflecting the mass–radius dependence, all the points are found to lie on a straight line with slope corresponding to $T_0 \approx 1.5 \times 10^7$ K. Campbell concludes from this that the Jeans condition of self-gravitational instability is valid.

We note that Jeans' formula was derived on the assumption of low gas temperature and that all the kinetic energy of the gas is used for particle heat movement. The radiation energy was not taken into account.

Because of absence of direct temperature measurements, the theoretically found high temperature values at very steep line slopes need other explanations. We must stress that in the observational data presented, the distance to the objects (in relative units) have been found with high degree of precision so that the experimentally derived constancy of the line slope should be trusted.

We interpret Campbell's data on the basis of our expression (6.22) where we consider the mass–radius relation to be dependent on electron temperature. That is why, contrary to Jeans, we write

$$\frac{m}{R_\mathrm{e}} = \frac{3kT_\mathrm{e}}{G\mu_\mathrm{e}}.$$

Now the value of the boundary surface temperature of Campbell's galaxies is $T_0 \approx 4000$ K. This value corresponds to the usual boundary temperatures of celestial bodies whose evolution goes according to the electron phase of the equilibrium.

Hence, the experimental data presented by Campbell in his paper confirms once more the validity of Eq. (6.22) and the assumption of the existence of two evolutionary phases for celestial bodies.

In connection with the interpretation of Campbell's data, it is possible to use Eq. (6.22) to obtain the limiting temperature which should be reached by a gaseous sphere in its evolution. We write (6.22) as

$$\frac{Gm}{c^2}\frac{1}{R} = \frac{3kT_\mathrm{e}}{\mu c^2} \quad \text{or} \quad \frac{R_\mathrm{g}}{R} = \frac{3kT_0}{\mu c^2}. \tag{6.29}$$

Hence, during the evolution of a gaseous sphere through the electron phase of equilibrium, when $R \rightarrow R_\mathrm{g} < T_0 \rightarrow \mu_\mathrm{e}c^2/3$ k or, equally,

$$3kT_0 \rightarrow \mu_\mathrm{e}c^2 \approx 0.5\,\mathrm{meV},$$
$$T \approx 5 \times 10^9\,\mathrm{K}.$$

This means that the temperature of the bodies approaches the electron temperature.

6.4 Bifurcation of a Dissipative System

In Chap. 4 we considered the dynamics of a dissipative system assuming that its evolution is a consequence of the loss of energy due to its radiation. Let us consider the problem in detail.

Jacobi's virial equation for a system was written as

$$\ddot{\Phi} = -A_0[1 + q(t)] + \frac{B}{\sqrt{\Phi}}, \tag{6.30}$$

where the function $A_o[1 + q(t)] = E - E_\gamma$ increases monotonically, reflecting the change of the total energy of a system as a function of time, and E_γ is the energy radiated up to time $t[E_\gamma > 0]$.

The solution of Eq. (6.30) was found to be

$$\begin{aligned} &- \arccos W + \arccos W_0 - \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}} = \sqrt{1 - W^2} \\ &+ \sqrt{1 - \frac{A_0 C}{2B^2}} \sqrt{1 - W_0^2} = \pm \frac{[2A_9(1 + q(t))]^{3/2}}{4B}(t - t_0). \end{aligned} \tag{6.31}$$

Equations of the discriminant curves which bound oscillations of the moment of inertia (Jacobi function) (see Fig. 4.5) are

$$\sqrt{I_{1,2}} = \frac{2B}{A_0[1 + q(t)]} \left\{ 1 \pm \sqrt{1 - \frac{A_0[1 + q(t)]C}{2B^2}} \right\}. \tag{6.32}$$

From analysis of the solution of Eq. (6.30) it follows that the dissipative system during its evolution must inevitably reach the state when its stability breaks; that moment (see Fig. 4.5) can be defined by the point O_b which is the physical bifurcation point. The position of the point can be defined by Eq. (6.32) as

$$\frac{2B^2}{A_0[1 + q(t_b)]} = C, \tag{6.33}$$

where $q(t_b)$ is a parameter of the bifurcation point which can be found from condition (6.33)

$$q(t_{\mathrm{b}}) = \frac{2B^2}{A_0 C} - 1. \tag{6.34}$$

The moment of inertia (Jacobi function) of the system corresponding to the bifurcation point, where the discriminant lines coincide, is

$$I_{\mathrm{b}} = \frac{B}{A_0\left(1 + \frac{2B^2}{A_0 C} - 1\right)} = \frac{C^2}{4B^2}. \tag{6.35}$$

To find the moment of time of t_{b},where the system reaches its bifurcation point, one must know the law of energy radiation of the body $q(t)$ or $E_\gamma(t)$, entering Eq. (6.30).

We give below our model solution for $E_\gamma(t)$.

The solution for the energy $E_\gamma(t)$ radiation up to t is based on the assumed existence of the proton and the electron phases of evolution for celestial bodies proposed in this chapter. On this basis, we have found a relationship between the body luminosity L and its radius R. During 'smooth' intervals of the body evolution, when $E_\gamma(t)$ is a continuous and monotonic function of time, the following relation holds:

$$\frac{Gm\mu_{\mathrm{p}}}{3k} = RT_0, \tag{6.36}$$

where μ_{p} is the mass of the particle (proton or electron) which provides the boundary heat equilibrium of the body; k is the Boltzmann constant; and T_0 is the gaseous sphere boundary temperature.

Let us write down the expression for the body luminosity L in relation to the time derivative of E_γ:

$$\frac{\mathrm{d}E_\gamma}{\mathrm{d}t} = L = 4\pi\sigma R^2 T_0^4 \tag{6.37}$$

where σ is the Stefan–Boltzmann constant.

Now we shall find an explicit expression for $E_\gamma(t)$ with the initial condition $E_\gamma(t_0)|_{t_0=0} = 0$.

Equation (6.37) between the limits 0 and t can be integrated with the help of (6.36):

$$E_\gamma(t) = \int_0^t 4\pi\sigma R^2 T_0^4 \mathrm{d}t = \int_0^t \frac{4\pi\sigma R^4 T_0^4}{R^2}\mathrm{d}t = \int_0^t \frac{4\pi\sigma\left(Gm\mu_{\mathrm{p}}\right)^4}{(3k)^4}\frac{1}{R^2}\mathrm{d}t = \int_0^t \frac{K}{R^2}\mathrm{d}t, \tag{6.38}$$

where $K = 4\pi\sigma\left(Gm\mu_{\mathrm{p}}\right)^4(3k)^4$.

Now let us make use of the expression (6.25) for the velocity of the gravitational contraction of the gaseous sphere v_c, which we had found earlier in this chapter:

$$\nu_c = \frac{dR}{dt} = \frac{32}{3}\frac{\pi\sigma}{Gm^2}\left(\frac{Gm\mu_p}{3k}\right)^4 \frac{\sqrt{\kappa}}{a}\sqrt[4]{\frac{R_1}{R}}, \tag{6.39}$$

Integrating this equation,

$$\int_0^R R^{1/4} dR = -\frac{32}{3}\frac{\pi\sigma}{Gm^2}\left(\frac{Gm\mu_p}{3k}\right)^4 \frac{1}{a}\sqrt[4]{\kappa^2 R_1}\int_0^t dt,$$

we obtain

$$\frac{4}{5}R^{5/4} - \frac{4}{5}R_0^{5/4} = -\frac{32}{3}\frac{\pi\sigma}{Gm^2}\left(\frac{Gm\mu_p}{3k}\right)^4 \frac{1}{a}\left(\sqrt[4]{\kappa^2 R_1}\right)t. \tag{6.40}$$

Then

$$R = \left(-Dt + R_0^{5/4}\right)^{4/5},$$

where

$$D = \frac{40\pi\sigma}{3Gm^2}\left(\frac{Gm\mu_p}{3k}\right)^4 \frac{1}{a}\sqrt[4]{\kappa^2 R_1}.$$

Finally, substituting the found expression for (6.40) into (6.38), we have

$$\begin{aligned} E_\gamma(t) &= \int_0^t \frac{K dt}{\left(-Dt + R_0^{5/4}\right)^{8/5}} = \frac{5K}{3D}\left[\left(R_0^{5/4} - Dt\right)^{-3/5} - R_0^{3/4}\right] \\ &= \frac{5}{3}\frac{K}{D}\left[\frac{1}{\left(R_0^{5/4} - Dt\right)} - \frac{1}{R_0^{3/4}}\right]. \end{aligned} \tag{6.41}$$

Thus we have obtained an expression in explicit form which we can use to calculate the energy loss by radiation during the time intervals of ‘smooth’ evolution of celestial bodies and hence find the parameters of the bifurcation point of a dissipative system.

6.5 Cosmochemical Effects

From the analysis of the solution of Eq. (6.30) for a dissipative system, we found that, because of energy loss, a celestial body reaches a bifurcation point, characterized by separation of its outer shell whose angular frequency coincides with frequency of virial oscillations. According to our theory of bifurcational creation of secondary bodies (in Alfvén's definition), some portion of the mass of the rotating primordial cloud reaches equilibrium relative to the inner force field of the whole cloud at the bifurcation point, and moves further in a Kepler's orbit. As a result, during the subsequent dissipation of energy, the primary body continues its contraction by means of redistribution of the mass density without a separated secondary body. This secondary body conserves the corresponding angular moment $M_1 = mv_1R_1 = m\,v_1^2/\omega$ which in fact is the kinetic energy divided by frequency of the interacted mass particles. In accordance with (6.3), the value of this tangential component of the kinetic energy is equal to half of the potential energy ($2\beta_t = \alpha_t$) at the moment of a secondary body separation.

It is commonly known that when both the gravitational and electromagnetic interactions are taken into account, the condition to attain an equilibrium state by some portion of the mass (secondary body) can be written in the form suggested by Chandrasekhar and Fermi (1953):

$$\int\limits_{(V)} \left[\rho\bar{v}^2 + 3p + \frac{H^2+E^2}{8\pi} - \frac{(\nabla U)^2}{8\pi G}\right]\mathrm{d}V = 0, \tag{6.42}$$

where ρ is the density of the substance of the secondary body; v the mean velocity; p the internal pressure; H and E are the components of the electromagnetic field; G the gravitational constant; V the volume of the system; and ∇U the gradient of the gravitational field.

Since the bifurcational point of a system is characterized by the zero amplitude of the virial oscillations, the kinetic terms in Eq. (6.42) are small compared to the mass terms. In this case, Eq. (6.42) can be rewritten as (Ferronsky et al. 1981a, b, 1996)

$$\int\limits_{(V)} \left[3p - \frac{(\nabla U)^2}{8\pi G}\right]\mathrm{d}V \approx 0$$

or

$$\int\limits_{(V)} 3p\mathrm{d}V \approx 0,1\frac{Gm^2}{R}, \tag{6.43}$$

where the coefficient 0.1 represents the electromagnetic component in expansion of the potential energy (6.43) found by astronomical observation of the equilibrium nebulae (Ferronsky et al. 1996).

The left-hand side of (6.43) is proportional to the energy of the Coulomb interactions of the charged particles (electrons, protons, ionized atoms and molecules). The right-hand side of this expression is proportional to the energy of the gravitational interaction of the particles.

Thus, assuming the separated secondary body to have mass m, radius R and the average mass of its constituent particles to be μ, expression (6.43) can be rewritten in the form of an equality of the energies of the gravitational and Coulomb interactions or Madelung's energy (Kittel 1968):

$$0,1\frac{Gm^2}{R} \propto \frac{m}{\mu}\frac{e^2}{R\sqrt[3]{\mu/m}}, \tag{6.44}$$

where $e = 4.8 \times 10^{-10}$ e.s.u. is the electron charge.

Expression (6.44) is the equivalent of

$$m\mu^2 \propto \frac{e^3}{G^{3/2}}. \tag{6.45}$$

The last expression relates the critical mass m_c of the separated secondary body to the averaged mass μ_a of its constituent particles (electron, proton, molecules), responsible for the hydrodynamic equilibrium of the body, as

$$m_c\mu_a^2 \propto \left(\frac{e^2}{G}\right)^{3/2} = \text{const} = 2 \times 10^{-16}\,\text{g}^3. \tag{6.46}$$

To illustrate this relationship, we determined the average values for the masses of the individual particles constituting the planets, stars and galaxies.

6.5.1 Planets

Table 6.1 shows critical masses of the constituent particles for the planets of the Solar System.

Thus, assuming that the bifurcation theory describes the formation of the Solar System correctly, the particles determining the hydrodynamic gas pressure in the case of the considered planet at the moment of their separation from the protosolar cloud could have been composed of such elements as H, He, O, Si, Mn, Fe in atomic or molecular form. The average masses of the particles obtained can be used as a criterion in the development of cosmochemical models of planets with a

Table 6.1 Critical and averaged masses of the constituent particle for the planets

Planets	m_c (g)	μ_a (g)	μ_a (aum)
Mercury	0.33×10^{27}	0.78×10^{-21}	469
Earth	5.97×10^{27}	0.18×10^{-21}	114
Jupiter	2×10^{30}	0.00×10^{-23}	6.02
Saturn	0.57×10^{30}	1.87×10^{-23}	11.3
Uranus	0.087×10^{30}	4.79×10^{-23}	28.8

complicated chemical composition at the moment of their separation from the protosolar cloud and also for the construction of their chemical evolution models.

6.5.2 *Stars*

From (6.46) the boundary values for all stellar critical masses can be found, corresponding to the masses of the proton and the electron—particles which can be responsible for the hydrodynamic pressure inside the stellar cloud at the moment of separation at the bifurcation point of the protogalactic cloud.

For the mass of the proton $\mu_p = 1.6 \times 10^{-24}$ g, $m_c = 10^{32}$ g.

For the mass of the electron $\mu_e = 0.9 \times 10^{-27}$ g, $m_c = 2 \times 10^{38}$ g.

In the case of $\mu_a = \sqrt{\mu_p \mu_a} = 0.4 \times 10^{-25}$ g, $m_c = 10^{35}$ g.

Therefore, considering a typical stellar mass to be $\sim 10^{33}$ g, we obtain in the framework of the bifurcation theory of formation of celestial bodies that the hydrodynamic equilibrium of the gas at the moment of separation of the protostellar cloud is supported both by electron and proton.

6.5.3 *Galaxies*

The presence of the factor $(e^2/G)^{3/2}$ in the right-hand side of (6.46) allows us to carry out the following transformations:

$$m_c \mu_a^2 = \left(\frac{e^2}{\hbar c}\right)^{3/2} \left(\frac{\hbar c}{G}\right) = \left(\frac{1}{137}\right)^{3/2} m_p^3, \tag{6.47}$$

where ħ is Planck's constant; c the velocity of light; and m_p the Planck's mass.

Thus, in the right-hand side of (6.47) there are two fundamental constants: the Planck mass m_p (2.2×10^{-5} g) and the fine-structure constant $\alpha = 1/137$. The presence of the constant α in the right-hand side of (6.47), being the universal constant of the weak and electromagnetic interactions, shows that this relation is applicable not only to electromagnetic but also to weak interactions. Then, putting the experimentally found values for the neutrino mass $\mu_v = 10^{-30}$ g (Shirkov 1980) into (6.44), we obtain

$$m_c = \frac{2 \times 10^{-16}}{(10^{-30})^2} = 2 \times 10^{44}\ \mathrm{g}. \tag{6.48}$$

This mass, following from (6.48), is a typical mass of galaxies. Therefore, in the framework of the bifurcation theory of formation of celestial bodies, the hydrodynamic equilibrium (6.41) of the substances of galaxies at the moment of their formation can be provided by the pressure of neutrinos.

6.5.4 Universe

In the framework of the virial oscillation theory, the evolution of the Universe can be described by a pulsating model (for c = constant) of the system of material elementary particles. Such a system exists for an indefinitely long time. The mass of the particle responsible for hydrodynamic equilibrium of the Universe at the moment of its maximal compression (singularity stage) can be obtained from the same expression (7.46). Assuming $m_c \approx 10^{56}$ g we obtain

$$\mu_a \approx 10^{-36}\ \mathrm{g}. \tag{6.49}$$

In the bifurcation theory the maximal average mass of particles in cosmic space can be determined from the condition $\mu_a = m_c$. Then,

$$\mu_{max} = 6 \times 10^{-6}\ \mathrm{g}.$$

This value is close to the Planck mass.

6.6 Radial Distribution of Mass Density and the Body's Inner Force Field

At present only the Earth has experimental data which allow us to interpret them with respect to radial distribution of the body's mass density. Taking into account our consideration of dynamics of celestial bodies as self-gravitating systems we assume that formation of the Earth's mass density distribution is typical at least for all the planets and satellites.

The existent idea about the radial mass density distribution of the Earth is based on interpretation of transmission velocity of the longitudinal and transverse seismic

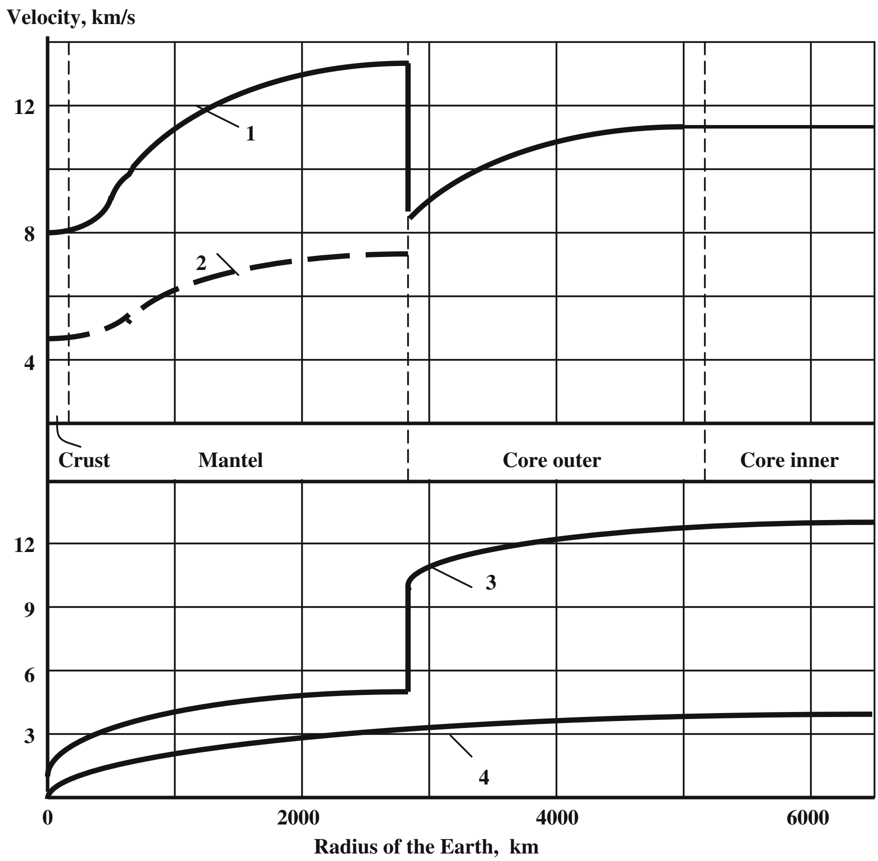


Fig. 6.2 Present-day interpretation of the curves of transmission velocities of longitudinal (*1*) and transverse (*2*) seismic waves, density (*3*), and hydrostatic pressure (*4*) in the earth

waves. Figure 6.2 presents the classic curve of transmission velocities of the longitudinal and transverse seismic waves in the Earth plotted after generalization of numerous experimental data (Jeffreys 1970; Melchior 1972; Zharkov 1978). The curves of the radial density and hydrostatic pressure distribution based on interpretation of the velocities of the longitudinal and transverse seismic waves are also shown.

The picture of the transmission velocities of the seismic waves was obtained by observations and therefore is realistic and correct. But interpretation of the obtained data was based on the idea of hydrostatic equilibrium of the Earth. It leads to incredibly high pressures in the core and high values of the mass density.

In accordance with Bullen's approach for interpretation of the seismic data, the density distribution is characterized by the following values (Bullen 1974; Melchior

1972; Zharkov 1978). The density of the crust rocks is 2.7–2.8 g/cm^3 and increases towards the centre by a certain curve up to $\sim$13.0 g/cm^3 with jumps at the Mohorovičić–Guttenberg discontinuity, between the upper and lower mantles, and on the border of the outer core. Within the core the values of the transverse seismic waves are equal to zero. Despite the jump of the longitudinal seismic wave velocity at the outer core border dropping down, Bullen accepted that the density increases towards the centre. It was done after his unsuccessful attempt to approximate the seismic data of the parabolic curve which gives a decrease of density in the core. Such a tendency is not consistent with the idea of iron core content. Bullen certainly had no idea that the radius of inertia and radius of gravity of the body do not coincide with its geometric centre of mass and, therefore, the maximum value of density is not located there. In accordance with our concept of the equilibrium condition of the planet and its dynamical parameters, the approach to interpretation of the seismic data related to the radial density and radial pressure distribution should be done on a new basis.

Now, when we accept the concept of dynamical equilibrium of the Earth and refuse its hydrostatic version, the basic idea to search for a solution of the problem seems to be the found relationship between the polar moment of inertia and the potential (kinetic) energy. The value of the structural form factor of the Earth's mean axial moment of inertia $\beta_{\perp}^2 = J_{\perp}/MR^2 = 0.3315$ found by artificial satellites (Zharkov 1978) should be taken as a starting point. The mean polar moment of inertia of the assumed spherical non-uniform planet is equal to $\beta^2 = (3/2)\beta_{\perp}^2 = 0.49725$. We accept this value for the development of the methodology.

Let us take as a basis the found mechanism of the shell separation with respect to the mass density which was presented in Sects. 5.5, 5.6, 5.7 and 5.8. The conditions and mechanism of the shell separation into radial and tangential components of the inner force field (by the Archimedes and Coriolis forces) represent continually acting effects and create physics for the Earth's structure formation. These effects explain the jumps between the shells observed by seismic data density. We take also into account the effect, according to which the velocity of the sound recorded by the transmission velocity of the longitudinal and transverse seismic waves quantitatively characterize the energy of the elastic deformation of the media and velocity of its transmission there (Ferronsky and Ferronsky 2010).

Applying the conception of Sect. 5.7, we accept that the non-uniformities of the spherical shells come together and, after their density becomes lower than that of the mean density of the inner sphere, move from the centre by the parabolic law because they interact according to the law $1/r^2$. So, we can find a probable law of the radial density distribution in the form

$$\rho(r) = \rho_0\left(ax^2 + bx + c\right), \tag{6.50}$$

where $x = r/R$ is the ratio of the running and the final radius of the planet; ρ_0 is the body's mean density; a, b, c are the numerical coefficients.

The numerical coefficients were selected for different densities for the upper shell and in such a way that the planet's total mass M would be constant, i.e.

$$M = 4\pi \int_0^R r^2 \rho(r) \mathrm{d}r = 4\pi \int_0^R r^2 \rho_0 (-a\frac{r^2}{R^2} + b\frac{r}{R} + c) \mathrm{d}r$$
$$= \frac{4}{3}\pi R^3 \rho_0 (-\frac{3}{5}a + \frac{3}{4}b + c).$$

Here the term (3/5)a + (3/4)b + c = 1 in the right-hand side of the expression allows us to calculate and plot the distribution density curves in a dimensionless form.

We accepted three most typical parabolas (6.51) which satisfy the condition of equality of their moment of inertia, found by artificial satellite data, namely, the axial moment of inertia $J_\perp = \beta_\perp^2 mR^2 = 0.3315\ mR^2$ or the polar moment of inertia $J = \beta^2 mR^2 = 0.4973\ mR^2$. In addition, the first relation in (6.51) represents the straight line for which the surface mass density and that in the centre correspond to the present-day version and to the form factor $\beta_\perp^2$. The fifth straight line represents the uniform spherical planet. The curve equations with selected numerical coefficients a, b and c are as follows:

$$\begin{aligned}
&1.\quad \rho(r) = \rho_0\left(-2\tfrac{r}{R} + 2.495\right), \quad a = 0, \quad \rho_s = 2.73\,\mathrm{g/cm^3};\\
&2.\quad \rho(r) = \rho_0\left(-1.51\tfrac{r^2}{R^2} + 0.016\tfrac{r}{R} + 1.894\right), \quad \rho_s = 2.08\,\mathrm{g/cm^3};\\
&3.\quad \rho(r) = \rho_0\left(-3.26\tfrac{r^2}{R^2} + 2.146\tfrac{r}{R} + 1.3465\right), \quad \rho_s = 1.28\,\mathrm{g/cm^3};\\
&4.\quad \rho(r) = \rho_0\left(-5.24\tfrac{r^2}{R^2} + 5.132\tfrac{r}{R} + 0.295\right), \quad \rho_s = 1.03224\,\mathrm{g/cm^3}.\\
&5.\quad \rho(r) = \rho_0 = \mathrm{const.}
\end{aligned} \tag{6.51}$$

Figure 6.3 shows all the curves of (6.51). They intersect the straight line 5 of the mean density in the common point which corresponds to the value $r/R = 0.61475$.

Using Eq. (6.51) and the found (by observations) form factor $\beta_\perp^2 = 0.3315$, the main dynamical parameters were calculated for all four curves. The calculations were done by the known formulae of the theory of interaction (Duboshin 1975) and taking into account the relations of (5.9) and (5.10) obtained in Sect. 5.2. These calculations are presented below for Eq. (6.51, 4), as an example.

The potential energy of the non-uniform sphere with the density distribution law $\rho(r)$ is found from the equation

$$U = 4\pi G \int_0^R r\rho(r) m(r) \mathrm{d}r, \tag{6.52}$$

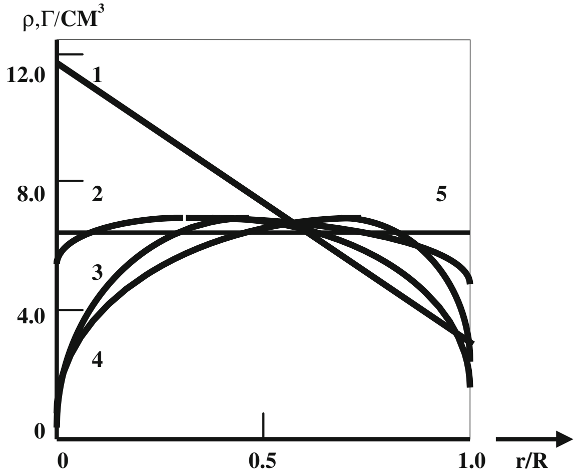


Fig. 6.3 Parabolic curves of radial density distribution calculated by Eq. (6.51)

where

$$\rho(r) = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right), \quad a = -5,24; \quad b = 5,132; \quad c = 0.295;$$

$$m(r) = 4\pi\int_0^r r^2\rho(r)\mathrm{d}r = 4\pi\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\mathrm{d}r$$
$$= \frac{4}{3}\pi r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right).$$

Then

$$U(r) = 4\pi G\int_0^R r\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\frac{4}{3}\pi r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)\mathrm{d}r$$
$$= \left(\frac{4}{3}\pi\rho_0\right)^2 GR^5\frac{R}{R}\left(\frac{1}{5}a^2 + \frac{81}{160}ab + \frac{9}{28}b^2 + \frac{24}{35}ac + \frac{7}{8}bc + \frac{3}{5}c^2\right) \tag{6.53}$$
$$= 0.0660143\frac{GM^2}{R}.$$

The form factor of the potential energy is $\alpha = r_g/R = 0.660143$, and the reduced radius of gravity is $r_g = \sqrt{0.660143R^2} = 0.8124918R$.

In accordance with (5.8), the potential energy of the non-uniform sphere is expanded into the components

$$U = U_0 + U_t + U_\gamma; \tag{6.54}$$

The potential energy of the uniform sphere is equal to

$$U_0 = \frac{3}{5}\frac{GM^2}{R}, \tag{6.55}$$

where form factors of potential and kinetic energies are equal to $\alpha_0 = 0.6$ and $\beta_0^2 = 0.6$.

In accordance with the second term of the right-hand side of Eq. 5.8, the tangential component of the non-uniform sphere is written as

$$U_t = -\frac{1}{2}4\pi G\int\limits_0^R r\rho_t(r)m_0(r)\mathrm{d}r, \tag{6.56}$$

where

$$\rho_t(r) = \rho(r) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right);$$

$$m_0(r) = 4\pi\int\limits_0^r r^2\rho_0\mathrm{d}r = \frac{4}{3}\pi\rho_0 r^3$$

The coefficient ½ in (6.56) is taken as the ratio of the second term of the right-hand side of Eqs. (5.8) and (5.9), as in this particular case the tangential component of the potential energy is determined through the tangential component of the kinetic energy and is equal to half its value. Then

$$U_t = -\frac{1}{2}4\frac{4}{3}(\pi\rho_0)^2 G\int\limits_0^R r^4\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)\mathrm{d}r$$
$$= -\frac{1}{2}\frac{GM^2}{R}\left(\frac{3}{7}a + \frac{1}{2}b + \frac{3}{5}c - \frac{3}{5}\right) = 0.0513571\frac{GM^2}{R}.$$

The form factors of the tangential components of the potential and kinetic energy are equal to $\alpha_t = 0.051357$ and $\beta_t^2 == 2 \times 0.051357 = 0.102714$.

