An Iterative Regularization Algorithm for the TV-Stokes in Image Processing

Leszek Marcinkowski[1(✉)] and Talal Rahman[2]

[1] Faculty of Mathematics, Informatics and Mechanics,
Institute of Applied Mathematics and Mechanics,
University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
L.Marcinkowski@mimuw.edu.pl

[2] Department of Computing, Mathematics and Physics,
Bergen University College, Inndalsveien 28, 5063 Bergen, Norway
Talal.Rahman@hib.no

Abstract. Image denoising is one of the fundamental problems in the image processing. In a PDE based approach for image processing, the simplest possible method for denoising is to solve the heat equation. However such a diffusion equation will destroy sharp edges in the image. An approach known for preserving the edges while denoising is called the classical Rudin-Osher-Fatemi (ROF) method based on the total variation (TV) regularization. Recently, an algorithm, also known as the TV-Stokes, based on two minimization steps involving the smoothing of the tangential field and then the reconstruction of the image has been proposed. The latter produces images without the blocky effect which we observe in the case of the ROF model. An iterative regularization method for the total variation based image restoration has recently been proposed giving significant improvement over the classical method in the quality of the restored image. In this paper we propose a similar algorithm for the TV-Stokes denoising algorithm.

Keywords: Iterative regularization · Total variation · TV-Stokes · Denoising

1 Introduction

Recovering an image from a noisy and blurry image is an inverse problem which is solved via variational methods, e.g. cf [1,7]. This requires the minimization of some energy functional.

By the Euler-Lagrange formulation it results into a set of nonlinear partial differential equations which are then solved using say, the gradient-descent iteration, see for instance [12] for the classical model of Rudin, Osher and Fatemi

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(ROF model), which is based on the total variation (TV) regularization of the intensity (gray level), and [9,11] for an improved model (TV-Stokes model) which is based on the total variation regularization of the tangential field of the intensity. The drawback of such algorithms is that their convergence is very slow, particularly for large images. There exist now algorithms which are much faster, those based on the dual formulation of the underlying models, see for instance [2,3,8]

An iterative regularization algorithm for the ROF model has recently been proposed, cf. [10], giving significant improvement over the classical method in the quality of the restored image. The main purpose of this paper is to propose a similar algorithm for the TV-Stokes model, and its dual formulation for faster convergence. The paper is organized as follows: in Sect. 2 we present the iterative regularization algorithm for the ROF model, and in Sect. 3 we propose a similar algorithm for the TV-Stokes model. In Sect. 4 we describe Chambolle's iteration for the dual formulation of the TV-Stokes model, which we use for the numerical experiments of Sect. 5.

2 Iterative Regularization for the TV Denoising

Let the noisy image d_0 represented as scalar $L^2(\Omega)$ function be given. The classical denoising method is based on the minimization problem:

$$\min_d \int_\Omega |\nabla d|\, d\mathbf{x} \;+\; \frac{\lambda}{2}\int_\Omega (d_0 - d)^2 d\mathbf{x}, \tag{1}$$

where λ is a constant which is used to balance between the smoothing of the image and the fidelity to the input image. It is difficult to know how to choose λ. An equivalent formulation of (1) is the following constrained minimization problem, cf. e.g. [4]:

$$\min_{\|d_0-d\|^2_{L^2}=\sigma^2} \int_\Omega |\nabla d|\, d\mathbf{x}, \tag{2}$$

where σ is the noise level. One often has a reasonable estimate of the noise level. In the original paper [12], a gradient projection method was used to solve (2). The method is known for its good edge preserving capability. It suffers however from its blocky effect on the resulting image. Not just that, it looses quite easily the high frequency part of the image as well. The recently proposed iterative regularization method [10], an algorithm which is based on the original TV denoising algorithm, has proven to give a much better result than the constrained denoising algorithm of ROF.

