Chapter 8
Real-Time Observables

Abstract Various real-time correlation functions are defined (Wightman, retarded, advanced, time-ordered, spectral). Their analytic properties are discussed, and general relations between them are worked out for the case of a system in thermal equilibrium. Examples are given for free scalar and fermion fields. A physically relevant spectral function related to a composite operator is analyzed in detail. The so-called real-time formalism is introduced, and it is shown how it can be used to compute the same spectral function that was previously determined with the imaginary-time formalism. The need for resummations in order to systematically determine spectral functions in weakly coupled systems is stated. The concept of Hard Thermal Loops (HTLs), which implement a particular resummation, is introduced. HTL-resummed gauge field and fermion propagators are derived. The main plasma physics phenomena that the HTL resummation captures are pointed out. A warning is issued that although necessary HTL resummation is in general not sufficient for obtaining a systematic weak-coupling expansion.

Keywords Wick rotation • Time ordering • Heisenberg operator • Wightman function • Retarded and advanced correlators • Kubo-Martin-Schwinger relation • Spectral representation • Sum rule • Analytic continuation • Density matrix • Schwinger-Keldysh formalism • Hard Thermal Loops • Landau damping • Plasmon • Plasmino • Dispersion relation

8.1 Different Green's Functions

We now move to a new class of observables including both a Minkowskian time t and a temperature T. Examples are production rates of weakly interacting particles from a thermal plasma; oscillation and damping rates of long-wavelength fields in a plasma; as well as transport coefficients of a plasma such as its electric and thermal conductivities and bulk and shear viscosities. We start by developing some aspects of the general formalism, and return to specific applications later on. Let us stress that we do remain in thermal equilibrium in the following, even though some of the results also apply to an off-equilibrium ensemble.

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Many observables of interest can be reduced to *2-point correlation functions* of elementary or composite operators. Let us therefore list some common definitions and relations that apply to such correlation functions [1–4].

We denote Minkowskian spacetime coordinates by $\mathcal{X} = (t, x^i)$ and momenta by $\mathcal{K} = (k^0, k^i)$, whereas their Euclidean counterparts are denoted by $X = (\tau, x^i)$, $K = (k_n, k_i)$. Wick rotation is carried out by $\tau \leftrightarrow it$, $k_n \leftrightarrow -ik^0$. Scalar products are defined as $\mathcal{K}{\cdot}\mathcal{X} = k_0 t + k_i x^i = k^0 t - \mathbf{k}{\cdot}\mathbf{x}$, $K{\cdot}X = k_n \tau + k_i x^i = k_n \tau - \mathbf{k}{\cdot}\mathbf{x}$. Arguments of operators denote implicitly whether we are in Minkowskian or Euclidean spacetime. In particular, Heisenberg-operators are defined as

$$\hat{O}(t,\mathbf{x}) \equiv e^{i\hat{H}t}\,\hat{O}(0,\mathbf{x})\,e^{-i\hat{H}t}\,, \quad \hat{O}(\tau,\mathbf{x}) \equiv e^{\hat{H}\tau}\,\hat{O}(0,\mathbf{x})\,e^{-\hat{H}\tau}\,. \tag{8.1}$$

The thermal ensemble is normally defined by the density matrix $\hat{\rho} = \mathcal{Z}^{-1}\exp(-\beta\hat{H})$, even though it is also possible to include a chemical potential, as will be done in Eq. (8.37). Expectation values of (products of) operators are defined through $\langle\cdots\rangle \equiv \mathrm{Tr}\big[\hat{\rho}\,(\cdots)\big]$.

Bosonic Case

We start by considering operators that are *bosonic* in nature, i.e. commuting (modulo possible contact terms). We denote the operators by $\hat{\phi}_\alpha$, $\hat{\phi}^\dagger_\beta$. These may be either elementary fields or composite operators built from them. In order to simplify the notation, functions and their Fourier transforms are to be recognized through the argument, $\mathcal{X}$ vs. $\mathcal{K}$.

We can define various classes of correlation functions. The "physical" (the origin of this terminology should become clear later on) correlators are defined as

$$\Pi^{>}_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\Big\langle \hat{\phi}_\alpha(\mathcal{X})\,\hat{\phi}^\dagger_\beta(0)\Big\rangle\,, \tag{8.2}$$

$$\Pi^{<}_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\Big\langle \hat{\phi}^\dagger_\beta(0)\,\hat{\phi}_\alpha(\mathcal{X})\Big\rangle\,, \tag{8.3}$$

$$\rho_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\Big\langle \frac{1}{2}\Big[\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0)\Big]\Big\rangle\,, \tag{8.4}$$

$$\Delta_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\Big\langle \frac{1}{2}\Big\{\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0)\Big\}\Big\rangle\,, \tag{8.5}$$

where $\Pi^{>}$ and $\Pi^{<}$ are called Wightman functions and ρ the *spectral function*, whereas Δ is sometimes referred to as the statistical correlator. We are implicitly assuming the presence of an UV regulator so that there are no short-distance singularities in the Fourier transforms.

The "retarded"/"advanced" correlators can be defined as

$$\Pi^R_{\alpha\beta}(\mathcal{K}) \equiv i \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \Big[\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0) \Big] \theta(t) \Big\rangle \,, \tag{8.6}$$

$$\Pi^A_{\alpha\beta}(\mathcal{K}) \equiv i \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle - \Big[\hat{\phi}_\alpha(\mathcal{X}), \hat{\phi}^\dagger_\beta(0) \Big] \theta(-t) \Big\rangle \,. \tag{8.7}$$

Note that since Π^R involves positive times only, $e^{ik^0 t} = e^{i[\mathrm{Re}\,k^0 + i\,\mathrm{Im}\,k^0]t} = e^{i\,\mathrm{Re}\,k^0 t} e^{-\mathrm{Im}\,k^0 t}$ is exponentially suppressed for $\mathrm{Im}\,k^0 > 0$. Therefore Π^R can be considered an analytic function of k^0 in the upper half of the complex k^0-plane (it can develop distribution-like singularities at the physical boundary $\mathrm{Im}\,k^0 \to 0^+$). Similarly, Π^A is an analytic function in the lower half of the complex k^0-plane. These turn out to be strong and useful properties, and do not apply to general correlation functions.

On the other hand, from the computational point of view one is often faced with "time-ordered" correlators,

$$\Pi^T_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \Big\langle \hat{\phi}_\alpha(\mathcal{X})\, \hat{\phi}^\dagger_\beta(0)\, \theta(t) + \hat{\phi}^\dagger_\beta(0)\, \hat{\phi}_\alpha(\mathcal{X})\, \theta(-t) \Big\rangle \,, \tag{8.8}$$

which appear in time-dependent perturbation theory at zero temperature, or with the "Euclidean" correlator

$$\Pi^E_{\alpha\beta}(K) \equiv \int_X e^{iK\cdot X} \Big\langle \hat{\phi}_\alpha(X)\, \hat{\phi}^\dagger_\beta(0) \Big\rangle \,, \tag{8.9}$$

which appears in non-perturbative formulations. Restricting to $0 \le \tau \le \beta$, the Euclidean correlator is also time-ordered, and can be computed with standard imaginary-time functional integrals. If the correlator is periodic [cf. text below Eq. (8.10)], then k_n is a *bosonic* Matsubara frequency.

It follows from Eq. (8.1), by using the cyclicity of the trace, that

$$\big\langle \hat{\phi}_\alpha(t - i\beta, \mathbf{x})\, \hat{\phi}^\dagger_\beta(0,\mathbf{0}) \big\rangle = \frac{1}{\mathcal{Z}} \mathrm{Tr} \Big[e^{-\beta\hat{H}} e^{\beta\hat{H}} \hat{\phi}_\alpha(t,\mathbf{x}) e^{-\beta\hat{H}} \hat{\phi}^\dagger_\beta(0,\mathbf{0}) \Big] = \big\langle \hat{\phi}^\dagger_\beta(0,\mathbf{0})\, \hat{\phi}_\alpha(t,\mathbf{x}) \big\rangle. \tag{8.10}$$

This is a configuration-space version of the so-called Kubo-Martin-Schwinger (KMS) relation, which relates $\Pi^>_{\alpha\beta}$ and $\Pi^<_{\alpha\beta}$ to each other, provided that we are in thermal equilibrium. If we set $t \to 0$ and keep $\mathbf{x} \neq \mathbf{0}$, then $\hat{\phi}_\alpha(0,\mathbf{x})$ and $\hat{\phi}^\dagger_\beta(0,\mathbf{0})$ commute with each other. In this case, the KMS relation implies that the integrand in Eq. (8.9) is a periodic function of τ, with periodicity defined in the same sense as around Eq. (1.41).

It turns out that *all* of the correlation functions defined can be related to each other in thermal equilibrium. In particular, all correlators can be expressed in terms of the spectral function, which in turn can be determined as a certain analytic continuation of the Euclidean correlator. In order to show this, we may first insert sets of energy eigenstates into the definitions of $\Pi^{>}_{\alpha\beta}$ and $\Pi^{<}_{\alpha\beta}$:

$$\begin{aligned}
\Pi^{>}_{\alpha\beta}(\mathcal{K}) &= \frac{1}{\mathcal{Z}}\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\mathrm{Tr}\Big[e^{-\beta\hat{H}+i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|}\hat{\phi}_\alpha(0,\mathbf{x})\,e^{-i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|}\hat{\phi}^\dagger_\beta(0,\mathbf{0})\Big] \\
&= \frac{1}{\mathcal{Z}}\sum_{m,n}\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} e^{(-\beta+it)E_m}e^{-itE_n}\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\,\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= \frac{1}{\mathcal{Z}}\int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}}\sum_{m,n} e^{-\beta E_m}\,2\pi\,\delta(k^0+E_m-E_n)\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\,\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle\;,
\end{aligned} \tag{8.11}$$

$$\begin{aligned}
\Pi^{<}_{\alpha\beta}(\mathcal{K}) &= \frac{1}{\mathcal{Z}}\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\mathrm{Tr}\Big[e^{-\beta\hat{H}}\underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|}\hat{\phi}^\dagger_\beta(0,\mathbf{0})\,e^{i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|}\hat{\phi}_\alpha(0,\mathbf{x})\,e^{-i\hat{H}t}\Big] \\
&= \frac{1}{\mathcal{Z}}\sum_{m,n}\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} e^{(-\beta-it)E_n}e^{itE_m}\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle\,\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle \\
&= \frac{1}{\mathcal{Z}}\int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}}\sum_{m,n} e^{-\beta E_n}\,2\pi\,\underbrace{\delta(k^0+E_m-E_n)}_{E_n=E_m+k^0}\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\,\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= e^{-\beta k^0}\,\Pi^{>}_{\alpha\beta}(\mathcal{K})\;.
\end{aligned} \tag{8.12}$$

This is a Fourier-space version of the KMS relation. Consequently

$$\rho_{\alpha\beta}(\mathcal{K}) \;=\; \frac{1}{2}[\Pi^{>}_{\alpha\beta}(\mathcal{K})-\Pi^{<}_{\alpha\beta}(\mathcal{K})] \;=\; \frac{1}{2}(e^{\beta k^0}-1)\Pi^{<}_{\alpha\beta}(\mathcal{K}) \tag{8.13}$$

and, conversely,

$$\Pi^{<}_{\alpha\beta}(\mathcal{K}) = 2n_{\mathrm{B}}(k^0)\rho_{\alpha\beta}(\mathcal{K})\;, \tag{8.14}$$

$$\Pi^{>}_{\alpha\beta}(\mathcal{K}) = 2\frac{e^{\beta k^0}}{e^{\beta k^0}-1}\rho_{\alpha\beta}(\mathcal{K}) \;=\; 2[1+n_{\mathrm{B}}(k^0)]\,\rho_{\alpha\beta}(\mathcal{K})\;, \tag{8.15}$$

where $n_{\mathrm{B}}(k^0) \equiv 1/[\exp(\beta k^0)-1]$ is the Bose distribution. Moreover,

$$\Delta_{\alpha\beta}(\mathcal{K}) = \frac{1}{2}\big[\Pi^{>}_{\alpha\beta}(\mathcal{K})+\Pi^{<}_{\alpha\beta}(\mathcal{K})\big] \;=\; \big[1+2n_{\mathrm{B}}(k^0)\big]\,\rho_{\alpha\beta}(\mathcal{K})\;. \tag{8.16}$$

Note that $1+2n_{\mathrm{B}}(-k^0) = -[1+2n_{\mathrm{B}}(k^0)]$, so that if ρ is odd in $\mathcal{K}\to-\mathcal{K}$, then Δ is even.

Inserting the representation

$$\theta(t) = i\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega t}}{\omega + i0^+} \tag{8.17}$$

into the definitions of Π^R, Π^A, in which the commutator is represented as an inverse transformation of Eq. (8.4), we obtain

$$\begin{aligned}
\Pi^R_{\alpha\beta}(\mathcal{K}) &= i\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\, 2\theta(t)\int_{\mathcal{P}} e^{-i\mathcal{P}\cdot\mathcal{X}}\rho_{\alpha\beta}(\mathcal{P}) \\
&= -2\int \mathrm{d}t\int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}p^0}{2\pi}\frac{e^{i(k^0-p^0-\omega)t}}{\omega+i0^+}\,\rho_{\alpha\beta}(p^0,\mathbf{k}) \\
&= -2\int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}p^0}{2\pi}\frac{2\pi\delta(k^0-p^0-\omega)}{\omega+i0^+}\,\rho_{\alpha\beta}(p^0,\mathbf{k}) \\
&= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{\pi}\frac{\rho_{\alpha\beta}(p^0,\mathbf{k})}{p^0-k^0-i0^+}\ ,
\end{aligned} \tag{8.18}$$

and similarly

$$\Pi^A_{\alpha\beta}(\mathcal{K}) = \int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{\pi}\frac{\rho_{\alpha\beta}(p^0,\mathbf{k})}{p^0-k^0+i0^+}\ . \tag{8.19}$$

Note that these can be considered to be limiting values from the upper half-plane for Π^R (since it is the combination $k^0 + i0^+$ that appears in the kernel) and from the lower half-plane for Π^A (since it is the combination $k^0 - i0^+$ that appears).

Making use of

$$\frac{1}{\Delta \pm i0^+} = \mathbb{P}\Big(\frac{1}{\Delta}\Big) \mp i\pi\delta(\Delta)\ , \tag{8.20}$$

and assuming that $\rho_{\alpha\beta}$ is real, we find

$$\mathrm{Im}\,\Pi^R_{\alpha\beta}(\mathcal{K}) = \rho_{\alpha\beta}(\mathcal{K})\ , \quad \mathrm{Im}\,\Pi^A_{\alpha\beta}(\mathcal{K}) = -\rho_{\alpha\beta}(\mathcal{K})\ . \tag{8.21}$$

Furthermore, the real parts of Π^R and Π^A agree, so that $-i[\Pi^R_{\alpha\beta} - \Pi^A_{\alpha\beta}] = 2\rho_{\alpha\beta}$.

Moving on to $\Pi^T_{\alpha\beta}$ and making use of Eqs. (8.14) and (8.15) as well as of Eq. (8.17), we find

$$\begin{aligned}
\Pi^T_{\alpha\beta}(\mathcal{K}) &= \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\int_{\mathcal{P}} e^{-i\mathcal{P}\cdot\mathcal{X}}\Big[\theta(t)2e^{\beta p^0}n_{\mathrm{B}}(p^0) + \theta(-t)2n_{\mathrm{B}}(p^0)\Big]\rho_{\alpha\beta}(\mathcal{P}) \\
&= 2i\int\mathrm{d}t\int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}p^0}{2\pi}\bigg[\frac{e^{i(k^0-p^0-\omega)t}}{\omega+i0^+}e^{\beta p^0} + \frac{e^{i(k^0-p^0+\omega)t}}{\omega+i0^+}\bigg]n_{\mathrm{B}}(p^0)\rho_{\alpha\beta}(p^0,\mathbf{k})
\end{aligned}$$

$$= 2i\int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}p^0}{2\pi}\left[\frac{2\pi\delta(k^0-p^0-\omega)}{\omega+i0^+}e^{\beta p^0}+\frac{2\pi\delta(k^0-p^0+\omega)}{\omega+i0^+}\right]n_{\rm B}(p^0)\rho_{\alpha\beta}(p^0,\mathbf{k})$$
$$= i\int\frac{\mathrm{d}p^0}{\pi}\left[\frac{e^{\beta p^0}}{k^0-p^0+i0^+}-\frac{1}{k^0-p^0-i0^+}\right]n_{\rm B}(p^0)\rho_{\alpha\beta}(p^0,\mathbf{k})$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{\pi}\frac{i\rho_{\alpha\beta}(p^0,\mathbf{k})}{k^0-p^0+i0^+}+2\rho_{\alpha\beta}(k^0,\mathbf{k})n_{\rm B}(k^0)$$
$$= -i\Pi^R_{\alpha\beta}(\mathcal{K})+\Pi^<_{\alpha\beta}(\mathcal{K})\,, \tag{8.22}$$

where in the penultimate step we inserted the identity $n_{\rm B}(p^0)e^{\beta p^0} = 1 + n_{\rm B}(p^0)$ as well as Eq. (8.20). Note that Eq. (8.22) can be obtained also directly from the definitions in Eqs. (8.3), (8.6) and (8.8), by inserting $1 = \theta(t)+\theta(-t)$ into Eq. (8.3). It can similarly be seen that $\Pi^T_{\alpha\beta} = -i\Pi^A_{\alpha\beta} + \Pi^>_{\alpha\beta}$.

We note that both sums on the second row of Eq. (8.11) are exponentially convergent for $0 < it < \beta$. Therefore we can formally relate the two functions

$$\left\langle\hat\phi_\alpha(\mathcal{X})\,\hat\phi^\dagger_\beta(0)\right\rangle \quad\text{and}\quad \left\langle\hat\phi_\alpha(X)\,\hat\phi^\dagger_\beta(0)\right\rangle \tag{8.23}$$

by a direct analytic continuation $t\to -i\tau$, or $it\to\tau$, with $0<\tau<\beta$. Thereby

$$\begin{aligned}\Pi^E_{\alpha\beta}(K) &= \int_X e^{iK\cdot X}\left[\int_{\mathcal{P}} e^{-i\mathcal{P}\cdot\mathcal{X}}\,\Pi^>_{\alpha\beta}(\mathcal{P})\right]_{it\to\tau}\\ &= \int_0^\beta \mathrm{d}\tau\, e^{ik_n\tau}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{2\pi}e^{-p^0\tau}\,\Pi^>_{\alpha\beta}(p^0,\mathbf{k})\\ &= \int_0^\beta \mathrm{d}\tau\, e^{ik_n\tau}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{2\pi}e^{-p^0\tau}\,\frac{2e^{\beta p^0}}{e^{\beta p^0}-1}\,\rho_{\alpha\beta}(p^0,\mathbf{k})\\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{\pi}\frac{\rho_{\alpha\beta}(p^0,\mathbf{k})}{1-e^{-\beta p^0}}\left[\frac{e^{(ik_n-p^0)\tau}}{ik_n-p^0}\right]_0^\beta\\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{\pi}\frac{\rho_{\alpha\beta}(p^0,\mathbf{k})}{1-e^{-\beta p^0}}\frac{e^{-\beta p^0}-1}{ik_n-p^0}\\ &\overset{p^0\to k^0}{=} \int_{-\infty}^{\infty}\frac{\mathrm{d}k^0}{\pi}\frac{\rho_{\alpha\beta}(k^0,\mathbf{k})}{k^0-ik_n}\,,\end{aligned} \tag{8.24}$$

where we inserted Eq. (8.15) for $\Pi^>(\mathcal{K})$, and changed orders of integration. This relation is called the *spectral representation* of the Euclidean correlator.

It is useful to note that Eq. (8.24) implies the existence of a simple "sum rule":

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}k^0}{\pi}\frac{\rho_{\alpha\beta}(k^0,\mathbf{k})}{k^0} = \int_0^\beta \mathrm{d}\tau\;\Pi^E_{\alpha\beta}(\tau,\mathbf{k})\,. \tag{8.25}$$

Here we set $k_n = 0$ and used the definition in Eq. (8.9) on the left-hand side of Eq. (8.24). The usefulness of the sum rule is that it relates integrals over Minkowskian and Euclidean correlators to each other. (Of course, we have implicitly assumed that both sides are integrable which, as already alluded to, necessitates a suitable ultraviolet regularization in the spatial directions.)

