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Canonical Representations for Hyperboloids: an Interaction with an Overalgebra

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Abstract. Canonical representations for the hyperboloid $\mathcal{X} = G/H$ where $G = \mathrm{SO}_0(p,q)$, $H = \mathrm{SO}_0(p,q-1)$, are defined as the restriction to G of maximal degenerate series representations of the overgroup $\widetilde{G} = \mathrm{SL}(n,\mathbb{R})$. We determine explicitly the interaction of Lie operators of $\widetilde{G}$ with operators intertwining canonical representations and representations of G associated with a cone.

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This paper continues the series of our papers [2–5], devoted to the interaction of Poisson and Fourier transforms associated with canonical representations and Lie operators of a larger group (an "overgroup").

This activity was inspired by Neretin's paper [7] (an old Mukunda paper [6] should be also mentioned) for the Lobachevsky plane G/K, where $G = \mathrm{SO}_0(2,1)$, $K = \mathrm{SO}(2)$. In these papers the authors essentially used the Plancherel formula for this manifold.

We use another approach. We use the notions of canonical representations, Poisson and Fourier transforms and "overgroups" and do not need any Plancherel formulae. Nevertheless, even in the framework of our version the computations of explicit formulae is a very difficult analytic problem. Earlier we already studied hyperboloids $\mathcal{X} = G/H$ with $G = \mathrm{SO}_0(p,q)$ and the overgroup $\widetilde{G} = \mathrm{SO}_0(p+1,q)$, see [2–4], and hyperboloids (Lobachevsky spaces) with $G = \mathrm{SO}_0(n-1,1)$ and $\widetilde{G} = \mathrm{SL}(n,\mathbb{R})$, see [5]. Now we consider the hyperboloid $\mathcal{X} = G/H$, where G is the pseudo-orthogonal group $\mathrm{SO}_0(p,q)$, $H = \mathrm{SO}_0(p,q-1)$, and the overgroup is $\widetilde{G} = \mathrm{SL}(n,\mathbb{R})$, $n = p+q$. This case is the most difficult. Notice that expressions of

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the interaction involve differential operators of the fourth, second and zero order (just as in [4]).

One of sources of getting canonical representations consists of the following. Let G be a semi-simple Lie group. We take some group $\widetilde{G}$ (overgroup) containing G such that G is a symmetric subgroup of $\widetilde{G}$, i.e., G is the fixed point subgroup of an involution. Let $\widetilde{P}$ be a maximal parabolic subgroup of $\widetilde{G}$. We take a series of representations $\widetilde{R}_\lambda$ of $\widetilde{G}$ induced by characters of $\widetilde{P}$, they can depend on some discrete parameters, we do not write them. As a rule, representations $\widetilde{R}_\lambda$ are irreducible. They act on functions on some compact manifold Ω (a flag space for $\widetilde{G}$). Denote by R_λ restrictions of $\widetilde{R}_\lambda$ to G:

$$R_\lambda = \widetilde{R}_\lambda \Big|_G .$$

We call these representations R_λ *canonical representations* of the group G. They act on functions on Ω.

Generally speaking, the manifold Ω is not a homogeneous space of the group G, this group has several orbits on Ω. Open G-orbits are semi-simple symmetric spaces G/H_i. Subgroups H_i can be not isomorphic. The manifold Ω is the closure of the union of open G-orbits.

Canonical representations R_λ give rise to *boundary representations*, related to boundaries of G-orbits G/H_i. There are two types of boundary representations. The boundary representations of the first type act on distributions concentrated at the union S of boundaries. The boundary representations of the second type act on jets transversal to S. These two types are dual to each other. Boundary representations are interesting both themselves and as a tool for the decomposition of canonical representations, they glue representations on separate G-orbits.

One can also consider another version of canonical representations: the restriction of these representations above to some open G-orbit in Ω. It is just the case we consider in this paper.

With the canonical representation R_λ, we associate Poisson transforms $P_{\lambda,\sigma}$ and Fourier transforms $F_{\lambda,\sigma}$. They are operators intertwining the representation R_λ with (irreducible) representations T_σ of G occurring in the decomposition of R_λ. Our aim is to find out how the Lie operators of $\widetilde{G}$ in $\widetilde{R}_\lambda$ (i.e., the representation $\widetilde{R}_\lambda$ of the Lie algebra $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$) interact with these transforms.