Given d_0, λ, and $v_0 = 0$. For $k = 0, 1, 2, \ldots$, find the minimizer d_{k+1} of the following minimization problem,

$$\min_d \int_\Omega |\nabla d|\, d\mathbf{x} \;+\; \frac{\lambda}{2}\int_\Omega (d_0 + v_k - d)^2 d\mathbf{x}, \tag{3}$$

and update

$$v_{k+1} = v_k + d_0 - d_{k+1}. \tag{4}$$

Algorithm: TV Iterative Regularization
Given d_0 and λ ;
Initialize counter: $k = 0$;
Set: $v_0 = 0$;
while *not converged* **do**
  Initialize counter: $n = 0$;
  Set: $u^0 = v_k + d_0$;
  while *not converged* **do**
    Calculate u^{n+1}:

$$\frac{u^{n+1} - u^n}{\Delta t} = \nabla \cdot \left(\frac{\nabla u^n}{|\nabla u^n|} \right) + \lambda(u^0 - u^n) \tag{5}$$

    Update counter: $n = n + 1$;
  end
  Set: $d_{k+1} = u^n$;
  Update:

$$v_{k+1} = v_k + d_0 - d_{k+1} \tag{6}$$

  Update counter: $k = k + 1$;
end

Algorithm 1. Iterative regularization for ROF denoising.

For stopping the iterative procedure, a reasonable criterion to use is the discrepancy principle, that is to stop the iteration the first time the residual $\|d_0 - d_k\|_{L^2}$ is of the same order as the noise level σ, cf. [10]. We know that the problem (3) has a unique solution. It is shown in [10] that d_k will converge to the original noisy image d_0 as we continue to iterate beyond the discrepancy point.

2.1 Discrete Algorithm

The algorithm consists of two loops. The first one, which will be the outer loop, we call it the k-loop. In each iteration of the k-loop, we need the minimizer of the classical ROF model, which we do by the descent technique, iterating over an artificial time step to steady state.

2.2 Discretization

For the time discretization we use an explicit scheme, where, in each time step, the nonlinear term is calculated using values from the previous time step and is therefore a known quantity. Each vertex of the rectangular grid corresponds to the position of a pixel or pixel center where the image intensity variable d is defined, cf. Fig. 1 (Right).

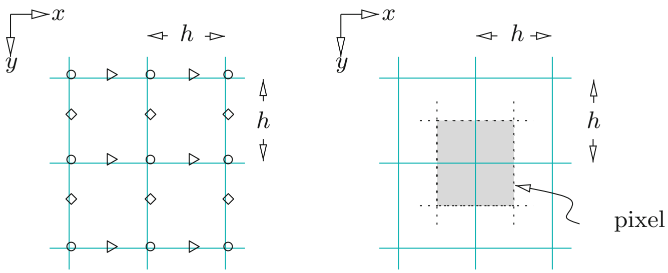

Fig. 1. Left: the computational grid with approximating points for the variables d, d_x, and d_y, represented by $\circ, \triangleright$, and $\diamond$, respectively. Right: mapping the computational grid onto the pixels.

For the space discretization, we approximate the derivatives by finite differences using the standard forward/backward difference operators $D_x^{\pm}$ and $D_y^{\pm}$, and the centered difference operators C_x^h and C_y^h, respectively in the x and y directions, as $D_x^{\pm} f = \pm\frac{f(x\pm h,y)-f(x,y)}{h}$, $D_y^{\pm} f = \pm\frac{f(x,y\pm h)-f(x,y)}{h}$, $C_y^h f = \frac{f(x+h,y)-f(x-h,y)}{2h}$, and $C_y^h f = \frac{f(x,y+h)-f(x,y-h)}{2h}$ for any function f, where h correspond to the $h-$spacing. We introduce two average operators A_x and A_y as $A_x f = (f(x,y) + f(x+h,y))/2$ and $A_y f = (f(x,y) + f(x,y+h))/2$.

The discrete approximation of (5), thus, takes the following form:

$$\nabla \cdot \left(\frac{\nabla u^n}{|\nabla u^n|}\right) + \lambda(u^0 - u^n) \approx D_x^- \left(\frac{D_x^+ u^n}{T_1^n}\right) + D_y^- \left(\frac{D_y^+ u^n}{T_2^n}\right) + \lambda(u^0 - u^n), \quad (7)$$

where T_1^n is defined as $T_1^n = \sqrt{\left(D_x^+ u^n\right)^2 + \left(A_x(C_y^h u^n)\right)^2 + \epsilon}$, and T_2^n as $T_2^n = \sqrt{\left(D_y^+ u^n\right)^2 + \left(A_y(C_x^h u^n)\right)^2 + \epsilon}$. Here ϵ is a small number.

