Geometric Methods in Physics. XXXIV Workshop 2015
Trends in Mathematics, 183–190

Howe's Correspondence and Characters

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Abstract. The purpose of this note is to explain how is Howe's correspondence used to construct irreducible unitary representations of low Gel'fand–Kirillov dimension and to recall and motivate a conjecture concerning the distribution characters of the representations involved.

Mathematics Subject Classification (2010). Primary: 22E45; secondary: 22E46, 22E30.

Keywords. Reductive dual pairs, Howe duality, characters, Gel'fand–Kirillov dimension.

1. Introduction

In this note we would like to shed some light at the wide open problem of understanding the distribution character Θ_Π, [8], of an irreducible unitary representation Π of a real reductive group G. The representations of low Gel'fand–Kirillov dimension are of special interest.

The notion of the Gel'fand–Kirillov dimension $GK \dim \Pi$ of an irreducible admissible representation Π of G (or rather of the corresponding Harish-Chandra module X_Π) was introduced in [21]. It is equal to one half times the Gel'fand–Kirillov dimension of the algebra $\mathcal{U}(\mathfrak{g})/\operatorname{Ann} X_\Pi$, a concept defined earlier in [7]. (See also [4] for more explanation.) Here $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G and $\operatorname{Ann} X$ is the annihilator of X_Π.

We explain why Howe's correspondence, [15], is a suitable tool for constructing irreducible unitary representations of low Gel'fand–Kirillov dimension and recall a conjecture concerning the distribution characters of the representations occurring in the correspondence [3].

2. The Weil representation

Let W be a vector space of finite dimension $2n$ over $\mathbb{R}$ with a non-degenerate symplectic form $\langle\cdot,\cdot\rangle$. Denote by $\mathrm{Sp} \subseteq \mathrm{GL(W)}$ the corresponding symplectic group.

Denote by $\mathfrak{sp}$ the Lie algebra of Sp. Fix a compatible positive complex structure J on W. Hence $J \in \mathfrak{sp}$ is such that $J^2 = -1$ (minus the identity in End(W)) and the symmetric bilinear form $\langle J\cdot, \cdot\rangle$ is positive definite on W.

For an element $g \in \mathrm{Sp}$, let $J_g = J^{-1}(g-1)$. Then its adjoint with respect to the form $\langle J\cdot, \cdot\rangle$ is $J_g^* = Jg^{-1}(1-g)$. In particular J_g and J_g^* have the same kernel. Hence the image of J_g is

$$J_g\mathrm{W} = (\mathrm{Ker}\, J_g^*)^\perp = (\mathrm{Ker}\, J_g)^\perp$$

where $\perp$ denotes the orthogonal complement with respect to $\langle J\cdot, \cdot\rangle$. Therefore, the restriction of J_g to $J_g\mathrm{W}$ defines an invertible element. Thus it makes sense to consider $\det(J_g)^{-1}_{J_g\mathrm{W}}$, the reciprocal of the determinant of the restriction of J_g to $J_g\mathrm{W}$. Let

$$\widetilde{\mathrm{Sp}} = \{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C}, \quad \xi^2 = i^{\dim(g-1)\mathrm{W}} \det(J_g)^{-1}_{J_g\mathrm{W}}\}\,. \tag{1}$$

Then there exists a 2-cocycle $C : \mathrm{Sp} \times \mathrm{Sp} \to \mathbb{C}$, such that $\widetilde{\mathrm{Sp}}$ is a group with respect to the multiplication

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1g_2, \xi_1\xi_2C(g_1, g_2))\,. \tag{2}$$

In fact, by [1, Lemma 52],

$$|C(g_1, g_2)| = \sqrt{\left|\frac{\det(J_{g_1})_{J_{g_1}\mathrm{W}} \det(J_{g_2})_{J_{g_2}\mathrm{W}}}{\det(J_{g_1g_2})_{J_{g_1g_2}\mathrm{W}}}\right|} \tag{3}$$

and by [1, Proposition 46 and formula (109)],

$$\frac{C(g_1, g_2)}{|C(g_1, g_2)|} = \chi(\frac{1}{8}\,\mathrm{sgn}(q_{g_1,g_2})), \tag{4}$$

where $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$, is a fixed unitary character of the additive group $\mathbb{R}$ and $\mathrm{sgn}(q_{g_1,g_2})$ is the signature of the symmetric form

$$\begin{aligned} q_{g_1,g_2}(u', u'') = {} & \frac{1}{2}\langle (g_1+1)(g_1-1)^{-1}u', u''\rangle \\ & + \frac{1}{2}\langle (g_2+1)(g_2-1)^{-1}u', u''\rangle, \quad u', u'' \in (g_1-1)\mathrm{W} \cap (g_2-1)\mathrm{W}. \end{aligned} \tag{5}$$

By the signature of a (possibly degenerate) symmetric form we understand the difference between the maximal dimension of a subspace where the form is positive definite and the maximal dimension of a subspace where the form is negative definite. The group $\widetilde{\mathrm{Sp}}$ is known as the metaplectic group.