In accordance with the third term in the right-hand side of Eq. (5.8), the dissipative component of the potential energy of the non-uniform sphere is

$$U_\gamma = 4\pi G \int_0^R r\rho_t(r) m_t(r)\mathrm{d}r, \tag{6.57}$$

where

$$\rho_t(r) = \rho(r) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right);$$

$$\begin{aligned} m_t(r) &= 4\pi\int_0^r r^2\rho_t(r)\mathrm{d}r = 4\pi\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)\mathrm{d}r \\ &= \frac{4}{3}\pi\rho_0 r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c - 1\right), \end{aligned}$$

then

$$\begin{aligned} U_\gamma &= 4\frac{4}{3}(\pi\rho_0)^2 G\int_0^R r^4\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c - 1\right)\mathrm{d}r \\ &= \frac{GM^2}{R}\left(\frac{1}{5}a^2 + \frac{81}{160}ab + \frac{24}{35}ac - \frac{24}{35}a + \frac{9}{28}b^2 + \frac{7}{8}bc - \frac{7}{8}b + \frac{3}{5}c^2 - \frac{5}{6}c + \frac{3}{5}\right) \\ &= 0.008786\frac{GM^2}{R}. \end{aligned} \tag{6.58}$$

So the value of the form factor of the dissipative component is $\alpha_\gamma = 0{,}008786$.

The radial distribution of the potential energy for interaction of a test mass point with the non-uniform sphere is

$$\begin{aligned} U(r) &= \frac{4\pi G}{r}\int_0^r r^2\rho(r)\mathrm{d}r + 4\pi G\int_r^R r\rho(r)\mathrm{d}r \\ &= \frac{4\pi G}{r}\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\mathrm{d}r \\ &\quad + 4\pi G\int_r^R r\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\mathrm{d}r \\ &= \frac{GMm_1}{R}\left(-\frac{3}{20}a\frac{r^4}{R^4} - \frac{1}{4}b\frac{r^3}{R^3} - \frac{1}{2}c\frac{r^2}{R^2} + \frac{3}{4a} + b + \frac{3}{2}c\right) \\ &= \frac{GMm_1}{R}\left(0.786\frac{r^4}{R^4} - 1.283\frac{r^3}{R^3} - 0.1475\frac{r^2}{R^2} + 1.6445\right). \end{aligned} \tag{6.59}$$

At $r/R = 0$, $\alpha_\nu(r) = 1.6445$; and at $r/R = 1$, $\alpha_\nu(r) = 1$.

The radial distribution of the interaction force of the test mass point with the non-uniform sphere is

$$\begin{aligned} q(r) &= -\frac{4\pi G}{r^2}\int_0^r r^2\rho(r)\mathrm{d}r = -\frac{4\pi G}{r^2}\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2}+b\frac{r}{R}+c\right)\mathrm{d}r \\ &= -\frac{GMm_1}{R^2}\left(\frac{3}{5}a\frac{r^3}{R^3}+\frac{3}{4}b\frac{r^2}{R^2}+c\frac{r}{R}\right) \\ &= -\frac{GMm_1}{R^2}\left(-3.144\frac{r^3}{R^3}+3.849\frac{r^2}{R^2}+0.295\frac{r}{R}\right) \end{aligned} \tag{6.60}$$

At $r/R = 0$, $\alpha_\gamma(r) = 0$; and at $r/R = 1$, $\alpha_\gamma(r) = 1$.

Table 6.2 demonstrates the results of the calculated dynamical parameters for all the density curves (6.51) and Fig. 6.4 shows the curves of radial distribution of the potential energy and gravity force for the test mass point.

We wish to evaluate all four curves of mass density distribution in order to recognize which one is closer to the real Earth. In this case we keep in mind that the observed density jumps can be obtained for any curve by approximation of its continuous section with the mean value for each shell.

Figure 6.4 shows that the radial density values are substantially different for each curve. It refers, first of all, to the surface and centre of the body. At the same time Table 6.2 demonstrates the complete identity of the dynamical parameters of all the non-uniform spheres. It means that a fixed value of the polar moment of inertia permits us to have a multiplicity of curves of the radial density distribution with

Table 6.2 Physical and dynamical parameters of the earth for the density distribution presented by Eq. (6.51)

Equation N^o	1	2	3	4
ρ_s, g/cm^3	2.76	2.08	1.65	1.03224
ρ_c, g/cm^3	13.8	10.455	6.315	1.6284
ρ_{max}, g/cm^3/km	13.8/0	10.455/0	8.26/2096	8.57/3122
$\beta^2{}_\perp$	0.3315	0.3315	0.3315	0.3315238
β^2	0.49725	0.49725	0.49725	0.49725858
β_t^2	0.10275	0.10275	0.102752	0.102714
α	0.660737	0.660737	0.660737	0.660143
α_t	0.051371	0.051371	0.0513714	0.0513571
α_γ	0.009366	0.009366	0.009366	0.0087859
r_g, км	5178.6	5178.7	5178.6	5176.4
r_m, км	4492.6	4492.6	4492.6	4492.7

Here ρ_s, ρ_c, ρ_{max} are the density on the sphere's surface, in the centre, and maximal accordingly; $\beta^2_\perp$, β^2, $\beta^2{}_t$ are the form factors of the axial, polar, and tangential components of the radius of inertia accordingly; α, α_t, α_γ are the form factors of the radial, tangential and dissipative components of the force function accordingly; r_g, r_m are the radiuses of the gravity and inertia

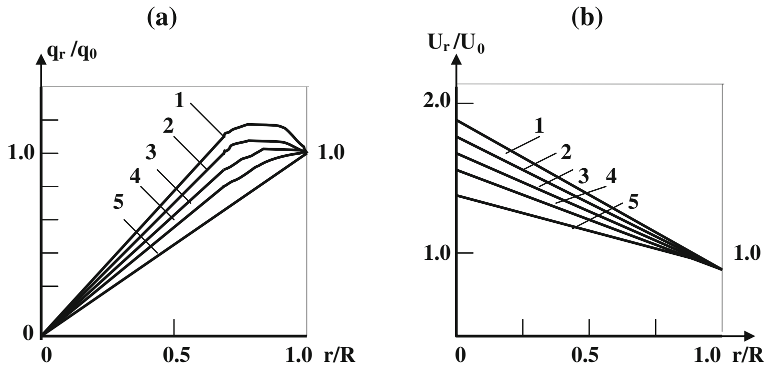


Fig. 6.4 The curves of the radial distribution of the potential energy (**a**) and gravity force (**b**) for the mass point test done by Eqs. (6.53) and (6.60)

identical dynamical parameters of the body. The found property of the non-uniform self-gravitating sphere proves the rigour of the discovered functional relationship between the potential (kinetic) energy and the polar moment of inertia of the sphere. This property, in turn, is explained by the energy conservation law of a body during its motion and evolution in the form of the dynamical equilibrium equation or generalized virial theorem.

If we accept the conditions of the mass density separation presented in Sects. 6.5, 6.6, 6.7 and 6.8, then the range of curves of the density distribution gives a principal picture of its evolutionary redistribution and can be applied for reconstruction of the Earth's history. It follows from Eq. (6.31) that the density value of each overlying shell of the created Earth should be higher than the mean density of the inner mass. Otherwise, such a shell cannot be retained and should be dispersed by the tidal forces. It follows from this that the planet's formation process should be strictly operated by the dynamical laws of motion in the form of the virial oscillations and accompanied by differentiation of the non-uniform shells. The model of a cyclonic vortex which was proposed by Descartes is the most acceptable from the point of view of the considered ideas of planets' and satellites' creation from a common nebula. This problem needs a separate consideration. We only note here that from the presented curves of radial density distribution the parabola (4) more closely reflects the present-day planet's evolution as fixed by observations. In this case location of the Earth's reduced inertia radius falls on the lower mantle and the reduced gravity radius on the upper mantle. The density maximum falls also on the lower mantle. Its value is found by ordinary means, namely, by taking the derivative from the density distribution law as equated to zero. From here $\rho_{max} = 8.57$ g/cm^3 is found to be at a distance of $r = 3.122$ km. It means that the density maximum comes close to the border of the outer core where, as seismic observations show, the main density jump occurs. Curve (4) corrects the values of the radial density distribution in the mantle and changes its earlier interpretation in the outer and inner cores. Because of zero values of the transverse velocities the

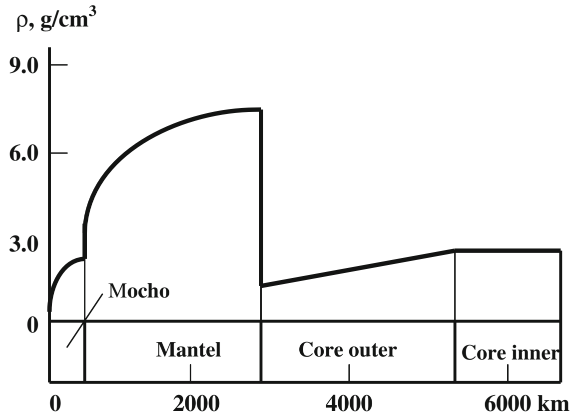


Fig. 6.5 Radial density distribution of the earth by the authors' interpretation

matter of the inner core has a uniform density structure and, from the point of view of the equilibrium state, seems to be in a gaseous state at a pressure of 1-2 atmospheres. Taking into account the location of the maximum density value, there is a reason to assume that the outer core matter stays in the liquid or supercritical gaseous stage. In any case, the density and pressure of the inner and outer core are much lower and should have values corresponding to the seismic wave velocities. On the basis of the equation of mass density differentiation (6.31) we interpret the density jumps observed (by seismic data) nearby Mohorovičić–Guttenberg and at the outer core borders as the borders of the shell's dynamical equilibrium. A shell which is found over that border appears in a suspended state due to action of the radial component of the gravitational pressure developed by the denser underlying shell. While the thickness of the suspended shell is growing it acquires its own equilibrium pressure (iceberg effect). The extremely high pressures in the Earth's interior, which follow from the hydrostatic equilibrium conditions, are impossible in its own force field.

The concept discussed above in relation to the Earth's density distribution is illustrated in Fig. 6.5.

The polar moment of inertia here is $r_{\mathrm{m}} = 3/2 r_{\mathrm{m}}^{\perp} = \sqrt{1.5 \times 0.3315 R^2} = 0.70516 R = 4493 \times 10^3\,\mathrm{m}$ and the radius of gravity is $r_{\mathrm{g}} = 0.8164$ R = 5201×10^3 m.

6.7 Oscillation Frequency and Angular Velocity of a Body Shell Rotation

Let us continue discussion about the nature of the Earth's dynamical parameters as an example. In order to determine numerical values of frequency of the virial oscillations and the angular velocities, which are the main dynamical parameters of the Earth's shells, we accept equation (4) of the density distribution (6.51) as the first approximation. All further relevant calculations can be made by applying this equation.

We know the mean values of the planet's density $\rho_0 = 5.519$ g/cm^3 and angular velocity of the upper shell $\omega_t = 7.29 \times 10^{-5}$ s^{-1}. Applying these values, the frequency and period of the virial oscillations, and the coefficient k_e of the tangential component of the inner forces can be found. In accordance with Eq. (5.28) the frequency of the upper shell is equal to

$$\omega_0(r) = \sqrt{\frac{4}{3}\pi G \rho_0(r)} = \sqrt{\frac{4}{3} 3.14 \times 6.67 \times 10^{-8} \times 5.519} = 1.24 \times 10^{-3} \mathrm{c}^{-1}.$$

The period of oscillation is found from the expression

$$T_\omega = \frac{2\pi}{\omega_0(r)} = \frac{6.28}{1.24 \mathrm{x} 10^{-3}} = 5060.4 \quad c = 1.405\ ч.$$

Unlike the usual expression for the first cosmic velocity in the form of $v_1 = \sqrt{GM/r}$, we used here the physical condition of the dynamical equilibrium at the Earth's surface between the inner gravitational pressure of interacting masses and the outer background pressure including atmospheric pressure.

Given below our own observation data on the near-surface atmospheric pressure and temperature oscillations at the near-surface layer and the results of the spectral analysis prove the above theoretical calculations of the planet's frequency of virial oscillations (Ferronsky and Ferronsky 2010).

Now, applying the known mean value of the Earth's angular velocity $\omega_t = 7.29 \times 10^{-5}$ s^{-1} and the known value of the frequency of virial oscillations for the upper shell $\omega_o = 1.24 \times 10^{-3}$ s^{-1} by Eq. (5.29) the coefficient k_e can be found

$$k_e = \frac{\omega_t^2}{\omega_0^2} = \frac{(7.29 \times 10^{-5})^2}{(1.24 \times 10^{-3})^2} = \frac{1}{289.33} = 0.003456.$$

The coefficient k_e is known in geodynamics as a parameter that shows the ratio between the centrifugal force at the Earth's equator and the acceleration of the gravity force there equal to $k_e = 1/289.37$ (Melchior 1972). The parameter is used to study the Earth's figure based on the Clairaut hydrostatic theory.

6.7.1 Thickness of the Upper Earth's Rotating Shell

It is known that the value of the mean linear velocity of the upper planet's shell is $v_e = 0.465$ km/c. We can find the thickness h_e at which the velocity v_e corresponds to the found frequency of radial oscillations of the shell $\omega_o = 1.24 \times 10^{-3}$ s$^{-1:}$

$$h_{\mathrm{e}} = \frac{v}{\omega_{\mathrm{o}}(r)} = \frac{0.465}{1.24 \times 10^{-3}} = 375\,\mathrm{km}. \tag{6.61}$$

Such is the thickness of the upper shell of the Earth which is rotating by forces in its own force field. It is assumed that the shell is found in the solid state. In reality it is known that the rigid shell has a thickness less than 50 km. The remaining more than 300-km-thick part of the shell has a viscous–plastic consistency, the density of which increases with depth. The border of the shell has a decreased density because of the melted substance due to high friction and saturation by a gaseous component. The border plays a role of some sort of spherical hinge. Because the density of the Earth's crust is lower than that of the underlying matter, then it occurs in the suspended state. During the oscillating motion the crust shells are affected by the alternating-sign acceleration and the inertial hydrostatic equilibrium.

6.7.2 Oscillation of the Earth's Shells

Let us obtain the expression of virial oscillations for the Earth's other shells by applying expression (4) of (6.51) for the radial density distribution. Write Eq. (5.28)

$$\omega_0(r) = \sqrt{\frac{4}{3}\pi G \rho_0(r)},$$

where

$$\begin{aligned}\rho_0(r) &= \frac{m_0(r)}{\frac{4}{3}\pi r^3} = \frac{4\pi \int_0^r r^2 \rho(r)\mathrm{d}r}{\frac{4}{3}\pi r^3} = \frac{\frac{4}{3}\pi r^3 \rho_0\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)}{\frac{4}{3}\pi r^3} \\ &= \rho_0\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right).\end{aligned}$$

Then

$$\begin{aligned}\omega_0(r) &= \sqrt{\frac{4}{3}\pi G \rho_0(r)} = \sqrt{\frac{4}{3}\pi G \rho_0\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)} \\ &= 1.24 \times 10^{-3}\sqrt{\left(-3.144\frac{r^2}{R^2} + 3.849\frac{r}{R} + 0.295\right)}.\end{aligned} \tag{6.62}$$

At $r/R = 0$ then $\omega_{\mathrm{o}}(r) = 0.6743 \times 10^{-3}\ \mathrm{s}^{-1}$; at $r/R = 1$ then $\omega_{\mathrm{o}}(r) = 1.24 \times 10^{-3}\ \mathrm{s}^{-1}$; at $\rho_{\max} = 8.57\ \mathrm{g/cm}^3$ $\omega_{\mathrm{o}}(r) = 1.486 \times 10^{-3}\ \mathrm{s}^{-1}$, where $r/R = 0.49$.

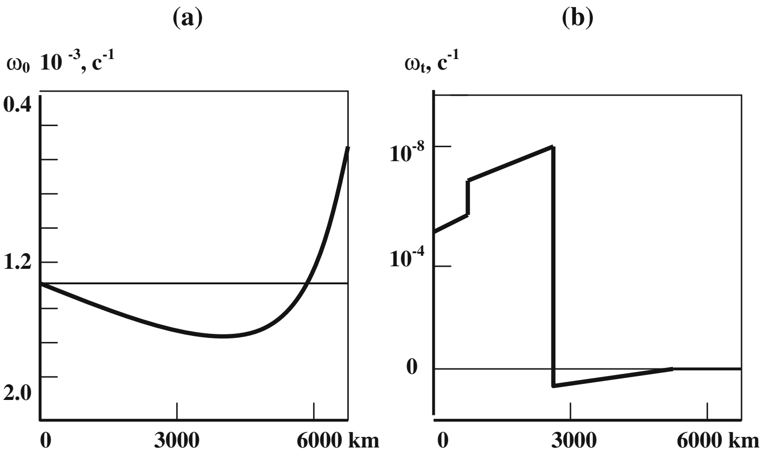


Fig. 6.6 Radial change in virial oscillation frequencies (**a**) and angular velocity of rotation (**b**) according to Eqs. (6.62)

Figure 6.6 shows changes in the virial oscillation frequencies of the Earth's shells.

6.7.3 *Angular Velocity of Shell Rotation*

Angular velocity of the Earth's shell rotations is determined from Eq. (5.28)

$$\begin{aligned}\omega_t(r) &= \sqrt{\frac{4}{3}\pi G\rho_t(r)} = \sqrt{\frac{4}{3}\pi G\rho_0(r)\left(\frac{3}{5}a\frac{r^2}{R^2}+\frac{3}{4}b\frac{r}{R}+c\right)k_e(r)} \\ &= \omega_0(r)\sqrt{\left(\frac{3}{5}a\frac{r^2}{R^2}+\frac{3}{4}b\frac{r}{R}+c\right)k_e(r)} \\ &= \omega_0(r)\sqrt{\left(-3.144\frac{r^2}{R^2}+3.8475\frac{r}{R}+0.295\right)k_e(r)},\end{aligned} \tag{6.63}$$

where $\omega_t(r)$ is the angular velocity of the shell rotation; $\omega_o(r)$ is the shell oscillation frequency which is determined by Eq. (6.62).

The geodynamic parameter $k_e(r)$, which expresses the ratio of the tangential component of the force field and the gravity force acceleration for the upper shell, is approximated as

$$k_e(r) = \frac{\omega_t^2(r)}{\omega_0^2(r)},$$

At $r/R = 1$ $k_e(r) = 0.003456$; at $r/R = 0$ $k_e(r) = 1$, $\omega_t(0) = \omega_o(0)$, i.e. the virial oscillation frequency corresponds to the gravity pressure of the uniform density masses. In this particular case we are interested in changes of the angular velocity of rotation of the upper (1000 km) and lower (up to the core border) mantle (2900 km) shells. Figure 7.6b shows the radial change of the angular velocity of rotation calculated by Eq. (6.63). It is seen that the angular velocity at the lower mantle—outer core is close to zero but changes its direction.

We emphasize once more that Eqs. (6.62) and (6.63) express the third Kepler law which determines radial distribution of both the virial oscillation frequencies and the angular velocities of rotation. Numerical values of these parameters are determined by the radial density distribution law. It also determines the density jumps which mark the effect of the shell's hydrostatic equilibrium.

6.8 The Nature of Precession, Nutation and Body's Equatorial Plane Obliquity

The most noteworthy effects of dynamics of the Earth and other bodies are the interrelated phenomena of the precession and nutation of the axis of rotation, tidal effects of the oceans, and atmosphere, the axial obliquity and declination of the plumb line and the gravity change at each point of the planet's outer force field. The present-day ideas about the nature of these phenomena were formed on the basis of the Earth's hydrostatic equilibrium and since old times were considered as effects of perturbation from the Sun, the Moon and other planets. All the above phenomena represent periodic processes and many observational and analytical works were done for their understanding and description. The present-day studies of these processes are still continuing to be specified and corrected. This is because such topical problems as correct time, ocean dynamics, short- and long-term weather and climate changes and other environmental changes are important for everyday human life.

Now, after it was found that the conditions of the hydrostatic equilibrium are not acceptable for study of the Earth's dynamics, we reconsider the nature of the phenomena by applying the concept of the planet's dynamical equilibrium and developing a novel approach to solving the problem.

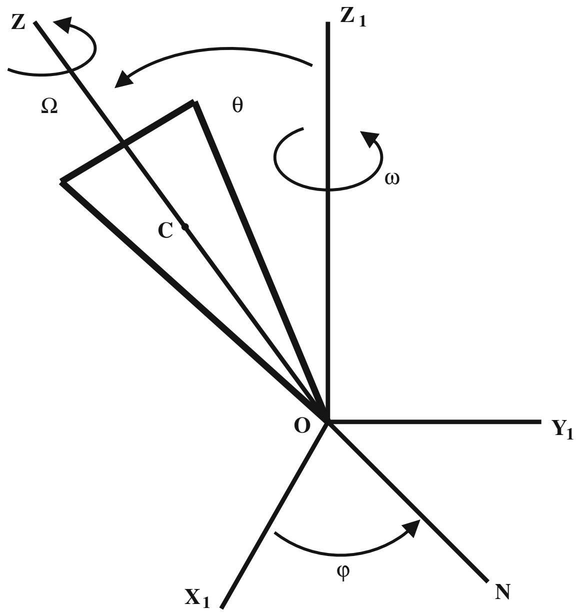
6.8.1 Phenomenon of Precession

The first discovered phenomenon was the precession of equinoxes. It was observed already in the second century BP by the Greek astronomer and mathematician Hipparchus. His discovery was based on comparison of longitudes of the far stars with the longitudes of the same stars determined 150 years ago by the other astronomers.

Inertial rotation of a symmetrical rigid body with a fixed point gives the classical explanation of precession. Such a motion of the body, presented on Fig. 6.7, includes its rotation with angular velocity Ω relative to the axis Oz, fixed in the body, and from rotation with angular velocity ω around the axis Oz_1. Here the axes x_1, y_1, z_1 are accepted to be immobile because motion of the body is considered just relative to them. The straight line ON perpendicular to the plane $z_1O\ z$ is called the line of nodes and angle $\psi = x_1ON$ is the precession angle. Together with precession, the body performs the nutation motions (axis wobbling) which cause changes in the nutation angle $\Theta = z_1Oz$.

Perturbation of the Earth's inertial rotation is considered as a result of the applied solar–moon force couple, the axis of which is at right angles to the rotation axis, the body turns around the third mutually perpendicular axis. The Earth is accepted as a rigid body oblate along the rotary axis. Newton's idea was that the spherical body has an equatorial bulge appearing as the result of the planet's oblateness. In this case the Sun attracts stronger the body's equatorial bulge and it tends to decrease the inclination of the Earth's equatorial plane to the ecliptic. The Moon affects analogously but two times as powerfully due to close distance. The common effect of the Sun and Moon on the equatorial excess of the rotating Earth mass leads to the rotary axis precession. Because the induced precession forces are continuously varying due to changes in the Sun and Moon position relative to the Earth, additional nutations (wobble) of the axis are observed during translational motion of the planet. In addition to the moon–solar precession, the effect of the other planets of about few tenths of an arc second is observed. The combined Earth precession rate

Fig. 6.7 Classical explanation of precession motion



is estimated to be equal to $\sim 50.3''$ per year or one complete rotation in $\sim 26{,}000$ years.

The theory of the precession and nutation of the Earth's axis of rotation based on the hydrostatics was developing in the works of D'Alembert, Laplace and Euler. The precession values were calculated by Bessel and Struve and under verification till now. Physical basis of the modern studies remains unchanged. The main accent in the studies is made on consideration of the elastic and rheological properties of the planet, effects of dynamics of the atmosphere and the oceans and dynamics of the liquid core, the probability of which is assumed (Jeffreys 1970; Munk and MacDonald 1964; Melchior 1972; Sabadini and Vermeersten 2004; Molodensky 1961; Magnitsky 1965).

6.8.2 Tidal Effects

The theory of the ocean tides was also presented first by Newton in his *Principia*, Proposition XXIV, Theorem XIX. He stated that the tides are caused by action of the Moon and the Sun. It follows from the Corollaries IX and XX (Proposition LXVI, Book I) that the sea should rise and subside twice per every lunar and twice per every solar day, and the highest tide in the free and deep seas should appear less than 6 hours after the tide body has passed the place meridian. And it happens like that along all the East Atlantic and Pacific shores. The effects of both tide bodies are summed up. At joining and opposing positions of the bodies their effects are summed up and provide the highest or lowest tide. Observation shows that the tide effect of the Moon is stronger than the Sun.

Modern studies in the theory of precession and nutation remain on the physical basis described by Newton. Besides, all the above phenomena are considered in close relationship and their amplitudes and periods are described by common equations which follow from the attraction theory (Melchior 1972).

The modern physical picture for explanation of the tidal interaction is presented as follows (Pariysky 1975). The tidal force is equal to a difference between any Moon-attracted place on the Earth (including the atmosphere, the oceans, and the solid body) and the same particle replaced to the centre of the planet (Fig. 6.8).

The normal tide forces are proportional to the mass of the Moon m and the distance to the centre of the Earth r, and to inverse cubic distance between the Moon and the Earth R, and zenith distance of the Moon z. The vertical component of the tide force per mass unit F_v is changing the gravity force into the value

$$F_{\mathrm{v}} = 3G\frac{mr}{R^3}\left(\cos^2 z - \frac{1}{3}\right), \tag{6.64}$$

where G is the gravity constant.

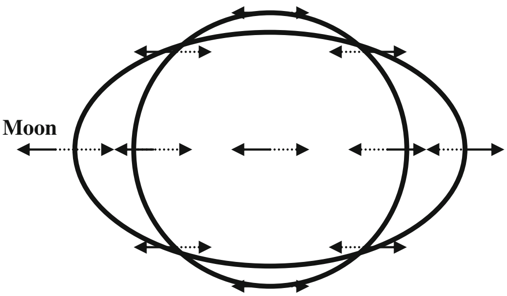


Fig. 6.8 Scheme of mass interaction between the moon and the earth for explanation of tidal effects (by Pariysky 1975)

The gravity force decreases by 0.1 mgal or by 10^{-7} of its value on the Earth's surface when the Moon stays in zenith or nadir, and increases twice when the Moon rises or sets.

The horizontal component of the tidal force is equal to zero when the Moon stays zenith, nadir and on the horizon. Its maximum value reaches 0.08 mgal at zenith distance of the Moon equal to 45°:

$$F_h = 3G\frac{mr}{R^3}\sin^2 z. \tag{6.65}$$

The tide force of the Sun is formed analogously. But because of distance, its value is 2.16 times less than that of the lunar one. Due to rotational and orbital motion of the Earth, the Moon and the Sun, the tide force of each point in the atmosphere, the oceans and the planet's surface continuously changes in time. The tables of integral values of the tide forces in the form of the sums of periodic components ($\sim$500 terms or more) calculated by the theory of motion of the Moon round the Earth and the Earth round the Sun were compiled.

By estimation of many authors the total tidal-slowing down of Earth's rotation amounts to 3.5 ms in 100 year. By astronomic observation the Earth rotation is accelerated by 1.5 ms per 100 year.

Note, that in the framework of the hydrostatic approach the problems of the nature of the obiquity of axis of the Earth's rotation to the ecliptic and the nature of the obliquity of axes of the Moon and the Sun to their orbit planes and their obliquity to the ecliptic are not discussed. These problems have not even a formulation.

6.8.3 The Nature of Perturbations Based on Dynamic Equilibrium

In the beginning, let us consider physical meaning of the gravitational perturbation for interacted volumetric (but not point) body masses. To the contrary of hydrostatics, where the measure of perturbation in the precession-nutation and the tidal phenomena is the perturbing force, in the dynamic approach that measure of perturbation is power's pressure. In Chap. 2 we came to the conclusion that the mass points and the vector forces as a physical and mathematical instrument in the problem solution of dynamics of the Earth in its own force field are inapplicable. This is because the outer vector central force field of the interacting volumetric masses expresses incorrectly dynamical effects of their interaction. As a result, the kinetic effect of interaction of the mass particles, namely the kinetic energy of their oscillation, is lost. And also the geometric centre of a body is accepted as the gravity centre and centre of the inertia (reaction). In dynamics it leads to wrong results and conclusions. In this connection we found that in dynamics, a self-gravitating body, the effect of gravitational interaction of mass particles should be considered as the power's pressure. In addition, in this case we are free in choice of a reference system. Our conclusion does not contradict Newton's physical ideas which are presented in Book I of his *Principia* where he says:

> I approach to state a theory about the motion of bodies tending to each other with centripetal forces, although to express that physically it should be called more correct as pressure. But we are dealing now with mathematics and in order to be understandable for mathematicians let us leave aside physical discussion and apply the force as its usual name.

Accepting the power pressure as an effect of gravitational interaction, we come to understand, in the considered problem of the mutual perturbations between the Earth, the Moon and the outer force fields of the bodies and between their inner force fields of the shells. Satellite observations show that the outer force field, induced by the Earth's mass, has 4π-outward direction of propagation and acquires a wave nature. We consider this outer wave force field as a physical media by which bodies transmit their energy. Thus, the Earth and other planets are held and move on the orbits by the power of the outer force field of the Sun. This statement is proved by the discovered Solar System bodies origin. This energy represents the integral effect of the Sun's mass interaction and this energy conserves by orbiting motion of separating bodies (see Tables 2.1 and 2.2 of Chap. 2). The frequency of the gravity interaction determines the border equality of energies for interacting bodies. In this case, equality of the frequencies is the condition of their dynamical equilibrium.

