Chapter 3
Interacting Scalar Fields

Abstract The key concepts of a perturbative or weak-coupling expansion are introduced in the context of evaluating the imaginary-time path integral representation for the partition function of an interacting scalar field. The issues of ultraviolet and infrared divergences are brought up. These problems are cured through renormalization and resummation, respectively.

Keywords Contraction • Propagator • Renormalization • Resummation • Ring diagrams • Ultraviolet and infrared divergences • Weak-coupling expansion • Wick's theorem

3.1 Principles of the Weak-Coupling Expansion

In order to move from a free to an interacting theory, we now include a quartic term in the potential in Eq. (2.4),

$$V(\phi) \equiv \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \ , \tag{3.1}$$

where $\lambda > 0$ is a dimensionless coupling constant. Thereby the Minkowskian and Euclidean Lagrangians become

$$\mathcal{L}_M = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 \ , \tag{3.2}$$

$$L_E = \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \ , \tag{3.3}$$

where repeated indices are summed over, irrespective of whether they are up and down or all down. The case with all indices down implies the use of Euclidean metric like in Eq. (2.7).

In the presence of $\lambda > 0$, it is no longer possible to determine the partition function of the system exactly, neither in the canonical formalism nor through a path integral approach. We therefore need to develop approximation schemes, which could in principle be either analytic or numerical. In the following we restrict our

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attention to the simplest analytic procedure which, as we will see, already teaches us a lot about the nature of the system.

In a weak-coupling expansion, the theory is solved by formally assuming that $\lambda \ll 1$, and by expressing the result for the observable in question as a (generalized) Taylor series in λ. The physical observable that we are interested in is the partition function defined according to Eq. (2.6). Denoting the free and interacting parts of the Euclidean action by

$$S_0 \equiv \int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} \left[\frac{1}{2} \partial_\mu \phi \, \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 \right] , \tag{3.4}$$

$$S_{\mathrm{I}} \equiv \lambda \int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} \left[\frac{1}{4} \phi^4 \right] , \tag{3.5}$$

the partition function can be written in the form

$$\begin{aligned} \mathcal{Z}^{\text{SFT}}(T) &= C \int \mathcal{D}\phi \, \exp\big(-S_0 - S_{\mathrm{I}}\big) \\ &= C \int \mathcal{D}\phi \, e^{-S_0} \left[1 - S_{\mathrm{I}} + \frac{1}{2} S_{\mathrm{I}}^2 - \frac{1}{6} S_{\mathrm{I}}^3 + \dots \right] \\ &= \mathcal{Z}^{\text{SFT}}_{(0)} \left[1 - \langle S_{\mathrm{I}} \rangle_0 + \frac{1}{2} \langle S_{\mathrm{I}}^2 \rangle_0 - \frac{1}{6} \langle S_{\mathrm{I}}^3 \rangle_0 + \dots \right] . \end{aligned} \tag{3.6}$$

Here,

$$\mathcal{Z}^{\text{SFT}}_{(0)} \equiv C \int \mathcal{D}\phi \, e^{-S_0} \tag{3.7}$$

is the free partition function determined in Sect. 2, and the expectation value $\langle \cdots \rangle_0$ is defined as

$$\langle \cdots \rangle_0 \; \equiv \; \frac{\int \mathcal{D}\phi \, [\cdots] \exp(-S_0)}{\int \mathcal{D}\phi \, \exp(-S_0)} \, . \tag{3.8}$$

With this result, the free energy density reads

$$\begin{aligned} \frac{F^{\text{SFT}}(T,V)}{V} &= -\frac{T}{V} \ln \mathcal{Z}^{\text{SFT}} \\ &= \frac{F^{\text{SFT}}_{(0)}}{V} - \frac{T}{V} \ln\Big(1 - \langle S_{\mathrm{I}} \rangle_0 + \frac{1}{2} \langle S_{\mathrm{I}}^2 \rangle_0 - \frac{1}{6} \langle S_{\mathrm{I}}^3 \rangle_0 + \dots \Big) \qquad (3.9) \\ &= \frac{F^{\text{SFT}}_{(0)}}{V} - \frac{T}{V} \bigg\{ -\langle S_{\mathrm{I}} \rangle_0 + \frac{1}{2} \Big[\langle S_{\mathrm{I}}^2 \rangle_0 - \langle S_{\mathrm{I}} \rangle_0^2 \Big] \\ &\quad - \frac{1}{6} \Big[\langle S_{\mathrm{I}}^3 \rangle_0 - 3 \langle S_{\mathrm{I}} \rangle_0 \langle S_{\mathrm{I}}^2 \rangle_0 + 2 \langle S_{\mathrm{I}} \rangle_0^3 \Big] + \dots \bigg\} , \qquad (3.10) \end{aligned}$$

where we have Taylor-expanded the logarithm, $\ln(1-x) = -x-x^2/2-x^3/3+\ldots$. The first term, $F^{\rm SFT}_{(0)}/V$, is given in Eq. (2.26), whereas the subsequent terms correspond to corrections of orders $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^2)$, and $\mathcal{O}(\lambda^3)$, respectively. As we will see, the combinations that appear within the square brackets in Eq. (3.10) have a specific significance: Eq. (3.10) is *simpler* than Eq. (3.6)!

For future reference, let us denote

$$f(T) \;\equiv\; \lim_{V\to\infty} \frac{F(T,V)}{V} \;, \tag{3.11}$$

where we have dropped the superscript "SFT" for simplicity. With this definition Eq. (3.10) can be compactly represented by the formula $f = f_{(0)} + f_{(\geq 1)}$, where

$$\begin{aligned} f_{(\geq 1)}(T) &= -\frac{T}{V}\Big\langle \exp(-S_{\rm I}) - 1 \Big\rangle_{0,\rm c} \\ &= \Big\langle S_{\rm I} - \frac{1}{2}S_{\rm I}^2 + \ldots \Big\rangle_{0,\rm c,\,drop\ overall\ \int_X} \;, \end{aligned} \tag{3.12}$$

where the subscript $(\ldots)_{\rm c}$ refers to "connected" contractions, the precise meaning of which is discussed momentarily, and an "overall $\int_X$" is dropped because it cancels against the prefactor T/V.

Inserting Eq. (3.5) into the various terms of Eq. (3.10), we are led to evaluate expectation values of the type

$$\langle \phi(X_1)\phi(X_2)\ldots\phi(X_n)\rangle_0 \;. \tag{3.13}$$

These can be reduced to products of free 2-point correlators, $\langle\phi(X_k)\phi(X_l)\rangle_0$, through the Wick's theorem, as we now discuss.

Wick's Theorem

Wick's theorem states that free (Gaussian) expectation values of any number of integration variables can be reduced to products of 2-point correlators, according to

$$\langle \phi(X_1)\phi(X_2)\ldots\phi(X_{n-1})\phi(X_n)\rangle_0 = \sum_{\text{all combinations}} \langle\phi(X_1)\phi(X_2)\rangle_0\cdots\langle\phi(X_{n-1})\phi(X_n)\rangle_0 \;. \tag{3.14}$$

Before applying this to the terms of Eq. (3.10), we briefly recall how the theorem can be derived with (path) integration techniques.

Let us assume that we can discretise spacetime such that the coordinates X only take a finite number of values, which in particular requires the volume to be finite. Then we can collect the values $\phi(X)$, $\forall X$, into a single vector v, and subsequently write the free action in the form $S_0 = \frac{1}{2} v^T A\, v$, where A is a matrix. Here, we assume that A^{-1} exists and that A is symmetric, i.e. $A^T = A$; then it also follows that $(A^{-1})^T = A^{-1}$.