This problem can be treated as a version of the classical problem on the action of a group (or a Lie algebra) in a basis that is an eigenbasis for some subgroup.

This theory can be considered as a new approach to representation theory of Lie algebras (and Lie groups): in this theory elements of a Lie algebra go to *differential-difference* operators.

In this paper we do not touch the decomposition problem for the canonical and boundary representations, which will be considered elsewhere. Also we do not consider the Fourier transform and do not discuss coefficients of the interactions, since this goes exactly as in [5].

Let us introduce some notation and conventions.

For a character of the group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ we use the following notation:

$$t^{\lambda,\nu} = |t|^\lambda (\operatorname{sgn} t)^\nu, \quad t \in \mathbb{R}^*, \ \lambda \in \mathbb{C}, \ \nu \in \mathbb{Z}.$$

This character depends on ν modulo 2 rather than ν itself.

For a manifold M, $\mathcal{D}(M)$ denotes the space of compactly supported infinitely differentiable complex-valued functions on M, with the usual topology.

For a representation of a Lie group, we retain the same symbol for the corresponding representations of its Lie algebra.

1. Pseudo-orthogonal group and hyperboloid

The group $G = \mathrm{SO}_0(p,q)$ is the connected component of the identity of the group of linear transformations of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$, $n = p+q$, preserving the bilinear form

$$[x,y] = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n.$$

The matrix of this form is $I = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, where

$$\lambda_1 = \cdots = \lambda_p = -1, \lambda_{p+1} = \cdots = \lambda_n = 1.$$

Let K be a subgroup of G consisting of elements g such that $g = IgI$. It is a maximal compact subgroup of G, it is isomorphic to $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

Let us denote by $\langle \cdot , \cdot \rangle$ the standard inner products in $\mathbb{R}^p$ and $\mathbb{R}^q$, let us denote by $| \cdot |$ and $\| \cdot \|$ corresponding norms in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. For a point $x \in \mathbb{R}^n$ written as the pair $x = (u,v)$, $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$, we denote $|x| = |u|$ and $\|x\| = \|v\|$ respectively.

We shall consider that G acts on $\mathbb{R}^n$ from the right: $x \mapsto xg$. In accordance with this we write vectors in the row form. Let $\mathcal{X}$ be the hyperboloid defined by equation $[x,x] = 1$, or $-|x|^2 + \|x\|^2 = 1$. The group G acts on it transitively. The stabilizer H of the point $x^0 = (0, \ldots, 0, 1)$ is $\mathrm{SO}_0(p, q-1)$, so that $\mathcal{X}$ is a homogeneous space G/H. It is more convenient for us to use another realization of the hyperboloid $\mathcal{X}$. Let us attach to a point $x \in \mathcal{X}$ the point $y = x/\|x\|$. Then $\mathcal{X}$ becomes a cylinder $\mathcal{Y}$, the direct product of the unit ball $B \subset \mathbb{R}^p$, defined by $|y| < 1$, and the unit sphere $S_2 \subset \mathbb{R}^q$, defined by $\|y\| = 1$.

Let dy be the Euclidean measure on $\mathcal{Y}$, then a G-invariant measure dx on $\mathcal{X}$ is

$$dx = [y,y]^{-n/2} dy.$$

The Lie algebra $\mathfrak{g}$ of the group G consists of matrices $X \in \mathrm{Mat}(n, \mathbb{R})$ satisfying the condition $X' = -XIX$, the prime means matrix transposition. A basis of $\mathfrak{g}$ is formed by matrices $L_{ij} = E_{ij} - \lambda_i \lambda_j E_{ji}$, $i < j$, where E_{ij} is the "matrix unit": it has 1 at the place (i,j) and 0 at other places.

2. Representations of G associated with a cone

Recall [1] some material about representations of the group G associated with a cone (class one representations). We use the "compact picture".

Denote by S the section of the cone $[x,x]=0$ in $\mathbb{R}^n$ by the sphere $x_1^2+\dots+x_n^2=2$. It consists of points s such that $|s|=\|s\|=1$. The section S is the direct product of two unit spheres $S_1\subset\mathbb{R}^p$ and $S_2\subset\mathbb{R}^q$, defined by equations $s_1^2+\dots+s_p^2=1$ and $s_{p+1}^2+\dots+s_n^2=1$ respectively. Let ds be the Euclidean measure on S.