Finally, the spectral representation in Eq. (8.24) can be inverted by making use of Eq. (8.20),

$$\rho_{\alpha\beta}(\mathcal{K}) = \frac{1}{2i}\,\text{Disc}\,\Pi^E_{\alpha\beta}(k_n \to -ik^0, \mathbf{k}) \tag{8.26}$$

$$\equiv \frac{1}{2i}\Big[\Pi^E_{\alpha\beta}(-i[k^0 + i0^+], \mathbf{k}) - \Pi^E_{\alpha\beta}(-i[k^0 - i0^+], \mathbf{k})\Big]\,. \tag{8.27}$$

Furthermore, a comparison of Eqs. (8.18) and (8.24) shows that

$$\Pi^R_{\alpha\beta}(\mathcal{K}) = \Pi^E_{\alpha\beta}(k_n \to -i[k^0 + i0^+], \mathbf{k})\,. \tag{8.28}$$

This last relation, which can be justified also through a more rigorous mathematical analysis [5], captures the essence of the analytic continuation from the imaginary-time (Matsubara) formalism to physical Minkowskian spacetime.

In the context of the spectral representation, Eq. (8.24), it will often be useful to note from Eq. (1.70), *viz.*

$$T\sum_{\omega_n}\frac{e^{i\omega_n\tau}}{\omega_n^2 + \omega^2} = \frac{n_\text{B}(\omega)}{2\omega}\Big[e^{(\beta-\tau)\omega} + e^{\tau\omega}\Big]\,, \tag{8.29}$$

that, for $0 < \tau < \beta$,

$$\begin{aligned} T\sum_{\omega_n}\frac{1}{k^0 - i\omega_n}e^{i\omega_n\tau} &= T\sum_{\omega_n}\frac{i\omega_n + k^0}{\omega_n^2 + (k^0)^2}e^{i\omega_n\tau} \\ &= (\partial_\tau + k^0)T\sum_{\omega_n}\frac{e^{i\omega_n\tau}}{\omega_n^2 + (k^0)^2} \\ &= \frac{n_\text{B}(k^0)}{2\,k^0}\Big[(-k^0 + k^0)e^{(\beta-\tau)k^0} + (k^0 + k^0)e^{\tau k^0}\Big] \\ &= n_\text{B}(k^0)e^{\tau k^0}\,. \end{aligned} \tag{8.30}$$

This relation turns out to be valid both for $k^0 < 0$ and $k^0 > 0$ [to show this, substitute $\omega_n \to -\omega_n$ and use Eq. (8.31)]. We also note that, again for $0 < \tau < \beta$,

$$T\sum_{\omega_n}\frac{1}{k^0 - i\omega_n}e^{-i\omega_n\tau} = T\sum_{\omega_n}\frac{1}{k^0 - i\omega_n}e^{i\omega_n(\beta-\tau)} = n_\text{B}(k^0)e^{(\beta-\tau)k^0}\,. \tag{8.31}$$

In particular, taking the inverse Fourier transform ($T\sum_{k_n} e^{-ik_n\tau}$) from the left-hand side of Eq. (8.24), and employing Eq. (8.31), we get the relation

$$\begin{aligned}
&\int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}}\Big\langle \hat{\phi}_\alpha(\tau,\mathbf{x})\,\hat{\phi}^\dagger_\beta(0,\mathbf{0})\Big\rangle \\
&= \int_{-\infty}^{\infty}\frac{\mathrm{d}k^0}{\pi}\rho_{\alpha\beta}(\mathcal{K})\,n_{\rm B}(k^0)e^{(\beta-\tau)k^0} \\
&= \int_0^{\infty}\frac{\mathrm{d}k^0}{\pi}\Bigg\{\frac{\rho_{\alpha\beta}(k^0,\mathbf{k})+\rho_{\alpha\beta}(-k^0,\mathbf{k})}{2}\frac{\sinh\Big[\Big(\frac{\beta}{2}-\tau\Big)k^0\Big]}{\sinh\Big(\frac{\beta}{2}k^0\Big)} \\
&\quad + \frac{\rho_{\alpha\beta}(k^0,\mathbf{k})-\rho_{\alpha\beta}(-k^0,\mathbf{k})}{2}\frac{\cosh\Big[\Big(\frac{\beta}{2}-\tau\Big)k^0\Big]}{\sinh\Big(\frac{\beta}{2}k^0\Big)}\Bigg\}\,, \qquad (8.32)
\end{aligned}$$

where we symmetrized and anti-symmetrized the "kernel" $n_{\rm B}(k^0)e^{(\beta-\tau)k^0}$ with respect to k^0. Normally (when $\hat{\phi}_\alpha$ and $\hat{\phi}^\dagger_\beta$ are identical) the spectral function is antisymmetric in $k^0 \to -k^0$, and only the second term on the last line of Eq. (8.32) contributes. Thereby we obtain a useful identity: if the left-hand side of Eq. (8.32) can be measured non-perturbatively on a Euclidean lattice with Monte Carlo simulations as a function of τ, then an "inversion" of Eq. (8.32) could lead to a non-perturbative estimate of the Minkowskian spectral function. Issues related to this inversion are discussed in [6].

Example: Free Boson

Let us illustrate the relations obtained with the example of a free propagator in scalar field theory:

$$\Pi^E(K) = \frac{1}{k_n^2+E_k^2} = \frac{1}{2E_k}\left(\frac{1}{ik_n+E_k}+\frac{1}{-ik_n+E_k}\right)\,, \qquad (8.33)$$

where $E_k = \sqrt{k^2+m^2}$. According to Eq. (8.28),

$$\begin{aligned}
\Pi^R(\mathcal{K}) &= \frac{1}{-(k^0+i0^+)^2+E_k^2} \\
&= -\frac{1}{\mathcal{K}^2-m^2+i\,\mathrm{sign}(k^0)0^+} \\
&= -\mathbb{P}\left(\frac{1}{(k^0)^2-E_k^2}\right)+\frac{i\pi}{2E_k}\Big[\delta(k^0-E_k)-\delta(k^0+E_k)\Big]\,, \qquad (8.34)
\end{aligned}$$

and according to Eq. (8.21),

$$\rho(\mathcal{K}) = \frac{\pi}{2E_k}\Big[\delta(k^0 - E_k) - \delta(k^0 + E_k)\Big] . \tag{8.35}$$

Finally, according to Eqs. (8.14) and (8.22),

$$\begin{aligned}
\Pi^T(\mathcal{K}) &= \mathbb{P}\left(\frac{i}{(k^0)^2 - E_k^2}\right) + \frac{\pi}{2E_k}\Big\{\delta(k^0 - E_k)\big[1 + 2n_{\rm B}(k^0)\big] \\
&\quad -\delta(k^0 + E_k)\big[1 + 2n_{\rm B}(k^0)\big]\Big\} \\
&= \mathbb{P}\left(\frac{i}{(k^0)^2 - E_k^2}\right) + \frac{\pi}{2E_k}\Big[\delta(k^0 - E_k) + \delta(k^0 + E_k)\Big]\big[1 + 2n_{\rm B}(|k^0|)\big] \\
&= \mathbb{P}\left(\frac{i}{(k^0)^2 - E_k^2}\right) + \pi\delta\Big((k^0)^2 - E_k^2\Big)\big[1 + 2n_{\rm B}(|k^0|)\big] \\
&= \frac{i}{(k^0)^2 - E_k^2 + i0^+} + 2\pi\delta\Big((k^0)^2 - E_k^2\Big)n_{\rm B}(|k^0|) \\
&= \frac{i}{\mathcal{K}^2 - m^2 + i0^+} + 2\pi\,\delta(\mathcal{K}^2 - m^2)\,n_{\rm B}(|k^0|) ,
\end{aligned} \tag{8.36}$$

where in the second step we made use of the identity $1+2n_{\rm B}(-E_k) = -[1+2n_{\rm B}(E_k)]$.

It is useful to note that Eq. (8.36) is closely related to Eq. (2.34). However, Eq. (2.34) is true in general, whereas Eq. (8.36) was derived for the special case of a free propagator; thus it is not always true that thermal effects can be obtained by simply replacing the zero-temperature time-ordered propagator by Eq. (8.36), even if surprisingly often such a simple recipe does function. We return to a discussion of this point in Sect. 8.3.

Fermionic Case

Let us next consider 2-point correlation functions built out of fermionic operators [1–4]. In contrast to the bosonic case, we take for generality the density matrix to be of the form

$$\hat{\rho} = \frac{1}{\mathcal{Z}}\exp[-\beta(\hat{H} - \mu\hat{Q})] , \tag{8.37}$$

where $\hat{Q}$ is an operator commuting with $\hat{H}$ and μ is the associated chemical potential.

We denote the operators appearing in the 2-point functions by $\hat{j}_\alpha$, $\hat{\bar{j}}_\beta$. They could be elementary field operators, in which case the indices α, β label Dirac and/or flavour components, but they could also be composite operators consisting of a product of elementary field operators. Nevertheless, we assume the validity of the relation

$$[\hat{j}_\alpha(t,\mathbf{x}),\hat{Q}] = \hat{j}_\alpha(t,\mathbf{x}) \ . \tag{8.38}$$

To motivate this, note that for $\hat{j}_\alpha \equiv \hat{\psi}_\alpha$, $\hat{\bar{j}}_\beta = \hat{\bar{\psi}}_\beta$, the canonical commutation relation of Eq. (4.33),

$$\{\hat{\psi}_\alpha(x^0,\mathbf{x}),\hat{\psi}^\dagger_\beta(x^0,\mathbf{y})\} = \delta^{(d)}(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta} \ , \tag{8.39}$$

and the expression for the conserved charge in Eq. (7.33),

$$\hat{Q} = \int_\mathbf{x} \hat{\bar{\psi}}\gamma_0\hat{\psi} = \int_\mathbf{x} \hat{\psi}^\dagger_\alpha\hat{\psi}_\alpha \ , \tag{8.40}$$

as well as the identity $[\hat{A},\hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} = \{\hat{A},\hat{B}\}\hat{C} - \hat{B}\{\hat{A},\hat{C}\}$, indicate that Eq. (8.38) is indeed satisfied for $\hat{\psi}_\alpha$. Equation (8.38) implies that

$$\begin{aligned} e^{\beta\mu\hat{Q}}\hat{j}_\alpha(t,\mathbf{x}) &= \sum_{n=0}^{\infty}\frac{1}{n!}(\beta\mu)^n(\hat{Q})^n\hat{j}_\alpha(t,\mathbf{x}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\beta\mu)^n\hat{j}_\alpha(t,\mathbf{x})(\hat{Q}-\hat{\mathbb{1}})^n \\ &= \hat{j}_\alpha(t,\mathbf{x})e^{\beta\mu\hat{Q}}e^{-\beta\mu} \ , \end{aligned} \tag{8.41}$$

and consequently that

$$\begin{aligned} \Big\langle\hat{j}_\alpha(t-i\beta,\mathbf{x})\,\hat{\bar{j}}_\beta(0,\mathbf{0})\Big\rangle &= \frac{1}{\mathcal{Z}}\mathrm{Tr}\Big[e^{-\beta(\hat{H}-\mu\hat{Q})}e^{\beta\hat{H}}\hat{j}_\alpha(t,\mathbf{x})e^{-\beta\hat{H}}\hat{\bar{j}}_\beta(0,\mathbf{0})\Big] \\ &= \frac{1}{\mathcal{Z}}\mathrm{Tr}\Big[\hat{j}_\alpha(t,\mathbf{x})e^{-\beta\mu}e^{-\beta(\hat{H}-\mu\hat{Q})}\hat{\bar{j}}_\beta(0,\mathbf{0})\Big] \\ &= \frac{1}{\mathcal{Z}}e^{-\mu\beta}\,\mathrm{Tr}\Big[\hat{j}_\alpha(t,\mathbf{x})e^{-\beta(\hat{H}-\mu\hat{Q})}\hat{\bar{j}}_\beta(0,\mathbf{0})\Big] \\ &= e^{-\mu\beta}\Big\langle\hat{\bar{j}}_\beta(0,\mathbf{0})\,\hat{j}_\alpha(t,\mathbf{x})\Big\rangle \ . \end{aligned} \tag{8.42}$$

This is a fermionic version of the KMS relation.

With this setting, we can again define various classes of correlation functions. The "physical" correlators are now set up as

$$\Pi^>_{\alpha\beta}(\mathcal{K}) \equiv \int_\mathcal{X} e^{i\mathcal{K}\cdot\mathcal{X}}\Big\langle\hat{j}_\alpha(\mathcal{X})\,\hat{\bar{j}}_\beta(0)\Big\rangle \ , \tag{8.43}$$

$$\Pi^<_{\alpha\beta}(\mathcal{K}) \equiv \int_\mathcal{X} e^{i\mathcal{K}\cdot\mathcal{X}}\Big\langle-\hat{\bar{j}}_\beta(0)\,\hat{j}_\alpha(\mathcal{X})\Big\rangle \ , \tag{8.44}$$

$$\rho_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \left\langle \frac{1}{2}\left\{ \hat{j}_\alpha(\mathcal{X}), \hat{\bar{j}}_\beta(0) \right\} \right\rangle , \tag{8.45}$$

$$\Delta_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \left\langle \frac{1}{2}\left[\hat{j}_\alpha(\mathcal{X}), \hat{\bar{j}}_\beta(0) \right] \right\rangle , \tag{8.46}$$

where $\rho_{\alpha\beta}$ is the spectral function. The retarded and advanced correlators can be defined as

$$\Pi^R_{\alpha\beta}(\mathcal{K}) \equiv i \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \left\langle \left\{ \hat{j}_\alpha(\mathcal{X}), \hat{\bar{j}}_\beta(0) \right\} \theta(t) \right\rangle , \tag{8.47}$$

$$\Pi^A_{\alpha\beta}(\mathcal{K}) \equiv i \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \left\langle - \left\{ \hat{j}_\alpha(\mathcal{X}), \hat{\bar{j}}_\beta(0) \right\} \theta(-t) \right\rangle . \tag{8.48}$$

On the other hand, the time-ordered correlation function reads

$$\Pi^T_{\alpha\beta}(\mathcal{K}) \equiv \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \left\langle \hat{j}_\alpha(\mathcal{X})\, \hat{\bar{j}}_\beta(0)\, \theta(t) - \hat{\bar{j}}_\beta(0) \hat{j}_\alpha(\mathcal{X})\, \theta(-t) \right\rangle , \tag{8.49}$$

whereas the Euclidean correlator is

$$\Pi^E_{\alpha\beta}(K) \equiv \int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} e^{(ik_n+\mu)\tau - i\mathbf{k}\cdot\mathbf{x}} \left\langle \hat{j}_\alpha(X) \hat{\bar{j}}_\beta(0) \right\rangle . \tag{8.50}$$

Note that the Euclidean correlator is time-ordered by definition ($0 \le \tau \le \beta$), and can be computed with standard imaginary-time functional integrals.

If the two operators in the integrand of Eq. (8.50) anticommute with each other at $t = 0$, then the KMS relation in Eq. (8.42) asserts that $\left\langle \hat{j}_\alpha(-i\beta, \mathbf{x})\, \hat{\bar{j}}_\beta(0, \mathbf{0}) \right\rangle = e^{-\mu\beta} \left\langle \hat{\bar{j}}_\beta(0, \mathbf{0})\, \hat{j}_\alpha(0, \mathbf{x}) \right\rangle = -e^{-\mu\beta} \left\langle \hat{j}_\alpha(0, \mathbf{x})\, \hat{\bar{j}}_\beta(0, \mathbf{0}) \right\rangle$. The additional term in the Fourier transform with respect to τ in Eq. (8.50) cancels the multiplicative factor $e^{-\mu\beta}$ at $\tau = \beta$, so that the τ-integrand is antiperiodic. Therefore the Matsubara frequencies k_n are fermionic.

We can establish relations between the different Green's functions just like in the bosonic case:

$$\begin{aligned}
\Pi^>_{\alpha\beta}(\mathcal{K}) &= \frac{1}{\mathcal{Z}} \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} \mathrm{Tr}\Big[e^{-\beta\hat{H}+i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|} e^{\beta\mu\hat{Q}} \hat{j}_\alpha(0, \mathbf{x})\, e^{-i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|} \hat{\bar{j}}_\beta(0, \mathbf{0}) \Big] \\
&= \frac{1}{\mathcal{Z}} \sum_{m,n} \int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} e^{(-\beta+it)E_m} e^{-itE_n} e^{-\beta\mu} \langle m| \hat{j}_\alpha(0, \mathbf{x}) e^{\beta\mu\hat{Q}} |n\rangle\, \langle n| \hat{\bar{j}}_\beta(0, \mathbf{0}) |m\rangle \\
&= \frac{1}{\mathcal{Z}} \int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} \sum_{m,n} e^{-\beta(E_m+\mu)}\, 2\pi\, \delta(k^0 + E_m - E_n) \langle m| \hat{j}_\alpha(0, \mathbf{x}) e^{\beta\mu\hat{Q}} |n\rangle\, \langle n| \hat{\bar{j}}_\beta(0, \mathbf{0}) |m\rangle \ ,
\end{aligned} \tag{8.51}$$

$$\Pi^{<}_{\alpha\beta}(\mathcal{K}) = -\frac{1}{\mathcal{Z}}\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}}\mathrm{Tr}\Big[e^{-\beta\hat{H}}e^{\beta\mu\hat{Q}} \underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|} \hat{\bar{j}}_\beta(0,\mathbf{0})\, e^{i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|} \hat{j}_\alpha(0,\mathbf{x})\, e^{-i\hat{H}t}\Big]$$

$$= -\frac{1}{\mathcal{Z}}\sum_{m,n}\int_{\mathcal{X}} e^{i\mathcal{K}\cdot\mathcal{X}} e^{(-\beta-it)E_n} e^{itE_m}\langle n|\hat{\bar{j}}_\beta(0,\mathbf{0})|m\rangle\,\langle m|\hat{j}_\alpha(0,\mathbf{x})e^{\beta\mu\hat{Q}}|n\rangle$$

$$= -\frac{1}{\mathcal{Z}}\int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}}\sum_{m,n} e^{-\beta E_n}\, 2\pi \underbrace{\delta(k^0+E_m-E_n)}_{E_n=E_m+k^0}\langle m|\hat{j}_\alpha(0,\mathbf{x})e^{\beta\mu\hat{Q}}|n\rangle\,\langle n|\hat{\bar{j}}_\beta(0,\mathbf{0})|m\rangle$$

$$= -e^{-\beta(k^0-\mu)}\,\Pi^{>}_{\alpha\beta}(\mathcal{K})\,. \tag{8.52}$$

Using the fact that $\rho_{\alpha\beta}(\mathcal{K}) = [\Pi^{>}_{\alpha\beta}(\mathcal{K}) - \Pi^{<}_{\alpha\beta}(\mathcal{K})]/2$, we subsequently obtain

$$\Pi^{>}_{\alpha\beta}(\mathcal{K}) = 2[1-n_{\mathrm{F}}(k^0-\mu)]\rho_{\alpha\beta}(\mathcal{K})\,, \quad \Pi^{<}_{\alpha\beta}(\mathcal{K}) = -2n_{\mathrm{F}}(k^0-\mu)\rho_{\alpha\beta}(\mathcal{K})\,, \tag{8.53}$$

where $n_{\mathrm{F}}(k^0) \equiv 1/[\exp(\beta k^0)+1]$ is the Fermi distribution. Moreover, the statistical correlator can be expressed as $\Delta_{\alpha\beta}(\mathcal{K}) = [1-2n_{\mathrm{F}}(k^0-\mu)]\rho_{\alpha\beta}(\mathcal{K})$.

