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Some Comments on Indistinguishable Particles and Interpretation of the Quantum Mechanical Wave Function

Gerald A. Goldin

Dedicated in memory of Professor Gérard G. Emch,
an inspiration, a mentor, and a friend

Abstract. This paper discusses some fundamental questions pertaining to the wave function description of multiparticle systems in quantum mechanics. Motivated by results from the study of diffeomorphism group representations, I outline a point of view addressing subtle issues often overlooked in standard, "textbook" answers to these questions.

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1. Questions motivating the discussion

Interesting questions arise in connection with the description of indistinguishable particles in quantum mechanics. Let us consider several of them:

1. What meaning should we ascribe to the wave function (for example, in a positional representation)?
2. How should we understand the construction of multiparticle states from single-particle ones?
3. How dependent are our descriptions on assumptions of strict linearity in quantum mechanics?

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4. What physical meaning attaches to the action of a group of permutations on particle coordinates?
5. What are the relationships among: (a) the exchange statistics of indistinguishable particles (Bose, Fermi, or other) expressed through a symmetry condition on the wave function; (b) configuration space topology; (c) self-adjoint extensions of densely-defined operators describing momentum, angular momentum, or energy; and (d) boundary conditions satisfied by wave functions? Which of these constructs are physically fundamental?
6. What are the implications for exotic particle statistics (e.g., anyons, nonabelian anyons, particles obeying parastatistics, configurations of extended objects, or particles in non-simply connected spaces)?

Various standard, easy answers (and some not-so-easy answers) to these questions are to be found in many textbooks and articles. But certain subtleties are overlooked in these answers, and I think there is something to be learned from probing more deeply. This paper is intended to highlight some important distinctions, and in so doing to stimulate possibly skeptical thinking about fundamental issues in quantum mechanics. I think that is something Gérard Emch would encourage us to do from time to time.

In a short presentation I can touch on only some of the above questions, and these these only partially; but I shall endeavor to provide a certain perspective from which to approach them. I cannot here include adequate references to the many researchers whose work should be cited; the reader is referred to more complete citations in [1, 2], and [3].

2. Positional representation of operators

In the conventional quantum mechanical description of a single particle, or of N particles, the interpretation of the wave function depends (of course) on how the observables are represented.

In a "positional" representation, the single-particle (complex- or spinor-valued) wave function is $\psi(x)$, where x coordinatizes physical space; the operators for position coordinates Q^j are represented by multiplication, $Q^j\psi(x) = x^j\psi(x)$; and the operators for momentum coordinates P^k are represented by differentiation, $P^k\psi(x) = -i\hbar(\partial/\partial x^k)\psi(x)$. In a "momentum" representation the single-particle wave function is $\tilde{\psi}(p)$, momentum coordinate operators are represented by multiplication, and those for position coordinates by differentiation. These are just two unitarily equivalent representations of the Heisenberg algebra, with the Fourier transform implementing the equivalence.

So to ask about an interpretation to be given to the wave function, we must first specify how some set of observables is being represented; otherwise, the question is not well posed. Here I focus on positional representations, partly because there is a fundamental sense in which actual measurements may be reduced to *sequences* of positional measurements (at different times) [4]. Then we need to describe the time-evolution of wave functions.

The time-evolution of the positional wave function $\psi(x;t)$ (for a single particle) is governed by a Schrödinger equation established by our representation of the Hamiltonian operator (corresponding to the energy observable). This time-evolution preserves the L^2 norm $\|\cdot\|$ of ψ. We then typically interpret $\psi(x;t)$ as a "probability amplitude;" i.e., $|\psi(x;t)|^2/\|\psi\|^2$ is the probability density for an idealized measurement localizing the particle in the vicinity of x at time t. To describe a sequence of two positional measurements, we must also specify the continued time-evolution after an outcome of the first (idealized) measurement. The initial condition after such a measurement localizes the particle in a region X at time t is often assumed to be the orthogonal projection of the wave function $\psi(x,t)$ onto the subspace having support in X.

But the interpretation of the single-particle wave function ψ in a positional representation does not end here. We must also say something about its phase. The interpretation of the phase of ψ depends further on how we choose to represent observables such as momentum and energy. After a *gauge transformation* $\psi'(x;t) = \exp[i\theta(x;t)]\,\psi(x;t)$, the representation is still positional, but the phase of ψ has been modified (so its interpretation must also change). Likewise, the representations of the Hamiltonian (energy) and momentum as differential operators have also been changed by the gauge transformation. We refer explicitly to these operators when we specify the gauge. While the modulus of ψ is gauge-invariant (under the usual gauge transformations of quantum mechanics), its phase is not.