The trick allowing us to evaluate integrals weighted by $\exp(-S_0)$ is to introduce a source vector b, and to take derivatives with respect to its components. Specifically, we define

$$\begin{aligned} \exp\Big[W(b)\Big] \quad &\equiv \quad \int \mathrm{d}v \, \exp\Big[-\frac{1}{2} v_i A_{ij} v_j + b_i v_i\Big] \\ &\overset{v_i \to v_i + A^{-1}_{ij} b_j}{=} \exp\Big[\frac{1}{2} b_i A^{-1}_{ij} b_j\Big] \int \mathrm{d}v \, \exp\Big[-\frac{1}{2} v_i A_{ij} v_j\Big] \,, \end{aligned} \tag{3.15}$$

where we made a substitution of integration variables at the second equality. We then obtain

$$\begin{aligned} \langle v_k v_l \ldots v_n \rangle_0 &= \frac{\int \mathrm{d}v \, (v_k v_l \ldots v_n) \exp\big[-\frac{1}{2} v_i A_{ij} v_j\big]}{\int \mathrm{d}v \, \exp\big[-\frac{1}{2} v_i A_{ij} v_j\big]} \\ &= \frac{\Big\{\frac{\mathrm{d}}{\mathrm{d}b_k} \frac{\mathrm{d}}{\mathrm{d}b_l} \cdots \frac{\mathrm{d}}{\mathrm{d}b_n} \exp\big[W(b)\big]\Big\}_{b=0}}{\exp\big[W(0)\big]} \\ &= \Big\{\frac{\mathrm{d}}{\mathrm{d}b_k} \frac{\mathrm{d}}{\mathrm{d}b_l} \cdots \frac{\mathrm{d}}{\mathrm{d}b_n} \exp\Big[\frac{1}{2} b_i A^{-1}_{ij} b_j\Big]\Big\}_{b=0} \\ &= \Big\{\frac{\mathrm{d}}{\mathrm{d}b_k} \frac{\mathrm{d}}{\mathrm{d}b_l} \cdots \frac{\mathrm{d}}{\mathrm{d}b_n} \Big[1 + \frac{1}{2} b_i A^{-1}_{ij} b_j \\ &\quad + \frac{1}{2}\Big(\frac{1}{2}\Big)^2 b_i A^{-1}_{ij} b_j \, b_r A^{-1}_{rs} b_s + \ldots\Big]\Big\}_{b=0} \,. \end{aligned} \tag{3.16}$$

Taking the derivatives in Eq. (3.16), we observe that:

- $\langle 1 \rangle_0 = 1$.
- If there is an odd number of components of v in the expectation value, the result is zero.
- $\langle v_k v_l \rangle_0 = A^{-1}_{kl}$.
- $\langle v_k v_l v_m v_n \rangle_0 = A^{-1}_{kl} A^{-1}_{mn} + A^{-1}_{km} A^{-1}_{ln} + A^{-1}_{kn} A^{-1}_{lm}$
  $= \langle v_k v_l \rangle_0 \langle v_m v_n \rangle_0 + \langle v_k v_m \rangle_0 \langle v_l v_n \rangle_0 + \langle v_k v_n \rangle_0 \langle v_l v_m \rangle_0$.
- At higher orders, we obtain a discretized version of Eq. (3.14).
- Since all the operations were purely combinatorial, removing the discretization does not modify the result, so that Eq. (3.14) holds also in the infinite volume and continuum limits.

Let us now use Eq. (3.14) in connection with Eq. (3.10). From Eqs. (2.26), (2.50) and (3.10), we read off the familiar leading-order result,

$$f_{(0)}(T) = J(m, T) \,. \tag{3.17}$$

At the first order, linear in λ, we on the other hand get

$$f_{(1)}(T) = \lim_{V\to\infty} \frac{T}{V} \langle S_\mathrm{I}\rangle_0 = \lim_{V\to\infty} \frac{T}{V} \int_0^\beta \mathrm{d}\tau \int_\mathbf{x} \frac{\lambda}{4} \langle \phi(X)\phi(X)\phi(X)\phi(X)\rangle_0 \,, \tag{3.18}$$

where we can now use Wick's theorem. Noting that due to translational invariance, $\langle\phi(X)\phi(Y)\rangle_0$ can only depend on $X-Y$, the spacetime integral becomes trivial, and we obtain

$$f_{(1)}(T) = \frac{3}{4}\lambda \,\langle\phi(0)\phi(0)\rangle_0\langle\phi(0)\phi(0)\rangle_0 \,. \tag{3.19}$$

Finally, at the second order, we get

$$\begin{aligned} f_{(2)}(T) &= \lim_{V\to\infty}\left\{-\frac{T}{2V}\left[\langle S_\mathrm{I}^2\rangle_0 - \langle S_\mathrm{I}\rangle_0^2\right]\right\} \\ &= \lim_{V\to\infty}\left\{-\frac{T}{2V}\left[\int_{X,Y}\left(\frac{\lambda}{4}\right)^2 \langle\phi(X)\phi(X)\phi(X)\phi(X)\,\phi(Y)\phi(Y)\phi(Y)\phi(Y)\rangle_0 \right.\right. \\ &\quad \left.\left. - \int_X \frac{\lambda}{4}\langle\phi(X)\phi(X)\phi(X)\phi(X)\rangle_0 \int_Y \frac{\lambda}{4}\langle\phi(Y)\phi(Y)\phi(Y)\phi(Y)\rangle_0\right]\right\} \,, \end{aligned} \tag{3.20}$$

where we have again denoted (cf. Eq. (2))

$$\int_X \equiv \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^d\mathbf{x} \,. \tag{3.21}$$

Upon carrying out the contractions in Eq. (3.20) according to Wick's theorem, the role of the "subtraction term", i.e. the second one in Eq. (3.20), becomes clear: it cancels all *disconnected* contractions where all fields at point X are contracted with other fields at the same point. In other words, the combination in Eq. (3.20) amounts to taking into account only the *connected* contractions; this is the meaning of the subscript c in Eq. (3.12). This combinatorial effect is caused by the logarithm in Eq. (3.10), i.e., by going from the partition function to the free energy.

As far as the connected contractions go, we obtain through a (repeated) use of Wick's theorem:

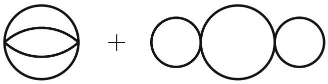
$$\begin{aligned} &\langle\phi(X)\phi(X)\phi(X)\phi(X)\,\phi(Y)\phi(Y)\phi(Y)\phi(Y)\rangle_{0,\mathrm{c}} \\ &\quad= 4\,\langle\phi(X)\phi(Y)\rangle_0\,\langle\phi(X)\phi(X)\phi(X)\,\phi(Y)\phi(Y)\phi(Y)\rangle_{0,\mathrm{c}} \\ &\qquad + 3\,\langle\phi(X)\phi(X)\rangle_0\,\langle\phi(X)\phi(X)\,\phi(Y)\phi(Y)\phi(Y)\phi(Y)\rangle_{0,\mathrm{c}} \\ &\quad= 4\times 3\,\langle\phi(X)\phi(Y)\rangle_0\,\langle\phi(X)\phi(Y)\rangle_0\,\langle\phi(X)\phi(X)\,\phi(Y)\phi(Y)\rangle_{0,\mathrm{c}} \\ &\qquad + 4\times 2\,\langle\phi(X)\phi(Y)\rangle_0\,\langle\phi(X)\phi(X)\rangle_0\,\langle\phi(X)\,\phi(Y)\phi(Y)\phi(Y)\rangle_{0,\mathrm{c}} \\ &\qquad + 3\times 4\,\langle\phi(X)\phi(X)\rangle_0\,\langle\phi(X)\phi(Y)\rangle_0\,\langle\phi(X)\,\phi(Y)\phi(Y)\phi(Y)\rangle_{0,\mathrm{c}} \end{aligned}$$

$$= 4 \times 3 \times 2 \langle\phi(X)\phi(Y)\rangle_0 \langle\phi(X)\phi(Y)\rangle_0 \langle\phi(X)\phi(Y)\rangle_0 \langle\phi(X)\phi(Y)\rangle_0 + (4 \times 3 + 4 \times 2 \times 3 + 3 \times 4 \times 3)\langle\phi(X)\phi(X)\rangle_0 \langle\phi(X)\phi(Y)\rangle_0 \times\langle\phi(X)\phi(Y)\rangle_0 \langle\phi(Y)\phi(Y)\rangle_0 \,. \tag{3.22}$$

Inspecting the 2-point correlators in this result, we note that they either depend on $X-Y$, or on neither X nor Y, the latter case corresponding to the contraction of fields at the same point. Thereby one of the spacetime integrals is trivial (just substitute $X \to X + Y$, and note that $\langle\phi(X+Y)\phi(Y)\rangle_0 = \langle\phi(X)\phi(0)\rangle_0$), and cancels against the factor $T/V = 1/(\beta V)$ in Eq. (3.20). In total, we then have

$$f_{(2)}(T) = -\Big(\frac{\lambda}{4}\Big)^2\bigg[12\int_X \{\langle\phi(X)\phi(0)\rangle_0\}^4 + 36\,\{\langle\phi(0)\phi(0)\rangle_0\}^2\int_X \{\langle\phi(X)\phi(0)\rangle_0\}^2\bigg]\,. \tag{3.23}$$

Graphically this can be represented as



(3.24)

where solid lines denote propagators, and the vertices at which they cross denote spacetime points, in this case X and 0.

