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Supergroup Actions and Harmonic Analysis

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Abstract. Kirillov's orbit philosophy holds for nilpotent Lie supergroups in a narrow sense, but due to the paucity of unitary representations, it falls short of being an effective tool of harmonic analysis in its present form. In this note, we survey an approach using families of coadjoint orbits which remedies this deficiency, at least in relevant examples.

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1. Introduction

The correspondence principle states that in the limit of large quantum numbers, quantum mechanics should reproduce classical mechanics. Quantization is the endeavour of reverse-engineering this correspondence in order to produce viable quantum models.

A prominent approach to this task is Geometric Quantization. Its notable strength lies in its ability to associate, with non-linear phase-space symmetries (a.k.a. symplectic Lie group actions), unitary symmetries of the quantum Hilbert space (a.k.a. unitary representations). Taking this ideology to extremes, one may entertain the idea that all irreducible unitary representations of some given Lie group G might be obtained by the quantization of some universal homogeneous symplectic G-spaces.

It is a famous result due to A.A. Kirillov [9] (partly reformulating earlier results due to J. Dixmier [8]) that this sanguine assumption is a hard fact, at least for nilpotent Lie groups.

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Theorem 1 (Kirillov 1962). *Let G be a simply-connected and connected nilpotent Lie group. There is a bijection between the isomorphism classes of irreducible unitary representations of G and the orbits of G in the coadjoint representation on $\mathfrak{g}^*$.*

Moreover, Kirillov showed that the regular representation of G decomposes in a natural fashion as a direct integral over the orbit space $\mathfrak{g}^*/G$. His ideas have been vastly extended and generalized, under the epithet of the "orbit method" or "orbit philosophy", thereby also shedding light on some older results. For example, the Peter–Weyl decomposition for a compact Lie group G can be obtained by applying Geometric Quantization to T^*G.

It is with these applications to harmonic analysis in mind that we will outline, in this survey, a new approach (developed jointly with J. Hilgert and T. Wurzbacher) to bring the orbit philosophy to fruition for Lie supergroups. Lie supergroups appear as the non-linear classical counterparts to the supersymmetries of quantum field theories, both fundamentally in high-energy physics and as effective symmetries of quasiparticles in condensed matter theory. The lack of a fully satisfactory theory of harmonic analysis for Lie supergroups is therefore a major drawback. In fact, as B. Kostant notes in a fundamental paper on the subject [11]: "[Lie supergroups are] likely to be [...] useful [objects] only insofar as one can develop a corresponding theory of harmonic analysis".

We reassess this basic problem and extend the basic notion of "orbits" by allowing for the presence of auxiliary parameters. This entails some necessary upgrades to the terminology, which we motivate and explain at length in this survey. As we show in examples, the resolution of the attendant technical difficulties dispels some of the basic limitations of the more traditional approaches, hopefully bringing us closer to the fulfilment of Kostant's vision.

Let us close this introduction with a synopsis of the article. After a pedestrian introduction to the wherewithal of supermanifolds in Section 2, we proceed to illustrate the failure of the orbit philosophy (in its traditional sense) for Lie supergroups in Section 3. We introduce our approach in Sections 4 and 5. In Section 6, we show how these ideas help to overcome some of the apparent limitations of the orbit philosophy, at least in some pertinent examples.

2. Supermanifolds in a nutshell

Supermanifolds arose in an attempt to define geometries supporting classical field theories which correspond to the bosonic and fermionic fields encountered in quantum field theory. In addition to the ordinary "even" (or bosonic) coordinates, such geometries allow for "odd" (or fermionic) coordinates which mutually anticommute and commute with their even counterparts.

Formally, such geometries are modelled by extending the algebra of (smooth, analytic, or holomorphic) functions to

$$\mathcal{F}(M)[\xi^1, \ldots, \xi^q] = \mathcal{F}(M) \otimes \bigwedge(\xi^1, \ldots, \xi^q)$$

where $\mathcal{F}(\cdots)$ is the algebra the ordinary functions and $\bigwedge(\cdots)$ denotes the Grassmann algebra in the generators ξ^μ. This is a *superalgebra*, *i.e.*, it admits a grading with respect to $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$.

Thus, a *superspace* X consists of the data of a) a topological space, denoted by X_0, and b) a sheaf of local algebras $\mathcal{O}_X$ on X_0 (where the ground field $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$). Here, $\mathcal{O}_X$ is an abstraction of the "algebra of functions", assigning to any open subset $U \subseteq X_0$ the "functions" defined on U, and the word "local" is a technical condition ensuring that the notion of the *value* (or "numerical part") of a function is well defined at every point $x \in U$.

