Chapter 7
Finite Density

Abstract The concept of a system at a finite density or, equivalently, at a finite chemical potential, is introduced. Considering first a complex scalar field, an imaginary-time path integral representation is derived for the partition function. The evaluation of the partition function reveals infrared problems, which are this time related to the phenomenon of Bose-Einstein condensation. A generic tool applicable to any scalar field theory, called the effective potential, is introduced in order to handle this situation. Subsequently the case of a Dirac fermion at a finite chemical potential is discussed. The concept of a susceptibility is introduced. The quark number susceptibility in QCD is evaluated up to second order in the gauge coupling.

Keywords Noether's theorem • Global symmetry • Bose-Einstein condensation • Condensate • Constrained effective potential • Susceptibility

7.1 Complex Scalar Field and Effective Potential

Let us consider a system which possesses some conserved global charge, Q. We assume the conserved charge to be *additive*, i.e. the charge can in principle have any (integer) value. Physical examples of possible Q's include:

- The baryon number B and the lepton number L. (In fact, within the classic Standard Model, the combination $B + L$ is not conserved because of an anomaly [1], so that strictly speaking only the linear combination $B-L$ is conserved; however, the rate of $B + L$ violation is exponentially small at $T < 160\,\mathrm{GeV}$ [2], so in this regime we can treat both B and L as separate conserved quantities.)
- If weak interactions are switched off (i.e., if we inspect phenomena at temperatures well below 50 GeV, time scales well shorter than 10^{-10} s, or distances well below 1 cm within the collision region of a particle experiment), then flavour quantum numbers such as the strangeness S are conserved. One prominent example of this is QCD thermodynamics, where one typically considers the chemical potentials of all quark flavors to be independent parameters.
- In non-relativistic field theories, the particle number N is conserved.

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- In some supersymmetric theories, there is a quantity called the R-charge which is conserved. (However this is normally a *multiplicative* rather than an *additive* charge. As discussed below, this leads to a qualitatively different behaviour.)

The case of a conserved Q turns out to be analogous to the case of gauge fields, treated in Sect. 5; indeed the introduction of a chemical potential, μ, as a conjugate variable to Q, is closely related to the introduction of the gauge field A_0 that was needed for imposing the Gauss law, "$Q = 0$". However, in contrast to that situation, we work in a grand canonical ensemble in the following, so that the quantum mechanical partition function is of the type

$$\mathcal{Z}(T, \mu) \equiv \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{Q})}\right] . \tag{7.1}$$

In Sect. 5 the projection operator $\delta_{\hat{Q},\hat{0}}$ was effectively imposed as

$$\delta_{\hat{Q},\hat{0}} = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} e^{i\theta\hat{Q}} , \tag{7.2}$$

where $\theta \propto A_0$ and we assumed the eigenvalues of $\hat{Q}$ to be integers. Comparing Eqs. (7.1) and (7.2), a chemical potential is seen to correspond, roughly speaking, to a *constant purely imaginary Euclidean gauge field* A_0.

Now, let us go back to classical field theory for a moment, and recall that if the system possesses a global U(1) symmetry, then there exists, according to Noether's theorem, a conserved current, $\mathcal{J}_\mu$. The integral of the zero component of the current, i.e. the charge density, over the spatial volume, defines the conserved charge,

$$Q \equiv \int_{\mathbf{x}} \mathcal{J}_0(t, \mathbf{x}) . \tag{7.3}$$

Conversely, one can expect that a system which does have a conserved global charge should also display a global U(1) symmetry in its field-theoretic description. Usually this is indeed the case, and we restrict to these situations in the following. (One notable exception is free field theory where, due to a lack of interactions, particle number is conserved even without a global symmetry. Another is that a discrete symmetry, $\phi \to -\phi$, may also lead to the concept of a generalized "parity", which acts as a *multiplicative* quantum number, with possible values ±1; however, in this case no non-trivial *charge density* $\rho = \langle\hat{Q}\rangle/V$ can be defined in the thermodynamic limit.)

As the simplest example of a system with an additive conserved charge and a global U(1) symmetry, consider a *complex scalar field*. The classical Lagrangian of a complex scalar field reads

$$\mathcal{L}_M = \partial^\mu\phi^*\partial_\mu\phi - V(\phi) , \tag{7.4}$$

where the potential has the form

$$V(\phi) \equiv m^2\phi^*\phi + \lambda(\phi^*\phi)^2 \ . \tag{7.5}$$

The system is invariant in the (position-independent) phase transformation

$$\phi \to e^{-i\alpha}\phi \ , \quad \phi^* \to e^{i\alpha}\phi^* \ , \tag{7.6}$$

where $\alpha \in \mathbb{R}$. The corresponding Noether current can be defined as

$$\begin{aligned} \mathcal{J}_\mu &\equiv \frac{\partial \mathcal{L}_M}{\partial(\partial^\mu\phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\partial \mathcal{L}_M}{\partial(\partial^\mu\phi^*)}\frac{\delta\phi^*}{\delta\alpha} \\ &= -\partial_\mu\phi^*\, i\phi + \partial_\mu\phi\, i\phi^* \\ &= -i[(\partial_\mu\phi^*)\phi - \phi^*\partial_\mu\phi] = -2\,\mathrm{Im}[\phi^*\partial_\mu\phi] \ . \end{aligned} \tag{7.7}$$

The overall sign (i.e., what we call particles and antiparticles) is a matter of convention; we could equally well have defined the global symmetry through $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$, and then $\mathcal{J}_\mu$ would have the opposite sign.