Nevertheless, a gauge-invariant (probability flux) current density may be constructed from the phase. Its specification becomes part of the physical interpretation of ψ. Thus we have, in a positional representation, the interpretation of the single-particle wave function as describing a probability density and flux density in the one-particle configuration space (often identified with the physical space), providing predictions for the distribution of outcomes of positional measurements.

Let us also remark that use of a positional representation does not rule out additional, "internal" degrees of freedom needed to describe observables such as components of the particle spin. Then ψ is no longer scalar-valued, but may take values in an inner product space carrying a representation of an internal symmetry group (a Lie group) associated with the particle.

3. Many-particle systems

The conventional procedure for describing many-particle systems is to write the wave function in the form $\psi(x_1,\ldots,x_N)$, even in the case of indistinguishable particles. That is, ψ is taken to be a complex-valued L^2 function on the space of *ordered* N-tuples of points (the particle coordinates) in the physical space.

When the particles are indistinguishable, one then imposes an *additional* condition of exchange symmetry. Conventionally, one then interprets ψ as a probability amplitude for finding (simultaneously) particle 1 at x_1, particle 2 at x_2, and so forth. This motivates the need for an exchange symmetry condition – since the particles are indistinguishable, the probability density for simultaneously finding particle j at x_j and particle k at x_k ($j \neq k$) must be the same as that of finding particle k at x_j and particle j at x_k.

But this conventional interpretation raises some difficulties. Even in the case of distinguishable particles (when no additional symmetry is imposed), the characterization of "particle k," for a specific k, depends on some other, not-yet-specified measurement to be taken (e.g., of the particle mass) which distinguishes one particle from another. Furthermore, actual measurements take place in the physical space, *not* in the configuration space. How should the latter limitation be expressed?

Returning to the situation of indistinguishable particles, the usual symmetry condition imposed relates $\psi(x_1, \ldots, x_N)$ to $\psi(x_{\sigma(1)} \ldots x_{\sigma(N)})$, where $\sigma \,\epsilon\, S_N$ (the symmetric group) is a permutation of the N indices. The relationship is by means of a unitary (typically, 1-dimensional) representation of S_N. The trivial representation characterizes bosons (totally symmetric wave functions), the alternating representation characterizes fermions (totally antisymmetric wave functions). A fundamental difficulty with this description, however, is that one has artificially labeled the indistinguishable particles with indices, and then introduced a symmetry to "undo" that step. What can this possibly mean physically?

An alternative approach is to refer to *unordered* configurations of particles in physical space, since the ordering is unnatural for distinguishable particles and unobservable for indistinguishable ones. Then a configuration is just an N-point subset of the spatial manifold M. Note that it is not necessary to include configurations where more than one particle occupy the same point. These form a Lebesgue measure zero set.

We write $\tilde{\gamma} = (x_1, \ldots, x_N)$ for an ordered configuration, and $\gamma = \{x_1, \ldots, x_N\}$ for an unordered configuration. Then $\tilde{\gamma} \to \gamma$ is a projection from the coordinate space $\tilde{\Gamma}^{(N)}$ (of ordered N-tuples of distinct points in physical space) to the configuration space $\Gamma^{(N)}$ (of N-point subsets of physical space).

It is natural to consider writing wave functions for identical particles on $\Gamma^{(N)}$ rather than $\tilde{\Gamma}^{(N)}$; indeed, $\Gamma^{(N)}$ is the physically relevant space. But we must then find a different way to characterize the exchange symmetry – to describe how bosons are to be distinguished from fermions, and what other particle statistics might be possible. This must now be done via representations of the *operators*, as there is no way available to impose a symmetry condition on wave functions on $\Gamma^{(N)}$.

We may also consider wave functions for distinguishable particles from this point of view. Then one is led quite naturally to the idea of *marked* configurations. A marked configuration is an N-point subset of a *bundle* B for which the base is the physical space M, and for which a fiber is a space in which additional values of particle attributes may be taken. This is discussed a little further below.

4. Diffeomorphism group representations and particle statistics

Taking seriously the comment that measurements occur in physical space (rather than configuration space), we observe that the mass density and momentum density operators form an infinite-dimensional Lie algebra of local currents modeled on physical space. This current algebra describes a natural class of kinematical observables. The group obtained by exponentiating the local currents is the group of compactly-supported diffeomorphisms of M. [5]

Let us take $M = \mathbf{R}^d$ ($d \geq 2$) for specificity. For a diffeomorphism ϕ of $\mathbf{R}^d$, one may write a unitary representation of the group on a space of wave functions $\psi(x_1, \ldots, x_N)$, $x_j \in \mathbf{R}^d$, as

$$[\hat{V}(\phi)\psi](x_1, \ldots, x_N) := \psi(\phi(x_1), \ldots, \phi(x_N))\Pi_{k=1}^{N}\sqrt{\mathcal{J}_\phi(x_k)}\,. \tag{1}$$

where $\mathcal{J}_\phi(x) = [d\mu_\phi/d\mu](x)$ is the Jacobian of ϕ at x (here μ is Lebesgue measure).