The most basic example of a superspace is obtained as above, *viz.*

$$\mathbb{A}^{p|q} = \big((\mathbb{A}^{p|q})_0, \mathcal{O}_{\mathbb{A}^{p|q}}\big),$$
$$(\mathbb{A}^{p|q})_0 := \Bbbk^p, \quad \mathcal{O}_{\mathbb{A}^{p|q}} := \mathcal{F}_{\Bbbk^p}(-, \mathbb{K}) \otimes \textstyle\bigwedge(\xi^1, \ldots, \xi^q).$$

Here, $\Bbbk \subseteq \mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{F}_{\Bbbk^p}(-, \mathbb{K})$ is the sheaf of $\mathbb{K}$-valued functions on $\Bbbk^p$ – where, according to our persuasion (which may vary over time), we take the liberty to consider smooth or $\Bbbk$-analytic functions. Given any superspace X and an open subset $U \subseteq X_0$, we may define the *open subspace* $X|_U$ on the set U to be the pair $(U, \mathcal{O}_X|_U)$.

Just as important as the notion of a "space" is the notion of a "map", incorporating central physical concepts such as trajectory, field, and gauge transformation. In local coordinates, maps of (smooth, analytic, or complex) manifolds take the form

$$y^\mu = \varphi^\mu(x^1, \ldots, x^n)$$

where on the right are arbitrary (smooth, analytic, or holomorphic) functions.

This is no different for supermanifolds; the only new distinction is between the parity (even/odd) assigned to the coordinates. Thus, grouping the coordinates according to their parity as $y = (v, \eta)$, $x = (u, \xi)$, "maps" of supermanifolds are of the form

$$\begin{aligned} v^a &= \varphi^a(u^1, \ldots, u^p, \xi^1, \ldots, \xi^q), \\ \eta^b &= \varphi^b(u^1, \ldots, u^p, \xi^1, \ldots, \xi^q), \end{aligned} \tag{1}$$

where again, the functions on the right are arbitrary – up to their parity, which is fixed by the left-hand side.

To make sense of this in our formal framework, we are faced with a conundrum: In order to speak of local coordinates, we need a notion of charts, so we need to know what a map is in the first place. The solution is to change perspective and consider maps as *devices which pull back functions*; the statement that for supermanifolds, the thus defined maps are indeed determined by the data in (1) (that is, by the pullback of coordinates), is then a non-trivial fact, due to D. Leites [12].

Thus, technically, a *morphism* $\varphi : X \longrightarrow Y$ of superspaces comprises the following data: a) a continuous map denoted by $\varphi_0 : X_0 \longrightarrow Y_0$ and b) a local morphism of superalgebra sheaves $\varphi^\sharp : \mathcal{O}_Y \longrightarrow (\varphi_0)_*(\mathcal{O}_X)$. That is, on any open set $V \subseteq Y_0$, to any function $f \in \mathcal{O}_Y(V)$ defined on V is assigned the pulled back function $\varphi^\sharp(f) \in \mathcal{O}_X(\varphi_0^{-1}(V))$ – whilst preserving the algebra structure and the grading. As before, the word "local" is a technical condition ensuring that the pullback preserves values, that is, that the equality $\varphi^\sharp(f)(x) = f(\varphi_0(x))$ holds whenever it makes sense.

As an example, consider the morphism $\varphi : \mathbb{A}^{1|2} \longrightarrow \mathbb{A}^{1|2}$ determined by the assignment

$$\varphi^\sharp(u) = u + \xi^1\xi^2, \quad \varphi^\sharp(\xi^b) = \xi^b \ (b = 1, 2).$$

Its effect on a general function $f = \sum_I f_I \xi^I \equiv f_\varnothing + f_1\xi^1 + f_2\xi^2 + f_{12}\xi^1\xi^2$ is

$$\varphi^\sharp(f) = f_\varnothing + \frac{\partial f_\varnothing}{\partial u}\xi^1\xi^2 + \sum\nolimits_{I \neq \varnothing} f_I \xi^I.$$

With these notions in place, we may now pose the following definition.

Definition 2. A *supermanifold* X is a superspace whose underlying topological space X_0 is Hausdorff and which is locally isomorphic to $\mathbb{A}^{p|q}$. Here, the latter statement means that for any $x \in X_0$, there are open sets U and V (with $x \in U$) and an isomorphism $X|_U \longrightarrow \mathbb{A}^{p|q}|_V$.