The first task, as always, is to write down a path integral expression for the partition function, Eq. (7.1). Subsequently, we may try to evaluate the partition function, in order to see what kind of phenomena take place in this system.

In order to write down the path integral, we start from the known expression of $\mathcal{Z}$ of a real scalar field ϕ_1 *without a chemical potential*, i.e. the generalization to field theory of Eq. (1.33):

$$\mathcal{Z} \propto \int_{\text{periodic}} \mathcal{D}\phi_1 \int \mathcal{D}\pi_1 \, \exp\biggl\{ -\int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} \biggl[\frac{1}{2}\pi_1^2 - i\pi_1\partial_\tau\phi_1 + \frac{1}{2}(\partial_i\phi_1)^2 + V(\phi_1) \biggr] \biggr\} \ , \tag{7.8}$$

where $\pi_1 = \partial\phi_1/\partial t$ (cf. the discussion in Sect. 2.1). Here the combination $\frac{1}{2}\pi_1^2 + \frac{1}{2}(\partial_i\phi_1)^2 + V(\phi_1)$ is nothing but the classical Hamiltonian density, $\mathcal{H}(\pi_1, \phi_1)$.

In order to make use of Eq. (7.8), let us rewrite the complex scalar field ϕ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $\phi_i \in \mathbb{R}$. Then

$$\partial^\mu\phi^*\partial_\mu\phi = \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 + \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 \ , \quad \phi^*\phi = \frac{1}{2}(\phi_1^2 + \phi_2^2) \ , \tag{7.9}$$

and the classical Hamiltonian density reads

$$\mathcal{H} = \frac{1}{2}\Bigl[\pi_1^2 + \pi_2^2 + (\partial_i\phi_1)^2 + (\partial_i\phi_2)^2 + m^2\phi_1^2 + m^2\phi_2^2\Bigr] + \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2 \ . \tag{7.10}$$

For the grand canonical ensemble, we need to add from Eqs. (7.3) and (7.7) the classical version of $-\mu\hat{Q}$ to the Hamiltonian, cf. Eq. (7.1):

$$\begin{aligned} -\mu Q &= \mu \int_{\mathbf{x}} \mathrm{Im}\Big[(\phi_1 - i\phi_2)(\partial_t\phi_1 + i\partial_t\phi_2)\Big] \\ &= \int_{\mathbf{x}} \mu(\pi_2\phi_1 - \pi_1\phi_2)\ . \end{aligned} \tag{7.11}$$

Since the charge can be expressed in terms of the canonical variables, nothing changes in the derivation of the path integral, and we can simply replace the Hamiltonian of Eq. (7.8) by the sum of Eqs. (7.10) and (7.11).

Finally, we carry out the Gaussian integrals over π_1, π_2:

$$\int \mathrm{d}\pi_1 \exp\bigg\{-\mathrm{d}^D X\left[\frac{1}{2}\pi_1^2 + \pi_1\Big(-i\frac{\partial\phi_1}{\partial\tau} - \mu\phi_2\Big)\right]\bigg\} = C \exp\bigg\{-\frac{1}{2}\mathrm{d}^D X\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2\bigg\}\ , \tag{7.12}$$

$$\int \mathrm{d}\pi_2 \exp\bigg\{-\mathrm{d}^D X\left[\frac{1}{2}\pi_2^2 + \pi_2\Big(-i\frac{\partial\phi_2}{\partial\tau} + \mu\phi_1\Big)\right]\bigg\} = C \exp\bigg\{-\frac{1}{2}\mathrm{d}^D X\left(\frac{\partial\phi_2}{\partial\tau} + i\mu\phi_1\right)^2\bigg\}\ . \tag{7.13}$$

Afterwards we go back to the complex notation, writing

$$\begin{aligned} &\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2 + \frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} + i\mu\phi_1\right)^2 \\ &\quad = \frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_2}{\partial\tau}\right)^2\right] - \mu \times \underbrace{i\left[\phi_2\frac{\partial\phi_1}{\partial\tau} - \phi_1\frac{\partial\phi_2}{\partial\tau}\right]}_{\phi\,\partial_\tau\phi^* - \phi^*\partial_\tau\phi} - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) \\ &\quad = [(\partial_\tau + \mu)\phi^*][(\partial_\tau - \mu)\phi]\ . \end{aligned} \tag{7.14}$$

In total, then, the path integral representation for the grand canonical partition function of a complex scalar field reads

$$\begin{aligned} \mathcal{Z}(T,\mu) = C \int_{\text{periodic}} \mathcal{D}\phi\, \exp\bigg\{ - \int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} \Big[& (\partial_\tau + \mu)\phi^*(\partial_\tau - \mu)\phi \\ & + \partial_i\phi^*\partial_i\phi + m^2\phi^*\phi + \lambda(\phi^*\phi)^2 \Big]\bigg\}\ . \end{aligned} \tag{7.15}$$

As anticipated, μ appears in a way reminiscent of an imaginary gauge field A_0.

