Novikov's Conjecture

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Abstract We describe Novikov's "higher signature conjecture," which dates back to the late 1960s, as well as many alternative formulations and related problems. The Novikov Conjecture is perhaps the most important unsolved problem in high-dimensional manifold topology, but more importantly, variants and analogues permeate many other areas of mathematics, from geometry to operator algebras to representation theory.

1 Origins of the Original Conjecture

The Novikov Conjecture is perhaps the most important unsolved problem in the topology of high-dimensional manifolds. It was first stated by Sergei Novikov, in various forms, in his lectures at the International Congresses of Mathematicians in Moscow in 1966 and in Nice in 1970, and in a few other papers [85–88]. For an annotated version of the original formulation, in both Russian and English, we refer the reader to [37]. Here we will try instead to put the problem in context and explain why it might be of interest to the average mathematician. For a nice book-length exposition of this subject, we recommend [66]. Many treatments of various aspects of the problem can also be found in the many papers in the collections [38, 39].

For the typical mathematician, the most important topological spaces are smooth manifolds, which were introduced by Riemann in the 1850s. However, it took about 100 years for the tools for classifying manifolds (except in dimension 1, which is trivial, and dimension 2, which is relatively easy) to be developed. The problem is that manifolds have no local invariants (except for the dimension); all manifolds of the same dimension look the same *locally*. Certainly many different manifolds were known, but how can one tell whether or not the known examples are "typical"? How can one distinguish one manifold from another?

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J.F. Nash, Jr., M.Th. Rassias (eds.), *Open Problems in Mathematics*,
DOI 10.1007/978-3-319-32162-2_11

With big leaps forward in topology in the 1950s, it finally became possible to answer these questions, at least in part. Here were a few critical ingredients:

1. the development of the theory of Reidemeister and Whitehead torsion and the related notion of "simple homotopy equivalence" (see [77] for a good survey of all of this);
2. the theory of characteristic classes of vector bundles, developed by Chern, Weil, Pontrjagin, and others;
3. the notion of cobordism, introduced by Thom [112], who also provided a method for computing it;
4. the Hirzebruch signature theorem $\operatorname{sign}(M) = \langle \mathscr{L}(M), [M] \rangle$ [54], giving a formula for the signature of an oriented closed manifold M^{4k} (this is the algebraic signature of the nondegenerate symmetric bilinear form $(x, y) \mapsto \langle x \cup y, [M] \rangle$ on H^{2k} coming from Poincaré duality), in terms of a certain polynomial $\mathscr{L}(M)$ in the rational Pontrjagin classes of the tangent bundle.

Using just these ingredients, Milnor [74] was able to show that there are at least 7 different diffeomorphism classes of 7-manifolds homotopy equivalent to S^7. (Actually there are 28 diffeomorphism classes of such manifolds, as Milnor and Kervaire [65] showed a bit later.) This and the major role played by items 2 and 4 on the above list[1] came as a big surprise, and showed that the classification of manifolds, even within a "standard" homotopy type, has to be a hard problem.

The final two ingredients came just a bit later. One was Smale's famous *h-cobordism theorem*, which was the main ingredient in his proof [109] of the high-dimensional Poincaré conjecture in the topological category. (In other words, if M^n is a smooth compact n-manifold, $n \geq 5$, homotopy equivalent to S^n, then M is homeomorphic to S^n, even though it may not be diffeomorphic to it.) But from the point of view of the general manifold classification program, Smale's important contribution was a criterion for telling when two manifolds really are diffeomorphic to one another. An h-cobordism between compact manifolds M and M' is a compact manifold with boundary W, such that $\partial W = M \sqcup M'$ and such that W has deformation retractions down to both M and M'. The h-cobordism theorem [76] says that if $\dim M = \dim M' \geq 5$ and if M, M', and W are simply connected, then W is diffeomorphic to $M \times [0, 1]$, and in particular, M and M' are diffeomorphic. The advantage of this is that diffeomorphisms between different manifolds are usually very hard to construct directly; it is much easier to construct an h-cobordism.

If one dispenses with simple connectivity, then an h-cobordism between M and M' need not be diffeomorphic to a product $M \times [0, 1]$. However, the s-cobordism theorem, due to Barden, Mazur, and Stallings, with simplifications due to Kervaire

[1] Spheres have stably trivial tangent bundle and no interesting cohomology, so one's first guess might be that the theory of vector bundles and the signature theorem might be irrelevant to studying homotopy spheres. Milnor, however, showed that one can construct lots of manifolds with the homotopy type of a 7-sphere as unit sphere bundles in rank-4 vector bundles over S^4. He also showed that the signature of an 8-manifold bounded by such a manifold yields lots of information about the homotopy sphere.

[64], says that the h-cobordisms themselves are classifiable by the Whitehead torsion $\tau(W, M)$, which takes values in the Whitehead group $\mathrm{Wh}(\pi)$, where $\pi = \pi_1(M)$, and all values in $\mathrm{Wh}(\pi)$ can be realized by h-cobordisms. (The Whitehead group is the quotient of the algebraic K-group $K_1(\mathbb{Z}\pi)$ by its "obvious" subgroup $\{\pm 1\} \times \pi_{\mathrm{ab}}$.) Thus an h-cobordism is a product if $\mathrm{Wh}(\pi) = 0$, which is the case for π free abelian, and in fact is conjectured to be the case if π is torsion-free. But for π finite, for example, $\mathrm{Wh}(\pi)$ is a finitely generated group of rank $r - q$, where r is the number of irreducible real representations of π, and q is the number of irreducible rational representations of π [77, Theorem 6.2]. This number $r - q$ is usually positive (for example, when π is finite cyclic, it vanishes only if $|\pi| = 1, 2, 3, 4, \text{or } 6$). Bass and Murthy have even shown [8] that there are finitely generated abelian groups π for which $\mathrm{Wh}(\pi)$ is not finitely generated.

The last major ingredient for the classification of manifolds is the method of *surgery*. Surgery on an n-manifold M^n means cutting out a neighborhood $S^k \times D^{n-k}$ of a k-sphere $S^k \hookrightarrow M$ (with trivial normal bundle) and replacing it by $D^{k+1} \times S^{n-k-1}$, which has the same boundary. This can be used to modify a manifold without changing its bordism class, and was first introduced by Milnor [75] and Wallace [117].

With the help of all of these techniques, Browder [20, 21] and Novikov [81, 82] finally introduced a general methodology for classifying manifolds in high dimensions. The method gave complete results for simply connected manifolds in dimensions ≥ 5, and only partial information in dimensions 3 and 4, which have their own peculiarities we won't discuss here. With the help of additional contributions by Sullivan [111], Novikov [86], and above all, Wall [115], this method grew into what we know today as *surgery theory*, codified by Wall in his book [116], which originally appeared in 1970. There are now fairly good expositions of the theory, for example in Ranicki's books [94, 95], in the book by Kreck and Lück [66], in the first half of Weinberger's book [119], and in Browder's colloquium lectures from 1977 [22], so we won't attempt to compete by going into details, which anyway would take far too many pages. Instead we will just outline enough of the ideas to set the stage for Novikov's conjecture.

As we indicated before, surgery theory addresses the uniqueness question for manifolds: given (closed and connected, say) manifolds M and M' of the same dimension n, when are they diffeomorphic (or homeomorphic)? It also addresses an existence question: given a connected topological space X (say a finite CW complex), when is it homotopy equivalent to a (closed) manifold?

A few necessary conditions are evident from a first course in topology. If M and M' are diffeomorphic, then certainly they are homotopy equivalent, and so they have the same fundamental group π. Furthermore, if a finite connected CW complex X has the homotopy type of a closed manifold, then it has to satisfy Poincaré duality, even in the strong sense of (possibly twisted) Poincaré duality of the universal cover with coefficients in $\mathbb{Z}\pi$. Homotopy equivalences preserve homology and cohomology groups and cup products, so an orientation-preserving homotopy equivalence also preserves the signature (in dimensions divisible by 4 when the signature is defined). However, these conditions are not nearly enough.

For one thing, for a homotopy equivalence to be homotopic to a diffeomorphism (or even a homeomorphism), it has to be *simple*, i.e., to have vanishing torsion in $\mathrm{Wh}(\pi)$. Depending on the fundamental group π, this may or may not be a serious restriction.

But the most serious conditions involve characteristic classes of the tangent bundle. Via a very ingenious argument using surgery theory and the Hirzebruch signature theorem, Novikov [83, 84] showed that the rational Pontrjagin classes of the tangent bundle of a manifold are preserved under homeomorphisms.[2] (Incidentally, Gromov [45, Sect. 7] has given a totally different short argument for this.) The rational Pontrjagin classes do *not* have to be preserved under homotopy equivalences. So if $\varphi\colon\ M \to M'$ is a homotopy equivalence not preserving rational Pontrjagin classes, it cannot be homotopic to a homeomorphism.

In the simply connected case, this is (modulo finite ambiguity) just about all: if $M' \to M$ is an orientation-preserving homotopy equivalence of closed simply connected oriented manifolds, the rational Pontrjagin classes of M' have to satisfy the constraint $\langle \mathscr{L}(M'), [M'] \rangle = \mathrm{sign}(M') = \mathrm{sign}(M)$ imposed by the Hirzebruch signature theorem, but otherwise they are effectively unconstrained (assuming the dimension of the manifold is at least 5).[3] And if the map does preserve rational Pontrjagin classes, then there are only finitely many possibilities for M' up to diffeomorphism.

When M is not simply connected, the situation is appreciably more complicated. Suppose one wants to check if two n-manifolds M and M' are diffeomorphic. As we indicated before, that means we need to have a simple homotopy equivalence $\varphi\colon\ M' \to M$. If φ were homotopic to a diffeomorphism, it would preserve the classes of the tangent bundles, so it's convenient to assume that φ has been promoted to a *normal map* $\varphi\colon\ (M', \nu') \to (M, \nu)$. Here ν and ν' are the stable normal bundles defined via the Whitney embedding theorem: if k is large enough ($n+1$ suffices), then M and M' have embeddings into Euclidean space $\mathbb{R}^{n+k}$, and any two such embeddings are isotopic, so the isomorphism class of the normal bundle ν or ν' for such an embedding is well defined. (Because of the Thom-Pontrjagin construction, it's better to work with the normal bundle than with the tangent bundle, but they contain the same information.) Being a normal map means that φ has been extended to a bundle map from ν' to ν, which we can assume is an isomorphism on fibers. The idea of trying to show that M and M' are diffeomorphic is to start with a *normal bordism* from φ to id_M, i.e., a manifold W^{n+1} with boundary $M \sqcup M'$ and a map $\Phi\colon\ W \to M \times [0,1]$ restricting to φ and to id_M on the two boundary components, and with a compatible map of bundles, and then to try to modify (W, Φ) by surgery

[2] The same does not hold for the torsion part of the Pontrjagin classes, as one can see from calculations with lens spaces [87, Sect. 3].