For local coordinates on spheres S_1 and S_2 we take variables s_i omitting one of them for either sphere, say, s_α for S_1 and s_β for S_2. The Laplace–Beltrami operators Δ_1 on S_1 and Δ_2 on S_2 are given respectively by formulae:

$$\Delta_1 = H_1 - D_1^2 - (p-2)D_1\,,$$
$$\Delta_2 = H_2 - D_2^2 - (q-2)D_2\,,$$

where

$$H_1=\sum\frac{\partial^2}{\partial s_i^2}\,,\ \ D_1=\sum s_i\frac{\partial}{\partial s_i}\,,$$
$$H_2=\sum\frac{\partial^2}{\partial s_j^2}\,,\ \ D_2=\sum s_j\frac{\partial}{\partial s_j}\,,$$

and derivatives with respect to s_α and s_β have to be omitted.

Let $\sigma\in\mathbb{C}$, $\varepsilon=0,1$. Let us denote by $\mathcal{D}_\varepsilon(S)$ the space of functions $\varphi\in\mathcal{D}(S)$ of parity ε: $\varphi(-s)=(-1)^\varepsilon\varphi(s)$. The representation $T_{\sigma,\varepsilon}$ of the group G acts on $\mathcal{D}_\varepsilon(S)$ by

$$(T_{\sigma,\varepsilon}(g)\varphi)(s)=\varphi\left(\frac{sg}{|sg|}\right)\cdot|sg|^\sigma.$$

If σ is not integer, then $T_{\sigma,\varepsilon}$ is irreducible and equivalent to $T_{2-n-\sigma,\varepsilon}$. For $X\in\mathfrak{g}$, differential operators $T_{\sigma,\varepsilon}(X)$ do not depend on ν, so we omit ε in the notation and write $T_\sigma(X)$.

Here are operators corresponding to basis elements L_{km}:

$$T_\sigma(L_{km})=A_{km},\quad 1\leqslant k<m\leqslant p \text{ or } \ p+1\leqslant k<m\leqslant n,$$
$$T_\sigma(L_{km})=\sigma\, s_k s_m+s_m B_k+s_k B_m,\ \ 1\leqslant k\leqslant p<m\leqslant n,$$

where

$$A_{km}=-s_m\frac{\partial}{\partial s_k}+s_k\frac{\partial}{\partial s_m},$$
$$B_k=\frac{\partial}{\partial s_k}-s_kD_1,\quad B_m=\frac{\partial}{\partial s_m}-s_mD_2,$$

as before, derivatives with respect to s_α and s_β have to be omitted.

3. Canonical representations

For an overgroup for the group G, we take the group $\widetilde{G} = \mathrm{SL}(n, \mathbb{R})$. Let $\lambda \in \mathbb{C}$, $\nu = 0, 1$. Denote by $\mathcal{D}_{\lambda,\nu}\big(\mathbb{R}^n\big)$ the space of functions $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying the following homogeneity condition:

$$f(tx) = t^{\lambda,\nu} f(x), \quad x \in \mathbb{R}^n \setminus \{0\},\ t \in \mathbb{R} \setminus \{0\}.$$

Representations $\widetilde{R}_{\lambda,\nu}$ of $\widetilde{G}$ act on $\mathcal{D}_{-\lambda-n,\nu}\big(\mathbb{R}^n\big)$ by translations:

$$\left(\widetilde{R}_{\lambda,\nu}(g) f\right)(x) = f(xg).$$

These representations form a maximal degenerate principal series.

Now let $\widetilde{\mathcal{Y}}$ be a manifold in $\mathbb{R}^n$, defined by $\|y\| = 1$ (it contains $\mathcal{Y}$). Restrict functions in $\mathcal{D}_{-\lambda-n,\nu}(\mathbb{R}^n)$ to $\widetilde{\mathcal{Y}}$. We obtain some space $\mathcal{D}_{-\lambda-n,\nu}(\widetilde{\mathcal{Y}})$ of functions f on $\widetilde{\mathcal{Y}}$ of parity ν:

$$f(-y) = (-1)^\nu f(y), \quad y \in \widetilde{\mathcal{Y}}.$$

In this space the representation $\widetilde{R}_{\lambda,\nu}$ acts as follows:

$$\big(\widetilde{R}_{\lambda,\nu}(g) f\big)(y) = f\left(\frac{yg}{\|yg\|}\right) \|yg\|^{-\lambda-n}, \quad g \in \widetilde{G}.$$

Restrict the representation $\widetilde{R}_{\lambda,\nu}$ of the group $\widetilde{G}$ to its subgroup G. Since G preserves the manifold $\mathcal{Y}$, we also restrict this representation to the space $\mathcal{D}_\nu(\mathcal{Y})$ of functions in $\mathcal{D}(\mathcal{Y})$ of parity ν. Let us call this restriction the *canonical representation*.