Notice that according to our persuasion (*i.e.*, our choice of function sheaf on the model space $\mathbb{A}^{p|q}$), we have defined the notion of *smooth*, *analytic*, or *complex* (*i.e.*, holomorphic) supermanifold.

A popular example of a supermanifold is obtained thus: Take any manifold X and define

$$\Pi TX := \big(X, \Omega_X^\bullet\big),$$

where $\Omega_X^\bullet$ is the sheaf of differential forms on X, with the exterior product as algebra multiplication and the $\mathbb{Z}_2$-grading induced by the degree of differential forms. This supermanifold is called the *parity-reversed tangent bundle* on X. Its main distinction is that it carries a canonical *odd vector field* – *i.e.*, a parity-reversing endomorphism of the function sheaf $\mathcal{O}_{\Pi TX} = \Omega_X^\bullet$ following a graded Leibniz rule – namely, the de Rham differential d.

Much of the local theory of manifolds goes through for supermanifolds without essential changes; in-depth accounts can be found in Refs. [6, 7, 12, 13]. In particular, supermanifolds admit direct products, and this allows us to define the notion of a Lie supergroup, generalizing that of a Lie group.

Definition 3. A *Lie supergroup* is a group object in the category whose objects are the supermanifolds and whose morphisms are the morphisms of superspaces. In other words, a Lie supergroup is the datum of a supermanifold G, together with morphisms $m : G \times G \longrightarrow G$, $1 : * \longrightarrow G$ (where $* = \mathbb{A}^{0|0}$ is the singleton space), and $i : G \longrightarrow G$, which respectively obey the defining equations of multiplication, unit element, and inverse in a group.

Similarly, a (left) action of a Lie supergroup G on a supermanifold X is a morphism $a : G \times X \longrightarrow X$ satisfying the defining equations of a group action on a set. A way to express this formally is to postulate the commutativity of the following diagrams:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \mathrm{id}_G} & G \times X \\ {\scriptstyle \mathrm{id}_G \times a}\downarrow & & \downarrow{\scriptstyle a} \\ G \times X & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{(1, \mathrm{id}_X)} & G \times X \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle a} \\ & & X \end{array}$$

that express, respectively, the associative and unit laws for the action.

An example of a Lie supergroup structure on $G = \mathbb{A}^{1|2}$ is obtained by writing its standard coordinates (u, ξ, η) in a matrix of the following shape:

$$\begin{pmatrix} 1 & u & -\xi \\ 0 & 1 & -\eta \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Matrix multiplication and inversion will then define a Lie supergroup structure. (The signs do not play a role here, but are vital in other contexts.) The Lie supergroup thus determined will be called the *Heisenberg supergroup with odd centre*. Explicitly, we have

$$m^\sharp(u) = u^1 + u^2, \quad m^\sharp(\xi) = \xi^1 + \xi^2 + u^1\eta^2, \quad m^\sharp(\eta) = \eta^1 + \eta^2,$$

and expressions for the inverse can be similarly derived. An example of an action of G on $X = \mathbb{A}^{2|1}$ is given by writing its standard coordinates (s, t, θ) in a column as follows:

$$\begin{pmatrix} s \\ t \\ -\theta \end{pmatrix},$$

and multiplying from the left by the matrix in Equation (2). It is immediate that this action fixes any point of the form $(0, t_0) \in \mathbb{A}^{2|1}_0 = \Bbbk^2$, although the coordinate t is not fixed, but instead mapped to $t + \eta\theta$.

A less contrived example of an action is obtained by integrating the odd vector field d on ΠTX (where X is any manifold) to an action of the additive Lie supergroup G of $\mathbb{A}^{0|1}$: If θ is the coordinate on $G = \mathbb{A}^{0|1}$, then the action morphism a is determined by

$$a^\sharp(\omega) := \omega + \theta d\omega$$

for any differential form ω on X. Notice that a_0 is the identity of X, but the action is far from trivial: The invariant functions on ΠTX are exactly the closed differential forms on X.

