The Prescribed Curvature Problem in Low Dimension

Giovanni Calvaruso

Dedicated to Jaime Muñoz Masqué on the occasion of his 65th birthday

Abstract We describe some recent results concerning the inverse curvature problem, that is, the existence and description of metrics with prescribed curvature, focusing on the low-dimensional homogeneous cases.

Keywords Homogeneous Lorentzian metrics · Ricci curvature · Segre types

1 Introduction

Geometric properties of a pseudo-Riemannian manifold (M, g) are encoded in its curvature, and usually expressed by some conditions on the curvature tensor itself. Starting from the metric tensor g, the curvature tensor R of (M, g) can be completely determined. The inverse problem, namely, to determine a pseudo-Riemannian manifold with assigned curvature, is known as the *prescribed curvature problem*, and it has been extensively studied. In this framework, two distinct problems naturally arise:

(i) Existence results: necessary and sufficient conditions for an assigned two-form on a manifold to be (locally) the curvature form of a pseudo-Riemannian metric.
(ii) Explicit examples of such a metric.

G. Calvaruso (✉)
Department of Mathematics and Physics "Ennio De Giorgi", University of Salento, 73100 Lecce, LE, Italy
e-mail: giovanni.calvaruso@unisalento.it

M. Castrillón López et al. (eds.), *Geometry, Algebra and Applications: From Mechanics to Cryptography*, Springer Proceedings in Mathematics & Statistics 161, DOI 10.1007/978-3-319-32085-4_4

The study of the first problem led to local existence theorems under very general hypotheses (see for example [6–11, 14] and references therein). In particular, as proved by DeTurck [6–8], if a symmetric (0, 2)-tensor $\mathscr{R}$ is analytic in a neighborhood of a point $x_0 \in \mathbb{R}^n$ and $\mathscr{R}^{-1}(x_0)$ exists, then there exists an analytic metric g, of any desired signature, such that $\mathscr{R} = \rho$ is the Ricci tensor of g in a neighborhood of x_0. The *Bianchi identity*

$$\mathrm{Bian}(g, \mathscr{R}) = g^{ab}\left(\mathscr{R}_{am;b} - \tfrac{1}{2}\mathscr{R}_{ab;m}\right) = 0$$

yields some restrictions for the 2-forms admissible as curvature forms. It is worth to emphasize the physical meaning of such restrictions. In fact, $\mathrm{Bian}(g, \mathscr{R}) = -\mathrm{div}(G\mathscr{R})$, where G is the gravitation operator $Gh = h_{ij} - \frac{1}{2}g_{ij}(g^{ab}h_{ab})$. In particular, $G\rho$ is the stress-energy tensor in Einstein's theory of gravitation [6].

In this framework, low-dimensional cases have some special properties. In fact, in dimension three every 2-form with values in a semi-simple Lie algebra is generically the curvature of a connection form locally [9, 10, 14]. Moreover, in dimension four, Bianchi's identities can be eliminated for a large class of Lie algebras (which strictly includes the semi-simple ones). Curvature forms can be then characterized as the solutions to a second-order partial differential system, which was proved in [11] to be formally integrable.

On the other hand, even in special cases, as in low dimension and for particularly simple forms of the curvature or the Ricci tensor, the second problem is still open (up to our knowledge). Moreover, it is a natural problem to look for *homogeneous* metrics of prescribed curvature, since they are the homogeneous models for metrics of the same dimension. Also with regard to the existence problem, the above cited Refs. [9–11, 14] showed the special role played by homogeneous examples (in particular, Lie groups and the corresponding Lie algebras).

In this framework, the three-dimensional case acquires a peculiar relevance, for several reasons. First of all, in dimension three the Ricci tensor completely determines the curvature. Moreover, a connected, simply connected, complete three-dimensional homogeneous manifold is either symmetric or isometric to some Lie group equipped with a left-invariant metric (we may refer to [13] for the Riemannian case and [1] for the Lorentzian one). Finally, with the obvious exceptions of $\mathbb{R} \times \mathbb{S}^2$ (Riemannian) and $\mathbb{R}_1 \times \mathbb{S}^2$ (Lorentzian), three-dimensional connected simply connected symmetric spaces are also realized in terms of suitable left-invariant metrics on Lie groups [2].

In this note we will illustrate how three-dimensional locally homogeneous Lorentzian metrics on $\mathbb{R}^3$ were constructed in [3] for all admissible Ricci operators, that is, for all real-valued matrices which can occur as the Ricci operator of a homogeneous Lorentzian three-manifold. To do so, we introduce a system of partial differential equations, whose solutions determine explicitly these Lorentzian metrics. Then, solutions are presented for *proper* Lorentzian models, that is, Lorentzian homogeneous three-spaces which do not have any counterpart in Riemannian geometry, since their Ricci operator is not diagonalizable. We also mention the fact that explicit examples for the wider class of *curvature homogeneous* Lorentzian three-manifolds were constructed in [4, 5], proving that for all Segre types of the Ricci operator, there exist examples of curvature homogeneous Lorentzian metrics in $\mathbb{R}^3$.

