Chapter 6
Low-Energy Effective Field Theories

Abstract The existence of a so-called infrared (IR) problem in relativistic thermal field theory is pointed out, both from a physical and a formal (imaginary-time) point of view. The notion of effective field theories is introduced, and the main issues related to their construction and use are illustrated with the help of a simple example. Subsequently this methodology is applied to the imaginary-time path integral representation for the partition function of non-Abelian gauge field theory. This leads to the construction of a dimensionally reduced effective field theory for capturing certain (so-called "static", i.e. time-independent) properties of QCD (or more generally Standard Model) thermodynamics in the high-temperature limit.

Keywords Bose enhancement • Effective theories • Electrostatic QCD • Hard and soft modes • Infrared divergences • Linde problem • Magnetostatic QCD • Matching • Matsubara zero mode • Power counting • Symmetries • Truncation

6.1 The Infrared Problem of Thermal Field Theory

Let us start by considering the types of integrals that appear in thermal perturbation theory. According to Eqs. (2.34) and (4.59), each new loop order (corresponding to an additional loop momentum) produces one of

$$\sum\!\!\!\!\!\!\int_P f(\omega_n, \mathbf{p}) = \int_{\mathbf{p}} \left\{ \frac{1}{2} \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{\mathrm{d}\omega}{2\pi} \, [f(\omega,\mathbf{p}) + f(-\omega,\mathbf{p})][1 + 2n_{\mathrm{B}}(i\omega)] \right\} , \quad (6.1)$$

$$\sum\!\!\!\!\!\!\int_{\{P\}} f(\omega_n, \mathbf{p}) = \int_{\mathbf{p}} \left\{ \frac{1}{2} \int_{-\infty-i0^+}^{+\infty-i0^+} \frac{\mathrm{d}\omega}{2\pi} \, [f(\omega,\mathbf{p}) + f(-\omega,\mathbf{p})][1 - 2n_{\mathrm{F}}(i\omega)] \right\} , \quad (6.2)$$

depending on whether the new line is bosonic or fermionic. The functions f here contain propagators and additional structures emerging from vertices; in the simplest case, $f(\omega, \mathbf{p}) \sim 1/(\omega^2 + E_p^2)$, where we denote $E_p \equiv \sqrt{p^2 + m^2}$.

Now, the structures which are the most important, or yield the largest contributions, are those where the functions f are largest. Let us inspect this question in terms of the left and right-hand sides of Eqs. (6.1) and (6.2).

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For bosons, the largest contribution on the left-hand side of Eq. (6.1) is clearly associated with the *Matsubara zero mode*, $\omega_n = 0$; in the case $f(\omega, \mathbf{p}) \sim 1/(\omega^2 + E_p^2)$, this gives simply

$$\left\{ T \sum_{\omega_n} f \right\}\bigg|_{\omega_n = 0} \sim \frac{T}{E_p^2} \,. \tag{6.3}$$

On the right-hand side, we on the other hand close the contour in the lower half-plane, whereby the largest contribution is associated with *Bose enhancement* around the pole $\omega = -iE_p$:

$$\begin{aligned} \{\ldots\} &\sim \frac{1}{2}\frac{-2\pi i}{2\pi}\frac{2}{-2iE_p}\Big[1 + 2n_{\mathrm{B}}(E_p)\Big] = \frac{1}{E_p}\left(\frac{1}{2} + \frac{1}{e^{E_p/T} - 1}\right) \\ &\approx \frac{1}{E_p}\left(\frac{1}{2} + \frac{1}{E_p/T + E_p^2/2T^2} + \ldots\right) = \frac{T}{E_p^2} + \mathcal{O}\Big(\frac{1}{T}\Big) \,. \end{aligned} \tag{6.4}$$

On the second row, we performed an expansion in powers of E_p/T, which is valid in the limit of high temperatures.

For fermions, there is no Matsubara zero mode on the left-hand side of Eq. (6.2), so that the largest terms have at most (i.e. for $E_p \ll \pi T$) the magnitude

$$\left\{ T \sum_{\{\omega_n\}} f \right\}\bigg|_{\omega_n = \pm \pi T} \sim \frac{T}{(\pi T)^2} \sim \frac{1}{\pi^2 T} \,. \tag{6.5}$$

Similarly, in terms of the right-hand side of Eq. (6.2), we can estimate

$$\begin{aligned} \{\ldots\} &\sim \frac{1}{2}\frac{-2\pi i}{2\pi}\frac{2}{-2iE_p}\Big[1 - 2n_{\mathrm{F}}(E_p)\Big] = \frac{1}{E_p}\left(\frac{1}{2} - \frac{1}{e^{E_p/T} + 1}\right) \\ &\approx \frac{1}{E_p}\left(\frac{1}{2} - \frac{1}{2 + E_p/T} + \ldots\right) = \mathcal{O}\Big(\frac{1}{T}\Big) \,. \end{aligned} \tag{6.6}$$

