Trends in Mathematics

Piotr Kielanowski
S. Twareque Ali
Pierre Bieliavsky
Anatol Odzijewicz
Martin Schlichenmaier
Theodore Voronov
Editors

Geometric Methods in Physics

XXXIV Workshop, Białowieża, Poland, June 28 – July 4, 2015

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Trends in Mathematics

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Editors
Piotr Kielanowski
Departamento de Física
CINVESTAV
Mexico City, Distrito Federal, Mexico

S. Twareque Ali
Department of Mathematics and Statistics
Concordia University
Montreal, Québec, Canada

Pierre Bieliavsky
Department de Mathematiques
Université Catholique de Louvain
Louvain-la-Neuve, Belgium

Anatol Odzijewicz
Institute of Mathematics
University of Białystok
Białystok, Poland

Martin Schlichenmaier
Mathematics Research Unit, FSTC
University of Luxembourg
Luxembourg-Kirchberg, Luxembourg

Theodore Voronov
School of Mathematics
University of Manchester
Manchester, United Kingdom

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Participants of the XXXIV WGMP

(Photo by Tomasz Goliński)

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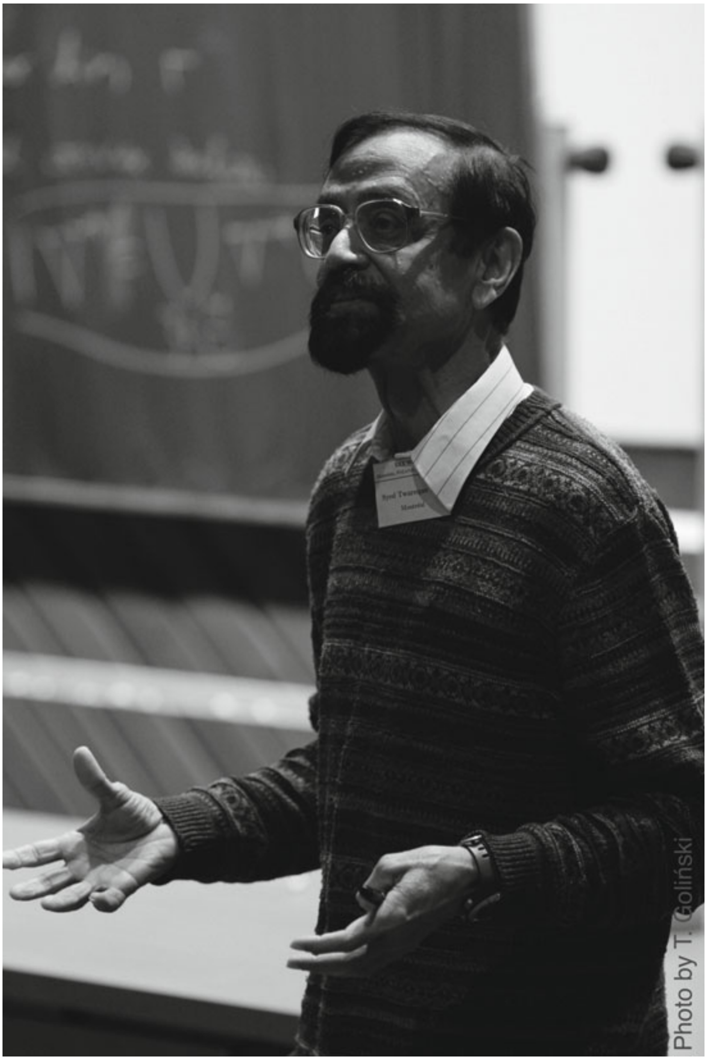

S. Twareque Ali (1942–2016)

As this volume went to press, we were saddened to learn of the sudden passing of our good friend and colleague, S. Twareque Ali. As a member of the Organizing Committee of the Workshop on Geometric Methods in Physics, and a participant each summer for 25 years, Twareque gave selflessly of his time and energy to ensure the success of the series. He will be long remembered for his scientific achievements, his generosity of spirit, and his devoted leadership.

Preface

The Workshop on Geometric Methods in Physics, also known as the "Białowieża Workshop", is an annual conference organized by the Department of Mathematical Physics at the Faculty of Mathematics and Computer Science of the University of Białystok in Poland. The idea of the conference is to bring together mathematicians and theoretical and mathematical physicists to discuss emerging ideas and developments in physics, which are important and require a mathematically precise formulation.

The Workshop, with open participation, is truly international and there are participants from many countries and almost all continents.

The range of topics discussed and the mathematical tools presented is always very ample. It includes descriptions of non-commutative systems, Poisson geometry, completely integrable systems, quantization, infinite-dimensional groups, supergroups and supersymmetry, quantum groups, Lie groupoids and algebroids and many more.

Antoinette and Gérard Emch during the XXV Workshop in Białowieża, 2006 (Photo by Tomasz Goliński)

The papers included in this volume are based on the plenary talks and other lectures given by the participants during the Workshop.

This year we had a special session dedicated to the memory of Gérard G. Emch, the outstanding mathematical physicist, who participated many times in our Workshops. Dr. Antoinette Emch, wife of Gérard Emch, gave a very interesting account of his efforts to understand and clarify the difference between Newton's and Leibniz' concepts of calculus.

The chapter *Representation Theory and Harmonic Analysis* contains the papers on groups, supergroups and group representations and also applications of group theoretical methods in mathematical and physical problems.

In the chapter *Quantum Mechanics and Integrable Systems* the discussed subjects comprise various properties of quantum systems, like supersymmetry, bound states or inverse scattering.

We also have two chapters *Algebraic Structures* and *Field Theory and Quantization*, which are devoted to discussions of new problems arising in quantum field theory and string theory and the new mathematical methods applied to such structures.

We conclude with a contribution of Bogdan Mielnik. Besides his strictly scientific interest Bogdan Mielnik likes to pinpoint some general problems of modern society and science. In his article he addresses possible obstacles which he sees for the future development of science. Being personal his observation and conclusion are nevertheless worth to be discussed in the community.

The Workshop in 2015, as in the previous years, was followed by the School on Geometry and Physics. It consisted of several mini-courses by top experts aimed mainly at young researchers and advanced students with the intention to help them to enter current research topics.

Białowieża, the traditional site of the Workshop, is a small village in eastern Poland at the border with Belarus. Białowieża is a place of remarkable and unspoiled beauty with an internationally known, unique National Park, containing the remnants of Europe's last primeval forest and the European bison reserve. These natural surroundings help to create a friendly atmosphere for discussions and collaboration.

The organizers of the Workshop gratefully acknowledge the financial support from the University of Białystok and the Belgian Science Policy Office (BELSPO), IAP Grant P7/18 DYGEST.

Finally, with great pleasure we thank the young researchers and graduate students from the University of Białystok for their indispensable help in the daily running of the Workshop.

The Editors January 2016

Part I

Quantum Structures
Gérard Emch in memoriam

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Gérard G. Emch

S. Twareque Ali

A special session, honouring the memory of Prof. Gérard G. Emch, was held on Tuesday, June 30, 2015. The sudden passing away of Gérard Emch (1936–2013), in his home in Gainesville, Florida, on March 5, 2013, left a pall of sadness over the mathematical physics community, his family, friends and colleagues and in particular the community surrounding the Bialowieza workshops. Emch had been a frequent participant at the Bialowieza meetings where, apart from contributing enormously to the scientific life of the meetings, he also endeared himself by

Prof. Gérard G. Emch (1936–2013)
(Photo by Tomasz Goliński)

his unique personality, incisive wit and cultural breadth. Among other contributions to the Bialowieza workshops, he co-edited a special volume entitled, *Twenty Years of Bialowieza: A Mathematical Anthology: Aspects of Differential Geometric Methods in Physics*, (Springer 2005), which was brought out to commemorate the twentieth anniversary of the Bialowieza meetings in 2001. During the 1996 and 2006 meetings, special sessions were organized to celebrate Emch's sixtieth and seventieth anniversaries. In the general scientific arena, Emch was an influential figure in contemporary mathematical physics, his work spanning the foundations of quantum mechanics, the algebraic approach to quantum physics and, during the last few years of his life, the history and philosophy of science. He was one of the pioneers in the axiomatic formulation of quaternionic quantum mechanics and the C^*-algebraic approach to quantum statistical mechanics, in particular quantum ergodic theory and quantum K-systems. His passing away has left an enormous void in the world of mathematical physics.

A number of Emch's former students, colleagues, friends, as well as his wife, attended the special session. Unfortunately not all of them managed to send in their contributions. We have collected together, in one section, the papers that were sent in. In particular, we include a paper based on a talk, given by Emch's wife, Antoinette, in which she reminisces about her life with Emch, the physicist, mathematician and philosopher, focusing in particular on his work on the history and philosophy of science. We dedicate this volume to the memory of Gérard G. Emch.

S. Twareque Ali

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The Gérard I knew for Sixty Years!

Antoinette Emch-Dériaz

Abstract. This paper is a very brief, and certainly not exhaustive, intellectual biography of Gérard G. Emch. The aim is to track and trace in his career recurring themes or subjects that led to the choice of his last years' research: which was to elucidate the philosophical difference between Newton's and Leibniz' conceptions of calculus as well as that behind the inventions of their methods.

It is not without trepidation and with much emotion that I stand here in this auditorium where Gérard stood so many times in the past. My purpose today is to bring to you a bit of what went on in his life and research since he spoke here in 2006.

Yet, before I dive into the last years of Gérard's research, I would like to recall some threads – recurring themes or particular subjects – that built the weft of his lifelong intellectual endeavor. I am now a historian, yet early in my life I was a scientist. As such I am curious about process and about how we get "there"; and this is why I want to elaborate on how Gérard got "there": that is, his last years' research on Isaac Newton (1642–1727), Gottfried Wilhelm Leibniz (1646–1716) and the philosophical differences behind their inventions of Calculus.

About 62 years ago, Gérard and I met for the first time; I had decided – with my father's blessings – to jump ship, leave the only girls' high school to join the Collège de Genève, founded by Calvin in 1559 to educate boys, at the time mostly for the ministry. Since the 1920s, this move was possible under two options: the classic and the scientific, which had courses in subjects not taught at the girls' high school. By the end of my two years in the scientific section of the Collège and approaching graduation, some of us decided to study more intensively Calculus and History in order to win some prizes offered to the graduating class. Gérard and I were among those who made this decision and it is how we started growing closer and finally dating following graduation. And sure we did win prizes! **First thread**.

We both enrolled at the Université de Genève in the Faculté des Sciences. We had many classes together, but not always, in particular, Gérard took some courses with Jean Piaget (1896–1980). It is well known that Piaget was interested in the acquisition of knowledge in children. Thus for someone interested in the teaching profession as Gérard was this was a natural. Piaget and his collaborators of the Institut Rousseau[1] had the whole system of the Genevan public schools at their disposal to gather data as we the children took years after years of tests that allowed Piaget to built his theory of genetic epistemology. In view of all the controversies about testing for grade-level learning that are currently raging in the USA, I will say that the tests were fun, that our schools or teachers were never ranked according to our results, and/or their financial support or salaries were never tied to them. Whatever scores we got, as far as I am concerned, they never affected my view of myself nor that of my teachers or parents. Yet Piaget's theories, especially those that led to what is called the "new math" and its teaching, are not exempt of criticism. Piaget's new math and the controversy it generated will be explored by Gérard in his later years. **Second thread**.

In 1959 we got married. Gérard was in the PhD program in Physics at the Université de Genève and I in the master's in biophysics. Joseph M. Jauch (1914–1974) had not yet arrived in Geneva; he came in 1960 and that changed Gérard's research direction toward more theoretical than solid states physics. At Jauch's urging, Gérard applied to the 1962 NATO Summer School in Theoretical Physics in Istanbul. There he met the stars of the day, including Eugene Wigner (1902–1995) who won the Physics Nobel Prize soon after. Gérard's questions and comments on how to simplify a proof or render it more elegant or even immediately generalizable, after some of Wigner's lectures, led the organizing committee to invite him to give a talk entitled, *On the introduction of the concept of superselection rules in Quantum Mechanics*. I brought that paper with me should anyone want to peruse it. Then, to Gérard's surprise, Wigner personally asked him to publish together on the subject in the proceedings. Too modest about his contribution, Gérard turned down the offer ... you can imagine his astonishment at what he had missed when the Nobels were announced in the Fall of 1962. Yet Wigner will re-appear in Gérard's career. **Third thread**.

In June of 1963, Gérard defended his PhD and Valentine Bargmann (1908–1989), who was for that year on sabbatical from Princeton University at the Federal Institute of Technology in Zurich, had agreed to be on the committee. After the defense, as we were celebrating with champagne, Bargmann offered Gérard a post-doctoral position at Princeton. Going to the USA was almost "de rigueur" at the time to obtain any kind of position or promotion in Swiss Universities; it even had an acronym IAG, that is "in Amerika gewesen". After the shock of an offer not done under the influence of too much champagne, we decided to give in and add the IAG to Gérard's credentials, while overlooking the consideration that we might never return permanently to Switzerland.

[1]named after the Genevan writer Jean-Jacques Rousseau (1712–1778) for his treatise Émile (1762) on education.

In September 1964, with two children in tow, we moved to Princeton and two years later to the University of Rochester, where most of Gérard's PhD students got their degrees; in 1984 I also got my PhD degree in intellectual history with an emphasis on the Enlightenment. As I had accompanied Gérard to some of his conferences, he reciprocated by attending the Eighteenth-Century Studies Meetings with me. At them, Gérard went to sessions on sciences or on music. At one annual meeting at MIT in 1981, in particular, where he listened to many talks on Newton, the quarrels of priorities with Leibniz, the wars on notation, and the politics of Newton's studies, he discovered that often the presenters did not know enough mathematics to buttress, even understand their cases, e.g., translating square by double! **Fourth thread**.

Today, STEM (science, technology, engineering, and mathematics) is at the forefront of university teaching and research. At the University of Rochester since the 1980s and still now (as I found out recently), the emphasis was on STEAM, that is to add the arts or the humanities to the program in what U of R called "clusters". This motivated Gérard and his philosopher colleague Henry E. Kyburg (1928–2007) to explore the possibilities of organizing weekly colloquiums in the Philosophy of science. They went to the then President O'Brien, whose specialty was Greek philosophy, with a padded yearly budget, sure to have it cut, and to their surprise O'Brien approved it and added: Come back next semester! And they did for five years until Gérard left to become chair of the Mathematics Department at the University of Florida in 1986. The colloquiums were held in the Physics Department auditorium; at first it was easy to find a seat, by the second year it was standing room only. **Fifth thread**.

Now, bringing these five threads together, I will show how their inter-play led to and informed Gérard's last research quest.

Wigner, again!

In the early 1990s, while Gérard was away at a conference, I picked up the phone and the caller asked for him. Dutifully I said that he was not home, that he would return in a few days, and that I would be happy to take a message. The caller identified himself as Jagdish Mehra and that he wanted Gérard to participate in the publishing of Wigner's complete works[2]. I got all of the information needed and waited for Gérard's return with great anticipation. But Gérard was not really interested. He had other projects on his mind, but this time I pressed him not to let the occasion slip away because he thought his contribution would not add value to this publication or was too small as he had done in Istanbul. I was determined not to let Gérard's self-abnegation prevail again. So he called Mehra back to accept the invitation. Mehra told him that his task was to annotate Wigner's more philosophical and reflective papers. Reading through hundreds of pages, Gérard

[2]The complete works, part A the scientific papers were edited by Arthur Wightman; Part B historical, philosophical, and socio-political papers by Jagdish Mehra were published by Springer-Verlag in 1993.

really found enjoyment in the process of discovering the maturing of Wigner's mind. Thus the nascent philosopher of science grew real roots in Gérard's life. He wrote the introduction to volume VI entitled: *Philosophical reflections and syntheses*. In his review of the eight volume Complete Works, the physicist/historian Silvan S. Schweber noted: "Volume VI, . . . , is introduced with a very helpful essay by Gerhard [sic] G. Emch . . . " and ". . . It would be wonderful if . . . this volume could be made available in an inexpensive paperback edition." And it was, the only one (Springer, 1997) in that series of eight volumes. The success of *Philosophical Reflections and Syntheses* as a paperback induced the Springer editor Beiglböck to approach Gérard about writing a book on "foundations". Gérard had been mulling on such a project, but did not feel completely confident he could bring it to fruition without a philosopher co-author close at hand. By then the Rochester connection with Kyburg was out-of-reach as we have been in Gainesville for almost 10 years. At the University of Florida there was a young assistant professor in the Philosophy Department whose specialty was philosophy of science. Would he be the one willing to bet his tenure on a book with a mathematical physicist and would he be the one to provide the know-how of writing philosophy? To find the answers to these questions and to test his knowledge of the field, Gérard decided to attend a Philosophy of Science Conference in Berlin on Einstein and relativity. This was just the experience Gérard needed to gain confidence in his philosophical abilities and to explore his presumed collaborator's credentials. The results were a book, *The Logic of Thermo-Statistical Physics* with Chuang Liu in 2002 and an invitation to be an All Souls College visiting fellow in 2004.

Calculus again!

All the while, at the back of Gérard's mind was "calculus". In Rochester, Gail Young, the Mathematics Department's chair had suggested to Gérard to write a text-book about teaching calculus that includes its foundation's context, because they had many conversations about the lack of historical and of philosophical perspectives on its development in the ones available. Gérard kept the suggestion in his to-do-list. As it happens in many places, the department's elder members do teach calculus, so it was at UF now for Gérard. He grew more and more frustrated with the assigned text-books that were more like cook-book recipes or mere turn the crank formulations. The memories of eighteenth-century studies meetings came back in force when I learned that a session on Madame du Châtelet (1706–1749) was in the making for the 1999 International Congress on the Enlightenment in Dublin/Ireland. Madame du Châtelet, had among many other things, translated Newton's third Latin edition (1726) of *Philosophiae Naturalis Principia Mathematica* into French, the only French complete translation to these days. The *Principia* first edition had appeared in 1687 and a second in 1713. Here was finally the occasion for us to put our expertise together, to fulfill an old and recurrent dream to study the intellectual pair Voltaire/du Châtelet. So began the trek with a presentation in Dublin on her translation, posthumously published in 1759, and how she dealt with a theory – calculus – still in the making and its weakness without falling into the quarrels of priorities or traps of notations. The 2006 tricentennial celebrations of Madame du Châtelet's birth allowed us to expand on her clarifications of the *Principia* that she elaborated in her commentaries on the original text, and published in the same volume as her translation. We picked it up again with her *Institution de physique*, published anonymously in 1740, which chronicled Madame du Châtelet's journey from a supporter of Leibniz to one of Newton. We had bought this leather-bound book way back to use it once upon a time. This original edition stayed on our shelves as a constant reminder of what today is, in American parlance, called a "bucket list"; at some point in time, its content would become a primary source for research. And that time came to Gérard when he retired from the University of Florida in 2005.

Free from teaching, which had become more and more burdensome because of mobility problems, Gérard was now able to immerse himself completely in research and writing. He had two projects on his mind. The first was a carry-over from his All Souls College fellowship: chapter 10: Quantum Statistical Physics in *Philosophy of Physics*, one of the volumes of the *Handbook of Philosophy of Science* (Elsevier, 2007). The second, which he pursued to his ultimate day, was "the why two hows of calculus". At first look, the *why two hows* corresponds to the different notations of Newton and Leibniz. The dot on top or the d in front. Much has been written on dot-age and d-ism, as plays on words to insinuate obsolescence or emergence. These somewhat ironic expressions were first coined in the nineteenth century by the astronomer John Frederick Herschel (1792–1871) and his student friends at Cambridge, who were annoyed by the enduring fuss over the by whom, when, and why one or the other notation was used or not use in Great Britain or on the Continent and whether or not the choice of notation determined creativity or stagnation in the pursuit of the sciences. Even national pride was invoked by some propagandists. Newton had been given a national funeral; Voltaire (1694–1778) had noted[3] that England knew how to honor his scientists in contrast to the Continent. Later it was insinuated that the towering figure of Newton had dimmed inventiveness in his followers. Herschel and his friends felt unjustly put upon and set apart from their Continental contemporaries for a mere dot.

But for Gérard this was just a superficial game for the sake of argumentation He had to seek a deeper meaning, perhaps rooted in the evolving political and economic contexts of Great Britain and of the Continent, probably and more easily apprehensible to him in the variant ways of thinking and of conceiving Calculus in Newton's and in Leibniz' writings. In Madame du Châtelet's commentary on the *Principia*, where she used throughout Leibniz' notation, a notation adopted by the Continental mathematicians as early as de L'Hôpital's treatise on analysis (1696), there was a hint when she had alluded to the geometry of the Ancients and the analysis of the Moderns. And that turned out to be the needed clue for Gérard to come to the conclusion that Newton constructs his theory on geometry, while Leibniz devises his on analysis, even if both tried to cover their tracts, which

[3]in his Fourteenth Letter concerning the English Nation (London, 1733).

makes it so hard to tease out the fundamentally different approaches that explain the *why two hows.*

There also intervenes a question of definition: what is meant by analysis for Newton or for Leibniz and what a casual reader understands it is. Since Francis Bacon (1561–1626) published his treatise *Novum Organum* in 1621, the pursuit of scientific explanations for natural phenomena had become more and more anchored on observation and experimentation, from which by induction one may discover their causes, and less and less justified by the Medieval notion that anything the human imagination could produce was possible; since God had allowed the thought, it had to exist in some form somewhere in order not to restrict God's omnipotence. Flight of fancy would not serve anymore as an answer. And that is what Newton called analysis, first to collect data, then to devise inductively probatory "principles", and finally to deduce future phenomena and verify their predictability power, for example, the shape of the earth or the return date of a comet. This method also referred as *probatio duplex* or double proof, Gérard used in his pursuit of the *why two hows.* Yet besides calculus, Newton's most important contribution was finally to put to rest the Aristotelian view of the two worlds, their motion governed by two sets of laws, the immutable incorruptible above and the decaying corrupt below. Just one cause, gravitation, explains motions in the sky and on earth, affirmed Newton, yet he was accused of using occult power in his attraction-at-a-distance explanation. Gravitation as description or explanation was also part of the Newton/Leibniz controversy.

The modern meaning of the term analysis traced back to René Descartes (1596–1650), who reversed Bacon's process to begin at the top with the famous *cogito ergo sum* and proceed to construct systematically a world based on analytical geometry. He tried to tackle natural phenomena such as the nature of light or how the planets stay in their orbits, but for Gérard's quest it is Descartes' influence on Leibniz that is important. While in Paris (1672–1676) on a diplomatic mission, Leibniz met the intelligentsia of the day, perfected his mathematical skills along Cartesian lines, and devised his version and notation of calculus which he published in 1684. As mentioned above, Leibniz' notation was quickly adopted by the Continent mathematicians over Newton's, which was published three years later. Leibniz also invented a calculating machine that brought an invitation to London in 1673 where he met acquaintances of Newton and, naturally, discussed mathematics with them. It is in recalling this visit that the participants in the quarrel over the invention of calculus priority found their pretext of accusing Leibniz of intellectual espionage, since he was as much a courtier than a philosopher, a man of the world as a mathematician; an almost antithesis to Newton's retiring personality, he could easily be accused of duplicity. But the quarrel per say was not Gérard's interest, his was in using *probatio duplex* to explore the mathematical and philosophical conflicts, not the priority one.

Philosophy again!

Now I turn to a paper Gérard wrote in 2007 entitled: *Three mathematical conflicts revisited in the light of probation duplex*

The abstract reads as follows:

Three mathematical conflicts illustrate the misunderstanding that may result from neglecting the methodological complementarity of (analysis versus synthesis) taught in the ancient probatio duplex. *These conflicts are:* (1) *the calculus wars between Newtonian and Leibizian tribes;* (2) *the misinterpretation of different intentions (explanation versus description) in promoting universal gravitation;* (3) *the attempts to conjugate the efforts of collaborators and disciples of Bourbaki and of Piaget toward a viable reform of the teaching of mathematics.*

To enter into all the details of the article would take too much time. I have brought a copy for anyone wanting to read it, yet I will read the introduction that Gérard wrote for it:

The extremely long duration of the calculus controversy between Newton, Leibniz, their respective disciples and their successors demands an explanation that involves more than the usual arguments of priority, notational effectiveness or national pride, rendered on the following statement, first published anonymously:

> *'By the help of the new* Analysis *Mr. Newton found out most of the Proposition in his* Principia Philosophia*: but because the Ancients for making things certain admitted nothing into Geometry before it was demonstrated synthetically, he demonstrated the Proposition synthetically, that the System of Heavens might be founded upon good Geometry. And it makes it now difficult for unskilled Men to see the Analysis by which those Propositions were found out.'* [*Newton,* 1715, 206].

This statement was condemned as a fake in the tribunal of Newton scholars; hence, we review first the reasons advanced to support this opinion. We propose in Section 3 *a revision of the trial in the light of probatio duplex; the ancient methodology, still recognized by Newton, is summarized in our Definition* 4*. Then, we examine in Section* 4 *the logical, methodological, and educational studies in Britain after Newton, especially during the first half of the nineteenth century; and show how and why a reconciliation with the Continent became possible. In Section* 5 *we discuss another manifestation of the different interpretation of* probatio duplex *which predicate the positions of Newton and Leibniz on universal gravitation. In Section* 6 *we move to the twentieth century and we examine the relation between the fundamental investigations of Bourbaki and of Piaget; the explorers discovered structural similarities between their programs and theoretical achievements, then tried but failed in the venture called new math. We emphasized how this episode re-enacts some of the eighteenth-century methodological misunderstanding we had exposed. Section* 7 *sums up our conclusions.*

The content of Subsection 4.3 entitled, "Three contributions that marked the British mathematical renaissance", addressed the work of George Green (1793–1841), George Biddell Airy (1801–1892), and William Rowan Hamilton (1805–1865). It was Gérard's most researched subject, which should be of no surprise. The last book Gérard was reading was *The Philosophical Breakfast Club* by Laura J. Snyder (Broadway Books, NY, 2011), which deals with how and who initiated this renaissance.

In Section 6, digging in the same vein of what motivates a renewal of activity after a dearth of achievement, Gérard studied the making and success of the Bourbakis and the link with Piaget's genetic epistemology in the new math. He came to the conclusion that its failing was due to the following:

The failure of the many new math initiatives proposed in the second half of the twentieth century, when would be reformers bypassed the explicit and repeated warnings of Bourbaki and of Piaget: neither the statements of abstract axioms nor the conduct of synthetic rigorous proofs can be assimilated by the novice students who have not been exposed first to a preparatory analysis of simple, elementary examples already familiar to them.

Gérard then concluded his paper by stating:

All three of our case-studies illustrates how theorems or theories, besides being correct become interesting by the strength and complexity of their connections in a wider web of knowledge. We showed that the complementarity of analysis and synthesis, the modern adaptations of the Ancients probatio duplex*, may help ensure correctness and relevance.*

This paper was circulated among colleagues, but never published, for circumstances which are not worth repeating here.