2 Locally Homogeneous Lorentzian Three-Manifolds

Let (M, g) be a connected Lorentzian three-manifold. We denote by ∇ the Levi-Civita connection of (M, g) and by R its curvature tensor, taken with the sign convention $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Since $\dim M = 3$, R is completely determined by the Ricci tensor ρ, defined by $\rho(X, Y)_p = \sum_{i=1}^{3} \varepsilon_i g(R(X, e_i)Y, e_i)$, where $\{e_i\}$ is a pseudo-orthonormal basis of T_pM and $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all i. Throughout the paper we shall assume that e_3 is *timelike*, that is, $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$.

Because of the symmetries of R, the Ricci tensor ρ is symmetric. Consequently, the *Ricci operator* Q, defined by $g(QX, Y) = \rho(X, Y)$, is self-adjoint. Thus, in the Riemannian case there exists an orthonormal basis diagonalizing Q, while for a Lorentzian manifold there exists a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, such that Q takes one of the following forms, called *Segre types*:

$$\text{Segre type } \{11, 1\} : \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{b} & 0 \\ 0 & 0 & \bar{c} \end{pmatrix}, \qquad \text{Segre type } \{1z\bar{z}\} : \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{b} & \bar{c} \\ 0 & -\bar{c} & \bar{b} \end{pmatrix},$$

$$\text{Segre type } \{21\} : \begin{pmatrix} \bar{a} & 0 & 0 \\ 0 & \bar{b} & \varepsilon \\ 0 & -\varepsilon & \bar{b} - 2\varepsilon \end{pmatrix}, \qquad \text{Segre type } \{3\} : \begin{pmatrix} \bar{b} & \bar{a} & -\bar{a} \\ \bar{a} & \bar{b} & 0 \\ \bar{a} & 0 & \bar{b} \end{pmatrix}.$$

If (M, g) is curvature homogeneous (in particular, locally homogeneous), then its Ricci operator Q has the same Segre type at every point $p \in M$ and there exists (at least, locally) a pseudo-orthonormal frame field $\{e_i\}$ such that Q is given by one of the expressions above, for some constants $\bar{a}$, $\bar{b}$ and $\bar{c}$. As in [1], we now put

$$\nabla_{e_i} e_j = \sum_k \varepsilon_j b^i_{jk} e_k, \tag{1}$$

for all indices i, j. Clearly, the functions b^i_{jk} determine completely the Levi-Civita connection, and conversely. As $\nabla g = 0$, we have

$$b^i_{kj} = -b^i_{jk}, \quad (\text{in particular, } b^i_{jj} = 0) \tag{2}$$

for all i, j, k. We now put

$$b^1_{12} = \alpha,\ b^1_{13} = \beta,\ b^1_{23} = \gamma,\ b^2_{12} = \kappa,\ b^2_{13} = \mu,\ b^2_{23} = \nu,\ b^3_{12} = \sigma,\ b^3_{13} = \tau,\ b^3_{23} = \psi. \tag{3}$$

By (1)–(3) we get

$$\begin{aligned} &[e_1, e_2] = -\varepsilon\alpha\, e_1 - \kappa\, e_2 + (\varepsilon\gamma - \mu)\, e_3, \quad [e_1, e_3] = -\beta\, e_1 - (\gamma + \sigma)\, e_2 - \tau e_3, \\ &[e_2, e_3] = (\varepsilon\sigma - \mu)\, e_1 - \nu\, e_2 - \varepsilon\psi\, e_3. \end{aligned} \tag{4}$$

Conversely, the functions (b^i_{jk}) are determined by (4) via the *Koszul formula* [12].

A locally homogeneous Lorentzian three-manifold admits (locally) a pseudo-orthonormal basis $\{e_i\}$, such that (4) holds with *constant* connection functions $\alpha, \dots, \psi$. Starting from (4), we compute the curvature components with respect to $\{e_1\}$ and, by contraction, the Ricci components. We get

$$\rho_{11} = -\alpha^2 - \kappa^2 + \beta\nu - \gamma\mu + \sigma(\gamma - \mu) + \beta^2 - \tau^2 - \gamma\sigma + \alpha\psi + \mu(\gamma - \sigma), \quad (5)$$
$$\rho_{22} = -\alpha^2 - \kappa^2 + \beta\nu - \gamma\mu + \sigma(\gamma - \mu) + \nu^2 - \psi^2 - \kappa\tau + \mu\sigma + \gamma(\mu + \sigma), \quad (6)$$
$$\rho_{33} = -\beta^2 + \tau^2 + \gamma\sigma - \alpha\psi - \mu(\gamma - \sigma) - \nu^2 + \psi^2 + \kappa\tau - \mu\sigma - \gamma(\mu + \sigma), \quad (7)$$
$$\rho_{12} = \beta(\gamma + \sigma) + \nu(\gamma - \sigma) - \tau(\alpha + \psi), \quad (8)$$
$$\rho_{13} = -\alpha(\mu + \sigma) - \nu(\kappa - \tau) - \psi(\mu - \sigma), \quad (9)$$
$$\rho_{23} = \alpha(\beta - \nu) + \kappa(\gamma + \mu) - \tau(\gamma - \mu). \quad (10)$$