It is instructive to write this in coordinates, say for $X = \mathbb{A}^1$. We have $\Pi TX \cong \mathbb{A}^{1|1}$ with the coordinates $(u, \xi = du)$ where u is the standard coordinate on X. Then

$$a^\sharp(u) = u + \theta\xi, \quad a^\sharp(\xi) = \xi. \tag{3}$$

This can again be realized by matrix multiplication:

$$\begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ -\xi \end{pmatrix} = \begin{pmatrix} u + \theta\xi \\ -\xi \end{pmatrix}.$$

3. The orbit nightmare for Lie supergroups

While the basic definitions in the theory of supermanifolds appear innocent enough, the simple examples discussed in the previous section may serve as an indication to all that is not well in the world of Lie supergroup actions. Nonetheless, one may still hope for a generalization of Kirillov's orbit philosophy to this universe.

In fact, quite some work has been done in this direction, beginning with B. Kostant, who, among other things, defined the coadjoint action of a Lie supergroup more or less simultaneously with the introduction of the latter concept [10]. He also defined homogeneous spaces of Lie supergroups, using the language of Lie–Hopf algebras. This was later recast in the language we are using here by Boyer–Sánchez-Valenzuela [4]. We briefly review the results.

Given an action a of a Lie supergroup G on a supermanifold X and a point $x \in X_0$, there is a natural notion of *isotropy supergroup.* (We will come back to this later.) As the above authors show, it is a closed subsupergroup G_x of G. Moreover, there is a natural supermanifold G/G_x and a surjective submersion $\pi : G \longrightarrow G\cdot x := G/G_x$ satisfying the obvious universal property. In particular, the inclusion $* \longrightarrow X$ of the point x factors through a natural G-equivariant injective immersion $G\cdot x \longrightarrow X$ – this is the *orbit of* x. In the case of the coadjoint action of G of $\mathfrak{g}^*$, the orbits $G \cdot f$ carry a natural supersymplectic structure invariant under the action.

Thus, coadjoint orbits are in place, and there is also a natural notion of *representation* for a Lie supergroup. For the case of $\Bbbk = \mathbb{K} = \mathbb{R}$, there is also a natural notion of *unitary representation* [5].

The following striking result of H. Salmasian [15] shows that these concepts are in unison, in perfect agreement with the orbit philosophy.

Theorem 4 (Salmasian 2010). *Let G be a simply-connected and connected nilpotent Lie supergroup. The orbits through points of the coadjoint action of G on $\mathfrak{g}^*$ are in bijection with the irreducible unitary representations of G up to parity and isomorphism.*

Let us apply to the simplest possible example, the additive group G of the affine supermanifold $\mathfrak{g} = \mathbb{A}^{0|q}$. The coadjoint action is trivial since G is Abelian. There is only one point of $\mathfrak{g}^*$, so there is only one orbit, the singleton space. On the other hand, there is up to parity and isomorphism only one irreducible unitary representation, namely, the trivial one.

While this confirms Salmasian's theorem and thereby in a narrow sense the orbit philosophy, it shows also that a decomposition of the regular representation on the space of functions $\mathcal{O}_G$ into unitary irreducibles is not conceivable in the traditional sense, as this representation is far from trivial.

What has gone wrong? The examples of Lie supergroup actions considered above suggest that orbits through ordinary points retain only an insufficient fraction of the information on the action. As we shall now argue, a remedy to this defect is to generalize the notion of points.

4. Points manifesto

In order to generalize the notion of points, it is first necessary to rephrase it. As we observed above, a point $x \in X_0$ of a supermanifold gives rise to a morphism $* \longrightarrow X$ from the singleton space $* = \mathbb{A}^{0|0}$ which assigns to a function on X its value at x. This actually sets up a bijection between the elements of X_0 and the morphisms $* \longrightarrow X$.

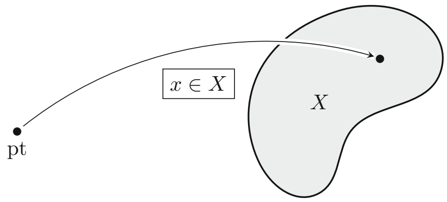

FIGURE 1. A point is a map from the singleton space.

The problem with such simple-minded ordinary points is that they have no "odd directions" with which to trace out those of X. So it is natural to allow them to acquire further degrees of freedom, that is, to replace the singleton $*$ by a general supermanifold T. This leads to the following notion.

Definition 5. Let X be a supermanifold. A *T-point of X*, where T is another supermanifold, is a morphism $x : T \longrightarrow X$. We write $x \in_T X$ and denote the set of T-points of X by $X(T)$.

Intuitively, a T-point of X is a family of points in X parametrised by the auxiliary space T – but this intuition has a limited validity, since a T-point carries more information than the range of the underlying map.