From 2007 on to his death in 2013, Gérard worked on his projected book on the historical and philosophical development of calculus. Here, he had to make choices of audiences: general educated public, scholars, or students, as well as the scope of the book in terms of chronology and territory. He could never really made up his mind as shown in the two tables of content he drew that are given below:

NEWTON'S DOT-AGE versus LEIBNIZ' D-ISM
a LESSON in METHODOLOGY
by
Gérard G. Emch

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Aug. 02, 2012

The Why Two Hows of Calculus:
an episode in the History of Ideas
by
Gérard G. Emch

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* * * * * * * **

And they were the sirens' calls of the internet, one search leading to another ad infinitum. This is the pleasure and peril of retirement. Gérard had the time, he thought, to research each and every link no matter how obscure and perhaps the more obscure the better. He accumulated about 10 meters of documentation neatly organized in filing cabinets, not counting the books on his shelves. Despite two partial knee replacements and two cataract removals to better his quality of life, Gérard had difficulty securing enough drive to pull out a finished product from what he had amassed ... or perhaps it was the thrill of the search that kept him going!

And in guise of epilogue and to evoke Gérard's sense of humor, here is what he told me one evening while we were recapping our day's activities:

Pour Newton comme pour Adam, la chute d'une pomme changea sa vie et la nôtre![4] (gge dixit Aug. 20, 2011)

Antoinette Emch-Dériaz
Gainesville, Florida, USA
e-mail: antemch@cox.net

[4]For Newton as for Adam, the fall of an apple changed his life and ours!

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Pseudo-bosons and Riesz Bi-coherent States

F. Bagarello

Abstract. After a brief review on $\mathcal{D}$-pseudo-bosons we introduce what we call *Riesz bi-coherent states*, which are pairs of states sharing with ordinary coherent states most of their features. In particular, they produce a resolution of the identity and they are eigenstates of two different annihilation operators which obey pseudo-bosonic commutation rules.

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1. Introduction

In a series of papers the notion of $\mathcal{D}$-pseudo bosons ($\mathcal{D}$-PBs) has been introduced and studied in many details. We refer to [1] for a recent review on this subject, and for more references. In particular, we have analyzed the functional structure arising from two operators a and b, acting on a Hilbert space $\mathcal{H}$ and satisfying, in a suitable sense, the pseudo-bosonic commutation rule $[a, b] = 1\!\!1$. Here $1\!\!1$ is the identity operator. We have shown how two biorthogonal families of eigenvectors of two non self-adjoint operators can be easily constructed, having real eigenvalues, and we have discussed how and when these operators are similar to a single self-adjoint number operator, and which kind of intertwining relations can be deduced. We have also seen that this setting is strongly related to physics, and in particular to PT-quantum mechanics [2, 3], since many models originally introduced in that context can be written in terms of $\mathcal{D}$-PBs.

In connection with $\mathcal{D}$-PBs, the notion of bicoherent states, originally introduced in [6], has been considered in some of its aspects, see [4, 5]. Since a and b are unbounded, several mathematical subtle points need to be considered when dealing with these states, as it is clear from the treatment in [5]. However, it is possible, and instructive, to consider a simpler situation, and this is exactly what we will do in this paper: more explicitly, we will adapt the notion of Riesz bases

This paper is dedicated to the memories of Gérard Emch and of Twareque Ali.

to coherent states, introducing what we can call *Riesz bicoherent states* (RBCS), and we will study some of their features.

This article is organized as follows: in the next section, to keep the paper self-contained, we review few facts on $\mathcal{D}$-PBs. In Section III we introduce our RBCS and analyze their properties, while our conclusions and plans for the future are discussed in Section IV.

2. A few facts on $\mathcal{D}$-PBs

We briefly review here few facts and definitions on $\mathcal{D}$-PBs. More details can be found in [1].

Let $\mathcal{H}$ be a given Hilbert space with scalar product $\langle .,. \rangle$ and related norm $\|.\|$. Let further a and b be two operators on $\mathcal{H}$, with domains $D(a)$ and $D(b)$ respectively, $a^\dagger$ and $b^\dagger$ their adjoint, and let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that $a^\sharp\mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\mathcal{D} \subseteq \mathcal{D}$, where $x^\sharp$ is x or $x^\dagger$. Of course, $\mathcal{D} \subseteq D(a^\sharp)$ and $\mathcal{D} \subseteq D(b^\sharp)$.

Definition 1. The operators (a, b) are $\mathcal{D}$-pseudo bosonic ($\mathcal{D}$-pb) if, for all $f \in \mathcal{D}$, we have

$$a\,b\,f - b\,a\,f = f. \tag{1}$$

Our working assumptions are the following:

Assumption $\mathcal{D}$-pb 1. – there exists a non-zero $\varphi_0 \in \mathcal{D}$ such that $a\,\varphi_0 = 0$.

Assumption $\mathcal{D}$-pb 2. – there exists a non-zero $\Psi_0 \in \mathcal{D}$ such that $b^\dagger\,\Psi_0 = 0$.

Then, if (a, b) satisfy Definition 1, it is obvious that $\varphi_0 \in D^\infty(b) := \cap_{k\geq 0} D(b^k)$ and that $\Psi_0 \in D^\infty(a^\dagger)$, so that the vectors

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n \varphi_0, \qquad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n \Psi_0, \tag{2}$$

$n \geq 0$, can be defined and they all belong to $\mathcal{D}$ and, as a consequence, to the domains of $a^\sharp$, $b^\sharp$ and $N^\sharp$, where $N = ba$. We further introduce $\mathcal{F}_\Psi = \{\Psi_n,\, n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\, n \geq 0\}$.

It is now simple to deduce the following lowering and raising relations:

$$\left\{\begin{array}{ll} b\,\varphi_n = \sqrt{n+1}\varphi_{n+1}, & n \geq 0, \\ a\,\varphi_0 = 0, \quad a\varphi_n = \sqrt{n}\,\varphi_{n-1}, & n \geq 1, \\ a^\dagger\Psi_n = \sqrt{n+1}\Psi_{n+1}, & n \geq 0, \\ b^\dagger\Psi_0 = 0, \quad b^\dagger\Psi_n = \sqrt{n}\,\Psi_{n-1}, & n \geq 1, \end{array}\right. \tag{3}$$

as well as the eigenvalue equations $N\varphi_n = n\varphi_n$ and $N^\dagger\Psi_n = n\Psi_n$, $n \geq 0$. In particular, as a consequence of these two last equations, choosing the normalization of φ_0 and Ψ_0 in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \tag{4}$$

for all $n, m \geq 0$. Hence $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$ are biorthogonal. Our third assumption is the following:

Assumption $\mathcal{D}$-pb 3. – $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$.

This is equivalent to requiring that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well, [7]. However, several physical models suggest to adopt the following weaker version of this assumption, [1]:

Assumption $\mathcal{D}$-pbw 3. – For some subspace $\mathcal{G}$ dense in $\mathcal{H}$, $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$-quasi bases.

This means that, for all f and g in $\mathcal{G}$,

$$\langle f, g\rangle = \sum_{n\geq 0} \langle f, \varphi_n\rangle \langle \Psi_n, g\rangle = \sum_{n\geq 0} \langle f, \Psi_n\rangle \langle \varphi_n, g\rangle , \tag{5}$$

which can be seen as a weak form of the resolution of the identity, restricted to $\mathcal{G}$. To refine further the structure, in [1] we have assumed that a self-adjoint, invertible, operator Θ, which leaves, together with Θ^{-1}, $\mathcal{D}$ invariant, exists: $\Theta\mathcal{D} \subseteq \mathcal{D}$, $\Theta^{-1}\mathcal{D} \subseteq \mathcal{D}$. Then we say that $(a, b^\dagger)$ are Θ-conjugate if $af = \Theta^{-1} b^\dagger\, \Theta\, f$, for all $f \in \mathcal{D}$. One can prove that, if $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{D}$-quasi bases for $\mathcal{H}$, then the operators $(a, b^\dagger)$ are Θ-conjugate if and only if $\Psi_n = \Theta\varphi_n$, for all $n \geq 0$. Moreover, if $(a, b^\dagger)$ are Θ-conjugate, then $\langle f, \Theta f\rangle > 0$ for all non zero $f \in \mathcal{D}$.

In the rest of the paper, rather than using Assumption $\mathcal{D}$-pbw 3, we will consider the following stronger version:

Assumption $\mathcal{D}$-pbs 3. – $\mathcal{F}_\varphi$ is a Riesz basis for $\mathcal{H}$.

This implies that a bounded operator S, with bounded inverse S^{-1}, exists in $\mathcal{H}$, together with an orthonormal basis $\mathcal{F}_e = \{e_n,\, n \geq 0\}$, such that $\varphi_n = Se_n$, for all $n \geq 0$. Then, because of the uniqueness of the basis biorthogonal to $\mathcal{F}_\varphi$, it is clear that $\mathcal{F}_\Psi$ is also a Riesz basis for $\mathcal{H}$, and that $\Psi_n = (S^{-1})^\dagger e_n$. Hence, putting $\Theta := (S^\dagger S)^{-1}$, we deduce that Θ is also bounded, with bounded inverse, is self-adjoint, positive, and that $\Psi_n = \Theta\varphi_n$, for all $n \geq 0$. Θ and Θ^{-1} can be both written as a series of rank-one operators. In fact, adopting the Dirac bra-ket notation, we have

$$\Theta = \sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n|, \qquad \Theta^{-1} = \sum_{n=0}^{\infty} |\varphi_n\rangle\langle\varphi_n|.$$

Of course both $|\Psi_n\rangle\langle\Psi_n|$ and $|\varphi_n\rangle\langle\varphi_n|$ are not projection operators[1] since, in general the norms of Ψ_n and φ_n are not equal to one.

Notice now that, calling $\mathcal{L}_\varphi$ and $\mathcal{L}_\Psi$ the linear span of $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ respectively, both sets are contained in $\mathcal{D}$ and dense in $\mathcal{H}$. Moreover, $\Theta : \mathcal{L}_\varphi \to \mathcal{L}_\Psi$, so that it is quite natural to imagine that Θ also maps $\mathcal{D}$ into itself. This is, in fact, ensured if both $S^\sharp$ and $(S^{-1})^\sharp$ map $\mathcal{D}$ into $\mathcal{D}$, condition which is satisfied in several explicit models, and for this reason will always be assumed here. Hence, both Θ and Θ^{-1}map $\mathcal{D}$ into itself. Of course, this assumption also guarantees that $e_n \in \mathcal{D}$, for all n.

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[1]Here $(|f\rangle\langle f|)\, g = \langle f, g\rangle\, f$, for all $f, g \in \mathcal{H}$.

The lowering and raising conditions in (3) for φ_n can be rewritten in terms of e_n as follows:

$$S^{-1}aSe_n = \sqrt{n}\, e_{n-1}, \qquad S^{-1}bSe_n = \sqrt{n+1}\, e_{n+1}, \tag{6}$$

for all $n \geq 0$. Notice that we are putting $e_{-1} \equiv 0$. It is now possible to check that

$$S^\dagger b^\dagger S^{-1} f = S^{-1}aSf, \qquad S^\dagger a^\dagger S^{-1} f = S^{-1}bSf,$$

for all $f \in \mathcal{D}$. Also, the first equation in (6) suggests to define an operator c acting on $\mathcal{D}$ as follows: $cf = S^{-1}aSf$. Of course, if we take $f = e_n$, we recover (6). Moreover, simple computations show that $c^\dagger$ satisfies the equality $c^\dagger f = S^{-1}bSf$, $f \in \mathcal{D}$, which again, taking $f = e_n$, produces the second equality in (6). These operators satisfy the canonical commutation relation (CCR) on $\mathcal{D}$: $[c, c^\dagger]f = f$, $\forall f \in \mathcal{D}$.

We end this section by noticing that, since each pair of biorthogonal Riesz bases are also $\mathcal{D}$-quasi bases, Proposition 3.2.3 of [1] implies that $(a, b^\dagger)$ are Θ-conjugate: $af = \Theta^{-1}b^\dagger\Theta f$, $\forall f \in \mathcal{D}$, and that Θ is positive, as we have already noticed because of its explicit form.

3. Riesz bicoherent states

In [4, 5] we have considered the notion of bicoherent states, and we have deduced some of their properties. Here we discuss a somehow stronger version of these states, which we call Riesz bicoherent states (RBCS).

We start by recalling that, calling $W(z) = e^{zc^\dagger - \overline{z}\, c}$, a *standard* coherent state is the vector

$$\Phi(z) = W(z)e_0 = e^{-|z|^2/2} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}}\, e_k. \tag{7}$$

Here c and $c^\dagger$ are operators satisfying the CCR, and $\mathcal{F}_e$ is the orthonormal basis related to these operators as shown in Section 2. The vector $\Phi(z)$ is well defined, and normalized, for all $z \in \mathbb{C}$. This is just a consequence of the fact that $W(z)$ is unitary, or, alternatively, of the fact that $\langle e_k, e_l\rangle = \delta_{k,l}$. Moreover,

$$c\,\Phi(z) = z\Phi(z), \qquad \text{and} \qquad \frac{1}{\pi}\int_{\mathbb{C}} d^2z |\Phi(z)\rangle\langle\Phi(z)| = \mathbb{1}.$$

It is also well known that $\Phi(z)$ saturates the Heisenberg uncertainty relation, which will not be discussed in this paper.

What is interesting to us here is whether the family of vectors $\{\Phi(z),\, z \in \mathbb{C}\}$ can be somehow generalized in order to recover similar properties, and if this generalization is related to the pseudo-bosonic operators a and b introduced in the previous section. For that, let us introduce the following operators:

$$U(z) = e^{zb - \overline{z}\, a}, \qquad V(z) = e^{za^\dagger - \overline{z}\, b^\dagger}. \tag{8}$$

Of course, if $a = b^\dagger$, then $U(z) = V(z)$ and the operator is unitary and essentially coincide with $W(z)$, with $a \equiv c$. However, the case of interest here is when $a \neq b^\dagger$. In [4, 5] we have introduced the vectors

$$\varphi(z) = U(z)\varphi_0, \qquad \Psi(z) = V(z)\,\Psi_0. \tag{9}$$

They surely exist if $z = 0$. We will see that, in the present working conditions, they are well defined in $\mathcal{H}$ for all $z \in \mathbb{C}$. A way to prove this result is to use the Baker–Campbell–Hausdorff formula which produces the identities

$$U(z) = e^{-|z|^2/2}\, e^{z\,b}\, e^{-\overline{z}\,a}, \qquad V(z) = e^{-|z|^2/2}\, e^{z\,a^\dagger}\, e^{-\overline{z}\,b^\dagger}.$$

Then,

$$\varphi(z) = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,\varphi_n, \qquad \Psi(z) = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,\Psi_n. \tag{10}$$

These clearly extend formula (7) for $\Phi(z)$. Now, [4], since $\|\varphi_n\| = \|Se_n\| \leq \|S\|$ and $\|\Psi_n\| = \|(S^{-1})^\dagger e_n\| \leq \|S^{-1}\|$, the two series converge for all $z \in \mathbb{C}$. Hence both $\varphi(z)$ and $\Psi(z)$ are defined everywhere in the complex plane. Incidentally we observe that this is different from what happens in [5], where $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are not assumed to be Riesz bases, and some estimate must be satisfied by $\|\varphi_n\|$ and $\|\Psi_n\|$. Also in view of possible applications, and in particular of the relation with Definition 3 below, it is interesting to show how to deduce the same result (i.e., $\varphi(z)$ and $\Psi(z)$ are defined everywhere) using a different strategy, assuming that a, b and c are related as in Section 2.

The key of this strategy is the following

Proposition 2. *With the above definitions the following equalities hold:*

$$U(z)f = SW(z)S^{-1}f, \qquad \textit{and} \qquad V(z)f = (S^{-1})^\dagger W(z)S^\dagger f \tag{11}$$

for all $f \in \mathcal{D}$.

Proof. We prove here the first equality. The second can be proved in a similar way.

First of all we can prove, by induction, that, for all $f \in \mathcal{D}$ and for all $k = 0, 1, 2, 3, \ldots$,

$$S\left(zc^\dagger - \overline{z}\,c\right)^k S^{-1}f = \left(zb - \overline{z}\,a\right)^k f. \tag{12}$$

This equality is evident for $k = 0$. This equality for $k = 1$ follows from the equations $cf = S^{-1}aSf$ and $c^\dagger f = S^{-1}bSf$, $f \in \mathcal{D}$. Now, assuming that this equation is satisfied for a given k, we have:

$$\begin{aligned} S\left(zc^\dagger - \overline{z}\,c\right)^{k+1} S^{-1}f &= S\left(zc^\dagger - \overline{z}\,c\right) S^{-1}S\left(zc^\dagger - \overline{z}\,c\right)^k S^{-1}f \\ &= S\left(zc^\dagger - \overline{z}\,c\right) S^{-1}\left(zb - \overline{z}\,a\right)^k f. \end{aligned}$$

Now, since $\left(zb - \overline{z}\,a\right)^k f \in \mathcal{D}$, it follows that

$$S\left(zc^\dagger - \overline{z}\,c\right) S^{-1}\left(zb - \overline{z}\,a\right)^k f = \left(zb - \overline{z}\,a\right)\left(zb - \overline{z}\,a\right)^k f = \left(zb - \overline{z}\,a\right)^{k+1} f.$$

Hence (12) follows. Notice that all the equalities above are well defined since $\mathcal{D}$ is stable under the action of all the operators involved in our computation.

Now, let us compute $SW(z)S^{-1}f$. Because of the boundedness of S, S^{-1} and $W(z)$, we have:

$$SW(z)S^{-1}f = S\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(zc^\dagger - \overline{z}\,c\right)^k\right)S^{-1}f$$
$$= \sum_{k=0}^{\infty}\frac{1}{k!}S\left(zc^\dagger - \overline{z}\,c\right)^k S^{-1}f = \sum_{k=0}^{\infty}\frac{1}{k!}\left(zb - \overline{z}\,a\right)^k f.$$

Then, since $SW(z)S^{-1}$ is bounded, the series $\sum_{k=0}^{\infty}\frac{1}{k!}\left(zb - \overline{z}\,a\right)^k f$ converges for all $z \in \mathbb{C}$ and for all $f \in \mathcal{D}$, and define $U(z)$ on $\mathcal{D}$. □

This proposition implies that, if S and S^{-1} are both bounded, the three *displacement operators* $U(z)$, $V(z)$ and $W(z)$ are *almost similar*, meaning with this that a similarity map S indeed exists, but the equalities in (11) makes only sense, in general, on $\mathcal{D}$ and not on the whole $\mathcal{H}$. This can be understood easily: while $W(z)$, S and S^{-1} are bounded operators, $U(z)$ and $V(z)$ in general are unbounded, so they cannot be defined in all of $\mathcal{H}$.

An immediate and interesting consequence of the equations in (11) is that $V(z)$ and $U(z)$ satisfy the following intertwining relation on $\mathcal{D}$:

$$SS^\dagger V(z)f = U(z)SS^\dagger f \tag{13}$$

for all $f \in \mathcal{D}$. This may be relevant, since this kind of relations have useful consequences in general. We refer to [8] for some results on intertwining operators. We will not insist on this aspect here, but still we want to stress that the operator doing the job, $SS^\dagger$, is close to $\Theta = S^\dagger S$, but S and $S^\dagger$ appear in the reversed order. Of course, these two operators coincide if S is self-adjoint.

Our results allow us to conclude now (once more, see formula (9)) that the two vectors in (9) are well defined for all $z \in \mathbb{C}$, and, more interesting, that

$$\varphi(z) = U(z)\varphi_0 = S\Phi(z), \qquad \Psi(z) = V(z)\Psi_0 = (S^{-1})^\dagger\Phi(z), \tag{14}$$

for all $z \in \mathbb{C}$. The proof is straightforward and will not be given here. We just notice that, in particular, these equations imply that $\varphi_0 \in D(U(z))$ and $\Psi_0 \in D(V(z))$, $\forall\, z \in \mathbb{C}$.

In analogy with the notion of Riesz bases, formula (14) suggests to introduce a general notion of RBCS:

Definition 3. A pair of vectors $(\eta(z), \xi(z))$, $z \in \mathcal{E}$, for some $\mathcal{E} \subseteq \mathbb{C}$, are called RBCS if a standard coherent state $\Phi(z)$, $z \in \mathcal{E}$, and a bounded operator T with bounded inverse T^{-1} exists such that

$$\eta(z) = T\Phi(z), \qquad \xi(z) = (T^{-1})^\dagger\Phi(z), \tag{15}$$

It is clear then that $(\varphi(z), \Psi(z))$ are RBCS, with $\mathcal{E} = \mathbb{C}$. It is easy to check that RBCS have a series of nice properties, which follow easily from similar properties of $\Phi(z)$. These properties are listed in the following proposition:

Proposition 4. *Let* $(\eta(z), \xi(z))$, $z \in \mathbb{C}$, *be a pair of RBCS. Then:*

(1) $$\langle \eta(z), \xi(z) \rangle = 1, \quad \forall\, z \in \mathbb{C}$$

(2) *For all* $f, g \in \mathcal{H}$ *the following equality* (resolution of the identity) *holds:*

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} d^2 z \, \langle f, \eta(z) \rangle \, \langle \xi(z), g \rangle \tag{16}$$

(3) *If a subset* $\mathcal{D} \subset \mathcal{H}$ *exists, dense in* $\mathcal{H}$ *and invariant under the action of* $T^\sharp$, $(T^{-1})^\sharp$ *and* $c^\sharp$, *and if the standard coherent state* $\Phi(z)$ *belongs to* $\mathcal{D}$, *then two operators* a *and* b *exist, satisfying* (1), *such that*

$$a\, \eta(z) = z \eta(z), \qquad b^\dagger \xi(z) = z \xi(z) \tag{17}$$

Proof. The first statement is trivial and will not be proved here. As for the second, due to the fact that both T and T^{-1} in Definition 3 are bounded, we have, for all $f, g \in \mathcal{H}$,

$$\langle f, g \rangle = \left\langle T^\dagger f, T^{-1} g \right\rangle = \frac{1}{\pi} \int_{\mathbb{C}} d^2 z \left\langle T^\dagger f, \Phi(z) \right\rangle \left\langle \Phi(z), T^{-1} g \right\rangle$$
$$= \frac{1}{\pi} \int_{\mathbb{C}} d^2 z \, \langle f, T\Phi(z) \rangle \left\langle (T^{-1})^\dagger \Phi(z), g \right\rangle = \frac{1}{\pi} \int_{\mathbb{C}} d^2 z \, \langle f, \eta(z) \rangle \, \langle \xi(z), g \rangle \,,$$

because of (15). To prove (3) we first observe that our assumption implies that the two operators a and b defined as $a = TcT^{-1}$ and $b = Tc^\dagger T^{-1}$ map $\mathcal{D}$ into $\mathcal{D}$, and that $[a, b] f = f$ for all $f \in \mathcal{D}$. The eigenvalue equations in (17) simply follow now from (15). □

It is interesting to notice that the resolution of the identity is valid in all of $\mathcal{H}$. This is true in the present settings, but we do not expect a similar result can be established if Assumption $\mathcal{D}$-pbs 3 is replaced with one of its weaker versions. We refer to [5] for some results concerning this situation. Concerning the saturation of the Heisenberg uncertainty relation, this cannot be recovered by these RBCS using the standard, self-adjoint, position and momentum operators q and p. However, if $q = \frac{1}{\sqrt{2}}(c + c^\dagger)$ and $p = i\,\frac{1}{\sqrt{2}}(c^\dagger - c)$ are replaced by $Q = \frac{1}{\sqrt{2}}(a + b)$ and $P = i\,\frac{1}{\sqrt{2}}(b - a)$, then we believe that a *deformed* version of the Heisenberg uncertainty relation involving these operators can, in fact, be saturated. This aspect will be discussed in a future paper, together with several examples of RBCS. Here we just consider a first simple example of these states, related to the harmonic oscillator.

An example from the harmonic oscillator. Let $\Phi(z)$ be a standard coherent state arising in the treatment of the quantum harmonic oscillator with Hamiltonian $H = c^\dagger c + \frac{1}{2}\,1\!\!1$, $[c, c^\dagger] = 1\!\!1$. In the coordinate representation this state, which we indicate here $\Phi_z(x)$, $z \in \mathbb{C}$ and $x \in \mathbb{R}$, is the solution of $c\,\Phi_z(x) = z\Phi_z(x)$. With a suitable choice of normalization we have

$$\Phi_z(x) = \frac{1}{\pi^{1/4}}\, e^{-\frac{1}{2}x^2 + \sqrt{2} z x - \Re(z)^2}.$$

Now, let $P = |e_0\rangle\langle e_0|$ be the orthogonal projector operator on the ground state $e_0(x) = \frac{1}{\pi^{1/4}}\, e^{-\frac{1}{2}x^2}$ of the harmonic oscillator. Then the operator $T = \mathbb{1} + iP$ is bounded, invertible, and its inverse, $T^{-1} = \mathbb{1} - \frac{1+i}{2}\, P$, is also bounded. Hence we can use formula (15) deducing that

$$\varphi_z(x) = T\Phi_z(x) = e_0(x)\left(e^{\sqrt{2}zx-\Re(z)^2} + ie^{-\frac{1}{2}|z|^2+\frac{i}{2}\Re(z)\Im(z)}\right),$$

while

$$\Psi_z(x) = (T^{-1})^\dagger\Phi_z(x) = e_0(x)\left(e^{\sqrt{2}zx-\Re(z)^2} - \frac{1-i}{2}\, e^{-\frac{1}{2}|z|^2+\frac{i}{2}\Re(z)\Im(z)}\right).$$

These are our RBCS, in coordinate representation. They both appear to be suitable deformations of the original vector $\Phi_z(x)$. It is not hard to imagine how to generalize this construction: it is enough to replace the operator P with some different orthogonal projector, for instance with the projector on a given normalized vector $u(x)$, $P_u = |u\rangle\langle u|$, $u(x) \neq e_0(x)$.

4. Conclusions

We have seen how bounded operators with bounded inverse can be used to construct not only Riesz biorthogonal bases, but also bicoherent states, having several properties which are similar to those of standard coherent states. More important, we have seen that these RBCS are naturally related to $\mathcal{D}$-PBs of a particular kind, the ones for which Assumption $\mathcal{D}$-pbs 3 holds true. It is clear that what we have discussed here is just the beginning of the story. There are several aspects of RBCS which deserve a deeper analysis. Among them, we cite the (maybe) most difficult: what does happen if Assumption $\mathcal{D}$-pbs 3 is not satisfied? And, more explicitly, what can be said when Assumption $\mathcal{D}$-pbw 3 is true? This is much harder, but possibly more interesting in concrete physical applications, since in this case, even if we can introduce a pair of bicoherent states [5], in general there is no bounded operator with bounded inverse mapping these states into a single standard coherent state. Moreover, we have several problems with the domain of the unbounded operators appearing in the game, and this, of course, requires more (and more delicate) mathematics.