For the components of the covariant derivative of ρ with respect to $\{e_i\}$, we find

$$\nabla_i \rho_{jk} = -\sum_t \left(\varepsilon_j b^i_{jt} \rho_{tk} + \varepsilon_k B^i_{kt} \rho_{tj}\right). \quad (11)$$

Observe that the connection functions $\alpha, \dots, \psi$ are not all independent. In fact, since (M, g) is locally homogeneous, its scalar curvature $r = \operatorname{tr} \rho$ is constant. The well-known *divergence formula* $dr = 2 \operatorname{div} \rho$ (see [12]) then implies $\sum_j \nabla_j \rho_{ij} = 0$, for all i, which, taking into account (11), gives some restrictions for the connection functions.

We end this section with the following classification result.

Theorem 1 ([1]) *A three-dimensional connected, simply connected complete homogeneous Lorentzian manifold (M, g) is either symmetric, or $M = G$ is a Lie group and g is left-invariant. Precisely, one of the following cases occurs:*

(I) If G is unimodular, then there exists a pseudo-orthonormal frame field $\{e_i\}$, with e_3 time-like, such that the Lie algebra of G is one of the following:

$$\mathfrak{g}_1: \ [e_1, e_2] = \alpha e_1 - \beta e_3, \ [e_1, e_3] = -\alpha e_1 - \beta e_2, \ [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \ \alpha \neq 0. \quad (12)$$

If $\beta \neq 0$, then G is $\widetilde{SL}(2, \mathbb{R})$, while $G = E(1, 1)$ when $\beta = 0$.

$$\mathfrak{g}_2: \ [e_1, e_2] = -\gamma e_2 - \beta e_3, \ [e_1, e_3] = -\beta e_2 + \gamma e_3, \ [e_2, e_3] = \alpha e_1, \quad \gamma \neq 0. \quad (13)$$

In this case, $G = \widetilde{SL}(2, \mathbb{R})$ if $\alpha \neq 0$, while $G = E(1, 1)$ if $\alpha = 0$.

$$\mathfrak{g}_3: \quad [e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1. \quad (14)$$

The following Table 1 lists all the Lie groups G which admit a Lie algebra $\mathfrak{g}_3$, according to the different possibilities for α, β and γ:

Table 1 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_3$

Lie group	(α, β, γ)	Lie group	(α, β, γ)
$\widetilde{SL}(2, \mathbb{R})$	$(+, +, +)$	$E(1, 1)$	$(+, -, 0)$
$\widetilde{SL}(2, \mathbb{R})$	$(+, -, -)$	$E(1, 1)$	$(+, 0, +)$
$SU(2)$	$(+, +, -)$	H_3	$(+, 0, 0)$
$\widetilde{E}(2)$	$(+, +, 0)$	H_3	$(0, 0, -)$
$\widetilde{E}(2)$	$(+, 0, -)$	$\mathbb{R}^3$	$(0, 0, 0)$

Table 2 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_4$

Lie group ($\varepsilon = 1$)	α	β	Lie group ($\varepsilon = -1$)	α	β
$\widetilde{SL}(2, \mathbb{R})$	$\neq 0$	$\neq 1$	$\widetilde{SL}(2, \mathbb{R})$	$\neq 0$	$\neq -1$
$E(1, 1)$	0	$\neq 1$	$E(1, 1)$	0	$\neq -1$
$E(1, 1)$	<0	1	$E(1, 1)$	>0	-1
$\widetilde{E}(2)$	>0	1	$\widetilde{E}(2)$	<0	-1
H_3	0	1	H_3	0	-1

$$\mathfrak{g}_4 : \ [e_1, e_2] = -e_2 + (2\varepsilon - \beta)e_3, \ [e_1, e_3] = -\beta e_2 + e_3, \ [e_2, e_3] = \alpha e_1, \ \varepsilon = \pm 1. \tag{15}$$

Table 2 describes all Lie groups G admitting a Lie algebra g_4.

(II) If G is non-unimodular, there exists a pseudo-orthonormal frame field $\{e_i\}$, with e_3 time-like, such that $\alpha + \delta \neq 0$ and the Lie algebra of G is one of the following:

$$\mathfrak{g}_5 : [e_1, e_2] = 0, \ [e_1, e_3] = \alpha e_1 + \beta e_2, \ [e_2, e_3] = \gamma e_1 + \delta e_2, \ \ \alpha\gamma + \beta\delta = 0. \tag{16}$$

$$\mathfrak{g}_6 : \ [e_1, e_2] = \alpha e_2 + \beta e_3, \ [e_1, e_3] = \gamma e_2 + \delta e_3, \ [e_2, e_3] = 0, \ \ \alpha\gamma - \beta\delta = 0. \tag{17}$$

$$\mathfrak{g}_7 : -[e_1, e_2] = [e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3, \quad [e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \ \alpha\gamma = 0. \tag{18}$$

3 The Basic System of Equations

We shall express Eqs. (5)–(10) via a system of PDE's, whose solutions give explicitly locally homogeneous Lorentzian metrics on $\mathbb{R}^3$ with the required curvature.