We could in principle go on with the third-order terms in Eq. (3.10). Again, it could be verified that the "subtraction terms" cancel all disconnected contractions, so that only the connected ones contribute to $f(T)$, and that one spacetime integral cancels against the explicit factor T/V. These features are of general nature, and hold at any order in the weak-coupling expansion.

In summary, Wick's theorem has allowed us to convert the terms in Eq. (3.10) to various structures made of the 2-point correlator $\langle\phi(X)\phi(0)\rangle_0$. We now turn to the properties of this function.

Propagator

The 2-point correlator $\langle\phi(X)\phi(Y)\rangle_0$ is usually called the *free propagator*. Denoting

$$\delta\!\!\!/(P+Q) \equiv \int_X e^{i(P+Q)\cdot X} = \beta\delta_{p_n+q_n,0}\,(2\pi)^d\delta^{(d)}(\mathbf{p}+\mathbf{q})\,, \tag{3.25}$$

where $P \equiv (p_n, \mathbf{p})$ and p_n are bosonic Matsubara frequencies, and employing the representation

$$\phi(X) \equiv \sumint_P \tilde{\phi}(P) e^{iP\cdot X} , \tag{3.26}$$

we recall from basic quantum field theory that the (Euclidean) propagator can be written as

$$\langle \tilde{\phi}(P)\tilde{\phi}(Q)\rangle_0 = \delta\!\!\!^-(P+Q)\, \frac{1}{P^2+m^2} , \tag{3.27}$$

$$\langle \phi(X)\phi(Y)\rangle_0 = \sumint_P e^{iP\cdot(X-Y)} \frac{1}{P^2+m^2} . \tag{3.28}$$

Before inserting these expressions into Eqs. (3.19) and (3.23), we briefly review their derivation, working in a finite volume V and proceeding like in Sect. 2.1.

First, we insert Eq. (3.26) into the definition of the propagator,

$$\langle \phi(X)\phi(Y)\rangle_0 = \sumint_{P,Q} e^{iP\cdot X + iQ\cdot Y} \langle \tilde{\phi}(P)\tilde{\phi}(Q)\rangle_0 , \tag{3.29}$$

as well as to the free action, S_0,

$$S_0 = \frac{1}{2}\sumint_P \tilde{\phi}(-P)(P^2+m^2)\tilde{\phi}(P) = \frac{1}{2}\sumint_P (P^2+m^2)|\tilde{\phi}(P)|^2 . \tag{3.30}$$

Here, we may further write $\tilde{\phi}(P) = a(P) + i\, b(P)$, with $a(-P) = a(P)$, $b(-P) = -b(P)$, and subsequently note that only half of the Fourier components are independent. We may choose these according to Eq. (2.13).

Restricting the sum to the independent components, and making use of the symmetry properties of $a(P)$ and $b(P)$, Eq. (3.30) becomes

$$S_0 = \frac{T}{V} \sum_{P_{\text{indep.}}} (P^2+m^2)[a^2(P) + b^2(P)] . \tag{3.31}$$

The Gaussian integral,

$$\frac{\int \mathrm{d}x\, x^2 \exp(-c\, x^2)}{\int \mathrm{d}x\, \exp(-c\, x^2)} = \frac{1}{2c} , \tag{3.32}$$

and the symmetries of $a(P)$ and $b(P)$ then imply the results

$$\langle a(P)\, b(Q)\rangle_0 = 0 \,, \tag{3.33}$$

$$\langle a(P)\, a(Q)\rangle_0 = (\delta_{P,Q} + \delta_{P,-Q})\, \frac{V}{2T} \frac{1}{P^2 + m^2} \,, \tag{3.34}$$

$$\langle b(P)\, b(Q)\rangle_0 = (\delta_{P,Q} - \delta_{P,-Q})\, \frac{V}{2T} \frac{1}{P^2 + m^2} \,, \tag{3.35}$$

where the δ-functions are of the Kronecker-type. Using these, the momentum-space propagator becomes

$$\begin{aligned} \langle \tilde{\phi}(P)\tilde{\phi}(Q)\rangle_0 &= \langle a(P)\, a(Q) + i\, a(P)\, b(Q) + i\, b(P)\, a(Q) - b(P)\, b(Q)\rangle_0 \\ &= \delta_{P,-Q}\, \frac{V}{T} \frac{1}{P^2 + m^2} = \beta \delta_{p_n + q_n, 0} V \delta_{\mathbf{p}+\mathbf{q},\mathbf{0}} \frac{1}{P^2 + m^2} \,, \end{aligned} \tag{3.36}$$

which in the infinite-volume limit (cf. Eq. (2.10)), *viz.*

$$\frac{1}{V}\sum_{\mathbf{p}} \longrightarrow \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \,, \quad V\delta_{\mathbf{p},\mathbf{0}} \longrightarrow (2\pi)^d \delta^{(d)}(\mathbf{p}) \,, \tag{3.37}$$

becomes exactly Eq. (3.27). Inserting this into Eq. (3.29) we also recover Eq. (3.28).

It is useful to study the behaviour of the propagator $\langle \phi(X)\phi(Y)\rangle_0$ at small and large separations $X - Y$. For this we may use the result of Eq. (1.70),

$$T\sum_{p_n} \frac{e^{ip_n\tau}}{p_n^2 + E^2} = \frac{1}{2E} \frac{\cosh\left[\left(\frac{\beta}{2} - \tau\right) E\right]}{\sinh\left[\frac{\beta E}{2}\right]} \,, \quad \beta = \frac{1}{T} \,, \quad 0 \le \tau \le \beta \,. \tag{3.38}$$

Even though this equation was derived for $0 \le \tau \le \beta$, it is clear from the left-hand side that we can extend its validity to $-\beta \le \tau \le \beta$ by replacing τ by $|\tau|$. Thereby, the propagator in Eq. (3.28) becomes

$$\begin{aligned} G_0(X - Y) &\equiv \langle \phi(X)\phi(Y)\rangle_0 \\ &= \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})} \frac{1}{2E_p} \frac{\cosh\left[\left(\frac{\beta}{2} - |x_0 - y_0|\right) E_p\right]}{\sinh\left[\frac{\beta E_p}{2}\right]} \Bigg|_{E_p \equiv \sqrt{p^2 + m^2}} \,, \end{aligned} \tag{3.39}$$

where we may set $Y = 0$ with no loss of generality.

Consider first short distances, $|\mathbf{x}|, |x_0| \ll \frac{1}{T}, \frac{1}{m}$. We may expect the dominant contribution in the Fourier transform of Eq. (3.39) to come from the regime

$|\mathbf{p}||\mathbf{x}| \sim 1$, so we assume $|\mathbf{p}| \gg T, m$. Then $E_p \approx p$ and $\beta E_p \approx p/T \gg 1$, and consequently,

$$\frac{\cosh\left[\left(\frac{\beta}{2} - |x_0|\right) E_p\right]}{\sinh\left[\frac{\beta E_p}{2}\right]} \approx \frac{\exp\left[\left(\frac{\beta}{2} - |x_0|\right) E_p\right]}{\exp\left[\frac{\beta E_p}{2}\right]} \approx e^{-|x_0|p} \,. \tag{3.40}$$

Noting that

$$\frac{1}{2p} e^{-|x_0|p} = \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi} \frac{e^{ip_0 x_0}}{p_0^2 + \mathbf{p}^2} \,, \tag{3.41}$$

this implies

$$G_0(X) \approx \int \frac{\mathrm{d}^{d+1}P}{(2\pi)^{d+1}} \frac{e^{iP\cdot X}}{P^2} \,, \tag{3.42}$$

with $P \equiv (p_0, \mathbf{p})$. We recognize this as the coordinate space propagator of a massless scalar field at zero temperature.