Another aspect, which was just touched in [5], but not here, and which surely deserves a deeper analysis, is the use of bicoherent states, of the Riesz type or not, in quantization procedures. This may be relevant in connection with non conservative systems, or with physical system described by non self-adjoint Hamiltonians.

Another interesting open problem, which has been widely considered for standard coherent states along the years, is to check if completeness can be recovered for some suitable discrete subset of RBCS, i.e., if we can fix a discrete lattice in $\mathbb{C}$, $\Lambda := \{z_j \in \mathbb{C},\ j \in \mathbb{N}\}$, such that the set $\{(\eta(z_j), \xi(z_j)),\ z_j \in \Lambda\}$ is rich enough to produce a resolution of the identity in $\mathcal{H}$. Stated in a different way, is it possible to extend the results deduced in [9] for standard coherent states to RBCS or to bicoherent states in general? We believe that this can in fact be done for RBCS, while for general bicoherent states this is not so evident.

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F. Bagarello
Dipartimento di Energia,
Ingegneria dell'Informazione
e Modelli Matematici
Facoltà di Ingegneria
Università di Palermo
I-90128 Palermo, Italy

and

I.N.F.N., Sezione di Torino
e-mail: fabio.bagarello@unipa.it
URL: www.unipa.it/fabio.bagarello

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Entropy of Completely Positive Maps and Applications to Quantum Information Theory

Ichiro Fujimoto

Dedicated to Professor Gérard G. Emch

Abstract. In scope of *CP-convexity theory*, we study the mathematical structure of quantum interactions, and propose new entropy for completely positive maps and information quantities which recover the natural meaning and inequalities in the quantum information theory.

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1. Introduction

In the algebraic approach in quantum physics (cf. G.G. Emch's monograph [5]), quantum interactions are described by contractive completely positive maps which is called *operations* (cf. [17]). This formulation provided a fundamental tool in quantum information theory, and in recent years it is even influential to other areas including cosmo physics and particle physics. So, it would be worth reviewing the basic notions and mathematical descriptions of these phenomena in this context.

Let us assume that we have a quantum state on a quantum system described by a density operator ρ on a Hilbert space H, and a quantum state on an exterior system described by a density operator σ on a Hilbert space K, so that initially we have the compound state $\omega_0 = \rho \otimes \sigma$ on the tensor product Hilbert space $H \otimes K$, which is a separable state with no quantum correlation between these subsystems. Suppose that there exists an interaction between the two subsystems for some time interval, then the compound state is changed to a state ω described by

$$\omega = U(\rho \otimes \sigma)U^*$$

where U is a unitary operator representing the quantum evolution of the joint system, which becomes now an entangled state with quantum correlation between subsystems. After the interaction, the states of the subsystems on H and K are described by $\mathrm{Tr}_K\omega$ and $\mathrm{Tr}_H\omega$ respectively. Then the *channels* φ^* on $T(H)$, and ϕ^* on $T(K)$, are defined by

$$\varphi^*(\rho) = \mathrm{Tr}_K U(\rho\otimes\sigma)U^* \quad \text{and} \quad \phi^*(\sigma) = \mathrm{Tr}_H U(\rho\otimes\sigma)U^*,$$

which are the duals of the *operations* φ on $B(H)$, and ϕ on $B(K)$ respectively, representing the change of observables. Recall from K. Kraus [17] that the unital operation φ is a completely positive map on $B(H)$ of the form

$$\varphi(a) = \sum\nolimits_i V_i^* a V_i \;\text{ for }\; a\in B(H) \;\text{ with }\; V_i\in B(H) \;\text{ such that }\; \sum\nolimits_i V_i^* V_i = I_H,$$

and ϕ has a similar representation. Actually, contractive CP-maps, such as each term in the above decomposition, are called *operations* in general.

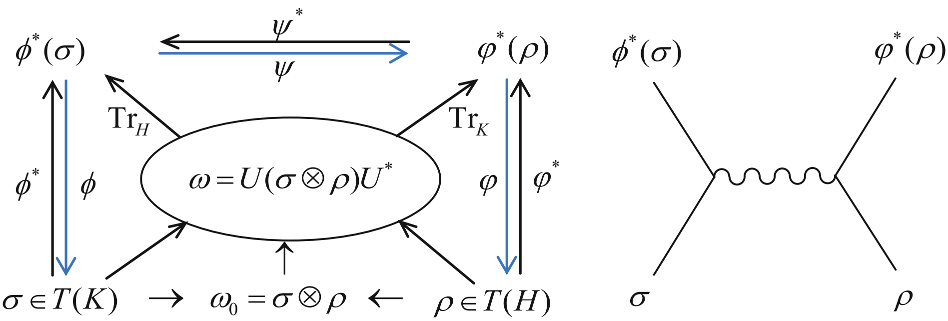

FIGURE 1. Quantum interactions

We can compare this to the standard description of quantum interactions in quantum field theory, which is a particular case where Hilbert spaces are Fock spaces, since every quantum interaction should be subject to this formulation as far as the systems are described by quantum theory.

We also note that the normal state ω on the tensor product $B(H)\otimes B(K)$ is represented by a normal completely positive map ψ_ω from $B(K)$ to $T(H)$ with $\mathrm{Tr}\,\psi_\omega(I_K) = 1$, i.e.,

$$\omega(a\otimes b) = \mathrm{Tr}(a\,\psi_\omega({}^t b)) \text{ for } a\in B(H) \text{ and } b\in B(K),$$

where we can observe that $\psi_\omega(I_K) = \varphi^*(\rho)$ and $\psi_\omega^*(I_H) = \phi^*(\sigma)$ in the above diagram. We then define the *correlation CP-map* ψ from $B(K)$ to $B(H)$ by

$$\psi(b) := \varphi^*(\rho)^{-\frac{1}{2}}\psi_\omega(b)\varphi^*(\rho)^{-\frac{1}{2}} \text{ for } b\in B(K),$$

where $\varphi^*(\rho)^{-\frac{1}{2}}$ is defined on the support of $\varphi^*(\rho)$. Then ψ is a unital CP-map from $B(s(\phi^*(\sigma))K)$ to $B(s(\varphi^*(\rho))H)$, where $s(\phi^*(\sigma))$ [resp. $s(\varphi^*(\rho))$] denotes the support projection of $\phi^*(\sigma)$ [resp. $\varphi^*(\rho)$], so that it can be represented as

$$\psi(b) = \sum\nolimits_j W_j^*\, b\, W_j \text{ where } W_j\in B(H,K) \text{ with } \sum\nolimits_j W_j^* W_j = I_{s(\varphi^*(\rho))H}.$$

We will see later that the *entanglement* of the state ω (the intensity of the quantum correlation) is equal to the *dissemination* of the channel φ.

It should be noted here that the CP-map φ depends on σ and U, ϕ depends on ρ and U, and ψ depends on ρ, σ and U. We also note that, when we focus on the operation φ, we can assume that σ is pure without loss of generality, i.e., the system K is closed before the interaction. In fact, we can consider the Hilbert space $\tilde{K} = K \otimes K$ and take a pure state $\tilde{\sigma}$ on $\tilde{K}$ (which we call a *purification* of σ), and a unitary $\tilde{U}$ on $H \otimes \tilde{K}$ such that $\tilde{U}|_{H\otimes K} = U$ and $\mathrm{Tr}_{\tilde{K}}\tilde{U}(\rho \otimes \tilde{\sigma})\tilde{U}^* = \varphi^*(\rho)$. In this situation, the correlation CP-map ψ can be derived by the symmetric arguments with those for φ.

In the second section, we overview the basic results in CP-convexity theory, which was initiated in [6]. The above diagram can be applied to the measurement process, where $B(K)$ is generalized for C*-algebra A (where we assume that A is unital in this note), and the set $Q_H(A)$ of contractive CP-maps from A to $B(H)$, which are the complementary CP-maps ϕ^c of ϕ, play the quantized state space of the system, which we call *CP-state space*. We then introduce an operator convexity in the CP-state space $Q_H(A)$, where φ is said to be a *CP-convex combination* of $\varphi_i \in Q_H(A)$ if

$$\varphi = \sum\nolimits_i S_i^* \, \varphi_i \, S_i \text{ with } S_i \in B(H) \text{ such that } \sum\nolimits_i S_i^* S_i = I_H,$$

which will be abbreviated by $\varphi = CP\text{-} \sum_i S_i^* \, \varphi_i \, S_i$.

We can thus develop a quantization of convexity theory, that is *CP-convexity theory*. We first show that A is *-isomorphic to the set of all $B(H)$-valued weakly continuous *CP-affine* functions on the CP-state space $Q_H(A)$ (cf. [7]). To identify the extreme elements of the CP-state space, we shall introduce two types of convexity, which inherently correspond to the algebraic structure and the statistical structure of the quantum system. In the first case, the extreme elements are just the set of irreducible representations $\mathrm{Irr}(A : H)$ of A on H if the dimension of H is large enough. On these extreme elements, we can realize the non-commutative Gelfand–Naimark theorem (cf. [9]), where A is *-isomorphic to the set of all $B(H)$-valued weakly uniformly continuous equivariant functions on the extreme elements $\mathrm{Irr}(A : H)$. In the latter case, where the CP-convex combination is considered for positive operator coefficients, the extreme elements are the set of *conditional transforms* from A to $B(H)$, which have the physical meaning of minimal interactions, including annihilations and creations in Fock Hilbert spaces. In [8], we developed *CP-measure* and its integration theory, and showed the CP-convexity version of Choquet's theorem, called *CP-Choquet theorem*. Therefore, we can view that the quantum interactions are the CP-measure distributions on the minimal interactions, so CP-convexity is essential to quantum interactions, which is no more possible to be described in scalar convexity. Furthermore, for $T(H)$-valued CP-maps in quantum information theory, this convexity is important since the CP-coefficients give rise to the notion of the *entanglement of formation* of the representing state.

In the third section, we apply our methods to find out the relations between the statistical and informational quantities. We shall introduce a new entropy for CP-maps which represents the statistical complexity including the entanglement information of the representing systems, form a concave function over the CP-maps, which recover the natural meaning and inequalities in the quantum information theory.

2. Quantization of convexity theory

We review some basic notations and results for CP-maps and CP-convexity. Recall that, by the Stinespring representation theorem, every CP-map $\psi \in CP(A, B(H))$ from a C*-algebra A to $B(H)$ can be represented as $\psi = V^*\pi V$ where π is a representation of A, and $V \in B(H, H_\pi)$ (cf. [2, 21]). We call ψ to be a *CP-state* if it is contractive, and denote by $Q_H(A)$ the set of all CP-states, i.e.,

$$Q_H(A) = \{\psi = V^*\pi V \in CP(A; B(H)); \|V\| \leq 1\},$$

and by $S_H(A)$ the set of all *unital* CP-states, i.e.,

$$S_H(A) = \{\psi = V^*\pi V; V^*V = I_H\}.$$

$\psi = V^*\pi V$ is *pure* iff π is irreducible, and we denote by $P_H(A)$ the set of all pure elements in $CP(A, B(H))$, and by $PS_H(A) = P_H(A) \cap S_H(A)$ the set of all unital pure CP-states. Recall also that $\mathrm{Rep}(A : H)$ [resp. $\mathrm{Rep}_c(A : H)$, $\mathrm{Irr}(A : H)$] represents the set of all [resp. cyclic, irreducible] representations of A on H (i.e., whose representation spaces are subspaces of H). We can show that

$$Q_H(A) = CP\text{-conv}\,\mathrm{Rep}_c(A : H)$$

if H is large enough, i.e., $\dim H \geq \alpha_c(A) := \sup\{\dim H_\pi; \pi \in \mathrm{Rep}_c(A)\}$, which guarantees that all cyclic representations of A are realized on H.

A function $\gamma : Q_H(A) \to B(H)$ is defined to be *CP-affine* if

$$\varphi = CP\text{-}\sum_i S_i^*\,\varphi_i\,S_i \quad \text{implies that} \quad \gamma(\varphi) = \sum_i S_i^*\,\gamma(\varphi_i)\,S_i.$$

We denote by $AC(Q_H(A), B(H))$ the set of all $B(H)$-valued BW-w continuous CP-affine functions on $Q_H(A)$, where the BW-topology is the point-wise weak operator topology in $Q_H(A)$. The following theorem generalizes Kadison's function representation theorem (cf. [15, 16]).

Theorem 1 (CP-duality Theorem). *Let A be a C*-algebra and H be a Hilbert space with* $\dim H \geq \alpha_c(A)$. *Then,*

$$A \cong AC(Q_H(A), B(H)) \quad (\text{*-isomorphism}).$$

Thus two C*-algebras A and B are *-isomorphic iff their CP-state spaces $Q_H(A)$ and $Q_H(B)$ are CP-affine BW-homeomorphic. This implies that CP-convexity captures C*-structure while the scalar convexity was limited in Jordan structure of the algebra (cf. [7])

The natural questions would arise: "What are the extreme elements of the CP-ctate space $Q_H(A)$ in the sense of CP-convexity?" Our first definition and its characterization of CP-extreme elements are as follows (cf. [10]).

Definition 1. A CP-state is defined to be *CP-extreme* if $\psi = CP\text{-}\sum_i v_i^* \psi_i v_i$ implies that ψ_i is unitarily equivalent to ψ. We denote by $D_H(A)$ the set of all CP-extreme states.

Theorem 2.

(i) *If* $\dim H = \infty$, *then* $D_H(A) = \mathrm{Irr}(A : H)$.
(ii) *If* $1 < \dim H < \infty$, *then* $D_H(A) = \mathrm{Irr}(A : H) \cup PS_H(A)$.
(iii) *If* $\dim H = 1$, *then* $D_H(A) = P(A)$.

We can now generalize the Gelfand–Naimark theorem [13] to the non-commutative case on the CP-extreme elements $\mathrm{Irr}(A : H)$ as follows. A function $\gamma : \mathrm{Irr}(A : H) \to B(H)$ is called *equivariant* if it preserves unitary equivalence. We shall denote by $A_u^E(\mathrm{Irr}(A : H), B(H))$ the set of all uniformly BW-w continuous $B(H)$-valued equivariant functions on $\mathrm{Irr}(A : H)$.

Theorem 3 (CP-Gelfand–Naimark Theorem). *Let A be a C^*-algebra and H be a Hilbert space with* $\dim H \geq \alpha_c(A)$. *Then,*

$$A \cong A_u^E(\mathrm{Irr}(A : H), B(H)) \quad (\text{*-isomorphism}).$$

This result sharpens Takesaki's duality theorem on *Rep*$(A : H)$ [4, 22]. For the proof and applications of this theorem, see [9].

On the other hand, we have another definition of CP-extreme states, restricting the CP-coefficients to positive operators.

Definition 2. A CP-state is defined to be *conditionally CP-extreme* if $\psi = CP - \sum_i v_i \psi_i v_i$ with $v_i \geq 0$, then $s(v_i)\psi_i s(v_i) = \psi$, where $s(v_i)$ denotes the support of v_i. We denote by $E_H(A)$ the set of all conditionally CP-extreme states.

Theorem 4. $E_H(A) = \{\psi = u^* \pi u \in P_H(A); u^* u = p_\psi\}$.

Thus conditional transforms are minimal interactions as expected, where CP-coefficients are partial isometries, and they contain the information of correlation in quantum information theory.

We note that Choquet's representation theorem (e.g., [1]) was generalized for CP-convexity context by introducing *CP-measure* (operation-valued measure) and its integration theory (see [8] for details).

Theorem 5 (CP-Choquet Theorem). *Let A and H be separable. Then, for any CP-state $\psi \in Q_H(A)$, there exists a CP-measure λ_ψ supported by $D_H(A)$ such that*

$$\psi(a) = \int_{D_H(A)} \hat{a} d\lambda_\psi \text{ for all } a \in A,$$

where we say that ψ is the barycenter of λ_ψ, i.e., $\psi = b(\lambda_\psi)$.

Thus, every quantum interaction is decomposed into minimal interactions where the distribution is represented by a CP-measure, so that CP-convexity is an essential tool for quantum interactions.

3. Quantization of information theory

It is well known that in quantum information theory we do not have the natural generalization of classical information quantities, such as joint entropy, mutual entropy, conditional entropies. For example, consider an entangled pure state with marginal entropies $H(A)$ and $H(B)$, then the joint entropy $H(A, B)$ satisfies $H(A, B) = 0 < H(A), H(B)$, which is impossible in the classical information theory. Moreover, the mutual entropy $I(A, B)$ is customarily defined by the relation $I(A, B) = H(A) + H(B) - H(A, B)$, so in this case $I(A, B) = H(A) + H(B) = 2H(A)$, which cannot happen in the classical theory. Also, note that the conditional entropy is defined by $H_B(A) = H(A) - I(A, B)$ in the classical case, but in this case $H_B(A) = -H(A) < 0$ which would be unacceptable. These situations are illustrated as follows:

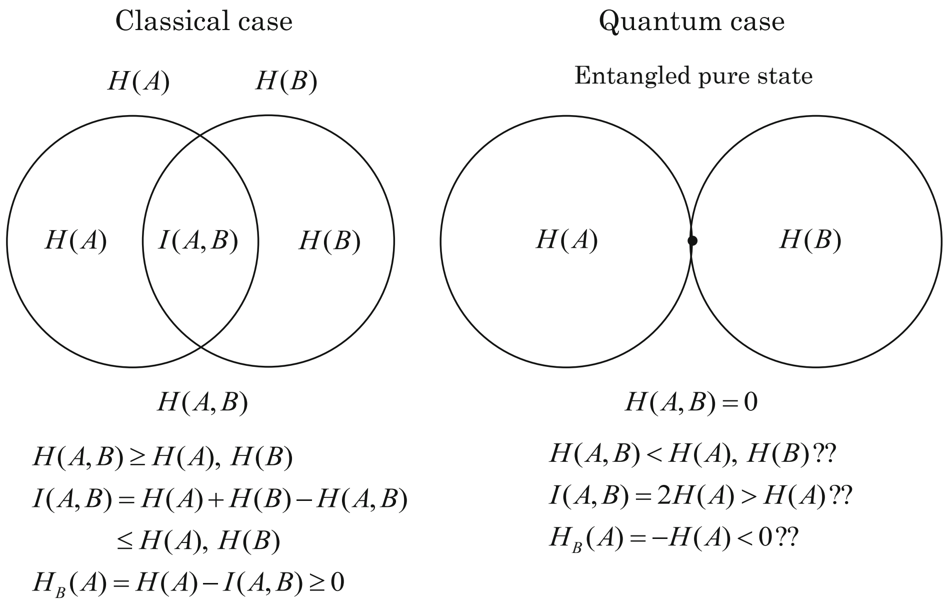

FIGURE 2. Comparison of information quantities

The purpose of this note is to define a new informational joint entropy $H(A, B)$ so that it should include the information from the entanglement of the bipartite system, it is symmetric with respect to A and B, it satisfies the inequality $H(A, B) \leq H(A) + H(B)$, and it is concave with respect to the representing state. Once $H(A, B)$ is constructed, then the natural generalization of other informational quantities would automatically follow.

Recall that there exists a one-to-one correspondence between a normal state ω on the compound system $B(K) \otimes B(H)$ and a normal completely positive map ψ_ω from $B(K)$ to $T(H)$ as we have seen in Introduction. Our scheme therefore can be reduced to find an appropriate definition of the entropy of the completely positive map ψ_ω. Recall that the notion of entropy is defined for atomic probability measure in scalar convexity theory, and that every operation is represented by a CP-convex combination, our problem will be reduced to define an appropriate entropy for such CP-measures. In this process, scalar coefficients in classical theory should be generalized for operator coefficients, which eventually means the process of *quantization* of information theory.

Now, let φ on $B(H)$ be an operation in the diagram in Figure 1, and then φ^* is a channel on $T(H)$, and let $\rho \in T(H)_1$, i.e., $\mathrm{Tr}\rho = 1$, which we call a *reference operator*. Let $\varphi_\rho = \rho^{1/2}\varphi\rho^{1/2} \in CP(B(H), T(H))$, and φ_ρ be decomposed as

$$\varphi_\rho = \sum_i v_i^* \cdot v_i = \sum_i |v_i|u_i^* \cdot u_i|v_i| = \sum_i p_i|\tilde{v}_i|u_i^* \cdot u_i|\tilde{v}_i|$$

where $p_i = \mathrm{Tr}\, v_i^* v_i > 0$ with $\sum p_i = 1$ and $\tilde{v}_i = p_i^{-1/2} v_i$ are the normalized coefficients such that $\tilde{v}_i^*\tilde{v}_i \in T(H)_1$ and $\sum_i p_i \tilde{v}_i^*\tilde{v}_i = \sum_i v_i^* v_i = \rho$. Note here that $u_i^* \cdot u_i$ are conditional transforms which represents minimal interactions. We shall denote by λ_{φ_ρ} the CP-measure corresponding to the above CP-decomposition, i.e., $\varphi_\rho = b(\lambda_{\varphi_\rho})$. For the notion of entropy of CP-maps, the following quantities are fundamental.

Definition 3.

(i) Let $S^L(\lambda_{\varphi_\rho}) := -\sum_i p_i \ln p_i$, and define

$$S^L_\rho(\varphi) := \inf_{\lambda_{\varphi_\rho}} \{S^L(\lambda_{\varphi_\rho}) : \varphi_\rho = b(\lambda_{\varphi_\rho})\},$$

which we call the *Lindblad entropy* of φ with respect to ρ.

(ii) Let $E(\lambda_{\varphi_\rho}) := \sum_i p_i S(\tilde{v}_i^*\tilde{v}_i)$, and call the *entanglement* of λ_{φ_ρ}. Then

$$E_\rho(\varphi) := \inf_{\lambda_{\varphi_\rho}} \{E(\lambda_{\varphi_\rho}) : \varphi_\rho = b(\lambda_{\varphi_\rho})\}$$

is called the *entanglement of formation* of φ with respect to ρ.

(iii) Let $\rho = \sum_k \mu_k P_k$ be a decomposition of ρ, and let

$$D(\varphi^*_\rho) := \inf \sum_k \mu_k S(\varphi^*_\rho(P_k)),$$

where inf is taken over all decomposition of ρ, which is called the *dissemination* of φ^* with respect to ρ.

(iv) Let $S^{op}(\lambda_{\varphi_\rho}) := -\sum_i Tr\, v_i^* v_i \ln v_i^* v_i$, and we define

$$S^{op}_\rho(\varphi) := \inf_{\lambda_{\varphi_\rho}} \{S^{op}(\lambda_{\varphi_\rho}) : \varphi_\rho = b(\lambda_{\varphi_\rho})\}$$

to be an *operator entropy* of φ with respect to ρ.

The entropy $S^L_\rho(\varphi)$ was originally defined by G. Lindblad [18] (see also [19]), which is called the *information exchange* in the field of quantum communications. The notion of entanglement of formation $E_\rho(\varphi)$ was introduced by [3] (cf. also [20]). We also note here the relations among the above-defined informational quantities. In the diagram in Figure 1, there exist states ω_{φ_ρ} and ω_{ψ_σ} corresponding to the

CP-maps φ_ρ and ϕ_σ respectively. Then, considering a pure state which corresponds to the unitary transform in the diagram, and assuming that σ is a pure state σ_0, we can deduce that there exists a partial isometry which connects $\phi^*(\sigma_0)$ and ω_{φ_ρ}, so that we have $S^L_\rho(\varphi) = S(\phi^*(\sigma_0))$. Similarly, from the unitary equivalence between ω_0 and ω which corresponds the correlation CP-map ψ, we conclude that $E_{\phi^*(\sigma_0)}(\psi) = D_\rho(\varphi^*)$. Observe also that

$$S^{op}(\lambda_{\varphi_\rho}) = -\sum\nolimits_i \mathrm{Tr} v_i^* v_i \ln v_i^* v_i = -\sum p_i \ln p_i + \sum\nolimits_i p_i S(\tilde{v}_i^* \tilde{v}_i).$$

Hence, we have

Theorem 6.

(i) $S^L_\rho(\varphi) = S(\phi^*(\sigma_0))$ *and* $E_{\phi^*(\sigma_0)}(\psi) = D_\rho(\varphi^*)$.
(ii) $S^{op}(\lambda_{\varphi_\rho}) = S^L(\lambda_{\varphi_\rho}) + E(\lambda_{\varphi_\rho})$, *so that* $S^{op}_\rho(\varphi) \geq S^L_\rho(\varphi) + E_\rho(\varphi)$.

Let A be the system described by ρ and B be the system described by $\varphi^*(\rho)$. We may consider $S^{op}_\rho(\varphi)$ or $S^L_\rho(\varphi) + E_\rho(\varphi)$ as a candidate of the joint entropy $H(A,B)$, however the inequality $H(A,B) \leq H(A) + H(B)$ may not be satisfied. Actually, we do not have a counterexample at the present, but we can easily find some cases where there exists a CP-decomposition λ_{φ_ρ} such that $S^{op}(\lambda_{\varphi_\rho}) > S(\rho) + S(\varphi^*(\rho))$. Another problem is that $S^{op}_\rho(\varphi)$ is not concave with respect to φ, since there is a counterexample.

We note here that $S^L_\rho(\varphi) = 0$ for all ρ iff φ is a conditionally CP-extreme, and $E(\varphi_\rho) = 0$ for all ρ iff φ is a separable CP-maps (i.e., all CP-coefficients are one-dimensional), and $S^{op}_\rho(\varphi) = 0$ for all ρ iff φ is an separable CP-extreme map, i.e., one-dimensional conditional transform, which we call an *atom*. This observation suggests that we may try to find a suitable separable state on the tensor space over $A \otimes B$ such that its partial traces are ρ and $\varphi^*(\rho)$, satisfying our requirements, i.e., inheriting the entanglement information, symmetric with respect to A and B, satisfying the triangle inequality, and concave with respect to φ. For this, let us consider the decomposition of φ_ρ again,

$$\varphi_\rho = \sum\nolimits_i v_i^* \cdot v_i = \sum\nolimits_i |v_i| u_i^* \cdot u_i |v_i| = \sum\nolimits_i p_i |\tilde{v}_i| u_i^* \cdot u_i |\tilde{v}_i|$$

where $\sum_i p_i \tilde{v}_i^* \tilde{v}_i = \sum v_i^* v_i = \rho$ and $\sum_i p_i \tilde{v}_i \tilde{v}_i^* = \sum v_i v_i^* = \varphi^*(\rho)$. Now, let $\rho_i := \tilde{v}_i^* \tilde{v}_i = \sum_j \alpha_{ij} P_{ij}$, $\hat{\rho}_i := \tilde{v}_i \tilde{v}_i^* = \sum_j \alpha_{ij} \hat{P}_{ij}$ be the spectral decompositions of ρ_i and $\tilde{\rho}_i$ respectively, and set $u_{ij} := u_i P_{ij}$, where we note that $\hat{\rho}_i = u_i \rho_i u_i^*$ and $\hat{P}_{ij} = u_i P_{ij} u_i^*$.