[3] A precise statement to this effect may be found in [31, Theorem 6.5]. It says for example that if M is a closed simply connected manifold and $\dim M$ is not divisible by 4, then for *any* set of elements $x_j \in H^{4j}(M, \mathbb{Q})$, $1 \le j \le \left\lfloor \frac{\dim M}{4} \right\rfloor$, there is a positive integer R such that for any integer m, there is a homotopy equivalence of manifolds $\varphi_m\colon\ M'_m \to M$ such that $p_j(M'_m) = \varphi_m^*\big(p_j(M) + mRx_j\big)$.

to make it into an s-cobordism. Once this is accomplished, then M and M' are diffeomorphic by the s-cobordism theorem. It turns out that doing the surgery is not difficult until one gets up to the middle dimension (if $n + 1$ is even) or the "almost middle" dimension $\lfloor \frac{n+1}{2} \rfloor$ (if $n + 1$ is odd). At this point a *surgery obstruction* appears, taking its value in a group $L_{n+1}(\mathbb{Z}\pi)$ constructed purely algebraically out of quadratic forms on $\mathbb{Z}\pi$. (Roughly speaking, the L-groups are groups of stable equivalence classes of forms on finitely generated projective or free $\mathbb{Z}\pi$-modules, and the type of the form—symmetric, skew-symmetric, etc.—depends only on the value of n mod 4. The original construction may be found in [116].) The existence problem (telling if one can find a manifold homotopy equivalent to a given finite complex with Poincaré duality) works in a very similar way, just down in dimension by 1, and the surgery obstruction in that case takes its values in $L_n(\mathbb{Z}\pi)$.

Ultimately, the result of this surgery process is to prove that there is a *surgery exact sequence* for computation of the *structure set* $\mathscr{S}(M)$, the set of (simple) homotopy equivalences $\varphi\colon M' \to M$, where M' is a smooth compact manifold, modulo equivalence. We say that two such maps $\varphi\colon M' \to M$ and $\varphi'\colon M'' \to M$ are equivalent if there is a commuting diagram

$$\begin{tikzcd} M' \arrow[rr, "\varphi"] \arrow[dr, "\cong"'] & & M \\ & M'' \arrow[ur, "\varphi'"'] & \end{tikzcd} .$$

The surgery exact sequence then takes the form

$$\cdots \xrightarrow{\alpha} L_{n+1}(\mathbb{Z}\pi) \dashrightarrow \mathscr{S}(M) \xrightarrow{\eta} \mathscr{N}(M) \xrightarrow{\alpha} L_n(\mathbb{Z}\pi) . \qquad (1)$$

Here $\mathscr{N}(M)$ is the set of *normal invariants*, the normal bordism classes of all normal maps $\varphi\colon (M', \nu') \to (M, \nu)$ (not necessarily homotopy equivalences as before) modulo linear automorphisms of ν. This can also be identified with homotopy classes of maps from M into a classifying space called G/O. If one works instead in the PL or the topological category, the same sequence (1) is valid, but G/O is replaced by G/PL or G/Top, which are easier to deal with,[4] and in fact look a lot like BO, the classifying space for real K-theory. The natural maps $G/O \to G/PL \to G/Top$ are rational homotopy equivalences. The map $\eta\colon \mathscr{S}(M) \to \mathscr{N}(M)$ sends a homotopy equivalence $\varphi\colon M' \to M$ to the associated normal data.

The groups $L_\bullet(\mathbb{Z}\pi)$ are 4-periodic, and only depend on the fundamental group and some "decorations" which we are suppressing here, which only affect the torsion. The map $\alpha\colon \mathscr{N}(M) \to L_n(\mathbb{Z}\pi)$ takes the bordism class of a normal map $\varphi\colon (M', \nu') \to (M, \nu)$ to its associated *surgery obstruction*. When this vanishes, exactness of (1) says we can lift φ to an element of $\mathscr{S}(M)$, or in other words, we

[4]Once the dimension is bigger than 4!

can do surgery to convert it to a homotopy equivalence. The dotted arrow from $L_{n+1}(\mathbb{Z}\pi)$ to $\mathscr{S}(M)$ signifies that the surgery group operates on $\mathscr{S}(M)$ (which is just a pointed set, not a group) and that two elements of the structure set have the same normal invariant if and only if they lie in the same orbit for the action of $L_{n+1}(\mathbb{Z}\pi)$.

The exact sequence (1) is closely related to an *algebraic surgery exact sequence*

$$\cdots \to L_{n+1}(\mathbb{Z}\pi) \to \mathscr{S}_n(M) \to H_n(M, \mathbb{L}(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}\pi) \tag{2}$$

constructed in [93, 95], where the map A, called the *assembly map*, corresponds to local-to-global passage. We will come back to this later.

For most groups π, the L-groups $L_\bullet(\mathbb{Z}\pi)$ are not easy to calculate, so a lot of the literature on surgery theory emphasizes things related to the exact sequence (1) which don't rely on explicit calculation of all the groups. For example, sometimes one can compare two related surgery problems, or rely on other invariants, such as η- and ρ-invariants for finite groups. These (as well as direct calculation from (1)) show that there are infinitely many manifolds with the homotopy type of $\mathbb{RP}^{4k+3}$, $k \geq 1$. In fact, it's shown in [27] that in dimension $4k+3$, $k \geq 1$, any closed manifold M with torsion in its fundamental group has infinitely many distinct manifolds simple homotopy-equivalent to it.

Now we are ready to explain Novikov's conjecture. We can rewrite the Hirzebruch signature theorem as saying that for a closed connected oriented manifold M, the 0-degree component of $\mathscr{L}(M) \cap [M]$ in $H_0(M, \mathbb{Q}) \cong \mathbb{Q}$ coincides with $\operatorname{sign} M$, which is preserved by orientation-preserving homotopy equivalences. The components of $\mathscr{L}(M) \cap [M]$ in other degrees have no such invariance property, and knowing them is equivalent to knowing the rational Pontrjagin classes. However, Novikov discovered in [83] (see [31, Theorem 2.1 and its proof] for a simplified version of his argument) that if $\pi_1(M) \cong \mathbb{Z}$, then the degree-1 component of $\mathscr{L}(M) \cap [M]$ is also an oriented homotopy invariant. This theorem is the simplest special case of Novikov's conjecture.

Definition 1.1. Let M be a closed connected oriented manifold, and let π be a countable discrete group (usually taken to be the fundamental group of M). Let $B\pi$ be a classifying space for π, a CW complex with contractible universal cover and fundamental group π, and let $f\colon M \to B\pi$ be a continuous map. (Up to homotopy, it's determined by the induced homomorphism $\pi_1(M) \to \pi$.) The associated *higher signature* of M is $f_*(\mathscr{L}(M) \cap [M]) \in H_\bullet(B\pi, \mathbb{Q})$.

Conjecture 1.2 (Novikov's Conjecture). *Any higher signature $f_*(\mathscr{L}(M) \cap [M]) \in H_\bullet(B\pi, \mathbb{Q})$ is always an oriented homotopy invariant. In other words, if M and M' are closed connected oriented manifolds and if $\varphi\colon M' \to M$ is*

(continued)

Conjecture 1.2 (continued)
an orientation-preserving homotopy equivalence and $f\colon M \to B\pi$*, then*

$$f_*(\mathscr{L}(M) \cap [M]) = (f \circ \varphi)_*(\mathscr{L}(M') \cap [M']) \in H_\bullet(B\pi, \mathbb{Q}).$$

The utility of the conjecture can be illustrated by an example.

Problem 1.3. Classify smooth compact 5-manifolds homotopy equivalent to $\mathbb{CP}^2 \times S^1$. (Note: the diffeomorphism classification of smooth 4-manifolds homotopy equivalent to $\mathbb{CP}^2$ is not known, since surgery breaks down in the smooth category in dimension 4. It is known by work of Freedman [41] that up to *homeomorphism*, there are exactly two closed topological 4-manifolds homotopy equivalent to $\mathbb{CP}^2$, but for the "exotic" one, the product with S^1 does not have a smooth structure.)

Proof. Suppose M is a smooth closed manifold of the homotopy type of $\mathbb{CP}^2 \times S^1$. There is a smooth map $f\colon M \to S^1$ inducing an isomorphism on π_1, and we can take this to be the map $f\colon M \to B\pi$, $\pi = \mathbb{Z}$, for the case of the conjecture proven by Novikov himself. So the conjecture implies that if $K = f^{-1}(\mathrm{pt})$, the inverse image of a regular value of f, then K has signature 1. This fixes the first Pontrjagin class of M. Furthermore, K being a smooth 4-manifold with signature 1, it is in the same oriented bordism class as $\mathbb{CP}^2$. From this we can get a normal bordism W^6 between M (with its stable normal bundle ν) and $\mathbb{CP}^2 \times S^1$ (with its stable normal bundle ξ). We plug into the surgery machine and try to do surgery to convert this to an h-cobordism (and thus automatically an s-cobordism, since $\mathrm{Wh}(\mathbb{Z}) = 0$). The surgery obstruction lives in $L_6(\mathbb{Z}[\mathbb{Z}])$. This group turns out to be $\mathbb{Z}/2$ (coming from the image of the Arf invariant in $L_6(\mathbb{Z}) \cong \mathbb{Z}/2$). So there are not a lot of possibilities. In fact one can show by studying the continuation of the sequence (1) to the left that M is diffeomorphic to $\mathbb{CP}^2 \times S^1$. But note that the key ingredient in the whole argument is the Novikov Conjecture, which pins down the first Pontrjagin class. □

2 Methods of Proof

Work on the Novikov Conjecture began almost as soon as the conjecture was formulated. Roughly speaking, methods fall into three different categories: topological, analytic, and algebraic. The *topological* approach began with Novikov's own work on the free abelian case of the conjecture, which we already mentioned in the case $\pi = \mathbb{Z}$, and which only uses transversality and basic homology theory. This method was generalized in work of Kasparov, Farrell-Hsiang, and Cappell [23, 33, 58], who used codimension-one splitting methods to deal with free abelian and poly-$\mathbb{Z}$ groups, and certain kinds of amalgamated free products.