Let us work out the properties of the free theory in the presence of μ. Going to momentum space with $P = (\omega_n, \mathbf{p})$, the quadratic part of the Euclidean action becomes

$$\begin{aligned} S_E^{(0)} &= \SumInt_P \tilde{\phi}^*(P)\Big[(-i\omega_n + \mu)(i\omega_n - \mu) + p^2 + m^2\Big]\tilde{\phi}(P) \\ &= \SumInt_P \tilde{\phi}^*(P)\tilde{\phi}(P)\Big[(\omega_n + i\mu)^2 + p^2 + m^2\Big] . \end{aligned} \tag{7.16}$$

We observe that the chemical potential induces a *shift of the Matsubara frequencies by a constant imaginary term* [this was the reason for considering the corresponding sum in Eq. (2.36)]. In particular, the propagator reads

$$\langle\tilde{\phi}(P)\tilde{\phi}^*(Q)\rangle_0 = \delta\!\!\!{}^-(P-Q)\frac{1}{(\omega_n + i\mu)^2 + p^2 + m^2} , \tag{7.17}$$

whereas the grand canonical free energy density (sometimes referred to as the grand potential in the literature) is obtained from Eqs. (2.44) and (2.49). We just need to replace $c \to i\mu$ and note that for a *complex* scalar field, all Fourier modes are independent, whereby the structures in Eqs. (2.44) and (2.49) are to be multiplied by a factor 2:

$$f(T,\mu) = \int_{\mathbf{p}} \bigg\{E_p + T\Big[\ln\Big(1 - e^{-\beta(E_p-\mu)}\Big) + \ln\Big(1 - e^{-\beta(E_p+\mu)}\Big)\Big]\bigg\}\bigg|_{E_p=\sqrt{p^2+m^2}} . \tag{7.18}$$

We may wonder how the existence of $\mu \neq 0$ affects the infrared problem of finite-temperature field theory, discussed in Sect. 6.1. In Sect. 2.3 we found that the high-temperature expansion ($T \gg m$) of Eq. (7.18) at $\mu = 0$ has a peculiar structure, because of a branch cut starting at $m^2 = 0$. From the second term in Eq. (7.18), we note that this problem has become *worse* in the presence of $\mu > 0$: the integrand is complex-valued if $\mu > m$, because then $\exp(-\beta(E_p - \mu)) > 1$ at small p. In an interacting theory, thermal corrections generate an effective mass $m_{\text{eff}}^2 \sim \lambda T^2$ [cf. Eq. (3.95)], which postpones the problem to a larger μ. Nevertheless, for large enough μ it still exists.

It turns out that there is a *physics consequence* from this infrared problem: the existence of *Bose-Einstein condensation*, to which we now turn.

Bose-Einstein Condensation

In order to properly treat complex scalar field theory with a chemical potential, two things need to be realized:

(i) In contrast to gauge field theory, the infrared problem exists even in the *non-interacting limit*. Therefore it cannot be cured by a perturbatively or non-perturbatively generated effective mass. Rather, it corresponds to a strong dependence of the properties of the system on the volume, so we should keep the volume finite to start with.

(ii) The chemical potential μ is a most useful quantity in theoretical computations, but it is somewhat "abstract" from a practical point of view; the physical properties of the system are typically best characterized not by μ but by the intensive variable conjugate to μ, i.e. the number density of the conserved charge. Therefore, rather than trying to give μ some specific value, we should fix the number density.

Motivated by point (i), let us put the system in a periodic box, $V = L_1 L_2 L_3$. The spatial momenta get discretized like in Eq. (2.9),

$$\mathbf{p} = 2\pi \left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3} \right) , \tag{7.19}$$

with $n_i \in \mathbb{Z}$. The mode with $\omega_n = 0, \mathbf{p} = \mathbf{0}$ will be called *the condensate*, and denoted by $\bar{\phi}$. Note that the condensate is a Matsubara zero mode but *in addition* a spatial zero mode.

We now rewrite the partition function of Eq. (7.15) as

$$\begin{aligned} \mathcal{Z}(T,\mu) &= \int_{-\infty}^{\infty} \mathrm{d}\bar{\phi} \left\{ \int_{\text{periodic}, P\neq 0} \mathcal{D}\phi' \, e^{-S_E[\phi = \bar{\phi} + \phi']} \right\} \\ &\equiv \int_{-\infty}^{\infty} \mathrm{d}\bar{\phi} \, \exp\left[-\frac{V}{T} V_{\text{eff}}(\bar{\phi}^* \bar{\phi}) \right] . \end{aligned} \tag{7.20}$$

Here ϕ' contains all modes with $P \neq 0$, and V_{eff} is called the *(constrained) effective potential*. The factor V/T is the trivial spacetime integral, $\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^d\mathbf{x}$.

Let us write down the effective potential explicitly for very weak interactions, $\lambda \approx 0$. It turns out that the limit $\lambda \to 0$ is subtle, so for the moment we keep λ non-zero in the zero-mode part. From Eqs. (7.15) and (7.16) we get

$$\begin{aligned} S_E[\phi = \bar{\phi} + \phi'] = {} & \frac{V}{T} \Big[(m^2 - \mu^2)\, \bar{\phi}^* \bar{\phi} + \lambda\, (\bar{\phi}^* \bar{\phi})^2 \Big] \\ & + \sum\!\!\!\!\!\!\!\int_{P \neq 0} \Big\{ \tilde{\phi}'^*(P) \tilde{\phi}'(P) \Big[(\omega_n + i\mu)^2 + p^2 + m^2 \Big] + \mathcal{O}(\lambda) \Big\} , \end{aligned} \tag{7.21}$$

where we made use of the fact that the crossterm between $\bar{\phi}$ and ϕ' vanishes, given that by definition ϕ' has no zero-momentum mode:

$$\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^d\mathbf{x}\, \phi' = 0 . \tag{7.22}$$