Let dw be the Lebesgue measure on W such that the volume of the unit cube with respect to this form is 1. (Since all positive complex structures are conjugate by elements of Sp, this normalization does not depend on the particular choice of J.) Let $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ be a complete polarization. We normalize the Lebesgue measures on X and on Y similarly.

Each element $K \in \mathcal{S}^*(\mathrm{X} \times \mathrm{X})$ defines an operator

$$\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$$

by

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x, x')v(x')\, dx'. \tag{6}$$

Here $\mathcal{S}(\mathrm{V})$ and $\mathcal{S}^*(\mathrm{V})$ denote the Schwartz space on the vector space V and the space of tempered distributions on V, respectively. The map $\mathrm{Op} : \mathcal{S}^*(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$ is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [12, Theorem 5.2.1].

Fix the unitary character $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$, and recall the Weyl transform

$$\begin{aligned}&\mathcal{K} : \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\mathrm{X} \times \mathrm{X})\\ &\mathcal{K}(f)(x, x') = \int_{\mathrm{Y}} f(x - x' + y)\chi\left(\frac{1}{2}\langle y, x + x'\rangle\right) dy \qquad (f \in \mathcal{S}(\mathrm{W})).\end{aligned} \tag{7}$$

Let

$$\chi_{c(g)}(u) = \chi\left(\frac{1}{4}\langle (g+1)(g-1)^{-1}u, u\rangle\right) \qquad (u = (g-1)w,\ w \in \mathrm{W}). \tag{8}$$

(In particular, if $g - 1$ is invertible on W, then $\chi_{c(g)}(u) = \chi(\frac{1}{4}\langle c(g)u, u\rangle$ where $c(g) = (g+1)(g-1)^{-1}$ is the usual Cayley transform.) For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}$ define

$$\Theta(\tilde{g}) = \xi, \qquad T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathrm{W}}, \qquad \omega(\tilde{g}) = \mathrm{Op} \circ \mathcal{K} \circ T(\tilde{g}), \tag{9}$$

where $\mu_{(g-1)\mathrm{W}}$ is the Lebesgue measure on the subspace $(g - 1)\mathrm{W}$ normalized so that the volume of the unit cube with respect to the form $\langle J\cdot, \cdot\rangle$ is 1. In these terms, $(\omega, L^2(\mathrm{X}))$ is the Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ.

3. Dual pairs

A real reductive dual pair is a pair of subgroups $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathsf{W})$ which act reductively on the symplectic space W, G' is the centralizer of G in Sp and G is the centralizer of G' in Sp, [13]. We shall be concerned with the irreducible pairs in the sense that there is no non-trivial direct sum decomposition of W preserved by G and G'. For brevity we shall simply call them dual pairs. They are listed in Table 1.

4. Howe's correspondence

For a member G of a dual pair, let $\mathcal{R}(\widetilde{\mathrm{G}}, \omega) \subseteq \mathcal{R}(\widetilde{\mathrm{G}})$ denote the subset of the representations which may be realized as quotients of $\mathcal{S}(X)$ by closed $\widetilde{\mathrm{G}}$-invariant subspaces. Let us fix a representation Π in $\mathcal{R}(\widetilde{\mathrm{G}}, \omega)$ and let $N_\Pi \subseteq \mathcal{S}(X)$ be the intersection of all the closed G-invariant subspaces $N \subseteq \mathcal{S}(X)$ such that Π is