Subsequent topological approaches to the conjecture have been based on *controlled topology* (if you like, a blend of analysis and topology since it amounts to topology with δ-ε estimates) or on various methods in stable homotopy theory. There is a lot more in this area than we can possibly summarize here, but it is discussed in detail in [37], which includes a long bibliography.

The *analytic* approach began with the important contribution of Lusztig [72]. The key idea here is to realize the higher signature of Definition 1.1 as the index of a family of elliptic operators, just as Atiyah and Singer [2, Sect. 6] had reproven Hirzebruch's signature theorem by realizing the signature as the index of a certain elliptic operator, now universally called the signature operator. (This is just the operator $d + d^*$ operating on differential forms, but with a grading on the forms coming from the Hodge $*$-operator.) A major step forward from the work of Lusztig came with the work of Mishchenko [78, 79] and Kasparov [57, 61, 62], who realized that one could generalize this construction by using "noncommutative" families of elliptic operators, based on a C^*-algebra completion $C^*(\pi)$ of the algebraic group ring $\mathbb{C}\pi$. Underlying this method was the idea [79, 99] that because of the inclusions $\mathbb{Z}\pi \hookrightarrow \mathbb{C}\pi \hookrightarrow C^*(\pi)$, there is a natural map $L_n(\mathbb{Z}\pi) \to L_n(C^*(\pi))$, and that because the spectral theorem enables one to diagonalize quadratic forms over a C^*-algebra, the L-groups and topological K-groups of a C^*-algebra essentially coincide. As we will see in the next section, the analytic approach to the Novikov conjecture is the one that has attracted the most recent attention, though there is still plenty of work being done on topological and algebraic methods.

Algebraic approaches to proving the Novikov conjecture depend on a finer understanding of the surgery exact sequence (1) and the L-groups. For a homotopy equivalence of manifolds $\varphi\colon\ M' \to M$, the difference $\varphi_*(\mathscr{L}(M') \cap [M']) - (\mathscr{L}(M) \cap [M]) \in H_\bullet(M, \mathbb{Q})$ is basically $\eta([M' \to M]) \otimes_{\mathbb{Z}} \mathbb{Q}$ in (1). The Novikov conjecture says that this should vanish when we apply $f_*, f\colon\ M \to B\pi$. Since we could also apply (1) with M replaced by $B\pi$ (at least if $B\pi$ can be chosen to be a manifold—but there is a way of getting around this), exactness in (1) shows that the Novikov Conjecture is equivalent to rational injectivity of the map α in (1), when we replace M by $B\pi$.

More precisely, we need to make use of an idea of Quinn [91], that the L-groups are the homotopy groups of a spectrum:

$$L_n(\mathbb{Z}\pi) = \pi_n(\mathbb{L}_\bullet(\mathbb{Z}\pi))$$

and that the map α in the surgery exact sequence (1) comes from an *assembly map* which is the induced map on homotopy groups of a map of spectra

$$A_M\colon\ M_+ \wedge \mathbb{L}_\bullet(\mathbb{Z}) \to \mathbb{L}_\bullet(\mathbb{Z}\pi).$$

This map factors (via $f\colon\ M \to B\pi$) through a similar map

$$A_\pi\colon\ B\pi_+ \wedge \mathbb{L}_\bullet(\mathbb{Z}) \to \mathbb{L}_\bullet(\mathbb{Z}\pi). \tag{3}$$

If A_π in (3) induces a rational injection on homotopy groups, then the Novikov Conjecture follows from exactness of (1). On the other hand, if A_π is not rationally injective, then one can construct an M and a higher signature for it that is not homotopy invariant. So the Novikov Conjecture is reduced to a statement which at least in principle is purely algebraic, as Ranicki in [93, 95] gives a purely algebraic construction of the surgery spectra and of the map A_π, leading to the exact sequence (2).[5]

3 Variations on a Theme

One of the most interesting features of the Novikov Conjecture is that it is closely related to a number of other useful conjectures. Some of these are known to be true, some are known to be false, and most are also unsolved. But even the ones that are false are false for somewhat subtle reasons, and still carry some "element of truth." Here we mention a number of these related conjectures and something about their status.

Conjecture 3.1 (Borel's Conjecture). *Any two closed aspherical* (*i.e., having contractible universal covers*) *manifolds M and M' with the same fundamental group are homeomorphic. In fact, any homotopy equivalence $\varphi\colon\ M' \to M$ of such manifolds is homotopic to a homeomorphism.*

This conjecture is known to have been posed informally by Armand Borel, before the formulation of Novikov's Conjecture, and was motivated by the Mostow Rigidity Theorem. It amounts to a kind of topological rigidity for aspherical manifolds. Note that if M is aspherical with fundamental group π and $n = \dim M \geq 5$, then we can take $M = B\pi$, and Borel's conjecture amounts to saying that in the surgery sequence (1) in the topological category, $\mathscr{S}(M)$ is just a single point, or by exactness, the assembly map A_π is an equivalence. This implies the Novikov Conjecture for π, but is stronger.

Incidentally, it is known now that the analogue of Borel's Conjecture, but with homeomorphism replaced by diffeomorphism, is false. The simplest

[5]It turns out that (2) coincides with the analogue of (1) in the topological, rather than smooth, category, but the difference between these is rather small since all homotopy groups of Top/O are torsion.

counterexample is with $M = T^7$, the 7-torus. Since a torus is parallelizable, Wall pointed out in [116, Sect. 15A] that the set of smooth structures on T^n compatible with the standard PL structure is parameterized by $[T^n, PL/O]$ (for $n \geq 5$). It is known that the classifying space PL/O is 6-connected and that (for $j \geq 7$) its jth homotopy group can be identified with the group Θ_j of smooth homotopy j-spheres.[6] Since $\Theta_7 \cong \mathbb{Z}/28$ by [65, 74], the differentiable structures on T^7 are parameterized by $[T^7, PL/O] \cong [T^7, K(\Theta_7, 7)] \cong H^7(T^7, \Theta_7) \cong \mathbb{Z}/28$ and there are 28 different differentiable structures on T^7. A series of counterexamples with negative curvature to the smooth Borel conjecture was constructed in [34, 35].

The fundamental group π of an aspherical manifold M (even if noncompact) has to be torsion-free, since if $g \in \pi$ has finite order $k > 1$, it would act freely on the universal cover $\widetilde{M}$, and $\widetilde{M}/\langle g \rangle$ would be a finite-dimensional model for $B\mathbb{Z}/k$, contradicting the fact that $\mathbb{Z}/k$ has homology in all positive odd dimensions. So Conjecture 3.1 can't apply to groups with torsion. In fact, the result of [27] shows that for groups with torsion, A_π in (3) is never an equivalence. We will come back to this shortly.

However, we have already mentioned the role of the Whitehead group, which comes from the algebraic K-theory of $\mathbb{Z}\pi$, in studying manifolds with fundamental group π. An important conjecture which we have already mentioned is:

Conjecture 3.2 (Vanishing of Whitehead Groups). *If* π *is torsion-free, then* $\mathrm{Wh}(\pi) = 0$.

Note that if Conjecture 3.2 fails and π is the fundamental group of a closed manifold M, then by the s-cobordism theorem, there is an h-cobordism W with $\partial W = M \sqcup (-M')$ which is not a product, and we have a homotopy equivalence $M' \to M$ which is not simple, hence Borel's Conjecture, Conjecture 3.1, fails for M.

More generally, one can ask what one can say about the algebraic K-theory of $\mathbb{Z}\pi$ in all degrees. Loday [69] constructed an assembly map $B\pi_+ \wedge \mathbb{K}(\mathbb{Z}) \to \mathbb{K}(\mathbb{Z}\pi)$, and this being an equivalence would say that all of the algebraic K-theory of $\mathbb{Z}\pi$ comes in some sense from homology of π and K-theory of $\mathbb{Z}$. This is known in some cases—for π free abelian, it follows from the "Fundamental Theorem of K-theory." The assembly map being an equivalence in degrees ≤ 1 for torsion-free groups π and $R = \mathbb{Z}$ implies Conjecture 3.2. The analogue of Novikov's Conjecture for K-theory is

[6] The group operation is the connected sum; inversion comes from reversing the orientation.

Conjecture 3.3 (Novikov Conjecture for K-Theory). *Let $R = \mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ and let π be a discrete group. Then the assembly map $B\pi_+ \wedge \mathbb{K}(R) \to \mathbb{K}(R\pi)$ induces an injection of rational homotopy groups.*

Conjecture 3.3 was proved (with $R = \mathbb{Z}$, the most important case) for groups π with finitely generated homology in [16]. It was also proved (without rationalizing) in [25], when π is a discrete, cocompact, torsion-free discrete subgroup of a connected Lie group. Subsequently, Carlsson and Pedersen [26] proved it (without rationalizing) for any group π for which there is a finite model for $B\pi$, such that the universal cover $E\pi$ of $B\pi$ admits a contractible metrizable π-equivariant compactification X such that compact subsets of $E\pi$ become small near the "boundary" $X \smallsetminus E\pi$. This was recently improved [92] to the case where there is a finite model for $B\pi$ and π has finite decomposition complexity, which is a tameness condition on π viewed as a metric space with the word length metric (for some finite generating set).