Thus, the canonical representation $R_{\lambda,\nu}$, $\lambda \in \mathbb{C}$, $\nu = 0, 1$, of the group G acts on the space $\mathcal{D}_\nu(\mathcal{Y})$ by

$$\big(R_{\lambda,\nu}(g) f\big)(y) = f\left(\frac{yg}{\|yg\|}\right) \|yg\|^{-\lambda-n}, \quad g \in G.$$

The Lie algebra $\widetilde{\mathfrak{g}}$ of the group $\widetilde{G}$ consists of matrices $X \in \mathrm{Mat}(n, \mathbb{R})$ with trace zero. It splits into the direct sum $\mathfrak{g} + \mathfrak{m}$ where $\mathfrak{m}$ is the space consisting of matrices X such that $X' = XIX$. Decompose $\mathfrak{m}$ into the direct sum of two subspaces: $\mathfrak{m} = \mathfrak{a} + \mathfrak{n}$ where $\mathfrak{a}$ is the subspace of diagonal matrices and $\mathfrak{n}$ consists of matrices with zero diagonal. A basis of $\mathfrak{n}$ is formed by matrices $M_{ij} = E_{ij} + \lambda_i \lambda_j E_{ji}$, $i < j$, or, more detailed, by matrices $M_{km} = E_{km} + E_{mk}$, $1 \leqslant k < m \leqslant p$ or $p+1 \leqslant k < m \leqslant n$, and $M_{kn} = E_{kn} - E_{nk}$, $1 \leqslant k \leqslant p < m \leqslant n$. The subalgebra $\mathfrak{a}$ is spanned by matrices $Y_{km} = E_{kk} - E_{mm}$, $1 \leqslant k < m \leqslant n$.

For $X \in \widetilde{\mathfrak{g}}$, differential operators $\widetilde{R}_{\lambda,\nu}(X)$ do not depend on ν, so we omit ν in the notation and write $\widetilde{R}_\lambda(X)$.

The centralizer of the group K in $\widetilde{\mathfrak{g}}$ is one-dimensional, a basis is the following matrix in $\mathfrak{a}$:

$$Y_0 = \frac{1}{n}\begin{pmatrix} qE_p & 0 \\ 0 & -pE_q \end{pmatrix}, \tag{1}$$

where E_k is the identity matrix of order k.

In particular, let us write $\widetilde{R}_\lambda(Y_0)$. Write y as a pair $y = (u,v)$, $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$. Then

$$\widetilde{R}_{\lambda,\nu}\left(\mathrm{e}^{tY_0}\right) f(u,v) = f\left(\mathrm{e}^t u, v\right) \mathrm{e}^{t(-\lambda-n)(-p)/n}.$$

Differentiating it with respect to t at the point $t=0$ and passing to polar coordinates: $u = r\omega$, $0 \leqslant r < 1$, $\omega \in S_1$, we obtain

$$\widetilde{R}_\lambda(Y_0) = \sum_{k=1}^p u_k \frac{\partial}{\partial u_k} + \frac{p}{n}(\lambda+n) = r\frac{\partial}{\partial r} + \frac{p}{n}(\lambda+n). \tag{2}$$

4. Interaction of the overalgebra with the Poisson transform

The Poisson transform $P_{\lambda,\nu;\sigma}$ is an operator $\mathcal{D}_\nu(S) \to C^\infty(\mathcal{Y})$, defined by

$$(P_{\lambda,\nu,\sigma}\varphi)(y) = a^{(-\lambda-\sigma-n)/2} \int_S [y,s]^{\sigma,\nu}\, \varphi(s)\, ds, \tag{3}$$

where

$$a = [y,y] = 1 - |y|^2.$$

It intertwines $T_{2-n-\sigma,\nu}$ and $R_{\lambda,\nu}$:

$$R_{\lambda,\nu}(g) P_{\lambda,\nu;\sigma} = P_{\lambda,\nu;\sigma} T_{2-n-\sigma,\nu}(g), \quad g \in G. \tag{4}$$

The integral converges absolutely for $\operatorname{Re}\sigma > -1$ and can be continued by analyticity to other λ, σ to a meromorphic function. Considered as a distribution, the function $(P_{\lambda,\nu,\sigma}\varphi)(y)$ has poles in σ (depending on λ) at points

$$\sigma = \lambda - 2k, \quad \sigma = 2 - n - \lambda + 2l,$$

where $k, l \in \{0,1,2,\ldots\}$. These poles are simple for generic λ.