The path integral over the latter term in Eq. (7.21) yields then Eq. (7.18); in the limit of a large volume, the omission of a single mode does not matter (its effect is $\propto (T/V)\ln(m^2-\mu^2)$). Thereby the effective potential reads

$$V_{\rm eff}(\bar\phi^*\bar\phi) = (m^2-\mu^2)\,\bar\phi^*\bar\phi + \lambda\,(\bar\phi^*\bar\phi)^2 + f(T,\mu) + \mathcal{O}\Big(\frac{1}{V},\lambda\Big)\,. \tag{7.23}$$

Physically, the first two terms correspond to the contribution of the particles that have formed a condensate, whereas the third term represents propagating particle modes in the plasma.

Now, if we go to the limit of very small temperatures, $T \ll m$, and assume furthermore that $|\mu| \le m$, which is required in order for Eq. (7.18) to be defined, then the thermal part of Eq. (7.18) vanishes. (It has a "non-relativistic" limiting value for $\mu \to m^-$, which scales as $-T^4(\frac{m}{2\pi T})^{\frac{3}{2}}$.) The vacuum contribution to Eq. (7.18) is on the other hand independent of T and μ, and can be omitted. Therefore,

$$V_{\rm eff}(\bar\phi^*\bar\phi) \approx (m^2-\mu^2)\,\bar\phi^*\bar\phi + \lambda\,(\bar\phi^*\bar\phi)^2\,. \tag{7.24}$$

The remaining task is to carry out the integral over $\bar\phi$ in Eq. (7.20). At this point we need to make contact with the particle number density. From $\mathcal{Z} = \mathrm{Tr}\,[\exp(-\beta\hat H + \beta\mu\hat Q)]$ and the definition of $V_{\rm eff}$ in Eq. (7.20), we obtain

$$\rho \equiv \frac{\langle\hat Q\rangle}{V} = \frac{T}{V}\frac{\partial\ln\mathcal{Z}}{\partial\mu} \tag{7.25}$$

$$= \frac{\int \mathrm{d}\bar\phi\, 2\mu\,\bar\phi^*\bar\phi\,\exp\Big[-\frac{V}{T}V_{\rm eff}(\bar\phi^*\bar\phi)\Big]}{\int \mathrm{d}\bar\phi\,\exp\Big[-\frac{V}{T}V_{\rm eff}(\bar\phi^*\bar\phi)\Big]} \equiv 2\mu\,\langle\bar\phi^*\bar\phi\rangle\,, \tag{7.26}$$

where $\langle\bar\phi^*\bar\phi\rangle$ is the expectation value of $\bar\phi^*\bar\phi$.

Let us consider a situation where we decrease the temperature, $T \ll m$, and attempt simultaneously to keep the particle number density, the left-hand side of Eq. (7.26), fixed. How should we choose μ in this situation? There are three possibilities:

(i) If $|\mu| < m$, the integrals in Eq. (7.26) can be carried out even for $\lambda \to 0^+$. In fact the result corresponds to the "propagator" of $\bar\phi$:

$$\lim_{\lambda\to 0^+}\rho = \frac{2\mu T}{V(m^2-\mu^2)}\,. \tag{7.27}$$

We note that if $T \to 0$, then $\rho \to 0$. This conflicts with our assumption that the number density stays constant; therefore this range of μ is not physically relevant for our situation.

(ii) If $|\mu| > m$, the integrals in Eq. (7.26) are defined only for $\lambda > 0$. For $V \to \infty$ they can be determined by the *saddle point approximation*:

$$V'_{\text{eff}}(\bar{\phi}^*\bar{\phi}) = 0 \quad \Rightarrow \quad \bar{\phi}^*\bar{\phi} = \frac{\mu^2 - m^2}{2\lambda} \quad \Rightarrow \quad \rho = \frac{\mu(\mu^2 - m^2)}{\lambda} \, . \tag{7.28}$$

We see that for $\lambda \to 0^+$, we need to send $\mu \to m^+$, in order to keep ρ finite.

(iii) According to the preceding points, the only possible choice at $\lambda = 0^+$ is $|\mu| = m$. For $\rho > 0$ we need to choose $\mu = m$. In this limit Eq. (7.26) can be expressed as

$$\rho = 2m\langle \bar{\phi}^*\bar{\phi} \rangle \, , \tag{7.29}$$

which should be thought of as a condition for the field $\bar{\phi}$.

Equation (7.29) manifests the phenomenon of *Bose-Einstein condensation* (at zero temperature in the free limit): the conserved particle number is converted into a non-zero scalar condensate.

It is straightforward to include the effects of a finite temperature in these considerations, by starting from Eq. (7.23) so that $-\partial_\mu f(T, \mu)$ gives another contribution to the charge density, and the effects of interactions, by keeping $\lambda > 0$. These very interesting developments go beyond the scope of the present lectures (cf. e.g. [3–5]). On the other hand, the concepts of a condensate and an effective potential will be met again in later chapters.