As we have already mentioned, for groups with torsion, the assembly map A_π of (3) is never an equivalence. For similar reasons, one also can't expect the K-theory assembly map to be an equivalence for groups with torsion. The correct replacement seems to be the following.[7]

Conjecture 3.4 (Farrell-Jones Conjecture). *Let π be a discrete group and let $\mathscr{F}$ be its family of virtually cyclic subgroups (subgroups that contain a cyclic subgroup of finite index). Such subgroups are either finite or else admit a surjection with finite kernel onto either $\mathbb{Z}$ or the infinite dihedral group $(\mathbb{Z}/2) * (\mathbb{Z}/2)$. Let $E_{\mathscr{F}}(\pi)$ denote the universal π-space with isotropy in $\mathscr{F}$. This is a contractible π-CW-complex X with all isotropy groups in $\mathscr{F}$ (for the π-action) and with X^H contractible for each $H \in \mathscr{F}$. It is known to be uniquely defined up to π-homotopy equivalence. Then the assembly maps*

$$H^\pi_\bullet(E_{\mathscr{F}}(\pi); \mathbb{L}(\mathbb{Z})) \to \mathbb{L}(\mathbb{Z}\pi) \quad \text{and} \quad H^\pi_\bullet(E_{\mathscr{F}}(\pi); \mathbb{K}(R)) \to \mathbb{K}(R\pi) \qquad (4)$$

are isomorphisms for $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or *$\mathbb{C}$.*

[7] Just for the experts: one needs to use the $-\infty$ decoration on the L-spectra here.

When π is torsion-free, (4) is just the assembly map (3) or its K-theory version, and the conjecture says that the assembly map is an equivalence. Conjecture 3.4 implies Conjectures 3.1, 1.2, and 3.3, even for groups with torsion, as well as Conjecture 3.2. More details on Conjecture 3.4 may be found in [70], in [66, Chaps. 19–24], or in [71]. The K-theory version of the conjecture has been proven in [7] for fundamental groups of manifolds of negative curvature and in [6] for hyperbolic groups, and both the K-theory and L-theory versions have been proven for certain groups acting on trees in [5, 107] and for cocompact lattice subgroups of Lie groups in [4]. Split injectivity of (4) has been proved for groups with finite quotient finite decomposition complexity (a condition weaker than that of [92]) in [63]. Rational injectivity of (4) holds under much weaker conditions; see for example [30].

Another variation on the Novikov Conjecture is to consider the situation where a finite group G acts on a manifold, and one wants to study G-equivariant invariants of M. Under suitable circumstances, one finds that the fundamental group of M leads to a certain extra amount of equivariant topological rigidity. To formulate the analogue of Conjecture 1.2, one needs a substitute for the homology L-class $\mathscr{L}(M) \cap [M]$. The easiest way to formulate this is in K-homology, since Kasparov [59, 60], following ideas of Atiyah and Singer, showed that an elliptic differential operator D on M naturally leads to a K-homology class $[D] \in K_\bullet(M)$ (see also [50] for an exposition), and when D is G-invariant, the class naturally lives in $K_\bullet^G(M)$. The image of $[D]$ in $K_\bullet^G(\mathrm{pt}) = R(G)$ under the map induced by $M \to \mathrm{pt}$ is the equivariant index $\mathrm{ind}_G D \in R(G)$ in the sense of Atiyah and Singer. When D is the signature operator, $\mathscr{L}(M) \cap [M]$ is basically (except for some powers of 2, not important here) the Chern character of $[D] \in K_\bullet(M)$, and so if $f\colon M \to B\pi$, the higher signature of Definition 1.1, is basically the Chern character of $f_*([D])$. That motivates the following.

Conjecture 3.5 (Equivariant Novikov Conjecture [105]). *Let M be a closed oriented manifold admitting an action of a finite group G, and suppose $f\colon M \to X$ is a G-equivariant smooth map to a finite G-CW complex which is G-equivariantly aspherical (i.e., X^H is aspherical for all subgroups H of G). Let $\varphi\colon M' \to M$ be a G-equivariant map of closed G-manifolds which, non-equivariantly, is a homotopy equivalence. Then if $[D_M]$ and $[D_{M'}]$ denote the equivariant K-homology classes of the signature operators on M and M', respectively,*

$$f_*([D_M]) = (f \circ \varphi)_*([D_{M'}]) \in K_\bullet^G(X).$$

Various generalizations and applications to rigidity theorems are possible (see for example [36, 104]), but we won't go into details here. Conjecture 3.4 was proven in [105] for X a closed manifold of nonpositive curvature and in [43] for X a Euclidean building, in both cases with G acting by isometries.

4 New Directions

The conjectures we discussed in Sect. 3 are fairly directly linked to the original Novikov Conjecture, and it is easy to see how they are connected with topological rigidity of highly connected manifolds. But in this section, we will discuss a number of other conjectures which grew out of work on Novikov's Conjecture but which go somewhat further afield, to the point where the connection with the original conjecture may not be immediately obvious. However, we will try to explain the relationships as we go along.

We have already mentioned the assembly map and the Farrell-Jones Conjecture (Conjecture 3.4), which gives a conjectural calculation of the L-groups $L_\bullet(\mathbb{Z}\pi)$ for a discrete group π. However, work on Novikov's Conjecture by analytic techniques (see Sect. 2) already required passing from the integral group ring to the complex group ring (this only affects 2-torsion in the L-groups) and then completing $\mathbb{C}\pi$ to a C^*-algebra. For C^*-algebras, L-theory is basically the same as topological K-theory, and even for real C^*-algebras, they agree after inverting 2 [99, Theorem 1.11]. So it's natural to ask if assembly can be used to compute the topological K-theory of $C^*(\pi)$. For the full group C^*-algebra this seems to be impossible, but for the *reduced* group C^*-algebra $C^*_r(\pi)$,[8] the completion of $\mathbb{C}\pi$ for its action on $L^2(\pi)$, there is a good guess for a purely topological calculation of $K_\bullet(C^*_r(\pi))$. (Here $K_\bullet$ denotes *topological* K-theory for Banach algebras, which satisfies Bott periodicity. This is much more closely related to L-theory, which is 4-periodic, than is algebraic K-theory in the sense of Quillen.) This guess is given by the *Baum-Connes Conjecture*, originally formulated in [9, 10] and further refined in [11] (see also [47] for a nice quick survey). The conjecture applies to far more than just discrete groups; it applies to locally compact groups, to such groups "with coefficients" (i.e., acting on a C^*-algebra), and even to groupoids [113]. In its greatest generality the conjecture is known to be false [48], though a patch which might repair it has been proposed [13]. However, the original version of the conjecture is still open, though the literature on the conjecture has grown to more than 300 items. To avoid having to talk about Kasparov's KK-theory, we will omit discussion of the conjecture with coefficients, and will just stick to the original conjecture for groups.

[8] It is known that the natural map $C^*(\pi) \twoheadrightarrow C^*_r(\pi)$ is an isomorphism if and only if π is amenable.

Conjecture 4.1 (Baum-Connes Conjecture). *Let G be a second countable locally compact group, and let $C^*_r(G)$ denote the completion of $L^1(G)$ for its action by left convolution on $L^2(G)$. Then there is a natural assembly map*

$$\mu\colon\ K^G_\bullet(\mathscr{E}G) \to K_\bullet(C^*_r(G)),$$

where $\mathscr{E}G$ is the universal proper G-space (a contractible space on which G acts properly), and this map is an isomorphism. If G has no nontrivial compact subgroups, then the assembly map simplifies to

$$\mu\colon\ K_\bullet(BG) \to K_\bullet(C^*_r(G)).$$

Proposition 4.2. *Conjecture 4.1 implies Conjecture 1.2.*

Proof. For this we take $G = \pi$ to be discrete and countable. For simplicity, we also work with the periodic L-theory spectra instead of the connective ones. (The difference only affects the bottom of the surgery sequence (1).) If π is torsion-free, the domain of μ is $K_\bullet(B\pi) = H_\bullet(B\pi;\mathbb{K}^{\mathrm{top}})$. But after inverting 2, $\mathbb{K}^{\mathrm{top}}$ is just a direct sum of two copies of $\mathbb{L}(\mathbb{Z})$, one of them shifted in degree by 2. So if Conjecture 4.1 holds for π and π is torsion-free, we have the commuting diagram

$$\begin{tikzcd}
H_\bullet(B\pi;\mathbb{L}(\mathbb{Z}))\otimes\mathbb{Q} \arrow[r, "A_\pi"] \arrow[dd, hook] & L_\bullet(\mathbb{Z}\pi)\otimes\mathbb{Q} \arrow[d] \\
& L_\bullet(C^*_r(\pi))\otimes\mathbb{Q} \arrow[d, "\cong"] \\
H_\bullet(B\pi;\mathbb{K}^{\mathrm{top}})\otimes\mathbb{Q} \arrow[r, "\mu", "\cong"'] & K_\bullet(C^*_r(\pi))\otimes\mathbb{Q}.
\end{tikzcd} \tag{5}$$

Diagram (5) immediately implies that the rational L-theory assembly map A_π [the same map as the map induced on rational homotopy groups by (3)] is injective.

If π is not torsion-free, then $\mathscr{E}\pi$ and $E\pi$ are not the same,[9] but there is always a π-equivariant map $E\pi \to \mathscr{E}\pi$. Thus we need only replace (5) by the diagram

[9] In the extreme case where π is a torsion group, $\mathscr{E}\pi = \mathrm{pt}$, while if π is nontrivial, $E\pi$ is necessarily infinite dimensional.

$$\begin{tikzcd}
H_\bullet(B\pi;\mathbb{L}(\mathbb{Z}))\otimes\mathbb{Q} \arrow[r] \arrow[rr, bend left, "A_\pi"] \arrow[dd, hook] & H_\bullet^\pi(\mathscr{E}\pi;\mathbb{L}(\mathbb{Z}))\otimes\mathbb{Q} \arrow[r] \arrow[dd, hook] & L_\bullet(\mathbb{Z}\pi)\otimes\mathbb{Q} \arrow[d] \\
 & & L_\bullet(C_r^*(\pi))\otimes\mathbb{Q} \arrow[d, "\cong"] \\
H_\bullet(B\pi;\mathbb{K}^{\mathrm{top}})\otimes\mathbb{Q} \arrow[r, "\alpha"] & H_\bullet^\pi(\mathscr{E}\pi;\mathbb{K}^{\mathrm{top}})\otimes\mathbb{Q} \arrow[r, "\mu", "\cong"'] & K_\bullet(C_r^*(\pi))\otimes\mathbb{Q}.
\end{tikzcd} \tag{6}$$

Since points in $\mathscr{E}\pi$ have finite isotropy, and since the π-map $\pi \twoheadrightarrow \pi/\sigma$, σ a finite subgroup of π, induces the map $\mathbb{Z} \hookrightarrow R(\sigma)$ on equivariant K-homology, a spectral sequence argument shows that the bottom left map α in (6) is injective, and so by a diagram chase, A_π is injective. $\square$

Thus Conjecture 4.1 (for the case of discrete groups) implies Conjecture 1.2. However, Conjecture 4.1 for *non-discrete* groups is also quite interesting and important. There are two main reasons for this:

1. There are "change of group methods" that enable one to pass from results for a group to results for a closed subgroup. Many of the significant early results on Novikov's Conjecture were proved by considering discrete groups π that embed in a Lie group (or p-adic Lie group) and then using these change of group methods to pass from the Lie group to the discrete subgroup.
2. The Baum-Connes Conjecture for connected Lie groups (also known as the Connes-Kasparov Conjecture) and the same conjecture for p-adic groups are both quite interesting in their own right, and say a lot about representation theory. For an introduction to this topic, see [11, 47]. For some of the more significant results, see [14, 67, 110, 118]. For recent applications to harmonic analysis on reductive groups, see [3, 73, 90, 102].