Given the estimates above, let us construct a dimensionless expansion parameter associated with the loop expansion. Apart from an additional propagator, each loop order also brings in an additional vertex or vertices; we denote the corresponding coupling by g^2, as would be the case in gauge theory. Moreover, the Matsubara summation involves a factor T, so we can assume that the expansion parameter contains the combination $g^2 T$. We now have to use the other scales in the problem to transform this into a dimensionless number. For the Matsubara zero modes, Eq. (6.3) tells us that we are allowed to use inverse powers of E_p or, after integration over the spatial momenta, inverse powers of m. Therefore, we can assume that for large temperatures, $\pi T \gg m$, the largest possible expansion parameter is

$$\epsilon_{\rm b} \sim \frac{g^2 T}{\pi m} \,. \tag{6.7}$$

For fermions, in contrast, Eq. (6.5) suggests that inverse powers of E_p or, after integration over spatial momenta, m, *cannot appear* in the denominator, even if $m \ll \pi T$; we are thus led to the estimate

$$\epsilon_{\rm f} \sim \frac{g^2 T}{\pi^2 T} \sim \frac{g^2}{\pi^2} \,. \tag{6.8}$$

In these estimates most numerical factors have been omitted for simplicity.

Assuming that we work in the weak-coupling limit, $g^2 \ll \pi^2$, we can thus conclude the following:

- *Fermions* appear to be purely *perturbative* in these computations concerning "static" observables, with the corresponding weak-coupling expansion proceeding in powers of g^2/π^2.
- Bosonic *Matsubara zero modes* appear to suffer from bad convergence in the limit $m \to 0$.
- The *resummations* that we saw around Eq. (3.94) for scalar field theory and in Sect. 5.3 for QCD produce an effective thermal mass, $m^2_{\rm eff} \sim g^2T^2$. Thus, we may expect the expansion parameter in Eq. (6.7) to become $\sim g^2T/(\pi g T) = g/\pi$. In other words, a small expansion parameter exists in principle if $g \ll \pi$, but the structure of the weak-coupling series is peculiar, with odd powers of g appearing.
- As we found in Eq. (5.101), colour-magnetic fields do not develop a thermal mass squared at $\mathcal{O}(g^2T^2)$. This might still happen at higher orders, so we can state that $m_{\rm eff} \lesssim g^2T/\pi$ for these modes. Thereby the expansion parameter in Eq. (6.7) reads $\epsilon_{\rm b} \gtrsim g^2T/g^2T = 1$. In other words, *colour-magnetic fields cannot be treated perturbatively*; this is known as the *infrared problem* (or "Linde problem") of thermal gauge theory [1].

The situation that we have encountered, namely that infrared problems exist but that they are related to particular degrees of freedom, is common in (quantum) field theory. Correspondingly, there is also a generic tool, called the *effective field theory* approach, which allows us to isolate the infrared problems into a simple Lagrangian, and treat them in this setting. The concept of effective field theories is not restricted to finite-temperature physics, but applies also at zero temperature, if the system possesses a *scale hierarchy*. In fact, the high-temperature case can be considered a special case of this, with the corresponding hierarchy often expressed as $g^2T/\pi \ll gT \ll \pi T$, where the first scale refers to the non-perturbative one associated with colour-magnetic fields. Given the generic nature of effective field theories, we first discuss the basic idea in a zero-temperature setting, before moving on to finite-temperature physics.

A Simple Example of an Effective Field Theory

Let us consider a Lagrangian containing two different scalar fields, ϕ and H, with masses m and M, respectively[1]:

$$L_{\rm full} \equiv \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu H\partial_\mu H + \frac{1}{2}M^2H^2 + g^2\phi^2H^2 + \frac{1}{4}\lambda\phi^4 + \frac{1}{4}\kappa H^4 \,. \tag{6.9}$$

We assume that there exists a hierarchy $m \ll M$ or, to be more precise, $m_{\rm R} \ll M_{\rm R}$, though we leave out the subscripts in the following. Our goal is to study to what extent the physics described by this theory can be captured by a simpler effective theory of the form

$$L_{\rm eff} = \frac{1}{2}\partial_\mu\bar\phi\,\partial_\mu\bar\phi + \frac{1}{2}\bar m^2\bar\phi^2 + \frac{1}{4}\bar\lambda\bar\phi^4 + \ldots \,, \tag{6.10}$$

where infinitely many higher-dimensional operators have been dropped.[2]

The main statement concerning the effective description goes as follows. Let us assume that $m \lesssim gM$ and that all couplings are parametrically of similar magnitude, $\lambda \sim \kappa \sim g^2$, and proceed to consider external momenta $P \lesssim gM$. Then the one-particle-irreducible Green's functions $\bar\Gamma_n$, computed within the effective theory, reproduce those of the full theory, Γ_n, with a relative error

$$\frac{\delta\bar\Gamma_n}{\bar\Gamma_n} \equiv \frac{|\bar\Gamma_n - \Gamma_n|}{\bar\Gamma_n} \lesssim \mathcal{O}(g^k) \,, \quad k > 0 \,, \tag{6.11}$$

if the parameters $\bar m^2$ and $\bar\lambda$ of Eq. (6.10) are tuned suitably. The number k may depend on the dimensionality of spacetime as well as on n, although a universal lower bound should exist. This lower bound can furthermore be increased by adding suitable higher-dimensional operators to $L_{\rm eff}$; in the limit of infinitely many such operators the effective description should become exact.