Definition 4. Let $\varphi^{at}_{\lambda_{\varphi_\rho}} := \sum_{ij} p_i \alpha_{ij} u_{ij}^* \cdot u_{ij}$, and define

$$S^{at}_\rho(\varphi) := \inf\{S^L(\varphi^{at}_{\lambda_{\varphi_\rho}}) : \varphi_\rho = b(\lambda_{\varphi_\rho})\}$$

to be the *atomic entropy* of φ with respect to ρ.

Then $S^{at}_\rho(\varphi)$ satisfies the requirements above, i.e., it includes the information both of the Lindblad entropy and the entanglement of formation of the bipartite system, symmetric with respect to ρ and $\varphi^*(\rho)$, and satisfies the inequality $S^{at}_\rho(\varphi) \leq S(\rho) + S(\varphi^*(\rho))$ since $\varphi^{at}_{\lambda_{\varphi_\rho}}$ is separable (cf. [14, 23]). Moreover, we can show that $S^{at}_\rho(\varphi)$ is concave with respect to φ, which is a desirable property as an entropy. We now propose to set $H(A,B) := S^{at}_\rho(\varphi)$ and $I(A,B) := S(\rho) + S(\varphi^*(\rho)) - S^{at}_\rho(\varphi)$. We can show that $S^{at}_\rho(\varphi) \leq S^{op}_\rho(\varphi)$, and then recover the desired inequalities for the new information quantities (cf. [12] for details).

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Ichiro Fujimoto
Department of Mathematics and Sciences
Kanazawa Institute of Technology
Nonoichi, Ishikawa 921-8501 Japan
e-mail: ifuji@neptune.kanazawa-it.ac.jp

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Some Comments on Indistinguishable Particles and Interpretation of the Quantum Mechanical Wave Function

Gerald A. Goldin

Dedicated in memory of Professor Gérard G. Emch,
an inspiration, a mentor, and a friend

Abstract. This paper discusses some fundamental questions pertaining to the wave function description of multiparticle systems in quantum mechanics. Motivated by results from the study of diffeomorphism group representations, I outline a point of view addressing subtle issues often overlooked in standard, "textbook" answers to these questions.

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Keywords. Current algebra, diffeomorphism groups, exotic statistics, indistinguishable particles, marked configuration spaces, permutation groups, quantum configurations, wave function description.

1. Questions motivating the discussion

Interesting questions arise in connection with the description of indistinguishable particles in quantum mechanics. Let us consider several of them:

1. What meaning should we ascribe to the wave function (for example, in a positional representation)?
2. How should we understand the construction of multiparticle states from single-particle ones?
3. How dependent are our descriptions on assumptions of strict linearity in quantum mechanics?

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4. What physical meaning attaches to the action of a group of permutations on particle coordinates?
5. What are the relationships among: (a) the exchange statistics of indistinguishable particles (Bose, Fermi, or other) expressed through a symmetry condition on the wave function; (b) configuration space topology; (c) self-adjoint extensions of densely-defined operators describing momentum, angular momentum, or energy; and (d) boundary conditions satisfied by wave functions? Which of these constructs are physically fundamental?
6. What are the implications for exotic particle statistics (e.g., anyons, non-abelian anyons, particles obeying parastatistics, configurations of extended objects, or particles in non-simply connected spaces)?

Various standard, easy answers (and some not-so-easy answers) to these questions are to be found in many textbooks and articles. But certain subtleties are overlooked in these answers, and I think there is something to be learned from probing more deeply. This paper is intended to highlight some important distinctions, and in so doing to stimulate possibly skeptical thinking about fundamental issues in quantum mechanics. I think that is something Gérard Emch would encourage us to do from time to time.

In a short presentation I can touch on only some of the above questions, and these these only partially; but I shall endeavor to provide a certain perspective from which to approach them. I cannot here include adequate references to the many researchers whose work should be cited; the reader is referred to more complete citations in [1, 2], and [3].

2. Positional representation of operators

In the conventional quantum mechanical description of a single particle, or of N particles, the interpretation of the wave function depends (of course) on how the observables are represented.

In a "positional" representation, the single-particle (complex- or spinor-valued) wave function is $\psi(x)$, where x coordinatizes physical space; the operators for position coordinates Q^j are represented by multiplication, $Q^j\psi(x) = x^j\psi(x)$; and the operators for momentum coordinates P^k are represented by differentiation, $P^k\psi(x) = -i\hbar(\partial/\partial x^k)\psi(x)$. In a "momentum" representation the single-particle wave function is $\tilde{\psi}(p)$, momentum coordinate operators are represented by multiplication, and those for position coordinates by differentiation. These are just two unitarily equivalent representations of the Heisenberg algebra, with the Fourier transform implementing the equivalence.

So to ask about an interpretation to be given to the wave function, we must first specify how some set of observables is being represented; otherwise, the question is not well posed. Here I focus on positional representations, partly because there is a fundamental sense in which actual measurements may be reduced to *sequences* of positional measurements (at different times) [4]. Then we need to describe the time-evolution of wave functions.

The time-evolution of the positional wave function $\psi(x;t)$ (for a single particle) is governed by a Schrödinger equation established by our representation of the Hamiltonian operator (corresponding to the energy observable). This time-evolution preserves the L^2 norm $\|\cdot\|$ of ψ. We then typically interpret $\psi(x;t)$ as a "probability amplitude;" i.e., $|\psi(x;t)|^2/\|\psi\|^2$ is the probability density for an idealized measurement localizing the particle in the vicinity of x at time t. To describe a sequence of two positional measurements, we must also specify the continued time-evolution after an outcome of the first (idealized) measurement. The initial condition after such a measurement localizes the particle in a region X at time t is often assumed to be the orthogonal projection of the wave function $\psi(x,t)$ onto the subspace having support in X.

But the interpretation of the single-particle wave function ψ in a positional representation does not end here. We must also say something about its phase. The interpretation of the phase of ψ depends further on how we choose to represent observables such as momentum and energy. After a *gauge transformation* $\psi'(x;t) = \exp[i\theta(x;t)]\,\psi(x;t)$, the representation is still positional, but the phase of ψ has been modified (so its interpretation must also change). Likewise, the representations of the Hamiltonian (energy) and momentum as differential operators have also been changed by the gauge transformation. We refer explicitly to these operators when we specify the gauge. While the modulus of ψ is gauge-invariant (under the usual gauge transformations of quantum mechanics), its phase is not.

Nevertheless, a gauge-invariant (probability flux) current density may be constructed from the phase. Its specification becomes part of the physical interpretation of ψ. Thus we have, in a positional representation, the interpretation of the single-particle wave function as describing a probability density and flux density in the one-particle configuration space (often identified with the physical space), providing predictions for the distribution of outcomes of positional measurements.

Let us also remark that use of a positional representation does not rule out additional, "internal" degrees of freedom needed to describe observables such as components of the particle spin. Then ψ is no longer scalar-valued, but may take values in an inner product space carrying a representation of an internal symmetry group (a Lie group) associated with the particle.

3. Many-particle systems

The conventional procedure for describing many-particle systems is to write the wave function in the form $\psi(x_1, \ldots, x_N)$, even in the case of indistinguishable particles. That is, ψ is taken to be a complex-valued L^2 function on the space of *ordered* N-tuples of points (the particle coordinates) in the physical space.

When the particles are indistinguishable, one then imposes an *additional* condition of exchange symmetry. Conventionally, one then interprets ψ as a probability amplitude for finding (simultaneously) particle 1 at x_1, particle 2 at x_2, and so forth. This motivates the need for an exchange symmetry condition – since the particles are indistinguishable, the probability density for simultaneously finding particle j at x_j and particle k at x_k ($j \neq k$) must be the same as that of finding particle k at x_j and particle j at x_k.

But this conventional interpretation raises some difficulties. Even in the case of distinguishable particles (when no additional symmetry is imposed), the characterization of "particle k," for a specific k, depends on some other, not-yet-specified measurement to be taken (e.g., of the particle mass) which distinguishes one particle from another. Furthermore, actual measurements take place in the physical space, *not* in the configuration space. How should the latter limitation be expressed?

Returning to the situation of indistinguishable particles, the usual symmetry condition imposed relates $\psi(x_1, \ldots, x_N)$ to $\psi(x_{\sigma(1)} \ldots x_{\sigma(N)})$, where $\sigma \,\epsilon\, S_N$ (the symmetric group) is a permutation of the N indices. The relationship is by means of a unitary (typically, 1-dimensional) representation of S_N. The trivial representation characterizes bosons (totally symmetric wave functions), the alternating representation characterizes fermions (totally antisymmetric wave functions). A fundamental difficulty with this description, however, is that one has artificially labeled the indistinguishable particles with indices, and then introduced a symmetry to "undo" that step. What can this possibly mean physically?

An alternative approach is to refer to *unordered* configurations of particles in physical space, since the ordering is unnatural for distinguishable particles and unobservable for indistinguishable ones. Then a configuration is just an N-point subset of the spatial manifold M. Note that it is not necessary to include configurations where more than one particle occupy the same point. These form a Lebesgue measure zero set.

We write $\tilde{\gamma} = (x_1, \ldots, x_N)$ for an ordered configuration, and $\gamma = \{x_1, \ldots, x_N\}$ for an unordered configuration. Then $\tilde{\gamma} \to \gamma$ is a projection from the coordinate space $\tilde{\Gamma}^{(N)}$ (of ordered N-tuples of distinct points in physical space) to the configuration space $\Gamma^{(N)}$ (of N-point subsets of physical space).

It is natural to consider writing wave functions for identical particles on $\Gamma^{(N)}$ rather than $\tilde{\Gamma}^{(N)}$; indeed, $\Gamma^{(N)}$ is the physically relevant space. But we must then find a different way to characterize the exchange symmetry – to describe how bosons are to be distinguished from fermions, and what other particle statistics might be possible. This must now be done via representations of the *operators*, as there is no way available to impose a symmetry condition on wave functions on $\Gamma^{(N)}$.

We may also consider wave functions for distinguishable particles from this point of view. Then one is led quite naturally to the idea of *marked* configurations. A marked configuration is an N-point subset of a *bundle* B for which the base is the physical space M, and for which a fiber is a space in which additional values of particle attributes may be taken. This is discussed a little further below.

4. Diffeomorphism group representations and particle statistics

Taking seriously the comment that measurements occur in physical space (rather than configuration space), we observe that the mass density and momentum density operators form an infinite-dimensional Lie algebra of local currents modeled on physical space. This current algebra describes a natural class of kinematical observables. The group obtained by exponentiating the local currents is the group of compactly-supported diffeomorphisms of M. [5]

Let us take $M = \mathbf{R}^d$ ($d \geq 2$) for specificity. For a diffeomorphism ϕ of $\mathbf{R}^d$, one may write a unitary representation of the group on a space of wave functions $\psi(x_1, \ldots, x_N)$, $x_j \in \mathbf{R}^d$, as

$$[\hat{V}(\phi)\psi](x_1, \ldots, x_N) := \psi(\phi(x_1), \ldots, \phi(x_N))\Pi_{k=1}^{N}\sqrt{\mathcal{J}_\phi(x_k)}\,. \tag{1}$$

where $\mathcal{J}_\phi(x) = [d\mu_\phi/d\mu](x)$ is the Jacobian of ϕ at x (here μ is Lebesgue measure).

Note that the representation is unitary, and the exchange symmetry of ψ is preserved. The representation $\hat{V}$ acting on the Hilbert space of totally symmetric wave functions is *unitarily inequivalent* to the representation acting on the Hilbert space of totally antisymmetric wave functions.

Alternatively, suppose we consider representing the diffeomorphism group on the space of *unordered* configurations, as suggested in earlier constructions. [5] To do this, we set

$$\begin{aligned}&[V(\phi)\psi](\{x_1, \ldots, x_N\})\\ &\quad := \chi_\phi(\{x_1, \ldots, x_N\})\psi(\{\phi(x_1), \ldots, \phi(x_N)\})\Pi_{k=1}^{N}\sqrt{\mathcal{J}_\phi(x_k)}\,,\end{aligned} \tag{2}$$

where χ obeys a 1-cocycle equation. Note that set brackets have replaced the parentheses. In a shorter way, we can write

$$[V(\phi)\psi](\gamma) := \chi_\phi(\gamma)\psi(\phi\gamma)\Pi_{x_k \epsilon \gamma}\sqrt{\mathcal{J}_\phi(x_k)}\,, \tag{3}$$

where γ denotes the unordered configuration.

In this construction, noncohomologous cocycles describe unitarily inequivalent representations. The information regarding particle statistics has been encoded in the cocycle (i.e., in how the observables are represented), not in the wave function symmetry! Thus we have a fundamental change in perspective on the meaning of the wave function itself. On the left-hand side of Eq. (1), the expression x_j (the jth entry in the N-tuple forming the argument of ψ) refers to the location of particle j. In Eq. (2), the expression x_j refers simply to the location of a particle – *any* particle. The subscript j has no intrinsic meaning; it is just a way to indicate that there are N elements in the configuration γ. No extraneous labeling has been introduced.

5. A comment about linearity vs. nonlinearity in quantum mechanics

In exploring the possibility of nonlinear modifications of quantum mechanics, it is of interest to examine the different ways in which the usual assumptions of linearity are introduced [3, 4, 6].

One assumption of linearity inheres in the conventional method for constructing a theory of composite systems from their components – in particular, constructing multiparticle states from single-particle states. The Hilbert space of states describing the composite system is normally taken to be the tensor product of the Hilbert spaces for the subsystems – i.e., the space constructed from linear combinations of product states. For indistinguishable particles, product states are replaced by symmetric or antisymmetric linear combinations of product states, leading to the symmetrized or antisymmetrized tensor product Hilbert space. Then configurations for the composite system are ordered N-tuples, as discussed above. Subsystem observables are extended by linearity from product states to the full Hilbert space.

But adopting the perspective suggested here, one begins naturally with (spatial) configurations for the subsystems (as subsets of the physical space, or subsets of bundles over the physical space). One then constructs the configurations for the composite system from generalized unions of these subsets. In particular, this leads to a direct construction of $\Gamma^{(N)}$ from N copies of $\Gamma^{(1)}$. The state-space for the composite system is the space of square-integrable functions on the composite configuration-space. Linearity need not be assumed in the construction (and there is no need for symmetrization or antisymmetrization of product states).

Without the initial assumptions of linearity, there is no obstacle to the discussion of the nonlinear gauge transformations introduced in [6]. Later, one can describe the quantum kinematics on this space of generalized unions by unitary representations of the group of compactly-supported diffeomorphisms of the physical space, identify irreducible representations, associate the particle statistics with inequivalent cocycles, and so forth.

6. Induced representations and the homotopy of configuration space

Select a particular configuration $\gamma \,\epsilon\, \Gamma^{(N)}$ and consider the stability subgroup K_γ. This is the group of those (compactly supported) diffeomorphisms of $\mathbf{R}^d$ which leave γ fixed. Note that a diffeomorphism can do this by implementing a permutation of the points in γ. For $d \geq 2$, there is thus a natural homomorphism from K_γ to S_N. A unitary representation of S_N thus defines a continuous unitary representation (CUR) of K_γ, which in turn induces a CUR of the diffeomorphism group.

Such an induced representation may be regarded as acting on a Hilbert space of *equivariant* wave functions on a covering space $\hat{\Gamma}^{(N)}$ of $\Gamma^{(N)}$ – or, equivalently, as acting directly on wave functions defined on $\Gamma^{(N)}$ but with a cocycle as in Eq. (3).

For $d \geq 3$, S_N is the fundamental group (first homotopy group) of $\Gamma^{(N)}$. The coordinate space $\tilde{\Gamma}^{(N)}$ defined earlier is then the universal covering space, and we recover the conventional description in terms of wave functions on ordered N-tuples.

For $d = 2$, however, the fundamental group of $\Gamma^{(N)}$ is the braid group B_N, and one obtains intermediate (or anyon) statistics [7] by inducing. This led to one of the early discoveries of the possibility of intermediate statistics for particles in two-space [8–10].

7. Label permutations and value permutations

Label permutations (also called index permutations) act on the indices of labeled particle coordinates, so that $\sigma \epsilon S_N$ takes x_k to $x_{\sigma(k)}$. The label permutation $\sigma_{(12)}$, for example, exchanges x_1 with x_2 in an ordered N-tuple, regardless of the actual values of the two variables.

Value permutations (in certain contexts, called wave function permutations) do not see the indices, but make reference to some specified ordering of points in the physical space M. In an ordered N-tuple, the value permutation $\sigma_{(12)}$ exchanges those entries having the two lowest values, regardless of where they occur in the N-tuple.

This distinction does not show up in 1-dimensional representations of S_N, so it is easily overlooked in discussing bosons and fermions. But it matters essentially if we want to consider higher-dimensional representations of S_N, describing particles satisfying parastatistics [11]. Furthermore, diffeomorphisms "see" only the values of the x_k, not the labels. Thus, whether they are acting in $\Gamma^{(N)}$ or a covering space, the relevant permutations are the value permutations. The inducing construction leading to anyon statistics involves discussion of homotopy classes of paths in configuration space, which refer to the values of the particle coordinates, not their labels (see also [10]). And it is clear why we require $d \geq 2$; in one dimension, a compactly supported diffeomorphism can never exchange two points on the real line.

8. Implications for exotic statistics

We have outlined a point of view that accommodates well the description of quantum configurations obeying statistics other than those of bosons and fermions. These include anyons and nonabelian anyons in two-space, distinguishable particles satisfying colored braid group statistics in two-space, and paraparticles when the spatial dimension is 2, 3, or more. The key unifying idea is the nontrivial homotopy of the respective configuration spaces, and how this allows particular classes of unitarily inequivalent diffeomorphism group representations modeled on those spaces.

Likewise, the quantum mechanics of configurations in physical spaces which themselves have nontrivial homotopy can be understood well from this point of view. A well-known example is the Aharonov–Bohm effect. Different self-adjoint extensions of densely-defined operators (describing, for example, kinetic angular momentum) have different spectra, and arise from different sets of boundary conditions satisfied by wave functions in their domains. These operators occur as the infinitesimal generators of the unitarily inequivalent group representations associated with the nontrivial homotopy.

This approach extends naturally to the study of infinite but locally finite particle configurations, as well as extended quantum configurations (embedded submanifolds or fractals in the physical space) and their internal symmetry – e.g., closed and open strings, vortex filaments and ribbons, or knotted configurations.

9. The meaning of the wave function and the notion of indistinguishability

We have seen that in a positional representation, the interpretation of ψ is quite different if we consider it to be defined on the space of unordered configurations (i.e., subsets of the physical space), rather than the space of ordered configurations. This point of view actually extends to the description of "distinguishable" particles via marked configurations.

Let us elaborate on this briefly. Consider a two-particle system, where the particles have distinct masses m and μ. Conventionally, one would interpret $\psi(x_1, x_2)$, as a probability amplitude for finding the first particle (the one with mass m) at x_1, and the second particle (having mass μ) at x_2. But ψ makes no explicit reference to these masses. Alternatively, consider (m, x) as an element of a real bundle B over the physical space M, with fiber $\mathbf{R}^+$. A generalized configuration is $\gamma = \{(m, x), (\mu, y)\}$, where $m, \mu \in \mathbf{R}^+$ and $x, y \in M$; and $\psi = \psi(\gamma)$. Now γ can be understood as describing "indistinguishable" particles with distinct spatial coordinates (when $x \neq y$) and distinct mass coordinates (when $m \neq \mu$).

Another way of saying this is that in the perspective taken here, particles can be "distinguished" by their coordinates. References to "the particle measured to have mass m" are analogous to "the particle measured to be in position x." The philosophical meaning of "indistinguishable," as well as the interpretation of the coordinates that appear as the argument of the wave function, thus change according to which view one chooses to take.

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Gerald A. Goldin
Departments of Mathematics and Physics
Rutgers University
New Brunswick, New Jersey, USA
e-mail: geraldgoldin@dimacs.rutgers.edu

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Hyperbolic Flows and the Question of Quantum Chaos

Geoffrey L. Sewell

To the Memory of Gérard Emch

Abstract. Hyperbolic flows, as formulated by Anosov, are the prototypes of chaotic evolutions in classical dynamical systems. Here we provide a concise updated account of their quantum counterparts originally formulated by Emch, Narnhofer, Thirring and Sewell within the operator algebraic setting of quantum theory; and we discuss their bearing on the question of quantum chaos.

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1. Introduction

Classical hyperbolic flows, as formulated by Anosov [1], are flows over smooth compact connected Riemannian manifolds that admit stable expanding and contracting foliations. Thus they are prototype examples of chaotic dynamical systems, in that orbits stemming from neighbouring points of their phase spaces diverge, generically, exponentially fast from one another.

In view of the fundamental character of both quantum ergodic theory [2–4] and quantum chaology [5, 6], it is natural to ask whether a formulation of a quantum counterpart of these flows is feasible. This question was addressed by Emch et al [7] in a treatment that overcame the obstacle imposed by the fact that quantum mechanics does not accommodate the differential geometric structures on which the classical treatment was based [1, 8]. In fact, their treatment was carried out within the framework of operator algebraic quantum theory [9, 10], wherein the observables of a model were represented by the self-adjoint elements of a $W^\star$-algebra and the non-commutative differential structure was carried by derivations of that algebra.

The present article is devoted to a concise updated account of the picture of quantum hyperbolic flows presented in Ref. [7]. Its essential content comprises a general formulation of these flows and their chaotic properties, together with concrete examples both of models for which chaos survives quantization and models for which it does not.

We start, in Section 2, with a brief account of the classical picture of hyperbolic flows. Here the generic model comprises a one-parameter group of diffeomorphisms of a manifold that satisfies a certain hyperbolicity condition. Prototype examples of these flows, which we provide, are the Arnold cat model and the geodesic flow over a compact Riemannian manifold of constant negative curvature.

In Section 3 we recast the classical model into the operator algebraic form given by the Gel'fand isomorphism. This enables us to express the hyperbolicity condition in terms of automorphisms of the resultant commutative algebra of observables.

In Section 4 we provide a simple passage from the classical commutative algebraic picture to the quantum non-commutative one, thereby formulatng the hyperbolicity condition for the quantum model in term of automorphisms of its algebra of observables. In particular we show that this condition implies the chaoticity of the quantum model in that the evolutes of neighbouring states, as represented by density matrices, diverge exponentially fast from one another.

In Section 5 we provide an explicit treatment of the quantum version of the Arnold cat model and prove that its hyperbolicity, and thus its chaotic property, survives the quantization.

Correspondingly, in Section 6 we provide an explicit treatment of the quantum version of the geodesic flow over a compact Riemannian manifold of negative curvature and show that, by contrast with the Arnold cat model, it violates the hyperbolicity condition. In other words, quantization of its original classical version destroys its hyperbolicity.

In Section 7 we generalize this result to arbitrary finite quantum Hamiltonian systems by showing that they cannot support hyperbolic flows.

We conclude in Section 8 with a brief discussion of the results presented here and their consequences for quantum chaology.

The Appendix is devoted to the proof of a key proposition involved in the formulation of the classical hyperbolicity condition of Section 2.

2. The classical picture

The classical model, Σ_{cl}, is given by a triple (M, μ, ϕ) [8], where M is a smooth, connected, compact Riemannian manifold, ϕ is a representation of $\mathbf{R}$ or $\mathbf{Z}$ in the diffeomorphisms of M, the representation being continuous in the former case; and μ is a ϕ-invariant probability measure on M. Thus ϕ and μ represent the dynamics and a stationary state, respectively, of the model. Specifically, for $m \in M$ and $t \in \mathbf{R}$ or $\mathbf{Z}$, $\phi_t m$ is the evolute of m at time t; and for measurable regions A of

M, $\mu(\phi_t A) = \mu(A)$. We denote the tangent space at the point m of M by $T(m)$ and note that, for fixed time t, the differential $d\phi_t$ of ϕ_t maps $T(m)$ into $T(\phi_t m)$.

In order to formulate the condition for the hyperbolicity of the dynamics of Σ_{cl} we first assume that M is equipped with vector fields $V_1, \ldots, V_n$, where $n = \dim(M)$ or $\big(\dim(M)-1\big)$ according to whether the time variable t is discrete or continuous[1]. It is assumed that at each point m of M these fields are linearly independent and that each V_j has a global integral curve $C_j(m) = \{m_j(s)|s\in\mathbf{R};\ m_j(0) = m\}$, given by the unique solution of the equation

$$m_j'(s) = V_j\big(m_j(s)\big);\ m_j(0) = m. \tag{1}$$

Thus, the curves $\{C_j(m)|m\in M\}$ are generated by the action on M of a one-parameter group $\{\theta_j(s)|s\in\mathbf{R}\}$ of diffeomorphisms, defined by the formula

$$\theta_j(s)m = m_j(s),\ \forall m\in M,\ s\in\mathbf{R}. \tag{2}$$

The orbits of the θ_j's are termed *horocycles.*We note here that the correspondence between the group θ_j and the vector field V_j is one-to-one since Eqs. (1) and (2) may be employed to define V_j in terms of θ_j by the formula

$$V_j(m) = \theta'(0)m\ \forall\ m\in M. \tag{3}$$

To establish consistency, we remark that this equation, together with the group property of θ_j, implies that

$$\begin{aligned} V_j\big(\theta_j(s)m\big) &= \theta_j'(0)\big(\theta_j(s)m\big) \\ &= \frac{\partial}{\partial t}\theta_j(t)\theta_j(s)m_{|t=0} = \frac{\partial}{\partial t}\theta_j(t+s)m_{|t=0} = \theta_j'(s)m, \end{aligned} \tag{4}$$

as demanded by Eqs. (1) and (2).

Definition. We term the dynamics of the model Σ_{cl} *hyperbolic* if the action of the differential of ϕ_t on the vector fields V_j takes the form

$$d\phi_t V_j(m) = V_j(\phi_t m)\mathrm{e}^{\lambda_j t}, \tag{5}$$

where the λ's are real numbers such that, for some positive integer r less than n, λ_j is positive for $j\in[1,r]$ and negative for $j\in[r+1,n]$.

Thus, if m' and m are neighbouring points of M whose difference, as represented on a chart at m, is $\sum_{j=1}^n a_j V_j(m)$, the hyperbolicity condition signifies that

$$\phi_t m - \phi_t m' \simeq \sum_1^n a_j V_j(\phi_t m)\mathrm{e}^{\lambda_j t}. \tag{6}$$

Hence, defining $T_+(m)$ (resp. $T_-(m)$) to be the subspace of $T(m)$ spanned by the vectors V_j for which λ_j is positive (resp. negative), the hyperbolicity condition is that the action of ϕ_t on neighbouring points of M serves to expand their separation exponentially fast if their relative displacement on a chart at m lies in $T_+(m)$ and contracts it if that displacement lies in $T_-(m)$. Thus the λ's are Lyapunov

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[1] The difference between n and $\dim(M)$ in the continuous case corresponds to the one dimensionality of the space generated by the velocity vector

exponents and, as some of them are positive, the hyperbolicity condition signifies that the flow is chaotic. The following Proposition will be proved in the Appendix.