Another direction arising out of both the controlled topology and the analytic approaches to Novikov's Conjecture leads to the so-called *coarse Baum-Connes Conjecture* [49, 96, 120]. This conjecture deals with the large-scale geometry of metric spaces X of bounded geometry (think of complete Riemannian manifolds with curvature bounds, or of finitely generated groups with a word-length metric). Roughly speaking, the coarse Novikov Conjecture says that indices of generalized elliptic operators capture all of the coarse (i.e., "large-scale") rational homology of such a space X.

Conjecture 4.3 (Coarse Baum-Connes and Novikov). *Let X be a uniformly contractible locally compact complete metric space of bounded*

(continued)

Conjecture 4.3 (continued)
geometry, in which all metric balls are compact. Let $KX_\bullet(X)$ be the coarse K-homology of X (the direct limit of the K-homologies of successively coarser Rips complexes) and let $C^*(X)$ be the C^*-algebra of locally compact, finite propagation operators on X. Then Roe defined a natural assembly map

$$\mu\colon\ KX_\bullet(X) \to K_*(C^*(X)). \tag{7}$$

The coarse Baum-Connes Conjecture is that μ is an isomorphism; the coarse Novikov Conjecture is that μ is rationally injective.

Positive results on Conjecture 4.3 may be found in [28, 29, 40, 42, 49, 96, 114, 120].

However, it is known that the conjecture fails in various situations [32, 48, 121], especially if one drops the bounded geometry assumption.

The coarse Baum-Connes conjecture implies the Novikov conjecture under mild conditions. To see this, suppose for example that there is a compact metrizable model Y for $B\pi$, and let $X = E\pi$ be its universal covering. Then there is a commutative diagram

$$\begin{tikzcd} K_*(B\pi) \arrow[r, "\alpha"] \arrow[d, "\cong"', "\mathrm{tr}"] & K_*(C_r^*(\pi)) \arrow[d, "\mathrm{tr}"] \\ \pi_*(\mathbb{K}X_*(X)^{h\pi}) \arrow[r, "\mu^{h\pi}"] & \pi_*(\mathbb{K}_*(C^*(X))^{h\pi}), \end{tikzcd}$$

where α is usual Baum-Connes assembly, μ is as in Conjecture 4.3, $h\pi$ denotes homotopy fixed points, and tr is a transfer map. Then μ being an isomorphism implies that $\mu^{h\pi}$ is an isomorphism, and so we get a splitting for α. Refinements of this argument, as well as generalizations of the coarse Baum-Connes conjecture, may be found in [80].

Thinking of $C_r^*(\pi)$ as being (up to Morita equivalence) the same thing as the fixed points of π on $C^*(X)$ also gives rise to a nice way of relating the surgery exact sequence (2) to the Baum-Connes assembly map. This was accomplished in the series of papers [51–53, 89], which set up a natural transformation from the surgery sequence to a long exact sequence where the C^*-algebraic assembly map corresponds to the L-theory assembly map in the original sequence. This gives an even more direct connection between coarse Baum-Connes and surgery theory.

Other "new directions" from Novikov's Conjecture arise from replacing the higher signature of Definition 1.1 with other sorts of "higher indices." For example, an important case is obtained by replacing $\mathscr{L}(M)$ with $\widehat{\mathscr{A}}(M)$, the total $\widehat{A}$ class.

This is again a certain polynomial in the rational Pontrjagin class, and has the property that when M is a spin manifold, $\widehat{\mathscr{A}}(M) \cap [M]$ is the Chern character of the class $[D]$ defined by the Dirac operator on M. (Here the reader doesn't need to know much about the Dirac operator D except for the fact that it's an elliptic first-order differential operator canonically defined on a Riemannian manifold with a spin structure.) It was pointed out by Lichnerowicz [68] that when M is closed and has positive scalar curvature, then the spectrum of D must be bounded away from 0, and thus $\operatorname{ind}(D) = \langle \widehat{\mathscr{A}}(M), [M] \rangle$ has to vanish. When M is not simply connected, a major strengthening of this is possible:

Conjecture 4.4 (Gromov-Lawson Conjecture [46]). *Let M be a connected closed spin Riemannian manifold of positive scalar curvature, let π be a discrete group, and let $f\colon M \to B\pi$ be a continuous map (determined up to homotopy by a homomorphism $\pi_1(M) \to \pi$). Then the higher $\widehat{A}$-genus $f_*(\widehat{\mathscr{A}}(M) \cap [M]) \in H_\bullet(B\pi, \mathbb{Q})$ vanishes.*

This conjecture is still open in general, but it is known to be closely related to Novikov's Conjecture. For example, it was shown in [97] that Conjecture 4.4 is true whenever the K-theory assembly map $K_\bullet(B\pi) \to K_\bullet(C_r^*(\pi))$ is rationally injective, and thus *a fortiori* whenever Conjecture 4.1 holds. It also can be deduced from certain cases of Conjecture 4.3, by a descent argument similar to the one above. The Lichnerowicz argument also applies to complete noncompact spin manifolds M of *uniformly* positive scalar curvature, and when Conjecture 4.3 holds, one gets obstructions to existence of such metrics living in $K_\bullet(C^*(X))$ whenever there is a coarse map $M \to X$.

Conjecture 4.4 can be refined to conjectures about necessary and sufficient conditions for positive scalar curvature. Here we just mention a few of several possible versions. For these it's necessary to go beyond ordinary homology and to consider KO-homology, the homology theory dual to the (topological) K-theory of real vector bundles. This theory is 8-periodic and has coefficient groups $KO_j = \mathbb{Z}$ when j is divisible by 4 (this part is detected by the Chern character to ordinary homology), $\mathbb{Z}/2$ when $j \equiv 1, 2 \pmod 8$, 0 otherwise. The class $[D]$ of the Dirac operator on a spin manifold M lives in $KO_n(M)$, $n = \dim(M)$. While the actual operator D depends on a choice of a Riemannian metric, the class $[D] \in KO_n(M)$ does not, so that the following conjecture makes sense.

Conjecture 4.5 (Gromov-Lawson-Rosenberg Conjecture). *Let M be a connected closed spin manifold with fundamental group π and Dirac operator D_M, and let $f\colon M \to B\pi$ be the classifying map for the universal cover. Let $A\colon KO_\bullet(B\pi) \to KO_\bullet(C^*_r(\pi))$ be the assembly map in real K-theory. Then M admits a Riemannian metric of positive scalar curvature if and only if $A \circ f_*([D_M]) = 0$ in $KO_n(C^*_r(\pi))$, $n = \dim M \geq 5$.*

The restriction to $n \geq 5$ is needed only to use surgery methods to construct a metric of positive scalar curvature when the obstruction vanishes; it is not needed to show that there is a genuine obstruction to positive scalar curvature when $A \circ f_*([D_M]) \neq 0$, which was proven in [98]. For the next conjecture, we need to introduce a choice of *Bott manifold*, a geometric representative for Bott periodicity in *KO*-homology. This is a simply connected closed spin manifold Bt^8 of dimension 8 with $\langle \widehat{\mathscr{A}}(\mathrm{Bt}^8), [\mathrm{Bt}^8] \rangle = 1$. It may be chosen to be Ricci flat. Since scalar curvature is additive on Riemannian products, Bt^8 being Ricci flat implies that taking a product with the Bott manifold does not change the scalar curvature.

Conjecture 4.6 (Stable Gromov-Lawson-Rosenberg Conjecture). *Let M be a connected closed spin manifold with fundamental group π and Dirac operator D_M, and let $f\colon M \to B\pi$ be the classifying map for the universal cover. Let Bt^8 be a Bott manifold as above. Then M **stably** admits a Riemannian metric of positive scalar curvature, in the sense that $M \times \overbrace{\mathrm{Bt}^8 \times \cdots \times \mathrm{Bt}^8}^{k}$ admits such a metric for some k, if and only if $A \circ f_*([D_M]) = 0$ in $KO_n(C^*_r(\pi))$, $n = \dim M$.*

There are simple implications

$$\text{Conj. 4.5} \Rightarrow \text{Conj. 4.6}, \quad \text{Conj. 4.6} + \text{injectivity of } A \Rightarrow \text{Conj. 4.4}.$$

The (very strong) Conjecture 4.5 is known to hold for especially nice groups, such as free abelian groups [98], hyperbolic groups of low dimension [55], and finite groups with periodic cohomology [18], but it fails in general [55, 108]. Conjecture 4.6 is weaker, and holds for all the known counterexamples to Conjecture 4.5. It was formulated and proven for finite groups in [103]. Subsequently, Stolz [unpublished] showed that it follows from the Baum-Connes Conjecture, Conjecture 4.1. For a survey on this entire field, see [100].

The last "new direction" we would like to discuss here comes from replacing the higher signature in Novikov's Conjecture by the higher Todd genus or the higher elliptic genus. This seems to be quite relevant for understanding the interaction between topological invariants and algebraic geometry invariants for algebraic varieties defined over $\mathbb{C}$.