At this point we make use of the $d + 1$-dimensional rotational symmetry of Euclidean spacetime, and choose $X = (x_0, \mathbf{x})$ to point in the direction of the component p_0. Then,

$$\begin{aligned}
\int \frac{\mathrm{d}^{d+1}P}{(2\pi)^{d+1}} \frac{e^{iP\cdot X}}{P^2} &= \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi} \frac{e^{ip_0|X|}}{p_0^2 + \mathbf{p}^2} \\
&= \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \frac{e^{-p|X|}}{2p} \\
&\overset{(2.61)}{=} \frac{1}{(2\pi)^d} \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^{\infty} \mathrm{d}p\, p^{d-2} e^{-p|X|} \\
&= \frac{\Gamma(d-1)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})|X|^{d-1}} \,,
\end{aligned} \tag{3.43}$$

from which, inserting $d = 3$ and $\Gamma(\frac{3}{2}) = \sqrt{\pi}/2$, we find

$$G_0(X) \approx \frac{1}{4\pi^2|X|^2} \,, \quad |X| \ll \frac{1}{T}, \frac{1}{m} \,. \tag{3.44}$$

The result is independent of T and m, signifying that at short distances (in the "ultraviolet" regime), temperature and masses do not play a role. We may further note that the propagator rapidly diverges in this regime.

Next, we consider the opposite limit of large distances, $x = |\mathbf{x}| \gg 1/T$, noting that the periodic temporal coordinate x_0 is always "small", i.e. at most $1/T$. We expect that the Fourier transform of Eq. (3.39) is now dominated by small momenta, $p \ll T$. If we simplify the situation further by assuming that we are also at very high temperatures, $m \ll T$, then $\beta E_p \ll 1$, and we can expand the hyperbolic functions in Taylor series, approximating $\cosh(\epsilon) \approx 1$, $\sinh(\epsilon) \approx \epsilon$. We then obtain from Eq. (3.39)

$$G_0(X) \approx T \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{p^2 + m^2} \,. \tag{3.45}$$

Note that the integrand here is also the $p_n = 0$ contribution from the left-hand side of Eq. (3.38). Setting $d = 3$,[1] and denoting $z \equiv \mathbf{p}\cdot\mathbf{x}/(px)$, the remaining integral can be worked out as

$$\begin{aligned} G_0(X) &\approx \frac{T}{(2\pi)^2} \int_{-1}^{+1} \mathrm{d}z \int_0^\infty \mathrm{d}p\, p^2 \frac{e^{-ipxz}}{p^2 + m^2} \\ &= \frac{T}{(2\pi)^2} \int_0^\infty \frac{\mathrm{d}p\, p^2}{p^2 + m^2} \frac{e^{ipx} - e^{-ipx}}{ipx} \\ &= \frac{T}{(2\pi)^2 ix} \int_{-\infty}^\infty \frac{\mathrm{d}p\, p\, e^{ipx}}{p^2 + m^2} \\ &= \frac{T\, e^{-mx}}{4\pi x} \,, \quad x \gg \frac{1}{T} \,. \end{aligned} \tag{3.46}$$

In the last step the integration contour was closed in the upper half-plane (recalling that $x > 0$).

We note from Eq. (3.46) that at large distances (in the "infrared" regime), thermal effects modify the behaviour of the propagator in an essential way. In particular, if we were to set the mass to zero, then Eq. (3.44) would be the exact behaviour at zero temperature, both at small and at large distances, whereas Eq. (3.46) shows that a finite temperature would "slow down" the long-distance decay to $T/(4\pi|\mathbf{x}|)$. In other words, we can say that at non-zero temperature the theory is more sensitive to infrared physics than at zero temperature.

[1]For a general d, $\int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{p^2+m^2} = (2\pi)^{-\frac{d}{2}} (\frac{m}{x})^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(mx)$, where K is a modified Bessel function.

3.2 Problems of the Naive Weak-Coupling Expansion

$\mathcal{O}(\lambda)$: *Ultraviolet Divergences*

We now proceed with the evaluation of the weak-coupling expansion for the free energy density in a scalar field theory, the first three orders of which are given by Eqs. (3.17), (3.19) and (3.23). Noting from Eqs. (2.54) and (3.28) that $G_0(0) = I(m,T)$, we obtain

$$f(T) = J(m,T) + \frac{3}{4}\lambda \, [I(m,T)]^2 + \mathcal{O}(\lambda^2) \,. \tag{3.47}$$

According to Eqs. (2.72) and (2.73), we have

$$J(m,T) = -\frac{m^4\mu^{-2\epsilon}}{64\pi^2}\left[\frac{1}{\epsilon} + \ln\frac{\bar{\mu}^2}{m^2} + \frac{3}{2} + \mathcal{O}(\epsilon)\right] + J_T(m) \,, \tag{3.48}$$

$$I(m,T) = -\frac{m^2\mu^{-2\epsilon}}{16\pi^2}\left[\frac{1}{\epsilon} + \ln\frac{\bar{\mu}^2}{m^2} + 1 + \mathcal{O}(\epsilon)\right] + I_T(m) \,, \tag{3.49}$$

where the finite functions $J_T(m)$ and $I_T(m)$ were evaluated in various limits in Eqs. (2.79), (2.80), (2.82) and (2.93).

Inserting Eqs. (3.48) and (3.49) into Eq. (3.47), we note that the result is, in general, *ultraviolet divergent*. For instance, restricting for simplicity to very high temperatures, $T \gg m$, and making use of Eq. (2.93),

$$I_T(m) \approx \frac{T^2}{12} - \frac{mT}{4\pi} + \mathcal{O}(m^2) \,, \tag{3.50}$$

the dominant term at $\epsilon \to 0$ reads

$$f(T) \approx -\frac{\mu^{-2\epsilon}}{64\pi^2\epsilon}\left\{m^4 + \lambda\left[\frac{1}{2}T^2m^2 - \frac{3}{2\pi}Tm^3 + \mathcal{O}(m^4)\right] + \mathcal{O}(\lambda^2)\right\} + \mathcal{O}(1) \,. \tag{3.51}$$

This result is clearly non-sensical; in particular the divergences depend on the temperature, i.e. cannot be removed by subtracting a T-independent "vacuum" contribution. To properly handle this issue requires *renormalization*, to which we return in Sect. 3.3.

$\mathcal{O}(\lambda^2)$: *Infrared Divergences*

Let us next consider the $\mathcal{O}(\lambda^2)$ correction to Eq. (3.47), given by Eq. (3.23). With the notation of Eq. (3.39), it can be written as

$$f_{(2)}(T) = -\frac{3}{4}\lambda^2 \int_X [G_0(X)]^4 - \frac{9}{4}\lambda^2 [I(m,T)]^2 \int_X [G_0(X)]^2 \,. \tag{3.52}$$

It is particularly interesting to inspect what happens if we take the particle mass m to be very small in units of the temperature, $m \ll T$.

As Eqs. (3.47), (2.82) and (3.50) show, at $\mathcal{O}(\lambda)$ the small-mass limit is perfectly well-defined. At the next order, we on the other hand must analyze the two terms of Eq. (3.52). Starting with the first one, we know from Eq. (3.44) that the behaviour of G_0 is independent of m at small x, and thus nothing particular happens for $x \ll T^{-1}$. On the other hand, for large x, G_0 is given by Eq. (3.46), and we may thus estimate the contribution of this region as

$$\int_{x \gtrsim \beta} [G_0(X)]^4 \sim \int_0^\beta \mathrm{d}\tau \int_{x \gtrsim \beta} \mathrm{d}^3\mathbf{x} \left(\frac{Te^{-mx}}{4\pi x} \right)^4 . \tag{3.53}$$

This integral is convergent even for $m \to 0$.

Consider then the second term of Eq. (3.52). Repeating the previous argument, we see that the long-distance contribution to the free energy density is proportional to the integral

$$\int_{x \gtrsim \beta} [G_0(X)]^2 \sim \int_0^\beta \mathrm{d}\tau \int_{x \gtrsim \beta} \mathrm{d}^3\mathbf{x} \left(\frac{Te^{-mx}}{4\pi x} \right)^2 . \tag{3.54}$$

If we now attempt to set $m \to 0$, we run into a linearly divergent integral. Because this problem emerges from large distances, we call this an *infrared divergence*.

In fact, it is easy to be more precise about the form of the divergence. We can namely write

$$\begin{aligned} \int_X [G_0(X)]^2 &= \int_X \SumInt_P \frac{e^{iP\cdot X}}{P^2 + m^2} \SumInt_Q \frac{e^{iQ\cdot X}}{Q^2 + m^2} \\ &= \SumInt_{PQ} \delta\!\!\!{}^-(P + Q) \frac{1}{(P^2 + m^2)(Q^2 + m^2)} \\ &= \SumInt_P \frac{1}{[P^2 + m^2]^2} \\ &= -\frac{\mathrm{d}}{\mathrm{d}m^2} I(m,T) \,. \end{aligned} \tag{3.55}$$

Inserting Eq. (3.50), we get

$$\int_X [G_0(X)]^2 = -\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}m} I(m,T) = \frac{T}{8\pi m} + \mathcal{O}(1) \;, \tag{3.56}$$

so that for $m \ll T$, Eq. (3.52) evaluates to

$$f_{(2)}(T) = -\frac{9}{4}\lambda^2 \frac{T^4}{144}\frac{T}{8\pi m} + \mathcal{O}(m^0) \;. \tag{3.57}$$

This indeed diverges for $m \to 0$.