We determine explicitly the interaction of the Poisson transform $P_{\lambda,\nu,\sigma}$ with Lie operators of the overgroup $\widetilde{G}$ in the representation $\widetilde{R}_{\lambda,\nu}$, i.e., with the representation $\widetilde{R}_{\lambda,\nu}$ of the Lie algebra $\widetilde{\mathfrak{g}}$ ("overalgebra") of the group $\widetilde{G}$.

We have to write explicitly the compositions $\widetilde{R}_\lambda(X)\, P_{\lambda,\nu,\sigma}$ where $X \in \widetilde{\mathfrak{g}}$. If $X \in \mathfrak{g}$, then, by (4), the answer is simple:

$$\widetilde{R}_\lambda(X)\, P_{\lambda,\nu,\sigma} = P_{\lambda,\nu;\sigma}\, T_{2-n-\sigma}(X).$$

Therefore, it is sufficient to take for X elements in the subspace $\mathfrak{m}$, see Section 3, for example, basis elements M_{km} and Y_{km}.

We write explicitly expressions of $\widetilde{R}_\lambda(X)P_{\lambda,\nu,\sigma}$ for the following elements $X \in \mathfrak{m}$: basis elements M_{km} in $\mathfrak{n}$ and Y_{km} in $\mathfrak{a}$. The crucial step is a computation of the composition $\widetilde{R}_\lambda(Y_0)P_{\lambda,\nu,\sigma}$, where Y_0 is the basis element in the centralizer of the group K, see (11). In order to find expressions for other $X \in \mathfrak{m}$, we use expressions for Y_0 and commutation relations.

Theorem 1. *Let σ be not a pole of the Poisson transform $P_{\lambda,\nu,\sigma}$. The operator $\widetilde{R}_\lambda(X)$, $X \in \mathfrak{m}$, interacts with this transform as follows:*

$$\widetilde{R}_\lambda(X)P_{\lambda,\nu;\sigma} = a(\lambda,\sigma)P_{\lambda,\nu;\sigma+2}\,K_\sigma(X) \\ + b(\lambda,\sigma)P_{\lambda,\nu;\sigma}\,E_\sigma(X) + c(\lambda,\sigma)P_{\lambda,\nu;\sigma-2}\,C(X), \quad (5)$$

where coefficients a, b, c are given by formulae:

$$a(\lambda,\sigma) = \frac{\lambda+\sigma+n}{(\sigma+1)(\sigma+2)(2\sigma+n-2)(2\sigma+n)}, \quad (6)$$

$$b(\lambda,\sigma) = \frac{2\lambda+n}{(2\sigma+n-4)(2\sigma+n)}, \quad (7)$$

$$c(\lambda,\sigma) = \frac{(\lambda-\sigma+2)\sigma(\sigma-1)}{(2\sigma+n-4)(2\sigma+n-2)}, \quad (8)$$

and $K_\sigma(X)$, $E_\sigma(X)$ and $C(X)$ are differential operators on S of order 4, 2 and 0 respectively (the operator $C(X)$ does not depend on σ) linearly depending on $X \in \mathfrak{m}$. In particular, for $X = Y_0$, see (1), we have

$$K_\sigma(Y_0) = (\Delta_1-\Delta_2)^2 + (2\sigma^2+2n\sigma+nq+2(p-q))\Delta_1 \\ + (2\sigma^2+2n\sigma+np-2(p-q))\Delta_2 \\ + (\sigma+2)(\sigma+p)(\sigma+q)(\sigma+n-2). \quad (9)$$

$$E_\sigma(Y_0) = \Delta_1 - \Delta_2 + \frac{p-q}{n}\sigma(\sigma+n-2), \quad (10)$$

$$C(Y_0) = 1, \quad (11)$$

Proof. We prove the theorem by straight computations of $\widetilde{R}_\lambda(X)P_{\lambda,\nu,\sigma}$ for basis elements $X \in \mathfrak{m}$.