7.2 Dirac Fermion with a Finite Chemical Potential

The Lagrangian of a Dirac fermion,

$$\mathcal{L}_M = \bar{\psi}_A (i\not{D}_{AB} - m\,\delta_{AB})\psi_B \, , \quad \not{D}_{AB} = \gamma^\mu(\delta_{AB}\partial_\mu - igA^a_\mu T^a_{AB}) \, , \tag{7.30}$$

possesses a *global symmetry*,

$$\psi_A \to e^{-i\alpha}\psi_A \, , \quad \bar{\psi}_A \to e^{i\alpha}\bar{\psi}_A \, , \tag{7.31}$$

in addition to the usual non-Abelian (local) gauge symmetry. Therefore there is a conserved quantity, and we can consider the behaviour of the system in the presence of a chemical potential.

The conserved Noether current reads

$$\begin{aligned} \mathcal{J}_\mu &= \frac{\partial \mathcal{L}_M}{\partial(\partial^\mu \psi_A)} \frac{\delta \psi_A}{\delta \alpha} \\ &= -\bar{\psi}_A \, i\gamma_\mu \, i\psi_A = \bar{\psi}_A \gamma_\mu \psi_A \, . \end{aligned} \tag{7.32}$$

The corresponding charge is $Q = \int_{\mathbf{x}} \mathcal{J}_0$, and as an operator it commutes with the Hamiltonian, $[\hat{H}, \hat{Q}] = 0$. Therefore, like with scalar field theory, we can treat the combination $\hat{H} - \mu\hat{Q}$ as an "effective" Hamiltonian, and directly write down the corresponding path integral, by adding

$$-\mu Q = -\mu \int_{\mathbf{x}} \bar{\psi}_A \gamma_0 \psi_A \tag{7.33}$$

to the Euclidean action. The path integral thereby reads

$$\mathcal{Z}(T,\mu) = \int_{\text{antiperiodic}} \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\;\exp\left\{-\int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} \bar{\psi}\,[\gamma_\mu D_\mu - \gamma_0\,\mu + m]\,\psi\right\}\;. \tag{7.34}$$

For perturbation theory, let us consider the quadratic part of the Euclidean action. Going to momentum space with $P = (\omega_n, \mathbf{p})$, we get

$$S_E^{(0)} = \SumInt_{\{P\}} \tilde{\bar{\psi}}(P)[i\gamma_0\,\omega_n + i\gamma_i p_i - \gamma_0\mu + m]\tilde{\psi}(P)\;. \tag{7.35}$$

Therefore, just like in Sect. 7.1, the existence of a chemical potential corresponds to a shift $\omega_n \to \omega_n + i\mu$ of the Matsubara frequencies.

Let us write down the free energy density of a single free Dirac fermion. Compared with a complex scalar field, there is an overall factor -2 (rather than -4 like in Eq. (4.52), where we compared with a real scalar field). Otherwise, the chemical potential appears in identical ways in Eqs. (7.16) and (7.35), so Eq. (4.55), $\sigma_\mathrm{f}(T) = 2\sigma_\mathrm{b}\left(\frac{T}{2}\right) - \sigma_\mathrm{b}(T)$, continues to apply. Employing it with Eq. (7.18) we get

$$\begin{aligned} f(T,\mu) &= -2\int_{\mathbf{p}} \left\{E_p + T\left[\ln\left(1 - e^{-2\beta(E_p-\mu)}\right) + \ln\left(1 - e^{-2\beta(E_p+\mu)}\right)\right.\right. \\ &\qquad\left.\left. - \ln\left(1 - e^{-\beta(E_p-\mu)}\right) - \ln\left(1 - e^{-\beta(E_p+\mu)}\right)\right]\right\} \\ &= -2\int_{\mathbf{p}} \left\{E_p + T\left[\ln\left(1 + e^{-\beta(E_p-\mu)}\right) + \ln\left(1 + e^{-\beta(E_p+\mu)}\right)\right]\right\}\;. \end{aligned} \tag{7.36}$$

The thermal part of this integral is well-defined for any μ; thus fermions do not suffer from infrared problems with $\mu \neq 0$, and do not undergo condensation (in the absence of interactions).

How About Chemical Potentials for Gauge Symmetries?

It was mentioned after Eq. (7.2) that a chemical potential has some relation to a gauge field A_0. However, in cases like QCD, a chemical potential has no colour structure (i.e. it is an identity matrix in colour space), whereas A_0 is a traceless matrix in colour space [cf. Eq. (7.30)]. On the other hand, in QED, A_0 is not traceless. In fact, in QED, the gauge symmetry is nothing but a local version of that in Eq. (7.31). We may therefore ask whether we can associate a chemical potential to the electric charge of QED, and what is the precise relation of A_0 and μ in this case.

Let us first recall what happens in such a situation physically. A non-zero chemical potential in QED corresponds to a system which is *charged*. Moreover, if we want to describe it perturbatively with the QED Lagrangian, we had better choose a system where the charge carriers (particles) are essentially free; such a system could be a metal or a plasma. In this situation, the free charge carriers interact repulsively with a long-range force, and hence all the net charge resides *on the surface*. In other words, the homogeneous "bulk" of the medium is *neutral* (i.e. has no free charge). The charged body as a whole has a non-zero electric potential, V_0, with respect to the ground.