The relation of Π^R, Π^A and Π^T to the spectral function can be derived in complete analogy with Eqs. (8.17)–(8.22). For brevity we only cite the final results:

$$\Pi^{R}_{\alpha\beta}(\mathcal{K}) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\frac{\rho_{\alpha\beta}(\omega,\mathbf{k})}{\omega-k^0-i0^+}\,, \quad \Pi^{A}_{\alpha\beta}(\mathcal{K}) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\frac{\rho_{\alpha\beta}(\omega,\mathbf{k})}{\omega-k^0+i0^+}\,, \tag{8.54}$$

$$\Pi^{T}_{\alpha\beta}(\mathcal{K}) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\frac{i\rho_{\alpha\beta}(\omega,\mathbf{k})}{k^0-\omega+i0^+} - 2n_{\mathrm{F}}(k^0-\mu)\rho_{\alpha\beta}(k^0,\mathbf{k})$$

$$= -i\,\Pi^{R}_{\alpha\beta}(\mathcal{K}) + \Pi^{<}_{\alpha\beta}(\mathcal{K})\,. \tag{8.55}$$

Note that when written in a "generic form", where no distribution functions are visible, the end results are identical to the bosonic ones. In addition, Eq. (8.55) can again be crosschecked using the right-hand sides of Eqs. (8.44), (8.47) and (8.49), and the alternative representation $\Pi^{T}_{\alpha\beta} = -i\Pi^{A}_{\alpha\beta} + \Pi^{>}_{\alpha\beta}$ also applies. The latter derivation implies that these "operator relations" apply even in a non-thermal situation, described by a generic density matrix (cf. Sect. 8.3).

Finally, writing the argument inside the τ-integration in Eq. (8.50) as a Wick rotation of the inverse Fourier transform of Eq. (8.43), inserting Eq. (8.53), and changing orders of integration, we get a spectral representation analogous to Eq. (8.24),

$$\Pi^{E}_{\alpha\beta}(K) = \int_0^\beta \mathrm{d}\tau\, e^{(ik_n+\mu)\tau}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{2\pi}e^{-p^0\tau}\,\Pi^{>}_{\alpha\beta}(p^0,\mathbf{k})$$

$$= \int_0^\beta \mathrm{d}\tau\, e^{(ik_n+\mu)\tau}\int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{2\pi}e^{-p^0\tau}\frac{2e^{\beta(p^0-\mu)}}{e^{\beta(p^0-\mu)}+1}\rho_{\alpha\beta}(p^0,\mathbf{k})$$

$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}p^0}{\pi}\frac{e^{\beta(p^0-\mu)}}{e^{\beta(p^0-\mu)}+1}\rho_{\alpha\beta}(p^0,\mathbf{k})\int_0^\beta \mathrm{d}\tau\, e^{(ik_n+\mu-p^0)\tau}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{\pi} \frac{e^{\beta(p^0-\mu)}}{e^{\beta(p^0-\mu)}+1} \rho_{\alpha\beta}(p^0,\mathbf{k}) \left[\frac{e^{(ik_n+\mu-p^0)\tau}}{ik_n+\mu-p^0} \right]_0^\beta \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{\pi} \frac{e^{\beta(p^0-\mu)}}{e^{\beta(p^0-\mu)}+1} \rho_{\alpha\beta}(p^0,\mathbf{k}) \frac{-e^{-\beta(p^0-\mu)}-1}{ik_n+\mu-p^0} \\
&\overset{p^0\to k^0}{=} \int_{-\infty}^{\infty} \frac{\mathrm{d}k^0}{\pi} \frac{\rho_{\alpha\beta}(k^0,\mathbf{k})}{k^0 - i[k_n - i\mu]} \ . \qquad (8.56)
\end{aligned}
$$

Like in the bosonic case, this relation can be inverted by making use of Eq. (8.20),

$$
\rho_{\alpha\beta}(\mathcal{K}) = \frac{1}{2i} \,\mathrm{Disc}\, \Pi^E_{\alpha\beta}(k_n - i\mu \to -ik^0, \mathbf{k}) \qquad (8.57)
$$

where the discontinuity is defined like in Eq. (8.27).

The fermionic Matsubara sum over the structure in Eq. (8.56) can be carried out explicitly. This could be verified by making use of Eq. (4.77), in analogy with the bosonic analysis in Eqs. (8.30) and (8.31), but let us proceed in another way for a change. We may recall (cf. footnote on p. 14) that

$$
T \sum_{\omega_n} e^{i\omega_n \tau} = \delta(\tau \bmod \beta) \ . \qquad (8.58)
$$

According to Eq. (4.55), viz. $\sigma_{\mathrm{f}}(T) = 2\sigma_{\mathrm{b}}\left(\frac{T}{2}\right) - \sigma_{\mathrm{b}}(T)$, we can thus write

$$
T \sum_{\{\omega_n\}} e^{i\omega_n \tau} = 2\delta(\tau \bmod 2\beta) - \delta(\tau \bmod \beta) \ . \qquad (8.59)
$$

Let us assume for a moment that $k^0 - \mu > 0$. Employing the representation

$$
\frac{1}{\alpha + i\beta} = \int_0^\infty \mathrm{d}s \, e^{-(\alpha+i\beta)s} \ , \quad \alpha > 0 \ , \qquad (8.60)
$$

and inserting subsequently Eq. (8.59), we get

$$
\begin{aligned}
T \sum_{\{\omega_n\}} \frac{1}{k^0 - \mu - i\omega_n} e^{i\omega_n\tau} &= \int_0^\infty \mathrm{d}s \, T \sum_{\{\omega_n\}} e^{i\omega_n\tau - k^0 s + \mu s + i\omega_n s} \\
&= \int_0^\infty \mathrm{d}s \, e^{-(k^0-\mu)s} \Big[2\delta(\tau + s \bmod 2\beta) - \delta(\tau + s \bmod \beta) \Big] \\
&= 2 \sum_{n=1}^{\infty} e^{-(k^0-\mu)(-\tau+2\beta n)} - \sum_{n=1}^{\infty} e^{-(k^0-\mu)(-\tau+\beta n)}
\end{aligned}
$$

$$= e^{(k^0-\mu)\tau}\Bigg[\underbrace{2\underbrace{\sum_{n=1}^{\infty}e^{-2\beta(k^0-\mu)n}}_{\frac{e^{-2\beta(k^0-\mu)}}{1-e^{-2\beta(k^0-\mu)}}}-\underbrace{\sum_{n=1}^{\infty}e^{-\beta(k^0-\mu)n}}_{\frac{e^{-\beta(k^0-\mu)}}{1-e^{-\beta(k^0-\mu)}}}}_{\frac{2}{(e^{\beta(k^0-\mu)}-1)(e^{\beta(k^0-\mu)}+1)}-\frac{1}{e^{\beta(k^0-\mu)}-1}}\Bigg]$$

$$= -e^{(k^0-\mu)\tau}n_{\rm F}(k^0-\mu)\;, \tag{8.61}$$

where we assumed $0 < \tau < \beta$. As an immediate consequence,

$$T\sum_{\{\omega_n\}}\frac{1}{k^0-\mu-i\omega_n}e^{-i\omega_n\tau} = -T\sum_{\{\omega_n\}}\frac{1}{k^0-\mu-i\omega_n}e^{i\omega_n(\beta-\tau)} = e^{(\beta-\tau)(k^0-\mu)}n_{\rm F}(k^0-\mu). \tag{8.62}$$

Furthermore, it is not difficult to show (by substituting $\omega_n \to -\omega_n$) that these relations continue to hold also for $k^0-\mu<0$.

As a consequence of Eq. (8.61), we note that

$$\begin{aligned}
T\sum_{\{\omega_n\}}\frac{e^{i(\omega_n+i\mu)\tau}}{(\omega_n+i\mu)^2+\omega^2} &= e^{-\mu\tau}T\sum_{\{\omega_n\}}e^{i\omega_n\tau}\frac{1}{(\omega-i\omega_n+\mu)(\omega+i\omega_n-\mu)} \\
&= e^{-\mu\tau}T\sum_{\{\omega_n\}}e^{i\omega_n\tau}\frac{1}{2\omega}\left[\frac{1}{\omega-\mu+i\omega_n}+\frac{1}{\omega+\mu-i\omega_n}\right] \\
&= \frac{e^{-\mu\tau}}{2\omega}\left[e^{-(\omega-\mu)\tau}n_{\rm F}(-\omega+\mu)-e^{(\omega+\mu)\tau}n_{\rm F}(\omega+\mu)\right] \\
&= \frac{e^{-\mu\tau}}{2\omega}\left[e^{(\beta-\tau)(\omega-\mu)}n_{\rm F}(\omega-\mu)-e^{\tau(\omega+\mu)}n_{\rm F}(\omega+\mu)\right] \\
&= \frac{1}{2\omega}\left[n_{\rm F}(\omega-\mu)e^{(\beta-\tau)\omega-\beta\mu}-n_{\rm F}(\omega+\mu)e^{\tau\omega}\right].
\end{aligned} \tag{8.63}$$

This constitutes a generalization of Eq. (4.77) to the case of a finite chemical potential.

Example: Free Fermion

We illustrate the relations obtained by considering the structure of the free fermion propagator in the presence of a chemical potential. With fermions, one has to be quite careful with definitions. Suppressing spatial coordinates and indices, Eq. (5.47) and the presence of a chemical potential *à la* Eq. (7.35) imply that the free propagator can be written in the schematic form (here A and B carry dependence on the spatial momentum and the Dirac matrices)

$$\langle \hat{\psi}(\tau)\, \hat{\bar{\psi}}(0) \rangle = T \sum_{\{p_n\}} e^{i(p_n + i\mu)\tau} \frac{-iA(p_n + i\mu) + B}{(p_n + i\mu)^2 + E^2} \,, \tag{8.64}$$

where an additional exponential has been inserted into the Fourier transform, in order to respect the KMS property in Eq. (8.42). The correlator in Eq. (8.50) then becomes

$$\begin{aligned} \Pi^E(k_n) &= \int_0^\beta \mathrm{d}\tau\, e^{(ik_n + \mu)\tau} T \sum_{\{p_n\}} e^{i(p_n + i\mu)\tau} \frac{-iA(p_n + i\mu) + B}{(p_n + i\mu)^2 + E^2} \\ &= \frac{iA(k_n - i\mu) + B}{(k_n - i\mu)^2 + E^2} \,. \end{aligned} \tag{8.65}$$

The analytic continuation in Eq. (8.57) yields the retarded correlator

$$\Pi^R(k^0) = \frac{A(k^0 + i0^+) + B}{-(k^0 + i0^+)^2 + E^2} = -\frac{Ak^0 + B}{(k^0)^2 - E^2 + i\,\mathrm{sign}(k^0)0^+} \,, \tag{8.66}$$

and its discontinuity gives

$$\begin{aligned} \rho(k^0) &= \pi(Ak^0 + B)\,\mathrm{sign}(k^0)\,\delta\big((k^0 - E)(k^0 + E)\big) \\ &= \pi(Ak^0 + B)\frac{\mathrm{sign}(k^0)}{2E}\Big[\delta(k^0 - E) + \delta(k^0 + E)\Big] \\ &= \frac{\pi}{2E}(Ak^0 + B)\Big[\delta(k^0 - E) - \delta(k^0 + E)\Big] \,. \end{aligned} \tag{8.67}$$

Any dependence on temperature and chemical potential has disappeared here. Note that (if B is odd in $\mathbf{k}$) ρ is even in $\mathcal{K} \to -\mathcal{K}$. From Eqs. (8.53) and (8.55), the time-ordered propagator can be determined after a few steps:

$$\begin{aligned} \Pi^T(k^0) &= (Ak^0 + B)\bigg\{ \frac{-i}{2E}\bigg(\frac{1}{E - k^0 - i0^+} + \frac{1}{E + k^0 + i0^+} \bigg) \\ &\quad - \frac{2\pi}{2E} n_{\mathrm{F}}(k^0 - \mu)\Big[\delta(k^0 - E) - \delta(k^0 + E)\Big]\bigg\} \\ &= \frac{Ak^0 + B}{2E}\bigg\{ -i\mathbb{P}\left(\frac{1}{E - k^0}\right) - i\mathbb{P}\left(\frac{1}{E + k^0}\right) \\ &\quad + \pi\delta(k^0 - E)\Big[1 - 2n_{\mathrm{F}}(k^0 - \mu)\Big] - \pi\delta(k^0 + E)\Big[1 - 2n_{\mathrm{F}}(k^0 - \mu)\Big]\bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{Ak^0+B}{2E}\bigg\{-i\mathbb{P}\left(\frac{2E}{E^2-(k^0)^2}\right) \\
&\quad + \pi\delta(k^0-E)\Big[1-2n_{\rm F}(k^0-\mu)\Big] + \pi\delta(k^0+E)\Big[1-2n_{\rm F}(-k^0+\mu)\Big]\bigg\} \\
&= \frac{Ak^0+B}{2E}\bigg\{-i\mathbb{P}\left(\frac{2E}{E^2-(k^0)^2}\right) + 2E\pi\delta\Big((k^0)^2-E^2\Big) \\
&\quad - 2\pi\Big[\delta(k^0-E)\,n_{\rm F}(k^0-\mu) + \delta(k^0+E)\,n_{\rm F}(-k^0+\mu)\Big]\bigg\} \\
&= (Ak^0+B)\bigg\{\frac{i}{\mathcal{K}^2-m^2+i0^+} - 2\pi\,\delta\big(\mathcal{K}^2-m^2\big)\,n_{\rm F}\Big(|k^0|-\text{sign}(k^0)\mu\Big)\bigg\}\ .
\end{aligned}
\tag{8.68}
$$

Medium effects are seen to reside in the on-shell part and, to some extent, one could hope to account for them simply by replacing free zero-temperature Feynman propagators by Eq. (8.68). The proper procedure, however, is to carry out the analytic continuation for the *complete observable* considered, and this may not always amount to the simple replacement of vacuum time-ordered propagators through Eq. (8.68), cf. Sect. 8.3.

8.2 From a Euclidean Correlator to a Spectral Function

As an application of the relations derived in Sect. 8.1, let us carry out an explicit 1-loop computation illustrating the steps.[1] The computation performed here will turn out to be directly relevant in the context of particle production, discussed in more detail in Sect. 9.3.

Our goal is to work out the leading non-trivial contribution to the spectral function of a right-handed lepton (N) that originates from its Yukawa interaction with Standard Model particles,

$$
\delta\mathcal{L}_M \;\equiv\; -h\bar{L}\,\tilde{\phi}\,a_{\rm R}N - h^*\bar{N}\tilde{\phi}^\dagger a_{\rm L}L\ . \tag{8.69}
$$

Here $\tilde{\phi} \equiv i\tau_2\phi^*$ is a conjugated Higgs doublet, L is a lepton doublet, $a_{\rm L} \equiv (1-\gamma_5)/2$ and $a_{\rm R} \equiv (1+\gamma_5)/2$ are chiral projectors, and h is a Yukawa coupling constant. The Higgs and lepton doublets have the forms

$$
\tilde{\phi} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_0+i\phi_3\\ -\phi_2+i\phi_1\end{pmatrix}\ ,\quad L = \begin{pmatrix}\nu\\ e\end{pmatrix} \equiv \begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix}\ , \tag{8.70}
$$

[1] A classic example of this kind of a computation can be found in [7]. It is straightforward to generalize the techniques to the 2-loop level, cf. e.g. [8]; at that order the novelty arises that there are infrared divergences in "real" and "virtual" parts of the result which only cancel in the sum.

where ϕ_μ, $\mu \in \{0, 1, 2, 3\}$, are real scalar fields. The neutral component ϕ_0 is the physical Higgs field, whereas the ϕ_i represent Goldstone modes after electroweak symmetry breaking.

Anticipating the results of Sect. 9.3, we consider the Euclidean correlator of the operators coupling to the right-handed lepton through the interaction in Eq. (8.69),

$$\Pi^E(K) \equiv \int_X e^{iK\cdot X} a_{\rm L} \big\langle (\tilde{\phi}^\dagger L)(X)\, (\bar{L}\tilde{\phi})(0) \big\rangle a_{\rm R} \,. \tag{8.71}$$

This has the form of Eq. (8.50); the coupling constant $|h|^2$ has been omitted for simplicity. The four-momentum K is *fermionic*. The operators in Eq. (8.71) are of a mixed "boson-fermion" type; similar computations will be carried out for "fermion-fermion" and "boson-boson" cases below, cf. Eqs. (8.134) and (8.178), respectively. The "boson-fermion" analysis is furthermore generalized to include a chemical potential around Eq. (8.180).

Inserting Eq. (8.70) and carrying out the contractions, we can rewrite Eq. (8.71) in the form

$$\begin{aligned}
\Pi^E(K) &= \frac{1}{2}\int_X e^{iK\cdot X} a_{\rm L} \langle \ell(X)\bar{\ell}(0)\rangle_0 \langle \phi(X)\phi(0)\rangle_0\, a_{\rm R} \\
&= \frac{1}{2}\int_X \sum\!\!\!\!\!\!\!\int_{\{P\}R} e^{i(K+P+R)\cdot X} a_{\rm L} \frac{-i\not{P} + m_\ell}{P^2 + m_\ell^2} \frac{1}{R^2 + m_\phi^2}\, a_{\rm R} \\
&= \frac{1}{2}\int_{\mathbf{p}} T \sum_{\{p_n\}} \frac{-i\not{P}\, a_{\rm R}}{p_n^2 + E_1^2} \frac{1}{(p_n + k_n)^2 + E_2^2} \,,
\end{aligned} \tag{8.72}$$

where we inserted the free scalar and fermion propagators, and denoted

$$E_1 \equiv \sqrt{\mathbf{p}^2 + m_\ell^2} \,, \quad E_2 \equiv \sqrt{(\mathbf{p}+\mathbf{k})^2 + m_\phi^2} \,. \tag{8.73}$$

Moreover the left and right projectors removed the mass term from the numerator. We have been implicit about the assignment of the masses m_ℓ, m_ϕ to the corresponding fields, as well as about the summation over the different field components, but for now no details of this kind are needed.