3 Iterative Regularization for the TV-Stokes

3.1 The TV-Stokes Denoising

Let the noisy image d_0 represented as scalar $L^2(\Omega)$ function be given. We compute $\tau_0 = \nabla^{\perp} d_0$. The algorithm is then defined in two steps, see [9,11]. In the first step, writing the tangent vector as $\tau = (v, u)$, we solve the following minimization problem:

$$\min_{\tau} \int_{\Omega} (|\nabla v| + |\nabla u|)\, d\mathbf{x} \; + \; \frac{\delta}{2} \int_{\Omega} |\tau - \tau_{\mathbf{0}}|^2 \; d\mathbf{x} \quad (8)$$

subject to $\nabla \cdot \tau = 0$, where δ is a constant which is used to balance between the smoothing of the tangent field and the fidelity to the input tangent field. Once we have the smoothed tangent field, we can get the corresponding normal field $\mathbf{n} = (u, -v)$. In the second step, we reconstruct our image by fitting it to the normal field through solving the following minimization problem:

$$\min_{d} \int_{\Omega} \left((|\nabla d| - \nabla d \frac{\mathbf{n}}{|\mathbf{n}|}\right) d\mathbf{x} + \frac{\lambda}{2} \int_{\Omega} (d_0 - d)^2 d\mathbf{x}. \quad (9)$$

As before, let $\|d - d_0\|_{L^2}^2 = \sigma^2$ be the estimated noise variance. This can be estimated using statistical methods. If the exact noise variance cannot be obtained, then an approximate value may be used. In which case, a larger value would result in over-smoothing and a smaller value would result in under-smoothing.

3.2 Iterative Regularization

Given $d_0, s_0 = 0, \delta$ and λ. For $k = 0, 1, 2, \ldots$, in the first step, we compute $\tau_0 = \nabla^{\perp}(d_0 + s_k)$, and we solve the following minimization problem.

$$\min_{\tau} \int_{\Omega} (|\nabla v| + |\nabla u|)\, d\mathbf{x} \;+\; \frac{\delta}{2} \int_{\Omega} |\tau - \tau_{\mathbf{0}}|^2 \; d\mathbf{x}, \tag{10}$$

subject to $\nabla \cdot \tau = 0$.

Once we have the smoothed tangent field, we get the corresponding normal field $\mathbf{n} = (u, -v)$. In the second step, we reconstruct our image by fitting it to the normal field through solving the following minimization problem.

$$\min_{d} \int_{\Omega} \left((|\nabla d| - \nabla d \frac{\mathbf{n}}{|\mathbf{n}|} \right) d\mathbf{x} \;+\; \frac{\lambda}{2} \int_{\Omega} (d_0 + s_k - d)^2 d\mathbf{x}, \tag{11}$$

and update $s_{k+1} = s_k + d_0 - d_{k+1}$. For stopping of the iterative procedure, we use the discrepancy principle, that is to stop the iteration the first time the residual $\|d_0 - d_k\|_{L^2}$ is of the same order as the noise level σ, cf. [10]. It is possible to show that d_k will converge to the original noisy imaged_0 as we continue to iterate beyond the discrepancy point.

3.3 Discrete Algorithm

The algorithm consists of two loops, the outer loop being the k-loop as before. In each iteration of the k-loop, the two minimizing steps of the TV-Stokes algorithms is performed. The discrete algorithm is in Algorithm 2 below.

3.4 Discretization

For the time discretization, we use an explicit scheme, where, in each time step, the nonlinear term is calculated using values from the previous time step and is therefore a known quantity. As before, each vertex of the rectangular grid corresponds to the position of a pixel or pixel center where the image intensity variable d is defined, cf. Fig. 1 (Right).

For the space discretization again we use a staggered grid, cf. Fig. 1 (Left). We approximate the derivatives by finite differences using the standard forward/backward difference operators $D_x^{\pm}$ and $D_y^{\pm}$, and the centered difference operators C_x^h and C_y^h, respectively in the x and y directions, as described in Sect. 2.