Let us try to understand how to reproduce this behaviour directly from the partition function, Eq. (7.34), adapted to QED:

$$\mathcal{Z}(T,\mu) = \int_{\text{b.c.}} \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \, \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} \left[\frac{1}{4} F_{\mu\nu}^2 + \bar\psi \Big(\gamma_0 (\partial_\tau - ieA_0 - \mu) + \gamma_i D_i + m \Big) \psi \right] \right\} . \tag{7.37}$$

The usual boundary conditions ("b.c.") over the time direction are assumed. The basic claim is that, according to the physical picture above, if we assume the system to be *homogeneous*, i.e. consider the "bulk" situation, then the partition function *should not depend on* μ. Indeed this would ensure the neutrality that we expect:

$$\rho = -\frac{\partial f}{\partial \mu} = 0 \, . \tag{7.38}$$

How does this arise?

The key observation is that we should again think of the system in terms of an *effective potential*, like in Eq. (7.20). The role of the condensate is now given to the field A_0; let us denote it by $\bar A_0$. The last integral to be carried out is

$$\mathcal{Z}(T,\mu) = \int_{-\infty}^{\infty} \mathrm{d}\bar A_0 \, \exp\left\{ -\frac{V}{T} V_{\text{eff}}(\bar A_0) \right\} . \tag{7.39}$$

Now, we can deduce from Eq. (7.37) that μ can only appear in the combination $-ie\bar A_0 - \mu$, so that $V_{\text{eff}}(\bar A_0) = f(\bar A_0 - i\mu/e)$. Moreover, we know from Eq. (6.36) that in a large volume and high temperature,

$$V_{\text{eff}}(\bar{A}_0) \approx \frac{1}{2} m_{\text{E}}^2 (\bar{A}_0 - i\mu/e)^2 + \mathcal{O}(\bar{A}_0 - i\mu/e)^4 \;, \tag{7.40}$$

where $m_{\text{E}}^2 \sim e^2T^2$. (The complete 1-loop V_{eff} could be deduced from Eq. (7.42) below, simply by substituting $\mu \to \mu + ie\bar{A}_0$ there.) In the infinite-volume limit, the integral in Eq. (7.39) can be carried out by making use of the *saddle point approximation*, like with Bose-Einstein condensation in Eq. (7.28). The saddle point is located in the complex plane at the position where $V'_{\text{eff}}(\bar{A}_0) = 0$, i.e. at $\bar{A}_0 = i\mu/e$. The value of the potential at the saddle point, as well as the second derivative and so also the Gaussian integral around it, are clearly independent of μ. This leads to Eq. (7.38).

It is interesting to note that the saddle point lies at a purely imaginary $\bar{A}_0$. Recalling the relation of Minkowskian and Euclidean A_0 from page 88, this corresponds to a real Minkowskian A_0. Thus there indeed is a real electric potential $V_0 \propto \mu$, just as we anticipated on physical grounds.

Appendix 1: Exact Results in the Free Massless Limit

The free energy density of a single Dirac fermion,

$$f(T,\mu) = -2 \sumint_{\{P\}} \Big\{ \ln[(\omega_n + i\mu)^2 + E_p^2] - \text{const.} \Big\} \;, \tag{7.41}$$

can be computed explicitly for the case $m = 0$ (i.e. $E_p = p$). We show that, subtracting the vacuum part, the result is

$$f(T,\mu) = -\left(\frac{7\pi^2 T^4}{180} + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2} \right) . \tag{7.42}$$

We start from Eq. (7.36), subtracting the vacuum term and setting $m = 0, d = 3$:

$$\begin{aligned} f(T,\mu) &= -2T \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left\{ \ln\left[1 + \exp\left(-\frac{p-\mu}{T}\right)\right] + \ln\left[1 + \exp\left(-\frac{p+\mu}{T}\right)\right] \right\} \\ &= -\frac{T^4}{\pi^2} \int_0^\infty \mathrm{d}x\, x^2 \left\{ \ln\left(1 + e^{-x+y}\right) + \ln\left(1 + e^{-x-y}\right) \right\} \;, \end{aligned} \tag{7.43}$$

where we set $x \equiv p/T$ and $y \equiv \mu/T$, and carried out the angular integration.

A possible trick now is to expand the logarithms in Taylor series,

$$\ln(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}\,, \quad |z| < 1\,. \tag{7.44}$$

Assuming $y > 0$, this is indeed possible with the second term of Eq. (7.43), whereas in the first term a direct application is not possible, because the series does not converge for all x. However, if $e^{-x+y} > 1$, we can write $1+e^{-x+y} = e^{-x+y}(1+e^{x-y})$, where $e^{x-y} < 1$. Thereby the Taylor expansion can be written as

$$\ln\!\left(1+e^{-x+y}\right) = \theta(x-y)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-xn}e^{yn} + \theta(y-x)\Big[y-x+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{xn}e^{-yn}\Big]\,. \tag{7.45}$$

Inserting this into Eq. (7.43), we get

$$\begin{aligned} f(T,\mu) &= -\frac{T^4}{\pi^2}\Bigg\{\int_0^y \mathrm{d}x\Bigg[yx^2-x^3+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^2\Big(e^{xn}e^{-yn}+e^{-xn}e^{-yn}\Big)\Bigg] \\ &\qquad + \int_y^{\infty}\mathrm{d}x\Bigg[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^2\Big(e^{-xn}e^{yn}+e^{-xn}e^{-yn}\Big)\Bigg]\Bigg\} \\ &= -\frac{T^4}{\pi^2}\Bigg\{\int_0^y \mathrm{d}x\Bigg[yx^2-x^3+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^2\Big(e^{xn}e^{-yn}-e^{-xn}e^{yn}\Big)\Bigg] \\ &\qquad + \int_0^{\infty}\mathrm{d}x\Bigg[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^2\Big(e^{-xn}e^{yn}+e^{-xn}e^{-yn}\Big)\Bigg]\Bigg\}\,. \end{aligned} \tag{7.46}$$