Now we can write the condition of the interacting force fields and find the physical border of such equilibrium between the Sun and the Earth in the form

$$\omega_s(R_s) = \omega_e(R_e), \tag{6.66}$$

where R_s and R_e are the radiuses of the Sun's and the Earth's outer force field where the energies carried by the frequencies are equal one to another.

Analogously, equilibrium of the field energy for the Earth and the Moon is written as

$$(\omega_e)R_e = \omega_m(R_m), \tag{6.67}$$

where R_e are R_m are the radiuses of the Earth's and the Moon's outer force field.

The mean value of the radius in (6.66) can be found from the equality of the frequencies of two bodies

$$\sqrt{\frac{GM_{\mathrm{s}}}{R_{\mathrm{s}}^3}} = \sqrt{\frac{GM_{\mathrm{e}}}{R_{\mathrm{e}}^3}}, \tag{6.68}$$

and

$$\mathrm{R_s} + R_{\mathrm{e}} = R_{\mathrm{se}}. \tag{6.69}$$

After transforming the equations and substituting numerical values of the solar mass $M_{\mathrm{s}} = 1.99 \times 10^{30}$ kg, mass of the Earth $M_{\mathrm{e}} = 5.976 \times 10^{24}$ kg and the mean distance between the two bodies $R_{\mathrm{se}} = 1.496 \times 10^{11}$ m, we obtain the cubic equation

$$R_{\mathrm{e}}^3 - 1.35 \times 10^8 R_{\mathrm{e}}^2 + 2.02 \times 10^{21} R_{\mathrm{e}} - 10^{34} = 0. \tag{6.70}$$

Compiling and solving analogous equations for the Earth's position in perihelion ($R_{\mathrm{pe}} = 1.471 \times 10^{11}$ m) and in aphelion ($R_{\mathrm{ae}} = 1.521 \times 10^{11}$ m), we can find the corresponding mean radius in the knots, perihelion and aphelion of the Earth's field equilibrium state:

$$R_{\mathrm{ek}} \approx 2.131 \times 10^9\,\mathrm{m}; \quad R_{\mathrm{ep}} \approx 2.1277 \times 10^9\,\mathrm{m}; \quad R_{\mathrm{ea}} \approx 2.1335 \times 10^9\,\mathrm{m}. \tag{6.71}$$

The corresponding energy caring by frequency of dynamical equilibrium of the Earth's field in the found points, using Eq. (5.43), is

$$\begin{gathered} \omega_{\mathrm{ek}} \approx 4.1183 \times 10^{-7}\,\mathrm{s}^{-1}; \quad \omega_{\mathrm{ep}} \approx 4.1374 \times 10^{-7}\,\mathrm{s}^{-1}; \\ \omega_{\mathrm{ea}} \approx 4.1038 \times 10^{-7}\,\mathrm{s}^{-1}. \end{gathered} \tag{6.72}$$

It follows from the equilibrium condition (6.66) that the found frequency values (6.72) for the Earth should coincide with the frequencies of the Sun's oscillations of the force field in the corresponding points of Earth's orbit.

By the same method the corresponding values of radiuses of the outer force field and the frequencies of the Moon locating in the Earth's force field can be found. According to (6.68) and (6.69) for the Moon at its mass $M_{\mathrm{m}} = 7.35 \times 10^{22}$ kg and at the mean value of distance between two bodies $R_{\mathrm{km}} = 3.844 \times 10^8$ m, in the perigee $R_{\mathrm{pm}} = 3.644 \times 10^8$ m and apogee $R_{\mathrm{am}} = 4.068 \times 10^8$ m, for the radius Eq. (6.70) is written in the form

$$R_{\mathrm{m}}^3 - 0.14 \times 10^8 R_{\mathrm{m}}^2 + 0.54 \times 10^{16} R_{\mathrm{m}} - 0.69 \times 10^{24} = 0. \tag{6.73}$$

The values of radiuses in the knots, perigee and apogee will be written as

$$R_{\mathrm{mk}} \approx 0.72 \times 10^{8}\,\mathrm{m}; \quad R_{\mathrm{mp}} \approx 0.724 \times 10^{8}\,\mathrm{m}; \quad R_{\mathrm{ma}} \approx 0.716 \times 10^{8}\,\mathrm{m} \tag{6.74}$$

The corresponding energy carried by frequencies in the above points of dynamical equilibrium of the Earth's field in the found points, using Eq. (5.43), is

$$\begin{gathered} \omega_{\mathrm{mk}} = 3.6242 \times 10^{-6}\,\mathrm{s}^{-1}; \quad \omega_{\mathrm{mp}} = 3.5942 \times 10^{-6}\,\mathrm{s}^{-1}; \\ \omega_{\mathrm{ma}} = 3.6546 \times 10^{-6}\,\mathrm{s}^{-1}. \end{gathered} \tag{6.75}$$

The found frequency values (6.75) for the Earth should coincide with the frequencies of the Moon's oscillations of the force field in each point of the Moon's orbit.

The above results mean that the Earth's dynamical equilibrium at its orbital motion around the Sun and the Moon's dynamic equilibrium at its motion around the Earth are determined by the frequency equality of the outer force field's of the two bodies in every point of their orbits. Because the frequency of oscillation in a given point of the force field is a function of the body's mass density distribution, then the inclination of the Earth's and the Moon's orbital plane to the equatorial plane of the Sun and the Earth are determined by asymmetry in mass density distribution of the two bodies. The observed inclination of the orbital planes is determined by asymmetry in mass density distribution of the Sun and the Earth. The observed parameters of the orbits and their inclination relative to the plane diameters of the Sun, the Earth and the Moon give a general view of the asymmetric distribution of the body's masses. In particular, the northern hemisphere of the Earth is more massive than the southern one. In the perihelion the northern hemisphere is turned to the less massive hemisphere of the Sun. So that, the polar oblateness of each body controls the location of its pericenter and apocenter, and the equatorial oblateness of each body responds to location of its nodes. Thus, the body motion in the outer force field of its parent occurs under strict conditions of dynamic equilibrium which is also the main condition of its separation. It follows from the condition of dynamic equilibrium that the orbital motion of the Earth and the Moon reflects asymmetry in mass density distribution of the Sun, the Earth and the Moon and asymmetry in the potential of the outer wave field distribution. Only the structure of the Sun's outer wave field controls the Earth's trajectory at the orbital motion and the Earth's force field manages the orbital motion of the Moon, but not vice versa or somehow else.

6.8.4 *Rotation of the Outer Force Field and the Nature of Precession and Nutation*

At the right time of motion of the bodies with outer wave fields, their mutual perturbations are transferred not directly from each body to the other one or from their shells, but through the outer fields by means of the corresponding active and reactive wave pressure of the interacting fields. There is an important dynamic effect of all the perturbations. This is the continuous change in the outer wave field of each body which proceeds from its non-uniform radial distribution of the mass density. As it was earlier shown, the non-uniform radial distribution of mass density initiates the differential rotation of the body shells. And, in accordance with Eqs. (5.27)–(5.28) expressing the third Kepler's law, the reduced body shells' perturbing effects are transferred to the other body by means of the outer wave field. So that the Sun, for instance, continuously through its outer wave field to the Earth, transfers all the perturbations resulting during rotation of the interacting masses of the shells. The Earth, in the framework of the energy conservation law, demonstrates all perturbations by changes in its orbit turns around the Sun (see below Fig. 6.9).

Earlier it was shown that in the case of non-uniform distribution of mass density the body's potential and kinetic energies have radial and tangential components which induce oscillation and rotation of the shells. It was defined by Eq. (6.61) that the observed daily rotation of the Earth concerns only the upper shell with thickness of ~ 375 km and reaches the nearby Mohorovičić discontinuity. By the same reasoning it is not difficult to find the thickness of the upper shells for the Sun and the Moon correspondingly equal to

$$h_s = \frac{v_s}{\omega_{0s}(R_s)} \approx \frac{2}{6.28 \times 10^{-4}} \approx 3180\,\text{km}, \tag{6.76}$$

$$h_m = \frac{v_m}{\omega_{0m}(R_m)} \approx \frac{4.56 \times 10^{-3}}{9.66 \times 10^{-4}} \approx 4.72\,\text{km}. \tag{6.77}$$

We do not know real values and angular velocities for the inner shells of the three bodies. These velocities have a direct interrelation with the observed changes

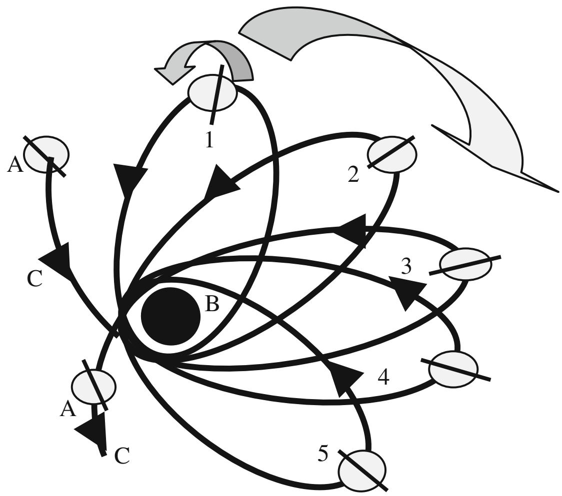


Fig. 6.9 Real picture of motion of body *A* in the force field of body *B*. Digits identify succession of turns of the body *A* moving around body *B* along the open orbit *C*

in parameters of the orbital motion of the Earth and the Moon including the retrograde motion of the orbital nodes and the apsidal line. In this connection let us try to understand first of all the nature of precession and nutation of the bodies from viewpoint of the dynamic approach.

It was noted above that, in accordance with the hydrostatic approach, precession of the equinoxes of the Earth is an effect of the net torque of the Moon and the Sun on the equatorial "bulge" aroused from gravitational attraction. The torque aspires to diminish inclination of the equatorial belt with surplus mass relative to the ecliptic and induce the retrograde motion of the nodal line. In addition, because the ratio of distance between the interacting bodies is changed, then the relationship between the forces is also changed. In this connection the precession is accompanied by nutation (wobbling) motion of the axes of rotation.

Analysis of orbits of the artificial satellite motion around the Earth shows that, in spite of absence of the equatorial "bulge" of mass, the apparatus demonstrates the precession effect. Its orbital plane has a clockwise rotation with retrograde motion of the nodal line. But a new explanation of the phenomenon is given. It appears that the retrograde motion of the nodal line associates with the Earth's equatorial and polar oblateness. The amplitude of the nodal line shift depends on the satellite orbit inclination to the Earth's equatorial plane. In the case of the poles' orbital plane the nodal line shift is completely absent. This is because the pole motion excludes both the polar and the equatorial oblatenesses of the Earth. The direction of motion of the apsidal line depends on the satellite's orbit inclination and is determined by the Lentz law.

It is also known that for other free-of-satellite planets the retrograde motion of the nodal line is also a characteristic phenomenon called the "secular perihelion shift". It was found from observation of Mercury, Venus, Earth and Mars that their secular perihelion shifts are decreased from $\sim 40''$ through $\sim 8.5''$, $\sim 5''$ to $\sim 1.5''$ accordingly (Chebotarev 1974).

All these facts imply that the explanation given for the satellites' precession depending on their orbital inclination to the ecliptic is correct. But the nature of this unique phenomenon, characteristic for all celestial bodies, are inconsistent with the hydrostatic approach and should be reconsidered, taking also into account the satellite observations.

The precession of the Earth, the Moon and the artificial satellites in the form of motion of an orbital plane towards the backward direction of the body's motion should be considered as a virtual explanation of the phenomenon. In fact, the orbit's plane is a geometric shape traced by the body. And there is no reason to consider its movement without the body itself. There is no difficulty to present the real body motion in space in two opposite directions synchronously. In particular, the actual picture of the Earth, the Moon and the satellite motion in counterclockwise direction and retrograde movement of the nodal line is shown in Fig. 6.9.

Here the satellite is moving in the counterclockwise direction along the unlocked elliptic orbit 1 in the continuously changing (perturbed by oblatenesses) force field of the planet. Because of the counterclockwise rotation of the Earth's mass, the satellite in perigee started to move on the orbit 2 and makes a shift in retrograde direction in the ascending and descending nodes. At the same time the eccentricity of the orbit 2 changes by a proper value. Analogously the body passes on orbit 3, 4, 5 and so on. The theory of dynamic equilibrium of the Earth explains the physics of the observed phenomenon as follows.

The dynamic equilibrium theory assumes that the Earth is a self-gravitating body, the interacting mass particles of which induce the inner and outer force fields. Separation of the planet's asymmetric shells results by the inner force field and depends on the law of the radial mass density distribution. The normal component of the body's power pressure provides oscillation, and the tangential component induces rotation of the shells having a different angular velocity. At the same time the mantle shells A and the outer shell of the core B may have the same (Fig. 6.10a) or opposite direction (Fig. 6.10b) of rotation depending on the radial mass density distribution.

The seismic data show that the inner core C has a uniform density distribution. Because of this, it does not rotate and its potential energy is realized in the form of oscillation of the interacting particles. The potential E of the outer force field is controlled by integral effect of the interacted masses of all the shells and presented by the reduced shell D having continuously changing power.

The energy of the Earth's outer force field is changed from the body surface in accordance with the $1/r$ law and at every r is continuously varied because of differences in the angular velocity of rotation of the shell's masses. This force field controls the direction and angular velocity of orbital motion of a satellite. Taking into account the non-uniform and asymmetric distribution of the masses of rotating shells, the change in the trajectory of the body motion is accompanied by a corresponding change in eccentricity of the orbit both at each and subsequent turns. Its maximum value is reached when the non-uniformities of the rotating masses coincide and the minimal value appears at the opposite position.

It is worth noting that the effect of retrograde motion of the nodal line of the Earth, the Moon and artificial satellites appears to be a common phenomenon

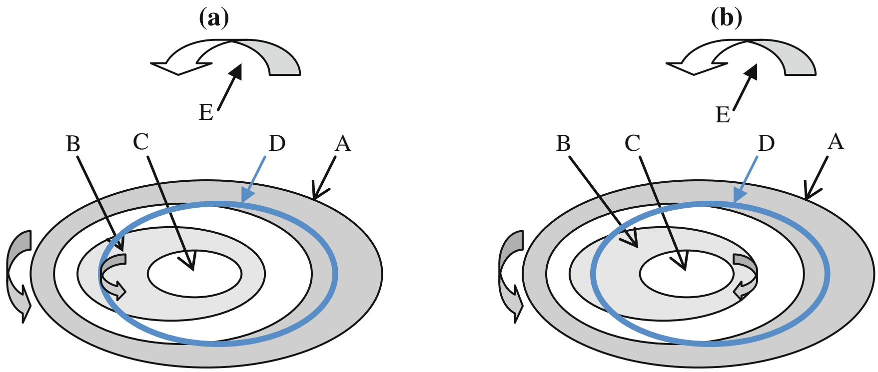


Fig. 6.10 Sketch of rotation of the earth's shells by action of the inner force field: *A* is the mantle shells; *B* is the outer core; *C* is the inner core; *E* is the outer force field; *D* is the reduced shell of the inner force field of the planet

because they are induced by the Sun and the Earth's outer force fields are changing with a finite velocity. The conclusion follows from here that the Sun has the same effects in its shell structure and motion. It is obvious that the other planets with their satellites have the same character of structure and motion.

If one takes into account the effect of a planet's orbital plane inclination to the equatorial plane of the Sun, then the above changes are found to follow the law of $1/r$. This observable fact proves our conclusion that the changes in the outer force field of a body are controlled by rotation of its reduced inner force shell (see the force shell D on Fig. 6.10). It explains why Mercury has maximal value of the "secular perihelion shift" between the other planets.

Thus, the Earth's orbital motion and retrograde movement of its nodal line are controlled by the Sun's dynamics of the masses through the outer force field. The Earth plays the same role for the Moon and the artificial satellites. As to the nutation motion, then its nature is related to the same peculiarities in the structure and motion of the bodies but the effects of their perturbations are fixed by the axis wobbling.

6.8.5 *The Nature of Possible Clockwise Rotation of the Outer Core of the Earth*

The question arises why the outer planet's core may have a clockwise rotation. It was shown in Sect. 2.6 that the law of radial density distribution determines the direction of a body's shell rotation.

It was found that in the case of uniform mass density distribution all energy of the mass interaction is realized in the form of oscillation of the interacting particles (Fig. 2.2a). If the density increases from the body's surface to the centre, then there are oscillations and counterclockwise rotation of shells (Fig. 2.2b). Increase of mass density from centre to surface leads to oscillation and clockwise rotation with different angular velocities of the body shells (Fig. 2.2c). Finally, the parabolic law of radial density distribution (Fig. 6.11), where the density increases from the surface and then it decreases, leads to oscillation and reverse directions of rotation.

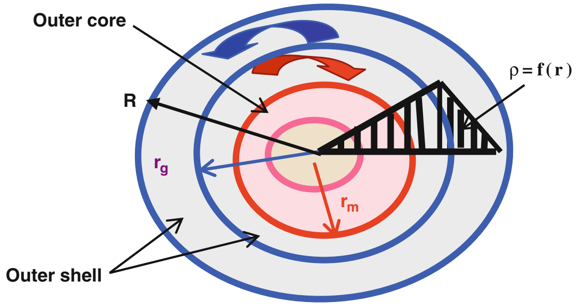


Fig. 6.11 Dependence of the parabolic law of radial density distribution on the shell rotation for the earth. Here r_m and r_g are the reduced radiuses of inertia and gravitation

Namely, the upper shells have a counterclockwise and the central shells clockwise rotation. The case demonstrated on Fig. 6.11, obviously, is characteristic for a self-gravitating body.

Note that direction of the body rotation depends on radial density distribution and corresponds with the Lenz right-hand or right-screw rule, well known in electrodynamics. Taking into account the observed effect of the retrograde motion of the satellite nodal line, the gravitational induction of the inner and outer force fields of the Earth has a common nature with electromagnetic induction noted earlier. Just Fig. 6.11 may explain the nature of the retrograde motion of the nodal line of a satellite orbit related to the finite velocity in the potential changes of the outer Earth's force field induced by the interacted mass particles. The continuous and opposite-directed movement of the asymmetric mass density distribution of the mantle and the outer core (Fig. 6.11) seems to be the physical cause of precession, nutation and variation of the inner and outer force fields observed by satellites. This idea is proved by the satellite data about the retrograde motion of the nodal line depending on inclination of its orbital plane with respect to the planet's equatorial plane.

It is worth recalling, from the literature, that the idea of dynamical effects of the probably liquid core of the Earth has been discussed among geophysicists for a long time (Melchior 1972).

6.8.6 *The Nature of the Earth's Orbit Plane Obliquity to the Sun's Equatorial Plane*

Celestial mechanics does not discuss the problem of obliquity of the planet's and satellite's orbit planes and accepts it as an observable fact. From the viewpoint of dynamical equilibrium creating and orbiting of the planets and satellites originated from the parental upper weightlessness shell with the first cosmic velocity. The separation could happen at any point of the body's surface depending on the stage of evolution and the radial mass density distribution. It is known from observation that in most cases it occurred in the parental equatorial zone. But there are observational data, from which some planets and satellites separated under higher angles of inclination. It is known from the experience of the artificial satellite launching that the angle of inclination to the Earth's equatorial plane is determined by the parameters of satellite motion and dynamical effects of the Earth's force field.

Unfortunately, up to now we fix inclination of orbital planes of all planets and even the Sun relative to the Earth's orbital plane accepted as ecliptic. This is a residual of the Ptolemaeus heliocentric system of the world, which was preserved in order to use the observational data. This is the cause of changes in virtual direction of the apsidal line at orbital inclination about 63° and the orbital plane rotation of the planets and satellites.

Despite this, we conclude that the Sun has a shell structure and its outer force field is rotating with angular velocity equal to the velocity of the Earth's retrograde motion of the knots. Thus, we find that the Earth's and planet's precession of the axes of rotation is the effect of difference in the Sun's velocity of the shells and correspondingly outer force field rotation. Taking into account observational data, the Earth's annual value in our time is equal to $\sim 50''$. But in a longer time scale this value is changing. This is because of changes in the period of rotation of the reduced shell of the inner force field of the Sun (see reduced shell D in Fig. 6.10). By the Earth's climatic changes, the period of rotation of the Sun's inner force field reduced shell changes is equal between $\sim$50,000 and $\sim$120,000 years (see below Fig. 6.12).

The problem of a body motion on non-closed rotating trajectory, shown in Fig. 6.9, in classical mechanics is known as a problem of the finite motion in the

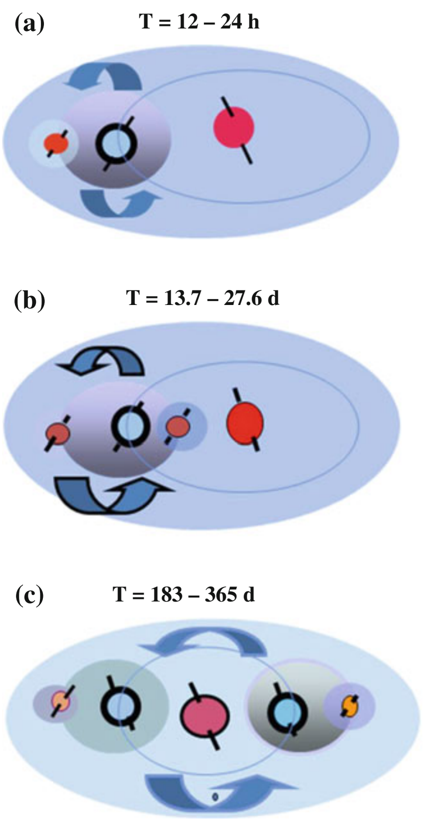


Fig. 6.12 Effects of inner and outer force fields of the three bodies (the sun, the earth and the moon) on the earth axes nutation: **a** diurnal and semidiurnal caused by the earth's upper shell rotation; **b** monthly and semimonthly initiated by the moon at its elliptic orbit revolution; **c** annual and semiannual caused by the non-uniform outer solar force field; **d** 18.6 year periodic nutation caused by the moon's precession (the outer force field rotation); **e** the Chandler's effect caused by the moon's yearly cycle

central force field in the domain restricted by the radiuses r_{max} and r_{min} from the aphelion to the perihelion (Landau and Lifshitz 1969). The trajectory can be closed after n turns at the condition of radius vector r tern on the angle Δφ, which is equal to the rational part from 2π, i.e. at $\Delta\varphi = 2\,\pi n_1/n_2$, but n_1 and n_2 should be equal to an integer.

In our case, by observation the annual precession of the Earth's axes of rotation is equal to $\Delta\varphi \approx 50''$. In addition, the uppermost light in the density shell of the Sun with thickness of 3200 km, which has the observed daily angular velocity of 14.4°, and also the Earth's rotating force field generate extra perturbation. So, strictly speaking, the Earth's trajectory remains non-closed. Thus, the above figures in first approximation can be used in practical geophysics for characterization of the integral rotation of masses of the Sun's shells.

The rotating upper Sun's shell appears to be an additional source of the Earth's perturbations, which developed in nutations of its upper shell. Speculative perception of the nutations is understood as a wobbling of the planet's axes of rotation in different timescales from daily to annual.

An analogous phenomenon has been observed for the Moon's motion around the Earth. Looking at the period of the main nutation, the integral period of rotation of all the planet's shells should be equal to 18.6 years. The angular velocity of rotation of reduced shell of the Earth's inner force field should be equal to 19.35° per year, 1.61° per month and 3.18′ per day. The period of the Moon's revolution around the Earth is 27.3 days. The Moon's daily angular orbital velocity is equal to 13.19°, and around its own axes of rotation is the same value. During one period of its turn the Moon delayed in the motion in arc distance $\Delta\varphi = 42.54'$, which we accent as a retrograde motion of the knots. The main period of the Earth's nutation seems to be the period of the Moon's precession of the axes. Because the daily and monthly time scales of motion of the Earth and the Moon do not coincide, then the arc values of their retrograde knots motion are continuously changing with a period of 18.6 years. The values of the daily, monthly and yearly nutations of the Earth's upper shell are correspondingly changing because of the Moon's perturbations.

6.8.7 The Nature of Chandler's Effect of the Earth Pole Wobbling

As it was noticed, changes in the planet's inner force field are observed in the form of nutation or wobbling of the axis of rotation. The axis itself reflects the dynamics of the upper planet's shell, the thickness of which, by our estimate, is about 375 km. The Moon is rotating about the Sun in the force field of the Earth which is perturbed by its natural satellite. Its maximum yearly perturbation should be the Chandler effect. The Moon's yearly cycle seems to be the ratio of the Earth's to the Moon's month (in days). Then this cycle is 365(30.5/27) ≈ 410 days (See Fig. 6.12).

6.8.8 The Nature of Obliquity of the Earth's Equatorial Plane to the Ecliptic

It is obvious that the obliquity of the planet's equatorial plane is related to the polar and equatorial oblateness of the Earth's masses. It follows from Eq. (5.28) that the obliquity, in turn, is determined by the tangential component of the inner force pressure generated by the non-uniform radial mass density distribution. This tangential component of the inner force field induces the inner field of the rotary moments, the energy of which was discussed above and presented in Table 6.2. The obliquity value can be obtained from the ratio of the potential energy of the uniform U_o and non-uniform U_t body of the same mass. Accepting this physical idea and the data of Table 6.2, we write and obtain

$$\cos\theta = \frac{U_0}{U_1} = \frac{\alpha_0}{\alpha_1} = \frac{0.6}{0.66} = 0.909, \quad \Theta = 24.5^\circ, \tag{6.78}$$

where α_o^2 and α_t^2 are the structural form factors taken from Table 6.2.

The error obtained in calculation of obliquity by formula (6.78) equal to about 1° or $\Delta\ \alpha_t^2 = 0.006$ can be explained by the accepted law of the continuous radial distribution of the planet's mass density.

Equation (6.78) expresses the integral effect of the obliquity of the planet's equatorial plane which is observed on the surface of the upper rotating shell. It was shown earlier that the observed obliquity is really an integral dynamical effect of the Earth's mass including the upper part of the Gutenberg shell. But being in a suspended state, relative to the other parts of the body, the upper shell is able to wobble as if on a hinge joint by perturbation from the Sun and the Moon. This effect of the upper shell wobbling gives an impression of the axial wobbling.

By the same cause the obliquity of the ecliptic with respect to the solar equator is determined by the Sun's polar and equatorial oblateness. The trajectory of the Earth's orbital motion at each point is controlled by the outer asymmetric solar force field in accordance with the dynamic equilibrium conditions. And only in the nodes, which are common points for equatorial oblateness of the Sun and the Earth, is the Huygens' effect of the innate initial conditions fixed by the third Kepler's law.

6.8.9 Tidal Interaction of Two Bodies

Let us consider the mechanism and effects of interaction of the outer force pressure of two bodies being in dynamic equilibrium. Come back to the mechanism and conditions of separation of a body mass with respect to its density when a shell with light density is extruded to the surface. Rewrite Eq. (5.31) for acceleration of the gravity force in points A and B of the two body shells (Fig. 6.1b) and their densities ρ_M and ρ:

$$q_{AB} = 4\pi G r\left(\frac{2}{3}\rho_M - \rho_m\right), \tag{6.79}$$

After the shell with density ρ_m appears on the outer surface of the body, the condition of its separation by Eq. (6.79) will be

$$\rho_M > 2/3\rho_m. \tag{6.80}$$

The gravitational pressure will replace the shell up to the radius $A + \delta A$, where the condition of its equilibrium reaches $\rho_M = \rho_m$. This condition is kept on the new border line between the body and its upper shell. Taking into account that the shell in any case has a thickness, then, by the Archimedes law, the body will be subject to its hydrostatic pressure. If the separated shell is non-uniform with respect to density, then a component of the tangential force pressure appears in it, and the secondary self-gravitating body (satellite) is formed. The new body will be kept on the orbit by the normal and equal tangential components of the outer force pressure. In this case the reaction of the normal gravitational pressure will be local and non-uniform. If the upper shell is uniform with respect to density, then the reaction of the normal gravitational pressure along the whole surface of the body and the shell remains uniform. In this case the separated shell remains in the form of a uniform ring.

The above schematic description of the physical picture of the separation and creation of a secondary body can be used for construction of a mechanism of the tidal phenomena in the oceans, the atmosphere and the upper solid shell at interaction between the Earth and the Moon. The outer gravitational pressure of the Moon, due to which it maintains itself in equilibrium on the orbit, at the same time renders hydrostatic pressure on the Earth's atmosphere, oceans and upper solid shell through its outer force field. This effect determines the tidal wave in the oceans and takes active part in formation and motion of cyclonic and anticyclonic vortexes. In accordance with the Pascal law, the reaction of the Moon's hydrostatic pressure is propagated within the total mass of the ocean water and forms two tidal bulges. Because the upper shell of the Earth is faster moving relative to motion of the Moon, the front tidal bulge appears ahead of the moving planet. Our perception of the ocean tides as an effect of attraction of the Moon appears to be speculative.