Fix a point $p \in M$ and consider a pseudo-orthonormal frame field $\{e_i\}$, satisfying (4) for some constants $\alpha, \dots, \psi$. Choose a surface S through p transversal to the lines generated by e_3, a local coordinates system (w, x) on S and a neighborhood U_p of p, sufficiently small that each $q \in U_p$ is situated on exactly one line generated by e_3 and passing through one point $\bar{q} \in S$. Choose an orientation of S and define the coordinate function y in U_p as the oriented distance of q from S along the corresponding line, that is, $y(q) = \text{dist}(q, \pi(q))$, where $\pi : U_p \to S$ is the corresponding projection. We also define $w(q) = w(\pi(q))$, $x(q) = x(\pi(q))$. In this way, a local coordinate system (w, x, y) is introduced in U_p. Observe that $e_3 = \partial/\partial y$ and the coframe $\{\omega_1, \omega_2, \omega_3\}$ of $\{e_1, e_2, e_3\}$ must take the form

$$\omega^1 = Adw + Bdx, \quad \omega^2 = Cdw + Ddx, \quad \omega^3 = Gdw + Hdx + dy, \tag{19}$$

for some functions A, B, C, D, G, H. Next, we introduce the connection forms $\omega^i_j = \sum_k \varepsilon_j b^i_{jk}\omega^k$, which completely determine the Levi-Civita connection, because $\nabla_{e_i}e_j = \sum_k \omega^k_j(e_i)e_k$, for all i, j. Moreover, from (1) we easily get

$$\omega^i_j + \varepsilon_i\varepsilon_j\omega^j_i = 0 \tag{20}$$

for all i, j (in particular, $\omega^i_i = 0$ for all i). The *structure equations* for ω^i_j give

$$d\omega^i + \sum_j \omega^i_j \wedge \omega^j = 0, \tag{21}$$

for all indices i. The curvature forms Ω^i_j are completely determined by

$$-d\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j. \tag{22}$$

By the definition of the Ricci tensor and taking into account (20) and (4), we obtain that (22) is equivalent to

$$\begin{aligned} d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 &= -R_{1212}\,\omega^1 \wedge \omega^2 - \rho_{23}\,\omega^1 \wedge \omega^3 + \rho_{13}\,\omega^2 \wedge \omega^3, \\ d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 &= \rho_{23}\,\omega^1 \wedge \omega^2 + R_{1313}\,\omega^1 \wedge \omega^3 - \rho_{12}\,\omega^2 \wedge \omega^3, \\ d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 &= -\rho_{13}\,\omega^1 \wedge \omega^2 - \rho_{12}\,\omega^1 \wedge \omega^3 + R_{2323}\,\omega^2 \wedge \omega^3. \end{aligned} \tag{23}$$

We then use (19) in (21). Also taking into account (3) and the divergence formula, we obtain that (21) is equivalent to the following system of nine PDE's:

$$\begin{array}{ll} A'_y = \beta A + (\mu + \sigma)C, & B'_y = \beta B + (\mu + \sigma)D, \\ C'_y = (\gamma - \sigma)A + \nu C, & D'_y = (\gamma - \sigma)B + \nu D, \\ G'_y = -\tau A - \psi C, & H'_y = -\tau B - \psi D, \\ B'_w - A'_x = \alpha\mathscr{D} - \beta\mathscr{E} - (\mu + \sigma)\mathscr{F}, & D'_w - C'_x = \kappa\mathscr{D} - (\gamma - \sigma)\mathscr{E} - \nu\mathscr{F}, \\ H'_w - G'_x = -(\gamma - \mu)\mathscr{D} + \tau\mathscr{E} + \psi\mathscr{F}, & \end{array} \tag{24}$$

where $\mathscr{D}, \mathscr{E}, \mathscr{F}$ are auxiliary functions, defined by

$$\mathscr{D} = AD - BC, \qquad \mathscr{E} = AH - BG, \qquad \mathscr{F} = CH - DG. \tag{25}$$

Observe that, because of (19), $\mathscr{D} = AD - BC \neq 0$ is a necessary and sufficient condition for linear independence of the ω^i. Starting from the connection functions b^i_{jk} of (M, g), by (24) we determine the functions $A, \ldots, H$ and so, explicit Lorentzian metrics on $\mathbb{R}^3$, with the same Levi-Civita connection of (M, g). Conversely, if $A, \ldots, H$ are known, then by (24) we can determine b^i_{jk}.