All the x-integrals can be carried out:

$$\int_0^y \mathrm{d}x\,(yx^2-x^3) = \left(\frac{1}{3}-\frac{1}{4}\right)y^4 = \frac{1}{12}y^4\,, \tag{7.47}$$

$$\int_0^y \mathrm{d}x\,x^2e^{\alpha x} = -\frac{2}{\alpha^3}+e^{\alpha y}\left(\frac{2}{\alpha^3}-\frac{2y}{\alpha^2}+\frac{y^2}{\alpha}\right)\,, \tag{7.48}$$

$$\int_0^{\infty}\mathrm{d}x\,x^2e^{-xn} = \frac{2}{n^3}\,. \tag{7.49}$$

Inserting these into Eq. (7.46) we get

$$f(T,\mu) = -\frac{T^4}{\pi^2}\Bigg\{\frac{y^4}{12}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Bigg[e^{-yn}\Bigg(-\frac{2}{n^3}+e^{yn}\Bigg(\frac{2}{n^3}-\frac{2y}{n^2}+\frac{y^2}{n}\Bigg)\Bigg)$$

$$-e^{yn}\left(\frac{2}{n^3}+e^{-yn}\left(-\frac{2}{n^3}-\frac{2y}{n^2}-\frac{y^2}{n}\right)\right)+e^{yn}\frac{2}{n^3}+e^{-yn}\frac{2}{n^3}\Bigg]\Bigg\}$$

$$= -\frac{T^4}{\pi^2}\left\{\frac{y^4}{12}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\frac{4}{n^3}+\frac{2y^2}{n}\right]\right\}, \tag{7.50}$$

where a remarkable cancellation took place. The sums can be carried out:

$$\eta(2) \equiv \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \zeta(2)-\frac{2}{2^2}\zeta(2)$$

$$= \frac{1}{2}\zeta(2) = \frac{\pi^2}{12}, \tag{7.51}$$

$$\eta(4) \equiv \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots = \zeta(4)-\frac{2}{2^4}\zeta(4)$$

$$= \frac{7}{8}\zeta(4) = \frac{7}{8}\frac{\pi^4}{90}. \tag{7.52}$$

Inserting into Eq. (7.50), we end up with

$$f(T,\mu) = -\frac{T^4}{\pi^2}\left\{\frac{y^4}{12}+\frac{\pi^2y^2}{6}+\frac{7\pi^4}{180}\right\}, \tag{7.53}$$

which after the substitution $y = \mu/T$ reproduces Eq. (7.42).

Appendix 2: Free Susceptibilities

Important characteristics of dense systems are offered by *susceptibilities*, which define fluctuations of the particle number in a grand canonical ensemble. For a Dirac fermion,

$$\chi_{\rm f} \equiv \lim_{V\to\infty}\frac{\langle\hat{N}^2\rangle-\langle\hat{N}\rangle^2}{V} = \lim_{V\to\infty}T\partial_\mu\left(\frac{\langle\hat{N}\rangle}{V}\right) = \lim_{V\to\infty}\frac{T^2\partial_\mu^2\ln\mathcal{Z}}{V} = -T\partial_\mu^2 f(T,\mu) \tag{7.54}$$

$$\overset{(7.36)}{=} 2T\int_{\mathbf{p}}\partial_\mu^2\Big\{E_p+T\Big[\ln\Big(1+e^{-\beta(E_p-\mu)}\Big)+\ln\Big(1+e^{-\beta(E_p+\mu)}\Big)\Big]\Big\}$$

$$= 2T\int_{\mathbf{p}}\partial_\mu\left\{\frac{1}{e^{\beta(E_p-\mu)}+1}-\frac{1}{e^{\beta(E_p+\mu)}+1}\right\}$$

$$= 2\int_{\mathbf{p}}\left\{\frac{e^{\beta(E_p-\mu)}}{[e^{\beta(E_p-\mu)}+1]^2}+\frac{e^{\beta(E_p+\mu)}}{[e^{\beta(E_p+\mu)}+1]^2}\right\}$$

$$= \frac{1}{\pi^2}\int_0^\infty \mathrm{d}p\, p^2 \Big\{ n_{\mathrm{F}}(E_p-\mu)\big[1-n_{\mathrm{F}}(E_p-\mu)\big] + n_{\mathrm{F}}(E_p+\mu)\big[1-n_{\mathrm{F}}(E_p+\mu)\big] \Big\} \,. \qquad (7.55)$$

In the massless limit, Eq. (7.42) directly gives $\chi_{\mathrm{f}} = \frac{T^3}{3} + \frac{\mu^2 T}{\pi^2}$. On the other hand, for $m \neq 0$, $\mu = 0$, one gets $\chi_{\mathrm{f}} = \frac{2m^2T}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}K_2\big(\frac{nm}{T}\big)$, where K_2 is a modified Bessel function. In the bosonic case of a complex scalar field, Eq. (7.18) similarly leads to

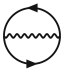
$$\chi_{\mathrm{b}} = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}p\, p^2 \Big\{ n_{\mathrm{B}}(E_p-\mu)\big[1+n_{\mathrm{B}}(E_p-\mu)\big] + n_{\mathrm{B}}(E_p+\mu)\big[1+n_{\mathrm{B}}(E_p+\mu)\big] \Big\} \,. \qquad (7.56)$$

In this case the massless limit is only relevant at $\mu = 0$ (otherwise the integrand is singular at $p = |\mu|$), where we obtain $\chi_{\mathrm{b}} = T^3/3$. For $m \neq 0$, the susceptibility at $\mu = 0$ can again be expressed in terms of modified Bessel functions, $\chi_{\mathrm{b}} = \frac{m^2 T}{\pi^2}\sum_{n=1}^{\infty} K_2\big(\frac{nm}{T}\big)$.