The essential issue in handling Eq. (8.72) is the treatment of the Matsubara sum. More generally, let us inspect the structure

$$\mathcal{F} \equiv T \sum_{\{p_n\}} \frac{f(ip_n, ik_n, \mathbf{v})}{[p_n^2 + E_1^2][(p_n + k_n)^2 + E_2^2]} \,, \tag{8.74}$$

where we assume that the book-keeping function f depends linearly on its arguments (this assumption will become crucial below), and $\mathbf{v}$ is a dummy variable representing spatial momenta. We can write

$$\mathcal{F} = T\sum_{\{p_n\}} T\sum_{r_n} \beta\,\delta(r_n - p_n - k_n)\frac{f(ip_n, ik_n, \mathbf{v})}{[p_n^2 + E_1^2][r_n^2 + E_2^2]}$$
$$= \int_0^\beta \mathrm{d}\tau\, e^{-ik_n\tau}\left\{T\sum_{\{p_n\}} e^{-ip_n\tau}\frac{f(ip_n, ik_n, \mathbf{v})}{p_n^2 + E_1^2}\right\}\left\{T\sum_{r_n}\frac{e^{ir_n\tau}}{r_n^2 + E_2^2}\right\}\,, \quad (8.75)$$

where we have used the relation

$$\beta\,\delta(r_n - p_n - k_n) = \int_0^\beta \mathrm{d}\tau\, e^{i(r_n - p_n - k_n)\tau}\,. \quad (8.76)$$

This way of handling the Matsubara sums is sometimes called the “Saclay method”, cf. e.g. [9, 10]. Now we can make use of Eqs. (8.29) and (8.63) and time derivatives thereof:

$$T\sum_{r_n}\frac{e^{ir_n\tau}}{r_n^2 + E_2^2} = \frac{n_{\mathrm{B}}(E_2)}{2E_2}\Big[e^{(\beta-\tau)E_2} + e^{\tau E_2}\Big]\,, \quad (8.77)$$

$$T\sum_{\{p_n\}}\frac{e^{\pm ip_n\tau}}{p_n^2 + E_1^2} = \frac{n_{\mathrm{F}}(E_1)}{2E_1}\Big[e^{(\beta-\tau)E_1} - e^{\tau E_1}\Big]\,, \quad (8.78)$$

$$T\sum_{\{p_n\}}\frac{ip_n e^{-ip_n\tau}}{p_n^2 + E_1^2} = \frac{n_{\mathrm{F}}(E_1)}{2E_1}\Big[E_1 e^{(\beta-\tau)E_1} + E_1 e^{\tau E_1}\Big]\,. \quad (8.79)$$

Accounting for the minus sign in Eq. (8.78) within the arguments of the linear function, we then get

$$\mathcal{F} = \int_0^\beta \mathrm{d}\tau\, e^{-ik_n\tau}\,\frac{n_{\mathrm{F}}(E_1)n_{\mathrm{B}}(E_2)}{4E_1E_2}$$
$$\times\Big\{e^{(\beta-\tau)(E_1+E_2)}f(E_1, ik_n, \mathbf{v})$$
$$+e^{(\beta-\tau)E_2+\tau E_1}f(E_1, -ik_n, -\mathbf{v})$$
$$+e^{(\beta-\tau)E_1+\tau E_2}f(E_1, ik_n, \mathbf{v})$$
$$+e^{\tau(E_1+E_2)}f(E_1, -ik_n, -\mathbf{v})\quad\Big\}\,. \quad (8.80)$$

As an example, let us focus on the third structure in Eq. (8.80); the other three follow in an analogous way. The τ-integral can be carried out, noting that k_n is fermionic:

$$\begin{aligned}\int_0^\beta \mathrm{d}\tau\, e^{\beta E_1} e^{\tau(-ik_n - E_1 + E_2)} &= \frac{e^{\beta E_1}}{-ik_n - E_1 + E_2}\Big[-e^{\beta(E_2 - E_1)} - 1\Big] \\ &= \frac{e^{\beta E_2} + e^{\beta E_1}}{ik_n + E_1 - E_2} \\ &= \frac{1}{ik_n + E_1 - E_2}\Big[n_{\rm B}^{-1}(E_2) + n_{\rm F}^{-1}(E_1)\Big] . \end{aligned} \quad (8.81)$$

Thus

$$\mathcal{F}\big|_{\rm 3rd} = \frac{1}{4E_1E_2}\Big[n_{\rm F}(E_1) + n_{\rm B}(E_2)\Big]\frac{f(E_1, ik_n, \mathbf{v})}{ik_n + E_1 - E_2} . \quad (8.82)$$

Finally we set $k_n \to -i(k^0 + i0^+)$ and take the imaginary part according to Eq. (8.57). Making use of Eq. (8.20), we note that

$$\frac{1}{2i}\Big[\frac{1}{k^0 + \Delta + i0^+} - \frac{1}{k^0 + \Delta - i0^+}\Big] = -\pi\delta(k^0 + \Delta) . \quad (8.83)$$

Thereby $1/(ik_n + E_1 - E_2)$ in Eq. (8.82) gets replaced with $-\pi\,\delta(k^0 + E_1 - E_2)$. Special attention needs to be paid to the possibility that k_n could also appear in the numerator in Eq. (8.82); however, we can then write

$$ik_n = \underbrace{ik_n + E_1 - E_2}_{\text{no discontinuity}} + E_2 - E_1 , \quad (8.84)$$

so that in total

$$\begin{aligned} &\mathrm{Im}\Big\{\mathcal{F}(ik_n \to k^0 + i0^+)\Big\}\Big|_{\rm 3rd} \\ &= -\frac{\pi}{4E_1E_2}\Big[n_{\rm F}(E_1) + n_{\rm B}(E_2)\Big]\delta(k^0 + E_1 - E_2) f(E_1, \underbrace{E_2 - E_1}_{k^0}, \mathbf{v}) \\ &= -\frac{2\pi\delta(k^0 + E_1 - E_2)}{8E_1E_2} f(E_1, k^0, \mathbf{v})\, n_{\rm F}^{-1}(k^0)\, \frac{e^{\beta E_1} + e^{\beta E_2}}{[e^{\beta(E_2 - E_1)} + 1](e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\ &= -\frac{2\pi\delta(k^0 + E_1 - E_2)}{8E_1E_2} f(E_1, k^0, \mathbf{v})\, n_{\rm F}^{-1}(k^0)\, \frac{e^{\beta E_1}[1 + e^{\beta(E_2 - E_1)}]}{[e^{\beta(E_2 - E_1)} + 1](e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\ &= -\frac{2\pi\delta(k^0 + E_1 - E_2)}{8E_1E_2} f(E_1, k^0, \mathbf{v})\, n_{\rm F}^{-1}(k^0)\, n_{\rm B}(E_2)[1 - n_{\rm F}(E_1)] . \end{aligned} \quad (8.85)$$

We have chosen to factor out $n_{\rm F}^{-1}(k^0)$ because in typical applications it gets cancelled against $n_{\rm F}(k^0)$, cf. Eq. (9.137). Moreover, we remember that $E_2 = \sqrt{m_\phi^2 + (\mathbf{p}+\mathbf{k})^2}$, and can therefore use the trivial identity

$$g(\mathbf{p}+\mathbf{k}) = \int_{\mathbf{p}_2} (2\pi)^d \delta^{(d)}(\mathbf{k}+\mathbf{p}-\mathbf{p}_2)\, g(\mathbf{p}_2) \tag{8.86}$$

to write the result in a somewhat more symmetric form (see below).

Let us now return to Eq. (8.72). We had there the object $i\tilde{\not{P}}$, which plays the role of the function f, and according to Eq. (8.85) becomes

$$i\tilde{\not{P}} = ip_n\gamma_0 + ip_j\gamma_j \to E_1\gamma^0 + ip_j(-i\gamma^j) \equiv \not{\mathcal{P}} \;, \tag{8.87}$$

where we made use of the definition of the Euclidean Dirac-matrices in Eq. (4.36) (Eq. (8.85) shows that any possible $i\tilde{\not{K}}$ can also be replaced by $\not{\mathcal{K}}$). Furthermore, two factors of $-1/2$ in Eq. (8.72) and (8.85) combine into $1/4$. Renaming also $\mathcal{P} \to \mathcal{P}_1$ and inserting Eq. (8.86), the spectral function finally becomes

$$\begin{aligned}
\rho(\mathcal{K}) = \frac{n_{\rm F}^{-1}(k^0)}{4} \int_{\mathbf{p}_1,\mathbf{p}_2} \frac{\not{\mathcal{P}}_1\, a_{\rm R}}{4E_1E_2} \\
\times \Big\{ (2\pi)^D \delta^{(D)}(\mathcal{P}_1 + \mathcal{P}_2 - \mathcal{K})\, n_{\rm F1} n_{\rm B2} \\
+ (2\pi)^D \delta^{(D)}(\mathcal{P}_1 - \mathcal{P}_2 - \mathcal{K})\, n_{\rm F1}(1+n_{\rm B2}) \\
+ (2\pi)^D \delta^{(D)}(\mathcal{P}_2 - \mathcal{P}_1 - \mathcal{K})\, n_{\rm B2}(1-n_{\rm F1}) \\
+ (2\pi)^D \delta^{(D)}(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{K})\, (1-n_{\rm F1})(1+n_{\rm B2}) \Big\} \;,
\end{aligned} \tag{8.88}$$

where the results of the other channels were added; $D \equiv d + 1$; and we denoted $n_{\rm Fi} \equiv n_{\rm F}(E_i)$, $n_{\rm Bi} \equiv n_{\rm B}(E_i)$. The graphs in Eq. (8.88) illustrate the various processes that the energy-momentum constraints correspond to, with a dashed line for ϕ, a solid for L, and a dotted for N. One immediate implication of these constraints is that for a positive k^0, the last of the four structures in Eq. (8.88) does not contribute at all. In general, depending on the particle masses, some of the other channels are also kinematically forbidden.

The physics lesson to draw from Eq. (8.88) is that the spectral function, as extracted here from an analytic continuation and cut of a Euclidean correlator, represents real scatterings of on-shell particles, whose distribution functions are given by the Bose and Fermi distributions. The Bose and Fermi distributions appear in a form reminiscent of a Boltzmann equation, save for the "external" line carrying the momentum $\mathcal{K}$ which appears differently (this is discussed in more detail in

Sect. 9.3). If we went to the 2-loop level, then there would also be virtual corrections, with the closed loops experiencing thermal modifications weighted by $n_{\rm B}$ or $-n_{\rm F}$.

As a final remark we note that the spectral function ρ has the important property that, in a CP-symmetric situation, it is even in $\mathcal{K}$:

$$\rho(-\mathcal{K}) = \rho(\mathcal{K}) \; . \tag{8.89}$$

(In contrast, bosonic spectral functions are odd in $\mathcal{K}$.) Let us demonstrate this explicitly with the 2nd channel in Eq. (8.88). Its energy-dependent part satisfies

$$\begin{aligned}
& n_{\rm F}^{-1}(k^0)\delta(E_1 - E_2 - k^0) n_{\rm F1}(1 + n_{\rm B2}) \stackrel{\mathcal{K}\to-\mathcal{K}}{\longrightarrow} n_{\rm F}^{-1}(-k^0)\delta(E_1 - E_2 + k^0) n_{\rm F1}(1 + n_{\rm B2}) \\
&\quad = \delta(E_1 - E_2 + k^0) \frac{(e^{-\beta k^0} + 1) e^{\beta E_2}}{(e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\
&\quad = \delta(E_1 - E_2 + k^0)(e^{\beta k^0} + 1) \frac{e^{\beta(E_2 - k^0)}}{(e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\
&\quad = \delta(E_1 - E_2 + k^0)\, n_{\rm F}^{-1}(k^0)\, \frac{e^{\beta E_1}}{(e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\
&\quad = n_{\rm F}^{-1}(k^0)\, \delta(E_2 - E_1 - k^0)\, n_{\rm B2}(1 - n_{\rm F1}) \; ,
\end{aligned} \tag{8.90}$$

which is exactly the structure of the 3rd channel. The spatial change $\mathbf{k} \to -\mathbf{k}$ only has an effect on the three-dimensional δ-function, turning it into that on the 3rd row of Eq. (8.88). Similarly, it can be checked that the 4th term goes over into the 1st term, and vice versa.

There are a number of general remarks to make about the determination of spectral functions of the type that we have considered here; these have been deferred to the end of Appendix A.

Appendix A: What If the Internal Lines Are Treated Non-Perturbatively?

Above we made use of tree-level propagators, but in general the propagators need to be resummed (cf. Sect. 8.4), and have a more complicated appearance. It is then useful to express them as in the spectral representation of Eq. (8.24). In particular, the scalar propagator can be written as

$$\langle \tilde\phi(K)\tilde\phi(Q)\rangle_0 = \frac{\delta(K+Q)}{k_n^2 + \mathbf{k}^2 + \Pi_{\rm S}(k_n, \mathbf{k})} = \delta(K+Q) \int_{-\infty}^{\infty} \frac{{\rm d}k^0}{\pi} \frac{\rho_{\rm S}(k^0, \mathbf{k})}{k^0 - ik_n} \; , \tag{8.91}$$

whereas the fermion propagator contains two possible structures in the chirally symmetric case of a vanishing mass (more general cases have been considered in [11]):

$$\begin{aligned}\langle\tilde{\psi}(K)\tilde{\bar{\psi}}(Q)\rangle_0 =&\delta(K-Q)\biggl[\frac{-ik_n\gamma_0}{k_n^2+\mathbf{k}^2+\Pi_{\rm W}(k_n,\mathbf{k})}+\frac{-ik_j\gamma_j}{k_n^2+\mathbf{k}^2+\Pi_{\rm P}(k_n,\mathbf{k})}\biggr]\\ =&\delta(K-Q)\biggl[ik_n\gamma_0\int_{-\infty}^{\infty}\frac{{\rm d}k^0}{\pi}\frac{\rho_{\rm W}(k^0,\mathbf{k})}{k^0-ik_n}+ik_j\gamma_j\int_{-\infty}^{\infty}\frac{{\rm d}k^0}{\pi}\frac{\rho_{\rm P}(k^0,\mathbf{k})}{k^0-ik_n}\biggr]. \quad (8.92)\end{aligned}$$

Here minus signs have been incorporated into the definitions of the spectral functions $\rho_{\rm W}$ and $\rho_{\rm P}$ for later convenience. Let us carry out the steps from Eq. (8.72) to (8.88) in this situation.

The structure in Eq. (8.74) now has the form

$$\mathcal{F}=T\sum_{\{p_n\}}\sum_{\rm F=W,P}\int_{-\infty}^{\infty}\frac{{\rm d}\omega_1}{\pi}\int_{-\infty}^{\infty}\frac{{\rm d}\omega_2}{\pi}\frac{f_{\rm F}(ip_n,ik_n,\mathbf{v})\,\rho_{\rm F}(\omega_1,\mathbf{p})\,\rho_{\rm S}(\omega_2,\mathbf{p}+\mathbf{k})}{[\omega_1-ip_n][\omega_2-i(p_n+k_n)]}\,, \quad (8.93)$$

where the book-keeping function $f_{\rm F}$ is again assumed to depend linearly on its arguments. We can write

$$\begin{aligned}\mathcal{F}=&\sum_{\rm F=W,P}\int_{-\infty}^{\infty}\frac{{\rm d}\omega_1\,{\rm d}\omega_2}{\pi^2}\,\rho_{\rm F}(\omega_1,\mathbf{p})\rho_{\rm S}(\omega_2,\mathbf{p}+\mathbf{k})\\ &\times T\sum_{\{p_n\}}T\sum_{r_n}\beta\,\delta(r_n-p_n-k_n)\frac{f_{\rm F}(ip_n,ik_n,\mathbf{v})}{[\omega_1-ip_n][\omega_2-ir_n]}\,. \quad (8.94)\end{aligned}$$

Employing Eqs. (8.76), (8.30) and (8.62), as well as the time derivative of the last one,[2]

$$T\sum_{\{p_n\}}\frac{ip_n}{\omega_1-ip_n}e^{-ip_n\tau}=-\frac{\rm d}{{\rm d}\tau}\Bigl[n_{\rm F}(\omega_1)e^{(\beta-\tau)\omega_1}\Bigr]=n_{\rm F}(\omega_1)\,\omega_1\,e^{(\beta-\tau)\omega_1}\,,\quad 0<\tau<\beta\,, \quad (8.95)$$

we get

$$\begin{aligned}\mathcal{F}=&\sum_{\rm F=W,P}\int_{-\infty}^{\infty}\frac{{\rm d}\omega_1\,{\rm d}\omega_2}{\pi^2}\,\rho_{\rm F}(\omega_1,\mathbf{p})\rho_{\rm S}(\omega_2,\mathbf{p}+\mathbf{k})\\ &\times\int_0^\beta{\rm d}\tau\,e^{-ik_n\tau}\,n_{\rm F}(\omega_1)\,n_{\rm B}(\omega_2)f_{\rm F}(\omega_1,ik_n,\mathbf{v})\,e^{(\beta-\tau)\omega_1+\tau\omega_2}\,. \quad (8.96)\end{aligned}$$

[2] We are somewhat sloppy here: a part of the sums leads to Dirac-δ's [cf. Eq. (8.59)], which can give a contribution to $\mathcal{F}$. That term is, however, independent of k_n and thus drops out when taking the discontinuity.

The τ-integral can now be carried out, noting that k_n is fermionic:

$$\begin{aligned}\int_0^\beta \mathrm{d}\tau\, n_{\mathrm{F}}(\omega_1)\, n_{\mathrm{B}}(\omega_2)\, e^{\beta\omega_1} e^{\tau(-ik_n-\omega_1+\omega_2)} &= \frac{n_{\mathrm{F}}(\omega_1)\, n_{\mathrm{B}}(\omega_2) e^{\beta\omega_1}}{-ik_n-\omega_1+\omega_2}\Big[-e^{\beta(\omega_2-\omega_1)}-1\Big] \\ &= \frac{n_{\mathrm{F}}(\omega_1)\, n_{\mathrm{B}}(\omega_2)}{ik_n+\omega_1-\omega_2}\Big[e^{\beta\omega_2}+e^{\beta\omega_1}\Big] \\ &= \frac{n_{\mathrm{F}}(\omega_1)\, n_{\mathrm{B}}(\omega_2)}{ik_n+\omega_1-\omega_2}\Big[n_{\mathrm{F}}^{-1}(\omega_1)+n_{\mathrm{B}}^{-1}(\omega_2)\Big] \\ &= \frac{1}{ik_n+\omega_1-\omega_2}\Big[n_{\mathrm{F}}(\omega_1)+n_{\mathrm{B}}(\omega_2)\Big]\,. \end{aligned} \tag{8.97}$$

Finally we set $k_n \to -i(k^0+i0^+)$ and take the discontinuity. The appearance of k_n inside f_{F} can be handled like in Eq. (8.84). Making use of Eq. (8.83), the denominator in Eq. (8.97) simply gets replaced with $(-\pi)$ times a Dirac δ-function, so that in total

$$\begin{aligned}&\mathrm{Im}\Big\{\mathcal{F}(ik_n\to k^0+i0^+)\Big\} = -\pi\sum_{\mathrm{F=W,P}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega_1\,\mathrm{d}\omega_2}{\pi^2}\,\rho_{\mathrm{F}}(\omega_1,\mathbf{p})\rho_{\mathrm{S}}(\omega_2,\mathbf{p}+\mathbf{k}) \\ &\times\Big[n_{\mathrm{F}}(\omega_1)+n_{\mathrm{B}}(\omega_2)\Big]\delta(k^0+\omega_1-\omega_2)f_{\mathrm{F}}(\omega_1,\omega_2-\omega_1,\mathbf{v}) \\ &\quad = -\frac{1}{2}\sum_{\mathrm{F=W,P}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega_1\,\mathrm{d}\omega_2}{\pi^2}\,\rho_{\mathrm{F}}(\omega_1,\mathbf{p})\rho_{\mathrm{S}}(\omega_2,\mathbf{p}+\mathbf{k}) \\ &\qquad\times 2\pi\,\delta(k^0+\omega_1-\omega_2)f_{\mathrm{F}}(\omega_1,k^0,\mathbf{v})\,n_{\mathrm{F}}^{-1}(k^0)\,n_{\mathrm{B}}(\omega_2)[1-n_{\mathrm{F}}(\omega_1)]\,,\end{aligned} \tag{8.98}$$

where we parallelled the steps in Eq. (8.85). Finally, making use of Eq. (8.86) and defining $\mathcal{P}_1 \equiv (\omega_1,\mathbf{p}) \equiv (\omega_1,\mathbf{p}_1)$, $\mathcal{P}_2 \equiv (\omega_2,\mathbf{p}_2)$, the spectral function corresponding to Eq. (8.88) becomes

$$\begin{aligned}\rho(\mathcal{K}) = -n_{\mathrm{F}}^{-1}(k^0)\int_{\mathcal{P}_1}\int_{\mathcal{P}_2}\Big[\omega_1\gamma^0\,\rho_{\mathrm{W}}(\mathcal{P}_1)+\not{\mathbf{p}}_1\,\rho_{\mathrm{P}}(\mathcal{P}_1)\Big]\,a_{\mathrm{R}}\,\rho_{\mathrm{S}}(\mathcal{P}_2) \\ \times\Big\{(2\pi)^D\delta^{(D)}(\mathcal{P}_2-\mathcal{P}_1-\mathcal{K})\,n_{\mathrm{B2}}(1-n_{\mathrm{F1}})\end{aligned}$$


$$\Big\}\,, \tag{8.99}$$

where $\not{\mathbf{p}}_1 \equiv p_{1j}\gamma^j$, $n_{\mathrm{F}i} \equiv n_{\mathrm{F}}(\omega_i)$ and $n_{\mathrm{B}i} \equiv n_{\mathrm{B}}(\omega_i)$. If we insert here the free spectral shape from Eq. (8.35), recalling the extra minus sign that was incorporated into ρ_{W} and ρ_{P} in Eq. (8.92), then it can be shown that this result goes over into Eq. (8.88), with the four channels originating from the on-shell points $\omega_i = \pm E_i$, $i = 1, 2$.

A few concluding remarks are in order:

- Expressions such as Eq. (8.99) are useful particularly if the scalar and fermion propagators are Hard Thermal Loop (HTL) resummed, cf. Sect. 8.4. In that case ρ_{W} and ρ_{P} are given by Eqs. (8.201) and (8.202), respectively.

- HTL resummed spectral functions contain in general two types of contributions. First of all, there are "pole contributions", represented by Dirac δ-functions. In these contributions the pole locations are shifted from the free vacuum spectral functions by thermal mass corrections. Consequently, kinematic channels which would be forbidden in vacuum (such as a $1 \to 2$ decay between three massless particles) may open up.
- The second type of HTL corrections originates from a "cut contribution". An HTL resummed fermion or gauge field spectral function $\rho(\omega, k)$ has a non-zero continuous part in the spacelike domain $k > |\omega|$. Physically, this originates from real $2 \leftrightarrow 1$ scatterings experienced by such off-shell fields. Inserted into Eq. (8.99) this turns the full process into a real $2 \to 2$ scattering, which tends to play an important role for the physics of nearly massless particles, because $2 \to 2$ processes are not kinematically suppressed even in the massless limit.