We now express the curvature conditions (23) using (19). Taking into account that the connection functions are constant, one can easily prove that (23) is equivalent to the following system of algebraic equations:

$$\begin{aligned}
&(U_3 + R_{1212})\mathscr{D} + (V_3 + \rho_{23})\mathscr{E} + (W_3 - \rho_{13})\mathscr{F} = 0,\\
&(U_2 - \rho_{23})\mathscr{D} + (V_2 - R_{1313})\mathscr{E} + (W_2 + \rho_{12})\mathscr{F} = 0,\\
&(U_1 + \rho_{13})\mathscr{D} + (V_1 + \rho_{12})\mathscr{E} + (W_1 - R_{2323})\mathscr{F} = 0,\\
&(V_3 + \rho_{23})A + (W_3 - \rho_{13})C = 0, \quad (V_3 + \rho_{23})B + (W_3 - \rho_{13})D = 0,\\
&(V_2 - R_{1313})A + (W_2 + \rho_{12})C = 0, \quad (V_2 - R_{1313})B + (W_2 + \rho_{12})D = 0,\\
&(V_1 + \rho_{12})A + (W_1 - R_{2323})C = 0, \quad (V_1 + \rho_{12})B + (W_1 - R_{2323})D = 0,
\end{aligned}$$

where we put

$$\begin{aligned}
U_1 &= \alpha(\gamma + \mu) - \kappa(\beta - \nu) - \psi(\gamma - \mu),\\
V_1 &= -\beta(\gamma + \sigma) - \nu(\gamma - \sigma) + \tau(\alpha + \psi),\\
W_1 &= -\nu^2 + \psi^2 + \kappa\tau - \mu\sigma - \gamma(\mu + \sigma),\\
U_2 &= \alpha(\beta - \nu) + \kappa(\gamma + \mu) - \tau(\gamma - \mu),\\
V_2 &= -\beta^2 + \tau^2 - \alpha\psi + \gamma\sigma - \mu(\gamma - \sigma),\\
W_2 &= -\beta(\mu + \sigma) - \nu(\mu - \sigma) - \psi(\kappa + \tau),\\
U_3 &= \alpha^2 + \kappa^2 - \beta\nu + \gamma\mu - \sigma(\gamma - \mu),\\
V_3 &= -\beta(\alpha + \psi) - \kappa(\gamma - \sigma) + \tau(\gamma + \sigma),\\
W_3 &= -\alpha(\mu + \sigma) - \nu(\kappa - \tau) - \psi(\mu - \sigma).
\end{aligned} \tag{26}$$

Comparing (8)–(10) with (26), we easily get $V_1 + \rho_{12} = 0$, $U_2 - \rho_{23} = 0$ and $W_3 - \rho_{13} = 0$. Hence, Eq. (26) reduce to

$$\begin{aligned}
&(U_3 + R_{1212})\mathscr{D} + (V_3 + \rho_{23})\mathscr{E} = 0, \quad (V_3 + \rho_{23})A = 0,\\
&(V_2 - R_{1313})\mathscr{E} + (W_2 + \rho_{12})\mathscr{F} = 0, \quad (V_3 + \rho_{23})B = 0,\\
&(V_2 - R_{1313})A + (W_2 + \rho_{12})C = 0, \quad (V_2 - R_{1313})B + (W_2 + \rho_{12})D = 0,\\
&(U_1 + \rho_{13})\mathscr{D} + (W_1 - R_{2323})\mathscr{F} = 0, \quad (W_1 - R_{2323})C = 0,\\
&(W_1 - R_{2323})D = 0.
\end{aligned} \tag{27}$$

In this way, we have proved the following result.

Theorem 2 *Given a locally homogeneous Lorentzian three-manifold* (M, g)*, having* $\mathscr{R} = (\rho_{ij})$ *as the matrix of Ricci components with respect to a suitable* *pseudo-orthonormal frame* $\{e_i\}$*, let* A, B, C, D, G, H *be smooth functions on* (w, x, y)*, satisfying the systems (24) and (27). Then, (19) determines a locally homogeneous Lorentzian metric* $\bar{g}$ *on* $\mathbb{R}^3$*, locally isometric to* (M, g) *(in particular, having the same curvature).*

4 Explicit Lorentzian Metrics in $\mathbb{R}^3$ with Prescribed Curvature

For each of the homogeneous models described by (12)–(18), we can now solve systems (24) and (27), providing explicit Lorentzian metrics on $\mathbb{R}^3$ which have exactly the Ricci tensor of the corresponding model. Curvature equations are remarkably simpler when the Ricci tensor is diagonal. This special case has been studied in [4]. Hence, we focus here on all the remaining cases, which do not have any correspondence with the Riemannian case. The Ricci tensor of all $3D$ Lie groups equipped with a left-invariant Lorentzian metric was calculated in [2] and can be easily obtained by direct calculation starting from (12)–(18). According to the results of [2], non-diagonal cases occur for the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_4$ and $\mathfrak{g}_7$.