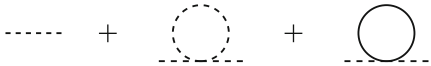
A weaker form of the effective theory statement, although already sufficiently strong for practical purposes, is that Green's functions are matched only "on-shell", rather than for arbitrary external momenta. This form of the statement is implemented, for instance, in the so-called non-perturbative Symanzik improvement program of lattice QCD [3] (for a nice review, see [4]).

It has been fittingly said that the effective theory assertion is almost trivial yet very difficult to prove. We will not attempt a formal proof here, but rather try to get an impression on how it arises, by inspecting with some care the 2-point Green's function of the light field ϕ. In the full theory, at 1-loop level, the inverse of this

[1] The discussion follows closely that in [2].

[2] If we also wanted to describe gravity with these theories, we could add a "fundamental" cosmological constant Λ in $L_{\rm full}$, and an "effective" cosmological constant $\bar\Lambda$ in $L_{\rm eff}$.

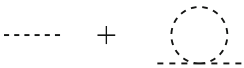
("amputated") quantity reads

$G^{-1} =$ 

$$= P^2 + m^2 + \Pi_l^{(1)}(0; m^2) + \Pi_h^{(1)}(0; M^2) \,, \tag{6.12}$$

where the dashed line represents the light field and the solid one the heavy field, while the subscripts l, h stand for light and heavy, respectively. The first argument of the functions $\Pi_l^{(1)}$, $\Pi_h^{(1)}$ is the external momentum; as the notation indicates, closed bubbles contain no dependence on it.

Within the effective theory, the same computation yields

$\bar{G}^{-1} =$ 

$$= P^2 + \bar{m}^2 + \bar{\Pi}_l^{(1)}(0; \bar{m}^2) \,. \tag{6.13}$$

The equivalence of all Green's functions at the on-shell point should imply the equivalence of pole masses, i.e. the locations of the on-shell points. By *matching* Eqs. (6.12) and (6.13), we see that this can indeed be achieved provided that

$$\bar{m}^2 = m^2 + \Pi_h^{(1)}(0; M^2) + \mathcal{O}(g^4) \,. \tag{6.14}$$

Note that within perturbation theory the matching is carried out "order-by-order": $\bar{\Pi}_l^{(1)}(0; \bar{m}^2)$ is already of 1-loop order, so inside it $\bar{\lambda}$ and $\bar{m}^2$ can be replaced by λ and m^2, respectively, given that the difference between $\bar{\lambda}$ and λ as well as $\bar{m}^2$ and m^2 is itself of 1-loop order.

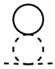
The situation becomes considerably more complicated once we go to the 2-loop level. To this end, let us analyze various types of graphs that exist in the full theory, and try to understand how they could be matched onto the simpler contributions within the effective theory.

First of all, there are graphs involving only light fields,

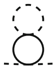
 . (6.15)

These can directly be matched with the corresponding graphs within the effective theory; as above, the fact that different parameters appear in the propagators (and vertices) is a higher-order effect.

Second, there are graphs which account for the "insignificant higher-order effects" that we omitted in the 1-loop matching, but that would play a role once we go to the 2-loop level:



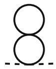
$$\Leftrightarrow\ (\bar{m}^2 - m^2)\,\frac{\partial \Pi_l^{(1)}(0;m^2)}{\partial m^2}\ , \tag{6.16}$$



$$\Leftrightarrow\ (\bar{\lambda} - \lambda)\,\frac{\partial \Pi_l^{(1)}(0;m^2)}{\partial \lambda}\ . \tag{6.17}$$

As indicated here, these two combine to reproduce (a part of) the 1-loop effective theory expression $\bar{\Pi}_l^{(1)}(0;\bar{m}^2)$ with 2-loop full theory accuracy.

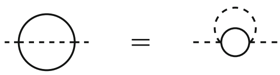
Third, there are graphs only involving heavy fields in the loops:



$$. \tag{6.18}$$

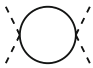
Obviously we can account for their effects by a 2-loop correction to $\bar{m}^2$.

Finally, there remain the most complicated graphs: structures involving both heavy and light fields, in a way that the momenta flowing through the two sets of lines do *not* get factorized:



$$. \tag{6.19}$$

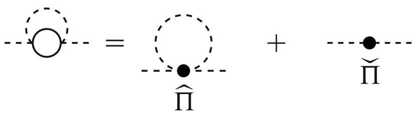
Naively, the representation on the right-hand side might suggest that this graph is simply part of the correction $(\bar{\lambda} - \lambda)\partial \Pi_l^{(1)}(0;m^2)/\partial\lambda$, just like the graph in Eq. (6.17). This, however, is *not* the case, because the substructure appearing,



$$, \tag{6.20}$$

is momentum-dependent, unlike the effective vertex $\bar{\lambda}$.

Nevertheless, it should be possible to split Eq. (6.19) into two parts, pictorially represented by



$$\tag{6.21}$$

$$\Leftrightarrow\quad \Pi^{(2)}_{mixed}(P^2;m^2,M^2) = \widehat{\Pi}^{(2)}_{mixed}(P^2;m^2,M^2) + \breve{\Pi}^{(2)}_{mixed}(P^2;m^2,M^2)\ . \tag{6.22}$$

The first part $\widehat{\Pi}^{(2)}$ is, *by definition*, characterized by the fact that it depends *non-analytically* on the mass parameter m^2 of the light field; therefore the internal ϕ field is *soft* in this part, i.e. gets a contribution from momenta $Q \sim m$. In this situation, the momentum dependence of Eq. (6.20) is of subleading importance. In other words, this part of the graph *does* contribute simply to $(\bar{\lambda} - \lambda)\partial\Pi_l^{(1)}(0; m^2)/\partial\lambda$, as we naively expected.