Let us outline the proof of formulae (5)–(11). Recall (Sections 2 and 3): we write $y \in \mathcal{Y}$ and $s \in S$ as pairs: $s = (\xi,\eta)$ and $y = (u,v)$, so that $[y,s] = -\langle u,\xi\rangle + \langle v,\eta\rangle$. Denote also $A = [y,s]$ and $z = \langle u,\xi\rangle$, $w = \langle v,\eta\rangle$, so that $A = -z + w$. We use polar coordinates $u = r\omega$, $0 \leqslant r < 1$, $\omega \in S_1$.

To simplify the notation, we omit the symbol ν in notation $t^{\lambda,\nu}$, so that, say, A^σ, $A^{\sigma\pm1}$ stand for $A^{\sigma,\nu}$, $A^{\sigma\pm1,\nu\pm1}$, respectively, etc. Then the Poisson transform (3) can be rewritten as

$$P_{\lambda,\nu,\sigma}\varphi = a^\mu \int_S A^\sigma \varphi(s)\,ds, \quad (12)$$

where

$$\mu = \frac{-\lambda-\sigma-n}{2}, \qquad a = 1 - r^2. \quad (13)$$

Let us apply to $a^\mu A^\sigma$ the operator $\widetilde{R}_\lambda(Y_0)$, see (2). Using (13), we obtain:

$$\widetilde{R}_\lambda(Y_0)\,(a^\mu A^\sigma) = (\lambda+\sigma+n)a^{\mu-1}A^\sigma - \sigma a^\mu \cdot w \cdot A^{\sigma-1} - q\,\frac{\lambda+n}{n}\,a^\mu A^\sigma, \quad (14)$$

so that, see (12), we get

$$\widetilde{R}_\lambda(Y_0)P_{\lambda,\nu,\sigma}\varphi = (\lambda+\sigma+n)P_{\lambda+2,\sigma}\varphi - \sigma P_{\lambda+1,\sigma-1}(w\cdot\varphi) - q\,\frac{\lambda+n}{n}\,P_{\lambda,\sigma}\varphi.$$

To prove (5)–(11), we have to present function (14) as a linear combination of the following functions:

$$\begin{aligned} &a^{\mu+1}A^{\sigma-2},\\ &a^{\mu}A^{\sigma},\quad a^{\mu}\Delta_i A^{\sigma},\\ &a^{\mu-1}A^{\sigma+2},\quad a^{\mu-1}\Delta_i A^{\sigma+2},\quad a^{\mu-1}\Delta_i\Delta_j A^{\sigma+2}, \end{aligned}$$

where $i,j\in\{1,2\}$. We do it by rather long computations, we omit them. Note only some relations: we use

$$\begin{aligned} \Delta_1 A^\sigma =\ & -\sigma(\sigma+p-2)A^\sigma + \sigma(2\sigma+p-3)\cdot w\cdot A^{\sigma-1}\\ & +\sigma(\sigma-1)A^{\sigma-2}(1-a-w^2),\\ \Delta_2 A^\sigma =\ & -\sigma(\sigma+q-2)A^\sigma - \sigma(2\sigma+q-3)\cdot z\cdot A^{\sigma-1}\\ & +\sigma(\sigma-1)A^{\sigma-2}(1-z^2).\\ \Delta_2\,(w\cdot A^\sigma) = w\cdot\Delta_2 A^\sigma &+ \Delta_2 A^{\sigma+1} - A\Delta_2 A^\sigma. \end{aligned}$$

□

Now let us go to other elements $X\in\mathfrak{m}$. We use expressions for $X=Y_0$ just found and commutation relations.

Suppose we know (5) with coefficients a,b,c given by (6), (7), (8) for an element $X\in\mathfrak{m}$ and want to find expressions $\widetilde{R}_\lambda(M)P_{\lambda,\nu;\sigma}$ for the element

$$M=[X,L]\in\mathfrak{m}, \tag{15}$$

where L is an element in $\mathfrak{g}$. From (15) we have

$$\widetilde{R}_\lambda(M) = \widetilde{R}_\lambda(X)R_\lambda(L) - R_\lambda(L)\widetilde{R}_\lambda(X).$$