Dual pair	$\mathbb{D}$	ι	$(\ ,\ )$	$(\ ,\ )'$	dim W	stable range
$\mathrm{GL}_m(\mathbb{D}),\ \mathrm{GL}_n(\mathbb{D})$	$\mathbb{R},\ \mathbb{C},\ \mathbb{H}$				$2nm\ \dim_{\mathbb{R}}(\mathbb{D})$	$m \leq \frac{n}{2}$
$\mathrm{O}_{p,q},\ \mathrm{Sp}_{2n}(\mathbb{R})$	$\mathbb{R}$	$\iota = 1$	+	-	$2n(p+q)$	$p+q \leq n$
$\mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{O}_{p,q}$	$\mathbb{R}$	$\iota = 1$	-	+	$2n(p+q)$	$2n \leq \min\{p,q\}$
$\mathrm{O}_p(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$	$\mathbb{C}$	$\iota = 1$	+	-	$4np$	$p \leq n$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_p(\mathbb{C})$	$\mathbb{C}$	$\iota = 1$	-	+	$4np$	$2n \leq \frac{p}{2}$
$\mathrm{U}_{p,q},\ \mathrm{U}_{r,s}$	$\mathbb{C}$	$\iota \neq 1$	+	-	$2(p+q)(r+s)$	$p+q \leq \min\{r,s\}$
$\mathrm{Sp}_{p,q},\ \mathrm{O}^*_{2n}$	$\mathbb{H}$	$\iota \neq 1$	+	-	$8n(p+q)$	$p+q \leq n$
$\mathrm{O}^*_{2n},\ \mathrm{Sp}_{p,q}$	$\mathbb{H}$	$\iota \neq 1$	-	+	$8n(p+q)$	$2n \leq \min\{p,q\}$

TABLE 1. Dual pairs

infinitesimally equivalent to $\mathcal{S}(X)/N$. This is a representation of both $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$. As such, it is infinitesimally isomorphic to

$$\Pi \otimes \Pi_1', \tag{10}$$

for some representation Π_1' of $\widetilde{\mathrm{G}}'$. Howe proved, [15, Theorem 1A], that Π_1' is a finitely generated admissible quasisimple representation of $\widetilde{\mathrm{G}}'$, which has a unique irreducible quotient $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$. Conversely, starting with $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$ and applying the above procedure with the roles of G and G′ reversed, we arrive at the representation $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$. The resulting bijection

$$\mathcal{R}(\widetilde{\mathrm{G}}, \omega) \ni \Pi \to \Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega) \tag{11}$$

is called Howe's correspondence, or local θ correspondence, for the pair (G, G′).

Recall the unnormalized moment map

$$\tau' : \mathsf{W} \to \mathfrak{g}'^*, \quad \tau'(w)(X) = \langle X(w), w \rangle \qquad (X \in \mathfrak{g}', w \in \mathsf{W}), \tag{12}$$

and the notion of the wave front set $WF(\Pi)$ of an irreducible admissible representation Π of a real reductive group G, [14], [20, Theorem 3.4]. Then, in terms of (11),

$$WF(\Pi') \subseteq \tau'(\mathsf{W}), \tag{13}$$

see [17, Corollary 2.8]. Since the wave front set is contained in the nilpotent cone $\mathcal{N}' \subseteq \mathfrak{g}'^*$, see [14, Proposition 1.2] and [20, Theorem 3.4], we actually have

$$WF(\Pi') \subseteq \tau'(\mathsf{W}) \cap \mathcal{N}' \tag{14}$$

Recall that, by [21, Theorem 1.1], [2, Theorem 4.1] and [20, Theorem C],

$$GK \dim(\Pi') = \frac{1}{2} \dim WF(\Pi'). \tag{15}$$

Hence we get a bound for the Gel'fand–Kirillov dimension of Π',

$$GK\,\dim(\Pi') \leq \frac{1}{2}\dim(\tau'(\mathsf{W}) \cap \mathcal{N}'). \tag{16}$$

One may realize the symplectic space as the tensor product of the defining modules for the groups G and G'. For example if $\mathrm{G} = \mathrm{O}_{p,q}$ and $\mathrm{G}' = \mathrm{Sp}_{2n}(\mathbb{R})$, then $\mathsf{W} = \mathbb{R}^{p+q} \otimes \mathbb{R}^{2n}$. Hence, roughly, the smaller the dimension of the defining module for the group G, the smaller the right-hand side of (16). One may compute this number for each dual pair using [5, Corollary 6.1.4] and [6, Table 3, page 456], but the formulas are not illuminating. We provide a sample in Table 2 below.

On the other hand, as shown in [16, Theorem A], for dual pairs in the stable range, with G-the smaller member (see table 1), if Π is unitary then so is Π'. (The case $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2n,2n}, \mathrm{Sp}_{2n})$ and Π trivial is excluded.) Later this fact was generalized beyond the stable range in [17, Theorem 3.1] and in [10, Theorem 1.1]. Thus Howe's correspondence provides a method for understanding irreducible unitary representations of classical groups of low Gel'fand–Kirillov dimension. What remains is to understand their characters and we propose an approach in the next section.