6.8.10 *Change in Climate as an Effect of Changes of the Earth's Orbit*

The above analysis of dynamical effects of the Earth's shells is based first of all on the data of satellite's orbit changes and measurements of the planet's force field. Unfortunately, a specific feature of an artificial satellite orbital motion is its artificial velocity which is ~ 16 times higher than the angular velocity of the upper Earth's shell. In this connection all its parameters of satellite motion are unnatural. So, we

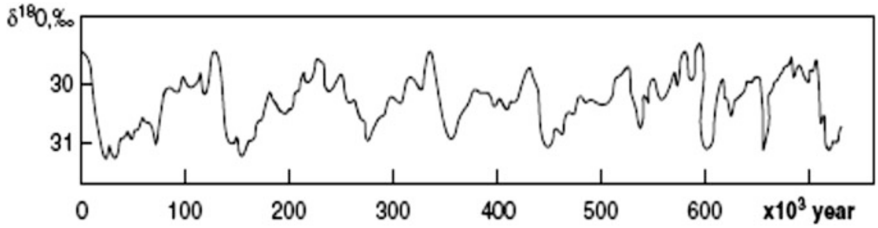


Fig. 6.13 Isotopic composition of oxygen in shells of mollusk *Globigerinoides Sacculifera* within time period 0–730,000 year (Emiliani 1978)

cannot directly divide the natural component of its nodal retrograde shift in order to get the total picture of perturbations which propagate the Earth's inner shells. This is an experimental problem.

But there are also long-term astronomical observations of the Earth's dynamics relative to the far stars, the results of which correspond to the presented ones. In addition, periodicity in rotation of asymmetric inner shells of the Sun can be fixed by climatic changes on the Earth over a long period of time. Such changes were being studied, for instance, by data of the oxygen isotopic composition in mollusk shells over a number of years. Figure 6.13 demonstrates the results of Emiliani (1978) who studied the core obtained during deep sea drilling in the Caribbean basin.

The author has obtained a picture of climate change in the Pleistocene era over 700,000 years. It is seen that the periods of climate change vary from 50,000 to 120,000 years. It means that the pure period of rotation of the asymmetric mass shells of the Sun is absent and the orbital trajectory has not been locked into place during the studied time.

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Chapter 7
The Unity of Electromagnetic and Gravitational Field of a Celestial Body and Centrifugal Mechanism of Its Energy Generation

Abstract In order to find solutions to our chosen problem, we have devised a novel idea based on the innate capacity of a body's energy for performing motion and the idea is discussed in this paper. Energy is our measure of the motion and interaction of particles of any kind of a body's matter. The various forms of energy are interconvertible and their sum for a system remains constant. The above unique properties of energy, with its oscillating mode of the motion in Jacobi's dynamics, make it possible to consider the nature of the electromagnetic and gravitational effects of celestial bodies as interconnected innate events. Applying the dynamical approach and the results obtained, it is shown that the nature of creation of the electromagnetic field and mechanism of its energy generation appears to be the effect of the volumetric gravitational oscillation of the body's masses. This effect is also characteristic for any celestial body. A number of tasks are considered in this chapter, namely: electromagnetic component of interacting masses, potential energy of the Coulomb interaction of mass particles, emission of electromagnetic energy by a celestial body as an electric dipole, quantum effects of generated electromagnetic energy and the nature of the star-emitted radiation spectrum. The relationship between the gravitational field (potential energy) and the polar moment of inertia of the Earth, discovered by artificial satellites, leads to understanding the nature and mechanism of a celestial body's energy generation as the force function of all the dynamical processes released in the form of oscillation and rotation of matter. Through the nature of energy, we understand the unity of forms of gravitational and electromagnetic interactions which, in fact, are the two sides of the same natural effect.

The hydromagnetic dynamo, the action of which is provided by the planet's liquid metal core or by solar gas plasma, is the most popular idea for explanation of a body's electromagnetic field generation. Its essence is in the motion of the conducting liquid core where self-excitation of the electric and magnetic poloidal (meridional) and toroidal (parallel) fields is observed. During the rotation of the inner planet's shells with different angular velocities, in the case of asymmetric thermal convection of the shell mass, the intensity of fields is increased. This condition, for example, for the Earth is achieved because the rotation and magnetic axes do not

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coincide and the thermal convection supposedly takes place. But the physically justified theory of the observed planets and solar phenomenon of electromagnetic field is absent. There is no explanation of mechanism of generation of the energy of this field, except for general physical principle of the mass and charge interaction. Also the ideas or hypotheses about source of refilling of the planet's energy which is spent for the gravitational and thermal irradiation are absent. The only source of the solar- and star-irradiated energy is accepted to be interior nuclear fusion. In this chapter, we discuss this problem from the position of Jacobi dynamics effects.

In order to find a solution of the problem, in this chapter, we discuss a novel idea based on the innate capacity of a body's energy for performing motion. As shown in Chaps. 2 and 3, energy is the measure of the motion and interaction of particles of any kind of a body's matter. The various forms of energy are interconvertible, and its sum for a system remains constant. The above unique properties of energy, with its oscillating mode of the motion in our dynamics, make it possible to consider the nature of the electromagnetic and gravitational effects of celestial bodies as interconnected events.

It was shown in Chap. 6 that the body's gravitational (potential) energy results in the body's matter volumetric pulsations, having an oscillating regime, frequencies of which depend on mass density. In our consideration, the planets and stars are accepted as self-gravitating bodies. Their dynamics is based on their own internal force field and the potential and kinetic energies are controlled by the energy of oscillation of the polar moment of inertia, that is, by interaction of the body's elementary particles.

Applying the dynamical approach and the results obtained, we show below that the nature of creation of the electromagnetic field and mechanism of its energy generation appears to be the effect of the volumetric gravitational oscillation of the body's masses. This effect is also characteristic for any celestial body.

7.1 Electromagnetic Component of the Interacting Masses

It was shown in Sect. 5.2 that the electromagnetic energy is a component of the expanded analytical expression of the potential energy. The expansion was done by means of the auxiliary function of the density variation relative to its mean value. The expression of the body's potential gravitational energy in the expanded form (5.8) was found as

$$U = \alpha \frac{GM^2}{R} = \left[\frac{3}{5} + 3\int_0^1 \psi x \mathrm{d}x + \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \right] \frac{GM^2}{R}, \tag{7.1}$$

where U is the potential energy of the gravitational interaction; α is the form-factor of the force function; G is gravity constant; M is the body mass; R is its radius and

$\Psi(s)$ is the auxiliary function of radial density distribution ρ_r relative to its mean value ρ_0

$$\psi(s) = \int_0^s \frac{(\rho_r - \rho_0)}{\rho_0} x^2 \mathrm{d}x. \tag{7.2}$$

where $s = r/R$ is the ratio of the running radius to the radius of the sphere R; ρ_0 is the mean density of the sphere of radius r; ρ_r is the radial density; x is the running coordinate; the value $(\rho_r - \rho_0)$ satisfies $\int_0^R (\rho_r - \rho_0) r^2 \mathrm{d}r = 0$ and the function $\Psi(1) = 0$.

We have considered and applied the two first right-hand side terms of Eq. (7.1). The third term in dimensionless form represents an additive part of the potential energy of the interaction of the non-uniformities between themselves, which was written as

$$\frac{9}{2}\lambda = \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \equiv \frac{9}{2}\int_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2 \mathrm{d}x. \tag{7.3}$$

where

$$\lambda = \int_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \geq 0.$$

The non-uniformities are determined as the difference between the given density of a spherical layer and the mean density of the body within the radius of the considered layer. For interpretation of the third term, we apply the analogy of electrodynamics (Ferronsky et al. 1996). For each particle, there is generated an external field, which determines its energy. The energies of some other interacting particles and their own charges are determined by this field. As far as the potential of the field is expressed by means of the Poisson's equation through the density of charge in the same point, the total energy can be presented in additive form through application of the squared field potential. If the body mass is considered as a moving system, then the Maxwell's radiation field applies.

In our solution, the dimensionless third term of the field energy is written as

$$\frac{9}{2}\lambda = \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \equiv \frac{9}{2}\int_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2 \mathrm{d}x \equiv \frac{9}{2}\int_0^1 E^2 \mathrm{d}V, \tag{7.4}$$

where $E = \Psi/x^2$ is a dimensionless form of the electromagnetic field potential which is a part of the gravitational potential; Ψ plays the role of the charge; $\mathrm{d}V = x^2\mathrm{d}x$ is the volume element in dimensionless form.

Table 7.1 Observational parameters of equilibrium nebulae

Parameters	Visible dark nebulae			
	Small globula	Large globula	Intermediate cloud	Large cloud
m/m_{Sun}	>0.1	3	8×10^2	1.8×10^4
R (pc)	0.03	0.25	100	20
n (п/cm^3)	$>4 \times 10^4$	1.6×10^3	100	20
$m/\pi R^2$ (g/cm^2)	$>10^{-2}$	3×10^{-3}	3×10^{-3}	3×10^{-3}

In order to determine the numerical value of λ, the calculations for a sphere with different laws of radial density distribution including the politropic model were done (Ferronsky et al. 1996). These models were used in our earlier numerical calculations of the form factors α and β. The results show that for the density distributions which have physical meaning (Dirac's envelopes, Gaussian and exponential distributions) and also for the politrops with index 1.5, the parameter λ has the same constant value. We interpret this fact for a steady-state dynamical system as evidence of the existence of equilibrium radiation between a celestial body and the external flow. The numerical value of the parameter λ is equal to 0.022. There is also an observational confirmation of this conclusion. Spitzer (1968) demonstrates observational results of nebulae of different mass and size in Table 7.1.

In this case, the energy of the equilibrium electromagnetic field of radiation is equal to

$$U_\gamma = \left(\frac{9}{2} \int_0^1 E^2 \mathrm{d}V \right) \frac{GM^2}{R} = 0.1 \frac{GM^2}{R}. \tag{7.5}$$

Thus, the virial approach to the problem solution of the Earth's global dynamics gives a novel idea about the nature of the planet's electromagnetic field. The energy of this field appears to be the component of the potential energy of the interacted masses. The question arises about the mechanism of the body's energy generation, which provides radiation in a wide range of the wave spectrum from radio through thermal and optical to x and γ rays.

7.2 Potential Energy of the Coulomb Interaction of Mass Particles

With the help of a model solution, we can show that for the Coulomb interactions of the charged particles, constituting a celestial body, the relationship between the potential energy of a self-gravitating system and its Jacobi function

$$U\sqrt{\Phi} = \text{const.} \tag{7.6}$$

remains to be true (Ferronsky et al. 1981).

Derivation of the expression for the potential energy of the Coulomb interactions of a celestial body is based on the concept of an atom following, for example, from the Tomas-Fermi model (Flügge 1971). In our problem, this approach does not result in limited conclusions since the expression for the potential energy, which we write, will be correct within a constant factor.

Let us consider a one component, ionized, quasi-neutral and gravitating gaseous cloud with a spherical symmetrical mass distribution and radius of the sphere R. We shall not consider here the problem of its stability, assuming that the potential energy of interaction of charged particles is represented by the Coulomb energy. Therefore, in order to prove relationship (7.6) it is necessary to obtain the energy of the Coulomb interactions of positively charged ions with their electron clouds.

Assume that each ion of the gaseous cloud has the mass number A_i and the order number Z and the function $\rho(r)$ expresses the law of mass distribution inside the gaseous cloud. The mass of the ion will be $A_i m_p$ (where $m_p = 4.8 \times 10^{-24}$ g is mass of the proton) and its total charge will be $+Ze$ (where $e = 4.8 \times 10^{-10}$ GCSE is an elementary charge). Then let the total charge of the electron cloud, which is equal to $-Ze$, be distributed around the ion in the spherically symmetrical volume of radius r_i with charge density $q_e(r_e)$, $r_e \in [0, r_i]$. Radius r_i of the effective volume of the ion may be expressed through the mass density distribution $\rho(r)$ by the relation

$$\frac{4}{3}\pi r_i^3 = \frac{A_i m_p}{\rho(r)}. \tag{7.7}$$

Then

$$r_i = \sqrt[3]{\frac{3A_i m_p}{4\pi\rho(r)}}. \tag{7.8}$$

Let us calculate the Coulomb energy U_c' per ion, using relation (7.8). Assuming that the charge distribution law in the effective volume of radius r_i is given, we may write U_c' in the form

$$U_c' = U_c^{(+)} + U_c^{(-)}, \tag{7.9}$$

where $U_c^{(-)}$ is the potential energy of the Coulomb repulsion of electrons inside the effective volume radius r_i; $U_c^{(+)}$ is the potential energy of attraction of the electron cloud to the positive ion.

Let the charge distribution law $q_e(r_e) = q_0(r_e)$ inside the electron cloud be given. Then normalization of the electron charge of the cloud, surrounding the ion, may be written in the form

$$-Ze = \int_0^{r_1} 4\pi q_e(r_e) r_e^2 \mathrm{d}r_e. \tag{7.10}$$

From expression (7.10), we may obtain the normalization constant q_0, which will depend on the given law of charge distribution, as

$$q_0 = -\frac{Ze}{4\pi \int_0^{r_1} r_e^2 f(r_e)\mathrm{d}r_e}. \tag{7.11}$$

Now it is easy to obtain expressions for $U_c^{(-)}$ and $U_c^{(+)}$ in the form

$$U_c^{(-)} = (4\pi)^2 q_0^2 \int_0^{r_1} r_e f(r_e)\mathrm{d}r_e \int_0^{r_1} \left(r_e'\right)^2 f\left(r_e'\right)\mathrm{d}r_e', \tag{7.12}$$

$$U_c^{(+)} = 4\pi Zeq_0 \int_0^{r_1} r_e f(r_e)\mathrm{d}r_e. \tag{7.13}$$

Finally, expression (7.8) for the potential energy U_c' corresponding to one ion may be rewritten using (7.11)–(7.13) in the form

$$U_c' = -e^2 Z^2 \left[\frac{\int_0^{r_1} r_e f(r_e)\mathrm{d}r_e}{\int_0^{r_1} r_e^2 f(r_e)\mathrm{d}r_e} - \frac{\int_0^{r_e} r_e f(r_e)\mathrm{d}r_e \int_0^{r_e} \left(r_e'\right) f\left(r_e'\right)\mathrm{d}r_e'}{\left(\int_0^{r_1} r_e^2 f(r_e)\mathrm{d}r_e\right)^2} \right]. \tag{7.14}$$

It is easy to see that in the right-hand side of Eq. (7.14) the expression enclosed in brackets determines the inverse value of some effective diameter of the electron cloud, which may be expressed through the form-factor α_i^2 of the ion radius r_i, that is,

$$\frac{\int_0^{r_1} r_e f(r_e)\mathrm{d}r_e}{\int_0^{r_1} r_e^2 f(r_e)\mathrm{d}r_e} - \frac{\int_0^{r_e} r_e f(r_e)\mathrm{d}r_e \int_0^{r_e} \left(r_e'\right) f\left(r_e'\right)\mathrm{d}r_e'}{\left(\int_0^{r_1} r_e^2 f(r_e)\mathrm{d}r_e\right)^2} = -\frac{\alpha_i}{r_i^2} \tag{7.15}$$

Table 7.2 Numerical values of the form factors α_i for different radial charge distribution of the electron cloud around the ion

The law of charge distribution*	α_i^2
$q_e(r_e) = q_0 = \text{const}$	0.9
$q_e(r_e) = q_0(1 - r_e/r_i)$	1.257
$q_e(r_e) = q_0\,(1 - r_e/r_i)^n$	$\frac{(n+3)\left(11n^2+41n+36\right)}{8(2n+3)(2n+5)}$
$q_e(r_e) = q_0\,(r_e/r_i)^n$	$\frac{(n+3)^2}{(n+2)(2n+5)}$
The same for $n \to \infty$	$\alpha_i^2 \to 1/2$

*Here q_0 is the charge value in the centre of the sphere; r_e is the parameter of radius, $r_e \in [0, r_i]$; n = 0, 1, 2, …. is an arbitrary number

Thus, expression (7.14), using (7.15), yields

$$-U_c' = \alpha \frac{e^2 Z^2}{r_i^2}. \tag{7.16}$$

The numerical values of the form factor α_i depending on the charge distribution $q_e(r_e)$ inside the electron cloud are given in Table 7.2 and their calculations were given in our work (Ferronsky et al. 1981).

Using expression (7.16), the total energy of the Coulomb interaction of particles may be written as

$$-U_c = 4\pi \int_0^R \frac{\rho(r)}{A_i m_p} U_c' r^2 \mathrm{d}r = \frac{3\alpha_i e^2 Z^2}{R} \int_0^R R r^2 \left(\frac{4\pi\rho(r)}{3A_i m_p}\right)^{4/3} \mathrm{d}r. \tag{7.17}$$

Introducing the form factor of the Coulomb energy α_i in expression (7.17), depending on the mass distribution in the gaseous cloud and on the charge distribution inside the effective volume of the ion, we obtain

$$-U_c = \alpha_c \frac{e^2 Z^2}{r_i^2} \left(\frac{m}{A_i m_p}\right)^{4/3}, \tag{7.18}$$

where

$$\alpha_c = \frac{3\alpha_i \int_0^R [(4\pi/3)\rho(r)]^{4/3} R r^2 \mathrm{d}r}{m^{4/3}},$$

$$m = \sum_{i=1}^N m_i = 4\pi \int_0^R r^2 \rho(r) \mathrm{d}r.$$

Since the total number of ions N in the gaseous cloud is equal to

$$N = \frac{m}{A_i m_p},$$

and the relation between the radius of the cloud and the radius of the ion may be obtained from the relationship of the corresponding volumes

$$\frac{4}{3}\pi R^3 = N\frac{4}{3}\pi r_i^3$$

then the expression (7.18) may be rewritten in the following form:

$$-U_c = \alpha_c \frac{N^{4/3} e^2 Z^2}{R^2} = \alpha_c^2 N \frac{e^2 Z^2}{r_i^2}. \tag{7.19}$$

Hence, the form factor entering the expression for the potential energy of the Coulomb interaction acquires the same physical meaning, which it has in the expression for the potential energy of the gravitational interaction of the masses considered in Sect. 2.7. It represents the effective shell to which the charges in the sphere are reduced, i.e.

$$\alpha_c = \frac{r_i}{r_{ei}}. \tag{7.20}$$

Taking into account that the moment of inertia of the body is $I = \beta^2 m R^2$, then the relation (7.6) can be written in the form

$$-U_c\sqrt{I} = \alpha_c N^{4/3}\frac{eZ^2}{R}\sqrt{\beta^2 m R^2} = \alpha_c \beta^2 N^{4/3} m^{1/2} e^2 Z^2 = \text{const.} \tag{7.21}$$

Since we have assumed that the mass of the system and its ion composition are constants, examination of Eq. (7.6) will be equivalent to analysis of the product of the form factors α_c and β. Equation (7.6) holds if

$$\alpha\beta = \frac{r_i}{r_{ei}} \approx \text{const.}$$

The results of the numerical calculations of the form factors α_c and β for different mass distribution in the cloud are shown in Table 7.3, and calculations were carried out in our work (Ferronsky et al. 1981). The values of the form factor α_i of the ion, the numerical value of which depends on the choice of charge distribution $q_e(r_e)$, are shown in Table 7.2.

In Table 7.3, the numerical values of the form facto α_c and the product of the form factors $\alpha_c\beta$ are given for the case of homogeneous distribution of the electron charge around an ion, that is, when $q_e(r_e)$ = const. From Table 7.3, it follows that for different laws of mass distribution, when the mass increases to the centre, the

Table 7.3 Numerical values of the form factors α_c and β product for different laws of radial mass distribution

The law of mass	α_c/α_i	α_c	β	$\alpha_c\,\beta$ at
Distribution*		при $\alpha_i = 0.9$**		$\alpha_i = 0.9$**
$\rho(r) = \text{const}$	1	0.9	0.6324	0.5692
$\rho(r) = \rho_0(1 - r/R)$	1.1303	1.0173	0.5163	0.5253
$\rho(r) = \rho_0(1 - r/R)^2$	1.3331	1.1998	0.4364	0.5236
$\rho(r) = \rho_0(1 - r/R)^3$	1.5510	1.3959	0.3779	0.5276
$\rho(r) = \rho_0(1 - r/R)^n$	$\frac{27}{\sqrt[3]{6}}\frac{[(n+1)(n+2)(n+3)]^{4/3}}{(4n+3)(4n+6)(4n+9)}$	$\frac{243}{10\sqrt[3]{6}}\frac{[(n+1)(n+2)(n+3)]^{4/3}}{(4n+3)(4n+6)(4n+9)}$	$\sqrt{\frac{8}{(n+4)(n+5)}}$	for $n \to \infty$, 0.5909
$\rho(r) = \rho_0(r/R)$	1.0159	0.9143	2/3	0.6095
$\rho(r) = \rho_0(r/R)^2$	1.0461	0.9415	0.6900	0.6497
$\rho(r) = \rho_0(r/R)^n$	$\frac{27}{10\sqrt[3]{3}}\frac{(n+3)^{4/3}}{(4n+9)}$	$\frac{81}{100\sqrt[3]{3}}\frac{(n+3)^{4/3}}{(4n+9)}$	$\sqrt{\frac{2n+3}{3n+5}}$ for $n \to \infty$, $\to \infty$	
$\rho(r) = \rho_0 e^{-K(r/R)}$	0.2321 K	0.2089 K	$2\sqrt{2}/k$	0.5909
$\rho(r) = \rho_0 e^{-K(r/R)_2}$	0.5907 $k^{1/2}$	0.5316 $K^{1/2}$	$(1/k)^{1/2}$	0.5316

*Here, ρ_0 is the normalization constant; r is the parameter of the radius $r \in [0, R]$; n and k are arbitrary numbers, $n = 0, 1, 2, \ldots$
**The value α_i corresponds to the homogeneous charge distribution in the electron cloud, surrounding the ion $q_e(r_e)$ = constant (see Table 7.2)

product of form factors α_c and β remains constant, and therefore Eq. (7.6) holds, with the same comments as were made previously.

From Eq. (7.20), it follows, however, that the form factor of the Coulomb energy α_c becomes infinite, when the volume occupied by the ions tends to zero. Correspondingly, the Coulomb energy in this case will also tend to infinity. In Table 7.3, there are two laws of mass distribution for which the last condition holds. They are $\rho(r) = \rho_0(1 - (r/R)^n$ for $n \to \infty$. When the particles of the system are gathering at the shell of the finite radius the energy of the Coulomb interaction tends to infinity, whereas the energy of gravitational interaction has a finite value. When the mass distribution is $\rho(r) = \rho_0(1 - (r/R)^n$, the form factor of gravitational and Coulomb energies are both finite. But the form factors of the Jacobi function of the system in this case tends to zero, a circumstance which provides the constancy of the product of the form factors α_c and β. This difference might play a decisive role in the evolution of the system.

In conclusion, we note that the results of the study on the relationship between the Jacobi function and the potential energy allows us to consider that the transfer from Jacobi's equation into the equations of virial oscillations is from the point of view of physics justified. This justification has been achieved in the framework of Newton and Coulomb interactions of the particles of the system.

7.3 Emission of Electromagnetic Energy by a Celestial Body as an Electric Dipole

In Chap. 4, we considered the solution of the virial equation of dynamical equilibrium for dissipative systems written in the form

$$\ddot{\Phi} = -A_0[1 - q(t)] + \frac{B}{\sqrt{\Phi}}. \tag{7.22}$$

Here the function of the energy emission $[1 - q(t)]$ was accepted on the basis of the Stefan–Boltzmann law without explanation of the nature of the radiation process. Now, after analysis of the relationship between the potential energy and the polar moment of inertia, considered in the previous section, and taking into account the observed relationship by artificial satellites, we try to obtain the same relation for the celestial body as an oscillating electric dipole (Ferronsky et al. 1987).

Equation (7.22) for a celestial body as a dissipative system can be rewritten as

$$X(t-t_o) = E_\gamma(t-t_o).$$

The electromagnetic field formed by the body is described by Maxwell's equations, which can be derived from Einstein's equations written for the energy-momentum tensor of electromagnetic energy. In this case, only the general property of the curvature tensor in the form of Bianchi's contracted identity is used. We recall briefly this derivation sketch (Misner et al. 1975).

Let us write Einstein's equation in geometric form:

$$G = 8\pi T, \tag{7.23}$$

where G is an Einstein tensor and T is an energy-momentum tensor.

In the absence of mass, the energy-momentum tensor of the electromagnetic field can be written in arbitrary coordinates in the equation

$$4\pi T^{\mu\nu} = F^{\mu\alpha} F^{\nu\beta} g_{\alpha\beta} - \frac{1}{4} g^{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}, \tag{7.24}$$

where $g_{\alpha\beta}$ is the metric tensor in co-ordinates and $F^{\mu\nu}$ the tensor of the electromagnetic field.

From Bianchi's identity

$$\nabla G \equiv 0, \tag{7.25}$$

where ∇ is a covariant 4–delta, follows the equation expressing the energy-momentum conservation law:

$$\nabla T \equiv 0, \tag{7.26}$$

In the component form, the equation is

$$F^{\mu\alpha}_{;\sigma} g_{\alpha\tau} F^{\sigma\tau} + F^{\mu\alpha}_{;\tau} g_{\alpha\sigma} F^{\tau\alpha} = g^{\mu\nu} \left(F_{\nu\tau;\sigma} + F_{\sigma\nu;\tau} \right) F^{\sigma\tau}. \tag{7.27}$$

After a series of simple transformations, we finally have

$$F^{\beta\nu}_{;\nu} = 0. \tag{7.28}$$

Here and above, the symbol ';' defines covariant differentiation.

To obtain the total power of the electromagnetic energy emitted by the body, Maxwell's equations should be solved, taking into account the motion of the charges constituting the body. In the general case, the expressions for the scalar and vector potentials are

$$4\pi\phi = \int\limits_{(V)} \frac{[\rho] \mathrm{d}V}{R}, \tag{7.29}$$

$$4\pi\overline{A} = \int\limits_{(V)} \frac{[j] \mathrm{d}V}{R}, \tag{7.30}$$

where ρ and j are densities of charge and current; $[j]$ denotes the retarding effect (i.e. the value of function j at the time moment $t - R/c$); R is the distance between the point of integration and that of observation and c the velocity of light.

In this case, however, it seems more convenient to use the Hertz vector Z of the retarded dipole $p(t - R/c)$ (Tamm 1976). The Hertz vector is defined as

$$4\pi Z = \frac{1}{R}\rho\left(t - \frac{R}{c}\right). \tag{7.31}$$

Electromagnetic field potentials of the Hertz dipole can be determined from the expressions

$$\phi = -\mathrm{div}Z, \tag{7.32}$$

$$\overline{A} = \frac{1}{c}\frac{\mathrm{d}Z}{\mathrm{d}t}. \tag{7.33}$$

Moreover, the Hertz vector satisfies the equation

$$\Upsilon Z \equiv \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)Z = 0, \tag{7.34}$$

where Υ is the d'Alembertian operator.

The intensities of the electric and magnetic fields E and H are expressed in terms $\overline{Z}$ by means of the equations

$$\overline{H} = \mathrm{rot}\dot{\overline{Z}}, \tag{7.35}$$

$$\overline{E} = \mathrm{grad}\,\mathrm{div}\overline{Z} - \frac{1}{c}\ddot{\overline{Z}}. \tag{7.36}$$

The radiation of the system can be described with the help of the Hertz vector of the dipole $\overline{p} = q\overline{r}$, where q is the charge and r the distance of the vector from the charge $(+q)$ to $(-q)$.

From the sense of the retardation of the dipole $p(t - R/c)$, we can write the following relations:

$$\frac{d\overline{\mathrm{p}}}{\partial R} = -\frac{1}{c}\dot{\overline{p}}, \quad \frac{\mathrm{d}^2\overline{p}}{\mathrm{d}R^2} = \frac{1}{c}\ddot{\overline{p}},$$

Then the components of the fields $\bar{\mathrm{E}}$ and $\bar{\mathrm{H}}$ of the dipole are as follows:

$$\mathrm{H}_\varphi = \frac{\sin\theta}{\mathrm{c}^2\mathrm{R}}\ddot{\overline{\mathrm{p}}}\left(\mathrm{t} - \frac{\mathrm{R}}{\mathrm{c}}\right), \tag{7.37}$$

$$E_\theta = \frac{\sin\theta}{c^2 R}\ddot{p}\left(t - \frac{R}{c}\right), \tag{7.38}$$

where θ is the angle between p and $\bar{R}$; $H_\varphi \perp E_\theta$ and $\perp R$; the other components of E and H in the wave zone are tending to zero quicker than $1/R$ in the limit $R \to \infty$.

The flax of energy (per unit area) is equal to

$$S = \frac{c}{4\pi}E_\theta H_\phi = \frac{1}{4\pi c^2}\frac{\sin\theta}{R^2}\left(\ddot{p}\right)^2. \tag{7.39}$$

The total energy radiated per unit time is given by

$$\oiint S d\sigma = \frac{2}{3c^3}\left(\ddot{p}\right)^2. \tag{7.40}$$

Thus, transforming the dissipative system to an electric dipole by means of the Hertz vector, we have reduced the task of a celestial body model construction to the determination of the dipole charges $+Q$ and $-Q$ through the effective parameters of the body.

This problem can be solved by equating expression (7.40) for the total radiation of a celestial body as an oscillating electric dipole. In addition, the relation for the black body radiation, expressed through effective parameters, is presented below in Sect. 7.5.

The expression (7.40) for the total rate of the electromagnetic radiation J of the electric dipole can be written in the form (Landau and Lifshitz 1973)

$$J = \frac{2}{3}\frac{Q^2}{c^3}\left(\ddot{r}\right), \tag{7.41}$$

where Q is the absolute value of each of the dipole charges, and r is the vector distance between the polar charges of the dipole. Its length in our case is equal to the effective radius of the body.