($\mathfrak{g}_1$) Comparing (12) with (4), we find that the connection functions of a locally homogeneous Lorentzian three-manifold described by (12) are given by

$$\alpha = \beta = -\nu = \psi = -a, \quad \gamma = -\mu = -\sigma = -\tfrac{b}{2}, \quad \kappa = \tau = 0, \tag{28}$$

where $a \neq 0$ and b are constant. Straightforward calculations (see also [1]) show that the Ricci tensor at any point is given by

$$\mathscr{R}_1 = \begin{pmatrix} -\frac{b^2}{2} & -ab & ab \\ -ab & -2a^2 - \frac{b^2}{2} & 2a^2 \\ ab & 2a^2 & \frac{b^2}{2} - 2a^2 \end{pmatrix}. \tag{29}$$

On the other hand, because of (28), Eq. (26) reduce to

$$\begin{array}{lll} U_1 = -ab, & U_2 = 2a^2, & U_3 = 2a^2 + \frac{b^2}{4}, \\ V_1 = ab, & V_2 = -2a^2 + \frac{b^2}{4}, & V_3 = ab, \\ W_1 = \frac{b^2}{4}, & W_2 = ab, & W_3 = ab. \end{array} \tag{30}$$

By (29) and (30) it follows at once that *all Eqs.* (27) *reduce to identities*, that is, under the assumption (28), the curvature conditions (27) are identically satisfied.

We now turn our attention to the connection equations (24). Again by (28), we obtain that (24) reduces to

$$\begin{array}{lll} A'_y = -aA + bC, & B'_y = -aB + bD, & C'_y = -bA + aC, \\ D'_y = -bB + aD, & G'_y = aC, & H'_y = aD, \\ B'_w - A'_x = -a\mathscr{D} + a\mathscr{E} - b\mathscr{F}, & D'_w - C'_x = b\mathscr{E} - a\mathscr{F}, & H'_w - G'_x = b\mathscr{D} - a\mathscr{F}. \end{array} \tag{31}$$

One can now find explicit solutions of the system (31). Different kinds of solutions are obtained according to the different possibilities for the sign of $a^2 - b^2$. Some explicit solutions of (31) are resumed in the following

Theorem 3 *Let $a \neq 0$ and b be two real constants and $\mathscr{R}_1$ any symmetric real matrix described by (29). Then, (19) determines a family of (locally isometric) locally homogeneous Lorentzian metrics on $\mathbb{R}^3[w, x, y]$ having $\mathscr{R}_1$ as the Ricci tensor at any point, where the functions A, B, C, D, G, H are the following:*
(i) When $b \neq 0$ and $a^2 - b^2 > 0$, we put $\eta = \sqrt{a^2 - b^2}$. Then

$$\begin{array}{ll} A = f\cosh(\eta y), & B = \theta\sinh(\eta y), \\ C = \frac{1}{b} f\,(a\cosh(\eta y) + \eta\sinh(\eta y)), & D = \frac{1}{b}\theta\,(\eta\cosh(\eta y) + a\sinh(\eta y)), \\ G = \frac{a}{b\eta} f\,(\eta\cosh(\eta y) + a\sinh(\eta y)) - \frac{1}{\theta\eta} f'_x, & \\ H = \frac{a}{b\eta}\theta\,(a\cosh(\eta y) + \eta\sinh(\eta y)), & \end{array}$$

for a real constant $\theta \neq 0$ and $f(w, x) = a_1(w)\cos(b\theta x) + a_2(w)\sin(b\theta x)$, where a_1, a_2 are two arbitrary one-variable functions. Corresponding solutions are found in [3] in the cases $a^2 = b^2$ and $a^2 - b^2 < 0$. In all the cases, the corresponding Lorentzian metric is defined in the open subset of $\mathbb{R}^3$ where $f \neq 0$.
(ii) When $b = 0$:

$$A = a_0(w)e^{-ay}, \quad B = b_0(x)e^{-ay}, \quad C = G = c_0(w)e^{ay}, \quad D = H = d_0(x)e^{ay},$$

where a_0, b_0, c_0, d_0 are arbitrary one-variable functions. The corresponding Lorentzian metric is defined in the open subset of $\mathbb{R}^3$ where $a_0(w)d_0(x) - b_0(x)c_0(w) \neq 0$.

($\mathfrak{g}_2$) The remaining cases can be treated similarly to the case $\mathfrak{g}_1$ above. So, for any of them, we shall only report the Ricci components, the equations for the connection functions and some explicit solutions. In the case of $\mathfrak{g}_2$, we have

$$\mathscr{R}_2 = \begin{pmatrix} -\frac{a^2}{2} - 2c^2 & 0 & 0 \\ 0 & \frac{a^2}{2} - ab & c(a - 2b) \\ 0 & c(a - 2b) & -\frac{a^2}{2} + ab \end{pmatrix}, \tag{32}$$

for three real constants a, b, c, and

$$\begin{array}{lll} A'_y = aC, & B'_y = aD, & C'_y = -bA, \\ D'_y = -bB, & G'_y = cA, & H'_y = cB, \\ B'_w - A'_x = -a\mathscr{F}, & D'_w - C'_x = c\mathscr{D} + b\mathscr{E}, & H'_w - G'_x = b\mathscr{D} - c\mathscr{E}. \end{array} \tag{33}$$

We present some solutions of (33) in the following

Theorem 4 *Given three real constants a, b, c and any symmetric real matrix $\mathscr{R}_2$ described by (32). Then, (19) gives a family of (locally isometric) locally homogeneous Lorentzian metrics on $\mathbb{R}^3[w, x, y]$ having $\mathscr{R}_2$ as the Ricci tensor at any point, where the functions A, B, C, D, G, H are the following:*