Proposition 1. *The hyperbolicity condition given by Eq.* (5) *is equivalent to the following one.*

$$\phi_t \theta_j(s) \phi_{-t} = \theta_j\big(s\mathrm{e}^{\lambda_j t}\big). \tag{7}$$

Example 1 (The Arnold Cat).[2] This is the model (M, ϕ, θ, μ), where

(i) M is the torus $[0,1)\ (\mathrm{mod}1)]^2$ with Euclidean metric;

(ii) the time variable t is discrete, its range being $\mathbf{Z}$, and the dynamical transformations are $\{\phi^n\ (:= \phi_n) | n \in \mathbf{Z}\}$, where

$$\phi = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}; \tag{8}$$

(iii) μ is the Lebesgue measure on the torus M; and

(iv) denoting the eigenvectors of ϕ by V_1 and V_2 and their respective eigenvalues by $k_1\ (> 1)$ and $k_2\ (< 1)$, θ is the pair of one-parameter groups θ_1 and θ_2 defined in terms of V_1 and V_2 by Eqs. (1) and (2). Thus

$$\theta_j(s)m = m + V_j s\ \big(\mathrm{mod}\ (1,1)\big)\ \forall\ m \in M,\ \ s \in \mathbf{R},\ \ j = 1, 2. \tag{9}$$

It now follows from these definitions that the model satisfies the hyperbolicity condition (7), with $\lambda_j = \ln(k_j)$.

Example 2 (Geodesic Flow on a Manifold of Negative Curvature [8]). This is a model of the free dynamics of a particle on a compact region of the Poincaré half-plane $\tilde{M} := \{(x, y) | x \in \mathbf{R}, y \in \mathbf{R}_+\}$, whose metric is given by the formula

$$ds^2 = y^{-2}(dx^2 + dy^2). \tag{10}$$

The points (x, y) of $\tilde{M}$ will sometimes be represented by the complex numbers $z := (x + iy)$.

The manifold $\tilde{M}$ is equipped with the symmetry group $G = SL(2, \mathbf{R})$ [11], which acts transitively on it. The elements g of this group are represented by two-by-two matrices with real-valued entries and unit determinant. Its actions on $\tilde{M}$ are given by the following formula. Denoting $g\ (\in G)$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$gz = \frac{(az + b)}{(cz + d)}. \tag{11}$$

We denote by K the subgroup of G whose elements leave the point i invariant. It then follows from the transitivity of G that G/K may be identified with the space $\tilde{M}$. Correspondingly, for a discrete co-compact non-abelian subgroup Γ, $\Gamma \backslash G/K$ is a compact manifold, $\hat{M}$, of constant negative curvature. Its unit tangent bundle, $T_1\hat{M} := M$ may then be identified with $\Gamma \backslash G$. We take this to be the phase space of the model.

[2]This model of automorphisms of the torus is often so termed because of Arnold's illustration [8] of their actions on a cat's face placed in the torus.

The dynamical group ϕ for the free geodesic motion of a particle on M is given by the formula [7, 11]

$$\phi_t m = m\xi(t), \tag{12}$$

where

$$\xi(t) = \begin{pmatrix} \exp(-t/2) & 0 \\ 0 & \exp(t/2) \end{pmatrix}. \tag{13}$$

We note that the measure $d\mu := y^{-2}dxdy$ is ϕ-invariant. Further, the horocyclic actions are given by the formulae

$$\theta_j(s)m = m\xi_j(s) \ \forall \ s\in\mathbf{R}, \ j = 1,2, \tag{14}$$

where

$$\xi_1(s) = \begin{pmatrix} 1 & s \\ 0 & s \end{pmatrix} \tag{15}$$

and

$$\xi_2(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}. \tag{16}$$

It follows directly from these formulae that the model satisfies the hyperbolicity condition (7).

3. The classical operator algebraic picture

As a first step towards a passage from the above classical picture to a corresponding quantum mechanical one, we now exploit the Gel'fand isomorphism, according to which the model (M,ϕ,μ) is equivalent to the $W^\star$ dynamic system $(\mathcal{A}_{cl},\alpha_{cl},\rho_{cl})$, where $\mathcal{A}_{cl}$ is the abelian $W^\star$ algebra of observables $L_\infty(M,d\mu)$, $\{\alpha_{cl}(t)|t\in\mathbf{R}\}$ is the one-parameter group of automorphisms of $\mathcal{A}_{cl}$ representing the dynamics of the model and given by the formula

$$[\alpha_{cl}(t)A](m) = A(\phi_{-t}m) \ \forall \ A\in\mathcal{A}_{cl}, \ m\in M, \ t\in\mathbf{R}, \tag{17}$$

and ρ_{cl} is the state on $\mathcal{A}_{cl}$ corresponding to the measure μ, i.e.,

$$\rho_{cl}(A) = \int A d\mu. \tag{18}$$

It follows immediately from these specifications that the ϕ-invariance of μ is equivalent to the α_{cl}-invariance of ρ_{cl}.

Furthermore the diffeomorphism groups θ_j correspond to representations $\sigma_{j,cl}$ of $\mathbf{R}$ in $\mathrm{Aut}(\mathcal{A}_{cl})$, given by the formula

$$[\sigma_{j,cl}(s)A](m) = A\big(\theta_j(-s)m\big) \ \forall \ A\in\mathcal{A}_{cl}, \ m\in M, \ s\in\mathbf{R}. \tag{19}$$

The hyperbolicity condition (7) is therefore equivalent to the following one.

$$\alpha_{cl}(t)\sigma_{j,cl}(s)\alpha_{cl}(-t) = \sigma_{j,cl}\big(se^{\lambda_j t}\big) \ \forall \ s\in\mathbf{R}, \ t\in\mathbf{R} \text{ or } \mathbf{Z}, \ j = 1,\dots,n. \tag{20}$$

4. The quantum picture

We assume that the generic quantum model corresponds to the algebraic picture of the classical one, but with the difference that the algebra of observables is non-commutative. Thus the quantum model is a triple $(\mathcal{A}, \alpha, \rho)$, where $\mathcal{A}$ is a $W^\star$-algebra, in general non-commutative, ρ is a normal state on $\mathcal{A}$ and $\{\alpha_t | t \in \mathbf{R} \text{ or } \mathbf{Z}\}$ is a one-parameter group of automorphisms of $\mathcal{A}$, which is continuous w.r.t. t in the former case, and ρ is a normal α-invariant state on $\mathcal{A}$. Furthermore, we assume that the model is equipped with n horocyclic actions, given by one-parameter groups $\{\sigma_j(s) | s \in \mathbf{R},\ j = 1, \ldots, n\}$ of $\mathcal{A}$ whose infinitesimal generators are linearly independent both of one another and of that of the group α in the case where the variable t runs through $\mathbf{R}$. Accordingly, we take the hyperbolicity condition to be the natural generalization of Eq. (20) for the possibly non-commutative case, i.e.,

$$\alpha_t \sigma_j(s) \alpha_{-t} = \sigma_j\left(s e^{\lambda_j t}\right)\ \forall\ s \in \mathbf{R},\ t \in \mathbf{R} \text{ or } \mathbf{Z},\ j = 1, \ldots, n, \tag{21}$$

where again λ is positive for $j = 1, \ldots, r$ and negative for $j = r + 1, \ldots, n$. This condition implies the following one for the duals, $\alpha_t^\star$ and $\sigma_j^\star(s)$, of α_t and $\sigma_j(s)$, in their actions on the normal states, $\mathcal{N}(\mathcal{A})$, on $\mathcal{A}$.

$$\alpha_{-t}^\star \sigma_j^\star(s) \alpha_t^\star = \sigma_j^\star\left(s e^{\lambda_j t}\right). \tag{22}$$

We denote by $\delta_j^\star$ the infinitesimal generator of the group $\sigma_j^\star$, in the $w^\star$ topology. It follows from this formula that its domain, $\mathcal{D}(\delta^\star)$, is stable under the group $\alpha^\star$ and that, if ρ_1 and ρ_2 are states in this domain, then

$$\|\delta^\star \alpha_t^\star(\rho_1 - \rho_2)\| = \|\delta^\star(\rho_1 - \rho_2)\| e^{\lambda_j t}. \tag{23}$$

Thus, in the quantum context, λ_j is a Lyapunov function that provides a measure of the speed at which the evolutes of ρ_1 and ρ_2 separate along the horocycle σ_j. Since some of the λ's are positive, this represents a chaoticity condition.

We shall show, in the following sections, that quantization does not affect the hyperbolic property of the Arnold cat model, but that it destroys that of the geodesic flow over the manifold of negative curvature; and that, in general, it does not admit chaos in finite Hamiltonian systems.

5. The quantum Arnold cat

In order to quantize the classical Arnold cat model, we start by expressing that model in a form readily amenable to quantization. Thus we first note that it follows from the definition of the classical algebra $\mathcal{A}_{cl}$ in Section 3 that this algebra is generated by the sinusoidal functions $\{W_{cl}(\nu) | \nu = (\nu_1, \nu_2) \in \mathbf{Z}^2\}$, defined by the formula

$$W_{cl}(\nu)[m] = \exp(2\pi i \nu . m)\ \forall\ \nu = (\nu_1, \nu_2) \in \mathbf{Z}^2, \tag{24}$$

where the dot denotes the Euclidean scalar product. Correspondingly, since μ is the Euclidean measure on the torus M, it follows from Eqs. (18) and (24) that

$$\rho_{cl}\big(W_{cl}(\nu)\big) = \delta_{\nu,0}, \tag{25}$$

where δ is the Kronecker delta. Moreover since, by Eq. (8), ϕ is Hermitean, it follows from Eqs. (17) and (24) that

$$\alpha_{cl}(t)W_{cl}(\nu) = W_{cl}\big(\phi_{-t}\nu\big)\ \forall\ t\in\mathbf{Z},\ \nu\in\mathbf{Z}^2; \tag{26}$$

while, by Eqs. (9), (19) and (24), the horocyclic actions for the model are given by the formula

$$\sigma_j(s)W_{cl}(\nu) = W_{cl}(\nu)\exp\big(2\pi i\nu.V_j s\big)\ \forall\ s\in\mathbf{R},\ \nu\in\mathbf{Z}^2. \tag{27}$$

Thus Eqs. (24)–(27) define the classical model. One may readily check that they satisfy the hyperbolicity condition (20), bearing in mind that V_j is the eigenvector of ϕ whose eigenvalue is $\exp(\lambda_j)$.

We now quantize the classical model by basing the algebra of observables on Weyl operators instead of the sinusoidal function W_{cl}. Thus, in order to construct $\mathcal{A}$, we start with an abstract algebra of elements $\{W(\nu)|\nu\in\mathbf{Z}^2\}$ which satisfy the Weyl condition that

$$W(\nu)W(\nu') = W(\nu+\nu')\exp\big(i\gamma\kappa(\nu,\nu')\big), \tag{28}$$

where κ is the symplectic form defined by the formula

$$\kappa(\nu,\nu') = \nu_1\nu_2' - \nu_2\nu_1' \tag{29}$$

and γ is a constant that plays the role of that of Planck. Thus the algebra $\mathcal{A}_0$ of the polynomials in the $W(\nu)$'s comprises just the linear combinations of them. We define ρ to be the positive normalized linear form on this algebra given by the precise analogue of the classical state ρ_{cl}, as given by Eq. (25), i.e.,

$$\rho\big(W(\nu)\big) = \delta_{\nu,0}. \tag{30}$$

We define the algebra of observables, $\mathcal{A}$, to be the strong closure of the GNS representation of $\mathcal{A}_0$ in the state ρ defined by this last equation. We then define the dynamical and horocyclic automorphisms, α and σ_j, by the canonical counterparts of the classical ones of Eqs. (26) and (27). Thus

$$\alpha(t)W(\nu) = W\big(\phi_{-t}\nu\big)\ \forall\ t\in\mathbf{Z},\ \nu\in\mathbf{Z}^2; \tag{31}$$

and

$$\sigma_j(s)W(\nu) = W(\nu)\exp\big(2\pi i\nu.V_j s\big)\ \forall\ s\in\mathbf{R},\ \nu\in\mathbf{Z}^2. \tag{32}$$

It follows from the last two formulae that the model satisfies the hyperbolicity condition (21). Thus we have established the following proposition.

Proposition 2. *The chaoticity of the flow of the Arnold cat model survives quantization.*

6. Quantum geodesic flow on a compact manifold of negative curvature

The model we now consider is the quantized version of that of Example 2 in Section 2, and it may be described as follows [7, 11]. Its $W^\star$-algebra of observables is $\mathcal{B}(\mathcal{H})$, the set of bounded operators in the Hilbert space $\mathcal{H} := L_2(\hat{M}, d\mu)$, where the measure $d\mu$ is defined following Eq. (13). The state space of the model comprises the normal states of $\mathcal{A}$ and its Hamiltonian is $-\Delta$, where Δ is the Laplace–Beltrami operator for the manifold $\hat{M}$. The dynamical automorphisms of the model are thus given by the formula

$$\alpha_t A = \exp(-i\Delta t) A \exp(i\Delta t) \ \forall \ A \in \mathcal{A}, \ t \in \mathbf{R}. \tag{33}$$

Moreover, the spectrum of Δ is discrete [12]. We denote by $\{f_k | k \in \mathbf{N}\}$ a complete orthonormal set of eigenvectors of this operator and by $\{e_k\}$ the corresponding set of its eigenvalues. We then define the operators F_{kl}, with $k, l \in \mathbf{N}$, by the equation

$$F_{kl} f_i = \delta_{li} f_k \ \forall k, l.i \in \mathbf{N}. \tag{34}$$

It follows now from Eqs. (33) and (34) that

$$\alpha_t F_{kl} = \exp(i\omega_{kl} t) F_{kl} \ \forall \ t \in \mathbf{R}, \ k, l \in \mathbf{N}, \tag{35}$$

where

$$\omega_{kl} = e_l - e_k. \tag{36}$$

We denote by $\mathcal{L}(F)$ the set of finite linear combinations of the F_{kl}'s. It follows from this definition that $\mathcal{L}(F)$ is closed with respect to involution and binary addition and multiplication. It is therefore a *-algebra, and it follows from our specifications that its strong closure is $\mathcal{A}$.

Proposition 3. *Under the above assumptions, the quantum geodesic flow on the manifold cannot be hyperbolic.*

We base the proof of this proposition on Lemmas 4 and 5 below.

Lemma 4. *Assume that the model satisfies the hyperbolicity condition with respect to horocyclic automorphisms $\sigma(\mathbf{R})$. Then it follows from the discreteness of the spectrum of Δ that any normal stationary state ρ of the model is σ-invariant.*

Assuming the result of this lemma, we denote the GNS triple for the state ρ by $(\mathcal{H}_\rho, \pi_\rho, \Phi_\rho)$ and define U_ρ and V_ρ to be the continuous unitary representations of $\mathbf{R}$ in $\mathcal{H}_\rho$ that implement the automorphisms α_t and $\sigma(s)$, respectively, according to the standard prescription

$$U_\rho(t)\pi_\rho(A)\Phi_\rho = \pi_\rho(\alpha_t A)\Phi_\rho \tag{37}$$

and

$$V_\rho(s)\pi_\rho(A)\Phi_\rho = \pi_\rho\big(\sigma(s)A\big)\Phi_\rho. \tag{38}$$

Hence, by the cyclicity of Φ_ρ and the hyperbolicity condition (21), as applied to the horocycle σ, that

$$U_\rho(t)V_\rho(s)U_\rho(-t) = V_\rho(se^{\lambda t}) \ \forall \ s,t\in\mathbf{R} \tag{39}$$

We define H_ρ to be the Hamiltonian operator in the GNS space, $\mathcal{H}_\rho$, according to the formula $U_\rho(t) = \exp(iH_\rho t)$.

Lemma 5. *Under the assumptions of Lemma* 4 *and with the subsequent definitions, the formula* (39) *implies that the spectrum of* H_ρ *is* $\mathbf{R}$.

Proof of Proposition 3 *assuming Lemmas* 4 *and* 5*.* Our strategy here is to infer from Lemma 4 that the assumption of hyperbolicity implies that the spectrum of H_ρ is discrete. Since this conflicts with Lemma 5, we conclude that that assumption is invalid.

We start by noting that, by Eqs. (35) and (37),

$$U_\rho(t)\pi_\rho(F_{kl})\Phi_\rho = \pi_\rho(F_{kl})\Phi_\rho\exp(i\omega_{kl}t). \tag{40}$$

Since ρ is a normal stationary state of the model, it follows from the definition of the vectors f_k that ρ corresponds to a density matrix of the form $\sum_{r\in\mathbf{N}} w_r P_r$, where the w_r's are non-negative numbers whose sum is unity and P_r $(= F_{rr})$ is the projection operator for the vector f_r. Hence

$$\langle\pi_\rho(A)\Phi_\rho, \pi_\rho(B)\Phi_\rho\rangle = \langle\rho;(A^\star B\rangle = \sum_{r\in\mathbf{N}} w_r(f_r, A^\star Bf_r) \ \forall \ A,B\in\mathcal{A}. \tag{41}$$

It follows from this formula and Eq. (34) that

$$\langle\pi_\rho(F_{kl})\Phi_\rho, \pi_\rho(F_{k'l'}\Phi_\rho\rangle = w_l\delta_{kk'}\delta_{ll'}. \tag{42}$$

Therefore, defining $D := \{(k,l)\in\mathbf{N}^2; w_l{\neq}0\}$ and

$$\Psi_{kl} = w_l^{-1/2}\pi_\rho(F_{kl})\Phi_\rho \ \forall \ (k,l)\in D, \tag{43}$$

the set of vectors $\{\Psi_{kl}|(k,l)\in D\}$ is orthonormal. It is also complete for the following reasons. By the definition (34) of the operators F_{kl}, the algebra $\mathcal{A}$ consists of linear combinations of these operators. Therefore, by the normality of the representation π_ρ, the algebra $\pi_\rho(\mathcal{A})$ consists of linear combinations of the operators $\pi_\rho(F_{kl})$. Hence by Eq. (43) and the cyclicity of Π_ρ with respect to that algebra, the set $\{\Psi_{kl}|(k,l)\in D\}$ of orthonormal vectors in $\mathcal{H}_\rho$ is complete.

Now, by Eqs. (40) and (43),

$$U_\rho(t)\Psi_{kl} = \Psi_{kl}\exp(i\omega_{kl}t) \ \forall \ (k,l)\in D.$$

and consequently, since $\{\Psi_{kl}|(k,l)\in D\}$ is an orthonormal basis in $\mathcal{H}_\rho$,

$$U_\rho(t) = \sum_{(k,l)\in D}\mathcal{P}_{kl}\exp(i\omega_{kl}t),$$

where $\mathcal{P}_{kl}$ is the projector for Ψ_{kl}. Hence

$$H_\rho = \sum_{(k,l)\in D}\omega_{kl}\mathcal{P}_{kl}, \tag{44}$$

and therefore the spectrum of H_ρ comprises the discrete set $\omega_{kl}|(k,l)\in D$. As this conflicts with Lemma 5, which was based on the assumption of a hyperbolic flow, we conclude that the model does not support such a flow. □

Proof of Lemma 4. By the hyperbolicity condition (21), as applied to the horocycle σ,

$$\langle \rho; \alpha_t \sigma(s) \alpha_{-t} F_{kl} \rangle = \langle \rho; \sigma\big(se^{\lambda t}\big) F_{kl} \rangle. \tag{45}$$

By Eq. (35) and the stationarity of ρ, the l.h.s. of this equation is equal to $\langle \rho : \sigma(s) F_{kl} \rangle \exp(i\omega_{lk} t)$. On the other hand, in the limit where $\lambda t \to -\infty$, it follows by continuity that the r.h.s. of Eq. (40) reduces to $\langle \rho; F_{kl} \rangle$. Compatibility of these expressions for the two sides of Eq. (40) implies that $\langle \rho; \sigma(s) F_{kl} \rangle$ and $\langle \rho; F_{kl} \rangle$ are equal to one another if $\omega_{kl} = 0$ and are both zero if $\omega_{kl} \neq 0$. Hence they are equal in all cases. In view of the normality of ρ and the strong density of $\mathcal{L}(F)$, this result implies that ρ is σ-invariant. □

Proof of Lemma 5. This is achieved in Ref. [7] on the basis of a version of Mackey's imprimitivity theorem. □

7. Generic non-hyperbolic flow of finite quantum Hamiltonian systems

The generic model of a finite quantum Hamiltonian system is not quite the same as the model presented in Section 4. Specifically it consists of a triple $(\mathcal{A}, \alpha, \mathcal{N})$ [10, 13], where $\mathcal{A}$ is the $W^\star$-algebra of bounded operators in a separable Hilbert space $\mathcal{H}$, $\mathcal{N}$ is the set of normal states on $\mathcal{A}$ corresponding to the density matrices in $\mathcal{H}$, and α is a representation of $\mathbf{R}$ in the automorphisms of $\mathcal{A}$ implemented by a unitary group whose infinitesimal generator is i times a self-adjoint operator H. Thus

$$\alpha_t A = \exp(iHt/\hbar) A \exp(-iHt/\hbar) \ \forall \ A \in \mathcal{A}, \ t \in \mathbf{R}. \tag{46}$$

Here H is the Hamiltonian of the model. In general, it is the sum of the kinetic and potential energies of its constituent particles and its spectrum is discrete. Note that these specifications do not include the assumption of a hyperbolicity assumption such as given by Eq. (21). In fact, the following proposition establishes the contrary of that assumption for this model.

Proposition 6. *Finite quantum Hamiltonian systems, as defined above, cannot support hyperbolic flows.*

Proof. This follows immediately from the discreteness of the spectrum of H by the same argument that led from Lemmas 4 and 5 to Prop. 3. □

8. Conclusions

The general picture of quantum hyperbolic flows, presented in Section 4, is the natural analogue of its algebraically cast classical counterpart and exhibits the chaotic property represented by Eq. (23). Moreover, this picture is realized by the quantum Arnold cat model. On the other hand, finite quantum Hamiltonian systems, including the geodesic flow over a compact manifold of constant negative curvature, do not support hyperbolic flows. This accords with a vast body of work on models for which chaos in classical systems is suppressed by quantization [5, 6]. Since, in those works, the classical chaos leaves its mark on the resultant quantum system in the form of certain scars on its eigenstates, we expect that this is also the case for the quantum Hamiltonian models treated here.

Appendix: Proof of Proposition 1

In order to derive Eq. (7) from Eq. (5), we start by defining

$$\tilde{m}_{j,t}(s) = \phi_t\theta_j\big(\exp(-\lambda_j t)s\big)\phi_{-t}m \ \forall s,t\in\mathbf{R},\ m\in M,\ j = 1,\dots,n \tag{A.1}$$

and inferring from this formula that, for fixed t and j,

$$\tilde{m}'_{j,t}(s) = \exp(-\lambda_j t)d\phi_t\theta'_{j,t}\big(\exp(-\lambda_j t)s\big)\phi_{-t}m. \tag{A.2}$$

Hence, by Eq. (1),

$$\tilde{m}'_{j,t}(s) = \exp(-\lambda_j t)d\phi_t V\big(\theta_{j,t}\big(\exp(-\lambda_j t)s\big)\phi_{-t}m.$$

and therefore, by Eqs. (5) and (A.1),

$$\tilde{m}'_{j,t}(s) = V\big(\tilde{m}_{j,t}(s)\big), \tag{A.3}$$

which signifies that $\tilde{m}_{j,t}(s)$ is the unique solution of Eq. (4), i.e., that $\tilde{m}_{j,t}(s) = \theta_j(s)m$. In view of Eq. (A.1), this implies that

$$\phi_t\theta_j\big(\exp(-\lambda_j t)s\big)\phi_{-t} = \theta_j(s),\ \forall\ s,t\in\mathbf{R},\ m\in M,\ j = 1,\dots,n,$$

which is equivalent to Eq. (7).

Conversely, in order to derive Eq. (5) from Eq. (7), we note that, in view of the formula (A.1), the latter equation signifies that $\tilde{m}_{j,t}(s) = m_j(s)$. Hence, by Eq. (1),

$$\tilde{m}'_{j,t}(s) = V\big(\tilde{m}_{j,t}(s)\big). \tag{A.4}$$

Furthermore, by Eq. (A.1), the l.h.s. of this formula is equal to

$$\frac{\partial}{\partial s}\phi_t\theta\big(\exp(-\lambda_j t)s\big)\phi_{-t}m = \exp(-\lambda_j t)d\phi_t\theta'_j\big(\exp(-\lambda_j t)s\big)\phi_{-t}m,$$

which, by Eq. (1), is equal to

$$\exp(-\lambda_j t)d\phi_t V\big(\theta_j\big(\exp(-\lambda_j t)s\big)\phi_{-t}m.$$

Hence, by Eq. (A.1), Eq. (A.4) reduces to the form

$$d\phi_t V\big(\theta_j\big(\exp(-\lambda_j t)s\big)\phi_{-t}m\big) = \exp(\lambda_j t)V\big(\tilde{m}_{j,t}(s)\big),$$

i.e., by Eq. (A.1),

$$d\phi_t V\big(\phi_{-t}\tilde{m}_{j,t}(s)\big) = \exp(\lambda_j t) V\big(\tilde{m}_{j,t}(s)\big).$$

Thus, putting

$$\hat{m} = \phi_{-t}\tilde{m}_{j,t}(s). \tag{A.5}$$

$$d\phi_t V(\hat{m}) = V(\phi_t \hat{m}).$$

Since, by Eqs. (A.1) and (A.5), the correspondence between m and $\hat{m}$ is one-to-one, this last equation is equivalent to Eq. (5). □

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Geoffrey L. Sewell
Department of Physics
Queen Mary University of London
Mile End Road, London E1 4NS, UK
e-mail: g.l.sewell@qmul.ac.uk

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A New Proof of the Helton–Howe–Carey–Pincus Trace Formula

Arup Chattopadhyay and Kalyan B. Sinha

Abstract. In this article, we give an alternative proof of the Helton–Howe–Carey–Pincus trace formula using Krein's trace formula.