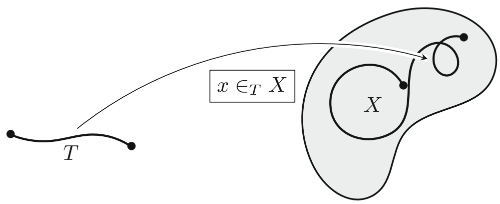

FIGURE 2. A T-point is a T-parameter family of points.

Working with T-points has many advantages: One is that it replaces the supermanifolds and their morphisms by sets and maps of sets.

Indeed, the supermanifold X is replaced by *sets* of T-points, for any T. Similarly, a morphism $\varphi : X \longrightarrow Y$ is replaced by the *maps* $X(T) \longrightarrow Y(T)$, defined by $x \longmapsto \varphi(x) := \varphi \circ x$. The Yoneda Lemma from category theory states that X is determined up to canonical isomorphism by the contravariant functorial assignment $T \longmapsto X(T)$, $(\psi : S \longrightarrow T) \longmapsto (x \longmapsto x \circ \psi)$, called the *functor of points*. Moreover, morphisms $X \longrightarrow Y$ are in bijection with natural transformations $(\varphi_T : X(T) \longrightarrow Y(T))$ of the functors of points.

Another advantage of T-points is that they lead to the notion of *base change*. We will not discuss this in all generality, but instead apply it to extending the notion of supergroup orbits, as we now explain.

5. Isotropies and orbits in families

Let us reconsider the notion of isotropy supergroups through ordinary points. Thus, let $a : G \times X \longrightarrow X$ be an action of a Lie supergroup G on a supermanifold X and let x be an ordinary point. The equation defining the isotropy can be written out in terms of T-points as follows: A T-point $g \in_T G$ is a T-point of the isotropy supergroup if and only if

$$g \cdot x = x.$$

Here, we write $g \cdot x$ for $a(g, x) = a \circ (g, x)$ and x is considered as a T-point of X *via* the composition

$$T \longrightarrow * \xrightarrow{x} X,$$

where the morphism $T \longrightarrow *$ is unique. Thus, the isotropy supergroup of Kostant and Boyer–Sánchez-Valenzuela is the supergroup G_x, unique up to canonical isomorphism, whose functor of points is

$$G_x(T) = \big\{ g \in_T G \bigm| g \cdot x = x \big\}.$$

An equivalent way to state this is that G_x is the fibre product of the point map $x : * \longrightarrow X$ and the orbit map $a_x : G = G \times * \longrightarrow X$, defined by $a_x := a \circ (\mathrm{id}_G \times x)$, *i.e.*, the following diagram is Cartesian:

$$\begin{array}{ccc} G_x & \longrightarrow & G \\ \big\downarrow & & \big\downarrow{\scriptstyle a_x} \\ * & \xrightarrow[x]{} & X. \end{array}$$

That is, any pair of morphisms to $*$ and G that lie over the same morphism to X factors uniquely through G_x.

In a similar vein, the orbit $G \cdot x := G/G_x$ is defined by the requirement that the following diagram is a coequaliser:

$$G \times G_x \underset{p_1}{\overset{m}{\rightrightarrows}} G \xrightarrow{\pi_x} G \cdot x.$$

That is, any morphism defined on G that yields the same morphism on $G \times G_x$ when composed with m and p_1 factors uniquely through π_x.

If now x is a T-point to start with, then the orbit map

$$a \circ (\mathrm{id}_G \times x) : G \times T \longrightarrow X$$

is defined on $G \times T$. (Actually, we prefer to put the T factor first, exchanging factors in the definition.) Thus, the Lie supergroup G gets replaced by a "family" of Lie supergroups $G_T = T \times G$. Formally, this is captured in the following definition.

Definition 6. A *superspace over* T is a morphism $X \longrightarrow T$. A morphism of superspaces over T is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ T & = & T. \end{array}$$

A *supermanifold over* T is a superspace over T locally isomorphic to the model space $\mathbb{A}_T^{p|q} := T \times \mathbb{A}^{p|q}$ with the projection onto T. ("Locally" here means locally in the domain.) A *Lie supergroup over* T is a group object in the category of supermanifolds over T and morphisms over T.

This definition actually makes sense for base superspaces T much more general than supermanifolds, see Ref. [2].