The Todd class $\mathscr{T}(M)$ is still another polynomial in characteristic classes, this time the rational Chern classes of a complex (or almost complex) manifold. Suppose for simplicity that M is a smooth projective variety over $\mathbb{C}$, viewed as a complex manifold via an embedding into some complex projective space. The Hirzebruch Riemann-Roch Theorem then says that

$$\langle \mathscr{T}(M), [M] \rangle = \chi(M, \mathscr{O}_M) = \sum_{j=0}^{n} (-1)^j \dim H^j(M, \mathscr{O}_M), \tag{8}$$

where $\mathscr{O}_M$ is the structure sheaf of M, the sheaf of germs of holomorphic functions, and n is the complex dimension of M. The right-hand side of (8) is called the *arithmetic genus*. (The original definition of the latter by algebraic geometers like Severi turned out to be $(-1)^n(\chi(M, \mathscr{O}_M) - 1)$, but the normalization here is a bit more convenient.) The left-hand side of (8) is called the *Todd genus*, and is known to be a birational invariant.[10] Once again, if one has a map $f : M \to B\pi$, then we can define the associated *higher Todd genus* as $f_*(\mathscr{T}(M) \cap [M]) \in H_\bullet(B\pi, \mathbb{Q})$.

Conjecture 4.7 (Algebraic Geometry Novikov Conjecture [101]). *Let M be a smooth complex projective variety, and let $f : M \to B\pi$ be a continuous map (for the topology of M as a complex manifold). Let $M' \overset{\varphi}{\dashrightarrow} M$ be a birational map. Then the corresponding higher Todd genera agree, i.e.,*

$$f_*(\mathscr{T}(M) \cap [M]) = (f \circ \varphi)_*(\mathscr{T}(M') \cap [M']) \in H_\bullet(B\pi, \mathbb{Q}).$$

Note the obvious similarity with Conjecture 1.2. However, unlike Novikov's original conjecture, this statement is actually a *theorem* [15, 19]. That follows from

[10] Recall that two varieties are said to be birationally equivalent if there are rational maps between them which are inverses of each. Since rational maps do not have to be everywhere defined (this is why we denote rational maps below by dotted lines), two varieties are birationally equivalent if and only if they have Zariski-open subsets which are isomorphic as varieties.

the fact that if $M' \overset{\varphi}{\dashrightarrow} M$ is a birational map, then $\varphi_*([D_{M'}]) = [D_M] \in K_0(M)$, where $[D_M]$ denotes the K-homology class of the Dolbeault operator, whose Chern character is $\mathscr{T}(M) \cap [M]$.[11] The corresponding statement for the signature operator is *not* true; a homotopy equivalence does not have to preserve the class of the signature operator. (However, the mod 8 reduction of this class *is* preserved [106].)

However, there is another similarity with Novikov's Conjecture which is pointed out in [101]. By [112, Théorème IV.17], $\Omega_\bullet$, the graded ring of cobordism classes of oriented manifolds, is, after tensoring with $\mathbb{Q}$, a polynomial ring in the classes of the complex projective spaces $\mathbb{CP}^{2k}$, $k \in \mathbb{N}$. Then if $I_\bullet$ is the ideal in $\Omega_\bullet$ generated by all $[M] - [M']$ with M and M' homotopy equivalent (in a way preserving orientation), Kahn [56] proved that $\Omega_\bullet / I_\bullet \cong \mathbb{Q}$, with the quotient map identified with the Hirzebruch signature. Similarly, $\Omega_\bullet^U$, the graded ring of cobordism classes of almost complex manifolds, is, after tensoring with $\mathbb{Q}$, a polynomial ring in the classes of all complex projective spaces, and the quotient of $\Omega_\bullet^U$ by the ideal generated by all $[M]-[M']$ with M and M' birationally equivalent smooth projective varieties is again $\mathbb{Q}$, this time with the quotient map identifiable with the Todd genus.

These results effectively say that, up to multiples, the signature is the only homotopy-invariant genus on oriented manifolds, and the arithmetic genus is the only birationally invariant genus on smooth projective varieties. But if one considers manifolds with large fundamental group, the situation changes. By [101, Theorem 4.1], a linear functional on $\Omega_\bullet(B\pi) \otimes \mathbb{Q}$ that is an oriented homotopy invariant must come from the higher signature, and by [101, Theorem 4.3], a linear functional on $\Omega_\bullet^U(B\pi) \otimes \mathbb{Q}$ that is a birational invariant must (under a certain technical condition satisfied in many cases) come from the higher Todd genus.

Finally, the papers [17, 24, 44] consider still more analogues of higher genera with the Todd genus replaced by the elliptic genus. The result of [17] is particularly nice; it is the exact analogue of Conjecture 4.7, but with the Todd genus replaced by the elliptic genus and with birational equivalence replaced by K-equivalence (a birational equivalence preserving canonical bundles).

Acknowledgements Work on this paper was partially supported by the United States National Science Foundation, grant number 1206159. I would like to thank Greg Friedman, Daniel Kasprowski, Andrew Ranicki, and Shmuel Weinberger for useful feedback on an earlier draft of this paper.

[11] It takes a bit of work to make sense of φ_* here, since φ may not be everywhere defined, but this can be done. The point is that by the factorization theorem for birational maps [1], we can factor φ into a sequence of blow-ups and blow-downs, and φ_* is clearly defined for a blow-down (since it is a continuous map) and is an isomorphism in this case by the Baum-Fulton-MacPherson variant of Grothendieck-Riemann-Roch [12]. In the case of a blow-up, let φ_* be given by the inverse of the map induced by the reverse blow-down.

References

1. Abramovich, D., Karu, K., Matsuki, K., Włodarczyk, J.: Torification and factorization of birational maps. J. Amer. Math. Soc. **15**(3), 531–572 (electronic) (2002). DOI 10.1090/S0894-0347-02-00396-X
2. Atiyah, M.F., Singer, I.M.: The index of elliptic operators. III. Ann. of Math. (2) **87**, 546–604 (1968)
3. Aubert, A.M., Baum, P., Plymen, R., Solleveld, M.: On the local Langlands correspondence for non-tempered representations. Münster J. Math. **7**, 27–50 (2014)
4. Bartels, A., Farrell, F.T., Lück, W.: The Farrell-Jones conjecture for cocompact lattices in virtually connected Lie groups. J. Amer. Math. Soc. **27**(2), 339–388 (2014). DOI 10.1090/S0894-0347-2014-00782-7
5. Bartels, A., Lück, W.: Isomorphism conjecture for homotopy K-theory and groups acting on trees. J. Pure Appl. Algebra **205**(3), 660–696 (2006). DOI 10.1016/j.jpaa.2005.07.020
6. Bartels, A., Lück, W., Reich, H.: The K-theoretic Farrell-Jones conjecture for hyperbolic groups. Invent. Math. **172**(1), 29–70 (2008). DOI 10.1007/s00222-007-0093-7
7. Bartels, A., Reich, H.: On the Farrell-Jones conjecture for higher algebraic K-theory. J. Amer. Math. Soc. **18**(3), 501–545 (2005). DOI 10.1090/S0894-0347-05-00482-0
8. Bass, H., Murthy, M.P.: Grothendieck groups and Picard groups of abelian group rings. Ann. of Math. (2) **86**, 16–73 (1967)
9. Baum, P., Connes, A.: Chern character for discrete groups. In: A fête of topology, pp. 163–232. Academic Press, Boston, MA (1988). DOI 10.1016/B978-0-12-480440-1.50015-0
10. Baum, P., Connes, A.: Geometric K-theory for Lie groups and foliations. Enseign. Math. (2) **46**(1–2), 3–42 (2000)
11. Baum, P., Connes, A., Higson, N.: Classifying space for proper actions and K-theory of group C^*-algebras. In: C^*-algebras: 1943–1993 (San Antonio, TX, 1993), *Contemp. Math.*, vol. 167, pp. 240–291. Amer. Math. Soc., Providence, RI (1994). DOI 10.1090/conm/167/1292018
12. Baum, P., Fulton, W., MacPherson, R.: Riemann-Roch and topological K theory for singular varieties. Acta Math. **143**(3–4), 155–192 (1979). DOI 10.1007/BF02392091
13. Baum, P., Guentner, E., Willett, R.: Expanders, exact crossed products, and the Baum-Connes conjecture. Ann. of K-Theory, **1**(2), 155–208 (2016). URL http://arxiv.org/abs/1311.2343.
14. Baum, P., Higson, N., Plymen, R.: A proof of the Baum-Connes conjecture for p-adic $\mathrm{GL}(n)$. C. R. Acad. Sci. Paris Sér. I Math. **325**(2), 171–176 (1997). DOI 10.1016/S0764-4442(97)84594-6
15. Block, J., Weinberger, S.: Higher Todd classes and holomorphic group actions. Pure Appl. Math. Q. **2**(4, Special Issue: In honor of Robert D. MacPherson. Part 2), 1237–1253 (2006). DOI 10.4310/PAMQ.2006.v2.n4.a13
16. Bökstedt, M., Hsiang, W.C., Madsen, I.: The cyclotomic trace and algebraic K-theory of spaces. Invent. Math. **111**(3), 465–539 (1993). DOI 10.1007/BF01231296
17. Borisov, L., Libgober, A.: Higher elliptic genera. Math. Res. Lett. **15**(3), 511–520 (2008). DOI 10.4310/MRL.2008.v15.n3.a10
18. Botvinnik, B., Gilkey, P., Stolz, S.: The Gromov-Lawson-Rosenberg conjecture for groups with periodic cohomology. J. Differential Geom. **46**(3), 374–405 (1997). URL http://projecteuclid.org/euclid.jdg/1214459973
19. Brasselet, J.P., Schürmann, J., Yokura, S.: Hirzebruch classes and motivic Chern classes for singular spaces. J. Topol. Anal. **2**(1), 1–55 (2010). DOI 10.1142/S1793525310000239
20. Browder, W.: Poincaré spaces, their normal fibrations and surgery. Invent. Math. **17**, 191–202 (1972)
21. Browder, W.: Surgery on simply-connected manifolds. Springer-Verlag, New York-Heidelberg (1972). Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65