If $-ab < 0$, we put $\eta = \sqrt{ab}$. Then

$$\begin{array}{lll} A = f\cos(\eta y), & B = \theta\sin(\eta y), & C = -\frac{\eta}{a} f\sin(\eta y), \\ D = \frac{\eta}{a}\theta\cos(\eta y), & G = \frac{c}{\eta} f\sin(\eta y) - \frac{1}{\theta\eta} f'_x, & H = -\frac{c}{\eta}\theta\cos(\eta y), \end{array}$$

where $f(w, x) = a_1(w)\cosh(\sqrt{\theta^2(b^2+c^2)}x) + a_2(w)\sinh(\sqrt{\theta^2(b^2+c^2)}\theta x)$, $\theta \neq 0$ is a real constant and a_1, a_2 are two arbitrary one-variable functions. The Lorentzian metric is defined on the open subset of $\mathbb{R}^3$ where $f \neq 0$. Corresponding solutions were found in [3] in the cases $ab < 0$, $a = 0$, $b = 0$.

($\mathfrak{g}_4$) For a locally homogeneous Lorentzian three-manifold described by (15), the Ricci components are given by

$$\mathscr{R}_4 = \begin{pmatrix} -\frac{a^2}{2} & 0 & 0 \\ 0 & \frac{a^2}{2} + 2\varepsilon(a-b) - ab + 2 & a + 2(\varepsilon - b) \\ 0 & a + 2(\varepsilon - b) & -\frac{a^2}{2} + ab + 2 - 2\varepsilon b \end{pmatrix}, \tag{34}$$

for two real constants a, b, and connection equations (24) become

$$\begin{array}{lll} A'_y = aC, & B'_y = aD & C'_y = -bA, \\ D'_y = -bB, & G'_y = A, & H'_y = B, \\ B'_w - A'_x = -a\mathscr{F}, & D'_w - C'_x = \mathscr{D} + b\mathscr{E}, & H'_w - G'_x = (b - 2\varepsilon)\mathscr{D} - \mathscr{E}. \end{array} \tag{35}$$

Some explicit solutions of (35) are given in the following

Theorem 5 *Given two real constants a, b and any symmetric real matrix $\mathscr{R}_4$ as in (34). Then, (19) describes a family of (locally isometric) locally homogeneous Lorentzian metrics on $\mathbb{R}^3[w, x, y]$ whose Ricci tensor at any point is $\mathscr{R}_4$, where the functions A, B, C, D, G, H are the following:*

If $ab < 0$, we put $\eta = \sqrt{-ab}$. Then,

$$\begin{array}{lll} A = f\cosh(\eta y), & B = \theta\sinh(\eta y), & C = \frac{\eta}{a} f\sinh(\eta y), \\ D = \frac{\eta}{a}\theta\cosh(\eta y), & G = \frac{1}{\eta} f\sinh(\eta y) - \frac{1}{\theta\eta} f'_x, & H = \frac{1}{\eta}\theta\cosh(\eta y), \end{array}$$

where

$$f(w, x) = \begin{cases} a_1(w)\cos(|\theta(b+\varepsilon)|x) + a_2(w)\sin(|\theta(b+\varepsilon)|x) & \text{if } b \neq -\varepsilon, \\ a_1(w)x + a_2(w) & \text{if } b = -\varepsilon, \end{cases}$$

for a real constant $\theta \neq 0$ and two arbitrary one-variable functions a_1, a_2. The Lorentzian metric is defined in the open subset of $\mathbb{R}^3$ where $f \neq 0$. Corresponding solutions were found in [3] in the cases $ab > 0$, $a = 0$, $b = 0$.

($\mathfrak{g}_7$) Consider a locally homogeneous Lorentzian three-manifold locally described by (18). Then, the Ricci components are given by

$$\mathscr{R}_7 = \begin{pmatrix} -\frac{c^2}{2} & 0 & 0 \\ 0 & ad - a^2 - bc + \frac{c^2}{2} & a^2 - ad + bc \\ 0 & a^2 - ad + bc & ad - a^2 - bc - \frac{c^2}{2} \end{pmatrix}, \tag{36}$$

where a, b, c, d are four real constants satisfying $ac = 0$.