Dual pair G, G'	$\dim\ \tau'(\mathsf{W}) \cap \mathcal{N}'$
$\mathrm{GL}_m(\mathbb{D}),\ \mathrm{GL}_n(\mathbb{D})$, $m \leq n$, $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$	$\dim \mathbb{D}\,(2mn - m - m^2)$
$\mathrm{O}_p,\ \mathrm{Sp}_{2n}(\mathbb{R})$, $p \leq n$	$2np - p^2 + p$
$\mathrm{O}_p,\ \mathrm{Sp}_{2n}(\mathbb{R})$, $p > n$	$n(n+1)$
$\mathrm{O}_{2p}(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$, $p \leq n$	$2(2n^2 - 2(n-p)^2 - 2(n-p))$
$\mathrm{O}_{2p+1}(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$, $p \leq n$	$2(2n^2 - 2(n-p)^2 + 2(n-p))$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_{2p}(\mathbb{C})$, $p \leq n$	$2(4pn - 2n - 2n^2)$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_{2p+1}(\mathbb{C})$, $p \leq n$	$2(4pn - 2n^2)$

TABLE 2. Examples of $\dim\ \tau'(\mathsf{W}) \cap \mathcal{N}'$

5. The Cauchy Harish-Chandra integral

The wave front set of the character Θ of the Weil representation is given by

$$\mathrm{WF}(\Theta) = \{(g, \xi) \in \widetilde{\mathrm{Sp}} \times \mathfrak{sp}^*;\ \xi \in \mathrm{WF}_1(\Theta),\ Ad(g)^*(\xi) = \xi\}, \tag{17}$$

where the fiber over the identity, $\mathrm{WF}_1(\Theta)$ is the closure of O_{min}, one of the two minimal non-zero nilpotent coadjoint orbits in $\mathfrak{sp}^*$. (The closure of the other minimal nilpotent orbit is in the wave front set of the contragredient Weil representation.) The formula is a key to a construction of an operator from the space of the invariant eigen-distributions on $\widetilde{\mathrm{G}}$ to the space of the invariant eigen-distributions on $\widetilde{\mathrm{G}}'$, [19], [3], assuming that the rank of G is less or equal to the rank of G′. We recall it below.

A maximal compact subgroup $\mathrm{K} \subseteq \mathrm{G}$ consists of the points fixed by a Cartan involution $\theta : \mathrm{G} \to \mathrm{G}$. Let $\mathrm{P} \subseteq \mathrm{G}$ be the subset of the elements $g \in \mathrm{G}$ such that $\theta(g) = g^{-1}$. Then $\mathrm{G} = \mathrm{KP}$. Any Cartan subgroup $\mathrm{H} \subseteq \mathrm{G}$ is conjugate to one which is invariant under θ. Thus let H be a θ-stable Cartan subgroup of G. Set $\mathrm{A} = \mathrm{H} \cap \mathrm{P}$. This is called the vector part of H, [22].

Denote by $\mathrm{A}' \subseteq \mathrm{Sp}$ the centralizer of A and let $\mathrm{A}'' \subseteq \mathrm{Sp}$ be the centralizer of A'. There is a measure $d\dot{w}$ on the quotient space $\mathrm{A}''\backslash\mathsf{W}$ defined by

$$\int_{\mathsf{W}} \phi(w)\, dw = \int_{\mathrm{A}''\backslash\mathsf{W}} \int_{\mathrm{A}''} \phi(aw)\, da\, d\dot{w}. \tag{18}$$

Let $\widetilde{\mathrm{A}'}$ be the preimage of A' in the metaplectic group. Recall, (9), the embedding

$$T : \widetilde{\mathrm{Sp}} \to S^*(\mathsf{W})\,.$$

The formula

$$Chc(f) = \int_{\mathrm{A}''\backslash\mathsf{W}} \int_{\widetilde{\mathrm{A}'}} f(g)T(g)(w)\, dg\, d\dot{w} \qquad (f \in C_c^\infty(\widetilde{\mathrm{A}'})), \tag{19}$$

where each consecutive integral is absolutely convergent, defines a distribution on $\widetilde{\mathrm{A}'}$, [19, Lemma 2.9]. Fix a regular element $h \in \mathrm{H}^{reg}$. Let $\widetilde{h}$ be an element in the preimage of h in the metaplectic group. The intersection of the wave front set of the distribution (19) with the conormal bundle of the embedding

$$\widetilde{\mathrm{G}'} \ni \widetilde{g} \to \widetilde{h}\widetilde{g'} \in \widetilde{\mathrm{A}''} \tag{20}$$

is empty (i.e., contained in the zero section), [19, Proposition 2.10]. Hence there is a unique restriction of the distribution (19) to $\widetilde{\mathrm{G}}$, denoted $Chc_{\widetilde{h}}$.