22. Browder, W.: Differential topology of higher-dimensional manifolds. In: Surveys on surgery theory, Vol. 1, *Ann. of Math. Stud.*, vol. 145, pp. 41–71. Princeton Univ. Press, Princeton, NJ (2000)
23. Cappell, S.E.: On homotopy invariance of higher signatures. Invent. Math. **33**(2), 171–179 (1976)
24. Cappell, S.E., Libgober, A., Maxim, L.G., Shaneson, J.L.: Hodge genera of algebraic varieties. II. Math. Ann. **345**(4), 925–972 (2009). DOI 10.1007/s00208-009-0389-6
25. Carlsson, G.: Bounded K-theory and the assembly map in algebraic K-theory. In: Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), *London Math. Soc. Lecture Note Ser.*, vol. 227, pp. 5–127. Cambridge Univ. Press, Cambridge (1995). DOI 10.1017/CBO9780511629365.004
26. Carlsson, G., Pedersen, E.K.: Controlled algebra and the Novikov conjectures for K- and L-theory. Topology **34**(3), 731–758 (1995). DOI 10.1016/0040-9383(94)00033-H
27. Chang, S., Weinberger, S.: On invariants of Hirzebruch and Cheeger-Gromov. Geom. Topol. **7**, 311–319 (electronic) (2003). DOI 10.2140/gt.2003.7.311
28. Chen, X., Wang, Q., Yu, G.: The maximal coarse Baum-Connes conjecture for spaces which admit a fibred coarse embedding into Hilbert space. Adv. Math. **249**, 88–130 (2013). DOI 10.1016/j.aim.2013.09.003
29. Chen, X., Wang, Q., Yu, G.: The coarse Novikov conjecture and Banach spaces with Property (H). J. Funct. Anal. **268**(9), 2754–2786 (2015). DOI 10.1016/j.jfa.2015.02.001
30. Cortiñas, G., Tartaglia, G.: Trace class operators, regulators, and assembly maps in K-theory. Doc. Math. **19**, 439–456 (2014). URL https://www.math.uni-bielefeld.de/documenta/vol-19/14.pdf
31. Davis, J.F.: Manifold aspects of the Novikov conjecture. In: Surveys on surgery theory, Vol. 1, *Ann. of Math. Stud.*, vol. 145, pp. 195–224. Princeton Univ. Press, Princeton, NJ (2000)
32. Dranishnikov, A.N., Ferry, S.C., Weinberger, S.: Large Riemannian manifolds which are flexible. Ann. of Math. (2) **157**(3), 919–938 (2003). DOI 10.4007/annals.2003.157.919
33. Farrell, F.T., Hsiang, W.C.: Manifolds with $\pi_1 = G \times_\alpha T$. Amer. J. Math. **95**, 813–848 (1973)
34. Farrell, F.T., Jones, L.E.: Negatively curved manifolds with exotic smooth structures. J. Amer. Math. Soc. **2**(4), 899–908 (1989). DOI 10.2307/1990898
35. Farrell, F.T., Jones, L.E.: Exotic smoothings of hyperbolic manifolds which do not support pinched negative curvature. Proc. Amer. Math. Soc. **121**(2), 627–630 (1994). DOI 10.2307/2160446
36. Ferry, S., Rosenberg, J., Weinberger, S.: Phénomènes de rigidité topologique équivariante. C. R. Acad. Sci. Paris Sér. I Math. **306**(19), 777–782 (1988)
37. Ferry, S.C., Ranicki, A., Rosenberg, J.: A history and survey of the Novikov conjecture. In: Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), *London Math. Soc. Lecture Note Ser.*, vol. 226, pp. 7–66. Cambridge Univ. Press, Cambridge (1995). DOI 10.1017/CBO9780511662676.003
38. Ferry, S.C., Ranicki, A., Rosenberg, J. (eds.): Novikov conjectures, index theorems and rigidity. Vol. 1, *London Mathematical Society Lecture Note Series*, vol. 226. Cambridge University Press, Cambridge (1995). Including papers from the conference held at the Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, September 6–10, 1993
39. Ferry, S.C., Ranicki, A., Rosenberg, J. (eds.): Novikov conjectures, index theorems and rigidity. Vol. 2, *London Mathematical Society Lecture Note Series*, vol. 227. Cambridge University Press, Cambridge (1995). Including papers from the conference held at the Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, September 6–10, 1993
40. Finn-Sell, M.: Fibred coarse embeddings, a-T-menability and the coarse analogue of the Novikov conjecture. J. Funct. Anal. **267**(10), 3758–3782 (2014). DOI 10.1016/j.jfa.2014.09.012
41. Freedman, M.H.: The topology of four-dimensional manifolds. J. Differential Geom. **17**(3), 357–453 (1982). URL http://projecteuclid.org/euclid.jdg/1214437136
42. Fukaya, T., Oguni, S.i.: Coronae of product spaces and the coarse Baum–Connes conjecture. Adv. Math. **279**, 201–233 (2015). DOI 10.1016/j.aim.2015.01.022

43. Gong, D.: Equivariant Novikov conjecture for groups acting on Euclidean buildings. Trans. Amer. Math. Soc. **350**(6), 2141–2183 (1998). DOI 10.1090/S0002-9947-98-01990-4
44. Gong, D., Liu, K.: Rigidity of higher elliptic genera. Ann. Global Anal. Geom. **14**(3), 219–236 (1996). DOI 10.1007/BF00054471
45. Gromov, M.: Positive curvature, macroscopic dimension, spectral gaps and higher signatures. In: Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993), *Progr. Math.*, vol. 132, pp. 1–213. Birkhäuser Boston, Boston, MA (1996). DOI 10.1007/s10107-010-0354-x
46. Gromov, M., Lawson Jr., H.B.: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. Inst. Hautes Études Sci. Publ. Math. **58**, 83–196 (1984) (1983). URL http://www.numdam.org/item?id=PMIHES_1983__58__83_0
47. Higson, N.: The Baum-Connes conjecture. In: Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Extra Vol. II, pp. 637–646 (electronic) (1998). URL https://www.math.uni-bielefeld.de/documenta/xvol-icm/08/08.html
48. Higson, N., Lafforgue, V., Skandalis, G.: Counterexamples to the Baum-Connes conjecture. Geom. Funct. Anal. **12**(2), 330–354 (2002). DOI 10.1007/s00039-002-8249-5
49. Higson, N., Roe, J.: On the coarse Baum-Connes conjecture. In: Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993), *London Math. Soc. Lecture Note Ser.*, vol. 227, pp. 227–254. Cambridge Univ. Press, Cambridge (1995). DOI 10.1017/CBO9780511629365.008
50. Higson, N., Roe, J.: Analytic K-homology. Oxford Mathematical Monographs. Oxford University Press, Oxford (2000). Oxford Science Publications
51. Higson, N., Roe, J.: Mapping surgery to analysis. I. Analytic signatures. K-Theory **33**(4), 277–299 (2005). DOI 10.1007/s10977-005-1561-8
52. Higson, N., Roe, J.: Mapping surgery to analysis. II. Geometric signatures. K-Theory **33**(4), 301–324 (2005). DOI 10.1007/s10977-005-1559-2
53. Higson, N., Roe, J.: Mapping surgery to analysis. III. Exact sequences. K-Theory **33**(4), 325–346 (2005). DOI 10.1007/s10977-005-1554-7
54. Hirzebruch, F.: The signature theorem: reminiscences and recreation. In: Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), pp. 3–31. Ann. of Math. Studies, No. 70. Princeton Univ. Press, Princeton, N.J. (1971)
55. Joachim, M., Schick, T.: Positive and negative results concerning the Gromov-Lawson-Rosenberg conjecture. In: Geometry and topology: Aarhus (1998), *Contemp. Math.*, vol. 258, pp. 213–226. Amer. Math. Soc., Providence, RI (2000). DOI 10.1090/conm/258/04066
56. Kahn, P.J.: Characteristic numbers and oriented homotopy type. Topology **3**, 81–95 (1965)
57. Kasparov, G.: Novikov's conjecture on higher signatures: the operator K-theory approach. In: Representation theory of groups and algebras, *Contemp. Math.*, vol. 145, pp. 79–99. Amer. Math. Soc., Providence, RI (1993). DOI 10.1090/conm/145/1216182
58. Kasparov, G.G.: The homotopy invariance of rational Pontrjagin numbers. Dokl. Akad. Nauk SSSR **190**, 1022–1025 (1970)
59. Kasparov, G.G.: Topological invariants of elliptic operators. I. K-homology. Izv. Akad. Nauk SSSR Ser. Mat. **39**(4), 796–838 (1975)
60. Kasparov, G.G.: The operator K-functor and extensions of C^*-algebras. Izv. Akad. Nauk SSSR Ser. Mat. **44**(3), 571–636, 719 (1980)
61. Kasparov, G.G.: Equivariant KK-theory and the Novikov conjecture. Invent. Math. **91**(1), 147–201 (1988). DOI 10.1007/BF01404917
62. Kasparov, G.G.: K-theory, group C^*-algebras, and higher signatures (conspectus). In: Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), *London Math. Soc. Lecture Note Ser.*, vol. 226, pp. 101–146. Cambridge Univ. Press, Cambridge (1995). DOI 10.1017/CBO9780511662676.007
63. Kasprowski, D.: On the K-theory of groups with finite decomposition complexity. Proc. Lond. Math. Soc. (3) **110**(3), 565–592 (2015). DOI 10.1112/plms/pdu062
64. Kervaire, M.A.: Le théorème de Barden-Mazur-Stallings. Comment. Math. Helv. **40**, 31–42 (1965)