In our elliptic motion model of the two equal masses, the vector $\bar{r}$ satisfies the equation

$$\ddot{r} = -Gm\frac{\bar{r}}{r^3}. \tag{7.42}$$

Thus, the total rate of the electromagnetic radiation of the dipole is

$$J = \frac{2}{3}\frac{Q^2}{c^3}\frac{(Gm)^2}{r^4}. \tag{7.43}$$

In order to obtain the average flux of electromagnetic energy radiation, a calculation of the value of the factor $1/r^4$ should be averaged during the time period of one oscillation. Using the angular momentum conservation law, we can replace time-averaging by angular averaging, taking into consideration that

$$\mathrm{dt} = \frac{mr^2}{2M}\mathrm{d}\varphi, \tag{7.44}$$

where M is angular momentum, and φ is the polar angle.

The equation of the elliptical motion is

$$\frac{1}{r} = \frac{1}{a(1-\varepsilon^2)}(1+\varepsilon\cos\,\varphi), \tag{7.45}$$

where a is the semi-major axis and ε is the eccentricity of the elliptical orbit.

The value of $1/r^4$ can be found by integration. In our case of small eccentricities, we neglect the value of ε^2 and write

$$\overline{\left(\frac{1}{r^4}\right)} = \frac{1}{a^4}. \tag{7.46}$$

Finally we obtain

$$\bar{J} = \frac{2}{3}\frac{Q^2}{c^3}\frac{Gm^2}{a^4}. \tag{7.47}$$

Earlier it was shown (Ferronsky et al. 1987) that

$$\bar{J} = 4\pi\sigma\frac{1}{a^2}A_c^4, \tag{7.48}$$

where σ is the Stefan–Boltzmann constant; $A_e = Gm\mu_e/3\ k$ is the electron branch constant; μ_e is the electron mass and k is the Boltzmann constant.

Equating relations (7.47) and (7.48), we find the expression for the effective charge Q as follows:

$$Q = \sqrt{6\pi\sigma}\frac{A_e^2}{cr_g}, \tag{7.49}$$

where $r_g = \mathrm{Gm/c^2}$ is the gravitation radius of the body.

We have thus demonstrated that it is possible to construct a simple model of the radiation emitted by a celestial body, using only the effective radius and the charge of the body. Moreover, a practical method of determining the effective charge using the body temperature from observed data is shown.

The logical question raised is, what is the mechanism of the energy generation of the bodies which they emitted in the wide range of oscillating frequencies spectrum. Let us consider this important question at least in first approximation.

7.4 Centrifugal and Quantum Effects of Generated Electromagnetic Energy

The problem of energy generation technology for human practical use has been solved long ago. In the beginning, it was understood how to transfer the wind and fair energy into the energy of translational and rotary motion. Later on, people have learned about production of the electric and atomic energies. Technology of the thermonuclear fusion energy generation is the next step. It is assumed that the Sun replenishes its emitted energy by the thermonuclear fusion of hydrogen, helium and carbon. The Earth's thermal energy loss is considered to be filled up by convection of the masses and thermal conductivity. But the source of energy for convection of the masses is not known.

The obtained solution of the problem of volumetric pulsations for a self-gravitating body based on their dynamical equilibrium creates a real physical basis on which to formulate and solve the problem. In fact, if a body performs gravitational pulsations (mechanical motions of masses) with strict parameters of contraction and expansion of any as much as desired small volume of the mass, then such a body, like a quantum generator, should generate electromagnetic energy by means of its transformation from mechanical form through the forced energy levels transitions and their inversion on both the atomic and nuclear levels. In short, the considered process represents transfer of mechanical energy of the mass pulsation into the energy of an electromagnetic field (Fig. 7.1).

An interpretation of the process can be presented as follows. While pulsating and acting in the regime of a quantum generator, the body should generate and emit coherent electromagnetic radiation. Its intensity and wave spectrum should depend on its body mass, radial density distribution and chemical (atomic) content. As it was shown in Sect. 5.4, the body with uniform density and atomic content provides pulsations of uniform frequency within the entire volume. In this case, the energy generated during the contraction phase will be completely absorbed at the expansion phase. The radiation appearing at the body's boundary surface must be in equilibrium with the outer flux of radiation. A phenomenon like this seems to be characteristic for the equilibrated galaxy nebulae and for the Earth's water vapour in anticyclonic atmosphere.

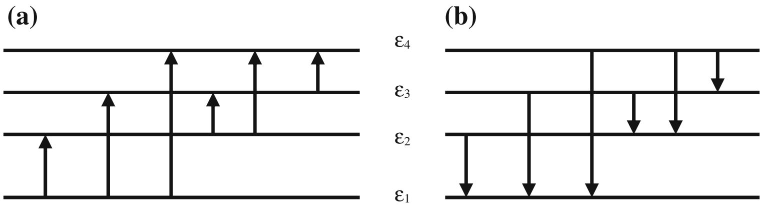

Fig. 7.1 Quantum transition of energy levels at contraction phase of the body mass (**a**) and inversion at the phase of its expansion (**b**); ε_1, ε_2, ε_3, ε_4 are levels of energy

The pulsation frequencies of the shell-structured bodies are different but steady for each shell density. In the case of density increase at the body centre, the radiation generating at the contraction phase will be partially absorbed by an overlying stratum at the expansion phase. The other part of radiation will be summed up and transferred to the body surface. That radiation forms an outer electromagnetic field and is equilibrated by interaction with the outer radiation flax. The rest of the non-equilibrated and more energetic in the spectrum of radiation moves to the space. The coherent radiation which reaches the boundary surface has a pertinent potential and wave spectrum depending on mass and atomic content of the interacted shells in accordance with the Moseley law. The Earth emits infrared thermal radiation in an optical short wave range of spectrum. The Sun and other stars cover the spectrum of electromagnetic radiation from radio-through optical, x and gamma ray of wave ranges. The observed spectra of star radiation show that total mass of a body takes part in generation and formation of surface radiation. According to the accepted parabolic low of density distribution of the Earth, it has maximum density value near the lower mantel boundary. The value of the outer core density has jump-like fall and the inner core density seems to be uniform up to the body centre. The discussed mechanism of the energy generation is justified by the observed seismic data of density distribution. It is assumed that the excess of generated electromagnetic energy from the outer core comes to the inner core and keeps there the pressure of dynamical equilibrium at the body pulsation during the entire time of the evolution. The parabolic distribution of density seems to be characteristic for most of the celestial bodies.

In connection with the discussed problem, it is worth considering the equilibrium conditions between radiation and matter on the body boundary surface.

7.5 The Nature of the Star-Emitted Radiation Spectrum

We assume that the Novas and Supernovas after explosion and collapse pass into neutron stars, white dwarfs, quasars, black holes and other exotic creatures which emit electromagnetic radiation in different ranges of the wave spectrum. The effects we have discussed in this book are based on dynamical equilibrium evolution of self-gravitating celestial bodies that allow the exotic stars to be interpreted from a new position. We consider the observed explosions of stars as a natural logical step of evolution related to their mass differentiation with respect to the density. The process is completed by separation of the upper 'light' shell. At the same time, the wave parameters of the generated energy of the star after shell separation are changed because of changes in density and atomic contents. As a result, the frequency intensity and spectrum of the coherent electromagnetic radiation on the boundary surface are changing. For example, instead of radiation in an optical range the coherent emission in x or gamma ray range takes its place. But the body's dynamical equilibrium should remain during all the time of evolution. The loss of the upper body shell leads to decrease of the angular velocity and increase of the oscillation frequency. The idea of the star gravitational collapse seems to be an effect of the hydrostatic equilibrium theory (Ferronsky 2005).

As to the high temperature on the body surface, the order of which from Rayleigh –Jeans' equation is 10^7 K and more, then in our interpretation as applying Eq. (6.22) for evolution of a star of solar mass at the electron phase (Fig. 7.1), the limiting temperate is $T_0 \to \mu_e c^2/3k$ or (Ferronsky et al. 1996)

$$3kT_0 \to \mu_e c^2 \approx 0.5\,\mathrm{MeV},$$

$$T \approx 5 \times 10^9\,\mathrm{K}.$$

This means that on the body surface the gas approaches to the electron temperature, because the velocity of its oscillating motion runs to c.

The energy is a quantitative measure of interaction and motion of all the forms of the matter. In accordance with the law of conservation, the energy does not disappear and does not appear itself. It only passes from one form to another. For a self-gravitating body the energy of mechanical oscillations, induced by the gravitational interactions, passes to electromagnetic energy of the radiation emission and vice versa. The process results by the induced quantum transition of the energy levels and their inversion. Here transition of the gravitational energy into electromagnetic and vice versa results in a self-oscillating regime. In the outer space of the body's border, the emitted radiation energy forms the equilibrium electromagnetic field. The non-equilibrium part of the energy in corresponding wave range of the spectrum is irradiated to outer space. The irreversible loss of the emitted energy is compensated by means of the binding energy (mass defect) at the fission and fusion of molecules, atoms and nuclei. The body works in the regime of a quantum generator. Those are conclusions that followed from the theory based on body dynamical equilibrium.

In conclusion, we wish to stress that the artificial satellites relationship we discovered between the gravitational field (potential energy) and the polar moment of inertia of the Earth leads to understanding the nature and mechanism of the planet's energy generation as the force function of all the dynamical processes that are released in the form of oscillation and rotation of the matter. Through the energy of nature we understand the unity of forms of the gravitational and electromagnetic interactions which, in fact, are the two sides of the same natural effect.

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Chapter 8
Creation and Decay of a Hierarchic Body System by Centrifugal Effects of the Potential Field Energy Interaction

Abstract All small and large celestial bodies appear to present some clots of energy in the form of condensed discrete infinitesimal particles. In this case, at some stage of the Universe's evolution, there was a common or a number of smaller clots of matter-energy. Once created, they started to decay. After decay they are created again. This phenomenon looks like the water cycle in nature, during which the initial "dark" energy is converted into condensate. The main part of that "dark" energy remains in the form of the background or of the force field. During the Universe expansion, which we observe, the initially condensed energy by the inner pressure is also expanding and emits energy in the form of discrete infinitesimal weightlessness particles. On the basis of the above-considered analysis of dynamical effects related to the origin of the Solar System bodies, one may note that the basic point of the process of initial condensate decay is the interaction of elementary particles and energy loss in the form of radiation emission. The interaction of particles results by their collision and crushing, which looks like acceleration and collision of the protons in a collider. The radiation is a flux of weightlessness with respect to the given body, particles. These particles have mass but it is defective, i.e. weightlessness with respect to the body matter. The radiated energy is the basis and content of a celestial body evolution. The loss of energy finally leads to differentiation of matter in density and to shell separation. The outer shell appears to be most light in density and at some stage of evolution its inner force field overtakes the weightless threshold with respect to the parental body. This is the way of outer shell separation and creation of a secondary body. Creation of a hierarchic subsystem of bodies like galaxy, star, planet and satellite in the scale of the whole Universe, in fact, is the process of body's weighted matter decay. The Universe expansion is the observable fact and the evidence of creation and decay of the weightless and weightlessness matter of the same energy by means of oscillating motion. As to the natural forces taking part in the evolutionary processes then, as it follows from the above-discussed problems, the weight (energy) of elementary particle force seems to be enough. All the other known forces appear to be the effects of energy interaction of elementary particles. During recent years, the scenario of the Big Bang has been discussed widely in connection with the Universe's origin. From the viewpoint of Jacobi's dynamics, the idea of a Big Bang corresponds to the stage of expansion in framework of Jacobi's pulsating

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model of the Universe. The experimental research is developed by the collider in CERN in search of the Higgins' boson, which is an elementary particle in the quantum field of the English physicist Peter Higgs. There is information about evidence of existence of such a scalar particle with a mass equivalent to ~ 125 GeV of energy. It is assumed that this is a fundamental particle of the Universe's creation at the Big Bang. If one accepts the idea of existence of the Universe's origin, then in the framework of Jacobi's dynamics its expansion should have a physical limit in time. This limit should be reached when all the hierarchic subsystems of bodies are decayed up to elementary scalar particles. After that, the stage of attraction (fall down) of the particles will start. The attraction of mass particles (electrons, nuclei of known and unknown elements) should continue up to their turn to expansion. The attraction process will be finished when inner pressure in the Universe's inner and outer fields come to equilibrium. After that, because of mass particle energy radiation, the process of mass particles decay, the expansion and creation of the hierarchic subsystem bodies will start again. Note that during the stage of Universe attraction, the interacting elementary particles at their synthesis into mass particles (electrons, nuclei and molecules) will absorb energy in the form of mass defects, which are used for binding the nuclei components on the basis of the equality of frequency oscillation. The process of the decay up to the level of elementary particles and attraction up to the stage of galaxies composed by atoms and molecules can continue infinitely long. In the light of a possible scenario of decay and creation of the Universe, the problem of creation of weighted mass particles (electrons and nuclei of atoms) by synthesis of elementary particles is of interest. In the framework of Jacobi's dynamics this problem, based on the effect of simultaneous collision of n particles, has a mathematical solution and is presented in this chapter.

Thus, we discovered an interesting natural phenomenon. All the small and large celestial bodies appear to be some clots of energy in the form of condensed discrete infinitesimal particles. It is possible to assume that at some initial stage, there was a common or a number of smaller clots of the matter-energy. Once created, they decayed. After decay they are created again. This phenomenon looks like the water cycle in nature, during which, as the water moisture, not all the initial "dark" energy is converted into condensate. The main part of that "dark" energy remains in the form of the background or of the force field. During the universe expansion, which we observe, the initially condensed energy by the inner pressure is also expanding and emits energy in the form of discrete infinitesimal particles.

On the basis of the above-considered analysis of dynamical effects related to the origin of the Solar System bodies, one may note that the basic point of the process of initial condensate decay is the interaction of elementary particles and energy loss in the form of radiation emission. The interaction of particles results by their collision and crushing, which looks like acceleration and collision of the protons in the collider. The radiation is a flux of weightlessness with respect to the given body particles. These particles have mass but it is defect, i.e. weightlessness with respect to the body matter. The radiated energy is the basis and content of a celestial body evolution. The loss of energy finally leads to differentiation of matter in density and to shell separation. The outer shell appears to be most light in density and at some stage of evolution its inner force field overtakes the weightless threshold with respect to the parental body. This is the way of the outer shell separation and creation of secondary body. Creation of the hierarchic subsystem of bodies like galaxy, star, planet, satellite in the scale of the whole universe, in fact, is the process of body's weighted matter decay. The universe's expansion is the observable fact and the evidence of creation and decay of the weightless and weightlessness matter of the same energy by means of oscillating motion.

The question is arising as to how long the Universe's expansion will continue. There are two options in the answer. The process will continue infinitely long or there is a time and physical limit for it. Infinite expansion needs infinite energy. There should be infinite number of Universes in the case of limiting our expansion. Or there is some new model of space geometry. We have no data for discussion of a problem like this.

In recent years, the scenario of the Big Bang in connection with the Universe's origin is widely discussed. From the viewpoint of Jacobi's dynamics, the idea of Big Bang corresponds to the stage of expansion in Jacobi's pulsating model of the Universe. The fundamental experimental research was developed by the collider in CERN in search of the Higgs' boson, which is an elementary particle in the quantum field of the English physicist Peter Higgs. There is information about evidence of existence of such scalar particles with a mass equivalent to ~ 125 GeV of energy. It is assumed that this is a fundamental particle of the Universe's creation according to the Big Bang theory.

If one accepts the idea of the existence of an oscillating Universe, then in the framework of Jacobi's dynamics its expansion should have a physical limit in time. This limit should be reached when all the hierarchic subsystems of bodies will have decayed up to elementary scalar particles. After that, the stage of attraction (fall down) of the particles will start. The attraction of mass particles, electrons, nuclei of known and unknown elements should continue up to their turn to expansion. The attraction process will be finished when inner pressure in the Universe's inner and outer fields will come to equilibrium. After that, because of mass particle energy radiation, the process of mass particles decay and the expansion and creation of the hierarchic subsystem bodies will start again.

Note that during the stage of universe attraction, the interacting elementary particles at their synthesis into mass particles (electrons, nuclei, molecules) absorb energy in the form of mass defect for binding of the nuclei components on the basis of the equality of frequency oscillation.

The process of decay up to the level of elementary particles and attraction up to the stage of galaxies composed by atoms and molecules can continue infinitely long.

In the light of a possible scenario of decay and creation of the Universe, the problem of creation of weighted mass particles (electrons and nuclei of atoms) by synthesis of elementary particles is of interest. In the framework of Jacobi dynamics, this problem is based on the effect of simultaneous collision of n particles; it has a mathematical solution and is presented below.

8.1 Relationship of the Jacobi Function and Potential Energy at Simultaneous Collision of n Particles

In the previous chapters, we have considered some general approaches to the formulation and solution of Jacobi's dynamics problems connected with the evolutionary processes of celestial bodies. For this purpose, we have transformed Jacobi's virial equations for conservative and non-conservative systems:

$$\ddot{\Phi} = 2E - U, \tag{8.1}$$

$$\ddot{\Phi} = 2E - U + X(t, \Phi, \dot{\Phi}) \tag{8.2}$$

into equations of virial oscillations in the form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \tag{8.3}$$

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + X(t, \Phi, \dot{\Phi}). \tag{8.4}$$

The transfer from Eqs. (8.1) and (8.2) to Eqs. (8.3) and (8.4) has been made by using the following relationship between the Jacobi function and potential energy:

$$U\sqrt{\Phi} = B = \text{const.} \tag{8.5}$$

As shown in Chap. 4, the validity of the relationship (8.5) for explicitly solved cases of the many-body problem in mechanics and physics is an obvious fact. Consequently, for example, in the case of the two-body problem which represents the conservative system, the solutions of Eq. (8.3) will be analogous to Keplerian equations of conic sections according to which the Jacobi function (or potential energy) changes with time. In the same manner, the solution of the generalized equation of virial oscillations (8.4) in celestial mechanics will correspond to a solution for the periodic motion in the two-body problem obtained by perturbation theory methods.

The validity of Eq. (8.5) for a many-body system, including the problem of synthesis of the mass points at simultaneous collision of n elementary particles, in the general case is not obvious despite the fact that both volumetric integral characteristics considered are functions of the distribution of mass density of a system.

In this chapter, we consider in detail the main physical aspect of the relationship between the Jacobi function and the potential energy of a system.

8.2 Asymptotic Limit of Simultaneous Collision of Mass Particles for a Conservative System

We take advantage of the results presented by Wintner (1941) in order to study the many-body problem. From such a study it follows that, within a conservative system of n mass points of arbitrary configuration interacting according to Newton's law, the following statement is valid.

If the motion of the material points of a system of arbitrary configuration has the consequence that all of them tend to simultaneous collision, then the relationship $U\sqrt{\Phi}$ approaches a constant value. This result obtained by Wintner supplements the general properties of conservative systems of material points interacting according to Newton's law when their number remains constant all the time. The condition of constancy of the number of mass points of a system is equivalent to that of the distance $\Delta_{ij} = |r_i - r_j|$ between any pair of points at any moment of time and should be $\Delta_{ij} > 0$, where r_i and r_j denote the 3-vectors of the co-ordinates of mass points in the barycentric co-ordinate system.

For such a system, from the analysis of Jacobi's virial Eq. (8.1) and the expression for the Jacobi function Φ,

$$\Phi = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j \Delta_{ij}^2, \tag{8.6}$$

for kinetic energy T

$$T = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j (\dot{r}_i - \dot{r}_j)^2, \tag{8.7}$$

and for potential energy U

$$U = -G \sum_{1 \le i < j \le n} \frac{m_i m_j}{\Delta_{ij}}. \tag{8.8}$$

Three inequalities were obtained that produce restrictions on the Jacobi function (or potential energy) and its derivatives. These inequalities can be written in the form

$$|\dddot{\Phi}| \le \eta (|\ddot{\Phi}| + 2|E|)^{5/2}, \tag{8.9}$$

$$(\ddot{\Phi} - 2E)\Phi^{1/2} \ge \mu > 0, \tag{8.10}$$

$$\ddot{\Phi} - E - \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi} \ge \frac{M^2}{4\Phi}, \tag{8.11}$$

where constants

$$\begin{aligned}
\eta &= \frac{\sqrt{2m}}{G} \sum_{1 \le i < j \le n} (m_i m_j)^{-3/2} > 0, \\
\mu &= \frac{G}{\sqrt{2m}} \sum_{1 \le i < j \le n} (m_i m_j)^{3/2} > 0, \\
M^2 &= C_1^2 + C_2^2 + C_3^2, \\
m &= \sum_{i=1} m_i,
\end{aligned}$$

where m_i is the mass of the ith point; $E = T + U$ is the total energy; C_1, C_2 and C_3, are projections of the angular momentum M on the axes.

The third inequality (8.11) is more complicated than the others as it contains the value M of the constant angular momentum besides the constant E, which is the total energy of the system.

It has been shown by Wintner (1941) that if the motion of material points of an arbitrary configuration system provides their simultaneous collision, then the system possesses zero angular momentum and a simultaneous collision will occur in a finite interval of time. In addition, the behaviour of the Jacobi function in the vicinity of the time moment t_o of simultaneous collision is defined by the following asymptotics:

$$\Phi \propto (t - t_0)^{4/3}, \tag{8.12}$$

$$\Phi \propto (t - t_0)^{1/3}, \tag{8.13}$$

$$\Phi \propto (t - t_0)^{-2/3}. \tag{8.14}$$

Following Wintner (1941), we introduce the definition of a central configuration which is needed for further consideration of the problem. If the positions of the material points in the system are such that the following relation is satisfied:

$$U_n = \sigma m_i r_i, \tag{8.15}$$

then the configuration of the system is called central.

Here, in Eq. (8.15)

$$\sigma = -\frac{U}{2\Phi}.$$

The definition (8.15) of the central configuration can be rewritten in equivalent form

$$(U^2\Phi)_{\mathrm{n}} = 0. \tag{8.16}$$

As proved by Wintner (1941), the important relation follows from asymptotics (8.12)–(8.14) at $t \rightarrow t_o$:

$$(U^2\Phi)_n \rightarrow 0 \tag{8.17}$$

which, together with the definition for the central configuration, leads to the following theorem.

Any arbitrary configuration of material points in the asymptotic time limit of simultaneous collisions of all the mass points tends to the central configuration.

It follows from this that

$$\lim_{t \rightarrow t_0} |U|\sqrt{\Phi} = \mathrm{const.} \tag{8.18}$$

This theorem (8.18) justifies the transformation of Jacobi's virial Eqs. (8.1) and (8.2) into equation of virial oscillations (8.3) and (8.4) within the framework of Newton's law of interaction of material points of a conservative system.

8.3 Asymptotic Limit of Simultaneous Collision of Mass Particles for a Non-conservative System

The model of a conservative system permits a limited number of problems to be solved. In reality, all natural systems are non-conservative. Study of the dynamics of such systems is the main object of the problem of evolution.

It is well-known from the observations described in the general course of physics by Kittel et al. (1965) that the gravitating systems in nature are contracting while losing part of their total energy through friction and electromagnetic radiation. From the kinematics point of view, this gravitational contraction is equivalent to the simultaneous collision of all n mass points of the system. We consider below the validity of the theorem expressed by Eq. (8.18) for non-conservative systems.

Let the motion of a system of n mass points occur by means of the gravitational interaction and Newtonean friction of the mass points. Then Jacobi's virial equation can be written as

$$\ddot{\Phi} = 2E(t) - U(t) - k\dot{\Phi}, \tag{8.19}$$

where $E(t)$ is the value of the total energy of the system at the moment of time t.

From analyzes of the equations of motion resulting in (3.23) it follows that

$$E(t) = E_0 - 2k\int\limits_{t_0}^{t} T(t)\mathrm{d}t = E_0[1 + q(t)],$$

where E_0 is the value of the total energy of the system at the initial moment of time t_o; $q(t)$ is a monotonically increasing function of time.

We also accept the condition of the constancy of the number of mass particles in the system, from which it follows that the distance between any pairs of points $\Delta_{ij} > 0$ and the following relation is correct:

$$|\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{ij}| \leq |\dot{r}_i - \dot{r}_j|.$$

In the framework of this essentially important condition which forbids paired, threefold and higher-fold collisions, we obtain three inequalities analogous to (8.9), (8.10) and (8.11). The inequalities are valid at any stage of the system's evolution and place restrictions on the Jacobi function and its derivatives.

From expression (8.8) for the potential energy of the system, the following inequalities can be written:

$$|\dot{U}| \ = |G\sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\Delta_{ij}^2}\dot{\Delta}_{ij}| \leq G\sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\Delta_{ij}^2}\left|\dot{r}_i - \dot{r}_j\right| \tag{8.20}$$

and

$$\frac{Gm_i m_j}{\Delta_{ij}} < -U,$$

where r_i and r_j are three vectors of co-ordinates of mass points in the barycentric co-ordinate system.

Substituting the last inequality into (8.20) we obtain

$$|\dot{U}| \leq \frac{U^2}{G}\sum_{1 \leq i < j \leq n} \frac{|\dot{r}_i - \dot{r}_j|}{m_i m_j}.$$

Since

$$m_i m_j(\dot{r}_i - r_j) \leq 2mT,$$

and assuming

$$\eta = \frac{1}{G}\sum_{1 \leq i < j \leq n} \frac{m^{1/2}}{(m_i m_j)^{3/2}},$$

we obtain

$$|\dot{U}| \leq U^2 \eta (2T)^{1/2}. \tag{8.21}$$

Then, using Eq. (8.19) in the form

$$U = 2E_0[1 + q(t)] - \ddot{\Phi} - k\dot{\Phi} \tag{8.22}$$

and the law of conservation of energy for a dissipative system

$$U + T = E_0[1 + q(t)], \tag{8.23}$$

we rewrite the inequality (8.21) in the form

$$\begin{aligned} |\, \dot{U} \,| &\leq \left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}\} + k|\dot{\Phi}|\right\}^2 \\ &\quad \times \eta\sqrt{2}\left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}\} + k|\dot{\Phi}|\right\}^{1/2} \\ &= \sqrt{2}\eta\left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}\} + k|\dot{\Phi}\right\}^{5/2}. \end{aligned} \tag{8.24}$$

Differentiating (8.22) with respect to time and substituting this into (8.24), we finally obtain the first inequality:

$$|\dddot{\Phi} + k\ddot{\Phi} - 2E_0\dot{q}(t)| \leq \sqrt{2}\eta\left\{2|E_0|[1 + q(t)] + |\ddot{\Phi}| + k|\dot{\Phi}|\right\}^{5/2}. \tag{8.25}$$

In the same way, it follows from (9.6) that

$$\Phi^{1/2} \geq \frac{1}{(2m)^{1/2}}(m_i m_j)^{1/2}\Delta_{ij}.$$

Then

$$\frac{\Phi^{1/2} m_i m_j}{\Delta_{ij}} \geq (2m)^{1/2}(m_i m_j)^{1/2}.$$

By virtue of (8.4) and (8.8),

$$\ddot{\Phi} + k\dot{\Phi} - 2E[1 - q(t)] = G \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\Delta_{ij}}.$$

The second inequality has the form

$$\ddot{\Phi} + k\dot{\Phi} - 2E[1 + q(t)]\Phi^{1/2} \geq \mu > 0, \tag{8.26}$$

where

$$\mu = \frac{G}{(2m)^{1/2}} \sum_{1 \le i < j \le n} (m_i m_j)^{3/2}.$$

Now, let us derive the third inequality following from the Cauchy–Bunyakovsky inequality, which is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Since

$r_i^2 = | r_i |^2$ и $| (r_i \cdot \dot{r}_i) | = (| r_i | \cdot |\dot{r}_i|)$,

and from the definition of the Jacobi function, one obtains

$$\Phi = \sum_{i=1}^{n} m_i(|r_i| \cdot |\dot{r}|).$$

Applying the Cauchy–Bunyakovsky inequality to this expression at

$$a_i = m_i^{1/2}|r| \text{ and } b_i = m_i^{1/2}|r|,$$

we can write

$$\dot{\Phi}^2 \le 2\Phi \sum_{i=1}^{n} m_i|\dot{r}_i|^2 = 2\Phi \sum_{i=1}^{n} \frac{m_i(r_i \cdot \dot{r}_i)^2}{r_i^2}.$$

Assuming

$$a_i = m_i^{1/2}|r_i|, \quad A_i = \frac{m_i^{1/2}[r_i X \dot{r}_i]}{|r_i|},$$

the vector of the angular momentum M is

$$M = \sum_{i=1}^{n} a_i A_i.$$

Then, in a similar may we write

$$M^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} A_i^2\right) \equiv 2\Phi \sum_{i=1}^{n} \frac{m_i[r_i X \dot{r}_i]}{r_i^2}.$$

The addition of the last two inequalities yields

$$\dot{\Phi}^2 + M^2 \le 2\Phi \sum_{i=1}^{n} \frac{m_i\left\{(r_i \cdot \dot{r}_i)^2 + [r_i X \dot{r}]^2\right\}}{r_i^2}.$$

But since

$$\left\{(r_i \cdot \dot{r}_i)^2 + [r_i X \dot{r}]^2\right\} = r_i^2 \cdot \dot{r}_i^2,$$

we have

$$\dot{\Phi}^2 + M^2 \le 2\Phi \sum_{i=1}^{n} m_i \dot{r}_i.$$

As Jacobi's equation can be written in the form

$$\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)] = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2,$$

after substitution of this into the right-hand side of the last inequality, we obtain

$$\dot{\Phi}^2 + M^2 \le 4\Phi\left\{\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)]\right\}.$$

Hence, the third inequality can be written

$$\ddot{\Phi} + k\dot{\Phi} - E_0[1 + q(t)] - \frac{\dot{\Phi}^2}{4\Phi} \ge \frac{M^2}{4\Phi}. \tag{8.27}$$

Let us now analyze the behaviour of the Jacobi function Φ and its derivatives. For this purpose, we introduce the auxiliary function $Q = Q(t)$, equal to

$$Q = k\dot{\Phi}\Phi^{1/2} - E_0[1 + q(t)]\Phi^{1/2} + \frac{1/4\dot{\Phi}^2 + 1/4M}{\Phi^{1/2}}, \tag{8.28}$$

where $\Phi^{1/2} > 0$.