A classic example of an HTL computation in which both "pole" and "cut" contributions play a role can be found in [12]. Further processes, contributing at the same order even though not accounted for just by using HTL spectral functions, have been discussed in [13]. A complete leading-order computation of the observable considered in the present section, related to right-handed fermions interacting with the Standard Model particles through Yukawa interactions, is presented in [14, 15], and a similar analysis for the production rate of photons from a QCD plasma can be found in [16, 17]. We return to some of these issues in Sect. 9.3.

8.3 Real-Time Formalism

In the previous section, we considered a particular spectral function, obtained from the Euclidean correlator in Eq. (8.71) through the basic relation in Eq. (8.57). The question may be posed, however, whether it really is necessary to go through Euclidean considerations at all. It turns out that, within perturbation theory, the answer is negative: in the so-called real-time formalism, real-time observables can be directly expressed as Feynman diagrams containing real-time propagators. The price to pay for this simplification is that the field content of the theory gets effectively "doubled" and, in a general situation, every propagator turns into a 2×2 matrix, and every vertex splits into multiple vertices.

A full-fledged formulation of the real-time formalism proceeds through the Schwinger-Keldysh or closed time-path framework; reviews can be found in [18, 19]. A frequently appearing concept is that of Kadanoff-Baym equations, which are analogues of Schwinger-Dyson equations within this formalism. In the following, we only provide a short motivation for the field doubling, and then demonstrate how the result of Eq. (8.88) can be obtained directly within the real-time formalism.

Basic Definitions

One advantage of the real-time formalism is that it also applies to systems out of equilibrium. In quantum statistical mechanics a general out-of-equilibrium situation is described by a *density matrix*, denoted by $\hat{\rho}(t)$. The density matrix is assumed normalized such that $\mathrm{Tr}\,(\hat{\rho}) = 1$, and statistical expectation values are defined as

$$\left\langle \hat{O}(t_1, \mathbf{x}_1)\, \hat{O}(t_2, \mathbf{x}_2)\, \ldots \right\rangle \equiv \mathrm{Tr}\left[\hat{\rho}(t)\, \hat{O}(t_1, \mathbf{x}_1)\, \hat{O}(t_2, \mathbf{x}_2)\, \ldots \right], \tag{8.100}$$

where $\hat{O}$ is a Heisenberg operator defined like in Eq. (8.1). The same 2-point functions as in Sect. 8.1 can be considered in this general ensemble, and some of the operator relations also continue to hold, such as $\Pi^T = -i\Pi^R + \Pi^< = -i\Pi^A + \Pi^>$.

An important difference between the out-of-equilibrium and equilibrium cases is that in the former situation the considerations leading to the KMS relation, cf. Eqs. (8.11) and (8.12) for the bosonic case, no longer go through. However, we can still work out the trace in Eq. (8.100) in a given basis and learn something from the outcome.

Consider the same Wightman function $\Pi^>$ as in Eq. (8.11). With a view of obtaining a perturbative expansion, we now choose as the basis not energy eigenstates, but rather eigenstates of elementary field operators; for the moment we denote these by $|\alpha_i\rangle$. Simplifying also the operator notation somewhat from that in Sect. 8.1, we can write

$$\begin{aligned}\Pi^>(t) &\equiv \mathrm{Tr}\left[\hat{\rho}(t)\, e^{i\hat{H}t}\, \hat{O}(0)\, e^{-i\hat{H}t}\, \hat{O}(0)\right] \\ &= \int \Pi_{i=1}^{5} \mathrm{d}\alpha_i\, \langle \alpha_1|\hat{\rho}(t)|\alpha_2\rangle\, \langle \alpha_2|e^{i\hat{H}t}|\alpha_3\rangle\, \langle \alpha_3|\hat{O}(0)|\alpha_4\rangle\, \langle \alpha_4|e^{-i\hat{H}t}|\alpha_5\rangle\, \langle \alpha_5|\hat{O}(0)|\alpha_1\rangle\, .\end{aligned} \tag{8.101}$$

If the operators $\hat{O}$ contain only the field operators $\hat{\alpha}$ and no conjugate momenta, then we can directly write $\langle \alpha_i|\hat{O}[\hat{\alpha}]|\alpha_j\rangle = O[\alpha_j]\delta_{\alpha_i,\alpha_j}$. For the time evolution, we insert the usual Feynman path integral,

$$\langle \alpha_4|e^{-i\hat{H}t}|\alpha_5\rangle = \int_{\alpha(0)=\alpha_5}^{\alpha(t)=\alpha_4} \mathcal{D}\alpha\, e^{i\mathcal{S}_M}\,, \tag{8.102}$$

while the "backward" time evolution $\langle \alpha_2|e^{i\hat{H}t}|\alpha_3\rangle$ is obtained from the Hermitian (complex) conjugate of this relation. Denoting the "forward-propagating" field interpolating between α_5 and α_4 now by ϕ_1, and that interpolating between α_3 and α_2 by ϕ_2, we thereby get

$$\Pi^>(t) = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2\, O[\phi_2(t)]\, O[\phi_1(0)] e^{i\mathcal{S}_M[\phi_1] - i\mathcal{S}_M[\phi_2]}\, \langle \phi_1(0)|\hat{\rho}(t)|\phi_2(0)\rangle\, . \tag{8.103}$$

Note that $\phi_1(t) = \phi_2(t) = \alpha_3 = \alpha_4$ in this example because t is the largest time value appearing; however $\phi_2(0) \neq \phi_1(0)$ and both are integrated over. It is helpful to use $\phi_2(t)$ rather than $\phi_1(t)$ inside $O[\phi_2(t)]$ in Eq. (8.103), because this makes it explicit that $O[\phi_2(t)]$ stands to the left of the operator $O[\phi_1(0)]$, as is indeed implied by the definition of the Wightman function $\Pi^>(t)$. One should think of the field ϕ_1 as corresponding to the operators positioned on the right and with time arguments increasing to the left, followed by ϕ_2 for the operators positioned on the left.

A similar computation for the other Wightman function yields

$$\Pi^<(t) = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \, O[\phi_2(0)] \, O[\phi_1(t)] e^{i\mathcal{S}_M[\phi_1] - i\mathcal{S}_M[\phi_2]} \, \langle \phi_1(0) | \hat{\rho}(t) | \phi_2(0) \rangle \; . \tag{8.104}$$

This time we have indicated the field with the largest time argument by $\phi_1(t)$ rather than $\phi_2(t)$, because the corresponding operator stands to the utmost right, i.e. closest to the origin of time flow. Note that within Eq. (8.104), $O[\phi_2(0)]$ and $O[\phi_1(t)]$ are just complex numbers and ordering plays no role (in the bosonic case), so we could also write $\Pi^<(t) = \langle O[\phi_1(t)] \, O[\phi_2(0)] \rangle$. Here $\langle \ldots \rangle$ refers to an expectation value in the sense of the Schwinger-Keldysh functional integral,

$$\langle \ldots \rangle \; \equiv \; \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \, (\ldots) \, e^{i\mathcal{S}_M[\phi_1] - i\mathcal{S}_M[\phi_2]} \, \langle \phi_1(0) | \hat{\rho}(t) | \phi_2(0) \rangle \; . \tag{8.105}$$

If $\hat{\rho}$ happens to be a time-independent thermal density matrix, $\hat{\rho} = e^{-\beta \hat{H}} / \mathcal{Z}$, then the remaining expectation value $\langle \phi_1(0) | \hat{\rho}(t) | \phi_2(0) \rangle$ can be represented as an imaginary-time path integral as was discussed for a scalar field in Sect. 2.1. For many formal considerations it is however not necessary to write down this part explicitly.

The lesson to be drawn from Eqs. (8.103) and (8.104) is that the two Wightman functions $\Pi^>$ and $\Pi^<$ are independent objects if $\hat{\rho}$ is non-thermal, and that representing them as path integrals necessitates a doubling of the field content of the theory ($\phi \to \{\phi_1, \phi_2\}$).

If we specialize to the case in which the operators in Eqs. (8.103) and (8.104) are directly elementary fields, rather than composite operators, then it is conventional to assemble these propagators into a 2×2 matrix. If we add a time-ordered structure,

$$\begin{aligned} & \theta(t_2 - t_1) \, \hat{\phi}(t_2) \, \hat{\phi}(t_1) + \theta(t_1 - t_2) \, \hat{\phi}(t_1) \, \hat{\phi}(t_2) \\ & \quad = \theta(t_2 - t_1) \, e^{i\hat{H}t_2} \, \hat{\phi}(0) \, e^{-i\hat{H}(t_2 - t_1)} \, \hat{\phi}(0) \, e^{-i\hat{H}t_1} \\ & \qquad + \theta(t_1 - t_2) \, e^{i\hat{H}t_1} \, \hat{\phi}(0) \, e^{-i\hat{H}(t_1 - t_2)} \, \hat{\phi}(0) \, e^{-i\hat{H}t_2} \; , \end{aligned} \tag{8.106}$$

then we need to represent the time evolution along the forward-propagating branch, denoted above by the field ϕ_1. Similarly, an anti-time-ordered propagator can be represented in terms of the ϕ_2-field. The general propagator is then

$$\begin{pmatrix} \langle\phi_1(t)\phi_1(0)\rangle & \langle\phi_1(t)\phi_2(0)\rangle \\ \langle\phi_2(t)\phi_1(0)\rangle & \langle\phi_2(t)\phi_2(0)\rangle \end{pmatrix} = \begin{pmatrix} \Pi^T_\phi(t) & \Pi^<_\phi(t) \\ \Pi^>_\phi(t) & \Pi^{\bar{T}}_\phi(t) \end{pmatrix}, \tag{8.107}$$

where $\bar{T}$ denotes anti-time-ordering. The action $\mathcal{S}_M[\phi_1] - \mathcal{S}_M[\phi_2]$ contains vertices for both types of fields, and the non-diagonal matrix structure of Eq. (8.107) implies that when interactions are included, both types of vertices indeed contribute to a given observable.

In the literature, the field basis introduced above is referred to as the 1/2-basis. There is another possible choice, referred to as the r/a-basis, which is beneficial for some practical computations. It is obtained by the linear transformation

$$\phi_r \equiv \frac{1}{2}(\phi_1 + \phi_2) \;, \quad \phi_a \equiv \phi_1 - \phi_2 \;. \tag{8.108}$$

Consequently, inserting the 1/2 propagators from Eq. (8.107), we get

$$\langle\phi_r(t)\phi_r(0)\rangle = \frac{1}{4}\Big(\Pi^T_\phi + \Pi^{\bar{T}}_\phi + \Pi^>_\phi + \Pi^<_\phi\Big) = \frac{1}{2}\big(\Pi^>_\phi + \Pi^<_\phi\big) = \Delta_\phi(t) \;, \tag{8.109}$$

$$\langle\phi_r(t)\phi_a(0)\rangle = \frac{1}{2}\Big(\Pi^T_\phi - \Pi^{\bar{T}}_\phi + \Pi^>_\phi - \Pi^<_\phi\Big) = \theta(t)\big(\Pi^>_\phi - \Pi^<_\phi\big) = -i\Pi^R_\phi(t) \;, \tag{8.110}$$

and similarly $\langle\phi_a(t)\phi_r(0)\rangle = -i\Pi^A_\phi(t)$ and $\langle\phi_a(t)\phi_a(0)\rangle = 0$.

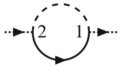
Among the advantages of the r/a-basis are that the aa propagator element vanishes, and that closed loops containing only the advanced $\langle\phi_a(t)\phi_r(0)\rangle$ or the retarded $\langle\phi_r(t)\phi_a(0)\rangle$ also vanish. In addition, the statistical function Δ_ϕ, containing the Bose distribution in the bosonic case [cf. Eq. (8.16)], is the only element surviving in the classical limit (because it is not proportional to a commutator), and may thus dominate the dynamics if we consider a soft regime $E \ll T$ such as in the situation described in Sect. 6.1 (cf. [20] for a detailed discussion).

Let us conclude by remarking that at higher orders of perturbation theory, the real-time formalism quickly becomes technically rather complicated, and for a long time only leading-order results existed. The past few years have, however, witnessed significant progress in the field, which is related in particular to the handling of soft contributions in the computations, as alluded to above. Examples of next-to-leading order computations can be found in [21, 22].

Practical Illustration

In order to illustrate how the real-time formalism works in practice, let us return to the 1-loop spectral function of the operator coupling to a right-handed fermion in the Standard Model, discussed in Sect. 8.2. According to Eq. (8.13), it suffices to consider the two Wightman functions, which by Eqs. (8.103) and (8.104) are related to 21 and 12-type Green's functions in the 1/2-basis. Starting with the latter, we are led to inspect the 1-loop graph

$$\Pi^{<}(\mathcal{K}) = \quad , \tag{8.111}$$

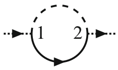


where the notation for the propagators follows Sect. 8.2. The numbers 1 and 2 indicate that the two vertices are of the 1 and 2-type, respectively—implying most importantly that both of the internal propagators in the graph must have the same ordering. In momentum space, we can then immediately write down a result for the graph,

$$\Pi^{<}(\mathcal{K}) = \frac{1}{2}\int_{\mathcal{P}} \Pi^{<}_{\ell}(\mathcal{P})\Pi^{<}_{\phi}(\mathcal{K}-\mathcal{P}) \; , \tag{8.112}$$

where, in analogy with Eq. (8.72), we have inserted an overall factor from the normalization of the fields and suppressed any coupling constants and sums over field indices. Similarly, for $\Pi^{>}$ we obtain

$$\Pi^{>}(\mathcal{K}) = \tag{8.113}$$



$$= \frac{1}{2}\int_{\mathcal{P}} \Pi^{>}_{\ell}(\mathcal{P})\Pi^{>}_{\phi}(\mathcal{K}-\mathcal{P}) \; . \tag{8.114}$$

The Wightman functions appearing here can be related to the corresponding spectral functions via [cf. Eqs. (8.14), (8.15) and (8.53)]

$$\Pi^{<}_{\ell} = -2n_{\rm F}\rho_{\ell} \; , \quad \Pi^{>}_{\ell} = 2(1-n_{\rm F})\rho_{\ell} \; , \quad \Pi^{<}_{\phi} = 2n_{\rm B}\rho_{\phi} \; , \quad \Pi^{>}_{\phi} = 2(1+n_{\rm B})\rho_{\phi} \; . \tag{8.115}$$

Thereby the full spectral function under consideration obtains the form

$$\begin{aligned} \rho(\mathcal{K}) &= \frac{1}{2}\Big[\Pi^{>}(\mathcal{K}) - \Pi^{<}(\mathcal{K})\Big] \\ &= \int_{\mathcal{P}_1,\mathcal{P}_2} (2\pi)^D \delta^{(D)}(\mathcal{P}_1+\mathcal{P}_2-\mathcal{K})\Big[1-n_{\rm F}(\omega_1)+n_{\rm B}(\omega_2)\Big]\,\rho_{\ell}(\omega_1,\mathbf{p}_1)\,\rho_{\phi}(\omega_2,\mathbf{p}_2) \; , \end{aligned} \tag{8.116}$$

where we have introduced a second momentum integration variable by inserting the relation

$$1 = \int_{\mathcal{P}_2} (2\pi)^D \delta^{(D)}(\mathcal{P}_1+\mathcal{P}_2-\mathcal{K}) \tag{8.117}$$

into the integral. We also denoted $\mathcal{P}_i \equiv (\omega_i, \mathbf{p}_i)$ here.

In order to make Eq. (8.116) more explicit, we insert the free spectral functions [cf. Eqs. (8.35) and (8.67)],

$$\rho_\phi(\omega_2,\mathbf{p}_2) \equiv \frac{\pi}{2E_2}\Big[\delta(\omega_2-E_2)-\delta(\omega_2+E_2)\Big]\,, \tag{8.118}$$

$$\rho_\ell(\omega_1,\mathbf{p}_1) \equiv \frac{\pi}{2E_1}\,a_{\rm L}\,\mathcal{P}\!\!\!/_1\,a_{\rm R}\Big[\delta(\omega_1-E_1)-\delta(\omega_1+E_1)\Big]\,, \tag{8.119}$$

where E_1 and E_2 are defined in accordance with Eq. (8.73) (but with spatial momenta adjusted as appropriate). Further re-organizing the phase space distributions in analogy with Eq. (8.85),

$$\delta(\omega_1+\omega_2-k^0)\big[1-n_{\rm F}(\omega_1)+n_{\rm B}(\omega_2)\big] \;=\; \delta(\omega_1+\omega_2-k^0)\,n_{\rm F}^{-1}(k^0)\,n_{\rm F}(\omega_1)\,n_{\rm B}(\omega_2)\,, \tag{8.120}$$

we arrive at the result

$$\begin{aligned}\rho(\mathcal{K}) = n_{\rm F}^{-1}(k^0)\int_{-\infty}^{\infty}\frac{{\rm d}\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{{\rm d}\omega_2}{2\pi}\int_{\mathbf{p_1},\mathbf{p_2}}(2\pi)^D\delta^{(D)}(\mathcal{P}_1+\mathcal{P}_2-\mathcal{K})\,n_{\rm F}(\omega_1)\,n_{\rm B}(\omega_2)\\ \times\frac{\pi^2}{4E_1E_2}\mathcal{P}\!\!\!/_1\,a_{\rm R}\Big[\delta(\omega_1-E_1)-\delta(\omega_1+E_1)\Big]\Big[\delta(\omega_2-E_2)-\delta(\omega_2+E_2)\Big]\,.\end{aligned} \tag{8.121}$$

If we now integrate over ω_1 and ω_2, re-adjust the notation so that $\mathcal{P}_i \equiv (E_i,\mathbf{p}_i)$, and in addition make the substitution $\mathbf{p}_i \to -\mathbf{p}_i$ where necessary, we obtain

$$\begin{aligned}\rho(\mathcal{K}) = \frac{n_{\rm F}^{-1}(k^0)}{4}\int_{\mathbf{p_1},\mathbf{p_2}}\frac{\mathcal{P}\!\!\!/_1\,a_{\rm R}}{4E_1E_2}\\ \times\Big\{(2\pi)^D\delta^{(D)}(\mathcal{P}_1+\mathcal{P}_2-\mathcal{K})\,n_{\rm F}(E_1)n_{\rm B}(E_2)\\ -(2\pi)^D\delta^{(D)}(\mathcal{P}_1-\mathcal{P}_2-\mathcal{K})\,n_{\rm F}(E_1)n_{\rm B}(-E_2)\\ +(2\pi)^D\delta^{(D)}(\mathcal{P}_1-\mathcal{P}_2+\mathcal{K})\,n_{\rm F}(-E_1)n_{\rm B}(E_2)\\ -(2\pi)^D\delta^{(D)}(\mathcal{P}_1+\mathcal{P}_2+\mathcal{K})\,n_{\rm F}(-E_1)n_{\rm B}(-E_2)\Big\}\,.\end{aligned} \tag{8.122}$$

This becomes identical with Eq. (8.88) upon using the relations

$$n_{\rm F}(-E_1) = 1-n_{\rm F}(E_1)\,,\quad n_{\rm B}(-E_2) = -1-n_{\rm B}(E_2)\,. \tag{8.123}$$

The above example confirms our expectation that with sufficient care Minkowskian (real-time) quantities may indeed be determined through the real-time formalism. The imaginary-time formalism is, however, equally valid for problems in thermal equilibrium, and applicable on the non-perturbative level as well. Within perturbation theory, the main difference between the two formalisms is that in the imaginary-time case Matsubara sums need to be carried out before taking the discontinuity, but there is only one expression under evaluation, whereas in the real-time case only integrations appear like in vacuum computations, with the price that there are more diagrams.

8.4 Hard Thermal Loops

For "static" observables, we realized in Sect. 3.2 that the perturbative series suffers from infrared divergences. However, as discussed in Sect. 6.1, in weakly-coupled theories these divergences can only be associated with bosonic Matsubara zero modes. They can therefore be isolated by constructing an effective field theory for the bosonic Matsubara zero modes, as we did in Sect. 6.2.