The second part $\breve{\Pi}^{(2)}$ is, by definition, analytic in the mass parameter m^2. We associate this with a situation where the internal ϕ is *hard*: even though its mass is small, it can have a large internal momentum $Q \sim M$, transmitted to it through interactions with the heavy modes. In this situation, the momentum dependence of Eq. (6.20) plays an essential role. At the same time, the fact that all internal momenta are hard, permits for a Taylor expansion in the small external momentum:

$$\breve{\Pi}^{(2)}_{mixed}(P^2; m^2, M^2) = \breve{\Pi}^{(2)}_{mixed}(0; m^2, M^2) + P^2 \frac{\partial}{\partial P^2} \breve{\Pi}^{(2)}_{mixed}(0; m^2, M^2)$$
$$+ \frac{1}{2}(P^2)^2 \frac{\partial^2}{\partial (P^2)^2} \breve{\Pi}^{(2)}_{mixed}(0; m^2, M^2) + \ldots \; . \tag{6.23}$$

The first term here represents a 2-loop correction to $\bar{m}^2$, just like the graph in Eq. (6.18), whereas the second term can be compensated for by a change of the normalization of the field $\bar{\phi}$. Finally, the further terms have the appearance of higher-order (derivative) operators, truncated from the structure shown explicitly in Eq. (6.10). Comparing with the leading kinetic term, the magnitude of the third term is very small,

$$\frac{g^4 \frac{(P^2)^2}{M^2}}{P^2} \lesssim g^6 \, , \tag{6.24}$$

for $P \lesssim gM$, justifying the truncation of the effective action up to a certain relative accuracy. The structures in Eq. (6.23) are collectively denoted by the 2-point "blob" in Eq. (6.21).

To summarize, we see that the explicit construction of an effective field theory becomes subtle at higher loop orders. Another illuminating example of the difficulties met with "mixed graphs" is given around Eq. (6.45) below. Nevertheless, we may formulate the following practical recipe for the effective field theory description of a Euclidean theory with a scale hierarchy:

(1) Identify the "light" or "soft" degrees of freedom, i.e. the ones that are *IR-sensitive*.
(2) Write down the most general Lagrangian for them, respecting all the *symmetries* of the system, and including local operators of arbitrary order.
(3) The parameters of this Lagrangian can be determined by *matching*:
   - Compute the same observable in the full and effective theories, applying the same UV-regularization and IR-cutoff.
   - Subtract the results.
   - The IR-cutoff should now disappear, and the result of the subtraction be analytic in P^2. This allows for a matching of the parameters and field normalizations of the effective theory.

- If the IR-cutoff does not disappear, the degrees of freedom, or the form of the effective theory, have not been correctly identified.

(4) *Truncate* the effective theory by dropping higher-dimensional operators suppressed by $1/M^k$, which can only give a relative contribution of order

$$\sim \left(\frac{m}{M}\right)^k \sim g^k \,, \tag{6.25}$$

where the dimensionless coefficient g parametrizes the scale hierarchy.

6.2 Dimensionally Reduced Effective Field Theory for Hot QCD

We now apply the effective theory recipe to the problem outlined at the beginning of Sect. 6.1, i.e. accounting for the soft contributions to the free energy density of thermal QCD. In this process, we follow the numbering introduced at the end of Sect. 6.1.

(1) Identification of the soft degrees of freedom. As discussed earlier, the soft degrees of freedom in perturbative Euclidean thermal field theory are the bosonic Matsubara zero modes. Since they do not depend on the coordinate τ, they live in $d = 3 - 2\epsilon$ spatial dimensions; for this reason, the construction of the effective theory is in this context called *high-temperature dimensional reduction* [5, 6]. For simplicity, we concentrate on the dimensional reduction of QCD in the present section, but within perturbation theory the same procedure can also be (and indeed has been) applied to the full Standard Model [7], as well as many extensions thereof.

(2) Symmetries. Since the heat bath breaks Lorentz invariance, the time direction and the space directions are not interchangeable. Therefore, the spacetime symmetries of the effective theory are merely invariances in *spatial rotations and translations*.

In addition, the full theory possesses a number of *discrete symmetries*: QCD is invariant in C, P and T separately. The effective theory inherits these symmetries, and it turns out that L_{eff} is symmetric in $\bar{A}_0 \to -\bar{A}_0$, where the low-energy fields are denoted by $\bar{A}_\mu$ (the symmetry $\bar{A}_0 \to -\bar{A}_0$ is absent if the C symmetry of QCD is broken by coupling the quarks to a chemical potential).