Then, differentiating (8.28) and using

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^{1/2}\right) = \frac{\dot{\Phi}}{2\Phi^{1/2}},$$

we obtain

$$\dot{Q} = \frac{1}{2}\frac{\dot{\Phi}}{\Phi^{1/2}}\left\{\ddot{\Phi} + k\dot{\Phi} - E_0[1+q(t)] - \frac{1}{4}\frac{M^2}{\Phi} - \frac{1}{4}\frac{\dot{\Phi}^2}{4\Phi}\right\} + \Phi^{1/2}\left[k\ddot{\Phi} - E_0\dot{q}(t)\right],$$

where $\Phi^{1/2} > 0$, $\dot{q}(t) > 0$ and, in agreement with (8.27),

$$\left\{\ddot{\Phi} + k\dot{\Phi} - E_0[1+q(t)] - \frac{1}{4}\frac{M^2}{\Phi} - \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi}\right\} \geq 0.$$

Let t_0 be the time of simultaneous collision of all the particles of the system. Then, for $t \rightarrow t_o$ ($t < t_o$), $\Phi \rightarrow 0$. Let us show that the necessary condition for existence of such t_0 for which $\Phi \rightarrow 0$ (if $t \rightarrow t_o$) is that the constant angular momentum M must be zero.

Note that if, for $t \rightarrow t_o$, $\Phi \rightarrow 0$, then all mutual $\Delta_{ij} = |r_i - r_j|$ also tend to zero, and the potential energy $U \rightarrow -\infty$.

Since

$$\ddot{\Phi} = 2E_0[1+q(t)] - U - k\dot{\Phi},$$

where E_0 = const, $|\dot{\Phi}| \rightarrow \infty$, $|q(t)|$, $|\dot{q}(t)| < \infty$, then, for $t \rightarrow t_o$, $\ddot{\Phi} \rightarrow \infty$. Thus, for t sufficiently close to t_0 we have $\ddot{\Phi} > 0$ and therefore the derivative $\dot{\Phi}$ increases and does not change its sign. Since $\Phi > 0$ and $\Phi \rightarrow 0$, then Φ is a monotonically decreasing function. It therefore follows from the expression for $\dot{Q}$ that the function Q in (8.28) for t sufficiently close to t_0 must decrease and its time limit for $t \rightarrow t_o$ might be $-\infty$, but cannot be $+\infty$. Moreover, it follows from the above statement that for $t \rightarrow t_0$ the limit of function (8.28) is

$$\lim_{t\rightarrow t_0} Q = \lim_{t\rightarrow t_0} \frac{1}{4}\frac{\dot{\Phi}^2 + M^2}{\Phi^{1/2}}, \tag{8.29}$$

but since $\Phi^{1/2} > 0$, the time limit (5.29) must be finite and non-negative. Hence, for $t \rightarrow t_o$ and $\Phi \rightarrow 0$ the value $M^2/\Phi^{1/2}$ must remain limited. Therefore, since M^2 = const, then $M \equiv 0$ and proof is completed.

The above analysis shows that, at $t \rightarrow t_o$ $\ddot{\Phi} \rightarrow \infty$, and it therefore follows from (8.25) that

$$|\dddot{\Phi} = 2E_0\dot{q}(t) + k\ddot{\Phi}| \leq \text{const}\left(|\ddot{\Phi}| + k|\dot{\Phi}|\right)^{5/2}. \tag{8.30}$$

Using the second inequality (8.26), it can be shown that if t_o is the time moment of simultaneous collision of all the particles of the system, then as $\Phi^{1/2} > 0$ at $t \rightarrow t_o$, the ratio $\dot{\Phi}/\Phi^{1/2}$ tends to a finite and positive limit.

In fact, as has been shown above, the limit (8.29) of the function (8.28) for $t \longrightarrow t_o$ has a finite value. Since $M = 0$,

$$\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}$$

will also be finite and non-negative. Let us show that this limit cannot be equal to zero.

Since for $t \longrightarrow t_o$, $M = 0$, $\Phi^{1/2} \longrightarrow 0$, then the function (8.28) and its limit (8.29) may be written in the form

$$Q = k\dot{\Phi}\Phi^{1/2} - E_0[1 - q(t)]\Phi^{1/2} + \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi^{1/2}}, \tag{8.31}$$

$$\mu_0 = \frac{1}{4}\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}, \tag{8.32}$$

where

$$\mu_0 = \lim_{t-t_0} Q\,.$$

From (8.31) we find that

$$2Q\Phi^{1/2} = k\dot{\Phi}\Phi - 2E_0[1 - q(t)]\Phi + \frac{1}{2}\dot{\Phi}^2.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(2Q\Phi^{1/2}\right) = \ddot{\Phi}\dot{\Phi} + k\ddot{\Phi}\Phi + 2k\dot{\Phi}^2 - 2E_0[1 - q(t)]\dot{\Phi} - 2E_0\Phi\dot{q}.$$

Let us carry out the integration between the limit t_o and $\bar{t}$ of the last relation where t_o has a fixed value and $\bar{t} \longrightarrow t_o$. We take into account that

$$\lim_{t-t_0} \Phi^{1/2} = 0,$$

$$\mu_0 = \frac{1}{4}\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}} < \infty.$$

Then, we write

$$2Q\Phi^{1/2} = \int_{t_0}^{\bar{t}} \left\{\left[\ddot{\Phi} - 2E_0(1 - q(t)) + 2k\dot{\Phi}\right]\dot{\Phi} + \left[2k\ddot{\Phi} - 2E_0\dot{q}\right]\Phi\right\}\mathrm{d}t.$$

As shown above, the derivative Φ retains its sign in the sufficiently small neighbourhood of point t_o. Since $\Phi \geq 0$ and $q > 0$, the positive constant μ in the inequality (8.26) will be such that in the sufficiently small neighbourhood of t_o we have

$$2|Q|\Phi^{1/2} \geq \int_{t_0}^{\bar{t}} \left\{ \frac{\mu}{\Phi^{1/2}} \dot{\Phi} + \left[2k\ddot{\Phi} - 2E_0\dot{q} \right] \Phi \right\} \mathrm{d}t.$$

The first integral to the right of this inequality being equal to $2\mu\Phi^{1/2}$ and $\Phi^{1/2} \to 0$ with $t \to t_o$, then, in the sufficient small neighbourhood of t_o, we have

$$2|Q|\,\Phi^{1/2} \geq 2\mu\Phi^{1/2} \quad \text{или} \quad |Q| \geq \mu.$$

Since $\mu > 0$, and taking into account the existence of the time limit (8.32), we have finished the proof of correctness of the inequality

$$\lim_{t \to t_0} \left(\frac{\dot{\Phi}}{\Phi^{1/2}} \right) > 0.$$

The above analysis allows us to obtain the following asymptotic relations for the Jacobi function when $t \to t_o$.

Since the limit

$$\mu_0 = \frac{1}{4} \lim_{t \to t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}$$

has a non-zero value, the function $\Phi = \Phi(t) > 0$ tends to zero as $t \to t_o$ in such a way that, in the neighbourhood of t_o, it is proportional to $(t - t_o)^{4/3}$ with a coefficient of proportionality of $((9/4)\mu_o)^{2/3}$, and one can differentiate this asymptotic relation with respect to t. Hence, the following asymptotic relations are satisfied:

$$\Phi \propto \left(3/2\mu_o^{1/2} \right)^{4/3} (t - t_0)^{4/3}, \tag{8.33}$$

$$\Phi \propto \left(12\mu_o^2 \right)^{1/3} (t - t_0)^{1/3}. \tag{8.34}$$

In fact, (8.34) follows from (8.33) not only from groundless differentiation, but actually from (8.33), if (8.32) is taken into account. The asymptotic relation (8.33) itself follows from (8.32), if we write the last relation in the form

$$\pm \frac{\mathrm{d}t}{\mathrm{d}\Phi} \propto \frac{1}{2} \mu_0^{-1/2} \Phi^{-1/4}$$

and then integrate it between the limits $\Phi = 0$ and $\Phi > 0$, but sufficiently close to $\Phi = 0$. Integration (but not differentiation) of such an asymptotic relation is always as allowed procedure and hence the asymptotic relations (8.33) and (8.34) are satisfied.

Let us show that besides (8.32), (8.33) and (8.34), the following asymptotic relations are also available:

$$\mu_0 = \pm \lim_{t \to t_0} \Phi^{1/2}\ddot{\Phi}, \tag{8.35}$$

$$\Phi \propto \left(2/3\mu_o^{1/2}\right)^{2/3}(t - t_0)^{-2/3} \tag{8.36}$$

To prove relation (8.35), we multiply (8.27) by $\Phi^{1/2}$. Assuming for $t \to t_o$ and $\Phi^{1/2} \to 0$, $|E_0|(1 + q(t)) < \infty$, $M \equiv 0$ and using (8.32), we find that the lower limit $\underline{\lim}\Phi^{1/2}\ddot{\Phi} \geq \mu_0$. Since (8.35) is equivalent to (8.36), this asymptotic relation will be proved, if the upper limit $\overline{\lim}\,\Phi^{1/2}\ddot{\Phi} \leq \mu_0$.

For the proof, we assume that $F = \left(\dot{\Phi}\right)^3$, so that

$$\ddot{F} = 6\dot{\Phi}\ddot{\Phi}^2 + 3\dot{\Phi}^2\dddot{\Phi}.$$

Then, with the aid of (8.30)

$$\left|\dddot{\Phi} - 2E_0\dot{q} + k\ddot{\Phi}\right| \leq \text{const}\left(|\ddot{\Phi}| + k|\dot{\Phi}|\right)^{5/2},$$

and expressing $\dot{\Phi}$ and $\ddot{\Phi}$ through the function $\dot{F} = \Phi^3$ and $F = 3\dot{\Phi}^2\ddot{\Phi}$, we find

$$\left|\ddot{F} + 6\dot{q}(t)F^{2/3}\right| < \text{const}\frac{\dot{F}^2 + \left(|\dot{F}|\right)^{5/2}}{|F|}.$$

On the right-hand side of this inequality, we find from (8.34) where $\dot{\Phi} = F^{1/3}$ that for $t \to t_o$

$$\left|\ddot{F} + 6\dot{q}(t)F^{2/3}\right| < \text{const}\frac{\dot{F}^2 + \left(|\dot{F}|\right)^{5/2}}{t - t_0}. \tag{8.37}$$

Finally, if v_o is a positive constant equal to $m\ (12\mu_0)^2$, then for $t \to t_o$

$$F \propto v_o(t \to t_o), \tag{8.38}$$

$$\underline{\lim}\,\dot{F} \geq v_o. \tag{8.39}$$

In fact, $F = \dot{\Phi}^3$ then (8.38) is equivalent to (8.34). At the same time, by virtue of the relation $v_o = (12\mu_0)^2$, $F = \dot{\Phi}^3$, $\dot{F} = 3\dot{\Phi}^2\ddot{\Phi}$ and (8.32), the inequality (8.39) is another form of the inequality $\underline{\lim}\,\Phi^{1/2}\ddot{\Phi} \geq \mu_0$ which we have already proved.

Therefore, we are bound to prove the inequality which can be written in the form $\overline{\lim}\dot{\Phi} \le \mu_0$ by analogy with (8.39). Hence, we must prove that the asymptotic relations (8.38) and (8.39) with the aid of the 'Tauberian condition' (8.37), yields the inequality $\lim \dot{F} \le v_o$ which denotes that $F \to v_o$. From this inequality and from (8.39) the existence of the succession of time intervals follows:

$$t_1^{\mathrm{I}} < t < t_1^{\mathrm{II}}, \ldots, t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$$

which tends to t_0 as $k \to \infty$ in such a way that whenever $t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$

$$0 < \upsilon_0 < p = \dot{F}\left(t_k^{\mathrm{I}}\right) < \dot{F}(t) < \dot{F}\left(t_k^{\mathrm{II}}\right) < q \tag{8.40}$$

where p and q are some fixed numbers which are chosen between the limits $\underline{\lim}\,\dot{F}$, $\overline{\lim}\dot{F}$ ($\le\infty$) of the conditions function $\lim \dot{F}(t)$. It is obvious that we can assume that t_o = 0. If we accept const = const (p^2 + $p^{5/2}$), then for any t in any of the time intervals $t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$, by virtue of (8.37) and (8.40), we find that the following inequality holds:

$$\left|\ddot{F}(t) + 6\dot{q}(t)F^{2/3}(t)\right| < \frac{\text{const}}{|t|}.$$

Since t tends to t_o = 0, increasing or decreasing, all t_k^{I} and t_k^{II} lie on the same side of t_o = 0. Integration of the inequality (6.40) between the limits t_k^{I} and t_k^{II} yields

$$\left|\dot{F}\left(t_k^{\mathrm{II}}\right) - \dot{F}\left(t_k^{\mathrm{I}}\right) + \int\limits_{t_k^{\mathrm{I}}}^{t_k^{\mathrm{II}}} 6\dot{q}(t)F^{2/3}(t)\mathrm{d}t\right| < \text{const } \log\left|\frac{t_k^{\mathrm{II}}}{t_k^{\mathrm{I}}}\right|.$$

By virtue of (8.40) the difference $\dot{F}\left(t_k^{\mathrm{II}}\right) - \dot{F}\left(t_k^{\mathrm{I}}\right)$ is equal to a positive constant $p - q$ and

$$\int\limits_{t_k^{\mathrm{I}}}^{t_k^{\mathrm{II}}} 6\dot{q}(t)F^{2/3}(t)\mathrm{d}t > 0.$$

Hence, the limit $\log|t_k^{\mathrm{II}}/t_k^{\mathrm{I}}|$, as $k \to \infty$, is greater than a certain positive number. For this reason, when $k \to \infty$, there exists a certain positive number λ which satisfies the relation

$$\frac{t_k^{\mathrm{II}}}{t_k^{\mathrm{I}}} > \lambda > 0. \tag{8.41}$$

Then with the aid of (8.38) it follows that

$$\frac{\left|F\left(t_k^{\mathrm{I}}\right)\right|}{\left|t_k^{\mathrm{I}}\right|} \to \nu_0 \text{ and } \frac{\left|F\left(t_k^{\mathrm{II}}\right)\right|}{\left|t_k^{\mathrm{II}}\right|} \to \nu_0,$$

since

$$t_k^{\mathrm{I}} \to t_0, \quad t_k^{\mathrm{II}} \to t_0, \quad t_0 = 0, \quad \nu = 0.$$

On the other hand, if k is sufficiently large, the following inequality is valid:

$$\frac{\left|F\left(t_k^{\mathrm{II}}\right)\right|}{\left|t_k^{\mathrm{II}}\right|} \cdot \left|\frac{\left|t_k^{\mathrm{II}}\right|}{t_k^{\mathrm{I}}}\right| - \frac{\left|F\left(t_k^{\mathrm{I}}\right)\right|}{\left|t_k^{\mathrm{II}}\right|} \cdot \left|\frac{\left|t_k^{\mathrm{II}}\right|}{F\left|t_k^{\mathrm{II}}\right|}\right| > p\left|\frac{\left|t_k^{\mathrm{II}}\right|}{\left|t_k^{\mathrm{I}}\right|} - 1\right|. \tag{8.42}$$

In fact, all t_k^{I} and t_k^{II} lie on the same side of t_o, and then

$$\left|\left|t_k^{\mathrm{II}}\right| - \left|t_k^{\mathrm{I}}\right|\right| = t_k^{\mathrm{II}} - t_k^{\mathrm{I}}.$$

Since $t_k^{\mathrm{I}} \to t_0$ and $t_k^{\mathrm{II}} \to t_0$, then for sufficiently large k all $F\left(t_k^{\mathrm{I}}\right)$, $F\left(t_k^{\mathrm{II}}\right)$ have the same sign. Hence, (8.42) can be written in the form

$$\left|\left|F\left(t_k^{\mathrm{II}}\right)\right| - \left|F\left(t_k^{\mathrm{I}}\right)\right|\right| > p\ \left|\left|t_k^{\mathrm{II}}\right| - \left|t_k^{\mathrm{I}}\right|\right|,$$

and is equivalent to the inequality

$$\left|F\left(t_k^{\mathrm{II}}\right) - F\left(t_k^{\mathrm{I}}\right)\right| > p\ \left|t_k^{\mathrm{II}} - t_k^{\mathrm{I}}\right|.$$

The validity of the last inequality is obvious, since by virtue of (8.40) for $t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$ we have $\dot{F}(t) > p > o$. Therefore, inequality (8.42) also holds.

From (8.42) in the limit $k \to \infty$ and with the aid of (8.41) where $\nu_o > 0$, we obtain the inequality

$$\nu_0\left|\lambda - \nu_0\nu_0^{-1}\right| = p|\lambda - 1|.$$

Finally, by virtue of (8.41),

$$\left|\lambda - \nu_0\nu_0^{-1}\right| = |\lambda - 1| > 0,$$

and hence $\nu_o \geq$ p. At the same time, according to (8.40), $p \geq \nu_o$.

On the other hand, by virtue of (8.40) $p \geq \nu_o$. The observed contradiction that the supposition we made at the beginning $(\overline{\lim}\dot{F} \gg \nu_0)$ is false. Thus, we have proved the validity of the increase inequality $\overline{\lim}\dot{F} \leq \nu_0$ and this completes the proof of the relations (8.35) and (8.36).

Let us now show that if the motion of n points with masses m_i in the time limit $t \to t_o$ produces their simultaneous collision, then the configuration of these n particles tends to central configuration (8.15) as $t \to t_o$. In the proof, we shall use the asymptotic relations (8.33), (8.34) and (8.36) and the Tauberian lemma, which states that if the function $g(u)$ has continuous derivatives $\dot{g}(u)$ and $\ddot{g}(u)$ for $u \to \infty$ and tends, as $u \to \infty$, to a finite limit and $\ddot{g}(u) < \text{const}$, then $\dot{g}(u) \to 0$.

There is no loss of generality in assuming that $t \to t_o \to 0$, so that $t \to t_o > 0$. Then, the asymptotic relations (8.33), (8.34) and (8.36) are simply equivalent to

$$t^{-4/3}\Phi \to \mu_1 > 0, \tag{8.43}$$

$$t\left(t^{-4\dot{/}3}\Phi\right) \to 0, \tag{8.44}$$

$$t\left(t^{-4\ddot{/}3}\Phi\right) \to 0 \tag{8.45}$$

where

$$\mu_1 = \left(\frac{3}{2}\mu_0^{1/2}\right)^{4/3} \quad \text{и} \quad t \to 0.$$

Since

$$\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2,$$

it follows from (8.43) that the time limit $t \to 0$ all n mass particles collide at the origin of the barycentric co-ordinate system $OXYZ$ in such a way that, for sufficiently small t, the linear dimensions of the configuration will be proportional to $t^{2/3}$. For this reason, we eliminate this factor $t^{2/3}$ simply by multiplying the unit of length by the factor $t^{-2/3}$. Then we consider, instead of the values

$$r_i, \quad \Delta_{ij} = |r_i - r_j|,$$

$$\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2, \tag{8.46}$$

$$U = -G\sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\Delta_{ij}}, \tag{8.47}$$

the corresponding values

$$
\begin{aligned}
&\bar{r} = t^{-2/3} r_i, \quad \bar{\Delta}_{ij} = \left|\bar{r}_i - \bar{r}_j\right| = t^{-2/3}\Delta_{ij},\\
&\bar{\Phi} = t^{-4/3}\Phi = \frac{1}{2}\sum_{\mathrm{i=1}}^{\mathrm{n}} m_i \bar{r}_i^2,\\
&U = t^{2/3}U = -G\sum_{1\le i<j\le n}\frac{m_i m_j}{\Delta_{ij}}.
\end{aligned}
$$

The procedure is permissible since the definition of the central configuration is an invariant relative scale transformation of all the co-ordinates $r_i \to \delta r_i$ where δ is an arbitrary non-zero factor. Then, the relation (8.15) is invalid for the fixed $t \neq 0$, but

$$\left(\bar{\Phi}\bar{U}^2\right)_{\bar{r}_i} = 0 \tag{8.48}$$

where $I = 1, 2, \ldots, n$ in the same limit $t \to 0$.

The proof of this theorem, the mathematically precise formulation of which is expressed by (8.48), has several stages.

First, we show that in the time limit $t \to 0$

$$\frac{4}{9}\bar{\Phi} + U \to 0, \tag{8.49}$$

and

$$\bar{\Delta}_{ij} > \mathrm{const} > 0. \tag{8.50}$$

Let us introduce a time transformation, changing t to $\bar{t} = -\ln t$ in such a way to have $\bar{t} \to \infty$ for $t \to 0$. Let this transformation be

$$t = \mathrm{e}^{-\bar{t}}. \tag{8.51}$$

Then, if the arbitrary function f depends on time t, we have

$$t\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{\mathrm{d}f}{\mathrm{d}t}, \tag{8.52}$$

$$t^2\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \frac{\mathrm{d}f}{\mathrm{d}t}. \tag{8.53}$$

With the aid of (8.51), (8.52) and (8.53) we rewrite the equation of motion

$$m_i\ddot{r}_i = -U_\eta$$

in the form

$$m_i\left(\ddot{\bar{r}}_i - \frac{1}{3}\dot{\bar{r}}_i - \frac{2}{9}\bar{r}_i\right) = -\bar{U}_{\bar{r}_i} - k\dot{\bar{r}}_i \tag{8.54}$$

where derivatives are written with respect to $\bar{t}$ and $U_{\bar{r}_i} = t^{4/3}U_{\bar{r}_i}$.

Similarly, let us rewrite the energy conservation law and Jacobi's virial equation in the form

$$\frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{\bar{r}}_i - \frac{2}{3}\bar{r}_i\right)^2 + \bar{U} = E_0[1 + q(t)]\mathrm{e}^{-2/3\bar{t}}, \tag{8.55}$$

$$\ddot{\bar{\Phi}} - \frac{5}{3}\dot{\bar{\Phi}} - \frac{4}{9}\bar{\Phi} = -U + 2E_0[1 + q(t)]\mathrm{e}^{-2/3\bar{t}}. \tag{8.56}$$

Assuming $f = \Phi$ in (8.52) and (8.53), we obtain relations which are valid in the time limit $\bar{t} \to \infty$ and similar to (8.43), (8.44), and (8.45) as $t \to 0$:

$$\Phi \to \mu_1 > 0, \tag{8.57}$$

$$\dot{\bar{\Phi}} \to 0, \tag{8.58}$$

$$\ddot{\bar{\Phi}} \to 0. \tag{8.59}$$

In the limit $\bar{\mathrm{t}} \to \infty$ from (8.56), where $E_0(1 + q(t))$ is finite, with the aid of (8.57), (8.58), and (8.59), it follows that (8.49) is valid. Moreover, it is obvious from (8.49) and (8.57) that the potential energy U tends to a finite value and hence (8.50) follows from (8.47).

Second, let us show that the time limit $\bar{t} \to +\infty$ $(t \to 0)$:

$$\dot{\bar{r}} \to 0, \tag{8.60}$$

$$\ddot{\bar{r}} < \text{const}, \tag{8.61}$$

$$\dddot{r} < \text{const}. \tag{8.62}$$

Note that (8.46) yields

$$\dot{\bar{\Phi}} = \sum_{i=1}^{n} m_i \dot{\bar{r}}_i \bar{r}_\mathrm{i}. \tag{8.63}$$

Then in the time limit $\bar{t} \to \infty$ and with the aid of (8.49) and (8.63), we obtain

$$\sum_{i=1}^{n} m_i \dot{\bar{r}}_i^2 \to 0,$$

which gives (8.60). Furthermore,

$$\bar{r} < \text{const}, \tag{8.64}$$

$$|\bar{U}_{r_i}| < \text{const}. \tag{8.65}$$

In fact, Eq. (8.64) follows from (8.57) by virtue of Eq. (8.46). At the same time, Eq. (8.65) follows from (8.47) and (8.50). Equation (8.56) follows from (8.54), (8.60), (8.64) and (8.65). Finally, by differentiating (8.56) with respect to $\bar{t}$ and then using (8.60) and (8.61), it is easy to see that for the proof of (8.62) it is sufficient to show the boundedness of the second derivatives of the functions $\bar{U}(\bar{r}_1, r_2, \ldots, r_n)$ in the time limit $t \to \infty$. But the boundedness of these derivatives follows obviously from (8.47), (8.50) and (8.64).

Finally, in accordance with (8.60) and (8.62), the Tauberian lemma is valid if we consider the function $g(u) = \dot{\bar{r}}_i$, where $u = \bar{t}$. Hence, not only $\dot{\bar{r}}_i \to 0$, but $\ddot{\bar{r}}_i \to 0$.

It follows, therefore, from (8.54) that

$$\frac{2}{9} m_i \bar{r}_i - U_{\bar{r}_i} \to 0.$$

Then, by virtue of (8.46)

$$\frac{2}{9} \bar{\Phi}_{\bar{\eta}} - U_{\bar{r}_i} \to 0.$$

From the last expression, with the aid of (8.49) and (8.57), it follows that

$$\left(\bar{\Phi}\bar{U}^2\right)_{\bar{r}_i} = \bar{\Phi}_{\bar{r}_i} U_{r_i}^2 + 2\bar{\Phi}\bar{U}\bar{U}_{\bar{r}_i} \to 0$$

and therefore

$$\left(\Phi U^2\right)_{\eta} \to 0$$

at $t \to 0$.

The last expression completes the proof of the theorem that an arbitrary non-conservative system tends to central configuration in the asymptotic limit of simultaneous collision of all its particles.

This theorem is the physical and mathematical basis for explanation of mechanism of the electrons and atomic nuclei creation, including the transuranium elements, during attraction of the universe and their decay at expansion.

8.4 Asymptotic Limit of Simultaneous Collision of Charged Particles of a System

The following analysis is given for a system consisting of a large number of charged material particles. The particles considered are positively charged nuclei of atoms and electrons.

The objective is to prove the statement that the arbitrary configuration of a system of charged particles interacting according to an inverse law (i.e. gravitational or Coulomb) in the asymptotic time limit of simultaneous collision of all the particles (for $t \longrightarrow t_o$) tends to a central configuration.

Using the definition of central configuration (8.15), (Wintner 1941), and assuming its uniqueness, the statement to be proved can be written in the form

$$\lim_{t \to t_0}\left(|U_\Sigma|\sqrt{\Phi}\right) = \text{const} \tag{8.66}$$

where $U_\Sigma = U + U_c$ is the potential energy of the system, which is equal to the sum of the gravitational potential energy of Coulomb interactions.

Using Wintner's method (Wintner 1941), we have previously studied the asymptotic time limit of (8.66) for conservative and non-conservative systems whose particles are interacting according to the law of gravitation. Since the relationship (8.66) is linear as a function of potential energy, we have to prove it only for Coulomb interactions of system particles. The proof given below for a non-conservative system is also based on Wintner's method, modified for the case of charged particles.

So, for a non-conservative system of n particles interacting according to the Coulomb law, let us write down in an inertial barycentric co-ordinate system the Jacobi function, functions of the potential and kinetic energies as well as the energy conservation law and Jabobi's virial equation as follows:

$$\Phi = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j \Delta_{ij}^2, \tag{8.67}$$

$$T = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i m_j (\dot{r}_i - \dot{r}_j)^2, \tag{8.68}$$

$$U = -G \sum_{1 \le i < j \le n} \frac{q_i q_j}{\Delta_{ij}}, \tag{8.69}$$

$$E = E(t) = E_0 - E_\gamma = T + U_c, \tag{8.70}$$

$$\ddot{\Phi} = 2E(t) - E_c \tag{8.71}$$

where $q_i = eZ_i$ is the charge of ith particle with mass m_i; $Z_i = -1, 1, +2, \ldots, N \leq n$; m is the total mass of the system; $E_\gamma < \infty$; $\dot{E}_\gamma < \infty$, that is, the total energy and the luminosity of the system at any time t are functions monotonically bounded from above.

The proof of the relationship (8.66) can easily be obtained from the asymptotic expressions for Jacobi function and its first and second derivatives as

$$\Phi \propto (t - t_0)^{4/3}, \tag{8.72}$$

$$\dot{\Phi} \propto (t - t_0)^{1/3}, \tag{8.73}$$

$$\ddot{\Phi} \propto (t - t_0)^{-2/3}. \tag{8.74}$$

where $t \longrightarrow t_o$, and t_o is the moment of simultaneous collision of the charged particles of the system.