It is clear that like the ultraviolet divergence in Eq. (3.51), the infrared divergence in Eq. (3.57) must be an artifact of some sort: the pressure and other thermodynamic properties of a plasma of weakly interacting massless scalar particles should be finite, as we know to be the case for a plasma of massless photons. We return to the resolution of this “paradox” in Sect. 3.4.

3.3 Proper Free Energy Density to $\mathcal{O}(\lambda)$: Ultraviolet Renormalization

In Sect. 3.2 we attempted to compute the free energy density $f(T)$ of a scalar field theory up to $\mathcal{O}(\lambda)$, but found a result which appeared to be ultraviolet (UV) divergent. Let us now show that, as must be the case in a renormalizable theory, the divergences disappear order-by-order in perturbation theory, if we *re-express* $f(T)$ *in terms of renormalized parameters*. Furthermore the renormalization procedure is identical to that at zero temperature.

In order to proceed, we need to change the notation somewhat. The zero-temperature parameters we employed before, i.e. m^2, λ, are now re-interpreted to be *bare parameters*, m_B^2, λ_B.[2] The expansion in Eq. (3.47) can then be written in the schematic form

$$f(T) = \phi^{(0)}(m_\mathrm{B}^2, T) + \lambda_\mathrm{B}\,\phi^{(1)}(m_\mathrm{B}^2, T) + \mathcal{O}(\lambda_\mathrm{B}^2) \;. \tag{3.58}$$

As a second step, we introduce the *renormalized parameters* m_R^2, λ_R. These could either be directly *physical quantities* (say, the mass of the scalar particle, and the scattering amplitude with particular kinematics), or quantities which are not directly physical, but are related to physical quantities by finite equations (say, so-called $\overline{\mathrm{MS}}$ scheme parameters). In any case, it is natural to choose the renormalized parameters such that in the limit of an extremely weak interaction, $\lambda_\mathrm{R} \ll 1$, they formally agree

[2] The temperature, in contrast, is a physical property of the system, and is not subject to any modification.

with the bare parameters. In other words, we may write

$$m_{\text{B}}^2 = m_{\text{R}}^2 + \lambda_{\text{R}} f^{(1)}(m_{\text{R}}^2) + \mathcal{O}(\lambda_{\text{R}}^2) \,, \tag{3.59}$$

$$\lambda_{\text{B}} = \lambda_{\text{R}} + \lambda_{\text{R}}^2 g^{(1)}(m_{\text{R}}^2) + \mathcal{O}(\lambda_{\text{R}}^3) \,, \tag{3.60}$$

where it is important to note that the renormalized parameters are defined at zero temperature (no T appears in these relations). The functions $f^{(i)}$ and $g^{(i)}$ are in general divergent in the limit that the regularization is removed; for instance, in dimensional regularization, they are expected to contain poles, such as $1/\epsilon$ or higher.

The idea now is to convert the expansion in Eq. (3.58) into an expansion in λ_{R} by inserting in it the expressions from Eqs. (3.59) and (3.60) and Taylor-expanding the result in λ_{R}. This produces

$$f(T) = \phi^{(0)}(m_{\text{R}}^2, T) + \lambda_{\text{R}}\left[\phi^{(1)}(m_{\text{R}}^2, T) + \frac{\partial \phi^{(0)}(m_{\text{R}}^2, T)}{\partial m_{\text{R}}^2} f^{(1)}(m_{\text{R}}^2)\right] + \mathcal{O}(\lambda_{\text{R}}^2) \,, \tag{3.61}$$

where we note that to $\mathcal{O}(\lambda_{\text{R}}^2)$ only the mass parameter needs to be renormalized.

To carry out renormalization in practice, we need to choose a *scheme*. We adopt here the so-called *pole mass scheme*, where m_{R}^2 is taken to be the physical mass squared of the ϕ-particle, denoted by m_{phys}^2. In Minkowskian spacetime, this quantity appears as an exponential time evolution,

$$e^{-iE_0 t} \equiv e^{-im_{\text{phys}} t} \,, \tag{3.62}$$

in the propagator of a particle at rest, $\mathbf{p} = \mathbf{0}$. In Euclidean spacetime, it on the other hand corresponds to an exponential fall-off, $\exp(-m_{\text{phys}}\tau)$, in the imaginary-time propagator. Therefore, in order to determine m_{phys}^2 to $\mathcal{O}(\lambda_{\text{R}})$, we need to compute the *full propagator*, $G(X)$, to $\mathcal{O}(\lambda_{\text{R}})$ at zero temperature.

The full propagator can be defined as the generalization of Eq. (3.39) to the interacting case:

$$\begin{aligned} G(X) &\equiv \frac{\langle \phi(X)\phi(0)\exp(-S_{\text{I}})\rangle_0}{\langle \exp(-S_{\text{I}})\rangle_0} \\ &= \frac{\langle \phi(X)\phi(0)\rangle_0 - \langle \phi(X)\phi(0)\, S_{\text{I}}\rangle_0 + \mathcal{O}(\lambda_{\text{B}}^2)}{1 - \langle S_{\text{I}}\rangle_0 + \mathcal{O}(\lambda_{\text{B}}^2)} \\ &= \langle \phi(X)\phi(0)\rangle_0 - \Big[\langle \phi(X)\phi(0)S_{\text{I}}\rangle_0 - \langle \phi(X)\phi(0)\rangle_0 \langle S_{\text{I}}\rangle_0\Big] + \mathcal{O}(\lambda_{\text{B}}^2) \,. \end{aligned} \tag{3.63}$$

We note that just like the subtractions in Eq. (3.10), the second term inside the square brackets serves to cancel disconnected contractions. Therefore, like in Eq. (3.12), we can drop the second term, if we replace the expectation value in the first one by $\langle \ldots \rangle_{0,\text{c}}$.

Let us now inspect the leading (zeroth order) term in Eq. (3.63), in order to learn how m_{phys} could most conveniently be extracted from the propagator. Introducing the notation

$$\int_P \equiv \lim_{T\to 0} \SumInt_P = \int \frac{\mathrm{d}^{d+1}P}{(2\pi)^{d+1}} \,, \tag{3.64}$$

and working in the $T = 0$ limit for the time being, the free propagator reads (cf. Eq. (3.28))

$$G_0(X) = \langle \phi(X)\phi(0) \rangle_0 = \int_P \frac{e^{iP\cdot X}}{P^2 + m^2} \,. \tag{3.65}$$

For Eq. (3.62), we need to project to zero spatial momentum, $\mathbf{p} = \mathbf{0}$; evidently this can be achieved by taking a spatial average of $G_0(X)$ via

$$\int_{\mathbf{x}} \langle \phi(\tau,\mathbf{x})\phi(0) \rangle_0 = \int \frac{\mathrm{d}p_0}{2\pi} \frac{e^{ip_0\tau}}{p_0^2 + m^2} \,. \tag{3.66}$$

We see that we get an integral which can be evaluated with the help of the Cauchy theorem and, in particular, that the exponential fall-off of the correlation function is determined by the pole position of the momentum-space propagator:

$$\int_{\mathbf{x}} \langle \phi(\tau,\mathbf{x})\phi(0) \rangle_0 = \frac{1}{2\pi} 2\pi i \frac{e^{-m\tau}}{2im} \,, \quad \tau \geq 0 \,. \tag{3.67}$$

Hence,

$$m^2_{\text{phys}}\Big|_{\lambda=0} = m^2 \,. \tag{3.68}$$

More generally, *the physical mass can be extracted by determining the pole position of the full propagator in momentum space, projected to* $\mathbf{p} = \mathbf{0}$.