Appendix 3: Finite Density QCD at Next-to-Leading Order

Extending the description of finite density systems to higher perturbative orders has become an actively studied topic particularly within QCD. An example is the susceptibility defined in Eq. (7.54), evaluated at $\mu = 0$, and the generalization thereof to the case of several quark flavours. These quantities probe finite density, but can nevertheless be compared with lattice QCD simulations that are well under control only at vanishing chemical potentials. The basic strategy in their evaluation follows the above leading-order computation in the sense that it is technically easier to first compute the entire free energy density at finite μ, and only afterwards to take derivatives with respect to μ [6]. As a by-product of evaluating susceptibilities at $\mu = 0$, we therefore obtain the behaviour of the pressure at finite density.

At 2-loop order, the μ-dependent part of the QCD free energy density gets contributions from one single diagram, namely the same as in Eq. (5.116). Like in Eq. (5.116) it is easy to see that in the limit of massless quarks (an approximation that significantly simplifies higher-order computations) this diagram can be written in the form



$$= d_A g^2\, \frac{d-1}{2} \sum\!\!\!\!\!\!\!\int_{\{P\}Q} \left[\frac{1}{\tilde{P}^2(\tilde{P}-Q)^2} - \frac{2}{\tilde{P}^2 Q^2} \right] , \qquad (7.57)$$

where $d_A \equiv N_{\mathrm{c}}^2 - 1$, we set $N_{\mathrm{f}} = 1$ and, in accordance with Eq. (7.35), the fermionic Matsubara frequencies have been shifted by $\omega_n \to \omega_n + i\mu \equiv \tilde{p}_n$. Both terms in this result clearly factorize into products of 1-loop sum-integrals that (up to the shift of the fermionic Matsubara frequencies) can be identified as the functions $I(0,T) = I_T(0)$ and $\tilde{I}(0,T) = \tilde{I}_T(0)$ studied in Sects. 2.3 and 4.2, respectively.

For completeness, let us next inspect the fermionic sum-integral

$$\tilde{I}(m=0,T,\mu,\alpha) \equiv \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{(\tilde{P}^2)^{\alpha}}, \tag{7.58}$$

following a strategy similar to that in Eq. (2.90). In other words we first perform the $3-2\epsilon$ -dimensional integral over the spatial momentum $\mathbf{p}$, and afterwards take care of the Matsubara sum. Applying the familiar result of Eq. (2.64), we obtain

$$\begin{aligned}\tilde{I}(m=0,T,\mu,\alpha) &= \frac{1}{(4\pi)^{3/2-\epsilon}}\frac{\Gamma(\alpha-3/2+\epsilon)}{\Gamma(\alpha)}\,T\sum_{k=-\infty}^{\infty}\frac{1}{\left[\left((2k+1)\,\pi T+i\mu\right)^2\right]^{\alpha-3/2+\epsilon}}\\ &= 2^{-2\alpha}\pi^{-2\alpha+3/2-\epsilon}T^{-2\alpha+4-2\epsilon}\,\frac{\Gamma(\alpha-3/2+\epsilon)}{\Gamma(\alpha)}\\ &\quad\times\left[\zeta\Big(2\alpha-3+2\epsilon,\frac{1}{2}-i\bar{\mu}\Big)+\zeta\Big(2\alpha-3+2\epsilon,\frac{1}{2}+i\bar{\mu}\Big)\right],\end{aligned} \tag{7.59}$$

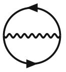
where $\bar{\mu} \equiv \mu/(2\pi T)$ and we have expressed the infinite sums in terms of the generalized (Hurwitz) zeta-function

$$\zeta(z,q) \equiv \sum_{n=0}^{\infty}\frac{1}{(q+n)^z}\,. \tag{7.60}$$

Specializing now to $\alpha = 1$ and dropping terms of $\mathcal{O}(\epsilon)$, we easily get

$$\tilde{I}(m=0,T,\mu,1) = -\frac{T^2}{24}-\frac{\mu^2}{8\pi^2}+\mathcal{O}(\epsilon)\,. \tag{7.61}$$

Plugging this and $I_T(0) = T^2/12$ into Eq. (7.57) produces



$$= \frac{d_A g^2 T^4}{576}\left[5+\frac{18\mu^2}{(\pi T)^2}+\frac{9\mu^4}{(\pi T)^4}\right]+\mathcal{O}(\epsilon)\,. \tag{7.62}$$

From here the next-to-leading order contribution to the quark number susceptibility can be extracted according to Eq. (7.54),

$$\chi_{\mathrm{f}}|_{\mu=0} = T^3\left(\frac{N_{\mathrm{c}}}{3}-\frac{d_A g^2}{16\pi^2}\right), \tag{7.63}$$

where we have added the appropriate colour factor to the leading-order term.

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