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1. Introduction

Notation. In the following, we shall use the notations given below: $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$, $\mathcal{B}_{sa}(\mathcal{H})$, $\mathcal{B}_1(\mathcal{H})$, $\mathcal{B}_{1+}(\mathcal{H})$, $\mathcal{B}_{1-}(\mathcal{H})$, $\mathcal{B}_2(\mathcal{H})$, $\mathcal{B}_p(\mathcal{H})$ denote a separable Hilbert space, set of bounded linear operators, set of bounded self-adjoint linear operators, set of trace class operators, set of positive trace class operators, set of negative trace class operators, set of Hilbert–Schmidt operators and Schatten-p class operators respectively with $\|.\|_p$ as the associated Schatten-p norm. Furthermore by $\sigma(A)$, $E_A(\lambda)$, $D(A)$, $\rho(A)$, we shall mean spectrum, spectral family, domain, resolvent set, and resolvent of a self-adjoint operator A respectively, and $\mathrm{Tr}(A)$ will denote the trace of a trace class operator A in $\mathcal{H}$. Also we denote the set of natural numbers and the set of real numbers by $\mathbb{N}$ and $\mathbb{R}$ respectively. The set $C(I)$ is the Banach space of continuous functions over a compact interval $I \subseteq \mathbb{R}$ with sup-norm $\|.\|_\infty$, and $C^n(I)$ $(n \in \mathbb{N} \cup \{0\})$, the space of nth continuously differentiable functions over a compact interval I with norm

$$\|f\|^n = \sum_{j=0}^{n} \|f^{(j)}\|_\infty \text{ for } f \in C^n(I)$$

and $f^{(j)}$ is the jth derivative of f (for $n = 0$, $C^n(I)$ is $C(I)$), and $L^p(\mathbb{R})$ the standard Lebesgue space. We shall denote $f^{(1)}$, the first derivative, as f'. Next we define the class $C_1^1(I) \subseteq C(I)$ as follows

$$C_1^1(I) = \left\{ f \in C(I) : \|f\|_1^1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{f}(\alpha)|(1+|\alpha|)d\alpha < \infty \right\},$$

where $\hat{f}$ is the Fourier transform of f; and it is easy to see that $C_1^1(I) \subseteq C^1(I)$ (since $\|.\|^1 \leq \|.\|_1^1$); and denote the set of all $f \in C_1^1(I)$ such that $f' \geq 0$ (≤ 0 respectively) by $C_{1+}^1(I)$ ($C_{1-}^1(I)$ respectively). Similarly we denote the set of all polynomials with complex coefficients in I by $\mathcal{P}(I)$ and the set of all $p \in \mathcal{P}(I)$ such that $p' \geq 0$ (≤ 0 respectively) by $\mathcal{P}_+(I)$ ($\mathcal{P}_-(I)$ respectively).

Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator, that is, $[T^*, T] \geq 0$. Set $T = X + \mathrm{i}Y$, where $X, Y \in \mathcal{B}_{sa}(\mathcal{H})$ and it is known that $\mathrm{Re}(\sigma(T)) = \sigma(X)$, $\mathrm{Im}(\sigma(T)) = \sigma(Y)$ [14], $[T^*, T] \in \mathcal{B}_1(\mathcal{H})$ if an additional assumption of finiteness of spectral multiplicity is assumed [2, 10, 13, 14]. A hyponormal operator T is said to be purely hyponormal if there exists no subspace $\mathcal{S}$ of $\mathcal{H}$ which is invariant under T such that the restriction of T to $\mathcal{S}$ is normal. For a purely hyponormal operator T, it is also known that its real and imaginary parts, that is, X and Y are spectrally absolutely continuous [10, 16].

(A) The main assumption of the whole paper is that $\boldsymbol{T = X + \mathrm{i}Y}$ is a purely hyponormal operator in $\boldsymbol{\mathcal{B}(\mathcal{H})}$ such that $\boldsymbol{[T^*, T] = -2\mathrm{i}[Y, X] = 2D^2 \in \mathcal{B}_{1+}(\mathcal{H})}$ and $\boldsymbol{\sigma(X) \cup \sigma(Y) \subseteq [a, b] \subseteq \mathbb{R}}$.

The details of the proofs of some of the stated results have been omitted for reasons of brevity and these can be found in an article by the authors [7].

2. Main section

Let us start with few lemmas which will be useful to prove our main result.

Lemma 1. *Let T satisfy* (**A**). *Then for $\psi \in C_1^1([a, b])$ or $\mathcal{P}([a, b])$,*

$$[\psi(Y), X] \in \mathcal{B}_1(\mathcal{H}) \quad \text{and} \quad -\mathrm{i}\mathrm{Tr}\{[\psi(Y), X]\} = \mathrm{Tr}\{\psi'(Y)D^2\}. \tag{1}$$

Similarly,

$$[Y, \psi(X)] \in \mathcal{B}_1(\mathcal{H}) \quad \text{and} \quad -\mathrm{i}\mathrm{Tr}\{[Y, \psi(X)]\} = \mathrm{Tr}\{\psi'(X)D^2\}. \tag{2}$$

Proof. Now for $\psi \in C_1^1([a, b])$ we have

$$\begin{aligned}[\psi(Y), X] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(\alpha)[\mathrm{e}^{\mathrm{i}\alpha Y}, X]d\alpha = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(\alpha)\left(\mathrm{e}^{\mathrm{i}\alpha Y}X - X\mathrm{e}^{\mathrm{i}\alpha Y}\right)d\alpha \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(\alpha)d\alpha \int_0^\alpha \mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha-\beta)Y}[Y, X]\mathrm{e}^{\mathrm{i}\beta Y}d\beta \\ &= -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(\alpha)d\alpha \int_0^\alpha \mathrm{e}^{\mathrm{i}(\alpha-\beta)Y}D^2\mathrm{e}^{\mathrm{i}\beta Y}d\beta.\end{aligned} \tag{3}$$

Since $D^2 \in \mathcal{B}_1(\mathcal{H})$ and $\int_{\mathbb{R}} |\hat{\psi}(\alpha)||\alpha| d\alpha < \infty$, then from the above equation (3) we conclude that

$$[\psi(Y), X] \in \mathcal{B}_1(\mathcal{H}) \text{ and } \|[\psi(Y), X]\|_1 \leq \|D^2\|_1 \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{\psi}(\alpha)||\alpha| d\alpha.$$

Moreover,

$$\begin{aligned} -\mathrm{i}\,\mathrm{Tr}\{[\psi(Y), X]\} &= \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(\alpha) d\alpha \int_0^\alpha \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}\alpha Y} D^2\} d\beta \\ &= \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{\mathbb{R}} \alpha \hat{\psi}(\alpha) d\alpha \, \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}\alpha Y} D^2\} \\ &= \mathrm{Tr}\{\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{i}\alpha \hat{\psi}(\alpha) \mathrm{e}^{\mathrm{i}\alpha Y} d\alpha \; D^2\} = \mathrm{Tr}\{\psi'(Y) D^2\}, \end{aligned}$$

where we have used the cyclicity of trace and the fact that

$$\psi'(\beta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{i}\alpha \hat{\psi}(\alpha) \mathrm{e}^{\mathrm{i}\alpha\beta} d\alpha.$$

By interchanging the role of X and Y in the above calculations, we conclude that

$$[Y, \psi(X)] \in \mathcal{B}_1(\mathcal{H}) \quad \text{and} \quad -\mathrm{i}\mathrm{Tr}\{[Y, \psi(X)]\} = \mathrm{Tr}\{\psi'(X) D^2\}.$$

By an identical calculation as above, we conclude that (1) and (2) are also true for $\psi \in \mathcal{P}([a,b])$. This completes the proof. □

Lemma 2. *Let T satisfy* (**A**)*. Then $-\mathrm{i}[\psi(Y), X] \in \mathcal{B}_{1\pm}(\mathcal{H})$ according as $\psi \in C^1_{1\pm}([a,b])$ or $\mathcal{P}_\pm([a,b])$ respectively. Similarly, $-\mathrm{i}[Y, \psi(X)] \in \mathcal{B}_{1\pm}(\mathcal{H})$ according as $\psi \in C^1_{1\pm}([a,b])$ or $\mathcal{P}_\pm([a,b])$ respectively.*

Proof. Let $\psi \in C^1_1([a,b])$. Then from equation (3) in Lemma 1, we have

$$-\mathrm{i}[\psi(Y), X] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{i}\hat{\psi}(\alpha) d\alpha \int_0^\alpha \mathrm{e}^{\mathrm{i}(\alpha-\beta)Y} D^2 \mathrm{e}^{\mathrm{i}\beta Y} d\beta.$$

Next by the spectral theorem for Y we get

$$-\mathrm{i}[\psi(Y), X] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{i}\hat{\psi}(\alpha) d\alpha \int_0^\alpha d\beta \int_a^b \int_a^b \mathrm{e}^{\mathrm{i}(\alpha-\beta)t} \mathrm{e}^{\mathrm{i}\beta t'} E^{(Y)}(dt) D^2 E^{(Y)}(dt'),$$

where $E^{(Y)}(.)$ is the spectral family of the self-adjoint operator Y. Note that $\mathcal{E}(\Delta \times \delta)(S) \equiv E^{(Y)}(\Delta) S E^{(Y)}(\delta)$ ($S \in \mathcal{B}_2(\mathcal{H})$ and $\Delta \times \delta \subseteq \mathbb{R} \times \mathbb{R}$) extends to a spectral measure (finite) on $\mathbb{R}^2$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$. Therefore by Fubini's theorem

$$\begin{aligned} -\mathrm{i}[\psi(Y), X] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{i}\hat{\psi}(\alpha) d\alpha \int_a^b \int_a^b \frac{\mathrm{e}^{\mathrm{i}\alpha t'} - \mathrm{e}^{\mathrm{i}\alpha t}}{\mathrm{i}(t'-t)} E^{(Y)}(dt) D^2 E^{(Y)}(dt')\} \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b \int_a^b \frac{\psi(t') - \psi(t)}{t'-t} E^{(Y)}(dt) D^2 E^{(Y)}(dt')\} \qquad (4) \\ &= \frac{1}{\sqrt{2\pi}} \int_{[a,b]^2} \tilde{\psi}(t,t') \mathcal{E}(dt \times dt')(D^2), \end{aligned}$$

where

$$\tilde{\psi}(t,t') = \begin{cases} \frac{\psi(t')-\psi(t)}{t'-t}, & \text{if } t \neq t', \\ \psi'(t), & \text{if } t = t'. \end{cases}$$

Note that for $\psi \in C^1_{1\pm}([a,b])$, we have $\tilde{\psi}(t,t') \geq 0$ (or ≤ 0) respectively for $t,t' \in [a,b]$ and hence from the equation (4) we conclude that $-\mathrm{i}[\psi(Y),X] \geq 0$ or $-\mathrm{i}[\psi(Y),X] \leq 0$ accordingly. Similarly by the same above calculations with X and Y interchanged we conclude that $-\mathrm{i}[Y,\psi(X)] \geq 0$ or ≤ 0 if $\psi \in C^1_{1+}([a,b])$ or $C^1_{1-}([a,b])$ respectively. The same conclusions follow similarly for $\psi \in \mathcal{P}_\pm([a,b])$. This completes the proof. □

Lemma 3. *Let T satisfy* (**A**). *Then $-\mathrm{i}[\psi(Y),\phi(X)] \in \mathcal{B}_{1\pm}(\mathcal{H})$ according as ψ and $\phi \in C^1_{1\pm}([a,b])$ or $\mathcal{P}_\pm([a,b])$ respectively.*

Proof. The proof the lemma is identical to that of Lemma 2 with $-\mathrm{i}[Y,\phi(X)]$ and $-\mathrm{i}[\psi(Y),X]$ replacing D^2 accordingly for ψ and $\phi \in C^1_{1\pm}([a,b])$ or $\mathcal{P}_\pm([a,b])$. □

As the title suggests, we shall next state Krein's theorem and study its consequences on commutators like $[\psi(Y),\phi(X)]$ for $\psi,\phi \in C^1_1([a,b])$ or $\mathcal{P}([a,b])$.

Proposition 4 (Krein's Theorem [11, 12, 17, 18]). *Let H and H_0 be two bounded self-adjoint operators in $\mathcal{H}$ such that $V = H - H_0 \in \mathcal{B}_1(\mathcal{H})$. Then there exists a unique $\xi_{H_0,H}(.) \in L^1(\mathbb{R})$ such that for $\phi \in C^1_1([a,b])$ or $\mathcal{P}([a,b])$, $\phi(H) - \phi(H_0) \in \mathcal{B}_1(\mathcal{H})$ and*

$$\mathrm{Tr}\{\phi(H) - \phi(H_0)\} = \int_a^b \phi'(\lambda)\xi_{H_0,H}(\lambda)d\lambda,$$

where $\sigma(H) \cup \sigma(H_0) \subseteq [a,b]$. Furthermore

$$\int_a^b |\xi_{H_0,H}(\lambda)|d\lambda \leq \|V\|_1; \int_a^b \xi_{H_0,H}(\lambda)d\lambda = \mathrm{Tr}\, V,$$

and if $V \in \mathcal{B}_{1+}(\mathcal{H})$ or $\mathcal{B}_{1-}(\mathcal{H})$, then $\xi_{H_0,H}(\lambda)$ is positive or negative respectively for almost all $\lambda \in [a,b]$.

Theorem 5. *Assume* (**A**). *Let ϕ and ψ be two complex-valued functions such that ϕ, $\psi \in C^1_1([a,b])$ or $\mathcal{P}([a,b])$. Then $[\psi(Y),\phi(X)]$ is a trace class operator and there exist unique $L^1(\mathbb{R})$-functions $\xi(t;\psi)$ and $\eta(\phi;\lambda)$ such that*

$$-\mathrm{i}\,\mathrm{Tr}\{[\psi(Y),\phi(X)]\} = \int_a^b \phi'(t)\xi(t;\psi)dt = \int_a^b \psi'(\lambda)\eta(\phi;\lambda)d\lambda. \tag{5}$$

Furthermore, if $\phi,\psi \in C^1_{1+}([a,b])$ or $\mathcal{P}_+([a,b])$, then $\xi(t;\psi)$, $\eta(\phi;\lambda) \geq 0$ for almost all $t,\lambda \in [a,b]$,

$$\int_a^b |\xi(t;\psi)|dt \leq \|{-\mathrm{i}}[\psi(Y),X]\|_1 \ , \ \int_a^b \xi(t;\psi)dt = \mathrm{Tr}\left(\psi'(Y)D^2\right)$$

and

$$\int_a^b |\eta(\phi;\lambda)|d\lambda \le \|-\mathrm{i}[Y,\phi(X)]\|_1 \ , \ \int_a^b \eta(\phi;\lambda)d\lambda = \mathrm{Tr}\left(\phi'(X)D^2\right).$$

Proof. At first assume that ϕ, ψ are real valued. Now let us consider the self-adjoint operators $H_0 = X$ and $H = \mathrm{e}^{\mathrm{i}\psi(Y)}X\mathrm{e}^{-\mathrm{i}\psi(Y)}$. Then

$$\begin{aligned} H - H_0 &= \int_0^1 \frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}s\psi(Y)}X\mathrm{e}^{-\mathrm{i}s\psi(Y)}\right)ds \\ &= \mathrm{i}\int_0^1 \mathrm{e}^{\mathrm{i}s\psi(Y)}[\psi(Y),X]\mathrm{e}^{-\mathrm{i}s\psi(Y)}ds \in \mathcal{B}_1(\mathcal{H}), \end{aligned} \tag{6}$$

by Lemma 1. On the other hand for $\psi, \phi \in C_1^1([a,b])$, a computation similar to that in (3) yields that

$$[\psi(Y),\phi(X)] = \int_{\mathbb{R}} \mathrm{i}\hat{\psi}(\alpha)d\alpha \int_0^\alpha \mathrm{e}^{\mathrm{i}(\alpha-\beta)Y}[Y,\phi(X)]\mathrm{e}^{\mathrm{i}\beta Y}d\beta \in \mathcal{B}_1(\mathcal{H}), \tag{7}$$

since $\int_{\mathbb{R}} |\hat{\psi}(\alpha)||\alpha|d\alpha < \infty$ and since $[Y,\phi(X)] \in \mathcal{B}_1(\mathcal{H})$, by Lemma 1. Similarly for ϕ, $\psi(t) = \sum\limits_{j=0}^{n} c_j t^j \in \mathcal{P}([a,b])$, we conclude that

$$[\psi(Y),\phi(X)] = \sum_{j=0}^{n} c_j[Y^j,\phi(X)] = \sum_{j=0}^{n} c_j \sum_{k=0}^{j-1} Y^{j-k-1}[Y,\phi(X)]Y^k \in \mathcal{B}_1(\mathcal{H}), \tag{8}$$

since $[Y,\phi(X)] \in \mathcal{B}_1(\mathcal{H})$, by Lemma 1. Thus by applying Proposition 4 for the above operators H, H_0 with the function ϕ, we conclude that there exists a unique function $\tilde{\xi}(t;\psi) \in L^1(\mathbb{R})$ such that $\phi(H) - \phi(H_0)$ is trace class and

$$\mathrm{Tr}\{\phi(H) - \phi(H_0)\} = \int_a^b \phi'(t)\tilde{\xi}(t;\psi)dt. \tag{9}$$

Furthermore from equation (6) we conclude that $H - H_0 \le 0$, since $\mathrm{i}[\psi(Y),X] \le 0$ by Lemma 2 for $\psi \in C_{1+}^1([a,b])$ or $\mathcal{P}_+([a,b])$. Therefore from Proposition 4 we also note that $\tilde{\xi}(t;\psi) \le 0$ for almost all $t \in [a,b]$. Now if we compute the left-hand side of (9), we get

$$\begin{aligned} &\mathrm{Tr}\{\phi(H) - \phi(H_0)\} = \mathrm{Tr}\{\phi(\mathrm{e}^{\mathrm{i}\psi(Y)}X\mathrm{e}^{-\mathrm{i}\psi(Y)}) - \phi(X)\} \\ &\quad = \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}\psi(Y)}\phi(X)\mathrm{e}^{-\mathrm{i}\psi(Y)}) - \phi(X)\} = \mathrm{i}\,\mathrm{Tr}\left\{\int_0^1 \mathrm{e}^{\mathrm{i}s\psi(Y)}[\psi(Y),\phi(X)]\mathrm{e}^{-\mathrm{i}s\psi(Y)}ds\right\} \\ &\quad = \mathrm{i}\,\mathrm{Tr}\{[\psi(Y),\phi(X)]\}, \end{aligned} \tag{10}$$

where for the second equality we have used functional calculus, for the third equality we have used equation (6) and for the last equality we have used the cyclicity of trace. Thus by combining (9) and (10) we have

$$-\mathrm{i}\,\mathrm{Tr}\{[\psi(Y),\phi(X)]\} = -\int_a^b \phi'(t)\tilde{\xi}(t;\psi)dt = \int_a^b \phi'(t)\xi(t;\psi)dt, \tag{11}$$

where $\xi(t;\psi) \equiv -\tilde{\xi}(t;\psi) \geq 0$. Next if we consider the operators $H_0 = Y$ and $H = e^{i\phi(X)} Y e^{-i\phi(X)}$. Then by repeating the above similar calculations we conclude that

$$-i\,\mathrm{Tr}\{[\psi(Y), \phi(X)]\} = \int_a^b \psi'(\lambda)\eta(\phi, \lambda) d\lambda, \tag{12}$$

where $\eta(\phi; \lambda) \geq 0$ for almost all $\lambda \in [a, b]$. Therefore the conclusion of the theorem follows from (11) and (12) for real-valued ϕ, ψ. The same above conclusions can be achieved for complex-valued functions $\phi, \psi \in C_1^1([a, b])$ or $\mathcal{P}([a, b])$ by decomposing

$$\phi = \phi_1 + i\phi_2 \quad \text{and} \quad \psi = \psi_1 + i\psi_2,$$

and by applying the conclusion of the theorem for real-valued functions ϕ_1, ϕ_2, ψ_1, ψ_2. By equation (6) of Proposition 4 and Lemma 1, it follows that

$$\int_a^b |\xi(t;\psi)| dt \leq \|H_0 - H\|_1 \leq \|-i[\psi(Y), X]\|_1$$

and

$$\int_a^b \xi(t;\psi) dt = \mathrm{Tr}(H_0 - H) = \mathrm{Tr}\left(-i[\psi(Y), X]\right) = \mathrm{Tr}\left(\psi'(Y) D^2\right).$$

The other results for η follows similarly. □

Remark 6. It is clear from equation (5) that both $\xi(t;\cdot)$ and $\eta(\cdot;\lambda)$ depend linearly on ψ' and ϕ' respectively and not on ψ and ϕ themselves as the left-hand side in (5) appears to. Therefore, to avoid confusion it is preferable to replace ψ', ϕ' by ψ and ϕ respectively, demand that ψ, $\phi \in \mathcal{P}([a, b])$, and consequently replace ψ, ϕ by their indefinite integrals $\mathcal{J}(\psi)$ and $\mathcal{J}(\phi)$ respectively. Thus the equation (5) now reads: For $\psi, \phi \in \mathcal{P}([a, b])$

$$\mathrm{Tr}\{-i\,[\mathcal{J}(\psi)(Y), \mathcal{J}(\phi)(X)]\} = \int_a^b \phi(t)\xi(t;\psi) dt = \int_a^b \psi(\lambda)\eta(\phi;\lambda) d\lambda, \tag{13}$$

where we have retained the earlier notation $\xi(t;\psi)$ and $\eta(\phi;\lambda)$. Furthermore, for almost all $t, \lambda \in [a, b]$, the maps

$$\mathcal{P}([a, b]) \ni \psi \longmapsto \xi(t;\psi) \in L^1(\mathbb{R}) \quad \text{and} \quad \mathcal{P}([a, b]) \ni \phi \longmapsto \eta(\phi;\lambda) \in L^1(\mathbb{R})$$

are positive linear maps. The next theorem gives L^1-estimates for $\xi(\cdot;\psi)$ and $\eta(\phi;\cdot)$ which allows one to extend these maps for all $\psi, \phi \in C([a, b])$.

Theorem 7. *Assume* (**A**).

(i) *Then*

$$\mathcal{P}([a, b]) \times \mathcal{P}([a, b]) \ni (\psi, \phi) \longmapsto \mathrm{Tr}\{-i\,[\mathcal{J}(\psi)(Y), \mathcal{J}(\phi)(X)]\}$$

can be extended as a positive linear map on $C([a, b]) \times C([a, b])$. *Furthermore if* Δ, $\Omega \in \mathrm{Borel}([a, b])$, *then*

$$\mathrm{Tr}\{-i\,[\mathcal{J}(\chi_\Delta)(Y), \mathcal{J}(\chi_\Omega)(X)]\} = \int_\Omega \xi(t;\Delta) dt = \int_\Delta \eta(\Omega;\lambda) d\lambda, \tag{14}$$

where we have written $\xi(t;\Delta)$ *for* $\xi(t;\chi_\Delta)$ *and* $\eta(\Omega;\lambda)$ *for* $\eta(\chi_\Omega;\lambda)$*. For almost all fixed* $t,\lambda\in[a,b]$*,* $\xi(t;\cdot)$ *and* $\eta(\cdot;\lambda)$ *are countably additive positive measures such that*

$$\int_a^b \xi(t;\Delta)dt = \mathrm{Tr}\left(\chi_\Delta(Y)D^2\right), \int_a^b \eta(\Omega;\lambda)d\lambda = \mathrm{Tr}\left(\chi_\Omega(X)D^2\right),$$

and

$$\int_a^b \xi(t;[a,b])dt = \int_a^b \eta([a,b];\lambda)d\lambda = \mathrm{Tr}(D^2).$$

(ii) *The set functions*

$$\mathrm{Borel}([a,b]) \ni \Delta \longmapsto \xi(t:\Delta) \quad \text{and} \quad \mathrm{Borel}([a,b]) \ni \Omega \longmapsto \eta(\Omega;\lambda)$$

are absolutely continuous with respect to the Lebesgue measures and the Radon–Nikodym derivatives satisfy:

$$\frac{\xi(t;d\lambda)}{d\lambda} = \frac{\eta(dt;\lambda)}{dt} \equiv r(t,\lambda) \geq 0$$

for almost all t,λ*, with* $\|r\|_{L^1([a,b]^2)} = \mathrm{Tr}(D^2)$*.*

(iii) *The statement of Theorem* 5 *now takes the form: For* $\psi,\phi\in C^1([a,b])$

$$\mathrm{Tr}\{-\mathrm{i}[\psi(Y),\phi(X)]\} = \int_{[a,b]^2} \phi'(t)\psi'(\lambda)r(t,\lambda)dtd\lambda, \tag{15}$$

with the unique non-negative $L^1([a,b]^2)$ *function* r*, which is sometimes called Carey–Pincus principal function.*

Proof. Let $\psi,\phi\in\mathcal{P}([a,b])$, then $\mathcal{J}(\psi)$ and $\mathcal{J}(\phi)$ are also polynomials. As in (8), a similar computation with $\psi,\phi\in\mathcal{P}([a,b])$ and if $\mathcal{J}(\phi)(t) = \sum_{j=0}^{n} c_j t^j$ leads to

$$\mathrm{Tr}\{-\mathrm{i}\,[\mathcal{J}(\psi)(Y),\mathcal{J}(\phi)(X)]\} = \mathrm{Tr}\{\phi(X)\,(-\mathrm{i}\,[\mathcal{J}(\psi)(Y),X])\} \tag{16}$$

and interchanging the role of X and Y (along with an associated negative sign) the above is equal to

$$\mathrm{Tr}\{-\mathrm{i}\,[\mathcal{J}(\psi)(Y),\mathcal{J}(\phi)(X)]\} = \mathrm{Tr}\{\psi(Y)\,(-\mathrm{i}\,[Y,\mathcal{J}(\phi)(X)])\}, \tag{17}$$

and all these expressions are also equal to (by Theorem 5)

$$\int_a^b \phi(t)\xi(t;\psi)dt = \int_a^b \psi(\lambda)\eta(\phi;\lambda)d\lambda$$

for respective ϕ and ψ. Now let $\phi=\phi_+-\phi_-$ and $\psi=\psi_+-\psi_-$, then $\phi_\pm$, $\psi_\pm$ are all non-negative. The domains of definitions of $\phi_\pm$ are open sets which are each a disjoint union of a countable collection of open intervals and furthermore, clearly $\mathrm{Supp}\,\phi_+\cap\mathrm{Supp}\,\phi_- = \{t\in[a,b] | \phi(t)=0\}$, which is a finite discrete set. Therefore ϕ_+ and ϕ_- and hence $|\phi| = \phi_+ + \phi_-$ are polynomials if $\phi\in\mathcal{P}([a,b])$. By Lemma 3,

$-\mathrm{i}[\psi(Y),\phi(X)] \in \mathcal{B}_{1\pm}(\mathcal{H})$ according as $\psi,\phi \in \mathcal{P}_{\pm}([a,b])$ respectively. Therefore by linearity, we have

$$\|-\mathrm{i}\,[\mathcal{J}(\psi)(Y),\mathcal{J}(\phi)(X)]\|_1 \le \mathrm{Tr}\{|\phi|(X)\,(-\mathrm{i}\,[\mathcal{J}(|\psi|)(Y),X])\} \tag{18}$$

and similarly

$$\|-\mathrm{i}\,[\mathcal{J}(\psi)(Y),\mathcal{J}(\phi)(X)]\|_1 \le \mathrm{Tr}\{|\psi|(Y)\,(-\mathrm{i}\,[Y,\mathcal{J}(|\phi|)(X)])\}. \tag{19}$$

Next by using the above estimates (18) and (19) we make two steps of approximations of characteristic functions of Borel sets $\Omega \in \mathrm{Borel}(\sigma(X))$, $\Delta \in \mathrm{Borel}(\sigma(Y))$ by continuous functions and polynomials to conclude part (i).