We then proceed to the second term in Eq. (3.63), keeping still $T = 0$:

$$\begin{aligned}
-\langle \phi(X)\phi(0)\, S_{\text{I}} \rangle_{0,\text{c}} &= -\frac{\lambda_{\text{B}}}{4} \int_Y \langle \phi(X)\phi(0)\ \phi(Y)\phi(Y)\phi(Y)\phi(Y) \rangle_{0,\text{c}} \\
&= -\frac{\lambda_{\text{B}}}{4} \int_Y 4\times 3\, \langle \phi(X)\phi(Y) \rangle_0\, \langle \phi(Y)\phi(0) \rangle_0\, \langle \phi(Y)\phi(Y) \rangle_0 \\
&= -3\lambda_{\text{B}} G_0(0) \int_Y G_0(Y) G_0(X-Y) \\
&= -3\lambda_{\text{B}} \int_P \frac{1}{P^2 + m_{\text{B}}^2} \int_Y \int_{Q,R} e^{iQ\cdot Y} e^{iR\cdot(X-Y)} \frac{1}{Q^2 + m_{\text{B}}^2} \frac{1}{R^2 + m_{\text{B}}^2} \\
&= -3\lambda_{\text{B}}\, I_0(m_{\text{B}}) \int_R \frac{e^{iR\cdot X}}{(R^2 + m_{\text{B}}^2)^2} \,.
\end{aligned} \tag{3.69}$$

Summing this expression together with Eq. (3.65), the full propagator reads

$$G(X) = \int_P e^{iP\cdot X}\left[\frac{1}{P^2+m_{\rm B}^2} - 3\lambda_{\rm B} I_0(m_{\rm B})\frac{1}{(P^2+m_{\rm B}^2)^2} + \mathcal{O}(\lambda_{\rm B}^2)\right]$$
$$= \int_P \frac{e^{iP\cdot X}}{P^2+m_{\rm B}^2+3\lambda_{\rm B} I_0(m_{\rm B})} + \mathcal{O}(\lambda_{\rm B}^2)\,, \quad (3.70)$$

where we have resummed a series of higher-order corrections in a way that is correct to the indicated order of the weak-coupling expansion.

The same steps that led us from Eqs. (3.66) to (3.68) now produce

$$m_{\rm phys}^2 = m_{\rm B}^2 + 3\lambda_{\rm B} I_0(m_{\rm B}) + \mathcal{O}(\lambda_{\rm B}^2)\,. \quad (3.71)$$

Recalling from Eq. (3.60) that $m_{\rm B}^2 = m_{\rm R}^2 + \mathcal{O}(\lambda_{\rm R})$, $\lambda_{\rm B} = \lambda_{\rm R} + \mathcal{O}(\lambda_{\rm R}^2)$, this relation can be inverted to give

$$m_{\rm B}^2 = m_{\rm phys}^2 - 3\lambda_{\rm R} I_0(m_{\rm phys}) + \mathcal{O}(\lambda_{\rm R}^2)\,, \quad (3.72)$$

which corresponds to Eq. (3.59). The function I_0, given in Eq. (2.73), furthermore diverges in the limit $\epsilon \to 0$,

$$I_0(m_{\rm phys}) = -\frac{m_{\rm phys}^2\mu^{-2\epsilon}}{16\pi^2}\left[\frac{1}{\epsilon} + \ln\frac{\bar{\mu}^2}{m_{\rm phys}^2} + 1 + \mathcal{O}(\epsilon)\right], \quad (3.73)$$

and we may hope that this divergence cancels those we found in $f(T)$.

Indeed, let us repeat the steps from Eqs. (3.58) to (3.61) employing the explicit expression for the free energy density from Eq. (3.47),

$$f(T) = J(m_{\rm B},T) + \frac{3}{4}\lambda_{\rm B}[I(m_{\rm B},T)]^2 + \mathcal{O}(\lambda_{\rm B}^2)\,. \quad (3.74)$$

Recalling from Eq. (2.52) that

$$I(m,T) = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}m}J(m,T) = 2\frac{\mathrm{d}}{\mathrm{d}m^2}J(m,T)\,, \quad (3.75)$$

we can expand the two terms in Eq. (3.74) as a Taylor series around $m_{\rm phys}^2$, obtaining

$$J(m_{\rm B},T) = J(m_{\rm phys},T) + (m_{\rm B}^2 - m_{\rm phys}^2)\frac{\partial J(m_{\rm phys},T)}{\partial m_{\rm phys}^2} + \mathcal{O}(\lambda_{\rm R}^2)$$
$$= J(m_{\rm phys},T) - \frac{3}{2}\lambda_{\rm R} I_0(m_{\rm phys}) I(m_{\rm phys},T) + \mathcal{O}(\lambda_{\rm R}^2)\,, \quad (3.76)$$

$$\lambda_{\text{B}}[I(m_{\text{B}},T)]^2 = \lambda_{\text{R}}[I(m_{\text{phys}},T)]^2 + \mathcal{O}(\lambda_{\text{R}}^2) \ , \tag{3.77}$$

where in Eq. (3.76) we inserted Eq. (3.72). With this input, Eq. (3.74) becomes

$$\begin{aligned} f(T) &= J(m_{\text{phys}},T) + \frac{3}{4}\lambda_{\text{R}}\Big[I^2(m_{\text{phys}},T) - 2I_0(m_{\text{phys}})\,I(m_{\text{phys}},T)\Big] + \mathcal{O}(\lambda_{\text{R}}^2) \\ &= \underbrace{\Big\{J_0(m_{\text{phys}}) - \frac{3}{4}\lambda_{\text{R}}I_0^2(m_{\text{phys}})\Big\}}_{T=0 \text{ part}} + \underbrace{\Big\{J_T(m_{\text{phys}}) + \frac{3}{4}\lambda_{\text{R}}I_T^2(m_{\text{phys}})\Big\}}_{T\neq 0 \text{ part}} + \mathcal{O}(\lambda_{\text{R}}^2) \ , \end{aligned} \tag{3.78}$$

where we inserted the definitions $J(m,T) = J_0(m) + J_T(m)$ and $I(m,T) = I_0(m) + I_T(m)$.

Recalling Eqs. (2.72) and (2.73), we observe that the first term in Eq. (3.78), the "$T = 0$ part", is still divergent. However, this term is independent of the temperature, and thus plays no role in thermodynamics. Rather, it corresponds to a *vacuum energy density* that only plays a physical role in connection with gravity. If we included gravity, however, we should also include a bare cosmological constant, Λ_{B}, in the bare Lagrangian; this would contribute additively to Eq. (3.78), and we could simply identify the physical cosmological constant as

$$\Lambda_{\text{phys}} \equiv \Lambda_{\text{B}} + J_0(m_{\text{phys}}) - \frac{3}{4}\lambda_{\text{R}}I_0^2(m_{\text{phys}}) + \mathcal{O}(\lambda_{\text{R}}^2) \ . \tag{3.79}$$

The divergences would now be cancelled by Λ_{B}, and Λ_{phys} would be finite.

In contrast, the second term in Eq. (3.78), the "$T \neq 0$ part", is finite: it contains the functions J_T, I_T for which we have analytically determined various limiting values in Eqs. (2.79), (2.80), (2.82) and (2.93), as well as general integral representations in Eqs. (2.76) and (2.77). Therefore all thermodynamic quantities obtained from derivatives of $f(T)$, such as the entropy density or specific heat, are manifestly finite. In other words, the temperature-dependent ultraviolet divergences that we found in Sect. 3.2 have disappeared through zero-temperature renormalization.

3.4 Proper Free Energy Density to $\mathcal{O}(\lambda^{\frac{3}{2}})$: Infrared Resummation

We now move on to a topic which is in a sense maximally different from the UV issues discussed in the previous section, and consider the limit where the physical mass of the scalar field, m_{phys}, tends to zero. With a few technical modifications, this would be the case (in perturbation theory) for, say, gluons in QCD. According to Eq. (3.72), this limit corresponds to $m_{\text{B}} \to 0$, since $I_0(0) = 0$; then we are faced with the infrared problem discussed in Sect. 3.2.