From the expressions (8.72), (8.73, and (8.74), the limit (8.66), which we are proving, follows from exact repetition of Wintner's arguments (Wintner 1941). However, Eqs. (8.72), (8.73) and (8.74) follows from the existence of the limits

$$\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}} = \mu_0 = \text{const} > 0, \tag{8.75}$$

$$\lim_{t \to t_0} \ddot{\Phi}\Phi^{1/2} = \eta_0 = \text{const} > 0. \tag{8.76}$$

The limits (8.75) and (8.76) may be obtained in future from analysis of the Jacobi function in the neighbourhood of t_o, using the auxiliary function

$$Q = -(E - E_\gamma)\Phi^{1/2} + \frac{1}{4}\frac{\dot{\Phi}^2 + M^2}{\Phi^{1/2}}$$

and the three inequalities, correct in the most general case, that is, not especially in the close neighbourhood of the point of simultaneous collision of particles. These inequalities are

$$\left|\dddot{\Phi} + 2E_\gamma\right| \leq \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right)^{5/2}\eta_0, \tag{8.77}$$

$$\left[\ddot{\Phi} - 2(E - E_\gamma)\Phi^{1/2}\right] \geq \eta_0 > 0, \tag{8.78}$$

$$\ddot{\Phi} - E + E_\gamma - \frac{\ddot{\Phi}^2}{4\Phi} \geq \frac{M^2}{4\Phi}, \tag{8.79}$$

where M is the angular moment of the system.

Let us prove inequalities (8.77), (8.78), and (8.79) for a system of particles interacting according to Coulomb law.

To prove the inequality (8.77), it is essential that the absolute value of the total potential energy of the system of particles is less than the absolute value of the energy of mutual interactions of any pair of charged particles, that is,

$$\frac{q_i q_j}{\Delta_{ij}} \le |U_c|. \tag{8.80}$$

Since

$$\left|\dot{U}_c\right| = \left| \sum_{1 \le i < j \le n} \frac{q_i q_j}{\Delta_{ij}^2} \frac{\mathrm{d}}{\mathrm{d}t} \Delta_{ij} \right| \le \sum_{1 \le i < j \le n} \frac{\left|q_i q_j\right|}{\Delta_{ij}^2} \left|\dot{r}_i - \dot{r}_j\right|,$$

and

$$\frac{1}{\Delta_{ij}} \le \frac{|U_c|}{\left|q_i q_j\right|^2},$$

then

$$\left|\dot{U}_c\right| \le |U_c|^2 \sum_{1 \le i < j \le n} \frac{\left|\dot{r}_i - \dot{r}_j\right|}{\left|q_i q_j\right|}.$$

Analogously, since

$$m_i m_j \left|\dot{r}_i - \dot{r}_j\right|^2 \le 2mT$$

and

$$m_i \ge \frac{|q_i|}{e} \mu_e,$$

then

$$2mT \ge \frac{\left|q_i q_j\right|}{e^2} \mu_e^2 \left|\dot{r}_i - \dot{r}_j\right|^2,$$

and therefore

$$\left|\dot{U}_c\right| \le \left|\dot{U}_c\right|^2 T^{1/2} \frac{(2m)^{1/2}}{\mu_e} \sum_{1 \le i < j \le n} \frac{1}{\left|q_i q_j\right|^{3/2}},$$

where μ_e is the electron mass.

From Jacobi's equation and the law of conservation of energy it follows that

$$\begin{aligned} |\dot{U}_c| &= |\dddot{\Phi} + 2\dot{E}_\gamma|, \\ |\dot{U}_c| &\le \left(|\ddot{\Phi}| + 2|E - E_\gamma|\right), \\ |T| &\le \left(|\ddot{\Phi}| + 2|E - E_\gamma|\right), \end{aligned}$$

and finally we obtain the first inequality

$$\begin{aligned} |\dddot{\Phi} + 2\dot{E}_\gamma| &\le \left(|\ddot{\Phi}| + 2|E - E_\gamma|\right)^{5/2} \eta_0, \\ \eta_0 &= \frac{(2m)^{1/2}}{\mu_c} e \sum_{1 \le i < j \le n} \frac{1}{(q_i q_j)^{3/2}} > 0. \end{aligned}$$

The second inequality (8.78) may be derived from Jacobi's equation

$$\ddot{\Phi} - 2(E - E_\gamma) = -U_c = -\sum_{1 \le i < j \le n} \frac{q_i q_j}{\Delta_{ij}} = |U_c| \ge \frac{|q_i q_j|}{\Delta_{ij}}$$

and the inequality following from the definition of the Jacobi function:

$$\begin{aligned} 2m\Phi &\ge m_i m_j \Delta_{ij}, \\ \frac{1}{\Delta_{ij}} &\ge \frac{(m_i m_j)^{1/2}}{(2m)^{1/2} \Phi^{1/2}}. \end{aligned}$$

Thus, finally we have

$$\left[\ddot{\Phi} - 2(E - E_\gamma)\right] \Phi^{1/2} \ge \frac{|q_i q_j| (m_i m_j)^{1/2}}{(2m)^{1/2}} = \mu_0 > 0.$$

The derivation of the third inequality (8.79) is based on the Cauchy–Bunyakovsky inequality

$$\left(\sum_{1 \le i \le n}^{n} a_i b_i\right)^2 \le \left(\sum_{1 \le i \le n}^{n} a_i^2\right) \left(\sum_{1 \le i \le n}^{n} b_i^2\right).$$

Substituting into it

$$a_i = m_i^{1/2} |r_i|, \; b_i = m_i^{1/2} \frac{\mathrm{d}}{\mathrm{d}t} |r_i,|$$

we have

$$\dot{\Phi} = \sum_{1 \le i \le n} m_i |r_i| \frac{\mathrm{d}}{\mathrm{d}t} |r_i|,$$

$$\left(\dot{\Phi}\right)^2 \le 2\Phi \sum_{1 \le i \le n} m_i \frac{\left(r_i \frac{\mathrm{d}}{\mathrm{d}t} r_i\right)^2}{|r_i|^2}.$$

Substituting as before

$$a_i = m_i^{1/2} |r_i|, \; b_i = m_i^{1/2} \frac{[r_i \dot{r}_i]}{|r_i|},$$

We obtain

$$M^2 \le 2\Phi \sum_{1 \le i \le n} \frac{m_i |r_i \dot{r}_i|^2}{|r_i|^2},$$

where M is the angular momentum of the system equal to

$$M = \sum_{1 \le i \le n} m_i [r_i \dot{r}_i].$$

Summing up the two inequalities just obtained, we have

$$\begin{aligned} \left(\dot{\Phi}\right)^2 + M^2 &\le 2\Phi \sum_{1 \le i \le n} \frac{m_i}{|r_i|^2} \left\{ (r_i \dot{r}_i)^2 + [r_i \dot{r}_i]^2 \right\} = 2\Phi \sum_{1 \le i \le n} m_i (\dot{r}_i)^2 = 4T\Phi \\ &= 4\Phi \left[\ddot{\Phi} - \left(E - E_\gamma\right) \right]. \end{aligned}$$

We finally obtain an expression for the third inequality (8.79)

$$\ddot{\Phi} - E + E_\gamma - \frac{\dot{\Phi}^2}{4\Phi} \ge \frac{M^2}{4\Phi}.$$

This ends the proof of the expression (8.66) for the Coulomb interactions of charged particles of the system in the asymptotic time limit of their simultaneous collision.

8.5 Relationship Between the Jacobi Function and Potential Energy for a System with High Symmetry

It was shown in Chaps. 2 and 6 that the relationship of the Jacobi's function and the potential energy

$$|U|\sqrt{\Phi} = B \tag{8.81}$$

does not change for different mass density distribution laws and configurations of the system. In this case, the Jacobi virial equation

$$\ddot{\Phi} = 2E - U, \tag{8.82}$$

by means of (8.81) we transfer into the equation of virial oscillations:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}. \tag{8.83}$$

However, even for a system with spherical symmetry and fixed mass, the value of (8.81) changes for different laws of distribution of the mass density $\rho(r)$ (where r is the radius of the shell with density $\rho(r)$; $r \in [0, \mathrm{R}]$. In this connection, transformation of Eq. (8.82) into (8.83) is possible only after special study, which is described below.

We pay special attention to the systems with high symmetry, namely spherical and elliptical. This is because most of the natural systems from galaxies, stars, planets and their satellites, and also liquids and DT targets for carrying out the nuclear synthesis works to atoms possess such symmetry. The systems with charged particles are also included here. We consider below the conditions which allow us to transform Eq. (8.82) into (8.83) for systems with spherical and elliptical symmetry.

8.5.1 Systems with Spherical Symmetry

Let us begin by consideration of the value of Eq. (8.81) for a spherical system. It is convenient to start such a study after rewriting the expressions for the Jacobi function and the potential energy in the form

$$\Phi = \frac{1}{2}\beta^2 m R^2, \tag{8.84}$$

$$U = -\alpha \frac{Gm^2}{R}, \tag{8.85}$$

where α^2 and β^2 are dimensionless form factors independent of radius R and mass m of the spherical system (see Sect. 2.7).

We now rewrite (8.81), using (8.84) and (8.85), as

$$B = \alpha\beta G m^{5/2}. \tag{8.86}$$

Use of form factors α and β allows us to show that the parameter B in (8.81) does not depend on the radius of the spherical system. The product of α and β depends on mass density distribution law $\rho(r)$ and does not depend on the total mass of the system. Hence, the problem of the study of changes of parameter B in (8.81) for an arbitrary spherical system is reduced to consideration of the dependence of the product of the αβ form factors on the mass density distribution law for the sphere with radius unity and mass unity. Let us consider such a sphere and calculate the value

$$a = \alpha\beta. \tag{8.87}$$

For the arbitrarily given law of density distribution $\rho(k)$, $k \in [0, 1]$, satisfying the condition

$$\int\limits_{(V)} \rho(k) \mathrm{d}V(k) = 1,$$

the volume of the sphere with radius unity is

$$V = \iiint\limits_{(V)} \mathrm{d}x, \mathrm{d}y, \mathrm{d}z = \int\limits_0^1 k^2 \mathrm{d}k \int\limits_0^{\pi} \sin\theta \mathrm{d}\theta \int\limits_0^{2\pi} \mathrm{d}\varphi = \frac{4}{3}\pi.$$

The volume of the sphere with radius k is

$$V(k) = \int\limits_0^k k'^2 \mathrm{d}k' \int\limits_o^{\pi} \sin\theta \mathrm{d}\theta \int\limits_0^{2\pi} \mathrm{d}\varphi = \frac{4}{3}\pi k^3 \tag{8.88}$$

The volume of the spherical shell with radius k and thickness dk is

$$\mathrm{d}V(k) = k^2 \mathrm{d}k \int\limits_o^{\pi} \sin\theta \mathrm{d}\theta \int\limits_0^{2\pi} \mathrm{d}\varphi = 4\pi k^2 \mathrm{d}k. \tag{8.89}$$

The mass of the spherical shell with radius k and thickness dk is

$$\mathrm{d}m(k) = 4\pi\rho(k)k^2\mathrm{d}k.$$

The mass of the sphere with radius k is

$$m(k) = 4\pi\int\limits_0^k \rho(k')(k')^2\mathrm{d}k'. \tag{8.90}$$

The mass of the sphere as a whole is

$$m = 4\pi\int\limits_0^1 \rho(k)k^2\mathrm{d}k = 1. \tag{8.91}$$

The polar moment of inertia of the shell with radius k and thickness dk is

$$\mathrm{d}I(k) = k^2\mathrm{d}m(k) = 4\pi\rho(k)k^4\mathrm{d}k,$$

and the Jacobi function of the sphere is

$$\Phi = \frac{4\pi}{2}\int\limits_0^1 \rho(k)k^4\mathrm{d}k. \tag{8.92}$$

We can write the expression for the form factor β from (8.84) using (8.91) and (8.92):

$$\beta = \sqrt{\frac{\Phi}{\frac{1}{2}m}} = \frac{\sqrt{\int_0^1 \rho(k)k^4\mathrm{d}k}}{\int_o^1 \rho(k)k^2\mathrm{d}k}. \tag{8.93}$$

The potential energy of the shell with radius k and thickness dk in the gravitational field of the sphere of radius k is

$$\mathrm{d}U(k) = -G\frac{m(k)\mathrm{d}m(k)}{k} = -G\frac{16\pi^2\rho(k)k^2\mathrm{d}k\int_o^k \rho(k')(k')^2\mathrm{d}k'}{k}.$$

The potential energy of the sphere as a whole is

$$U = -16\pi^2 G\int\limits_0^1 \rho(k)k\mathrm{d}k\int\limits_0^k \rho(k')(k')^2\mathrm{d}k'. \tag{8.94}$$

We can write the expression for the form factor α using (8.95), (8.91) and (8.94) as

$$\alpha = -\frac{U}{Gm^2}\frac{\int_0^1 \rho(k)k\mathrm{d}k \int_o^k \rho(k')(k')^2\mathrm{d}k'}{\left(\int_o^1 \rho(k)k^2\mathrm{d}k\right)^2}. \tag{8.95}$$

Finally, the product of form factors α and β represents the functional of the function of mass density distribution $\rho(k)$

$$a = \alpha\beta = \frac{\int_0^1 k\rho(k)\mathrm{d}k \int_0^k \rho(k')(k')^2\mathrm{d}k' \sqrt{\int_0^1 \rho(k)k^4\mathrm{d}k}}{\left(\int_o^1 \rho(k)k^2\mathrm{d}k\right)^{5/2}}. \tag{8.96}$$

The values of the form factors α and β and of their product $\alpha^2\beta$ for different formal laws of mass density distribution are given in Table 8.1. The numerical calculations of this table can be found in our paper (Ferronsky et al. 1978).

It can be seen from Table 8.1 that the form factor β changes from 0 to 1: $\beta \in [0, 1]$. It reaches the value of unity in the case when the entire mass of the sphere is distributed within its outer shell (at $k = 1$). The minimal value of the form factor β must be when the entire mass concentrates in the centre of the sphere (at $k = 0$). But if we do not place any strong restrictions on the function $\rho(k)$, i.e. in the general case, nothing can be said about the changing interval of the value $a = \alpha\beta$ (8.85). It is only possible to note that $a = \alpha\beta$ always has a positive value. From Table 8.1, it can also be assumed that the value of a is more then $(3/5)^{3/2} \approx 0.46$, which corresponds to the homogeneous distribution of the mass density within the sphere. It is known also from Chap. 5 that the homogeneous sphere, while contracting under gravitational forces, conserves its homogeneity up to the moment of simultaneous collision of all its particles. Thus, according to Wintner's terminology, a uniform body appears to be the central configuration.

Table 8.1 Numerical values of form factors α and β and their product $\alpha^2\beta$ for various formal laws of radial mass density distribution of the spherical system

Law of mass density distribution $\rho(k)$, $k \in [0, 1]$	α	$\beta_\perp^2$	β^2	αβ
$\rho(r) = \rho_0$	0.6	0.4	0.6	0.46
$\rho(r) = \rho_0(1 - \mathrm{k})$	0.728	0.27	0.4	0.47
$\rho(r) = \rho_0(1 - \mathrm{k}^2)$,	0.7142	0.29	0.42	0.46
$\rho(r) = \rho_0(1 - \mathrm{k})^n$	$\frac{(5T+8)(T+3)^2}{8(2n+3)(2n+5)}$	$\frac{8}{(n+4)(n+5)}$	$\frac{12}{(n+4)(n+5)}$	At $n \to \infty$, 0.54
$\rho(r) = \rho_0 k^n$	$\frac{n+3}{2n+5}$	$\frac{2n+9}{6n+15}$	$\frac{n+3}{2n+5}$	At $n \to \infty$, 0.5
$\rho(r) = \rho_0\delta(1 - k)$	0.5	0.67	1.0	0.5

The sphere expands and then (the time is reversible in classical physics) becomes homogeneous again. So, in accordance with the definitions given in the previous section, the homogeneous sphere appears to be the central configuration. Applying the main idea of the central configuration theorem discussed above in the general case, we assume the following qualitative picture of the evolution of a heterogeneous spherical system. During the contraction of the system the $\alpha^2\beta$ decreases and tends to the quantity $(3/5)^{3/2}$, reaching this value at the moment of simultaneous collision of all the particles. If the expansion starts before the moment of simultaneous collision of the matter (at the neighbourhood of singularity), the value of $\alpha^2\beta$ again increases. Thus, there is a case of perturbed virial oscillations of the system. This case is known in the literature as 'stormy relaxation' of a gaseous sphere and is described quantitatively by the following equation of change of value of $|U|\sqrt{\Phi}$ (Ferronsky et al. 1984):

$$U\sqrt{\Phi} = B - k\Phi$$

where B = const, and k is also constant.

This law of change of value of $|U|\sqrt{\Phi}$ will be considered in detail in this chapter, which is devoted to astrophysics applications. Here, we only note that a mechanism that drives the matter of a system towards simultaneous collision forces a loss of energy through radiation. So, for conservative systems, the equation of virial oscillations has the form:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} - \frac{k\dot{\Phi}}{\sqrt{\Phi}}.$$

The term $k\dot{\Phi}/\sqrt{\Phi}$ is part of the perturbation function. It does not lead to the loss of total energy of the system, and we can call it internal friction.

8.5.2 Poytropic Gas Sphere Model

The laws of mass density distribution in the previous section were considered formally, neglecting the requirement of hydrodynamic stability of the system. However, it is well known that for the many really existing celestial gas bodies, a polytropic model in the central domain is a good one.

Let us study the value of the form factors α and β and their product $\alpha\beta$ for the polytropic gas sphere model at various quantities of polytropic index. The equation of state for a gas sphere is

$$\frac{\mathrm{d}p(k)}{\mathrm{d}k} = -G\frac{m(k)\rho(k)}{k^2}, \tag{8.97}$$

where $p(k)$ is the gas pressure; $\rho(k)$ is the mass density of the gas, and G is the gravitational constant.

Using Eq. (8.97) we can rewrite it for the sphere with radius k and mass m in the form

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}\left|\frac{k^2}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}\right| = -4\pi G\rho(k). \tag{8.98}$$

This is one of the basic equations in the theory of the internal structure of the stars used up to now.

It is assumed that for polytropic models, the two independent characteristics in Eq. (8.98), namely pressure $p(k)$ and mass density $\rho(k)$, are linked by the relationship:

$$p(k) = C\rho^b(k), \tag{8.99}$$

where C and b are constants.

From (8.99) it follows that

$$\frac{1}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k} = C\frac{b}{b-1}\frac{\mathrm{d}\rho^{b-1}(k)}{\mathrm{d}k}. \tag{8.100}$$

Substituting (8.100) into (8.98) and introducing specification

$$\rho^{b-1}(k) = u(k)n = \frac{1}{b-1} \tag{8.101}$$

we obtain

$$C(1+n)\frac{1}{k^2}\frac{\mathrm{d}}{\mathrm{d}k}\left|k^2\frac{\mathrm{d}u(k)}{\mathrm{d}k}\right| = 4\pi G u^n(k). \tag{8.102}$$

Eq. (8.102) can be simplified if dimensionless variables $\Theta(x) = u(x)/u_o$ and $x = \lambda k$ are introduced. Here, u_o is the value $u(k)$ in the centre of the sphere, i.e. at $k = 0$. The coefficient λ is selected with the condition that, after substitution of the function $\Theta(x)$ into (8.102), all the constants should be cancelled. Then, the following relationship for λ can be obtained:

$$C(1+n)\lambda^2 = 4\pi G u_0^{n-1} \tag{8.103}$$

and Eq. (8.102), known as the Emden equation, takes the form

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left|x^2\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\right| = -\Theta^n(x). \tag{8.104}$$

It is obvious that for $x = 0$ the function $\Theta(x)$, known as the Emden function, should satisfy two conditions

$$\Theta(x)|_{x=0} = 1, \frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}|_{x=0} = 0. \tag{8.105}$$

We now obtain the expression for the form factor α for a sphere with polytropic index n. For this purpose, we write the expression of potential energy in the form

$$U = -G \int \frac{m(k)\mathrm{d}m(k)}{k}.$$

Using Eq. (8.97) for the gas sphere and the expression for d$m(k)$, we rewrite (8.105) as follows:

$$U = \int \frac{k}{\rho(k)} \frac{\mathrm{d}p(k)}{\mathrm{d}k} \mathrm{d}m(k) = 4\pi \int k^3 \mathrm{d}p(k). \tag{8.106}$$

After integration by parts of the right-hand side of (8.105) we obtain

$$U = -12\pi \int_0^1 k^2 p(k)\mathrm{d}k. \tag{8.107}$$

On the other hand, (8.105) can be rewritten in the form

$$U = -\frac{G}{2} \int \frac{\mathrm{d}m^2(k)}{k}.$$

Integrating the right-hand side of the last relationship by parts, we obtain

$$U = -\frac{G}{2} \frac{m^2(k)}{k} |_{k=0}^{k=1} - \frac{G}{2} \int \frac{m^2(k)\mathrm{d}k}{k^2}. \tag{8.108}$$

The integral in the right-hand side of (8.108) is transformed with the help of (8.97) as follows

$$-\frac{G}{2} \int \frac{m^2(k)\mathrm{d}k}{k^2} = \frac{1}{2} \int \frac{m(k)}{\rho(k)} \frac{\mathrm{d}p(k)}{\mathrm{d}k} \mathrm{d}k.$$

Thus, using (8.100), we obtain

$$-\frac{G}{2} \int \frac{m^2(k)\mathrm{d}k}{k^2} = \frac{1}{2} \int m(k) C \frac{b}{b-1} \mathrm{d}\rho^{b-1}(k)$$

and, integrating by parts, we have

$$
\begin{aligned}
-\frac{G}{2}\int \frac{m^2(k)\mathrm{d}k}{k^2} &= \frac{1}{2}C\frac{b}{b-1}\rho^{b-1}(k)m(k)\Big|_{k=0}^{k=1} - \frac{1}{2}\int C\frac{b}{b-1}\rho^{b-1}(k)4\pi k^2\rho(k)\mathrm{d}k \\
&= -\frac{1}{2}\int (n+1)4\pi k^2 p(k)\mathrm{d}k.
\end{aligned}
\tag{8.109}
$$

Substituting (8.109) into (8.108), we obtain the second expression for the potential energy:

$$
U = -\frac{G}{2} - \frac{4\pi(n+1)}{2}\int_0^1 k^2\rho(k)\mathrm{d}k, \tag{8.110}
$$

where the condition $m(1) = 1$ has been taken into account.

Solving the system of Eqs. (8.110) and (8.107) with respect to U, we find that

$$
U = -G\frac{3}{5-n},
$$

and hence

$$
\alpha = \frac{3}{5-n}. \tag{8.111}
$$

Now, we derive the expression for the form factor β. For this purpose, we write the Jacobi function expression for a polytropic sphere

$$
\Phi = \frac{4\pi}{2}\int_0^1 k^4\rho(k)\mathrm{d}k = \frac{4\pi}{2}\int_0^{x_1}\frac{\Theta^n(x)x^4\mathrm{d}x}{\lambda^5}, \tag{8.112}
$$

where x_1 is the first root of the equation $\Theta(x) = 0$.

Let us specify

$$
\nu = \int_0^{x_1}\Theta^n(x)x^4\mathrm{d}x
$$

And, taking into account (8.103), we write

$$
C(1+n)\lambda^2 = 4\pi G u_0^{n-1}.
$$

Then

$$\Phi = \frac{4\pi\nu}{2}\frac{u_0^n}{\lambda^5} = \frac{4\pi\nu}{2}\frac{[C(1+n)]n/n-1}{(4\pi G)n/n-1}\lambda^{(5-3n)/n-1}. \quad (8.113)$$

Now, we obtain the second expression for the Jacobi function using the condition of Eq. (8.99) at the border surface of the sphere, i.e. at $k = 1$. Then

$$\frac{1}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}|_{k=1} = -\frac{Gm(k)}{k^2}|_{k=1} \quad (8.114)$$

and

$$m(k)k^2\Big|_{k=1} = -\frac{k^4}{G}\frac{1}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}|_{k=1}.$$

The left-hand side of Eq. (8.114), taking into account (8.100) and (8.101), is

$$\frac{1}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}|_{\mathrm{k}=1} = C\frac{b}{b-1}\frac{\mathrm{d}\rho^{b-1}(k)}{\mathrm{d}k} = C(n-1)\frac{\mathrm{d}u(k)}{\mathrm{d}k}. \quad (8.115)$$

Finally, we obtain

$$\begin{aligned}\Phi &= \frac{1}{2}\beta^2 m(k)k^2|_{k=1} = -\frac{1}{2}\beta^2\frac{C(n+1)}{G}k^4\frac{\mathrm{d}u(k)}{\mathrm{d}k}|_{k=1}\\ &= -\frac{1}{2}\beta^2\frac{C(n+1)}{G}u_0\frac{x^4}{\lambda^3}\frac{\mathrm{d}\Theta(x)}{\mathrm{d}k}|_{x=x_1}.\end{aligned}$$

Or when using (8.103),

$$\Phi = \frac{1}{2}\pi\beta^2\frac{C(1+n)^{n/n-1}}{(4\pi G)^{n/n-1}}\lambda^{(5-3n)/n-1}\left|x^4\frac{d\Theta(x)}{\mathrm{d}k}\right|\Big|_{x=x_1} \quad (8.116)$$

Dividing (8.116) by (8.113), we obtain

$$\beta = \sqrt{\frac{\nu}{\left[-x^4\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\right]|_{x=x_1}}}. \quad (8.117)$$

We calculated the values of α and β and their product $\alpha\beta$ using the data for ν, x_1, and

$$\frac{-x^2\mathrm{d}\Theta(x)}{\mathrm{d}x}\Big|_{x=x_1}$$

at different polytropic index values, taken from Chandrasekhar (1939, 1942). The calculated data are shown in Table 8.2. It is interesting to note that in the framework of the really existing physical laws of mass density distribution ρ(*k*), the quantity $\alpha^2\beta$ changes within the narrow limits despite the fact that each of the form factors α and β varies almost three times more the variation of the polytropic index from 0 to 3.5.

8.5.3 A System with Elliptical Symmetry

We have shown in the previous section that the property of the central configurations consisting in the constancy of the product αβ holds for a system with spherical symmetry.

Now we prove that this property holds for elliptical symmetry with an ellipsoidal mass distribution. Moreover, we show that among all the configurations, only an ellipsoidal mass distribution possesses this property of central configurations.

Let us write the equation of the general ellipsoid with semi-axes *a*, *b*, *c*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c_2} = 1, \tag{8.118}$$

where *x*, *y*, *z* are the Cartesian co-ordinates of the surface of this ellipsoid.

The equation of a set of similar ellipsoidal shells of this ellipsoid with the ellipsoidal mass distribution ρ(*x*) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c_2} = k^2, \tag{8.119}$$

where $k \in [0,\ 1]$ is a parameter of the homogeneous ellipsoidal shell.

The gravitational potential inside this ellipsoidal shell is equal to a constant at an arbitrary point (*x*, *y*, *z*),

$$F(x,y,z) = -\frac{Gm_s}{2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \tag{8.120}$$

where m_s is the mass of the shell; *u* is a parameter of integration.

We write down the form factor α_e of the potential energy *U* of this ellipsoid as

$$\alpha_c = \frac{aU}{Gm^2}, \tag{8.121}$$

Table 8.2 Numerical values of form factors α and β and their product αβ for different values of polytropic index n

Index n	α^2	x_1	$-x^2 \frac{B\Theta(x)}{\mathrm{d}x}$	ν	β	αβ
0	0.6	2.45	4.9	17.63	0.77	0.46
1	0.75	3.14	3.14	12.15	0.62	0.465
1.5	0.87	3.63	2.71	11.12	0.55	0.475
2	1.0	4.35	2.41	10.61	0.48	0.482
3	1.5	6.89	2.01	10.85	0.34	0.502
3.5	2.0	9.53	1.89	11.74	0.26	0.52

where a is a semi-major axis in the equatorial plane; m is total mass.

The volume of an ellipsoid bounded by the surface (8.119) with the parameter k is

$$V(k) = \frac{4}{3}\pi abck^3. \tag{8.122}$$

The volume of the thin shell bounded by ellipsoidal surfaces with the parameters k and $k + \mathrm{d}k$ is

$$\mathrm{d}V(k) = 4\pi abck^2\mathrm{d}k. \tag{8.123}$$

The mass of this shell is expressed as

$$\mathrm{d}m_s(k) = 4\pi abck^2\rho(k)\mathrm{d}k. \tag{8.124}$$

Then the total mass of the ellipsoid is

$$m = 4\pi abc\int_0^1 k^2\rho(k)\mathrm{d}k. \tag{8.125}$$

The mass of an ellipsoid bounded by the surface with the parameter k is

$$m(k) = 4\pi abc\int_o^k (k')^2\rho(k')\mathrm{d}k'. \tag{8.126}$$

Using the reciprocation theorem (Duboshin 1975), we write the potential energy of the ellipsoid in the form

$$U = -\int_0^1 m(k)\mathrm{d}F(k). \tag{8.127}$$

The gravitational potential inside the thin shell bounded by an elliptical surface with parameters k and k + dk (8.120) is

$$\mathrm{d}F(k) = 2\pi Gabck\rho(k)\mathrm{d}k\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \tag{8.128}$$

Now, we write the expression for the form factor α_e using the corresponding values of U and m as

$$\begin{aligned}\alpha_e &= -\frac{aU}{Gm^2} = \frac{a}{2}\frac{\int_0^1 k\rho(k)\mathrm{d}k\int_0^k (k')^2\rho(k')\mathrm{d}k'}{\left[\int_0^1 k^2\rho(k)\mathrm{d}k\right]^2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\\ &= \alpha\frac{a}{2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},\end{aligned} \tag{8.129}$$

where α is the potential energy form factor corresponding to the radial mass distribution law ρ(k).