Note that the representation is unitary, and the exchange symmetry of ψ is preserved. The representation $\hat{V}$ acting on the Hilbert space of totally symmetric wave functions is *unitarily inequivalent* to the representation acting on the Hilbert space of totally antisymmetric wave functions.

Alternatively, suppose we consider representing the diffeomorphism group on the space of *unordered* configurations, as suggested in earlier constructions. [5] To do this, we set

$$\begin{aligned}&[V(\phi)\psi](\{x_1, \ldots, x_N\})\\ &\quad := \chi_\phi(\{x_1, \ldots, x_N\})\psi(\{\phi(x_1), \ldots, \phi(x_N)\})\Pi_{k=1}^{N}\sqrt{\mathcal{J}_\phi(x_k)}\,,\end{aligned} \tag{2}$$

where χ obeys a 1-cocycle equation. Note that set brackets have replaced the parentheses. In a shorter way, we can write

$$[V(\phi)\psi](\gamma) := \chi_\phi(\gamma)\psi(\phi\gamma)\Pi_{x_k \epsilon \gamma}\sqrt{\mathcal{J}_\phi(x_k)}\,, \tag{3}$$

where γ denotes the unordered configuration.

In this construction, noncohomologous cocycles describe unitarily inequivalent representations. The information regarding particle statistics has been encoded in the cocycle (i.e., in how the observables are represented), not in the wave function symmetry! Thus we have a fundamental change in perspective on the meaning of the wave function itself. On the left-hand side of Eq. (1), the expression x_j (the jth entry in the N-tuple forming the argument of ψ) refers to the location of particle j. In Eq. (2), the expression x_j refers simply to the location of a particle – *any* particle. The subscript j has no intrinsic meaning; it is just a way to indicate that there are N elements in the configuration γ. No extraneous labeling has been introduced.

5. A comment about linearity vs. nonlinearity in quantum mechanics

In exploring the possibility of nonlinear modifications of quantum mechanics, it is of interest to examine the different ways in which the usual assumptions of linearity are introduced [3, 4, 6].

One assumption of linearity inheres in the conventional method for constructing a theory of composite systems from their components – in particular, constructing multiparticle states from single-particle states. The Hilbert space of states describing the composite system is normally taken to be the tensor product of the Hilbert spaces for the subsystems – i.e., the space constructed from linear combinations of product states. For indistinguishable particles, product states are replaced by symmetric or antisymmetric linear combinations of product states, leading to the symmetrized or antisymmetrized tensor product Hilbert space. Then configurations for the composite system are ordered N-tuples, as discussed above. Subsystem observables are extended by linearity from product states to the full Hilbert space.

But adopting the perspective suggested here, one begins naturally with (spatial) configurations for the subsystems (as subsets of the physical space, or subsets of bundles over the physical space). One then constructs the configurations for the composite system from generalized unions of these subsets. In particular, this leads to a direct construction of $\Gamma^{(N)}$ from N copies of $\Gamma^{(1)}$. The state-space for the composite system is the space of square-integrable functions on the composite configuration-space. Linearity need not be assumed in the construction (and there is no need for symmetrization or antisymmetrization of product states).

Without the initial assumptions of linearity, there is no obstacle to the discussion of the nonlinear gauge transformations introduced in [6]. Later, one can describe the quantum kinematics on this space of generalized unions by unitary representations of the group of compactly-supported diffeomorphisms of the physical space, identify irreducible representations, associate the particle statistics with inequivalent cocycles, and so forth.

6. Induced representations and the homotopy of configuration space

Select a particular configuration $\gamma \in \Gamma^{(N)}$ and consider the stability subgroup K_γ. This is the group of those (compactly supported) diffeomorphisms of $\mathbf{R}^d$ which leave γ fixed. Note that a diffeomorphism can do this by implementing a permutation of the points in γ. For $d \geq 2$, there is thus a natural homomorphism from K_γ to S_N. A unitary representation of S_N thus defines a continuous unitary representation (CUR) of K_γ, which in turn induces a CUR of the diffeomorphism group.

Such an induced representation may be regarded as acting on a Hilbert space of *equivariant* wave functions on a covering space $\hat{\Gamma}^{(N)}$ of $\Gamma^{(N)}$ – or, equivalently, as acting directly on wave functions defined on $\Gamma^{(N)}$ but with a cocycle as in Eq. (3).