The situation is more complicated in the case of real-time observables discussed in the present chapter. Indeed, as Eq. (8.26) shows, the dependence on *all* Matsubara modes is needed in order to carry out the analytic continuation leading to the spectral function, even if we were only interested in its behaviour at small frequencies $|k^0| \ll \pi T$. (The same holds also in the opposite direction: as the sum rule in Eq. (8.25) shows, the information contained in the Matsubara zero mode is spread out to *all* k^0's in the Minkowskian formulation.) Therefore, it is non-trivial to isolate the soft/light degrees of freedom for which to write down the most general effective Lagrangian.[3]

Nevertheless, it turns out that the dimensionally reduced effective field theory of Sect. 6.2 *can* to some extent be generalized to real-time observables as well. In the case of QCD, the generalization is known as the *Hard Thermal Loop* effective theory. The effective theory dictates what kind of resummed propagators should be used for instance in the computation of Sect. 8.2 in order to alleviate infrared problems appearing in perturbative computations. An example of a computation showing that (logarithmic) infrared divergences get cancelled this way can be found in [23].

More precisely, Hard Thermal Loops (HTL) can operationally be defined via the following steps that refer to the computation of 2 or higher-point functions [24–27]:

- Consider "soft" external frequencies and momenta: $|k^0|, |\mathbf{k}| \sim gT$.
- Inside the loops, sum over all Matsubara frequencies p_n.
- Subsequently, integrate over "hard" spatial loop momenta, $|\mathbf{p}| \gtrsim \pi T$, Taylor-expanding the result to leading non-trivial order in $|k^0|/|\mathbf{p}|, |\mathbf{k}|/|\mathbf{p}|$.

[3] This continues to be so in the real-time formalism, introduced in Sect. 8.3; for a discussion see [20].

The soft momenta $|k^0|, |\mathbf{k}|$ are the analogues of the small mass m considered in Sect. 6.1, and the scale $\sim \pi T$ plays the role of the heavy mass M. According to Eq. (6.25), the parametric error made through a given truncation might be expected to be $\sim (g/\pi)^k$ with some $k > 0$, however as will be discussed below this is unfortunately not guaranteed to be the case in general.

In order to illustrate the procedure, let us compute the gluon self-energy in this situation. The computation is much like that in Sect. 5.3, except that now we keep the external momentum (K) non-zero while carrying out the Matsubara sum, because the full dependence on k_n is needed for the analytic continuation. It is crucial to take $k^0, \mathbf{k}$ soft only *after the analytic continuation*.

As a starting point, we take the gluon self-energy in Feynman gauge, $\Pi_{\mu\nu}(K)$, as defined in Eq. (5.64). This will be interpreted as being a part of an "effective action",

$$S_{\text{eff}} = \sum\!\!\!\!\!\!\!\int_K \frac{1}{2}\tilde{A}^a_\mu(K)\Big[K^2\delta_{\mu\nu} - K_\mu K_\nu + \frac{1}{\xi}K_\mu K_\nu + \Pi_{\mu\nu}(K)\Big]\tilde{A}^a_\nu(-K) + \ldots \,. \tag{8.124}$$

Summing together results from Eqs. (5.69), (5.74), (5.77), (5.89) and (5.96), setting the fermion mass to zero for simplicity, and expressing the spacetime dimensionality as $D \equiv d + 1$, the 1-loop self-energy reads

$$\begin{aligned}\Pi_{\mu\nu}(K) = {} & \frac{g^2 N_c}{2} \sum\!\!\!\!\!\!\!\int_P \frac{\delta_{\mu\nu}\big[-4K^2 + 2(D-2)P^2\big] + (D+2)K_\mu K_\nu - 4(D-2)P_\mu P_\nu}{P^2(K-P)^2} \\ & - g^2 N_f \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{\delta_{\mu\nu}\big[-K^2 + 2P^2\big] + 2K_\mu K_\nu - 4P_\mu P_\nu}{P^2(K-P)^2} \,. \end{aligned} \tag{8.125}$$

The bosonic part is discussed in Appendix A; here we focus on the fermionic part.

Consider first the spatial components, Π_{ij}. Shifting $P \to K - P$ in one term, we can write

$$\Pi^{(f)}_{ij}(K) = -g^2 N_f \int_{\mathbf{p}} T \sum_{\{p_n\}} \Big[\frac{2\delta_{ij}}{P^2} + \frac{-K^2\delta_{ij} + 2k_i k_j - 4p_i p_j}{P^2(K-P)^2}\Big] \,. \tag{8.126}$$

For generality we assume that, like in Eq. (8.64), the Matsubara frequency is of the form

$$p_n \to \tilde{p}_n \equiv \omega_n + i\mu \,, \quad \omega_n = 2\pi T\Big(n + \frac{1}{2}\Big) \,. \tag{8.127}$$

The Matsubara sum can now be carried out, in analogy with the procedure described in Sect. 8.2. Denoting

$$E_1 \equiv |\mathbf{p}| \,, \quad E_2 \equiv |\mathbf{p} - \mathbf{k}| \,, \tag{8.128}$$

we can read from Eq. (8.63) that

$$T\sum_{\{\omega_n\}}\frac{1}{(\omega_n+i\mu)^2+E_1^2} = \frac{1}{2E_1}\Big[n_{\rm F}(E_1-\mu)e^{\beta(E_1-\mu)} - n_{\rm F}(E_1+\mu)\Big]$$
$$= \frac{1}{2E_1}\Big[1-n_{\rm F}(E_1-\mu)-n_{\rm F}(E_1+\mu)\Big]\,. \quad (8.129)$$

It is somewhat more tedious to carry out the other sum. Proceeding in analogy with the analysis following Eq. (8.74) and denoting the result by $\mathcal{G}$, we get

$$\mathcal{G} = T\sum_{\{p_n\}}\frac{1}{[\tilde{p}_n^2+E_1^2][(k_n-\tilde{p}_n)^2+E_2^2]} \quad (8.130)$$
$$= T\sum_{\{p_n\}}T\sum_{\{r_n\}}\beta\,\delta(\tilde{r}_n+k_n-\tilde{p}_n)\frac{1}{[\tilde{p}_n^2+E_1^2][\tilde{r}_n^2+E_2^2]}$$
$$= \int_0^\beta \mathrm{d}\tau\, e^{ik_n\tau}\left\{T\sum_{\{p_n\}}\frac{e^{-i\tilde{p}_n\tau}}{\tilde{p}_n^2+E_1^2}\right\}\left\{T\sum_{\{r_n\}}\frac{e^{i\tilde{r}_n\tau}}{\tilde{r}_n^2+E_2^2}\right\}\,, \quad (8.131)$$

where we used the trick in Eq. (8.76). The sums can be carried out by making use of Eq. (8.63),

$$T\sum_{\{r_n\}}\frac{e^{i\tilde{r}_n\tau}}{\tilde{r}_n^2+E_2^2} = \frac{1}{2E_2}\Big[n_{\rm F}(E_2-\mu)e^{(\beta-\tau)E_2-\beta\mu} - n_{\rm F}(E_2+\mu)e^{\tau E_2}\Big]\,, \quad (8.132)$$

$$T\sum_{\{p_n\}}\frac{e^{-i\tilde{p}_n\tau}}{\tilde{p}_n^2+E_1^2} = -e^{\mu\beta}T\sum_{\{p_n\}}\frac{e^{i\tilde{p}_n(\beta-\tau)}}{\tilde{p}_n^2+E_1^2}$$
$$= \frac{1}{2E_1}\Big[n_{\rm F}(E_1+\mu)e^{(\beta-\tau)E_1+\beta\mu} - n_{\rm F}(E_1-\mu)e^{\tau E_1}\Big]\,, \quad (8.133)$$

where in the latter equation attention needed to be paid to the fact that Eq. (8.63) only applies for $0\le\tau\le\beta$ and that there is a shift due to the chemical potential in $\tilde{p}_n$.

Inserting these expressions into Eq. (8.131) and carrying out the integral over τ, we get

$$\mathcal{G} = \int_0^\beta \mathrm{d}\tau\, e^{ik_n\tau}\frac{1}{4E_1E_2}\Big\{n_{\rm F}(E_1+\mu)n_{\rm F}(E_2-\mu)e^{(\beta-\tau)(E_1+E_2)}$$
$$-n_{\rm F}(E_1+\mu)n_{\rm F}(E_2+\mu)e^{\tau(E_2-E_1)+\beta(E_1+\mu)}$$

$$
\begin{aligned}
&-n_F(E_1-\mu)n_F(E_2-\mu)e^{\tau(E_1-E_2)+\beta(E_2-\mu)} \\
&+n_F(E_1-\mu)n_F(E_2+\mu)e^{\tau(E_1+E_2)}\Big\} \\
= \frac{1}{4E_1E_2}\Big\{&n_F(E_1+\mu)n_F(E_2-\mu)\frac{1}{ik_n-E_1-E_2}\Big[1-e^{\beta(E_1+E_2)}\Big] \\
&-n_F(E_1+\mu)n_F(E_2+\mu)\frac{1}{ik_n+E_2-E_1}\Big[e^{\beta(E_2+\mu)}-e^{\beta(E_1+\mu)}\Big] \\
&-n_F(E_1-\mu)n_F(E_2-\mu)\frac{1}{ik_n+E_1-E_2}\Big[e^{\beta(E_1-\mu)}-e^{\beta(E_2-\mu)}\Big] \\
&+n_F(E_1-\mu)n_F(E_2+\mu)\frac{1}{ik_n+E_1+E_2}\Big[e^{\beta(E_1+E_2)}-1\Big]\Big\} \\
= \frac{1}{4E_1E_2}\Big\{&\frac{1}{ik_n-E_1-E_2}\Big[n_F(E_1+\mu)+n_F(E_2-\mu)-1\Big] \\
&+\frac{1}{ik_n+E_2-E_1}\Big[n_F(E_2+\mu)-n_F(E_1+\mu)\Big] \\
&+\frac{1}{ik_n+E_1-E_2}\Big[n_F(E_1-\mu)-n_F(E_2-\mu)\Big] \\
&+\frac{1}{ik_n+E_1+E_2}\Big[1-n_F(E_1-\mu)-n_F(E_2+\mu)\Big]\Big\} \, .
\end{aligned}
\tag{8.134}
$$

At this point we could carry out the analytic continuation $ik_n \to k^0 + i0^+$, but it will be convenient to postpone it for a moment; we just need to keep in mind that after the analytic continuation, ik_n becomes a *soft* quantity.

The next step is to Taylor-expand to leading order in $k^0, \mathbf{k}$. To this end we can write

$$
E_1 = p \equiv |\mathbf{p}| \, , \quad E_2 = |\mathbf{p}-\mathbf{k}| \approx p - k_i \frac{\partial}{\partial p_i}|\mathbf{p}| = p - k_i v_i \, , \tag{8.135}
$$

where

$$
v_i \equiv \frac{p_i}{p} \, , \quad i \in \{1,2,3\} \, , \tag{8.136}
$$

are referred to as the *velocities of the hard particles.*

It has to be realized that a Taylor expansion is sensible only in terms in which there is a thermal distribution function providing an external scale T and thereby guaranteeing that the integral obtains its dominant contributions from hard momenta, $p \sim \pi T$. We cannot Taylor-expand in the vacuum part, which has no scale with respect to which to expand. It can, however, be separately verified that the vacuum part vanishes as a power of $k^0, \mathbf{k}$, which is consistent with the fact that there is no gluon mass in vacuum. Here we simply omit the temperature-independent part.

With these approximations, the function $\mathcal{G}$ reads

$$\begin{aligned}\mathcal{G} \approx \frac{1}{4p^2}\Bigg\{ & \frac{1}{2p}\Big[-n_\mathrm{F}(p+\mu) - n_\mathrm{F}(p-\mu)\Big] \\ & + \frac{1}{ik_n - \mathbf{k}\cdot\mathbf{v}}(-\mathbf{k}\cdot\mathbf{v})n'_\mathrm{F}(p+\mu) \\ & + \frac{1}{ik_n + \mathbf{k}\cdot\mathbf{v}}(+\mathbf{k}\cdot\mathbf{v})n'_\mathrm{F}(p-\mu) \\ & + \frac{1}{2p}\Big[-n_\mathrm{F}(p-\mu) - n_\mathrm{F}(p+\mu)\Big]\Bigg\} + \mathcal{O}(k^0, \mathbf{k}) \ . \end{aligned} \tag{8.137}$$

Now we insert Eqs. (8.129) and (8.137) into Eq. (8.126). Through the substitution $\mathbf{p} \to -\mathbf{p}$ (whereby $\mathbf{v} \to -\mathbf{v}$), the 3rd row in Eq. (8.137) can be put in the same form as the 2nd row. Furthermore, terms containing k_n or $\mathbf{k}$ in the numerator in Eq. (8.126) are seen to be of higher order. Thereby

$$\begin{aligned}\Pi^{(\mathrm{f})}_{ij}(K) \approx -g^2 N_\mathrm{f} \int_\mathbf{p} \Bigg\{ & \frac{\delta_{ij}}{p}\Big[-n_\mathrm{F}(p+\mu) - n_\mathrm{F}(p-\mu)\Big] \\ & - \frac{p_i p_j}{p^2}\frac{1}{p}\Big[-n_\mathrm{F}(p+\mu) - n_\mathrm{F}(p-\mu)\Big] \\ & - \frac{p_i p_j}{p^2}\frac{ik_n - \mathbf{k}\cdot\mathbf{v} - ik_n}{ik_n - \mathbf{k}\cdot\mathbf{v}}\Big[n'_\mathrm{F}(p+\mu) + n'_\mathrm{F}(p-\mu)\Big]\Bigg\} \\ = -g^2 N_\mathrm{f} \int_\mathbf{p} \Bigg\{ & \frac{-\delta_{ij}}{p}\Big[n_\mathrm{F}(p+\mu) + n_\mathrm{F}(p-\mu)\Big] \\ & + \frac{v_i v_j}{p}\Big[n_\mathrm{F}(p+\mu) + n_\mathrm{F}(p-\mu)\Big] \\ & - v_i v_j \Big[n'_\mathrm{F}(p+\mu) + n'_\mathrm{F}(p-\mu)\Big] \\ & + \frac{v_i v_j \, ik_n}{ik_n - \mathbf{k}\cdot\mathbf{v}}\Big[n'_\mathrm{F}(p+\mu) + n'_\mathrm{F}(p-\mu)\Big]\Bigg\} \ . \end{aligned} \tag{8.138}$$

The remaining integration can be factorized into a radial and an angular part,

$$\int_\mathbf{p} = \int_p \int \mathrm{d}\Omega_v \ , \tag{8.139}$$

where the *angular integration* goes over the directions of $\mathbf{v} = \mathbf{p}/p$, and is normalized to unity:

$$\int \mathrm{d}\Omega_v \equiv 1 \; . \tag{8.140}$$

Then, the following identities can be verified (for Eqs. (8.141) and (8.143) details are given in Appendix C; Eq. (8.142) is a trivial consequence of rotational symmetry and $\mathbf{v}^2 = 1$):

$$\int_p \Big[n'_{\mathrm{F}}(p+\mu) + n'_{\mathrm{F}}(p-\mu) \Big] = -(d-1) \int_p \frac{1}{p} \Big[n_{\mathrm{F}}(p+\mu) + n_{\mathrm{F}}(p-\mu) \Big] \; , \tag{8.141}$$

$$\int \mathrm{d}\Omega_v \, v_i v_j = \frac{\delta_{ij}}{d} \; , \tag{8.142}$$

and, for $d = 3$,

$$\int_p \frac{1}{p} \Big[n_{\mathrm{F}}(p+\mu) + n_{\mathrm{F}}(p-\mu) \Big] \stackrel{d=3}{=} \frac{1}{4} \left(\frac{T^2}{3} + \frac{\mu^2}{\pi^2} \right) . \tag{8.143}$$

The integration

$$\int \mathrm{d}\Omega_v \frac{v_i v_j}{ik_n - \mathbf{k} \cdot \mathbf{v}} \tag{8.144}$$

can also be carried out (cf. Appendix C) but we do not need its value for the moment.

With these ingredients, Eq. (8.138) becomes

$$\begin{aligned} \Pi^{(\mathrm{f})}_{ij}(K) &= -g^2 N_\mathrm{f} \int_p \frac{1}{p} \Big[n_{\mathrm{F}}(p+\mu) + n_{\mathrm{F}}(p-\mu) \Big] \\ &\quad \times \left\{ \delta_{ij} \left(-1 + \frac{1}{d} + \frac{d-1}{d} \right) - (d-1) \int \mathrm{d}\Omega_v \frac{v_i v_j \, ik_n}{ik_n - \mathbf{k} \cdot \mathbf{v}} \right\} \\ &= g^2 N_\mathrm{f} (d-1) \int_p \frac{1}{p} \Big[n_{\mathrm{F}}(p+\mu) + n_{\mathrm{F}}(p-\mu) \Big] \int \mathrm{d}\Omega_v \frac{v_i v_j \, ik_n}{ik_n - \mathbf{k} \cdot \mathbf{v}} . \end{aligned} \tag{8.145}$$

Including also gluons and ghosts, the complete result reads

$$\Pi_{ij}(K) = m_{\mathrm{E}}^2 \int \mathrm{d}\Omega_v \frac{v_i v_j \, ik_n}{ik_n - \mathbf{k} \cdot \mathbf{v}} + \mathcal{O}(ik_n, \mathbf{k}) \; , \tag{8.146}$$

where m_E is the generalization of the *Debye mass* in Eq. (5.102) to the case of a fermionic chemical potential,

$$m_{\rm E}^2 \equiv g^2(d-1)\int_p \frac{1}{p}\Big\{N_{\rm f}\Big[n_{\rm F}(p+\mu)+n_{\rm F}(p-\mu)\Big]+(d-1)N_{\rm c}n_{\rm B}(p)\Big\} \tag{8.147}$$

$$\overset{d=3}{=} g^2\left[N_{\rm f}\left(\frac{T^2}{6}+\frac{\mu^2}{2\pi^2}\right)+\frac{N_{\rm c}T^2}{3}\right]. \tag{8.148}$$

Equation (8.146), known for QED since a long time [28–30], is a remarkable expression. Even though it is of $\mathcal{O}(1)$ is we count ik_n and $\mathbf{k}$ as quantities of the same order, it depends non-trivially on the ratio $ik_n/|\mathbf{k}|$. In particular, for $k^0 = ik_n \to 0$, i.e. *in the static limit*, Π_{ij} vanishes. This corresponds to the result in Eq. (5.100), i.e. that spatial gauge field components do not develop a thermal mass at 1-loop order. On the other hand, for $0 < |k^0| < |\mathbf{k}|$, it contains both a real and an imaginary part, cf. Eqs. (8.221) and (8.225). The imaginary part is related to the physics of *Landau damping*: it means that spacelike gauge fields can lose energy to hard particles in the plasma through real $2 \leftrightarrow 1$ scatterings.

So far, we were only concerned with the spatial part Π_{ij}. An interesting question is to generalize the computation to the full self-energy $\Pi_{\mu\nu}$. Fortunately, it turns out that all the information needed can be extracted from Eq. (8.146), as we now show.

Indeed, the self-energy $\Pi_{\mu\nu}$, obtained by integrating out the hard modes, must produce a structure which is gauge-invariant in "soft" gauge transformations, and therefore it must obey a Slavnov-Taylor identity and be *transverse* with respect to the external four-momentum. However, the meaning of transversality changes from the case of zero temperature, because the heat bath introduces a preferred frame, and thus breaks Lorentz invariance. More precisely, we can now introduce *two different* projection operators,

$$\mathbb{P}^{\rm T}_{\mu\nu}(K) \equiv \delta_{\mu i}\delta_{\nu j}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right), \tag{8.149}$$

$$\mathbb{P}^{\rm E}_{\mu\nu}(K) \equiv \delta_{\mu\nu}-\frac{K_\mu K_\nu}{K^2}-\mathbb{P}^{\rm T}_{\mu\nu}(K), \tag{8.150}$$

which both are four-dimensionally transverse,

$$\mathbb{P}^{\rm T}_{\mu\nu}(K)\,K_\nu = \mathbb{P}^{\rm E}_{\mu\nu}(K)\,K_\nu = 0, \tag{8.151}$$

and of which $\mathbb{P}^{\rm T}_{\mu\nu}(K)$ is in addition three-dimensionally transverse,

$$\mathbb{P}^{\rm T}_{\mu i}(K)\,k_i = 0. \tag{8.152}$$

The two projectors are also orthogonal to each other, $\mathbb{P}^{\rm E}_{\mu\alpha}\mathbb{P}^{\rm T}_{\alpha\nu} = 0$.