It is easy to see from Eq. (8.129) that when $a = b$ we obtain the value of the form factor α_e for the ellipsoid of rotation

$$\alpha_e = \alpha\frac{\arcsin e}{e}. \tag{8.130}$$

Since

$$e = \sqrt{\frac{a^2-c^2}{a^2}} \in |0, 1|,$$

then

$$\alpha_e \in \left[,\frac{\pi}{2}\alpha\right].$$

When $a > b > c$, Eq. (8.129) be (Janke et al. 1960)

$$\begin{aligned}\alpha_e &= \alpha\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}\\ &= \alpha\frac{a}{\sqrt{a^2-c^2}}F\left(\arcsin\sqrt{\frac{a^2-c^2}{a^2}}, \sqrt{\frac{a^2-b^2}{a^2-c^2}}\right).\end{aligned}$$

Writing

$$\arcsin\sqrt{\frac{a^2-c^2}{a^2}}=\arcsin e_1=\varphi \quad и \quad \sqrt{\frac{a^2-b^2}{a^2-c^2}}=\frac{e_2}{e_1}=\sin\alpha=f\,,$$

we obtain

$$\alpha_e=\alpha\frac{F(\varphi,f)}{\sin\varphi}, \tag{8.131}$$

where $F(\varphi, f)$ is an incomplete elliptical integral of the first degree in the normal Legendre form. If $e_1 < 0.999$ and $0 < e_2 < e_1$, the function $F(\varphi,f)\sin^{-1}\varphi \in [1.000;$ $3.999]$ (Janke et al. 1960). When the arguments φ and f increase, the function $F(\varphi,f)\sin^{-1}\varphi$ also increases continuously.

Let us now consider the form factor β, which may be written

$$\beta=\left[\frac{\Phi}{ma^2}\right]^{1/2}. \tag{8.132}$$

Obviously, β can be obtained by over the parameter $k \in [0, 1]$, if one writes the Jacobi function for the homogeneous thin shell bounded by the surfaces within the parameters k and k + dk and with mass distribution $\rho(k)$ in the integrand.

Since the Jacobi function for a homogeneous ellipsoid with mass density ρ_o is

$$\Phi=\frac{2}{15}\pi abc\rho_0\left(a^2+b^2+c^2\right), \tag{8.133}$$

the Jacobi function for a thin ellipsoid shell may be written

$$\mathrm{d}\Phi(k)=\frac{2}{3}\pi abc\rho(k)k^4\mathrm{d}k\left(a^2+b^2+c^2\right). \tag{8.134}$$

Consequently, the Jacobi function Φ of the ellipsoid is equal to

$$\Phi=\frac{2}{3}\pi abc\left(a^2+b^2+c^2\right)\int_0^1\rho(k)k^4\mathrm{d}k. \tag{8.135}$$

Finally, using (8.135) and (8.125), Eq. (8.132) for the form factor β will be

$$\beta_e=\left[\frac{a^2+b^2+c^2}{3a^2}\frac{\int_0^1\rho(k)k^4\mathrm{d}k}{\int_0^1\rho(k)k^2\mathrm{d}k}\right]^{1/2}=\beta\left[\frac{a^2+b^2+c^2}{3a^2}\right]^{1/2}, \tag{8.136}$$

where β is a form factor of the Jacobi function of the system with radial mass distribution $\rho(k)$ and the expression

$$\left[\frac{a^2+b^2+c^2}{3a^2}\right]^{1/2} \in \left[\frac{1}{\sqrt{3}}, 1\right].$$

So, the value a_e is equal to

$$a_e = \alpha_e \beta_e = a\frac{F(\varphi, f)}{\sin\varphi}\left[\frac{a^2+b^2+c^2}{3a^2}\right]^{1/2}. \tag{8.137}$$

Now, it can be shown that the property (8.137) of the product $\alpha^2\beta$ constancy is possessed only by systems with elliptical symmetry and ellipsoidal mass density distribution. This means that for such systems the form factors α and β may be expressed as a product of corresponding for factors of the sphere and terms depending on the form of the boundary surface.

For this proof we consider an arbitrary system with a similar law of mass distribution $\rho(k)$, $k \in |0, 1|$ and the boundary surface S. Then, since we consider only one-dimensional $\rho(k)$, mass density will be constant on any surface with a fixed parameter k and similar to S. The area of this surface is

$$S'(k) = Sk^2. \tag{8.138}$$

If the volume of the body is equal to V, then the volume of the part of the body bounded by the surface $S'(k)$ is

$$V'(k) = Vk^3, \tag{8.139}$$

and its mass is

$$m(k) = V\int_0^1 k^2\rho(k)\mathrm{d}k. \tag{8.140}$$

Let us introduce the Cartesian co-ordinate system $OXYZ$ with an origin coinciding with the centre of similarity. Let us denote by h in the equatorial plane OXY the longest distance from the centre of similarity to the boundary and assume that the form factor α_e^2 of the body can be expressed as a product of the form factors of the potential energy α for the radial mass density distribution law and some term $\Delta(S)$ depending on the form of the boundary surface

$$\alpha_e = -\frac{Uh}{Gm^2} = \alpha\Delta(S) = \frac{\int_0^1 k\rho(k)\mathrm{d}k \int_0^k (k')^2\rho(k')\mathrm{d}k'}{\left(\int_0^1 k^2\rho(k)\mathrm{d}k\right)^2}\Delta(S). \tag{8.141}$$

From Eq. (8.141) we can obtain the potential energy

$$U = -\frac{Gm^2}{h}\alpha\Delta(S) = -\frac{GV}{h}\int_0^1 k\rho(k)\mathrm{d}k\Delta(S). \tag{8.142}$$

Since the terms G, V, H, $\Delta(S)$ do not depend on the parameter k, let us put them into the integrand and write

$$\frac{GV}{h}k\rho(k)\Delta(S)\mathrm{d}k = F(k).$$

Then Eq. (8.142) may be written as

$$U = -\int_0^1 m(k)\mathrm{d}F(k). \tag{8.143}$$

Comparing Eqs. (8.143) and (8.127), one can see that Eq. (8.143) is an equation for the reciprocation theorem, whose validity is based on the constancy of the gravitational potential d$F(k)$ inside the thin shell bounded by similar and similarly situated surfaces with parameters k and k + dk. But, as shown in the work of Dive (1931), where one can find rigorous proof of the reverse Newton theorem, only ellipsoidal shells possess such a property. Therefore, the body with the one-dimensional mass distribution law $\rho(k)$ for which the form factor α_e is equal to the product of the form factor of the sphere and some term depending on the form of the boundary surface $\Delta(S)$ must satisfy the equation of the ellipsoid (8.118).

8.5.4 System with Charged Particles

In Sect. 7.2, it was shown by a modelling solution that for the Coulomb interactions of charged particles, constituting a system, Eq. (8.5) holds with the same conditions as the previous models discussed above.

Considering a one-component ionized quasi-neutral and self-gravitating gaseous cloud with spherically symmetric mass density distribution, we found that the form-factors in expression for the potential energy of the Coulomb interaction have the same physical meaning, which has the gravity mass interaction. It represents the shell to which the sphere of charges is reduced.

The task about the Coulomb potential energy of the interacting charged particles proves legality of solution of the Jacobi virial equation for study of the celestial body's electromagnetic effects.

But it follows from Eq. 7.3 that the form factor α_c of the Coulomb energy becomes an infinite value when an ion's volume tends to zero, in this case the

Coulomb energy tends also to infinity. In Table 7.3 there are two laws of the mass density distribution for which the last condition holds. These laws are: $\rho(r) = \rho_o(r/R)^n$ and $\rho(r) = \rho_o(1 - r/R)^n$ at $n \to \infty$. When particles come together in the shell of infinite radius, the Coulomb interaction energy becomes infinitely large. When the mass density distribution law is $\rho(r) = \rho_0(1 - r/R)^n$, then the form factors of the gravity and Coulomb energy have a finite value. In this case, the form-factors of Jacobi's function of a system make the constant $a = \alpha_c\beta$ equal to zero. This effect can play a decisive role in evolution of the system.

We note in conclusion that the analysis of relationship between the Jacobi function and potential energy from a physical viewpoint justify transfer from Jacobi's Eqs. (8.1) and (8.2) to equations of virial oscillation (8.3) and (8.4). At the same time it is possible to meet deviation of B in Eq. (8.5) from the constant value because of small effects of perturbations, which can take part with evolution of heterogenic systems.

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Chapter 9
Conclusions

Let us summarize the obtained results of the work. The discovered nature of the planets and satellites orbital motion with the first cosmic velocity of their proto-parents prove the correctness of the introduced fundamentals for dynamics of the natural systems based on dynamical equilibrium. The conditions of hydrostatical equilibrium used earlier for such problem solution should be considered as an incorrect approach. This is precisely the reason that the nature of Newton's gravity force continues to be actively discussed even now. Solution of this problem appeared to be ordinary. Physical meaning of gravitation of a self-gravitating system and its rotation follows from the centrifugal effect of energy of its interacting elementary particles. The force of gravitation is the first derivative in time from the inner energy of the interacting elementary matter particles of the system. The energy itself being the measure of that interaction, in the general case, is the second derivative in time from the moment of inertia of the system. In the case of a uniform system, the energy is equal to the first derivative in time from the moment of inertia. That is why Einstein's equivalency principle is correct only for uniformity in density systems (see Chap. 5).

The only change of force as the measure of interaction by energy brings together the nature of gravity and electromagnetic interactions. And if one takes into account the inner and outer force fields, then both interactions become equivalent in their capacity.

It was found that the process of the planets and satellites creation by separation from the parental bodies is coupled with conditions of the Universe's expansion. It was precisely in these conditions that the hierarchic subsystems appeared. The inner energy of the parental bodies is released only in expansion conditions. In reality, the process of a subsystem creation is only a part of a more general process of decay of a system of elementary particles accompanied by release of bonded energy. The bullet points of dynamical effects of this process are self-gravitation ('weightness') and weightlessness. The process of the Universe's attraction appears to be the next stage of its evolution. If the law of energy conservation is upheld, then this stage must come. It starts with creation of mass particles including

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electrons, nuclei of atoms and their isotopes by synthesis of the elementary weightlessness scalar particles. The bullet point of the dynamic effect of this process should be the simultaneous collision of n elementary particles with creation of a mass particle. Synthesis of the scalar elementary particles will be accompanied by absorption of energy from the force field for binding the particles. The hydrostatic pressure also takes part in the process of attraction of the oscillating system in the framework of Archimedes' law.

It can be noted that the problem of creation of the solar system bodies considered here, despite its apparent complexity, is a very simple and natural process. The difficulty is in perception of the new physical conception, based on the effect of inner energy irradiation, which in fact is an observable phenomenon.

It might be surprising that the integral approach to the description of dynamics of natural systems, which has a number of obvious advantages, has been underdeveloped compared with the differential hydrostatic approach. However, if we consider the development of the apparatus of mathematical physics from this viewpoint, the picture changes completely.

In fact, as soon as the concept of the field was formulated—even though initially this concept was a purely mathematical one (e.g. of the electrostatic and magnetic fields)—Gauss' theorem relating to the flux of a field vector through a closed surface was put forward. This integral characteristic of a field enclosed within a surface is an invariant of the field. In the case of electrostatics, it is charge which gives rise to the field.

The concept of vector flux through a closed surface has been generalized and developed. For example, such a generalization is Stokes' theorem relating to the circulation of a vector around a closed circuit, which can be used to identify vortex sources in vector fields. These theorems, which by their very nature are distinctly integral ones, have served as the basis for the whole mathematical theory of continuum mechanics, the electromagnetic theory of Maxwell and Poisson's theory of Newtonian gravitation.

Thus, the development of the mathematical apparatus of physics has taken the course of the integral approach to the description of natural phenomena. The concepts of divergence and the rotor introduced in this connexion have served as instruments for finding the sources and sinks of a field and its vortices.

However, the idea of the continuity of a field, which gave rise to these concepts itself, placed a limit on them, because the size of the region in which the charge was enclosed by a surface had to tend to zero. The Gaussian surface integral was thus replaced by divergence as a differential operation.

Circulation was similarly replaced by the rotor as a differential operation. It is these operations which are used in Maxwell's field theory. This is because of the erroneous idea that the electric charges giving rise to the field are themselves continuous quantities distributed over the volume and also over the surface of dielectrics and conductors. The theorems of Gauss and Stokes are therefore limited to volumes shrinking to nil, and the theory became a purely differential one. This situation was later improved by Lorentz, who introduced into the field discrete charge points of finite magnitude scattered in empty space. According to his theory, Maxwell's equations remain applicable in the empty space between the small regions enclosing point singularities. On the closed surfaces surrounding these regions containing field singularities, the solutions to the field equations satisfy integral conditions. The flax of the field vector through these surfaces is equal to the sum of discrete charges enclosed by the total surface.

With the solution averaged over space, Lorentz' theory led to Maxwell's theory, which was in fact his objective. This is how the integral approach to the description of natural phenomena came into being.

The same approach was used by Einstein in the interpretation of his general theory of relativity and for deriving the equations of motion of matter in accordance with Newton's theory from his own equations.

It is, of course, well known that Einstein constructed his general theory of relativity as a relativistic theory of gravitation. For this, he first wrote Newton's equations in the form of field equations using Poisson's equation, and then gave the latter a relativistic, generalized character.

Einstein went further and abandoned inertial counting system, which had been accorded a position of privilege. Thus, the invariance was no longer assumed to be Lorentzian but universal in relation to any improper continuous transformation. Here, use was also made of Lorentz' idea, which we have mentioned earlier, of the discrete nature of the distribution of matter. Matter is concentrated in point singularities of a field, and between them there is empty space, for which Einstein's field equations hold true. The equations are not satisfied at singular points, which must be surrounded by closed surfaces. For the latter, the integral relations of Gauss in turn hold true, i.e. the flux of the field through these surfaces is equal to the charges found inside them. It should be emphasized once again that the actual fields inside these regions need not satisfy the conditions of Einstein's equations.

Einstein's theory is, therefore, by its very nature and because of the basis on which it is constructed, an integral one. This fact is not usually realized, which is why we draw attention to it. It is by this condition, which in mathematical terms amounts to the requirement that the divergence of the original tensor should become exactly nil, that the nature of Einstein's tensor is uniquely defined. Such a tensor is one, the divergence of which is twice the contracted Bianchi identity for the Riemann curvature tensor.

If all the singularities of a field are surrounded by small spheres, in the space between them the field will everywhere be regular and its equations can be expanded in descending series in terms of the reciprocals of the velocities of light. Equating the coefficients in terms of the same powers, we obtain a series of equations. Every such system contains new quantities not found in the previous systems and is easily solved.

The motion of singularities (i.e. of particles) is determined by virtue of the fact that the left-hand sides of the systems of equations being solved satisfy four identities. The right-hand side of these equations must therefore also satisfy these identities or, with the singularities taken into account, the integral conditions. In the absence of singularities these conditions are automatically satisfied and provide nothing new. But if they are present, they determine the equations of motion.

Einstein followed all calculations through and obtained Newton's equations. This method can also be used when gravitational and magnetic fields exist simultaneously, and the result of the calculation is positive. In this way, Einstein showed that even the classical interaction of mass points is caused by the non-linearity of the field equations. This fact is usually emphasized, but the role of integral conditions tends not to be mentioned.

Einstein's equations therefore contain Newton's equations and thus also their solutions and combinations.

Jacobi's virial equation is derived from Newton's equations and consequently must itself be contained in Einstein's equations. However, it is not immediately apparent whether Newton's or Jacobi's equation is the more fundamental. Newton's equations were obtained by Einstein from his second-order equations by approximation. Jacobi's equation was obtained from Einstein's by the method of oscillation moments, also in second-order but by an exact method. This makes Jacobi's equation the more fundamental one; moreover, unlike Newton's equation, it remains integral and dynamical in nature.

As we have mentioned, the way in which the whole problem is formulated gives Jacobi's moment equation an exact solution, closed from which in fact solves the problem itself. In the case of the Universe the problem is also one of a non-steady-state nature. A clever solution to this problem was found earlier by Friedmann. His solution is a solution to Jacobi's equation or to the smoothed Einstein equation. This is an analogue of Maxwell's equation in the form of a smoothed Lorentzian equation for charge points.

For the empty space between point singularities, an anisotropic solution to Einstein's equation has been found (also by indirect means). This solution is Kasner's metric. Analysis of this metric shows that the empty space being considered pulsates. It is compressed on two axes, expands on one, and vice versa. Since this solution has been obtained for the case of space without matter, i.e. without its interaction, so that the law of interaction is without significance, the oscillatory nature of processes in nature is universal. The solution, however, is a formal one and its physical significance needs to be elucidated.

In fact, in Newton's well-known law of gravitation for two masses it is assumed that these are mass points. Otherwise, the inverse-square law ceases to apply to their interaction. This in turn contravenes the law of remote screening mentioned in Chap. 1, which makes it impossible for approximately isolated (conservative) systems to exist.

The law of gravitation thus permits the existence of infinitely small radii of curvature and thereby of an infinitely large curvature of space-time, i.e. of singularities. There are other examples of motion towards or away from a singularity, such as the formation of stars and planets, the expansion of the Universe, etc. Newton's law of gravitation therefore non-explicitly reflects the conditions for the existence of singularities, and the generalization of his theory by Einstein retains and, on the basis of the principle of equivalence, clearly demonstrates these singularities.

Singularities are therefore an empirical fact. So what are they?

In accordance with Einstein's theory, curvature is produced by mass. Consequently, empty-space time is not abstract emptiness but a physical vacuum with its own structure and also an analogue of mass, which in fact reflects Kasner's solution to Einstein's equation. This view is now widely held. In most models, a vacuum is considered to be a quantum-mechanical system of virtual particles and to behave in a way similar to an elastic medium. Belinsky et al. (1970) studied the behavior of Einstein's equation for non-empty space-time but near a singularity. They showed that with increasing proximity to (distance from) the singularity, a moment is approached at which the vacuum curvature exceeds the curvature from matter and the solution to Einstein's equation again becomes Kasner's solution.

Its solution, however, is a case of uniform—although anisotropic—space-time. Belinsky, Lifshitz and Khalatnikov also examined the case of inhomogeneous space-time and came to the conclusion that the nature of the solution was the same but that the Kasner parameters were dependent on co-ordinates and time.

In the case of further evolution of the Kasner solution with expansion of space away from the singularity, the original anisotropic space is gradually converted into isotropic space, i.e. into the Friedmann model, which is a solution to the second-order virial equation.

The oscillatory law of the dynamics of natural processes is thus a universal law of nature. It should, however, be noted that into all the approaches mentioned above, the concept of finite time and of a beginning of time counting has been introduced. In some models, there is also the concept of the end of the world. Only in one of them (in which the average density of matter for the space being considered is strictly determined) do the periodically alternating processes of expansion and contraction infinitely occurs. It is this mode which is determined by the solution to Jacobi's virial equation.

A special feature of the Kasner solution for the general anisotropic case of space-time is the appearance in it of dependence of metric coefficients of time in accordance with the $|t|^{2/3}$ law, where t is a time interval. This law was found for the most general case in which there is no external symmetry, i.e. no symmetry which is not associated only with the internal arrangement of singularities.

The sources of the important relation $|t|^{2/3}$ go back to Kepler, who found experimentally the law in accordance with which the squares of the periods of rotation of bodies of the solar system are the cubes of the semi-axes of the ellipses in which they undergo motion.

It was pointed out in Chap. 6 that in Newton's theory about the attraction of mass points such a law is also found to be asymptotic for the case in which n bodies collide simultaneously. It was also shown there that within this asymptotic limit, the simultaneous collision of n bodies leads to a homologous configuration. And for it in turn the condition of the applicability of Jacobi's general virial equation with two functions holds true. Thus, using a solution of the Kasner type, the applicability of Jacobi's virial equation within the asymptotic limit of simultaneous collision between n bodies which was found earlier for Newton's theory is extended to the case of the solution of Einstein's general equation. This indicates the universal nature of Jacobi's virial equation in dynamics.

Let us note a further important aspect of the solutions under consideration, which relates to the change of Kasner epochs. Their number is infinitely independent of whether the world has a beginning and end. This occurs as a result of a decrease in the duration of an individual epoch as a singularity is approached.

Let us now consider yet another aspect of the fundamental nature of Jacobi's virial equation. As we have already pointed out, Newton's law of gravitation permits the existence of a curvature in space-time, which is derived from Einstein's theory. However, there is one fundamental difference between the two theories. According to Newton, the gravitational interaction is a long-range one, corresponding to an infinite velocity of propagation of the interaction. Einstein assumes a short-range interaction. It is propagated at finite velocity (at the velocity of light). Consequently, Newton's theory is formulated in terms of Euclidian geometry. Nevertheless, with both theories space-time is distorted.

Newton's theory is constructed on the basis of a simple empirical law of Kepler's and does not make use of another empirical law, namely the principle of equivalence derived from the experiments of Eötvös.

So what common ground is there between the theories?

The fact is that Newton's theory is constructed as Newtonian mechanics plus his own law of gravitation. In Newton's mechanics there are three axioms, but the type of interaction is not determined; this is done experimentally. In generalized Newton's theory, it is the mechanics that should have been generalized and not the type of interaction.

With Einstein the type of interaction is replaced by the principle of equivalence. The mechanics, on the other hand, are generalized in accordance with the principle of the invariance of equations. Long-range interaction is thus not involved here, and the type of interaction makes no difference.

Jacobi's virial equation, which was obtained from Newton's equations, also does not so much generalize the type of interaction law the way that this was done in his (Jacobi's) conclusions, as taken into account the mass defect (potential energy). It is therefore linked with the principle of equivalence. The mass defect, in turn, is determined by a system that has already been formed and, consequently, does not depend on the type of interaction during the process of formation (long-range or short-range).

As was thought by Wintner, Jacobi's virial equation therefore reflects the type of interaction law only integrally over the whole period of time in which the mass defect is formed. Also, if there is no delay, as in the case of Newton's long-range law of gravitation, it will be simultaneously a specific and instantaneous type of interaction, as pointed out by Wintner.

If a delay does take place, for example, in accordance with Einstein's short-range interaction law, instead of a specific, instantaneous type of interaction, the equation will include an expression which has been strongly averaged over time, and the dependence on the type of interaction will cease to be of significance. It will be replaced by an assertion about the dependence on instantaneous mass or on the mass defect which has built up over a long time.

This is the answer to the question posed. At the same time, the strength of the Jacobi equation is evident. Since in the general theory of relativity the usual problems in the framework of a short time interval—and even the classical two-body problem—are not solved, the enormous practical significance of solving Jacobi's virial equation becomes obvious. The fact that there are oscillations even in empty-space time indicates the exclusively fundamental nature of this equation.

Moreover, it has now become obvious that Jacobi's virial equation, which was obtained from Newton's equations is a particular case of more general virial equation derived from Einstein's equations. This equation will thus be studied from the most general global points of view, namely that of empty space time, which will not be called a vacuum, and that of models of an evolving Universe. It should be noted here that the models that have so far been developed from Jacobi's equation of an open, a closed and a pulsating Universe have been obtained automatically as its natural solutions as a function of the source data—the quantities of total moment and mass defect. In this case, all possible types of solution are encompassed, and the question of the completeness of the set of possible models of the Universe is thereby solved.

Let us now consider an example which demonstrates the use of the integral approach for constructing a complete closed theory based on Hooke's law. The theory concerned is the theory of elasticity.

In this theory, for any volume of a continuum, only quantities and parameters which are integral from the point of view of an external observer are considered, namely deformation, stress and modulus of elasticity. The elements of the volume interact through their surface. A quantitative measure of their interaction is provided by strains, and a quantitative measure of the results of interaction by relative changes in the external dimensions of elements, in other words, their deformation. The internal structure of the material is demonstrated quantitatively by means of integral parameters, namely the mass density, the modulus of elasticity and the Poisson coefficient.

The interaction between the element of interest of a body and the external world takes place through external surface and volume forces. The external surface forces act only on the surfaces of the whole body and not on that of any of its elements. External volume forces amount to the application of surface forces to the surfaces of any element, and thereby to tensions. Here, external surface forces do not come into the equilibrium equations but into the boundary conditions of the problem and are thus excluded as forces.

It is important to stress this point. It was mentioned earlier by Hertz, who set himself the problem of constructing a system of mechanics without forces. The fact that he was relatively unsuccessful is because in his days Minkowski's idea about the unity of space-time was as yet unknown. The link between static and dynamics was not as clear as it would be after Minkowski.

Hooke's theory is a strictly linear one. It considers two states of an object, the initial and the final states before and after application of forces. One of them is generally the equilibrium state. If these two states of one and the same system occur at different times, displacement deformations are replaced by velocity deformations. In this case, the approach followed takes the form of the theory of viscous or liquid media of gases.

For a fluid, Hooke's law is written in the form of Pascal's law. This way of writing it expresses the condition of equilibrium of the medium, where the stresses on the main axes are equal to the pressure of the fluid. Another condition of equilibrium for a fluid is the law of the conservation of matter.

If in the context of Hooke's law, we move to the point of view of Minkowskian unified space-time, and effect a Lorentzian transformation from a stationary system of co-ordinates to a moving one, the equilibrium conditions in accordance with Pascal's law or in the form of any other Hookean tension law can be expressed as Euler equations and as an equation of the continuity of the medium. Here it is important to note that, when deriving the Euler equations of motion, it is not obligatory to use Newton's second law of mechanics and that a Hooke's system equilibrium equation can be used.

Nor are any dynamic laws used to justify the Minkowski approach, which is based directly on experimental values and is considered to be valid.

It should be noted that, in the context of Hooke's law, a rigorous solution can be found to Jacobi's virial equation for conservative system. In this case, Hooke's law determines the constancy of the product of potential energy and of the Jacobi function; this constancy is written in the form $|U|\sqrt{\Phi} = aGm^{5/2}$.

In this relation, the coefficient a = αβ (which stands for the product of form factors included in the expressions for the potential energy and the Jacobi function) acts as a modulus of the dynamic elasticity of the system. It remains a constant and reflects the constancy of the law of mass density distribution of the system within the limits of its elastic deformations with virial oscillations. The deformation of the system is characterized by its integral parameter the Jacobi function—and the stresses are determined by the term $Gm^{5/2}/U$. As a result, the virial pulses of the system will be strictly periodic, and the deformations will be found to be elastic and therefore reversible.

On this basis, it was shown in Chap. 7 that the parameters of the virial oscillations of the Earth, which are detected, can be used as if the Earth were an elastic body for determining its potential energy. This option remains open for when natural systems are being examined in the framework of other models of continuousand discrete media.

We have mentioned a number of aspects of the universality of Jacobi dynamics in the examination of natural systems. We shall now consider the prospects for solving a number of practical problems in the context of the dynamic approach.

One of these problems is that of the dynamics of the Solar System, of its evolution and its origin. In Chaps. 1, 6 and 7, we made a first step and obtained the basic common solution on creation of celestial bodies and their systems. It appears that any rank of new celestial body (from galaxy to meteorite and even to molecule and atom) is born by self-gravitating parents in consequence of loss of its energy by radiation. It means that the stage of self-gravitation and separation must be changed by the stage of gravitation and joining of the matter. Thus, the present day stage of the expansion of the Universe after total separation of the matter should come to the stage of its contraction and gathering. Generally speaking, our Universe is a closed pulsating and perpetual system. More new detailed solutions in this direction are desirable.

The problem of the dynamics of bodies in the Solar System is a traditional practical problem of classical celestial mechanics. The key to it is solution of the two-body problem, one of the bodies being the Sun, while the other is one of the planets of the system. The influence of other planets is taken into account using the methods of perturbation theory.

In the context of the dynamic approach, a new problem of dynamics of the self-gravitating Earth and its interaction with the Sun and the Moon were considered in Chap. 7. The existence of normal and tangential components of the potential and kinetic energies of a self-gravitating body made it possible to understand the mechanism of separation of the body's shells, their oscillation and rotation by the inner force field. It was understood that the induced outer force field, which has all the properties of the electromagnetic field, acquires the property to conserve the irradiated energy and potential in the orbital motion of its secondary body. But because of limited velocity of propagation of the changing potential the orbital trajectory is found to be open. This fact is proved both by artificial satellites and the observed precession of all the planets and the moons. The discovered important effect makes it possible to interpret inner structure of the Sun, the Earth, the Moon and other celestial bodies. Also, it raises the problem of improving of Kepler's approximation of the Earth's and other body's orbits which are found to be too rough (Severny 1988).

In our opinion, the dynamics of the microcosm is a very interesting field for application of Jacobi dynamics. This book takes only the first step in this direction. It is shown that Jacobi dynamics is also applicable for the solution of this type of problem. An attractive idea is to use the dynamic approach for studying the physics of molecules, atoms and nuclei as dissipative systems, which might lead to discovery of many interesting effects.

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