Finally, consider the *gauge symmetry* from Eq. (5.5):

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \,. \tag{6.26}$$

Since we now restrict to *static* (i.e. τ-independent) fields, U should not depend on τ, either, and the effective theory should be invariant under

$$\bar{A}'_i = U\bar{A}_i U^{-1} + \frac{i}{g} U\partial_i U^{-1} \;, \tag{6.27}$$

$$\bar{A}'_0 = U\bar{A}_0 U^{-1} \;. \tag{6.28}$$

In other words, the spatial components $\bar{A}_i$ remain gauge fields, whereas the temporal component $\bar{A}_0$ has turned into a *scalar field in the adjoint representation* (cf. Eq. (5.9)).

With these ingredients, we can postulate the general form of the effective Lagrangian. It is illuminating to start by simply writing down the contribution of the soft degrees of freedom to the full Yang-Mills Lagrangian, Eq. (5.34). Noting from Eq. (5.32), *viz.*

$$F^a_{0i} \equiv \partial_\tau A^a_i - \mathcal{D}^{ab}_i A^b_0 \;, \tag{6.29}$$

that in the static case $F^a_{i0} = \mathcal{D}^{ab}_i A^b_0$, we end up with

$$L_E = \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} (\mathcal{D}^{ab}_i A^b_0)(\mathcal{D}^{ac}_i A^c_0) \;. \tag{6.30}$$

At this point, it is convenient to note that

$$T^a \mathcal{D}^{ab}_i A^b_0 = \partial_i A_0 + g f^{acb} T^a A^c_i A^b_0 = \partial_i A_0 - ig[A_i, A_0] = [D_i, A_0] \;, \tag{6.31}$$

where $D_i = \partial_i - igA_i$ is the covariant derivative in the fundamental representation. Thereby we obtain as the "tree-level" terms of our effective theory the structure

$$L^{(0)}_{\text{eff}} = \frac{1}{4} \bar{F}^a_{ij} \bar{F}^a_{ij} + \text{Tr}\,\{[\bar{D}_i, \bar{A}_0][\bar{D}_i, \bar{A}_0]\} \;, \tag{6.32}$$

where we have now replaced $A_\mu \to \bar{A}_\mu$.

Next, we complete the tree-level structure by adding all mass and interaction terms allowed by symmetries. In this process, it is useful to proceed in order of increasing dimensionality, whereby we obtain in the three lowest orders:

$$\text{dim} = 2: \;\; \text{Tr}\,[\bar{A}^2_0] \;; \tag{6.33}$$

$$\text{dim} = 4: \;\; \text{Tr}\,[\bar{A}^4_0] \;, \quad (\text{Tr}\,[\bar{A}^2_0])^2 \;; \tag{6.34}$$

$$\text{dim} = 6: \;\; \text{Tr}\,\{[\bar{D}_i, \bar{F}_{ij}][\bar{D}_k, \bar{F}_{kj}]\} \;, \ldots \;. \tag{6.35}$$

In the last case, we have only shown one example operator, while many others are listed in [8]. Note also that for $N_c = 2$ and 3, there exists a linear relation between the two operators of dimensionality 4, but from $N_c = 4$ onwards they are fully independent.

Combining Eqs. (6.32)–(6.34), we can write the effective action in the form

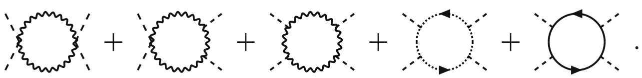
$$S_{\text{eff}} = \frac{1}{T} \int_{\mathbf{x}} \left\{ \frac{1}{4} \bar{F}^a_{ij} \bar{F}^a_{ij} + \text{Tr}\,([\bar{D}_i, \bar{A}_0][\bar{D}_i, \bar{A}_0]) + \bar{m}^2 \text{Tr}\,[\bar{A}_0^2] \right. \\ \left. + \bar{\lambda}^{(1)} (\text{Tr}\,[\bar{A}_0^2])^2 + \bar{\lambda}^{(2)} \text{Tr}\,[\bar{A}_0^4] + \ldots \right\} . \tag{6.36}$$

The prefactor $1/T$, appearing like in *classical* statistical physics, comes from the integration $\int_0^\beta \mathrm{d}\tau$, since none of the soft fields depend on τ. This theory is referred to as EQCD, for "Electrostatic QCD". Note that in the presence of a finite chemical potential, cf. Sect. 7, charge conjugation symmetry is broken and the additional operator $i\bar{\gamma}\text{Tr}\,[\bar{A}_0^3]$ appears in the effective action [9].

(3) Matching. If we restrict to 1-loop order, then the matching of the parameters in Eq. (6.36) is rather simple, as explained around Eq. (6.14): we just need to compute Green's functions for the soft fields with vanishing external momenta, with the heavy modes appearing in the internal propagators. For the parameter $\bar{m}^2$, this is furthermore precisely the computation that we carried out in Sect. 5.3, so the result can be directly read off from Eq. (5.102):

$$\bar{m}^2 = g^2 T^2 \left(\frac{N_c}{3} + \frac{N_f}{6} \right) + \mathcal{O}(g^4 T^2) . \tag{6.37}$$

The parameters $\bar{\lambda}^{(1)}$, $\bar{\lambda}^{(2)}$ can, in turn, be obtained by considering 4-point functions with soft modes of A_0 on the external legs, and non-zero Matsubara modes in the loop:



(6.38)