In the limit of a small mass, we can employ high-temperature expansions for the functions $J(m,T)$ and $I(m,T)$, given in Eqs. (2.92) and (2.95). Employing Eqs. (3.47) and (3.57), we write the leading terms in the small-$m_{\rm B}$ expansion as

$$\mathcal{O}(\lambda_{\rm B}^0): \quad f_{(0)}(T) = J(m_{\rm B},T) = -\frac{\pi^2 T^4}{90} + \frac{m_{\rm B}^2 T^2}{24} - \frac{m_{\rm B}^3 T}{12\pi} + \mathcal{O}(m_{\rm B}^4) \,, \tag{3.80}$$

$$\begin{aligned} \mathcal{O}(\lambda_{\rm B}^1): \quad f_{(1)}(T) &= \frac{3}{4}\lambda_{\rm B}[I(m_{\rm B},T)]^2 \\ &= \frac{3}{4}\lambda_{\rm B}\left[\frac{T^2}{12} - \frac{m_{\rm B}T}{4\pi} + \mathcal{O}(m_{\rm B}^2)\right]^2 \\ &= \frac{3}{4}\lambda_{\rm B}\left[\frac{T^4}{144} - \frac{m_{\rm B}T^3}{24\pi} + \mathcal{O}(m_{\rm B}^2 T^2)\right] \,, \end{aligned} \tag{3.81}$$

$$\mathcal{O}(\lambda_{\rm B}^2): \quad f_{(2)}(T) = -\frac{9}{4}\lambda_{\rm B}^2 \frac{T^4}{144}\frac{T}{8\pi m_{\rm B}} + \mathcal{O}(m_{\rm B}^0) \,. \tag{3.82}$$

Let us inspect, in particular, *odd powers of* m_B, which according to Eqs. (3.80)–(3.82) are becoming increasingly important as we go further in the expansion. We remember from Sect. 2.3 that odd powers of $m_{\rm B}$ are necessarily associated with contributions from the Matsubara zero mode. In fact, the odd power in Eq. (3.80) is directly the zero-mode contribution to Eq. (2.87),

$$\delta_{\rm odd} f_{(0)} = J^{(n=0)} = -\frac{m_{\rm B}^3 T}{12\pi} \,. \tag{3.83}$$

The odd power in Eq. (3.81) on the other hand originates from a cross-term between the zero-mode contribution and the leading non-zero mode contribution to $I(0,T)$:

$$\delta_{\rm odd} f_{(1)} = \frac{3}{2}\lambda_{\rm B} \times I'(0,T) \times I^{(n=0)} = -\frac{\lambda_{\rm B} m_{\rm B} T^3}{32\pi} \,. \tag{3.84}$$

Finally, the small-$m_{\rm B}$ divergence in Eq. (3.82) comes from a product of two non-zero mode contributions and a particularly infrared sensitive zero-mode contribution:

$$\delta_{\rm odd} f_{(2)} = \frac{9}{4}\lambda_{\rm B}^2 \times [I'(0,T)]^2 \times \frac{{\rm d}I^{(n=0)}}{{\rm d}m_{\rm B}^2} = -\frac{\lambda_{\rm B}^2 T^5}{8^3 \pi m_{\rm B}} \,. \tag{3.85}$$

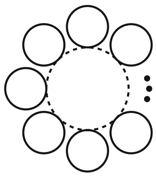
Comparing these structures, we see that the "expansion parameter" related to odd powers is

$$\frac{\delta_{\rm odd} f_{(1)}}{\delta_{\rm odd} f_{(0)}} \sim \frac{\delta_{\rm odd} f_{(2)}}{\delta_{\rm odd} f_{(1)}} \sim \frac{\lambda_{\rm B} T^2}{8 m_{\rm B}^2} \,. \tag{3.86}$$

Thus, if we try to set $m_{\rm B}^2 \to 0$ (or even just $m_{\rm B}^2 \ll \lambda_{\rm B} T^2/8$), the loop expansion shows no convergence.

In order to cure the problem with the infrared (IR) sensitivity of the loop expansion, our goal now becomes to *identify and sum the divergent terms to all orders*. We may then expect that the complete sum obtains a form where we can set $m_{\rm B}^2 \to 0$ without meeting divergences. This procedure is often referred to as *resummation*.

Fortunately, it is indeed possible to identify the problematic terms. Equations (3.83)–(3.85) already suggest that at order N in $\lambda_{\rm B}$, they are associated with terms containing N non-zero mode contributions $I'(0,T)$, and *one zero-mode contribution*. Graphically, this corresponds to a single loop formed by a zero-mode propagator, dressed with N non-zero mode "bubbles". Such graphs are usually called "ring" or "daisy" diagrams, and can be illustrated as follows (the dashed line is a zero-mode propagator, solid lines are non-zero mode propagators):



(3.87)

To be more quantitative, we consider Eq. (3.12) at order $\lambda_{\rm B}^N$. A straightforward combinatorial analysis then gives

$$f(T) = \Big\langle S_{\rm I} - \frac{1}{2} S_{\rm I}^2 + \ldots + \frac{(-1)^{N+1}}{N!} S_{\rm I}^N \Big\rangle_{0,\text{c, drop overall} \int_X}$$

$$\Rightarrow \frac{(-1)^{N+1}}{N!}\left(\frac{\lambda_{\rm B}}{4}\right)^N \Big\langle \underbrace{\phi\,\phi\,\phi\,\phi}_{6} \;\; \underbrace{\phi\,\phi\,\phi\,\phi}_{2(N-1)} \;\; \underbrace{\phi\,\phi\,\phi\,\phi}_{2(N-2)} \;\; \cdots \;\; \phi\,\phi\,\phi\,\phi \Big\rangle_{0,\ldots}$$

$$= \frac{(-1)^{N+1}}{N!}\left(\frac{\lambda_{\rm B}}{4}\right)^N 6^N \underbrace{[2(N-1)][2(N-2)]\ldots[2]}_{2^{N-1}(N-1)!} \Big[\underbrace{\frac{T^2}{12}}_{I'(0,T)}\Big]^N \underbrace{T\int \frac{{\rm d}^d\mathbf{p}}{(2\pi)^d}\left(\frac{1}{p^2+m_{\rm B}^2}\right)^N}_{\text{zero-mode part}}, \tag{3.88}$$

where we have indicated the contractions from which the various factors originate. Let us compute the zero-mode part for the first few orders, omitting for simplicity terms of $\mathcal{O}(\epsilon)$:

$$N = 1: \int_{\mathbf{p}} \frac{1}{p^2 + m_{\rm B}^2} = -\frac{m_{\rm B}}{4\pi} = \frac{\rm d}{{\rm d}m_{\rm B}^2}\left(-\frac{m_{\rm B}^3}{6\pi}\right),$$

$$N = 2: \int_{\mathbf{p}} \frac{1}{(p^2 + m_{\rm B}^2)^2} = -\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\left(-\frac{m_{\rm B}}{4\pi}\right) = -\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\left(-\frac{m_{\rm B}^3}{6\pi}\right),$$

$$\text{generally}: \int_{\mathbf{p}} \frac{1}{(p^2 + m_{\rm B}^2)^N} = -\frac{1}{N-1}\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\int_{\mathbf{p}} \frac{1}{(p^2 + m_{\rm B}^2)^{N-1}}$$

$$= \left(\frac{-1}{N-1}\right)\left(\frac{-1}{N-2}\right)\cdots\left(\frac{-1}{1}\right)\left(\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\right)^{N-1}\int_{\mathbf{p}} \frac{1}{p^2 + m_{\rm B}^2}$$

$$= \frac{(-1)^N}{(N-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\right)^N\left(\frac{m_{\rm B}^3}{6\pi}\right). \tag{3.89}$$

Combining Eqs. (3.88) and (3.89), we get

$$\delta_{\rm odd} f_{(N)} = \frac{(-1)^{N+1}}{N!}\left(\frac{3\lambda_{\rm B}}{2}\right)^N 2^{N-1}(N-1)!\left(\frac{T^2}{12}\right)^N T\frac{(-1)^N}{(N-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\right)^N\left(\frac{m_{\rm B}^3}{6\pi}\right)$$

$$= -\frac{T}{2}\frac{1}{N!}\left(\frac{\lambda_{\rm B}T^2}{4}\right)^N\left(\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\right)^N\left(\frac{m_{\rm B}^3}{6\pi}\right). \tag{3.90}$$

As a crosscheck, it can easily be verified that this expression reproduces Eqs. (3.83)–(3.85).

Now, owing to the fact that Eq. (3.90) has precisely the right structure to correspond to a Taylor expansion, we can sum the contributions in Eq. (3.90) to all orders, obtaining

$$\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{\lambda_{\rm B}T^2}{4}\right)^N\left(\frac{\mathrm{d}}{\mathrm{d}m_{\rm B}^2}\right)^N\left(-\frac{m_{\rm B}^3 T}{12\pi}\right) = -\frac{T}{12\pi}\left(m_{\rm B}^2 + \frac{\lambda_{\rm B}T^2}{4}\right)^{\frac{3}{2}}. \tag{3.91}$$

We observe that a "miracle" has happened: in Eq. (3.91) the limit $m_{\rm B}^2 \to 0$ can be taken without divergences. But there is a surprise: setting the mass parameter to zero, we arrive at a contribution of $\mathcal{O}(\lambda_{\rm B}^{3/2})$, rather than $\mathcal{O}(\lambda_{\rm B}^2)$ as naively expected in Sect. 3.2. In other words, infrared divergences modify qualitatively the structure of the weak-coupling expansion.