65. Kervaire, M.A., Milnor, J.W.: Groups of homotopy spheres. I. Ann. of Math. (2) **77**, 504–537 (1963)
66. Kreck, M., Lück, W.: The Novikov conjecture, Geometry and algebra, *Oberwolfach Seminars*, vol. 33. Birkhäuser Verlag, Basel (2005).
67. Lafforgue, V.: K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. Invent. Math. **149**(1), 1–95 (2002). DOI 10.1007/s002220200213
68. Lichnerowicz, A.: Spineurs harmoniques. C. R. Acad. Sci. Paris **257**, 7–9 (1963)
69. Loday, J.L.: K-théorie algébrique et représentations de groupes. Ann. Sci. École Norm. Sup. (4) **9**(3), 309–377 (1976)
70. Lück, W.: K- and L-theory of group rings. In: Proceedings of the International Congress of Mathematicians. Volume II, pp. 1071–1098. Hindustan Book Agency, New Delhi (2010)
71. Lück, W., Reich, H.: The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In: Handbook of K-theory. Vol. 1, 2, pp. 703–842. Springer, Berlin (2005). DOI 10.1007/978-3-540-27855-9_15. URL http://k-theory.org/handbook/2-0703-0842.pdf
72. Lusztig, G.: Novikov's higher signature and families of elliptic operators. J. Differential Geometry **7**, 229–256 (1972)
73. Mendes, S., Plymen, R.: Base change and K-theory for $\mathrm{GL}(n)$. J. Noncommut. Geom. **1**(3), 311–331 (2007). DOI 10.4171/JNCG/9
74. Milnor, J.: On manifolds homeomorphic to the 7-sphere. Ann. of Math. (2) **64**, 399–405 (1956)
75. Milnor, J.: A procedure for killing homotopy groups of differentiable manifolds. In: Proc. Sympos. Pure Math., Vol. III, pp. 39–55. American Mathematical Society, Providence, R.I (1961)
76. Milnor, J.: Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J. (1965)
77. Milnor, J.: Whitehead torsion. Bull. Amer. Math. Soc. **72**, 358–426 (1966)
78. Mishchenko, A.S.: Infinite-dimensional representations of discrete groups, and higher signatures. Izv. Akad. Nauk SSSR Ser. Mat. **38**, 81–106 (1974)
79. Mishchenko, A.S.: C^*-algebras and K-theory. In: Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), *Lecture Notes in Math.*, vol. 763, pp. 262–274. Springer, Berlin (1979)
80. Mitchener, P.D.: The general notion of descent in coarse geometry. Algebr. Geom. Topol. **10**(4), 2419–2450 (2010). DOI 10.2140/agt.2010.10.2419. URL http://msp.org/agt/2010/10-4/p18.xhtml
81. Novikov, S.P.: A diffeomorphism of simply connected manifolds. Dokl. Akad. Nauk SSSR **143**, 1046–1049 (1962)
82. Novikov, S.P.: Homotopically equivalent smooth manifolds. I. Izv. Akad. Nauk SSSR Ser. Mat. **28**, 365–474 (1964)
83. Novikov, S.P.: Rational Pontrjagin classes. Homeomorphism and homotopy type of closed manifolds. I. Izv. Akad. Nauk SSSR Ser. Mat. **29**, 1373–1388 (1965)
84. Novikov, S.P.: Topological invariance of rational classes of Pontrjagin. Dokl. Akad. Nauk SSSR **163**, 298–300 (1965)
85. Novikov, S.P.: Pontrjagin classes, the fundamental group and some problems of stable algebra. In: Amer. Math. Soc. Translations Ser. 2, Vol. 70: 31 Invited Addresses (8 in Abstract) at the Internat. Congr. Math. (Moscow, 1966), pp. 172–179. Amer. Math. Soc., Providence, R.I. (1968)
86. Novikov, S.P.: Algebraic construction and properties of Hermitian analogs of K-theory over rings with involution from the viewpoint of Hamiltonian formalism. Applications to differential topology and the theory of characteristic classes. I. II. Izv. Akad. Nauk SSSR Ser. Mat. **34**, 253–288; ibid. 34 (1970), 475–500 (1970)
87. Novikov, S.P.: Pontrjagin classes, the fundamental group and some problems of stable algebra. In: Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), pp. 147–155. Springer, New York (1970)

88. Novikov, S.P.: Analogues hermitiens de la K-théorie. In: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 39–45. Gauthier-Villars, Paris (1971)
89. Piazza, P., Schick, T.: The surgery exact sequence, K-theory and the signature operator. Ann. of K-Theory, **1**(2), 109–154 (2016). URL http://arxiv.org/abs/1309.4370, DOI 10.2140/akt.2016.1.109.
90. Plymen, R.J.: Reduced C^*-algebra of the p-adic group $\mathrm{GL}(n)$. II. J. Funct. Anal. **196**(1), 119–134 (2002). DOI 10.1006/jfan.2002.3980
91. Quinn, F.: A geometric formulation of surgery. In: Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), pp. 500–511. Markham, Chicago, Ill. (1970)
92. Ramras, D.A., Tessera, R., Yu, G.: Finite decomposition complexity and the integral Novikov conjecture for higher algebraic K-theory. J. Reine Angew. Math. **694**, 129–178 (2014). DOI 10.1515/crelle-2012-0112
93. Ranicki, A.: The total surgery obstruction. In: Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), *Lecture Notes in Math.*, vol. 763, pp. 275–316. Springer, Berlin (1979)
94. Ranicki, A.: Algebraic and geometric surgery. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford (2002). DOI 10.1093/acprof:oso/9780198509240.001.0001. Oxford Science Publications
95. Ranicki, A.A.: Algebraic L-theory and topological manifolds, *Cambridge Tracts in Mathematics*, vol. 102. Cambridge University Press, Cambridge (1992)
96. Roe, J.: Coarse cohomology and index theory on complete Riemannian manifolds. Mem. Amer. Math. Soc. **104**(497), x+90 (1993). DOI 10.1090/memo/0497
97. Rosenberg, J.: C^*-algebras, positive scalar curvature, and the Novikov conjecture. Inst. Hautes Études Sci. Publ. Math. **58**, 197–212 (1984) (1983). URL http://www.numdam.org/item?id=PMIHES_1983__58__197_0
98. Rosenberg, J.: C^*-algebras, positive scalar curvature, and the Novikov conjecture. III. Topology **25**(3), 319–336 (1986). DOI 10.1016/0040-9383(86)90047-9
99. Rosenberg, J.: Analytic Novikov for topologists. In: Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), *London Math. Soc. Lecture Note Ser.*, vol. 226, pp. 338–372. Cambridge Univ. Press, Cambridge (1995). DOI 10.1017/CBO9780511662676.013
100. Rosenberg, J.: Manifolds of positive scalar curvature: a progress report. In: Surveys in differential geometry. Vol. XI, *Surv. Differ. Geom.*, vol. 11, pp. 259–294. Int. Press, Somerville, MA (2007). DOI 10.4310/SDG.2006.v11.n1.a9
101. Rosenberg, J.: An analogue of the Novikov conjecture in complex algebraic geometry. Trans. Amer. Math. Soc. **360**(1), 383–394 (2008). DOI 10.1090/S0002-9947-07-04320-6
102. Rosenberg, J.: Structure and applications of real C^*-algebras. In: R.S. Doran, E. Park (eds.) Operator Algebras and Their Applications: A Tribute to Richard V. Kadison. Contemporary Math., Amer. Math. Soc., 2016. URL http://arxiv.org/abs/1505.04091
103. Rosenberg, J., Stolz, S.: A "stable" version of the Gromov-Lawson conjecture. In: The Čech centennial (Boston, MA, 1993), *Contemp. Math.*, vol. 181, pp. 405–418. Amer. Math. Soc., Providence, RI (1995). DOI 10.1090/conm/181/02046
104. Rosenberg, J., Weinberger, S.: Higher G-indices and applications. Ann. Sci. École Norm. Sup. (4) **21**(4), 479–495 (1988). URL http://www.numdam.org/item?id=ASENS_1988_4_21_4_479_0
105. Rosenberg, J., Weinberger, S.: An equivariant Novikov conjecture. K-Theory **4**(1), 29–53 (1990). DOI 10.1007/BF00534192. With an appendix by J. P. May
106. Rosenberg, J., Weinberger, S.: The signature operator at 2. Topology **45**(1), 47–63 (2006). DOI 10.1016/j.top.2005.06.001
107. Roushon, S.K.: The isomorphism conjecture in L-theory: graphs of groups. Homology Homotopy Appl. **14**(1), 1–17 (2012). DOI 10.4310/HHA.2012.v14.n1.a1
108. Schick, T.: A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. Topology **37**(6), 1165–1168 (1998). DOI 10.1016/S0040-9383(97)00082-7

109. Smale, S.: Generalized Poincaré's conjecture in dimensions greater than four. Ann. of Math. (2) **74**, 391–406 (1961)
110. Solleveld, M.: Periodic cyclic homology of reductive p-adic groups. J. Noncommut. Geom. **3**(4), 501–558 (2009). DOI 10.4171/JNCG/45
111. Sullivan, D.P.: Triangulating and smoothing homotopy equivalences and homeomorphisms. Geometric Topology Seminar Notes. In: The Hauptvermutung book, *K-Monogr. Math.*, vol. 1, pp. 69–103. Kluwer Acad. Publ., Dordrecht (1996). DOI 10.1007/978-94-017-3343-4_3
112. Thom, R.: Quelques propriétés globales des variétés différentiables. Comment. Math. Helv. **28**, 17–86 (1954). URL http://retro.seals.ch/digbib/view?rid=comahe-002:1954:28::8
113. Tu, J.L.: The Baum-Connes conjecture for groupoids. In: C^*-algebras (Münster, 1999), pp. 227–242. Springer, Berlin (2000) www.e-periodica.ch/digbib/view?pid=com-001:1954:28::8
114. Tu, J.L.: The coarse Baum-Connes conjecture and groupoids. II. New York J. Math. **18**, 1–27 (2012). URL http://nyjm.albany.edu:8000/j/2012/18_1.html
115. Wall, C.T.C.: Surgery of non-simply-connected manifolds. Ann. of Math. (2) **84**, 217–276 (1966)
116. Wall, C.T.C.: Surgery on compact manifolds, *Mathematical Surveys and Monographs*, vol. 69, second edn. American Mathematical Society, Providence, RI (1999). DOI 10.1090/surv/069. Edited and with a foreword by A. A. Ranicki
117. Wallace, A.H.: Modifications and cobounding manifolds. Canad. J. Math. **12**, 503–528 (1960)
118. Wassermann, A.: Une démonstration de la conjecture de Connes-Kasparov pour les groupes de Lie linéaires connexes réductifs. C. R. Acad. Sci. Paris Sér. I Math. **304**(18), 559–562 (1987)
119. Weinberger, S.: The topological classification of stratified spaces. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL (1994)
120. Yu, G.: Coarse Baum-Connes conjecture. K-Theory **9**(3), 199–221 (1995). DOI 10.1007/BF00961664
121. Yu, G.: The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. Invent. Math. **139**(1), 201–240 (2000). DOI 10.1007/s002229900032