The discrete approximation of (15)–(17) are as follows

$$\frac{v^{n+1}-v^n}{\Delta t} = D_x^-\left(\frac{D_x^+ v^n}{T_1^n(v)}\right) + D_y^-\left(\frac{D_y^+ v^n}{T_2^n(v)}\right) + \delta(v^0 - v^n) + D_x^- q^n \qquad (12)$$

$$\frac{u^{n+1}-u^n}{\Delta t} = D_x^-\left(\frac{D_x^+ u^n}{T_1^n(u)}\right) + D_y^-\left(\frac{D_y^+ u^n}{T_2^n(u)}\right) + \delta(u^0 - u^n) + D_y^- q^n \qquad (13)$$

$$\frac{q^{n+1}-q^n}{\Delta t} = D_x^+ v^n + D_y^+ u^n \qquad (14)$$

where $T_1^n(u)$ is defined as $T_1^n(u) = \sqrt{\left(D_x^+ u^n\right)^2 + \left(A_x(C_y^h u^n)\right)^2 + \epsilon}$ and $T_2^n(u)$ as $T_2^n(u) = \sqrt{\left(D_y^+ u^n\right)^2 + \left(A_y(C_x^h u^n)\right)^2 + \epsilon}$. Analogously, we define $T_1^n(v)$ and $T_2^n(v)$ by replacing u with v.

Algorithm: TV-Stokes iterative regularization
Given d_0, δ and λ ;
Initialize counter: $k = 0$;
Set: $s_0 = 0$;
while *not converged* **do**
  Initialize counter: $n = 0$;
  Set: $w^0 = d_0 + s_k$ and $(v^0, u^0) = \nabla^\perp w^0$, $q^0 = 0$;
  while *not converged* **do**
    Calculate $\tau^{n+1} = (v^{n+1}, u^{n+1})$:

$$\frac{v^{n+1}-v^n}{\Delta t} = \nabla\cdot\left(\frac{\nabla v^n}{|\nabla v^n|}\right) - \delta\left(v^n - v^0\right) + \frac{\partial q^n}{\partial x} \qquad (15)$$

$$\frac{u^{n+1}-u^n}{\Delta t} = \nabla\cdot\left(\frac{\nabla u^n}{|\nabla u^n|}\right) - \delta\left(u^n - u^0\right) + \frac{\partial q^n}{\partial y} \qquad (16)$$

$$\frac{q^{n+1}-q^n}{\Delta t} = \frac{\partial v^n}{\partial x} + \frac{\partial u^n}{\partial y} \qquad (17)$$

    Update counter: $n = n + 1$;
  end
  Set $\mathbf{n} = (u^{n+1}, -v^{n+1})$;
  while *not converged* **do**
    Calculate w^{n+1}:

$$\frac{w^{n+1}-w^n}{\Delta t} = \nabla\left(\frac{\nabla d}{|\nabla d|} - \frac{\mathbf{n}}{|\mathbf{n}|}\right) + \lambda(w^0 - w^n) \qquad (18)$$

    Update counter: $n = n + 1$;
  end
  Set: $d_{k+1} = w^{n+1}$;
  Update:

$$s_{k+1} = s_k + d_0 - d_{k+1} \qquad (19)$$

  Update counter: $k = k + 1$;
end

Algorithm 2. Iterative regularization for TV Stokes denoising.

The discrete approximation of (18) is defined as follows

$$\frac{w^{n+1} - w^n}{\Delta t} = D_x^- \left(\frac{D_x^+ w^n}{T_3^n} - n_1 \right) + D_y^- \left(\frac{D_y^+ w^n}{T_4^n} - n_2 \right) + \lambda(w^0 - w^n) \quad (20)$$

where T_3^n is defined as $T_3^n = \sqrt{\left(D_x^+ w^n\right)^2 + \left(A_x(C_y^h w^n)\right)^2 + \epsilon}$ and T_4^n as $T_4^n = \sqrt{\left(D_y^+ w^n\right)^2 + (A_y(C_x^h w^n)^2 + \epsilon}$. and n_k, for $k = 1$ and 2, respectively as $n_1 = \frac{u}{\sqrt{u^2 + (A_x(A_y v))^2) + \epsilon}}$ and $n_2 = \frac{-v}{\sqrt{v^2 + (A_y(A_x u))^2) + \epsilon}}$.