With these notions, the definition of the isotropy supergroup through x is immediate: It is the fibre product of $(p_1, x) : T \longrightarrow X_T = T \times X$ and

$$a_x := (p_1, a \circ (\mathrm{id}_G \times x) \circ (1\,2)) : G_T = T \times G \longrightarrow X_T = T \times X,$$

with $(1\,2)$ denoting the flip. Thus, it makes the following diagram Cartesian:

$$\begin{array}{ccc} G_x & \longrightarrow & G_T \\ \downarrow & & \downarrow{\scriptstyle a_x} \\ T & \xrightarrow[(p_1,x)]{} & X_T. \end{array}$$

In terms of the functor of points, we have

$$G_x(R) := \big\{ (t, g) \in_R G_T \;\big|\; g \cdot x(t) = x(t) \big\}$$

for any supermanifold R over T. Here, recall that $x(t) = x \circ t$.

The notion of isotropies through T-points was defined by Mumford [14] in the context of group schemes. By the Yoneda Lemma, it is clear that G_x is indeed a Lie supergroup over T if only it exists as a supermanifold over T.

A tame example of an action is given by $G = \mathrm{GL}(2, \mathbb{R})$ acting naturally on $X = \mathbb{A}^2$ (where $\mathbb{K} = \mathbb{R}$). For $T = \mathbb{A}^1$ and

$$x(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

we obtain

$$G_x = \left\{ \left(t, \begin{pmatrix} 1 + s\cos t \sin t & s\cos^2 t \\ -s\sin^2 t & 1 - s\cos t \sin t \end{pmatrix} \right) \;\middle|\; s, t \in \mathbb{R} \right\}.$$

In this case, the isotropy supergroup exists and is a Lie group over $\mathbb{A}^1$.

On the other hand, consider the action of $G = \mathbb{A}^{0|1}$ on $X = \mathbb{A}^{1|1}$ defined in Equation (3). Let $T = \mathbb{A}^{0|1}$ with standard coordinate τ and define x by

$$x^\sharp(u) = 0, \quad x^\sharp(\xi) = \tau.$$

In this case, G_x does not exist as a supermanifold over T. However, it does exist as a superspace which is *locally finitely generated* in the sense of Ref. [2]. One computes easily that

$$G_x = \big(*, \mathbb{K}[\theta, \tau]/(\theta\tau)\big).$$

In any case, if G_x exists, then one may define $G\cdot x := G/G_x$ by the requirement that the following diagram is a coequaliser:

$$G_T \times_T G_x \underset{p_1}{\overset{m}{\rightrightarrows}} G_T \xrightarrow{\pi_x} G \cdot x,$$

provided this exists as a supermanifold over T.

In order to understand when the isotropy supergroup exists, we need a piece of data encoding the geometry of the action.

Definition 7. Let $a : G \times X \longrightarrow X$ be an action of a Lie supergroup G on a supermanifold X. For $v \in \mathfrak{g}$, the *fundamental vector field* of v is the unique vector field a_v on X such that

$$(v \otimes 1) \circ a^\sharp = (1 \otimes a_v) \circ a^\sharp.$$

The *fundamental distribution* $\mathscr{A}_\mathfrak{g}$ is the $\mathscr{O}_X$-submodule of the tangent sheaf $\mathscr{T}_X$ spanned by the fundamental vector fields.

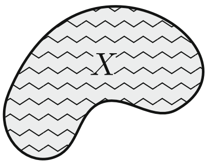

FIGURE 3. The fundamental distribution $\mathscr{A}_\mathfrak{g}$.

The following theorem is proved in Ref. [3].

Theorem 8 (Alldridge–Hilgert–Wurzbacher 2015). *The following are equivalent:*

(i) *The isotropy supergroup G_x exists as a Lie supergroup over T.*

(ii) *The orbit morphism $a_x : G_T \longrightarrow X_T$ has constant rank over T. (That is, its tangent map on the tangent sheaf over T has a locally free cokernel.)*

(iii) *The pullback $x^*\mathscr{A}_\mathfrak{g}$ is locally direct in $x^*\mathscr{T}_X$.*

In this case, $G \cdot x$ is a supermanifold over T and the canonical morphism $G \cdot x \longrightarrow X_T$ is an injective immersion.

We remark that the notion of constant rank morphisms of supermanifolds is much more subtle than for manifolds; in particular, it is not implied by the weaker condition that the rank of the tangent map on the level of tangent spaces is constant.