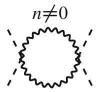
These graphs are clearly of $\mathcal{O}(g^4)$ and, using the same notation as in Eq. (5.102), the actual values of the two parameters read [10, 11]

$$\bar{\lambda}^{(1)} = \frac{g^4}{4\pi^2} + \mathcal{O}(g^6) , \quad \bar{\lambda}^{(2)} = \frac{g^4}{12\pi^2}(N_c - N_f) + \mathcal{O}(g^6) . \tag{6.39}$$

The gauge coupling $\bar{g}$ appearing in $\bar{D}_i$ and $\bar{F}^a_{ij}$ is of the form $\bar{g}^2 = g^2 + \mathcal{O}(g^4)$ and needs to be matched as well [12, 13]. If there are non-zero chemical potentials μ_i in the problem, the same is true for $\bar{\gamma} = \sum_{i=1}^{N_f} \mu_i\, g^3/(3\pi^2) + \mathcal{O}(g^5)$ [9].

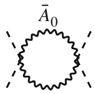
(4) Truncation of higher-dimensional operators. The most non-trivial part of any effective theory construction is the quantitative analysis of the error made, when operators beyond a given dimensionality are dropped. In other words, the challenge is to determine the constant k in Eq. (6.11). We illustrate this by considering the error made when dropping the operator in Eq. (6.35).

First of all, we need to know the parametric magnitude of the coefficient with which the neglected operator would enter L_{eff}, if it were kept. The operator of Eq. (6.35) could be generated through the momentum dependence of graphs like



$$\sim \quad \frac{g^2}{T^2}(\partial_i \bar{F}^a_{ij})^2 \, , \tag{6.40}$$

where the dashed lines now stand for the spatial components of the gauge field, $\bar{A}_i$. If we drop this term, the corresponding Green's function will not be computed correctly; however, it still has *some* value, namely that which would be obtained within the effective theory via the graph



$$\sim \quad g^2(\partial_i \bar{F}^a_{ij})^2 T \int_{\mathbf{p}} \frac{1}{(p^2 + \bar{m}^2)^3} \sim \frac{g^2 T}{\bar{m}^3}(\partial_i \bar{F}^a_{ij})^2 \, . \tag{6.41}$$

Here, we have noted that to account for the momentum dependence of the graph, represented by the derivative ∂_i in front of $\bar{F}^a_{ij}$, one needs to Taylor-expand the integral to the first non-trivial order in external momentum, explaining why the propagator is raised to power three in Eq. (6.41). An explicit computation further shows that the coefficient in Eq. (6.41) comes with a negative sign, but this has no significance for our general discussion.

Next, we note that the value of the Green's function within the (truncated) effective theory, Eq. (6.41), is in fact *larger* than what the contribution of the omitted operator would have been, cf. Eq. (6.40)! Therefore, the error made through the omission of Eq. (6.40) is *small*:

$$\frac{\delta \bar{\Gamma}}{\bar{\Gamma}} \sim \frac{g^2}{T^2}\frac{\bar{m}^3}{g^2 T} \sim \left(\frac{\bar{m}}{T}\right)^3 \sim g^3 \, . \tag{6.42}$$

In other words, for the Green's function considered and the dimensionally reduced effective theory of hot QCD truncated beyond dimension 4, we can expect that the relative accuracy exponent of Eq. (6.11) takes the value $k = 3$ [14].

Having now completed the construction of the effective theory of Eq. (6.36), we can take a further step: the field $\bar{A}_0$ is massive, and can thus be integrated out, should we wish to study distance scales longer than $1/\bar{m}$. Thereby we arrive at an even simpler effective theory,

$$S'_{\text{eff}} = \frac{1}{T}\int_{\mathbf{x}} \left\{ \frac{1}{4} \bar{\bar{F}}^a_{ij} \bar{\bar{F}}^a_{ij} + \ldots \right\} \, , \tag{6.43}$$

referred to as MQCD, for "Magnetostatic QCD". It is important to realize that this theory, i.e. three-dimensional Yang-Mills theory (up to higher-order operators such as the one in Eq. (6.35)), only has one parameter, the gauge coupling. Furthermore, if the fields $\bar{\bar{A}}_i^a$ are rescaled by an appropriate power of $T^{1/2}$, $\bar{\bar{A}}_i^a \to \bar{\bar{A}}_i^a T^{1/2}$, then the coefficient $1/T$ in Eq. (6.43) disappears. The coupling constant squared that appears afterwards is $\bar{\bar{g}}^2 T$, and this is the only scale in the system. Therefore all dimensionful quantities (correlation lengths, string tension, free energy density, ...) must be proportional to an appropriate power of $\bar{\bar{g}}^2 T$, with a non-perturbative coefficient. This is the essence of the non-perturbative physics pointed out by Linde [1].[3]

The implication of the above setup for the properties of the weak-coupling expansion is the following. Consider a generic observable $\mathcal{O}$, with an expectation value of the form

$$\langle \mathcal{O} \rangle \sim g^m T^n [1 + \alpha\, g^r + \ldots] \,. \tag{6.44}$$

There are now four distinct possibilities:

(i) r is even, and α is determined by the heavy scale $\sim \pi T$ and is purely perturbative. This is the case for instance for the leading correction to the free energy density $f(T)$, cf. Eq. (5.118).