(ii) From the equality, for $\Omega \in \mathrm{Borel}(\sigma(X))$, $\Delta \in \mathrm{Borel}(\sigma(Y))$, $\int\limits_{\Omega} \xi(t;\Delta)dt = \int\limits_{\Delta} \eta(\Omega;\lambda)d\lambda$ with ξ and η both non-negative, it follows that both $\xi(t;\cdot)$ and $\eta(\cdot;\lambda)$ are absolutely continuous with respect to the respective Lebesgue measures, and we set

$$r(t,\lambda) = \frac{\xi(t;d\lambda)}{d\lambda} = \frac{\eta(dt;\lambda)}{dt} \ge 0.$$

The uniqueness of r follows from the equation (15) and the fact that $r \in L^1([a,b]^2)$ and that it has compact support. □

Next, we want to compute the trace of

$$\mathrm{Tr}\{[\alpha(X)\psi(Y),\phi(X)]\} = \mathrm{Tr}\{\alpha(X)[\psi(Y),\phi(X)]\}$$

and by symmetry between X and Y,

$$\mathrm{Tr}\{[\alpha(X)\psi(Y),\beta(Y)]\} = -\,\mathrm{Tr}\{\psi(Y)[\beta(Y),\alpha(X)]\},$$

where $\phi,\psi,\alpha,\beta \in \mathcal{P}([a,b])$, which constitutes the next theorem.

Theorem 8. *Let T satisfy* (**A**). *Let $\phi,\psi,\alpha,\beta \in \mathcal{P}([a,b])$. Then*

$$[\alpha(X)\psi(Y),\phi(X)] \in \mathcal{B}_1(\mathcal{H})$$

and

$$-\,\mathrm{i}\,\mathrm{Tr}\{[\alpha(X)\psi(Y),\phi(X)]\} = \int_{[a,b]^2} -J(\alpha\psi,\phi)(t,\lambda)r(t,\lambda)dtd\lambda, \tag{20}$$

where r is the function obtained in Theorem 7 and

$$J(\alpha\psi,\phi)(t,\lambda) = \frac{\partial}{\partial t}(\alpha(t)\psi(\lambda))\frac{\partial}{\partial\lambda}(\phi(t)) - \frac{\partial}{\partial\lambda}(\alpha(t)\psi(\lambda))\frac{\partial}{\partial t}(\phi(t))$$

is the Jacobian of $\alpha\psi$ and ϕ in $[a,b]\times[a,b] \equiv [a,b]^2$. Similarly,

$$[\alpha(X)\psi(Y),\beta(Y)] \in \mathcal{B}_1(\mathcal{H})$$

and

$$-\,\mathrm{i}\,\mathrm{Tr}\{[\alpha(X)\psi(Y),\beta(Y)]\} = \int_{[a,b]^2} -J(\alpha\psi,\beta)(t,\lambda)r(t,\lambda)dtd\lambda, \tag{21}$$

where r is the function obtained in Theorem 7 and

$$J(\alpha\psi,\phi)(t,\lambda) = \frac{\partial}{\partial t}(\alpha(t)\psi(\lambda))\frac{\partial}{\partial\lambda}(\beta(\lambda)) - \frac{\partial}{\partial\lambda}(\alpha(t)\psi(\lambda))\frac{\partial}{\partial t}(\beta(\lambda))$$

is the Jacobian of $\alpha\psi$ and β in $[a,b]^2$.

Proof. Using (7) we say that $[\alpha(X)\psi(Y),\phi(X)] = \alpha(X)[\psi(Y),\phi(X)] \in \mathcal{B}_1(\mathcal{H})$. Next from (8) we conclude for $\psi,\phi \in \mathcal{P}([a,b])$ that

$$\begin{aligned} -\mathrm{i}\,\mathrm{Tr}\{[\psi(Y),\phi(X)]\} &= -\mathrm{i}\,\mathrm{Tr}\{\phi'(X)[\psi(Y),X]\} \\ &= \int_a^b \phi'(t)\,\mathrm{Tr}\left(E^{(X)}(dt)\{-\mathrm{i}[\psi(Y),X]\}\right), \end{aligned} \tag{22}$$

where we have used spectral theorem for the self-adjoint operator X and $E^{(X)}(.)$ is the spectral family of X. On the other hand from Theorem 7(iii) we conclude that

$$-\mathrm{i}\,\mathrm{Tr}\{[\psi(Y),\phi(X)]\} = \int_{[a,b]^2} \phi'(t)\psi'(\lambda)r(t,\lambda)dtd\lambda, \tag{23}$$

for $\psi,\phi \in \mathcal{P}([a,b])$. Therefore combining (22) and (23) we get

$$\int_a^b \phi'(t)\,\mathrm{Tr}\left(E^{(X)}(dt)\{-\mathrm{i}[\psi(Y),X]\}\right) = \int_a^b \phi'(t)\left(\int_a^b \psi'(\lambda)r(t,\lambda)d\lambda\right)dt, \tag{24}$$

for $\psi,\phi \in \mathcal{P}([a,b])$. Since $\Delta \longrightarrow \mathrm{Tr}\left(E^{(X)}(\Delta)\{-\mathrm{i}[\psi(Y),X]\}\right)$ ($\Delta \subseteq \mathbb{R}$, a Borel subset of $\mathbb{R}$) is a complex measure with finite total variation and $r \in L^1[a,b]^2$ and since the equality (24) is true for every $\phi \in \mathcal{P}([a,])$, it follows that the measure $\Delta \longrightarrow \mathrm{Tr}\left(E^{(X)}(\Delta)\{-\mathrm{i}[\psi(Y),X]\}\right)$ is absolutely continuous with respect to the Lebesgue measure and

$$\mathrm{Tr}\left(E^{(X)}(dt)\{-\mathrm{i}[\psi(Y),X]\}\right) = \left(\int_a^b \psi'(\lambda)r(t,\lambda)d\lambda\right)dt. \tag{25}$$

As in (16), a similar computation with $\psi,\phi \in \mathcal{P}([a,b])$ and if $\phi(\lambda) = \sum_{j=0}^{n} b_j\lambda^j$ leads to

$$-\mathrm{i}\,\mathrm{Tr}\{[\alpha(X)\psi(Y),\phi(X)]\} = \int_a^b \alpha(t)\phi'(t)\,\mathrm{Tr}\left(E^{(X)}(dt)\{-\mathrm{i}[\psi(Y),X]\}\right), \tag{26}$$

where we have used the cyclicity of trace and the spectral theorem for the self-adjoint operator X and $E^{(X)}(.)$ is the spectral family of X. Thus by combining (25) and (26) we conclude that

$$-\mathrm{i}\,\mathrm{Tr}\{[\alpha(X)\psi(Y),\phi(X)]\} = \int_{[a,b]^2} -J(\alpha\psi,\phi)(t,\lambda)r(t,\lambda)dtd\lambda.$$

Next by interchanging the role of X and Y in the above calculations, we can establish equation (21). This completes the proof. □

Remark 9. If T satisfy $(\mathbf{A})$, then the conclusion of the above Theorem 8 also can be achieved for $\phi, \psi, \alpha, \beta \in C_1^1([a,b])$.

The next theorem replaces effectively the so-called "Wallach's Collapse Theorem" [13].

Theorem 10. *Let T be as in the statement of Theorem 8. Let $\phi, \psi, \alpha, \beta \in \mathcal{P}([a,b])$. Then the following is true*

$$-\mathrm{i}\,\mathrm{Tr}\{[\alpha(X)\psi(Y), \phi(X)\beta(Y)]\} = \int_{[a,b]^2} -J(\alpha\psi, \phi\beta)(t,\lambda) r(t,\lambda) dt d\lambda,$$

where

$$J(\alpha\psi, \phi\beta)(t,\lambda) = \frac{\partial}{\partial t}(\alpha(t)\psi(\lambda))\frac{\partial}{\partial \lambda}(\phi(t)\beta(\lambda)) - \frac{\partial}{\partial \lambda}(\alpha(t)\psi(\lambda))\frac{\partial}{\partial t}(\phi(t)\beta(\lambda))$$

is the Jacobian of $\alpha\psi$ and $\phi\beta$ in $[a,b]^2$.

Proof. By using the cyclicity of trace and the fact that $[\phi(X), \beta(Y)] \in \mathcal{B}_1(\mathcal{H})$, we conclude that

$$\begin{aligned} &-\mathrm{i}\,\mathrm{Tr}\{[\alpha(X)\psi(Y), \phi(X)\beta(Y)]\} \\ &\quad = -\mathrm{i}\,\mathrm{Tr}\{[\alpha(X)(\psi\beta)(Y), \phi(X)]\} - \mathrm{i}\,\mathrm{Tr}\{[(\alpha\phi)(X)\psi(Y), \beta(Y)]\}, \end{aligned} \tag{27}$$

and the right-hand side of the above equality is equal to

$$\int_{[a,b]^2} -J(\alpha\psi, \phi\beta)(t,\lambda) r(t,\lambda) dt d\lambda,$$

by using Theorem 8. □

Now we are in a position to state our main result, the Helton–Howe–Carey–Pincus trace formula [1, 8, 9, 13].

Theorem 11. *Let $\Psi(t,\lambda) = \sum_{j=1}^{n} c_j \alpha_j(t)\psi_j(\lambda)$ and $\Phi(t,\lambda) = \sum_{k=1}^{m} d_j \phi_k(t)\beta_k(\lambda)$, $(m, n \in \mathbb{N})$ and α_j, ψ_j, ϕ_j, β_j are all in $\mathcal{P}([a,b])$. Then $-\mathrm{i}\,[\Psi(X,Y), \Phi(X,Y)] \in \mathcal{B}_1(\mathcal{H})$ and*

$$\mathrm{Tr}\{-\mathrm{i}\,[\Psi(X,Y), \Phi(X,Y)]\} = \int_{[a,b]^2} J(\Psi, \Phi)(t,\lambda) r(t,\lambda) dt d\lambda.$$

Proof. The proof follows easily by applying Theorem 10 and the fact that

$$\mathrm{Tr}\{-\mathrm{i}\,[\Psi(X,Y), \Phi(X,Y)]\} = \sum_{j=1}^{n}\sum_{k=1}^{m} c_j d_k\, \mathrm{Tr}\{-\mathrm{i}\,[\alpha_j(X)\psi_j(Y), \phi_k(X)\beta_k(Y)]\}. \quad \square$$

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Arup Chattopadhyay
A. Chattopadhyay: Indian Institute of Technology Guwahati
Department of Mathematics
Guwahati- 560059
Kamrup, Assam, India
e-mail: 2003arupchattopadhyay@gmail.com, arupchatt@iitg.ernet.in

Kalyan B. Sinha
K.B. Sinha: J.N. Centre for Advanced Scientific Research
and Indian Institute of Science
Bangalore, India
e-mail: kbs@jncasr.ac.in

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Quasi-classical Calculation of Eigenvalues: Examples and Questions

Tomoyo Kanazawa and Akira Yoshioka

To the memory of Gérard G. Emch

Abstract. We discuss the Maslov quantization condition, especially a method of quasi-classical calculation of energy levels of Schrödinger operators. The method gives an approximation of eigenvalues of operators in general. We give several concrete examples of Schrödinger operators to which the quasi-classical calculation gives the correct eigenvalues and pose some open problems.

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Introduction

Maslov introduces the so-called Maslov index and the quantization condition for Lagrangian submanifolds and studies the "asymptotic solutions" of the eigenvalue problems in quantum mechanics [7]. The Malsov quantization condition can be regarded as a generalization of the Bohr quantization rule. By means of the quantization condition we can obtain good approximate eigenvalues of Schrödinger operators (see, for example, [4, 8–10]).

On the contrary, there exist several concrete quantum mechanical systems where we obtain exact eigenvalues and multiplicities by means of the Maslov quantization condition (see, for example, [1]). Thus, as far as these systems are concerned, we need not to consider the operator theory to obtain the exact quantum mechanical energy levels and their multiplicities. What we need is only classical mechanics, invariant Lagrangian submanifolds and Maslov's quantization condition.

Our question is:

Why there is such a coincidence?

As far as we know, we have no mathematical proof of the coincidence at present.

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In this note, we explain a concept of quasi-classical calculation of eigenvalues of Schrödinger operators. We also show examples for which the Maslov quantization condition gives exact eigenvalues.

We should also mention the paper of Leray [6], inspired by Malsov's theory, where he constructed a theory of a Lagrangian analysis and treated such a kind of concrete examples.

Maslov quantization condition

Let θ be the canonical 1-form of the cotangent bundle T^*M of a smooth manifold M and $\pi : T^*M \to M$ be the canonical projection. We consider the symplectic manifold $(T^*M, d\theta)$. Consider a Lagrangian submanifold L of $(T^*M, d\theta)$. The Maslov quantization condition is then written as

$$\frac{1}{2\pi\hbar}\int_c \theta - \frac{1}{4}\langle m_L, [c]\rangle \in \mathbb{Z},$$

where $\hbar$ (Planck constant) is a positive parameter, $[c] \in H_1(L, \mathbb{Z})$ and m_L is the Maslov class of L.

Example 1: Harmonic oscillator

We explain here how the Maslov quantization condition determines discrete energy levels of a Hamiltonian. We consider the case where $M = \mathbb{R}$ and then the cotangent bundle is $T^*M = T^*\mathbb{R} = \mathbb{R}^2$. We write points as $(x, p) \in \mathbb{R}^2$. Then the cotangent bundle has the canonical symplectic form $d\theta$, where $\theta = pdx$. We consider a Hamiltonian function $H = \frac{1}{2}(p^2 + x^2)$ of the harmonic oscillator.

Now we consider a level set of the function H for every constant $E > 0$, such that

$$L(E) = \left\{(x, p) \in \mathbb{R}^2 \mid H(x, p) = E\right\}.$$

The level set $L(E)$ is a Lagrangian submanifold of $(\mathbb{R}^2, d\theta)$. We consider the Maslov quantization condition for the Lagrangian submanifold $L(E)$. The equation of motion is

$$\dot{x} = p,\ \dot{p} = -x$$

and an orbit in $L(E)$ is

$$c_E\colon\ x(t) = x_0 \cos t + p_0 \sin t, \quad p(t) = p_0 \cos t - x_0 \sin t,$$

where (x_0, p_0) is a point in $L(E)$ and then $E = H(x_0, p_0) = \frac{1}{2}(p_0^2 + x_0^2)$. Hence the action integral along c_E is

$$\int_{c_E} \theta = \int_0^{2\pi} p(t)\dot{x}(t)dt = \frac{1}{2}(p_0^2 + x_0^2)\ 2\pi = 2\pi E.$$

As to the Maslov index, we prepare the following lemma.

We consider the symplectic manifold $(T^*\mathbb{R}^n, d\theta)$. Let

$$H_1(x, p),\ H_2(x, p), \ldots, H_n(x, p)$$

be smooth functions on a domain D in $T^*\mathbb{R}^n$. Suppose they are in involution, or Poisson commuting each other. We denote their level set by

$$L(c_1, c_2, \ldots, c_n) = \{(x,p) \in D \mid H_1(x,p) = c_1, H_2(x,p) = c_2, \ldots, H_n(x,p) = c_n\}.$$

We put $H = (H_1, H_2, \ldots, H_n)$ and define $n \times n$ matrices by

$$H_x = \left(\frac{\partial H_j}{\partial x_k}\right), \ H_p = \left(\frac{\partial H_j}{\partial p_k}\right), \ j, k = 1, 2, \ldots, n.$$

Then we have (see [11])

Lemma 1. *The Maslov form on $L(c_1, c_2, \ldots, c_n)$ is given explicitly as*

$$m_L = \frac{1}{\pi}\, d\left(\arg\det(H_p + iH_x)\right).$$

For the harmonic oscillator $H(x,p) = 1/2\ (x^2 + p^2)$, we have

$$\det(H_p + iH_x) = p + ix.$$

Hence, on the curve

$$c_E \colon x(t) = x_0 \cos t + p_0 \sin t,\ \ p(t) = p_0 \cos t - x_0 \sin t$$

we see $m_L = (1/\pi)\ d\left(\arg\det(H_p + iH_q)\right) = (1/\pi)\ dt$, and then the Maslov index for c_E is

$$\langle m_L, [c_E]\rangle = \int_{c_E} m_L = \frac{1}{\pi}\int_0^{2\pi} dt = 2$$

Then the Maslov quantization condition for $L(E)$ becomes

$$\frac{1}{2\pi\hbar}\int_c \theta - \frac{1}{4}\langle m_L, [c]\rangle = \frac{E}{\hbar} - \frac{1}{2} \in \mathbb{Z}$$

and the level set $L(E)$ satisfies the Malsov quantization condition if and only if the parameter E is given as

$$E = E_n = \left(n + \frac{1}{2}\right)\hbar,\ \ n = 0, 1, 2, \ldots,$$

which gives exactly the eigenvalues of the Schrödinger operator of the harmonic oscillator.

Example 2: the hydrogen atom

In this section, we see that the Maslov quantization condition determines the eigenvalues of the Schrödinger operator of the hydrogen atom, the angular momentum operator and the Lenz operator, and also determines multiplicities of the eigenspaces for the hydrogen atom.

The operators of the hydrogen atom, the angular momentum operator and the Lenz operator are respectively given by

$$\widehat{H}\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right) = -\frac{\hbar^2}{2}\triangle - \frac{1}{|x|},\ |x| = \left(\sum_{k=1}^{3} x_k^2\right)^{1/2}$$

$$\widehat{l_1}\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right) = x_2\widehat{p}_3 - x_3\widehat{p}_2,$$

$$\widehat{e}_1\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right) = x_2\widehat{p}_1\widehat{p}_2 + x_3\widehat{p}_1\widehat{p}_3 - x_1(\widehat{p}_2^2 + \widehat{p}_3^2) + \widehat{p}_1 + \frac{x_1}{|x|},$$

where $\triangle$ is the 3-dimensional Laplacian and $\widehat{p}_k = \frac{\hbar}{i}\frac{\partial}{\partial x_k}$, $k = 1, 2, 3$. These operators are mutually commuting. Denote the corresponding Hamiltonian functions of $\widehat{H}$, $\widehat{l_1}$ and $\widehat{e}_1$ by

$$H(x,p) = \frac{1}{2}|p|^2 - \frac{1}{|x|},$$

$$l_1(x,p) = x_2p_3 - x_3p_2,$$

$$e_1(x,p) = p_1\langle x,p\rangle - x_1|p|^2 + \frac{x_1}{|x|},$$

where $(x,p) \in T^*(\mathbb{R}^3\backslash 0)$ and $\langle x,p\rangle = \sum_{k=1}^{3} x_kp_k$. It is easy to see that the functions H, l_1 and e_1 are in involution, or Poisson commuting, with respect to the canonical Poisson bracket. We consider the level set of H, l_1 and e_1 such that

$$L(\overline{E}, \overline{l}_1, \overline{e}_1) = \left\{ \begin{matrix} (x,p) \in T^*(\mathbb{R}^3 - 0) \mid H(x,p) = -E\ (E > 0) \\ l_1(x,p) = \overline{l}_1, e_1(x,p) = \overline{e}_1 \end{matrix} \right\}.$$

The functions H, l_1 and e_1 satisfy a priori inequality (see [11, Proposition 1.1])

$$1/\sqrt{-2H(x,p)} \geq |l_1(x,p)| + |e_1(x,p)|/\sqrt{-2H(x,p)}$$

for any $(x,p) \in T^*(\mathbb{R}^3 - 0)$. For parameters $(\overline{E}, \overline{l}_1, \overline{e}_1)$ satisfying an inequality such that

$$1/\sqrt{2\overline{E}} > |\overline{l}_1| + \left(|\overline{e}_1|/\sqrt{2\overline{E}}\right)$$

it is easy to see that the level sets $L(\overline{E}, \overline{l}_1, \overline{e}_1)$ are compact. Then we have

Proposition 2. *The level sets $L(\overline{E}, \overline{l}_1, \overline{e}_1)$ are Lagrangian submanifolds and are diffeomorphic to* 3 *torus generically.*

Now we calculate action integrals and also the Maslov indices along certain closed curves c_1, c_2, c_3 on $L(\overline{E}, \overline{l}_1, \overline{e}_1)$ which generate $H_1(L(\overline{E}, \overline{l}_1, \overline{e}_1), \mathbb{Z})$ as before, and by a direct calculation we can check the Malsov quantization condition. We see

$$c_1 : \frac{1}{2\pi\hbar}\int_{c_1}\theta - \frac{1}{4}\langle m_L, [c_1]\rangle = \frac{1}{2\pi\hbar}\left(\frac{1}{\sqrt{2\overline{E}}} + \frac{|\overline{e}_1|}{\sqrt{2\overline{E}}} + \overline{l}_1\right) - \frac{1}{2} \in \mathbb{Z},$$

$$c_2\colon \frac{1}{2\pi\hbar}\int_{c_2}\theta - \frac{1}{4}\langle m_L,[c_2]\rangle = \frac{1}{2\pi\hbar}\left(\frac{1}{\sqrt{2\overline{E}}} - \frac{|\overline{e}_1|}{\sqrt{2\overline{E}}} + \overline{l}_1\right) - \frac{1}{2} \in \mathbb{Z},$$
$$c_3\colon \frac{1}{2\pi\hbar}\int_{c_3}\theta - \frac{1}{4}\langle m_L,[c_3]\rangle = \frac{\overline{l}_1}{\hbar} \in \mathbb{Z}.$$

Then we have ([11])

Theorem 3. *$L(\overline{E},\overline{l}_1,\overline{e}_1)$ satisfies the Maslov quantization condition if and only if*

$$\overline{E} = \overline{E}_n = \frac{1}{2n^2\hbar^2},\ \overline{l}_1 = \overline{l}_{1,m} = m\hbar,\ \overline{e}_1 = \overline{e}_{1,n_1,n_2} = \frac{n_1 - n_2}{n},$$

where

$$n = n_1 + n_2 + |m| + 1,\ n, n_1, n_2, m \in \mathbb{Z}, n_1, n_2 \geq 0.$$

Theorem 4. *The numbers $E_n, \overline{l}_{1,m}$ and $\overline{e}_{1,n_1,n_2}$ are just equal to the eigenvalues of the operators $\widehat{H}$, $\widehat{l}_1$ and $\widehat{e}_1$, respectively. Moreover, for each $\overline{E}_n = \frac{1}{2n^2\hbar^2}$, the number of level sets $L(\overline{E}_n,\overline{l}_1,\overline{e}_1)$ satisfying the Maslov quantization condition $n = n_1 + n_2 + |m| + 1,\ n_1, n_2 \geq 0$ is equal to the multiplicity of the eigenspace of $\widehat{H}$ belonging to E_n.*

Example 3: MIC-Kepler problem

The MIC-Kepler problem is the Kepler problem under the influence of Dirac's magnetic monopole. The quantized MIC-Kepler problem is formulated and solved by Iwai–Uwano as follows [3]: For every $m \in \mathbb{Z}$, Dirac's monopole field is defined by a closed two-form on $\mathbb{R}^3\setminus\{0\}$

$$\widetilde{\Omega}_m = -(m/2)|\widetilde{x}|^{-3}(\widetilde{x}_1\, d\widetilde{x}_2 \wedge d\widetilde{x}_3 + \widetilde{x}_3\, d\widetilde{x}_1 \wedge d\widetilde{x}_2 + \widetilde{x}_2\, d\widetilde{x}_3 \wedge d\widetilde{x}_1),$$

where $\widetilde{x} = (\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3) \in \mathbb{R}^3\setminus\{0\}$ and $|\widetilde{x}| = \left(\widetilde{x}_1^2 + \widetilde{x}_2^2 + \widetilde{x}_3^2\right)^{1/2}$. A simple calculation yields $\int_{S^2}\widetilde{\Omega}_m = 2\pi m$, where S^2 is the unit two-sphere and $\widetilde{\Omega}_m$ is an integral. Then we have a complex line bundle E_m over $\mathbb{R}^3\setminus\{0\}$ with a Hermitian inner product $\langle\ ,\ \rangle_m$ and a linear connection ∇^m with the curvature form $\widetilde{\Omega}_m$. The Hamiltonian of the quantized MC-Kepler problem is given by

$$\widehat{H}_m = -\frac{\hbar^2}{2}\sum_{j=1}^{3}\left(\nabla_j^m\right)^2 + \frac{(m/2)^2}{2|\widetilde{x}|^2} - \frac{1}{|\widetilde{x}|},$$

where ∇_j^m stands for the covariant derivative in the direction of $\partial/\partial\widetilde{x}_j$, $j = 1,2,3$. The operator $\widehat{H}_m$ has mutually commuting operators

$$\widehat{l}_{m,1} = \frac{\hbar}{i}\left(\widetilde{x}_2\nabla_3^m - \widetilde{x}_3\nabla_2^m\right) + \frac{(m/2)}{|\widetilde{x}|}\widetilde{x}_1,$$
$$\widehat{e}_{m,1} = \frac{\hbar}{2i}\left(\widehat{l}_{m,2}\nabla_3^m - \widehat{l}_{m,3}\nabla_2^m - \nabla_2^m\widehat{l}_{m,3} + \nabla_3^m\widehat{l}_{m,2}\right) + \frac{\widetilde{x}_1}{|\widetilde{x}|}.$$

The eigenvalue problem is exactly solved by Iwai–Uwano [3] as follows. Consider a non-negative integer n subject to the condition: $|m| \leq n$ and $n-m$ is even. Then the eigenvalues of $\widehat{H}_m$ and their multiplicities are

$$E_n^{(m)} = -\frac{2}{(n+2)^2\hbar^2}, \quad \frac{(n-m+2)(n+m+2)}{4},$$

respectively.