The theorem subsumes the previous results by Kostant and Boyer–Sánchez-Valenzuela. Moreover, it explains why the isotropy does not exist in the example before Definition 7: the fundamental distribution is spanned by the differential $d = \xi \frac{\partial}{\partial u}$, and its pullback along x is spanned by $\tau \left(x^\sharp \circ \frac{\partial}{\partial u}\right)$, so is not a direct summand.

This phenomenon is not restricted to actions of Lie supergroups. The action on $X = \mathbb{A}^{0|1}$ of the additive group of $\mathbb{A}^1$ that is generated by the even vector field $\xi \frac{\partial}{\partial \xi}$ is also an example where the theorem's assumption fails.

For the particular case of the coadjoint action of G on $\mathfrak{g}^*$, whenever an orbit through a T-point $f \in_T \mathfrak{g}^*$ exists, it carries a symplectic structure (à la Kirillov–Kostant–Souriau), as the following theorem from Ref. [3] shows.

Theorem 9 (Alldridge–Hilgert–Wurzbacher 2015). *Let $f \in_T \mathfrak{g}^*$ be a T-point of $\mathfrak{g}^*$. If the orbit morphism a_f with respect to the coadjoint action $a = \mathrm{Ad}^*$ of G has constant rank, then the coadjoint orbit $G \cdot f$ carries a canonical invariant supersymplectic structure ω_f over T.*

Here, a supersymplectic structure is a non-degenerate super-antisymmetric bilinear form on the tangent sheaf over T (whose sections are vector fields along the fibres of the projection onto T) that is closed for the relative differential $d_{X/T}$.

We emphasize two points: a) The definition of ω_f is the standard one (the precise formulation is somewhat technical since one has to handle the sheaves correctly), and b) in previous attempts by G.M. Tuynman [16] to handle coadjoint orbits through T-points in a more *ad hoc* fashion, it was necessary to consider symplectic forms that where no longer homogeneous with respect to parity. This difficulty disappears in our systematic treatment.

6. Applications to harmonic analysis

We now illustrate in some examples how the point of view introduced in the two previous sections resolves some of the issues around the orbit philosophy for Lie supergroups.

We fix a Lie supergroup G and a T-point f of $\mathfrak{g}^*$. We think of representations of G_T as families over T. For several reasons, the simplest (and most general) way to phrase its representation theory is in terms of contravariant functors on the category $\mathbf{SMan}_T$ of supermanifolds over T. One basic such functor is $\mathcal{O}$, defined by

$$\mathcal{O}(U) := \Gamma(\mathcal{O}_{U,\bar{0}}), \quad \mathcal{O}(f : U \longrightarrow U') := f^\sharp : \mathcal{O}(U') \longrightarrow \mathcal{O}(U).$$

Here, Γ denotes global sections and the subscript $(-)_{\bar{0}}$ the even part. Then $\mathcal{O}$ is a ring object in the category of contravariant functors on $\mathbf{SMan}_T$. The functor of points of G_T is a group object in this category.

Definition 10. A *representation* $(\mathcal{H},\pi)$ *of* G_T consists of an $\mathcal{O}$-module object $\mathcal{H}$ in the category of contravariant functors on $\mathbf{SMan}_T$ and an $\mathcal{O}$-linear action

$$\pi : G_T \times \mathcal{H} \longrightarrow \mathcal{H}.$$

Let $\mathfrak{h}$ be an $\mathcal{O}_T$-subalgebra of $\mathfrak{g}_T := \mathcal{O}_T \otimes \mathfrak{g}$ (preferably, one that is polarizing in some sense). Then we define a representation $\big(Q(f,\mathfrak{h}),\pi_f^{\mathfrak{h}}\big)$ of G_T as follows. On $(t : U \longrightarrow T) \in \mathbf{SMan}_T$, we define

$$Q(f,\mathfrak{h})(t) := \big\{\psi \in \Gamma(\mathcal{O}_{U\times_T G_T,\bar{0}}) \bigm| \forall v \in \Gamma((t^*\mathfrak{h})_{\bar{0}}) : R_v = -i\langle f(t),v\rangle\psi\big\}.$$

Here, R denotes the right regular representation (by right translation). On morphisms $\varphi : U \longrightarrow U'$ over T, we set

$$Q(f,\mathfrak{h})(\varphi) := (\varphi \times_T \mathrm{id}_{G_T})^\sharp.$$

The action $\pi_f^{\mathfrak{h}}$ is given by restriction of the left regular representation, *viz.*

$$\pi_f^{\mathfrak{h}}(g)\psi := \psi(-,g^{-1}(-)) = \big((\mathrm{id}_U \times_T m)\circ((\mathrm{id}_U,g^{-1})\times_T \mathrm{id}_G)\big)^\sharp(\psi)$$

for $\psi \in Q(f,\mathfrak{h})(t)$ and $g \in_U G$. When the coadjoint orbit $G\cdot f$ exists as a supermanifold over T, $Q(f,\mathfrak{h})$ can be seen to define an $\mathcal{O}_T$-module. But it makes sense as a functor in any case.