(ii) r is odd, and α is determined by the intermediate scale $\sim gT$, being still purely perturbative. This is the case for the next-to-leading order corrections to many real-time quantities in thermal QCD, for instance to the heavy quark diffusion coefficient [17].

(iii) $m + r$ is even, and α is non-perturbatively determined by the soft scale $\sim g^2 T/\pi$. This is the case e.g. for the next-to-leading order correction to the physical Debye screening length [8, 9] and for one of the subleading corrections to $f(T)$ in a non-Abelian plasma [1, 18].

(iv) $r > k$, and α can only be determined correctly by adding higher-dimensional operators to the effective theory.

A few final remarks are in order:

- We have seen that the omission of higher-order operators in the construction of an effective theory usually leads to a small error, since the same Green's function is produced with a larger coefficient within it. It could happen, however, that there is some approximate symmetry in the full theory, which *becomes exact* within the effective theory, if we truncate its derivation to a given order. For instance, many Grand Unified Theories violate baryon minus lepton number ($B - L$), whereas in the classic Standard Model this is an exact symmetry, to be broken only by some higher-dimensional operator [19, 20]. Therefore, if such a Grand Unified Theory

[3] In contrast, topological configurations such as instantons, which play an important role for certain non-perturbative phenomena in vacuum, only play a minor role at finite temperatures [15], save for special observables where the anomalous $\mathrm{U_A}(1)$ breaking dominates the signal (cf. [16] and references therein). The reason is that the Euclidean topological susceptibility (measuring topological "activity") vanishes to all orders in perturbation theory, and is numerically small.

represented a true description of Nature and we considered $B-L$ violation within the classic Standard Model, we would make an *infinitely large* relative error.

- There are several reasons why effective theories constitute a useful framework. First of all, they allow us to justify and extend resummations such as those discussed in Sect. 3.4 systematically to higher orders in the weak-coupling expansion. As mentioned below Eqs. (3.93) and (5.118), this has led to the determination of many subsequent terms in the weak-coupling series. Second, effective theories permit for a simple non-perturbative study of the infrared sector affected by the Linde problem; examples are provided by [9, 18, 21, 22], and further ones will be encountered below.
- When proceeding to higher orders in the matching computations, they are often most conveniently formulated in the so-called *background field gauge* [23], rather than in the covariant gauge of Eq. (5.40), cf. e.g. [24].

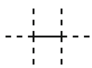
Appendix: Subtleties Related to the Low-Energy Expansion

Let us consider the full theory

$$L_{\rm full} \equiv \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu H\partial_\mu H + \frac{1}{2}M^2H^2 + \frac{1}{6}\gamma H\phi^3\,. \tag{6.45}$$

For simplicity (more precisely, in order to avoid ultraviolet divergences), we assume that the dimensionality of spacetime is 3, i.e. $d = 2 - 2\epsilon$ in our standard notation, and moreover work at zero temperature, like in Sect. 6.1. We then take the following steps:

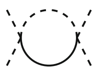
(i) Integrating out H in order to construct an effective theory, we compute the graph

 . (6.46)

After Taylor-expanding the result in external momenta, we write down all the corresponding operators.

(ii) We focus on the 4-point function of the $\bar{\phi}$ field at vanishing external momenta, and determine the contributions of the operators computed in step (i) to this Green's function.

(iii) Finally we consider directly the full theory graph

 (6.47)

at vanishing external momenta. Comparing with the Taylor-expanded result obtained from step (ii), we demonstrate how a "careless" Taylor expansion can lead to wrong results.

The construction of the effective theory proceeds essentially as in Eq. (3.12), except that only the H-field is now integrated out. We get from here

$$\begin{aligned}
S_{\text{eff}} &\approx \left\langle -\frac{1}{2}S_{\text{I}}^2 \right\rangle_{H,\text{c}} \\
&= -\frac{\gamma^2}{72}\int_{X,Y}\phi^3(X)\phi^3(Y)\langle H(X)H(Y)\rangle_0 \\
&= -\frac{\gamma^2}{72}\int_{X,Y}\phi^3(X)\phi^3(Y)\int_P \frac{e^{iP\cdot(X-Y)}}{P^2+M^2} \\
&= -\frac{\gamma^2}{72}\int_{X,Y}\phi^3(X)\phi^3(Y)\int_P e^{iP\cdot(X-Y)}\left[\sum_{n=0}^{\infty}\frac{(-1)^n(P^2)^n}{(M^2)^{n+1}}\right] \\
&= -\frac{\gamma^2}{72}\int_{X,Y}\phi^3(X)\phi^3(Y)\left[\sum_{n=0}^{\infty}\frac{(\nabla_X^2)^n}{(M^2)^{n+1}}\right]\delta(X-Y) \\
&= -\frac{\gamma^2}{72}\int_X \sum_{n=0}^{\infty}\phi^3(X)\,\frac{(\nabla_X^2)^n}{(M^2)^{n+1}}\,\phi^3(X)\,,
\end{aligned} \tag{6.48}$$

where an expansion was carried out assuming $P^2 \ll M^2$, and partial integrations were performed at the last step.