Harish-Chandra's Regularity Theorem, [9, Theorem 2], implies that the character of an irreducible representation coincides with a function multiplied by the Haar measure. Thus for $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}})$ we may consider the following integral

$$\int_{\widetilde{\mathrm{H}^{reg}}} \Theta_\Pi(\widetilde{h}^{-1})|\det(Ad(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}}|Chc_{\widetilde{h}}(f)\, d\widetilde{h} \qquad (f \in C_c^\infty(\widetilde{\mathrm{G}'})). \tag{21}$$

In fact, this integral is absolutely convergent, [19, Theorem 2.14].

Recall the Weyl–Harish-Chandra integration formula

$$\int_{\widetilde{\mathrm{G}}} f(g)\, dg = \sum \frac{1}{|\mathcal{W}(\mathrm{H},\mathrm{G})|} \int_{\widetilde{\mathrm{H}^{reg}}} |\det(Ad(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}}| \int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{H}}} f(g\widetilde{h}g^{-1})\, d\dot{g}\, d\widetilde{h}, \tag{22}$$

where $\mathcal{W}(\mathrm{H},\mathrm{G})$ is the Weyl group of H in G and the summation is over a maximal family of mutually non-conjugate (θ-stable) Cartan subgroups $\widetilde{\mathrm{G}}$. In terms of (22), set

$$\Theta'_\Pi(f) = C_\Pi \sum \frac{1}{|\mathcal{W}(\mathrm{H},\mathrm{G})|} \int_{\widetilde{\mathrm{H}^{reg}}} \Theta_\Pi(\widetilde{h}^{-1})|\det(Ad(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}}|Chc_{\widetilde{h}}(f)\, d\widetilde{h}, \tag{23}$$

where C_Π is a non-zero constant. This is an invariant distribution on $\widetilde{\mathrm{G}'}$. Hence a finite linear combination of irreducible characters, see [11, page 52]. In fact, with the appropriate normalization of all the measures involved, [3, Theorem 4], Θ'_Π is an invariant eigen-distribution whose infinitesimal character is equal to the one obtained from the infinitesimal character of Θ_Π by ([18, Theorem 1.19]). There are reasons to believe that (for an appropriate constant C_Π) Θ'_Π coincides with the character of the representation Π'_1, (10). Since quite often, $\Pi'_1 = \Pi'$, the above construction could explain Howe's correspondence on the level of characters in the sense that knowing the character of the representation of the small group gives a formula for the character of the representation of the large group. Though the conjecture holds in many cases, see for example [19], there is no proof of the equality $\Theta'_\Pi = \Theta_{\Pi'_1}$ in general. In the next section we recall how is our conjecture related to the classical Cauchy Determinantal Identity.

6. The pair $\mathrm{G} = \mathrm{U}_p$, $\mathrm{G}' = \mathrm{U}_s$

In this case $\Pi'_1 = \Pi'$, $\Theta'_\Pi = \Theta_{\Pi'}$ and the formula (23) coincides with the following equality

$$\int_{\widetilde{\mathrm{G}}} \int_{\widetilde{\mathrm{G}'}} f(g')\Theta(g'g)\Theta_\Pi(g^{-1})\, dg'\, dg = \int_{\widetilde{\mathrm{G}'}} \Theta_{\Pi'}(g')f(g')\, dg',$$

where each consecutive integral is absolutely convergent. This is an explicit version of the following equality of distributions

$$\int_{\widetilde{\mathrm{G}}} \Theta(g'g)\Theta_\Pi(g^{-1})\, dg = \Theta_{\Pi'}(g'), \tag{24}$$

which is equivalent to the First Fundamental Theorem of Classical Invariant Theory. By restricting to the maximal tori one sees that, for $r = s$, (24) is equivalent to the Cauchy Determinantal Identity:

$$\det\left(\frac{1}{1 - h_i h'_j}\right) = \frac{\prod_{i<j}(h_i - h_j) \cdot \prod_{i<j}(h'_i - h'_j)}{\prod_{i<j}(1 - h_i h'_j)}. \tag{25}$$

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