Setting finally $m_{\rm B}^2 \to 0$ everywhere, and collecting all finite terms from Eqs. (3.80), (3.81) and (3.91), we find the correct expansion of $f(T)$ in the massless limit,

$$f(T) = -\frac{\pi^2 T^4}{90} + \frac{\lambda_{\rm B}T^4}{4\times 48} - \frac{T}{12\pi}\left(\frac{\lambda_{\rm B}T^2}{4}\right)^{3/2} + \mathcal{O}(\lambda_{\rm B}^2 T^4) \tag{3.92}$$

$$= -\frac{\pi^2 T^4}{90}\left[1 - \frac{15}{32}\frac{\lambda_{\rm R}}{\pi^2} + \frac{15}{16}\left(\frac{\lambda_{\rm R}}{\pi^2}\right)^{\frac{3}{2}} + \mathcal{O}(\lambda_{\rm R}^2)\right], \tag{3.93}$$

where at the last stage we inserted $\lambda_{\rm B} = \lambda_{\rm R} + \mathcal{O}(\lambda_{\rm R}^2)$.

It is appropriate to add that despite the complications we have found, higher-order corrections can be computed to Eq. (3.93). In fact, as of today, the coefficients of the seven subsequent terms, of orders $\mathcal{O}(\lambda_{\rm R}^2)$, $\mathcal{O}(\lambda_{\rm R}^{5/2}\ln\lambda_{\rm R})$, $\mathcal{O}(\lambda_{\rm R}^{5/2})$, $\mathcal{O}(\lambda_{\rm R}^3\ln\lambda_{\rm R})$, $\mathcal{O}(\lambda_{\rm R}^3)$, $\mathcal{O}(\lambda_{\rm R}^{7/2})$, and $\mathcal{O}(\lambda_{\rm R}^8\ln\lambda_{\rm R})$, are known [1, 2]. This progress is possible due to the fact that the resummation of higher-order contributions that we carried out explicitly in this section can be implemented more elegantly and systematically with so-called *effective field theory methods*. We return to this general procedure in Sect. 6, but some flavour can be obtained by organizing the above computation in yet another way, outlined in the appendix below.

Appendix: An Alternative Method for Resummation

In this appendix we show that the previous resummation can also be implemented through the following steps:

(i) Following the computation of $m^2_{\rm phys}$ in Eq. (3.71) but working now at finite temperature, we determine a specific T-dependent pole mass in the $m_{\rm B} \to 0$ limit. The result can be called an *effective thermal mass*, $m^2_{\rm eff}$.

(ii) We argue that in the weak-coupling limit ($\lambda_{\rm R} \ll 1$), the thermal mass is important only for the Matsubara zero mode [3].

(iii) Writing the Lagrangian (for $m_{\rm B}^2 = 0$) in the form

$$L_E = \underbrace{\frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{1}{2}m^2_{\rm eff}\,\phi^2_{n=0}}_{L_0} + \underbrace{\frac{1}{4}\lambda_{\rm B}\phi^4 - \frac{1}{2}m^2_{\rm eff}\,\phi^2_{n=0}}_{L_{\rm I}}\,, \tag{3.94}$$

we treat L_0 as the free theory and $L_{\rm I}$ as an interaction of order $\lambda_{\rm R}$. With this reorganization of the theory, we write down the contributions $f_{(0)}$ and $f_{(1)}$ to the free energy density, and check that we obtain a well-behaved perturbative expansion that produces a result agreeing with what we got in Eq. (3.92).

Starting with the effective mass parameter, the computation proceeds precisely like the one leading to Eq. (3.71), with just the replacement $\int_P \to \SumInt_P$. Consequently,

$$m^2_{\rm eff} = \lim_{m_{\rm B}^2\to 0}\Big[m_{\rm B}^2 + 3\lambda_{\rm B} I(m_{\rm B}, T)\Big] = 3\lambda_{\rm B} I(0,T) = \frac{\lambda_{\rm R} T^2}{4} + \mathcal{O}(\lambda_{\rm R}^2)\,. \tag{3.95}$$

We note that for the *non-zero* Matsubara modes, with $\omega_n \neq 0$, we have $m^2_{\rm eff} \ll \omega_n^2$ in the weak-coupling limit $\lambda_{\rm R} \ll (4\pi)^2$, so that the thermal mass plays a subdominant role in the propagator. In contrast, for the Matsubara zero mode, $m^2_{\rm eff}$ modifies the propagator significantly for $p^2 \ll m^2_{\rm eff}$, removing any infrared divergences. This observation justifies the fact that the thermal mass was only introduced for the $n = 0$ mode in Eq. (3.94).

With our new reorganization, the free propagators become different for the Matsubara zero ($\tilde{\phi}_{n=0}$) and non-zero ($\tilde{\phi}'$) modes:

$$\langle \tilde{\phi}'(P)\tilde{\phi}'(Q)\rangle_0 = \not{\delta}(P+Q)\,\frac{1}{\omega_n^2+p^2}\,, \tag{3.96}$$

$$\langle \tilde{\phi}_{n=0}(P)\tilde{\phi}_{n=0}(Q)\rangle_0 = \not{\delta}(P+Q)\,\frac{1}{p^2+m_{\text{eff}}^2}\,. \tag{3.97}$$

Consequently, Eq. (3.17) gets replaced with

$$\begin{aligned} f_{(0)}(T) &= \sum\!\!\!\!\!\!\!\!\int_P{}' \frac{1}{2}\ln(P^2) + T\int_{\mathbf{p}} \frac{1}{2}\ln(p^2+m_{\text{eff}}^2) - \text{const.} \\ &= J'(0,T) + J^{(n=0)}(m_{\text{eff}},T) \\ &= -\frac{\pi^2 T^4}{90} - \frac{m_{\text{eff}}^3 T}{12\pi}\,. \end{aligned} \tag{3.98}$$

In the massless first term, the omission of the zero mode made no difference. Similarly, with $f_{(1)}$ now coming from L_{I} in Eq. (3.94), Eq. (3.19) is modified into

$$\begin{aligned} f_{(1)}(T) &= \frac{3}{4}\lambda_{\text{B}}\langle\phi(0)\phi(0)\rangle_0\langle\phi(0)\phi(0)\rangle_0 - \frac{1}{2}m_{\text{eff}}^2\langle\phi_{n=0}(0)\phi_{n=0}(0)\rangle_0 \\ &= \frac{3}{4}\lambda_{\text{B}}\Big[I'(0,T) + I^{(n=0)}(m_{\text{eff}},T)\Big]^2 - \frac{1}{2}m_{\text{eff}}^2 I^{(n=0)}(m_{\text{eff}},T) \\ &= \frac{3}{4}\lambda_{\text{B}}\left[\frac{T^4}{144} - \frac{m_{\text{eff}}T^3}{24\pi} + \frac{m_{\text{eff}}^2T^2}{16\pi^2}\right] + \frac{1}{2}m_{\text{eff}}^2\frac{m_{\text{eff}}T}{4\pi}\,. \end{aligned} \tag{3.99}$$

Inserting Eq. (3.95) into the last term of Eq. (3.99), we see that this contribution precisely cancels against the linear term within the square brackets. As we recall from Eq. (3.84), the linear term was part of the problematic series that needed to be resummed. Combining Eqs. (3.98) and (3.99), we instead get

$$f(T) = -\frac{\pi^2T^4}{90} + \frac{3}{4}\lambda_{\text{R}}\frac{T^4}{144} - \frac{m_{\text{eff}}^3T}{12\pi} + \mathcal{O}(\lambda_{\text{R}}^2)\,, \tag{3.100}$$

which agrees with Eq. (3.93).

The cancellation that took place in Eq. (3.99) can also be verified at higher orders. In particular, proceeding to $\mathcal{O}(\lambda_{\text{R}}^2)$, it can be seen that the structure in Eq. (3.85) gets cancelled as well. Indeed, the resummation of infrared divergences that we carried out explicitly in Eq. (3.91) can be fully captured by the reorganization in Eq. (3.94).

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