If $c = 0$, then, (24) reduces to

$$\begin{array}{ll} A'_y = aA, & B'_y = aB, \\ C'_y = G'_y = bA + dC, & D'_y = H'_y = bB + dD, \\ B'_w - A'_x = a\mathscr{D} - a\mathscr{E}, & D'_w - C'_x = H'_w - G'_x = b\mathscr{D} - b\mathscr{E} - d\mathscr{F}, \end{array} \tag{37}$$

while if $c \neq 0$, then $a = 0$ and the system (24) reduces to

$$\begin{array}{ll} A'_y = cC, & B'_y = cD, \\ C'_y = G'_y = bA + dC, & D'_y = H'_y = bB + dD, \\ B'_w - A'_x = -c\mathscr{F}, & D'_w - C'_x = H'_w - G'_x = b\mathscr{D} - b\mathscr{E} - d\mathscr{F}. \end{array} \tag{38}$$

Some solutions of (37) and (38) are given in the following

Theorem 6 *Given three real constants a, b, d and any symmetric real matrix $\mathscr{R}_7$ described by (36). Then, (19) gives a family of (locally isometric) locally homogeneous Lorentzian metrics on $\mathbb{R}^3[w, x, y]$ having $\mathscr{R}_7$ as the Ricci tensor at any point, where the functions A, B, C, D, G, H are the following:*

(I) When $c = 0$:

$$\begin{array}{ll} A = a_0(w)e^{ay}, & B = b_0(x)e^{ay}, \\ C = G = e^{dy}(c_0(w) + \frac{b}{a-d}a_0(w)e^{(a-d)y}), & D = H = e^{dy}(d_0(x) + \frac{b}{a-d}b_0(x)e^{(a-d)y}), \end{array}$$

where a_0, b_0, c_0, d_0 are arbitrary one-variable functions. The Lorentzian metric is defined in the open subset of $\mathbb{R}^3$ where $a_0(w)d_0(x) - b_0(x)c_0(w) \neq 0$.

(II) When $a = 0 \neq c$: if $\Delta = d^2 + 4bc > 0$, let $\lambda_1 \neq \lambda_2$ be the solutions of $\lambda^2 - d\lambda - bc = 0$. Then,

$$\begin{array}{ll} A = k_1(w)e^{\lambda_1 y} + k_2(w)e^{\lambda_2 y}, & B = h_1(x)e^{\lambda_1 y} + h_2(x)e^{\lambda_2 y}, \\ C = G = \frac{1}{c}(k_1(w)\lambda_1 e^{\lambda_1 y} + k_2(w)\lambda_2 e^{\lambda_2 y}), & D = H = \frac{1}{c}(h_1(x)\lambda_1 e^{\lambda_1 y} + h_2(x)\lambda_2 e^{\lambda_2 y}), \end{array}$$

where k_1, k_2, h_1, h_2 *are four arbitrary one-variable functions, and the Lorentzian metric is defined on the open subset of* $\mathbb{R}^3$ *where* $k_1(w)h_2(x) - k_2(w)h_1(x) \neq 0$*. Corresponding solutions exist when* $\Delta = 0$ *and when* $\Delta < 0$ *(see [3]).*

Acknowledgments Partially supported by the Ministry of Economy and Competitiveness, Spain, project MTM2011-22528.

References

1. Calvaruso, G.: Homogeneous structures on three-dimensional Lorentzian manifolds. J. Geom. Phys. **57**, 1279–1291 (2007); Addendum: J. Geom. Phys. **58**, 291–292 (2008)
2. Calvaruso, G.: Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds. Geometriae Dedicata **127**, 99–119 (2007)
3. Calvaruso, G.: Three-dimensional homogeneous Lorentzian metrics with prescribed Ricci tensor. J. Math. Phys. **48**, 123518, 17 p. (2007)
4. Calvaruso, G.: Pseudo-Riemannian 3-manifolds with prescribed distinct constant Ricci eigenvalues. Differ. Geom. Appl. **26**, 419–433 (2008)
5. Calvaruso, G.: Curvature homogeneous Lorentzian three-manifolds. Ann. Glob. Anal. Geom. **36**, 1–17 (2009)
6. DeTurck, D.M.: The equation of prescribed Ricci curvature. Bull. Am. Math. Soc. **3**, 701–704 (1980)
7. DeTurck, D.M.: Existence of metrics with prescribed Ricci curvature: local theory. Invent. Math. **65**, 179–207 (1981)
8. DeTurck, D.M.: The Cauchy problem for Lorentz metrics with prescribed Ricci curvature. Compos. Math. **48**, 327–349 (1983)
9. DeTurck, D.M., Goldschmidt, H., Talvacchia, J.: Connections with prescribed curvature and Yang-Mills currents: the semi-simple case. Ann. Sci. Ecole Norm. S. **24**, 57–112 (1991)
10. DeTurck, D.M., Goldschmidt, H., Talvacchia, J.: Local existence of connections with prescribed curvature. In: Differential Geometry, Global Analysis, and Topology. Halifax, NS (1990), pp. 13-25. In: CMS Conf. Proc., vol. 12, Amer. Math. Soc., Providence, RI (1991)
11. Muñoz-Masqué, J., Pozo Coronado, L.M., Sánchez Rodríguez, I. J.: The prescribed curvature problem in dimension four. Math. Pure Appl. **92**, 599–612 (2009)
12. O'Neill, B.: Semi-Riemannian Geometry. Academic Press, New York (1983)
13. Sekigawa, K.: On some three-dimensional curvature homogeneous spaces. Tensor **31**, 87–97 (1977)
14. Tsarev, S.P.: Which 2-forms are locally curvature forms? Funct. Anal. Appl. **16**, 235–237 (1982)