4 Chambolle's Algorithm

In this section we present a dual approach for solving our TV Stokes iterative regularization, cf. [5,6,8]. We consider the image in $L^2(\Omega)$ be approximated on the regular mesh and be represented as $d \in \mathbb{R}^{N \times N}$. The derivative matrices, corresponding to the u and v, are then computed naturally from d using appropriate finite differences, which again constitute the pair of matrices corresponding to the tangential vector $\tau = (v, u)$.

4.1 First Step

In the first step, we consider the minimization problem (10):

$$\min_{\nabla \tau = 0} \int_\Omega (|\nabla v| + |\nabla u|)\, d\mathbf{x} \ + \ \frac{\delta}{2} \int_\Omega |\tau - \tau_\mathbf{0}|^2 \ d\mathbf{x}. \quad (21)$$

Using a dual formulation of the TV norm we can write

$$\int_\Omega (|\nabla v| + |\nabla u|)\, d\mathbf{x} = \max_\mathbf{G} \int_\Omega \langle \tau, \nabla \cdot \mathbf{G} \rangle \, dx,$$

where $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, and $\mathbf{G} = (\mathbf{g}_1, \mathbf{g}_2)^T$ is the dual variable such that $\mathbf{g}_i \in C_c^1(\Omega)^2$ and $|\mathbf{g}_i|_\infty \leq 1$. Using this, (10) can be reformulated as

$$\min_{\nabla \tau = 0} \max_\mathbf{G} \int_\Omega \langle \tau, \nabla \cdot \mathbf{G} \rangle d\mathbf{x} \ + \ \frac{\delta}{2} \int_\Omega |\tau - \tau_\mathbf{0}|^2 \ d\mathbf{x}. \quad (22)$$

Here $\nabla \cdot \mathbf{G} = (\nabla \cdot \mathbf{g}_1, \nabla \cdot \mathbf{g}_2)^T$. We define the orthogonal projection Π_Y onto $Y = \{\tau : \nabla \cdot \tau = 0\}$ as

$$\Pi_Y \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} - \nabla \triangle^\dagger \nabla \cdot \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \quad (23)$$

We note that $\nabla \cdot \tau = 0$ is equivalent to $\Pi_Y \tau = \tau$; using this, and exchanging min and max, we get

$$\max_\mathbf{G} \min_\tau \int_\Omega \langle \tau, \Pi_Y \nabla \cdot \mathbf{G} \rangle d\mathbf{x} \ + \ \frac{\delta}{2} \int_\Omega |\tau - \tau_\mathbf{0}|^2 \ d\mathbf{x}. \quad (24)$$

Minimizing with respect to τ we get

$$\tau = \tau_0 - \frac{1}{\delta} \Pi_Y \nabla \cdot \mathbf{G}. \tag{25}$$

Substituting it back, we obtain the dual problem:

$$\min_{\mathbf{G}} \int_{\Omega} |\Pi_Y \nabla \cdot \mathbf{G} - \delta\tau_0|^2 \, dx. \tag{26}$$

This problem can be solved using Chambolle's fixed point iteration (cf. [3]):

$$\mathbf{G}^{n+1} = \frac{\mathbf{G}^n + \Delta t \nabla[\Pi_Y \nabla \cdot \mathbf{G} - \delta\tau_0]}{1 + \Delta t \nabla[\Pi_Y \nabla \cdot \mathbf{G} - \delta\tau_0]} \tag{27}$$

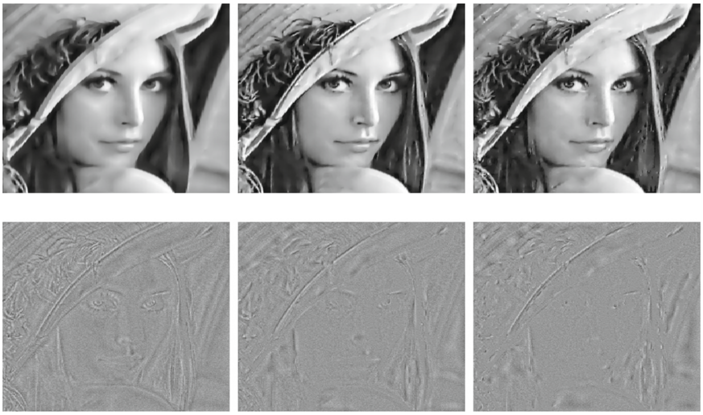

Fig. 2. Denoising of Lena image, with noise level ≈ 8, $\delta = .16$ and $\mu = 0.20$.