Using Eq. (6.48), we can extract the corresponding contribution to the 4-point function at vanishing momenta:

$$\begin{aligned}
&\left\langle \tilde\phi(0)\tilde\phi(0)\tilde\phi(0)\tilde\phi(0)e^{-S_{\text{eff}}}\right\rangle \\
&\Rightarrow \frac{\gamma^2}{72}\left\langle \tilde\phi(0)\tilde\phi(0)\tilde\phi(0)\tilde\phi(0)\int_{P_1,\ldots,P_6}\delta(\Sigma_i P_i)\tilde\phi(P_1)\ldots\tilde\phi(P_6)\right\rangle \\
&\quad\times\sum_{n=0}^{\infty}\frac{[-(P_4+P_5+P_6)^2]^n}{(M^2)^{n+1}} \\
&= \frac{\gamma^2}{72}\times 6\times(2\times 3\times 2+3\times 4\times 2)\int_{P_1,\ldots,P_6}\delta(\Sigma_i P_i) \\
&\quad\times\sum_{n=0}^{\infty}\frac{[-(P_4+P_5+P_6)^2]^n}{(M^2)^{n+1}}
\end{aligned}$$

$$\times\langle\tilde{\phi}(0)\tilde{\phi}(P_1)\rangle_0\,\langle\tilde{\phi}(0)\tilde{\phi}(P_2)\rangle_0\,\langle\tilde{\phi}(0)\tilde{\phi}(P_5)\rangle_0\,\langle\tilde{\phi}(0)\tilde{\phi}(P_6)\rangle_0\,\langle\tilde{\phi}(P_3)\tilde{\phi}(P_4)\rangle_0$$
$$= 3\gamma^2\frac{\hat{\delta}(0)}{(\bar{m}^2)^4}\int_{P_3}\frac{1}{P_3^2+\bar{m}^2}\sum_{n=0}^{\infty}\frac{(-P_3^2)^n}{(M^2)^{n+1}}, \tag{6.49}$$

where we denoted by $\bar{m}$ the mass of the effective-theory field $\bar{\phi}$ and by $\tilde{\phi}$ its Fourier representation. Furthermore we noted that the result vanishes unless the fields $\tilde{\phi}(P_i)$ are contracted so that one of the momenta P_4, P_5 and P_6 remains an integration variable. The integrals appearing in the result can be carried out in dimensional regularization; for instance, the two leading terms read

$$n=0:\ \frac{1}{M^2}\int_{P_3}\frac{1}{P_3^2+\bar{m}^2}=\frac{1}{M^2}\left(-\frac{\bar{m}}{4\pi}\right), \tag{6.50}$$

$$n=1:\ -\frac{1}{M^4}\int_{P_3}\frac{P_3^2}{P_3^2+\bar{m}^2}=\frac{\bar{m}^2}{M^4}\int_{P_3}\frac{1}{P_3^2+\bar{m}^2}=-\frac{1}{M^4}\frac{\bar{m}^3}{4\pi}, \tag{6.51}$$

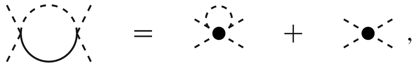
where we made use of Eq. (2.86) and of the vanishing of scale-free integrals in dimensional regularization. We note that the terms get smaller with increasing n, apparently justifying *a posteriori* the Taylor expansion we carried out above.

Let us finally carry out the integral corresponding to Eq. (6.47) exactly. The contractions remain as above, and we simply need to replace the integral in Eq. (6.49) by

$$\begin{aligned}\int_{P_3}\frac{1}{P_3^2+\bar{m}^2}\frac{1}{P_3^2+M^2} &= \int_{P_3}\frac{1}{M^2-\bar{m}^2}\left[\frac{1}{P_3^2+\bar{m}^2}-\frac{1}{P_3^2+M^2}\right]\\ &= \frac{1}{M^2-\bar{m}^2}\left(\frac{-1}{4\pi}\right)(\bar{m}-M)\\ &= \frac{1}{4\pi(M+\bar{m})}\\ &= \frac{1}{4\pi M}\left(1-\frac{\bar{m}}{M}+\frac{\bar{m}^2}{M^2}-\frac{\bar{m}^3}{M^3}+\ldots\right).\end{aligned} \tag{6.52}$$

Comparing Eqs. (6.50) and (6.51) with Eq. (6.52), we note that by carrying out the Taylor expansion, i.e. the naive matching of the effective theory parameters, we missed the leading contribution in Eq. (6.52). The largest term we found, Eq. (6.50), is only next-to-leading in Eq. (6.52). It furthermore appears that we missed all even powers of $\bar{m}$ in the sum of Eq. (6.52).

The reason for the problem encountered is the same as in Eq. (6.21): it again has to be taken into account that the light fields ϕ can also carry large momenta $P_3 \sim M$, in which case a Taylor expansion of $1/(P_3^2+M^2)$ is not justified. Rather, we have to view Eq. (6.47) in analogy with Eq. (6.21),



(6.53)

where the first term corresponds to a naive replacement of Eq. (6.46) by a momentum-independent 6-point vertex, and the second term to a contribution from hard ϕ-modes to an effective 4-point vertex. In accordance with our discussion around Eq. (6.21), we see that the result of Eq. (6.50) (and more generally Eq. (6.49)) is indeed non-analytic in the parameter $\bar{m}^2$, whereas the supplementary terms in Eq. (6.52) that the naive Taylor expansion missed are analytic in it.

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