For $d \geq 3$, S_N is the fundamental group (first homotopy group) of $\Gamma^{(N)}$. The coordinate space $\tilde{\Gamma}^{(N)}$ defined earlier is then the universal covering space, and we recover the conventional description in terms of wave functions on ordered N-tuples.

For $d = 2$, however, the fundamental group of $\Gamma^{(N)}$ is the braid group B_N, and one obtains intermediate (or anyon) statistics [7] by inducing. This led to one of the early discoveries of the possibility of intermediate statistics for particles in two-space [8–10].

7. Label permutations and value permutations

Label permutations (also called index permutations) act on the indices of labeled particle coordinates, so that $\sigma \epsilon S_N$ takes x_k to $x_{\sigma(k)}$. The label permutation $\sigma_{(12)}$, for example, exchanges x_1 with x_2 in an ordered N-tuple, regardless of the actual values of the two variables.

Value permutations (in certain contexts, called wave function permutations) do not see the indices, but make reference to some specified ordering of points in the physical space M. In an ordered N-tuple, the value permutation $\sigma_{(12)}$ exchanges those entries having the two lowest values, regardless of where they occur in the N-tuple.

This distinction does not show up in 1-dimensional representations of S_N, so it is easily overlooked in discussing bosons and fermions. But it matters essentially if we want to consider higher-dimensional representations of S_N, describing particles satisfying parastatistics [11]. Furthermore, diffeomorphisms "see" only the values of the x_k, not the labels. Thus, whether they are acting in $\Gamma^{(N)}$ or a covering space, the relevant permutations are the value permutations. The inducing construction leading to anyon statistics involves discussion of homotopy classes of paths in configuration space, which refer to the values of the particle coordinates, not their labels (see also [10]). And it is clear why we require $d \geq 2$; in one dimension, a compactly supported diffeomorphism can never exchange two points on the real line.

8. Implications for exotic statistics

We have outlined a point of view that accommodates well the description of quantum configurations obeying statistics other than those of bosons and fermions. These include anyons and nonabelian anyons in two-space, distinguishable particles satisfying colored braid group statistics in two-space, and paraparticles when the spatial dimension is 2, 3, or more. The key unifying idea is the nontrivial homotopy of the respective configuration spaces, and how this allows particular classes of unitarily inequivalent diffeomorphism group representations modeled on those spaces.

Likewise, the quantum mechanics of configurations in physical spaces which themselves have nontrivial homotopy can be understood well from this point of view. A well-known example is the Aharonov–Bohm effect. Different self-adjoint extensions of densely-defined operators (describing, for example, kinetic angular momentum) have different spectra, and arise from different sets of boundary conditions satisfied by wave functions in their domains. These operators occur as the infinitesimal generators of the unitarily inequivalent group representations associated with the nontrivial homotopy.

This approach extends naturally to the study of infinite but locally finite particle configurations, as well as extended quantum configurations (embedded submanifolds or fractals in the physical space) and their internal symmetry – e.g., closed and open strings, vortex filaments and ribbons, or knotted configurations.

9. The meaning of the wave function and the notion of indistinguishability

We have seen that in a positional representation, the interpretation of ψ is quite different if we consider it to be defined on the space of unordered configurations (i.e., subsets of the physical space), rather than the space of ordered configurations. This point of view actually extends to the description of "distinguishable" particles via marked configurations.

Let us elaborate on this briefly. Consider a two-particle system, where the particles have distinct masses m and μ. Conventionally, one would interpret $\psi(x_1, x_2)$, as a probability amplitude for finding the first particle (the one with mass m) at x_1, and the second particle (having mass μ) at x_2. But ψ makes no explicit reference to these masses. Alternatively, consider (m, x) as an element of a real bundle B over the physical space M, with fiber $\mathbf{R}^+$. A generalized configuration is $\gamma = \{(m, x), (\mu, y)\}$, where $m, \mu \in \mathbf{R}^+$ and $x, y \in M$; and $\psi = \psi(\gamma)$. Now γ can be understood as describing "indistinguishable" particles with distinct spatial coordinates (when $x \neq y$) and distinct mass coordinates (when $m \neq \mu$).

Another way of saying this is that in the perspective taken here, particles can be "distinguished" by their coordinates. References to "the particle measured to have mass m" are analogous to "the particle measured to be in position x." The philosophical meaning of "indistinguishable," as well as the interpretation of the coordinates that appear as the argument of the wave function, thus change according to which view one chooses to take.

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Gerald A. Goldin
Departments of Mathematics and Physics
Rutgers University
New Brunswick, New Jersey, USA
e-mail: geraldgoldin@dimacs.rutgers.edu