Multiplying this equality by $P_{\lambda,\nu,\sigma}$ from the right and using (4) with L instead of g, we obtain expression (5) for $\widetilde{R}_\lambda(M)$ where

$$\begin{aligned} K_\sigma(M) &= K_\sigma(X)T_{2-n-\sigma}(L) - T_{-n-\sigma}(L)K_\sigma(X),\\ E_\sigma(M) &= E_\sigma(X)T_{2-n-\sigma}(L) - T_{2-n-\sigma}(L)E_\sigma(X),\\ C(M) &= C(X)T_{2-n-\sigma}(L) - T_{4-n-\sigma}(L)C(X). \end{aligned}$$

For example, for $M=M_{kn}$ we take $X=Y_0$, $L=L_{kn}$; for $M=Y_{kn}$ we take $X=(1/2)M_{kn}$ $L=L_{kn}$ and so on. Expanded expressions for operators $K_\sigma(M)$ turn out to be rather cumbersome, we reduce them to products (compositions) of differential operators. Omitting long analytical computations, let us bring the result.

Introduce the following differential operators on S:

$$Z_k(\sigma) = s_k(\Delta_1 - \Delta_2) + (2\sigma + n)B_k - (\sigma + n - 2)(\sigma + p)s_k,$$
$$V_m(\sigma) = s_m(\Delta_1 - \Delta_2) - (2\sigma + n)B_m + (\sigma + n - 2)(\sigma + q)s_m,$$

where $k = 1, \dots, p$, $m = p + 1, \dots, n$.

Then we have

for $X = M_{km}$ $(k \leqslant p < m)$:

$$\begin{aligned} K_\sigma(M_{km}) &= -Z_k(\sigma+1)V_m(\sigma) - V_m(\sigma+1)Z_k(\sigma),\\ E_\sigma(M_{km}) &= -s_m Z_k(\sigma) - s_k V_m(\sigma),\\ C(M_{km}) &= -2s_k s_m; \end{aligned}$$

for $X = Y_{km}$ $(k \leqslant p < m)$:

$$\begin{aligned} K_\sigma(Y_{km}) &= V_m(\sigma)V_m(\sigma+1) + Z_k(\sigma)Z_k(\sigma+1),\\ E_\sigma(Y_{km}) &= s_k Z_k(\sigma) + s_m V_m(\sigma),\\ C(Y_{km}) &= s_k^2 + s_m^2; \end{aligned}$$

for $X = Y_{km}$ $(k < m \leqslant p)$:

$$\begin{aligned} K_\sigma(Y_{km}) &= Z_k(\sigma+1)Z_k(\sigma) - Z_m(\sigma+1)Z_m(\sigma),\\ E_\sigma(Y_{km}) &= s_k Z_k(\sigma) - s_m Z_(\sigma),\\ C(Y_{km}) &= s_k^2 - s_m^2; \end{aligned}$$

for $X = M_{km}$ $(1 \leqslant k < m \leqslant p)$:

$$\begin{aligned} K_\sigma(M_{km}) &= Z_k(\sigma+1)Z_m(\sigma) + Z_m(\sigma+1)Z_k(\sigma),\\ E_\sigma(M_{km}) &= s_k Z_m(\sigma) + s_m Z_k(\sigma),\\ C(M_{km}) &= 2s_k s_m. \end{aligned}$$

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References

[1] V.F. Molchanov, *Representations of a pseudo-orthogonal group that are connected with a cone*, Mat. Sbornik, 1970, **81**, No. 3, 358–375.

[2] V.F. Molchanov, *Canonical representations and overgroups*, Amer. Math. Soc. Transl., Ser. 2, 2003, **210**, 213–224.

[3] V.F. Molchanov, *Canonical representations and overgroups for hyperboloids of one sheet and Lobachevsky spaces*, Acta Appl. Math., **86** (2005), 115–129.

[4] V.F. Molchanov, *Canonical representations and overgroups for hyperboloids*, Funkts. Analiz Prilozh., 2005, **39**, No. 4, 48–61.

[5] V.F. Molchanov, *Canonical representations on Lobachevsky spaces: an interaction with an overalgebra*, Acta Appl. Math., 2007, **99** No. 3, 321–337.

[6] N. Mukunda, *Unitary representations of the homogeneous Lorentz group in an* $O(2,1)$ *basis*, J. Math Phys., 1968, **9**, No. 1, 50–61.

[7] Yu.A. Neretin, *Action of overalgebra in Plancherel decomposition and shift operators in imaginary direction*, Izv. Ross. Akad. Nauk, Ser. Mat., 2002, **66**, No. 5, 171–182.

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