With the above projectors, we can write

$$\Pi_{ij}(K) \;=\; m_{\rm E}^2 \int {\rm d}\Omega_v \frac{v_i v_j\, ik_n}{ik_n - \mathbf{k}\cdot\mathbf{v}} \;\equiv\; \mathbb{P}^{\rm T}_{ij}(K)\,\Pi_{\rm T}(K) + \mathbb{P}^{\rm E}_{ij}(K)\,\Pi_{\rm E}(K)\,. \tag{8.153}$$

Note that this decomposition applies for $(\ldots)_{ij} \to (\ldots)_{\mu\nu}$ as well. Contracting Eq. (8.153) with δ_{ij} and with $k_i k_j$ leads to the equations

$$m_{\rm E}^2\, ik_n L = (d-1)\,\Pi_{\rm T} + \left(1 - \frac{k^2}{k_n^2 + k^2}\right)\Pi_{\rm E}\,, \tag{8.154}$$

$$m_{\rm E}^2 \int {\rm d}\Omega_v \frac{(\mathbf{k}\cdot\mathbf{v})^2 ik_n}{ik_n - \mathbf{k}\cdot\mathbf{v}} = 0\,\Pi_{\rm T} + \left(k^2 - \frac{(k^2)^2}{k_n^2 + k^2}\right)\Pi_{\rm E}\,, \tag{8.155}$$

where

$$L \equiv \int {\rm d}\Omega_v \frac{1}{ik_n - \mathbf{k}\cdot\mathbf{v}}\,. \tag{8.156}$$

The integral on the left-hand side of Eq. (8.155) can furthermore be written as

$$\begin{aligned}\int {\rm d}\Omega_v \frac{(\mathbf{k}\cdot\mathbf{v})^2 ik_n}{ik_n - \mathbf{k}\cdot\mathbf{v}} &= \int {\rm d}\Omega_v \frac{(-\mathbf{k}\cdot\mathbf{v} + ik_n - ik_n)(-\mathbf{k}\cdot\mathbf{v})ik_n}{ik_n - \mathbf{k}\cdot\mathbf{v}} \\ &= (ik_n)^2 \int {\rm d}\Omega_v \frac{\mathbf{k}\cdot\mathbf{v}}{ik_n - \mathbf{k}\cdot\mathbf{v}} \\ &= (ik_n)^2\Big[-1 + ik_n L\Big]\,,\end{aligned} \tag{8.157}$$

where we have in the second step dropped a term that vanishes upon angular integration. Solving for $\Pi_{\rm T}$, $\Pi_{\rm E}$ and subsequently inserting the expression for L from Eq. (8.213), we thus get

$$\Pi_{\rm T}(K) \;=\; \frac{m_{\rm E}^2}{d-1}\left\{-\frac{k_n^2}{k^2} + \frac{K^2}{k^2}\, ik_n L\right\} \tag{8.158}$$

$$\overset{d=3}{=} \frac{m_{\rm E}^2}{2}\left\{\frac{(ik_n)^2}{k^2} + \frac{ik_n}{2k}\left[1 - \frac{(ik_n)^2}{k^2}\right]\ln\frac{ik_n + k}{ik_n - k}\right\}\,, \tag{8.159}$$

$$\Pi_{\rm E}(K) \;=\; \frac{m_{\rm E}^2 K^2}{k^2}(1 - ik_n L) \tag{8.160}$$

$$\overset{d=3}{=} m_{\rm E}^2\left[1 - \frac{(ik_n)^2}{k^2}\right]\left[1 - \frac{ik_n}{2k}\ln\frac{ik_n + k}{ik_n - k}\right]. \tag{8.161}$$

Equations (8.159) and (8.161) have a number of interesting limiting values. For $ik_n \to 0$ but with $k \neq 0$, $\Pi_{\rm T} \to 0$, $\Pi_{\rm E} \to m_{\rm E}^2$. This corresponds to the physics of

Debye screening, familiar to us from Eq. (5.101). On the contrary, if we consider homogeneous but time-dependent waves, i.e. take $k \to 0$ with $ik_n \neq 0$, it can be seen that $\Pi_{\rm T}, \Pi_{\rm E} \to m_{\rm E}^2/3$. This genuinely Minkowskian structure in the resummed self-energy corresponds to *plasma oscillations*, or *plasmons*.

We can also write down a resummed gluon propagator: in a general covariant gauge, where the tree-level propagator has the form in Eq. (5.45) and the static Feynman gauge propagator the form in Eq. (5.101), we get

$$\langle A_\mu^a(X)A_\nu^b(Y)\rangle_0 = \delta^{ab} \SumInt_K e^{iK\cdot(X-Y)}\left[\frac{\mathbb{P}^{\rm T}_{\mu\nu}(K)}{K^2+\Pi_{\rm T}(K)} + \frac{\mathbb{P}^{\rm E}_{\mu\nu}(K)}{K^2+\Pi_{\rm E}(K)} + \frac{\xi\, K_\mu K_\nu}{(K^2)^2}\right], \tag{8.162}$$

where ξ is the gauge parameter.

If the propagator of Eq. (8.162) is used in practical applications, it is often useful to express it in terms of the *spectral representation*, cf. Eq. (8.24). The spectral function appearing in the spectral representation can be obtained from Eq. (8.27), where now $1/[K^2 + \Pi_{\rm T(E)}(K)]$ plays the role of $\Pi^E_{\alpha\beta}$. After analytic continuation, $ik_n \to k^0 + i0^+$,

$$\frac{1}{K^2+\Pi_{\rm T(E)}(k_n,\mathbf{k})} \to \frac{1}{-(k^0+i0^+)^2+\mathbf{k}^2+\Pi_{\rm T(E)}(-i(k^0+i0^+),\mathbf{k})}\,, \tag{8.163}$$

where

$$\Pi_{\rm T}(-i(k^0+i0^+),\mathbf{k}) = \frac{m_{\rm E}^2}{2}\left\{\frac{(k^0)^2}{k^2} + \frac{k^0}{2k}\left[1-\frac{(k^0)^2}{k^2}\right]\ln\frac{k^0+k+i0^+}{k^0-k+i0^+}\right\}\,, \tag{8.164}$$

$$\Pi_{\rm E}(-i(k^0+i0^+),\mathbf{k}) = m_{\rm E}^2\left[1-\frac{(k^0)^2}{k^2}\right]\left[1-\frac{k^0}{2k}\ln\frac{k^0+k+i0^+}{k^0-k+i0^+}\right]. \tag{8.165}$$

For $|k^0| > k$, $\Pi_{\rm T}, \Pi_{\rm E}$ are real, whereas for $|k^0| < k$, they have an imaginary part. Denoting $\eta \equiv \frac{k^0}{k}$, a straightforward computation (utilizing the fact that $\ln z$ has a branch cut on the negative real axis) leads to the spectral functions $\rho_{\rm T(E)} \equiv {\rm Im}\big(\frac{1}{K^2+\Pi_{\rm T(E)}}\big)_{ik_n\to k^0+i0^+}$, where

$$\rho_{\rm T}(\mathcal{K}) = \begin{cases} \dfrac{\Gamma_{\rm T}(\eta)}{\Sigma_{\rm T}^2(\mathcal{K})+\Gamma_{\rm T}^2(\eta)}\,, & |\eta|<1\,, \\ \pi\,{\rm sign}(\eta)\,\delta(\Sigma_{\rm T}(\mathcal{K}))\,, & |\eta|>1\,, \end{cases} \tag{8.166}$$

$$(\eta^2-1)\rho_{\rm E}(\mathcal{K}) = \begin{cases} \dfrac{\Gamma_{\rm E}(\eta)}{\Sigma_{\rm E}^2(\mathcal{K})+\Gamma_{\rm E}^2(\eta)}\,, & |\eta|<1\,, \\ \pi\,{\rm sign}(\eta)\;\delta(\Sigma_{\rm E}(\mathcal{K}))\,, & |\eta|>1\,. \end{cases} \tag{8.167}$$

Here we have introduced the well-known functions [28–30]

$$\Sigma_{\rm T}(\mathcal{K}) \equiv -\mathcal{K}^2 + \frac{m_{\rm E}^2}{2}\left[\eta^2 + \frac{\eta(1-\eta^2)}{2}\ln\left|\frac{1+\eta}{1-\eta}\right|\right], \tag{8.168}$$

$$\Gamma_{\rm T}(\eta) \equiv \frac{\pi m_{\rm E}^2\,\eta(1-\eta^2)}{4}, \tag{8.169}$$

$$\Sigma_{\rm E}(\mathcal{K}) \equiv k^2 + m_{\rm E}^2\left[1 - \frac{\eta}{2}\ln\left|\frac{1+\eta}{1-\eta}\right|\right], \tag{8.170}$$

$$\Gamma_{\rm E}(\eta) \equiv \frac{\pi m_{\rm E}^2\,\eta}{2}. \tag{8.171}$$

The essential structure is that in each case there is a "plasmon" pole, i.e. a δ-function analogous to the δ-functions in the free propagator of Eq. (8.35) but displaced by an amount $\propto m_{\rm E}^2$, as well as a cut at $|k^0| < k$, representing Landau damping.

So far, we have only computed the resummed gluon propagator. A very interesting question is whether also an *effective action* can be written down, which would then not only contain the inverse propagator like Eq. (8.124), but also new vertices, in analogy with the dimensionally reduced effective theory of Eq. (6.36). Such effective vertices are needed for properly describing how the soft modes interact with each other. Note that since our observables are now non-static, the effective action should be gauge-invariant also in time-dependent gauge transformations.

Most remarkably, such an effective action can indeed be found [31, 32]. We simply cite here the result for the gluonic case. Expressing everything in Minkowskian notation (i.e. after setting $ik_n \to k^0$ and using the Minkowskian A_0^a), the effective Lagrangian reads

$$\mathcal{L}_M = -\frac{1}{2}\mathrm{Tr}\,[F_{\mu\nu}F^{\mu\nu}] + \frac{m_{\rm E}^2}{2}\int \mathrm{d}\Omega_v\,\mathrm{Tr}\left[\left(\frac{1}{\mathcal{V}\cdot\mathcal{D}}\,\mathcal{V}^\alpha F_{\alpha\mu}\right)\left(\frac{1}{\mathcal{V}\cdot\mathcal{D}}\,\mathcal{V}^\beta F_\beta{}^\mu\right)\right]. \tag{8.172}$$

Here $\mathcal{V} \equiv (1, \mathbf{v})$ is a light-like four-velocity, and $\mathcal{D}$ represents the covariant derivative in the adjoint representation.

Several remarks on Eq. (8.172) are in order:

- A somewhat tedious analysis, making use of the velocity integrals listed in Eqs. (8.216)–(8.224) below, shows that in the static limit the second term in Eq. (8.172) reduces to the mass term in Eq. (6.36) (modulo Wick rotation and the Minkowskian vs. Euclidean convention for A_0^a).
- In the static limit, we found quarks to always be infrared-safe, but this situation changes after the analytic continuation. Therefore a "dynamical" quark part should be added to Eq. (8.172) [31, 32]; some details are given in Appendix B.
- Equation (8.172) has the unpleasant feature that it is *non-local*: derivatives appear in the denominator. This we do not usually expect from effective theories. Indeed, if non-local structures appear, it is difficult to analyze what kind of higher-

order operators have been omitted and, hence, what the relative accuracy of the effective description is.

In some sense, the appearance of non-local terms is a manifestation of the fact that the proper infrared degrees of freedom have not been identified. It turns out that the HTL theory can be reformulated by introducing additional degrees of freedom, which gives the theory a local appearance [20, 33–35] (for a pedagogic introduction see [36]). However the reformulation contains classical on-shell particles rather than quantum fields, whereby it continues to be difficult to analyze the accuracy of the effective description.

- We arrived at Eq. (8.172) by integrating out the hard modes, with momenta $p \sim \pi T$. However, like in the static limit, the theory still has multiple dynamical momentum scales, $k \sim gT$ and $k \sim g^2T/\pi$. It can be asked what happens if the momenta $k \sim gT$ are also integrated out. This question has been analyzed in the literature, and leads indeed to a simplified (local) effective description [37–41], which can be used for non-perturbatively studying observables only sensitive to "ultrasoft" momenta, $k \sim g^2T/\pi$.
- Remarkably, for certain light-cone observables, "sum rules" can be established which allow to reduce gluonic HTL structures to the dimensionally reduced theory [15, 42, 43].[4] This is an important development, because the dimensionally reduced theory can be studied with standard non-perturbative techniques [44].

Appendix A: Hard Gluon Loop

Here a few details are given concerning the handling of the gluonic part of Eq. (8.125). We follow the steps from Eq. (8.126) onwards. The spatial part of the self-energy can be written as

$$\Pi^{(b)}_{ij}(K) = \frac{g^2 N_c}{2} \SumInt_P \left\{ (D-2)\left[\frac{2\delta_{ij}}{P^2} + \frac{k_i k_j - 4p_i p_j}{P^2(K-P)^2} \right] - 4\,\frac{k^2\delta_{ij} - k_i k_j}{P^2(K-P)^2} \right\}, \tag{8.175}$$

[4] Picking out one spatial component and denoting it by $k_\parallel$, so that $\mathbf{k} \equiv (k_\parallel, \mathbf{k}_\perp)$, the sum rules can be expressed as [15, 42, 43]

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}k_\parallel}{2\pi} \left\{ \frac{\rho_\mathrm{T}(k_\parallel, \mathbf{k})}{k_\parallel} - \frac{\rho_\mathrm{E}(k_\parallel, \mathbf{k})}{k_\parallel} \right\} \frac{k_\perp^4}{k_\perp^2 + k_\parallel^2} = \frac{1}{2}\,\frac{m_\mathrm{E}^2}{k_\perp^2 + m_\mathrm{E}^2}\,, \tag{8.173}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}k_\parallel}{2\pi}\, k_\parallel \left\{ \rho_\mathrm{P}(k_\parallel, \mathbf{k}) - \rho_\mathrm{W}(k_\parallel, \mathbf{k}) \right\} = \frac{1}{4}\,\frac{m_\ell^2}{k_\perp^2 + m_\ell^2}\,, \tag{8.174}$$

where ρ_T, ρ_E, ρ_W and ρ_P are the spectral functions from Eqs. (8.166), (8.167), (8.201) and (8.202), respectively.

where all terms containing k_i in the numerator are subleading. The bosonic counterpart of Eq. (8.129) [cf. Eq. (8.29)] reads

$$T \sum_{p_n} \frac{1}{p_n^2 + E_1^2} = \frac{1}{2E_1} \Big[1 + 2n_{\rm B}(E_1) \Big] , \tag{8.176}$$

whereas Eqs. (8.130)–(8.134) get replaced with

$$\mathcal{G}' \equiv T \sum_{p_n} \frac{1}{[p_n^2 + E_1^2][(k_n - p_n)^2 + E_2^2]} \tag{8.177}$$

$$\begin{aligned} &= \frac{1}{4E_1E_2} \bigg\{ \frac{1}{ik_n - E_1 - E_2} \Big[-n_{\rm B}(E_1) - n_{\rm B}(E_2) - 1 \Big] \\ &+ \frac{1}{ik_n + E_2 - E_1} \Big[n_{\rm B}(E_1) - n_{\rm B}(E_2) \Big] \\ &+ \frac{1}{ik_n + E_1 - E_2} \Big[n_{\rm B}(E_2) - n_{\rm B}(E_1) \Big] \\ &+ \frac{1}{ik_n + E_1 + E_2} \Big[1 + n_{\rm B}(E_1) + n_{\rm B}(E_2) \Big] \bigg\} . \end{aligned} \tag{8.178}$$

We observe that the bosonic results can be obtained from the fermionic ones simply by setting $n_{\rm F} \to -n_{\rm B}$. The expansions of Eqs. (8.135)–(8.137) proceed as before, although one must be careful in making sure that the IR behaviour of the Bose distribution still permits a Taylor expansion in powers of the external momentum. The partial integration identity in Eq. (8.141) can in addition be seen to retain its form, so that, effectively,

$$\mathcal{G}' \to \frac{n_{\rm B}(p)}{2p^3} \bigg[1 - (D-2) \frac{\mathbf{k} \cdot \mathbf{v}}{ik_n - \mathbf{k} \cdot \mathbf{v}} \bigg] = \frac{n_{\rm B}(p)}{2p^3} \bigg[D - 1 - (D-2) \frac{ik_n}{ik_n - \mathbf{k} \cdot \mathbf{v}} \bigg] . \tag{8.179}$$

The final steps are like in Eq. (8.145) and lead to Eq. (8.146), with $m_{\rm E}^2$ as given in Eq. (8.147).

Appendix B: Fermion Self-Energy

Next, we consider a Dirac fermion at a finite temperature T and a finite chemical potential μ, interacting with an Abelian gauge field (this is no restriction at the current order: for a non-Abelian case simply replace $e^2 \to g^2 C_{\rm F}$, where $C_{\rm F} \equiv (N_{\rm c}^2 - 1)/(2N_{\rm c})$). The action is of the form in Eq. (7.34) with $D_\mu = \partial_\mu - ieA_\mu$. To second order in e, the "effective action", or generating functional, takes the form $S_{\rm eff} =$

$S_0 + \langle S_\mathrm{I} - \frac{1}{2}S_\mathrm{I}^2 + \mathcal{O}(e^3)\rangle_{1\mathrm{PI}}$, where S_0 is the quadratic part of the Euclidean action and S_I contains the interactions. Carrying out the Wick contractions, this yields

$$S_\mathrm{eff} = \sum\!\!\!\!\!\!\!\int_{\{K\}} \bar{\tilde{\psi}}(\tilde{K})\left[i\tilde{\not{K}} + m + e^2 \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{\gamma_\mu(-i\tilde{\not{P}} + m)\gamma_\mu}{(\tilde{P}^2 + m^2)(\tilde{P} - \tilde{K})^2} + \mathcal{O}(eA_\mu) \right]\tilde{\psi}(\tilde{K}) \,, \tag{8.180}$$

where we have for simplicity employed the Feynman gauge, and $\tilde{P}, \tilde{K}$ are fermionic Matsubara momenta where the zero component contains the chemical potential as indicated in Eq. (8.127): $\tilde{k}_n \equiv k_n + i\mu$. In the momentum $\tilde{P} - \tilde{K}$, carried by the gluon, the chemical potential drops out.