In practice we compute an approximation of Π_Y using the following discrete gradient, discrete divergence and discrete Laplace operator. For $d \in \mathbb{R}^{N\times N}$ representing an image on a 2D grid let

$$\nabla^h d = (dD^T, Dd)^T, \quad \nabla^h \cdot (p_1, p_2) = -p_1 D - D^T p_2, \tag{28}$$

where D is differentiation matrix. Then $\triangle^h = -dDD^T - D^T Dd$ and the discrete projection becomes: $\Pi_Y^h = I - \nabla^h(\triangle^h)^\dagger \nabla^h$. Because we know SVD of $\triangle^h$, thus the action of $(\triangle^h)^\dagger$ can be computed using discrete cosine and sine matrices with the aid of the Fast Fourier Transform requiring only $O(N^2 \log_2(N))$ operations.

4.2 Second Step

In the second step, we have an unconstrained minimization problem (11). Using the dual formulation of the TV norm, the problem can be reformulated as

$$\min_{d} \max_{\mathbf{g}\in C_c^1(\Omega)^2:|\mathbf{g}|_\infty \le 1} \int_{\Omega} d\, \nabla \cdot \left(\mathbf{g} + \frac{\mathbf{n}}{|\mathbf{n}|}\right) \, d\mathbf{x} \; + \; \frac{\lambda}{2} \int_{\Omega} (d_0 + s_k - d)^2 d\mathbf{x}. \tag{29}$$

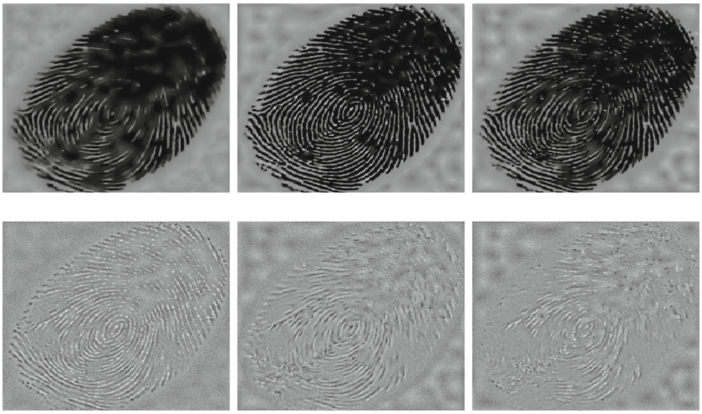

Fig. 3. Denoising of fingerprint image, with noise level ≈ 6.4, $\delta = .16$ and $\mu = 0.20$.

Exchanging the min and max, and minimizing with respect to d, we get

$$d = d_0 + s_k - \frac{1}{\lambda}\nabla \cdot \left(\mathbf{g} + \frac{\mathbf{n}}{|\mathbf{n}|}\right). \tag{30}$$

Substituting it back, we obtain the dual problem:

$$\min_{\mathbf{g}} \int_{\Omega} \left|\lambda(d_0 + s_k) - \nabla \cdot \left(\mathbf{g} + \frac{\mathbf{n}}{|\mathbf{n}|}\right)\right|^2 d\mathbf{x}. \tag{31}$$

Using Chambolle's fixed point iteration we get

$$\mathbf{g}^{n+1} = \frac{\mathbf{g}^n + \Delta t \nabla[\nabla \cdot \left(\mathbf{g}^n + \frac{\mathbf{n}}{|\mathbf{n}|}\right) - \lambda(d_0 + s_k)]}{1 + \Delta t \nabla[\nabla \cdot \left(\mathbf{g}^n + \frac{\mathbf{n}}{|\mathbf{n}|}\right) - \lambda(d_0 + s_k)]}. \tag{32}$$

5 Numerical Results

The algorithm has been applied to the Lena and the fingerprint image, and the results are shown in Figs. 2 and 3, respectively, showing three iterations of the iterative regularization algorithm, with denoised images in the first row and their corresponding difference images (difference between the noisy image and the denoised image) in the second row. The preliminary results shown in the figures proves that he proposed algorithm works well.

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