Let us come back to the most basic example of the Abelian supergroup $G = \mathbb{A}^{0|q}$. Recall from Section 3 that this could not be handled satisfactorily in the traditional approach.

In this case, we have $G_f = G_T$, $G\cdot f = T$, and the Kirillov–Kostant–Souriau form ω_f is zero. Thus, the only reasonable choice for $\mathfrak{h}$ is $\mathfrak{g}_T$. For any $(t : U \longrightarrow T) \in \mathbf{SMan}_T$, the $\mathcal{O}(t)$-module $Q(f,\mathfrak{h})$ is generated by

$$\psi_t = e^{-i\sum_j t_j\xi^j}.$$

Here, t is determined by $t^\sharp(\xi_j) = t_j$, where (ξ_j) is a basis of $\mathfrak{g}$ and (ξ^j) the dual basis. That is, $Q(f,\mathfrak{h})$ comes from a rank $1|0$ locally free $\mathcal{O}_T$-module (or vector bundle on T). The action on ψ_t is given by

$$\pi_f^{\mathfrak{h}}(g)\psi_t = e^{i\langle t,g\rangle}\psi_t.$$

These representations suffice for a decomposition of the regular representation of G. In fact, taking $T = \mathfrak{g}^*$ and $f = \mathrm{id}_T$, one of them is enough.

Proposition 11 (Alldridge–Hilgert–Wurzbacher 2015). *Denote* $\pi_f^{\mathfrak{h}}$ *by* π. *For any superfunction* h *on* G, *we have*

$$\int_T D(\theta)\,\operatorname{str}\pi(h) = (-1)^{n(n+1)/2} i^n h_0(0),$$

where $\pi(h)$ *is defined by*

$$\pi(h) := \int_G D(\xi)\,h\pi,$$

and the integrals are Berezin integrals, cf. *Refs.* [7, 12, 13].

This is just the standard inversion formula for the fermionic Fourier transform. In our framework, it acquires an interpretation as the decomposition of the function algebra as an odd direct integral of representations.

We end our discussion by a brief account of the representation theory of the Clifford supergroup G in our framework. Recall that G is the simply-connected Lie supergroup with Lie superalgebra $\mathfrak{g} = \langle x_j, y_j, z | j = 1, \dots, q \rangle$ where x_j, y_j are odd, z is even, and the bracket is given by

$$[x_j, y_j] = z$$

with all other relations zero. If we choose $T = \mathbb{A}^1 \setminus 0$ and $f = z^*$, where (x^j, y^j, z^*) is the dual basis, then the orbit $G \cdot f$ exists as a supermanifold over T. We choose $\mathfrak{h} = \big\langle x_1, \dots, x_q, z \big\rangle_{\mathcal{O}_T}$. Then we obtain the following nice characterization of the representation attached to the orbit $G \cdot f$.

Proposition 12 (Alldridge–Hilgert–Wurzbacher 2015). *The representation $\pi = \pi_f^{\mathfrak{h}}$ on $Q(f, \mathfrak{h})$ is the bundle of spinor modules over T of central character $-it$.*

This result can also be reached by other methods, but it is still delightful to see that the spinor module naturally comes out of our construction. Furthermore, this fits nicely together with the following result from Ref. [1].

Theorem 13 (Alldridge–Hilgert–Laubinger 2013). *For any f contained the Schwartz space $\mathscr{S}(G)$, we have the Fourier inversion*

$$f(1) = \frac{(-1)^q}{2\pi} \int_{\mathbb{A}^1} \frac{Dt}{(2t)^{\lfloor (q+1)/2 \rfloor}} \, \tau\big(\pi(f)\big), \ \ \tau = \begin{cases} \mathrm{str} & 2 \mid q, \\ 2^{(1-q)/2} e^{-i\pi/4} \, \mathrm{tr}(\epsilon \cdot) & 2 \nmid q. \end{cases}$$

While a number of issues remain open, these examples may serve as a motivation to study Kirillov's orbit method for Lie supergroups from the more general vantage point that we have suggested here.

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