The Dirac structures appearing in Eq. (8.180) can be simplified: $\gamma_\mu\gamma_\mu = D\,\mathbb{1}_{4\times4}$, $\gamma_\mu\tilde{\not{P}}\gamma_\mu = (2-D)\tilde{\not{P}}$. Denoting

$$f(i\tilde{p}_n, \mathbf{v}) \equiv i(D-2)\tilde{\not{P}} + D\,m\,\mathbb{1}_{4\times4} \tag{8.181}$$

where $\mathbf{v}$ is a dummy variable for both $\mathbf{p}$ and m; as well as

$$E_1 \equiv \sqrt{p^2 + m^2} \,, \quad E_2 \equiv \sqrt{(\mathbf{p} - \mathbf{k})^2} \,, \tag{8.182}$$

we are led to consider the sum [a generalization of Eq. (8.74)]

$$\mathcal{F} \equiv T \sum_{\{p_n\}} \frac{f(i\tilde{p}_n, \mathbf{v})}{[\tilde{p}_n^2 + E_1^2][(\tilde{p}_n - \tilde{k}_n)^2 + E_2^2]} \,. \tag{8.183}$$

We can now write

$$\begin{aligned} \mathcal{F} &= T\sum_{\{p_n\}} T\sum_{r_n} \beta\,\delta(\tilde{p}_n - \tilde{k}_n - r_n) \frac{f(i\tilde{p}_n, \mathbf{v})}{[\tilde{p}_n^2 + E_1^2][r_n^2 + E_2^2]} \\ &= \int_0^\beta \mathrm{d}\tau\, e^{-i\tilde{k}_n\tau} \left\{ T\sum_{\{p_n\}} e^{i\tilde{p}_n\tau} \frac{f(i\tilde{p}_n, \mathbf{v})}{\tilde{p}_n^2 + E_1^2} \right\} \left\{ T\sum_{r_n} \frac{e^{-ir_n\tau}}{r_n^2 + E_2^2} \right\} \,, \end{aligned} \tag{8.184}$$

where we used a similar representation as before,

$$\beta\,\delta(\tilde{p}_n - \tilde{k}_n - r_n) = \int_0^\beta \mathrm{d}\tau\, e^{i(\tilde{p}_n - \tilde{k}_n - r_n)\tau} \,. \tag{8.185}$$

Subsequently Eqs. (8.29) and (8.63) and their time derivatives can be inserted:

$$T\sum_{r_n} \frac{e^{-ir_n\tau}}{r_n^2 + E_2^2} = \frac{n_\mathrm{B}(E_2)}{2E_2}\left[e^{(\beta-\tau)E_2} + e^{\tau E_2} \right] , \tag{8.186}$$

$$T\sum_{\{p_n\}} \frac{e^{i\tilde{p}_n\tau}}{\tilde{p}_n^2+E_1^2} = \frac{1}{2E_1}\Big[n_{\rm F}(E_1-\mu)e^{(\beta-\tau)E_1-\beta\mu} - n_{\rm F}(E_1+\mu)e^{\tau E_1}\Big] , \quad (8.187)$$

$$T\sum_{\{p_n\}} \frac{i\tilde{p}_n e^{i\tilde{p}_n\tau}}{\tilde{p}_n^2+E_1^2} = -\frac{1}{2}\Big[n_{\rm F}(E_1-\mu)e^{(\beta-\tau)E_1-\beta\mu} + n_{\rm F}(E_1+\mu)e^{\tau E_1}\Big] . \quad (8.188)$$

Thereby we obtain

$$\begin{aligned}\mathcal{F} = \int_0^\beta {\rm d}\tau\, e^{-i\tilde{k}_n\tau}\frac{n_{\rm B}(E_2)}{4E_1E_2}\Big\{&n_{\rm F}(E_1-\mu)e^{(\beta-\tau)(E_1+E_2)-\beta\mu}f(-E_1,\mathbf{v}) \\ &+n_{\rm F}(E_1-\mu)e^{(\beta-\tau)E_1+\tau E_2-\beta\mu}f(-E_1,\mathbf{v}) \\ &+n_{\rm F}(E_1+\mu)e^{(\beta-\tau)E_2+\tau E_1}f(-E_1,-\mathbf{v}) \\ &+n_{\rm F}(E_1+\mu)e^{\tau(E_1+E_2)}f(-E_1,-\mathbf{v})\Big\} . \end{aligned} \quad (8.189)$$

As an example, let us focus on the second structure in Eq. (8.189). The τ-integral can be carried out, noting that $\tilde{k}_n$ is fermionic:

$$\begin{aligned}\int_0^\beta {\rm d}\tau\, e^{\beta(E_1-\mu)}e^{\tau(-i\tilde{k}_n-E_1+E_2)} &= \frac{e^{\beta(E_1-\mu)}}{-i\tilde{k}_n-E_1+E_2}\Big[-e^{\beta(E_2-E_1+\mu)}-1\Big] \\ &= \frac{e^{\beta E_2}+e^{\beta(E_1-\mu)}}{i\tilde{k}_n+E_1-E_2} \\ &= \frac{1}{i\tilde{k}_n+E_1-E_2}\Big[n_{\rm B}^{-1}(E_2)+n_{\rm F}^{-1}(E_1-\mu)\Big] . \end{aligned} \quad (8.190)$$

The inverse distribution functions nicely combine with those appearing explicitly in Eq. (8.189):

$$\begin{aligned}\mathcal{F} = \frac{1}{4E_1E_2}\Big\{&\frac{f(-E_1,\mathbf{v})}{i\tilde{k}_n+E_1+E_2}\Big[1+n_{\rm B}(E_2)-n_{\rm F}(E_1-\mu)\Big] \\ &+\frac{f(-E_1,\mathbf{v})}{i\tilde{k}_n+E_1-E_2}\Big[n_{\rm F}(E_1-\mu)+n_{\rm B}(E_2)\Big] \\ &+\frac{f(E_1,\mathbf{v})}{i\tilde{k}_n-E_1+E_2}\Big[-n_{\rm F}(E_1+\mu)-n_{\rm B}(E_2)\Big] \\ &+\frac{f(E_1,\mathbf{v})}{i\tilde{k}_n-E_1-E_2}\Big[-1-n_{\rm B}(E_2)+n_{\rm F}(E_1+\mu)\Big]\Big\} . \end{aligned} \quad (8.191)$$

We now make the assumption, akin to that leading to Eq. (8.137), that all four components of the (Minkowskian) external momentum $\mathcal{K}$ are small compared with the loop three-momentum $p = |\mathbf{p}|$, whose scale is fixed by the temperature and the chemical potential (this argument does not apply to the vacuum terms which are omitted; they amount e.g. to a radiative correction to the mass parameter m). Furthermore, in order to simplify the discussion, we assume that the (renormalized) mass parameter is small compared with T and μ. Thereby the "energies" of Eq. (8.182) become

$$E_1 \approx p + \frac{m^2}{2p} + \mathcal{O}\Big(\frac{m^4}{p^3}\Big) , \quad E_2 \approx p - \mathbf{k}\cdot\mathbf{v} + \mathcal{O}\Big(\frac{k^2}{p}\Big) \tag{8.192}$$

where again

$$\mathbf{v} \equiv \frac{\mathbf{p}}{p} . \tag{8.193}$$

Combining Eqs. (8.181) and (8.191) with Eq. (8.192), and noting that (for $m \ll p$)

$$f(\pm E_1, \mathbf{v}) \approx (D-2)(\pm\gamma^0 + v_i\gamma^i)p , \tag{8.194}$$

where we returned to Minkowskian conventions for the Dirac matrices [cf. Eq. (4.36)], it is easy to see that the dominant contribution, of order $1/\mathcal{K}$, arises from the 2nd and 3rd terms in Eq. (8.191) which contain the difference $E_1 - E_2$ in the denominator. Writing $-\mathbf{v}\cdot\boldsymbol{\gamma} \equiv v_i\gamma^i$ and substituting $\mathbf{v} \to -\mathbf{v}$ in the 3rd term, Eq. (8.180) becomes $S_\text{eff}^{(0)} = \SumInt_{\{\tilde K\}} \bar{\tilde\psi}(\tilde K)[i\tilde{\not K} + m + \Sigma(\tilde K)]\tilde\psi(\tilde K)$, where the superscript indicates that terms of $\mathcal{O}(eA_\mu)$ have been omitted, and

$$\Sigma(\tilde K) \approx -m_\text{F}^2 \int \mathrm{d}\Omega_v \frac{\gamma_0 + \mathbf{v}\cdot\boldsymbol{\gamma}}{i\tilde k_n + \mathbf{k}\cdot\mathbf{v}} . \tag{8.195}$$

Here we have defined

$$m_\text{F}^2 \equiv \frac{(D-2)e^2}{4} \int_\mathbf{p} \frac{1}{p}\Big[2n_\text{B}(p) + n_\text{F}(p+\mu) + n_\text{F}(p-\mu)\Big] \tag{8.196}$$

$$\overset{D=4}{=} e^2\bigg(\frac{T^2}{8} + \frac{\mu^2}{8\pi^2}\bigg) , \tag{8.197}$$

and carried out the integrals for $D = 4$ (the bosonic part gives $2\int_\mathbf{p} n_\text{B}(p)/p = T^2/6$; the fermionic part is worked out in Appendix C). The angular integrations can also be carried out, cf. Eqs. (8.219) and (8.220) below.

Next, we want to determine the corresponding spectral representation. As discussed in connection with the example following Eq. (8.64), sign conventions are tricky with fermions. Our $S_\text{eff}^{(0)}$ defines the inverse propagator, representing therefore a generalization of the object in Eq. (8.64), with the frequency variable appearing as $\tilde k_n = k_n + i\mu$. Aiming for a spectral representation directly in terms of this variable, needed in Eq. (8.92), we define the analytic continuation as $i\tilde{k}_n \to \omega$ where ω has a small positive imaginary part. Carrying out the angular integrals in Eq. (8.195) as explained in Appendix C, the analytically continued inverse propagator becomes (we set $m \to 0$)

$$\not{K} + \Sigma(-i\omega, \mathbf{k}) = \omega\gamma_0\left[1 - \frac{m_{\rm F}^2}{2k\omega}\ln\frac{\omega + k}{\omega - k}\right] - \mathbf{k}\cdot\boldsymbol{\gamma}\left[1 + \frac{m_{\rm F}^2}{k^2}\left(1 - \frac{\omega}{2k}\ln\frac{\omega + k}{\omega - k}\right)\right]. \tag{8.198}$$

Introducing the concept of an "asymptotic mass" $m_\ell^2 \equiv 2m_{\rm F}^2$ and denoting $L \equiv \frac{1}{2k}\ln\frac{\omega+k}{\omega-k}$, the corresponding spectral function reads

$$\mathrm{Im}\Big\{\big[\not{K} + \Sigma(-i\omega, \mathbf{k})\big]^{-1}\Big\} = \not{\rho}(\omega, \mathbf{k}) , \tag{8.199}$$

$$\rho \equiv (\omega\rho_{\rm W}, \mathbf{k}\rho_{\rm P}) , \tag{8.200}$$

$$\rho_{\rm W} = \mathrm{Im}\left\{\frac{1 - \frac{m_\ell^2 L}{2\omega}}{\big[\omega - \frac{m_\ell^2 L}{2}\big]^2 - \big[k + \frac{m_\ell^2(1-\omega L)}{2k}\big]^2}\right\} , \tag{8.201}$$

$$\rho_{\rm P} = \mathrm{Im}\left\{\frac{1 + \frac{m_\ell^2(1-\omega L)}{2k^2}}{\big[\omega - \frac{m_\ell^2 L}{2}\big]^2 - \big[k + \frac{m_\ell^2(1-\omega L)}{2k}\big]^2}\right\} . \tag{8.202}$$

These are well-known results [11, 29], generalized to the presence of a finite chemical potential [45]; note that the chemical potential only appears "trivially", inside m_ℓ, without affecting the functional form of the momentum dependence. The corresponding "dispersion relations", relevant for computing the "pole contributions" mentioned below Eq. (8.99), have been discussed in the literature [46] and can be shown to comprise two branches. There is a novel branch, dubbed a "plasmino" branch, with the peculiar property that

$$\omega \approx m_{\rm F} - \frac{k}{3} + \frac{k^2}{3m_{\rm F}} < m_{\rm F} , \quad k \ll m_{\rm F} . \tag{8.203}$$

If the zero-temperature mass m is larger than $m_{\rm F}$, the plasmino branch decouples [47]. For large momenta, the dispersion relation of the normal branch is of the form

$$\omega \approx k + \frac{m_\ell^2}{2k} , \quad k \gg m_\ell , \tag{8.204}$$

which explains why m_ℓ is called an asymptotic mass. A comprehensive discussion of the dispersion relation in various limits can be found in [48].

Appendix C: Radial and Angular Momentum Integrals

We compute here the radial and angular integrals defined in Eqs. (8.141)–(8.144).

For generality, and because this is necessary in loop computations, it is useful to keep the space dimensionality open for as long as possible. Let us recall that the dimensionally regularized integration measure can be written as

$$\int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \to \frac{4}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d-1}{2})} \int_0^\infty \mathrm{d}p\, p^{d-1} \int_{-1}^{+1} \mathrm{d}z\,(1-z^2)^{\frac{d-3}{2}} , \tag{8.205}$$

where $d \equiv D-1$ and $z = \mathbf{k}\cdot\mathbf{p}/(kp)$ parametrizes an angle with respect to some external vector. An important use of Eq. (8.205) is that it allows us to carry out partial integrations with respect to both p and z. If the integrand is independent of z, the z-integral yields

$$\int_{-1}^{+1} \mathrm{d}z\,(1-z^2)^{\frac{d-3}{2}} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} , \tag{8.206}$$

and we then denote (cf. Eq. (2.61), now divided by $(2\pi)^d$)

$$c(d) \equiv \frac{2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})} , \tag{8.207}$$

so that $\int_{\mathbf{p}} = \int_p \equiv c(d)\int_0^\infty \mathrm{d}p\, p^{d-1}$.

Now, Eq. (8.141) can be verified through partial integration as follows:

$$\begin{aligned}\int_p \frac{1}{p}\Big[n_{\mathrm{F}}(p+\mu)+n_{\mathrm{F}}(p-\mu)\Big] &= c(d)\int_0^\infty \mathrm{d}p\,\frac{\mathrm{d}p}{\mathrm{d}p}\,p^{d-2}\Big[n_{\mathrm{F}}(p+\mu)+n_{\mathrm{F}}(p-\mu)\Big] \\ &= -(d-2)\,c(d)\int_0^\infty \mathrm{d}p\,p^{d-2}\Big[n_{\mathrm{F}}(p+\mu)+n_{\mathrm{F}}(p-\mu)\Big] \\ &\quad -c(d)\int_0^\infty \mathrm{d}p\,p^{d-1}\Big[n'_{\mathrm{F}}(p+\mu)+n'_{\mathrm{F}}(p-\mu)\Big] .\end{aligned} \tag{8.208}$$

Moving the first term to the left-hand side leads directly to Eq. (8.141).

In order to derive the explicit expression in Eq. (8.143), we set $d = 3$; then a possible starting point is a combination of Eqs. (7.36) and (7.42):

$$\begin{aligned}-f(T,\mu) &= 2\int_p \Big\{ p + T\Big[\ln\Big(1+e^{-\frac{p-\mu}{T}}\Big) + \ln\Big(1+e^{-\frac{p+\mu}{T}}\Big)\Big]\Big\} \\ &\overset{d=3}{=} \frac{7\pi^2T^4}{180} + \frac{\mu^2T^2}{6} + \frac{\mu^4}{12\pi^2} .\end{aligned} \tag{8.209}$$

Taking the second partial derivative with respect to μ, we get

$$
\begin{aligned}
-\frac{\partial^2 f(T,\mu)}{\partial\mu^2} &= 2T\int_p \frac{\partial^2}{\partial\mu^2}\Big[\ln\Big(1+e^{-\frac{p-\mu}{T}}\Big)+\ln\Big(1+e^{-\frac{p+\mu}{T}}\Big)\Big] \\
&= 2T\int_p \frac{\mathrm{d}^2}{\mathrm{d}p^2}\Big[\ln\Big(1+e^{-\frac{p-\mu}{T}}\Big)+\ln\Big(1+e^{-\frac{p+\mu}{T}}\Big)\Big] \\
&\overset{d=3}{=} -4T\int_p \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}p}\Big[\ln\Big(1+e^{-\frac{p-\mu}{T}}\Big)+\ln\Big(1+e^{-\frac{p+\mu}{T}}\Big)\Big] \qquad (8.210) \\
&= \frac{T^2}{3}+\frac{\mu^2}{\pi^2}\,, \qquad (8.211)
\end{aligned}
$$

where in the penultimate step we carried out one partial integration. On the other hand, the integral in Eq. (8.210) can be rewritten as

$$
\begin{aligned}
&-4T\int_p \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}p}\Big[\ln\Big(1+e^{-\frac{p-\mu}{T}}\Big)+\ln\Big(1+e^{-\frac{p+\mu}{T}}\Big)\Big] \\
&= -4T\int_p \frac{1}{p}\Big[\frac{e^{-\frac{p-\mu}{T}}}{1+e^{-\frac{p-\mu}{T}}}+\frac{e^{-\frac{p+\mu}{T}}}{1+e^{-\frac{p+\mu}{T}}}\Big]\Big(-\frac{1}{T}\Big) \\
&= 4\int_p \frac{1}{p}\Big[n_{\mathrm{F}}(p+\mu)+n_{\mathrm{F}}(p-\mu)\Big]\,. \qquad (8.212)
\end{aligned}
$$

Equations (8.211) and (8.212) combine into Eq. (8.143).

As far as angular integrals go [such as the one in Eq. (8.144)], we start with the simplest structure, defined in Eq. (8.156):

$$
\begin{aligned}
L(K) \equiv \int \mathrm{d}\Omega_v \frac{1}{ik_n-\mathbf{k}\cdot\mathbf{v}} &\overset{d=3}{=} \frac{1}{4\pi}2\pi\int_{-1}^{+1}\mathrm{d}z\,\frac{1}{ik_n-kz} \\
&= -\frac{1}{2k}\int_{-1}^{+1}\mathrm{d}z\,\frac{\mathrm{d}}{\mathrm{d}z}\ln(ik_n-kz) \\
&= \frac{1}{2k}\ln\frac{ik_n+k}{ik_n-k}\,. \qquad (8.213)
\end{aligned}
$$

Further integrals can then be obtained by making use of rotational symmetry. For instance,

$$
\int \mathrm{d}\Omega_v \frac{v_i}{ik_n-\mathbf{k}\cdot\mathbf{v}} = k_i f(ik_n,k)\,, \qquad (8.214)
$$

where, contracting both sides with $\mathbf{k}$,

$$f(ik_n, k) = \frac{1}{k^2}\int \mathrm{d}\Omega_v \, \frac{\mathbf{k}\cdot\mathbf{v}}{ik_n - \mathbf{k}\cdot\mathbf{v}} = \frac{1}{k^2}\left[-1 + ik_n \int \mathrm{d}\Omega_v \, \frac{1}{ik_n - \mathbf{k}\cdot\mathbf{v}}\right] . \quad (8.215)$$

Another trick, needed for having higher powers in the denominator, is to take derivatives of Eq. (8.213) with respect to ik_n.

Without detailing further steps, we list the results for a number of velocity integrals that can be obtained this way. Let us change the notation at this point: we replace ik_n by $k^0 + i0^+$, as is relevant for retarded Green's functions ($i0^+$ is not shown explicitly), and introduce the light-like four-velocity $\mathcal{V} \equiv (1, \mathbf{v})$. Then the integrals read ($d = 3$; $i, j = 1, 2, 3$)

$$\int \mathrm{d}\Omega_v = 1 \,, \quad (8.216)$$

$$\int \mathrm{d}\Omega_v \, v^i = 0 \,, \quad (8.217)$$

$$\int \mathrm{d}\Omega_v \, v^i v^j = \frac{1}{3}\delta^{ij} \,, \quad (8.218)$$

$$\int \mathrm{d}\Omega_v \, \frac{1}{\mathcal{V}\cdot\mathcal{K}} = L(\mathcal{K}) \,, \quad (8.219)$$

$$\int \mathrm{d}\Omega_v \, \frac{v^i}{\mathcal{V}\cdot\mathcal{K}} = \frac{k^i}{k^2}\Big[-1 + k^0 L(\mathcal{K})\Big] \,, \quad (8.220)$$

$$\int \mathrm{d}\Omega_v \, \frac{v^i v^j}{\mathcal{V}\cdot\mathcal{K}} = \frac{L(\mathcal{K})}{2}\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right) + \frac{k^0}{2k^2}\Big[1 - k^0 L(\mathcal{K})\Big]\left(\delta^{ij} - \frac{3k^i k^j}{k^2}\right) , \quad (8.221)$$

$$\int \mathrm{d}\Omega_v \, \frac{1}{(\mathcal{V}\cdot\mathcal{K})^2} = \frac{1}{\mathcal{K}^2} \,, \quad (8.222)$$

$$\int \mathrm{d}\Omega_v \, \frac{v^i}{(\mathcal{V}\cdot\mathcal{K})^2} = \frac{k^i}{k^2}\left[\frac{k^0}{\mathcal{K}^2} - L(\mathcal{K})\right] , \quad (8.223)$$

$$\int \mathrm{d}\Omega_v \, \frac{v^i v^j}{(\mathcal{V}\cdot\mathcal{K})^2} = \frac{1}{2\mathcal{K}^2}\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right) - \frac{1}{2k^2}\left[1 - 2k^0 L(\mathcal{K}) + \frac{(k^0)^2}{\mathcal{K}^2}\right]\left(\delta^{ij} - \frac{3k^i k^j}{k^2}\right) , \quad (8.224)$$

where $\mathcal{V}\cdot\mathcal{K} = k^0 - \mathbf{v}\cdot\mathbf{k}$, and

$$L(\mathcal{K}) = \frac{1}{2k}\ln\frac{k^0 + k + i0^+}{k^0 - k + i0^+} \overset{|k^0|\ll k}{\approx} -\frac{i\pi}{2k} + \frac{k^0}{k^2} + \frac{(k^0)^3}{3k^4} + \ldots \,. \quad (8.225)$$

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