On the other hand, the corresponding classical mechanical system is the following. The symplectic manifold is $(T^*(\mathbb{R}^3\setminus\{0\}),\ \sigma_m)$, where

$$\sigma_m = \sum_{j=1}^{3} d\widetilde{p}_j \wedge d\widetilde{x}_j + \pi^*\widetilde{\Omega}_m, \ (\widetilde{x},\widetilde{p}) \in T^*(\mathbb{R}^3\setminus\{0\}) = (\mathbb{R}^3\setminus\{0\}) \times \mathbb{R}^3$$

and $\pi : T^*(\mathbb{R}^3\setminus\{0\}) \to \mathbb{R}^3\setminus\{0\}$ is the canonical projection. The classical Hamiltonian of the operators $\widehat{H}_m, \widehat{l}_{m,1}$ and $\widehat{e}_{m,1}$ are respectively given by

$$\begin{aligned}
H_m(\widetilde{x},\widetilde{p}) &= \frac{1}{2}|\widetilde{p}|^2 + \frac{(m/2)^2}{2|\widetilde{x}|^2} - \frac{1}{|\widetilde{x}|},\\
l_{m,1}(\widetilde{x},\widetilde{p}) &= \widetilde{x}_2\widetilde{p}_3 - \widetilde{x}_3\widetilde{p}_2 + \frac{(m/2)}{|\widetilde{x}|}\widetilde{x}_1,\\
e_{m,1}(\widetilde{x},\widetilde{p}) &= -\widetilde{x}_1|\widetilde{p}|^2 + \widetilde{p}_1\langle\widetilde{x},\widetilde{p}\rangle + \frac{(m/2)(\widetilde{x}_2\widetilde{p}_3 - \widetilde{x}_3\widetilde{p}_2)}{|\widetilde{x}|} + \frac{\widetilde{x}_1}{|\widetilde{x}|}.
\end{aligned}$$

The classical Hamiltonian functions $H_m, l_{m,1}$ and $e_{m,1}$ are in involution, and we consider their level sets

$$L(\overline{E},\overline{l}_{m,1},\overline{e}_{m,1}) = \left\{ \begin{matrix} (\widetilde{x},\widetilde{p}) \in T^*(\mathbb{R}^3-\{0\}) \mid H_m(\widetilde{x},\widetilde{p}) = -\overline{E},\ (\overline{E}>0) \\ l_{m,1}(\widetilde{x},\widetilde{p}) = \overline{l}_{m,1},\ e_{m,1}(\widetilde{x},\widetilde{p}) = \overline{e}_{m,1} \end{matrix} \right\}.$$

By a similar way as in the Kepler problem, we see that the parameters $(\overline{E},\overline{l}_{m,1},\overline{e}_{m,1})$ satisfy a certain natural inequality, and generically the level set $L(\overline{E},\overline{l}_{m,1},\overline{e}_{m,1})$ is diffeomorphic to 3-torus.

Iwai–Uwano [2] showed that the classical MIC-Kepler problem is obtained by the Marsden–Weinstein reduction by $U(1)$ action on the cotangent bundle $(T^*(\mathbb{R}^4-\{0\}), d\theta)$. Using this structure Yoshioka–Ii [12] defined a quantization condition on the symplectic manifold $(T^*(\mathbb{R}^3\setminus\{0\}),\ \sigma_m)$ which is regarded as a $U(1)$-reduction of the Maslov quantization condition on the symplectic manifold $(T^*(\mathbb{R}^4-\{0\}),\ d\theta)$.

Similarly as before, we can check the quantization condition for the level set $L(\overline{E},\overline{l}_{m,1},\overline{e}_{m,1})$ and we obtain ([12])

Theorem 5. *The Lagrangian submanifold $L(\overline{E},\overline{l}_{m,1},\overline{e}_{m,1})$ satisfies the quantization condition if and only if the parameters $(\overline{E},\overline{l}_{m,1},\overline{e}_{m,1})$ coincide with the eigenvalues of the corresponding Hamiltonian operators. For each eigenvalue $\overline{E} = -E_n^{(m)} = \frac{2}{(n+2)^2\hbar^2}$, the number of the Lagrangian submanifolds $L(\overline{E},\overline{l}_{m,1},\overline{e}_{m,1})$ satisfying the quantization condition is equal to the multiplicity of the operator $\widehat{H}_m$.*

Example 4: The Bochner-Laplacian associated with the harmonic connection on the line bundle over $\mathbb{C}P^n$

In this section, we consider quasi-classical eigenvalues of the Bochner-Laplacian associated with the harmonic connection on the line bundle over $\mathbb{C}P^n$. The harmonic connection is given as follows (see [5]).

We provide $\mathbb{C}^{n+1} = \{z = (z_0, \ldots, z_n)\}$ with the Hermitian inner product

$$\langle z, z' \rangle = \sum_{j=0}^{n} z_j \overline{z}_j$$

and the real inner product $\langle z, z' \rangle_R = \mathrm{Re}\, \langle z, z' \rangle$. Consider the $2n+1$-dimensional sphere with radius 2,

$$S^{2n+1}_{[2]} = \{z = (z_0, \ldots, z_n) \mid \langle z, z \rangle_R = 4\},$$

which is endowed with the canonical Riemannian metric g_S induced from $\langle z, z' \rangle_R$. The action of $U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ on $S^{2n+1}_{[2]}$ denoted by R is given by

$$R(\lambda)z = \lambda z, \quad \lambda \in U(1),\ z \in S^{2n+1}_{[2]}.$$

As a quotient space, we get the principal fibre bundle (Hopf fibre bundle) $\nu_P : S^{2n+1}_{[2]} \to \mathbb{C}P^n$. We fix a Riemannian metric g on $\mathbb{C}P^n$ so that ν_P is a Riemannian submersion. Define a connection on $S^{2n+1}_{[2]}$ by means of the Riemannian metric g_S such that $\beta = g_S(\gamma, *)/|\gamma|$, where γ is the fundamental vector field on $S^{2n+1}_{[2]}$ of the action R. Its curvature form is denoted by Ω. For every $m \in \mathbb{Z}$, we consider a $U(1)$ action ρ on $\mathbb{C}$ such that

$$\rho(\lambda)w = \lambda^m w, \quad \lambda \in U(1), w \in \mathbb{C}.$$

We then have a Hermitian line bundle $(E_m, \langle\ ,\ \rangle_m)$ associated with $S^{2n+1}_{[2]}$ by ρ_m. The metric connection $\widetilde{d}_m$ induced by β is called the *harmonic connection* in $(E_m, \langle\ ,\ \rangle_m)$. We denote by D_m the Bochner-Laplacian associated with $\widetilde{d}_m$. The eigenvalues and their multiplicities are already known ([5]).

Proposition 6. *The eigenvalues of D_m are*

$$\lambda_m^{(k)} = (k + |m|/2)(k + |m|/2 + n) - m^2/4,\ k = 0, 1, 2, \ldots$$

and the multiplicity of $\lambda_m^{(k)}$ is

$$\begin{bmatrix} k+|m|+n \\ k+|m| \end{bmatrix} \begin{bmatrix} k+n \\ k \end{bmatrix} - \begin{bmatrix} k+|m|+n-1 \\ k+|m|-1 \end{bmatrix} \begin{bmatrix} k+n-1 \\ k-1 \end{bmatrix}.$$

We consider the corresponding quasi-classical calculation. Consider the cotangent bundle $\pi : T^*\mathbb{C}P^n \to \mathbb{C}P^n$. We denote the energy Hamiltonian of g by H. We consider a symplectic structure on $T^*\mathbb{C}P^n$ such that $\sigma_m = d\theta + \pi^* m\Omega$, where θ is the canonical 1-form of $T^*\mathbb{C}P^n$. The function H is completely integrable and we take certain functions $H_1, \dots, H_{2n-1}, H_{2n} = H$, which are Poisson commuting. Similarly as before, we consider a level set

$$L(E_1, \dots, E_{2n}) = \{p \in T^*\mathbb{C}P^n \mid H_j(p) = E_j,\ j = 1, 2, \dots, 2n\}.$$

We check the quantization condition directly for the level sets $L(E_1, \cdots, E_{2n})$ and we obtain ([13])

Theorem 7. *The quasi-classical eigenvalues of H are*

$$\begin{aligned}\widetilde{\lambda}_m^{(k)} &= (k + |m|/2)(k + |m|/2 + n) - m^2/4 + n^2/4 \\ &= \lambda_m^{(k)} + n^2/4,\ k = 0, 1, 2, \dots\end{aligned}$$

Remark 8. As to multiplicities, we have that for each k the number of

$$L(E_1, \dots, E_{2n})$$

satisfying the quantization condition is equal to the number of tuples of integers

$$(\gamma_1, \dots, \gamma_{n-1}, p_0, p_1, \dots, p_n)$$

such that

$$\sum_{l=0}^{n} p_l = m, \quad k \geq \gamma_{n-1} \geq \cdots \geq \gamma_1 \geq \left(\sum_{l=0}^{n} |p_l| - |m| \right) /2.$$

(For details, see [13].) We can check directly the number of tuples is just equal to the multiplicities of the kth eigenvalue of the Bochner-Laplacian $\lambda_m^{(k)}$ for every k.

Question

Now we would like to ask:

- Can we find other examples which have such coincidence?
- Can we prove mathematically why such coincidence occurs?

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Tomoyo Kanazawa and Akira Yoshioka
Department of Mathematics
Faculty of Science
Tokyo University of Science
Kagurazaka 1-3, Shinjyuku-ku
Tokyo, 162-8601, Japan
e-mail: a26795@rs.tus.ac.jp
yoshioka@rs.kagu.tus.ac.jp

Part II

Representation Theory and Harmonic Analysis

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Supergroup Actions and Harmonic Analysis

Alexander Alldridge

Abstract. Kirillov's orbit philosophy holds for nilpotent Lie supergroups in a narrow sense, but due to the paucity of unitary representations, it falls short of being an effective tool of harmonic analysis in its present form. In this note, we survey an approach using families of coadjoint orbits which remedies this deficiency, at least in relevant examples.

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1. Introduction

The correspondence principle states that in the limit of large quantum numbers, quantum mechanics should reproduce classical mechanics. Quantization is the endeavour of reverse-engineering this correspondence in order to produce viable quantum models.

A prominent approach to this task is Geometric Quantization. Its notable strength lies in its ability to associate, with non-linear phase-space symmetries (a.k.a. symplectic Lie group actions), unitary symmetries of the quantum Hilbert space (a.k.a. unitary representations). Taking this ideology to extremes, one may entertain the idea that all irreducible unitary representations of some given Lie group G might be obtained by the quantization of some universal homogeneous symplectic G-spaces.

It is a famous result due to A.A. Kirillov [9] (partly reformulating earlier results due to J. Dixmier [8]) that this sanguine assumption is a hard fact, at least for nilpotent Lie groups.

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Theorem 1 (Kirillov 1962). *Let G be a simply-connected and connected nilpotent Lie group. There is a bijection between the isomorphism classes of irreducible unitary representations of G and the orbits of G in the coadjoint representation on $\mathfrak{g}^*$.*

Moreover, Kirillov showed that the regular representation of G decomposes in a natural fashion as a direct integral over the orbit space $\mathfrak{g}^*/G$. His ideas have been vastly extended and generalized, under the epithet of the "orbit method" or "orbit philosophy", thereby also shedding light on some older results. For example, the Peter–Weyl decomposition for a compact Lie group G can be obtained by applying Geometric Quantization to T^*G.

It is with these applications to harmonic analysis in mind that we will outline, in this survey, a new approach (developed jointly with J. Hilgert and T. Wurzbacher) to bring the orbit philosophy to fruition for Lie supergroups. Lie supergroups appear as the non-linear classical counterparts to the supersymmetries of quantum field theories, both fundamentally in high-energy physics and as effective symmetries of quasiparticles in condensed matter theory. The lack of a fully satisfactory theory of harmonic analysis for Lie supergroups is therefore a major drawback. In fact, as B. Kostant notes in a fundamental paper on the subject [11]: "[Lie supergroups are] likely to be [...] useful [objects] only insofar as one can develop a corresponding theory of harmonic analysis".

We reassess this basic problem and extend the basic notion of "orbits" by allowing for the presence of auxiliary parameters. This entails some necessary upgrades to the terminology, which we motivate and explain at length in this survey. As we show in examples, the resolution of the attendant technical difficulties dispels some of the basic limitations of the more traditional approaches, hopefully bringing us closer to the fulfilment of Kostant's vision.

Let us close this introduction with a synopsis of the article. After a pedestrian introduction to the wherewithal of supermanifolds in Section 2, we proceed to illustrate the failure of the orbit philosophy (in its traditional sense) for Lie supergroups in Section 3. We introduce our approach in Sections 4 and 5. In Section 6, we show how these ideas help to overcome some of the apparent limitations of the orbit philosophy, at least in some pertinent examples.

2. Supermanifolds in a nutshell

Supermanifolds arose in an attempt to define geometries supporting classical field theories which correspond to the bosonic and fermionic fields encountered in quantum field theory. In addition to the ordinary "even" (or bosonic) coordinates, such geometries allow for "odd" (or fermionic) coordinates which mutually anticommute and commute with their even counterparts.

Formally, such geometries are modelled by extending the algebra of (smooth, analytic, or holomorphic) functions to

$$\mathcal{F}(M)[\xi^1,\dots,\xi^q] = \mathcal{F}(M) \otimes \textstyle\bigwedge(\xi^1,\dots,\xi^q)$$

where $\mathcal{F}(\cdots)$ is the algebra the ordinary functions and $\bigwedge(\cdots)$ denotes the Grassmann algebra in the generators ξ^μ. This is a *superalgebra*, *i.e.*, it admits a grading with respect to $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$.

Thus, a *superspace* X consists of the data of a) a topological space, denoted by X_0, and b) a sheaf of local algebras $\mathcal{O}_X$ on X_0 (where the ground field $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$). Here, $\mathcal{O}_X$ is an abstraction of the "algebra of functions", assigning to any open subset $U \subseteq X_0$ the "functions" defined on U, and the word "local" is a technical condition ensuring that the notion of the *value* (or "numerical part") of a function is well defined at every point $x \in U$.

The most basic example of a superspace is obtained as above, *viz.*

$$\mathbb{A}^{p|q} = \big((\mathbb{A}^{p|q})_0, \mathcal{O}_{\mathbb{A}^{p|q}}\big),$$
$$(\mathbb{A}^{p|q})_0 := \Bbbk^p, \quad \mathcal{O}_{\mathbb{A}^{p|q}} := \mathcal{F}_{\Bbbk^p}(-,\mathbb{K}) \otimes \textstyle\bigwedge(\xi^1, \ldots, \xi^q).$$

Here, $\Bbbk \subseteq \mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{F}_{\Bbbk^p}(-,\mathbb{K})$ is the sheaf of $\mathbb{K}$-valued functions on $\Bbbk^p$ – where, according to our persuasion (which may vary over time), we take the liberty to consider smooth or $\Bbbk$-analytic functions. Given any superspace X and an open subset $U \subseteq X_0$, we may define the *open subspace* $X|_U$ on the set U to be the pair $(U, \mathcal{O}_X|_U)$.

Just as important as the notion of a "space" is the notion of a "map", incorporating central physical concepts such as trajectory, field, and gauge transformation. In local coordinates, maps of (smooth, analytic, or complex) manifolds take the form

$$y^\mu = \varphi^\mu(x^1, \ldots, x^n)$$

where on the right are arbitrary (smooth, analytic, or holomorphic) functions.

This is no different for supermanifolds; the only new distinction is between the parity (even/odd) assigned to the coordinates. Thus, grouping the coordinates according to their parity as $y = (v, \eta)$, $x = (u, \xi)$, "maps" of supermanifolds are of the form

$$\begin{aligned} v^a &= \varphi^a(u^1, \ldots, u^p, \xi^1, \ldots, \xi^q), \\ \eta^b &= \varphi^b(u^1, \ldots, u^p, \xi^1, \ldots, \xi^q), \end{aligned} \tag{1}$$

where again, the functions on the right are arbitrary – up to their parity, which is fixed by the left-hand side.

To make sense of this in our formal framework, we are faced with a conundrum: In order to speak of local coordinates, we need a notion of charts, so we need to know what a map is in the first place. The solution is to change perspective and consider maps as *devices which pull back functions*; the statement that for supermanifolds, the thus defined maps are indeed determined by the data in (1) (that is, by the pullback of coordinates), is then a non-trivial fact, due to D. Leites [12].

Thus, technically, a *morphism* $\varphi : X \longrightarrow Y$ of superspaces comprises the following data: a) a continuous map denoted by $\varphi_0 : X_0 \longrightarrow Y_0$ and b) a local morphism of superalgebra sheaves $\varphi^\sharp : \mathcal{O}_Y \longrightarrow (\varphi_0)_*(\mathcal{O}_X)$. That is, on any open set $V \subseteq Y_0$, to any function $f \in \mathcal{O}_Y(V)$ defined on V is assigned the pulled back function $\varphi^\sharp(f) \in \mathcal{O}_X(\varphi_0^{-1}(V))$ – whilst preserving the algebra structure and the grading. As before, the word "local" is a technical condition ensuring that the pullback preserves values, that is, that the equality $\varphi^\sharp(f)(x) = f(\varphi_0(x))$ holds whenever it makes sense.

As an example, consider the morphism $\varphi : \mathbb{A}^{1|2} \longrightarrow \mathbb{A}^{1|2}$ determined by the assignment

$$\varphi^\sharp(u) = u + \xi^1\xi^2, \quad \varphi^\sharp(\xi^b) = \xi^b \ (b = 1, 2).$$

Its effect on a general function $f = \sum_I f_I \xi^I \equiv f_\varnothing + f_1\xi^1 + f_2\xi^2 + f_{12}\xi^1\xi^2$ is

$$\varphi^\sharp(f) = f_\varnothing + \frac{\partial f_\varnothing}{\partial u}\xi^1\xi^2 + \sum\nolimits_{I \neq \varnothing} f_I \xi^I.$$

With these notions in place, we may now pose the following definition.

Definition 2. A *supermanifold* X is a superspace whose underlying topological space X_0 is Hausdorff and which is locally isomorphic to $\mathbb{A}^{p|q}$. Here, the latter statement means that for any $x \in X_0$, there are open sets U and V (with $x \in U$) and an isomorphism $X|_U \longrightarrow \mathbb{A}^{p|q}|_V$.

Notice that according to our persuasion (*i.e.*, our choice of function sheaf on the model space $\mathbb{A}^{p|q}$), we have defined the notion of *smooth*, *analytic*, or *complex* (*i.e.*, holomorphic) supermanifold.

A popular example of a supermanifold is obtained thus: Take any manifold X and define

$$\Pi TX := \big(X, \Omega_X^\bullet\big),$$

where $\Omega_X^\bullet$ is the sheaf of differential forms on X, with the exterior product as algebra multiplication and the $\mathbb{Z}_2$-grading induced by the degree of differential forms. This supermanifold is called the *parity-reversed tangent bundle* on X. Its main distinction is that it carries a canonical *odd vector field* – *i.e.*, a parity-reversing endomorphism of the function sheaf $\mathcal{O}_{\Pi TX} = \Omega_X^\bullet$ following a graded Leibniz rule – namely, the de Rham differential d.

Much of the local theory of manifolds goes through for supermanifolds without essential changes; in-depth accounts can be found in Refs. [6, 7, 12, 13]. In particular, supermanifolds admit direct products, and this allows us to define the notion of a Lie supergroup, generalizing that of a Lie group.

Definition 3. A *Lie supergroup* is a group object in the category whose objects are the supermanifolds and whose morphisms are the morphisms of superspaces. In other words, a Lie supergroup is the datum of a supermanifold G, together with morphisms $m : G \times G \longrightarrow G$, $1 : * \longrightarrow G$ (where $* = \mathbb{A}^{0|0}$ is the singleton space), and $i : G \longrightarrow G$, which respectively obey the defining equations of multiplication, unit element, and inverse in a group.

Similarly, a (left) action of a Lie supergroup G on a supermanifold X is a morphism $a : G \times X \longrightarrow X$ satisfying the defining equations of a group action on a set. A way to express this formally is to postulate the commutativity of the following diagrams:

$$
\begin{array}{ccc}
G \times G \times X & \xrightarrow{m \times \mathrm{id}_G} & G \times X \\
{\scriptstyle \mathrm{id}_G \times a}\downarrow & & \downarrow{\scriptstyle a} \\
G \times X & \xrightarrow[a]{} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{(1,\mathrm{id}_X)} & G \times X \\
& \searrow\!\!\!\searrow & \downarrow{\scriptstyle a} \\
& & X
\end{array}
$$

that express, respectively, the associative and unit laws for the action.

An example of a Lie supergroup structure on $G = \mathbb{A}^{1|2}$ is obtained by writing its standard coordinates (u, ξ, η) in a matrix of the following shape:

$$
\begin{pmatrix} 1 & u & -\xi \\ 0 & 1 & -\eta \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}
$$

Matrix multiplication and inversion will then define a Lie supergroup structure. (The signs do not play a role here, but are vital in other contexts.) The Lie supergroup thus determined will be called the *Heisenberg supergroup with odd centre*. Explicitly, we have

$$
m^\sharp(u) = u^1 + u^2, \quad m^\sharp(\xi) = \xi^1 + \xi^2 + u^1\eta^2, \quad m^\sharp(\eta) = \eta^1 + \eta^2,
$$

and expressions for the inverse can be similarly derived. An example of an action of G on $X = \mathbb{A}^{2|1}$ is given by writing its standard coordinates (s, t, θ) in a column as follows:

$$
\begin{pmatrix} s \\ t \\ -\theta \end{pmatrix},
$$

and multiplying from the left by the matrix in Equation (2). It is immediate that this action fixes any point of the form $(0, t_0) \in \mathbb{A}_0^{2|1} = \Bbbk^2$, although the coordinate t is not fixed, but instead mapped to $t + \eta\theta$.

A less contrived example of an action is obtained by integrating the odd vector field d on ΠTX (where X is any manifold) to an action of the additive Lie supergroup G of $\mathbb{A}^{0|1}$: If θ is the coordinate on $G = \mathbb{A}^{0|1}$, then the action morphism a is determined by

$$
a^\sharp(\omega) := \omega + \theta d\omega
$$

for any differential form ω on X. Notice that a_0 is the identity of X, but the action is far from trivial: The invariant functions on ΠTX are exactly the closed differential forms on X.

It is instructive to write this in coordinates, say for $X = \mathbb{A}^1$. We have $\Pi TX \cong \mathbb{A}^{1|1}$ with the coordinates $(u, \xi = du)$ where u is the standard coordinate on X. Then

$$
a^\sharp(u) = u + \theta\xi, \quad a^\sharp(\xi) = \xi. \tag{3}
$$

This can again be realized by matrix multiplication:

$$
\begin{pmatrix} 1 & -\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ -\xi \end{pmatrix} = \begin{pmatrix} u + \theta\xi \\ -\xi \end{pmatrix}.
$$

3. The orbit nightmare for Lie supergroups

While the basic definitions in the theory of supermanifolds appear innocent enough, the simple examples discussed in the previous section may serve as an indication to all that is not well in the world of Lie supergroup actions. Nonetheless, one may still hope for a generalization of Kirillov's orbit philosophy to this universe.

In fact, quite some work has been done in this direction, beginning with B. Kostant, who, among other things, defined the coadjoint action of a Lie supergroup more or less simultaneously with the introduction of the latter concept [10]. He also defined homogeneous spaces of Lie supergroups, using the language of Lie–Hopf algebras. This was later recast in the language we are using here by Boyer–Sánchez-Valenzuela [4]. We briefly review the results.

Given an action a of a Lie supergroup G on a supermanifold X and a point $x \in X_0$, there is a natural notion of *isotropy supergroup.* (We will come back to this later.) As the above authors show, it is a closed subsupergroup G_x of G. Moreover, there is a natural supermanifold G/G_x and a surjective submersion $\pi : G \longrightarrow G{\cdot}x := G/G_x$ satisfying the obvious universal property. In particular, the inclusion $* \longrightarrow X$ of the point x factors through a natural G-equivariant injective immersion $G{\cdot}x \longrightarrow X$ – this is the *orbit of* x. In the case of the coadjoint action of G of $\mathfrak{g}^*$, the orbits $G \cdot f$ carry a natural supersymplectic structure invariant under the action.

Thus, coadjoint orbits are in place, and there is also a natural notion of *representation* for a Lie supergroup. For the case of $\Bbbk = \mathbb{K} = \mathbb{R}$, there is also a natural notion of *unitary representation* [5].

The following striking result of H. Salmasian [15] shows that these concepts are in unison, in perfect agreement with the orbit philosophy.

Theorem 4 (Salmasian 2010). *Let G be a simply-connected and connected nilpotent Lie supergroup. The orbits through points of the coadjoint action of G on $\mathfrak{g}^*$ are in bijection with the irreducible unitary representations of G up to parity and isomorphism.*

Let us apply to the simplest possible example, the additive group G of the affine supermanifold $\mathfrak{g} = \mathbb{A}^{0|q}$. The coadjoint action is trivial since G is Abelian. There is only one point of $\mathfrak{g}^*$, so there is only one orbit, the singleton space. On the other hand, there is up to parity and isomorphism only one irreducible unitary representation, namely, the trivial one.

While this confirms Salmasian's theorem and thereby in a narrow sense the orbit philosophy, it shows also that a decomposition of the regular representation on the space of functions $\mathcal{O}_G$ into unitary irreducibles is not conceivable in the traditional sense, as this representation is far from trivial.

What has gone wrong? The examples of Lie supergroup actions considered above suggest that orbits through ordinary points retain only an insufficient fraction of the information on the action. As we shall now argue, a remedy to this defect is to generalize the notion of points.

4. Points manifesto

In order to generalize the notion of points, it is first necessary to rephrase it. As we observed above, a point $x \in X_0$ of a supermanifold gives rise to a morphism $* \longrightarrow X$ from the singleton space $* = \mathbb{A}^{0|0}$ which assigns to a function on X its value at x. This actually sets up a bijection between the elements of X_0 and the morphisms $* \longrightarrow X$.

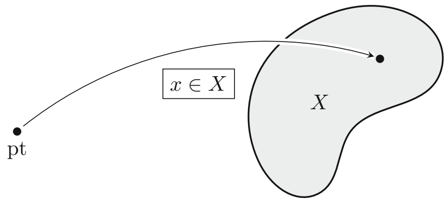

FIGURE 1. A point is a map from the singleton space.

The problem with such simple-minded ordinary points is that they have no "odd directions" with which to trace out those of X. So it is natural to allow them to acquire further degrees of freedom, that is, to replace the singleton $*$ by a general supermanifold T. This leads to the following notion.

Definition 5. Let X be a supermanifold. A *T-point of X*, where T is another supermanifold, is a morphism $x : T \longrightarrow X$. We write $x \in_T X$ and denote the set of T-points of X by $X(T)$.

Intuitively, a T-point of X is a family of points in X parametrised by the auxiliary space T – but this intuition has a limited validity, since a T-point carries more information than the range of the underlying map.

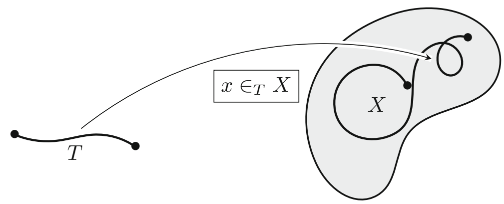

FIGURE 2. A T-point is a T-parameter family of points.

Working with T-points has many advantages: One is that it replaces the supermanifolds and their morphisms by sets and maps of sets.

Indeed, the supermanifold X is replaced by *sets* of T-points, for any T. Similarly, a morphism $\varphi : X \longrightarrow Y$ is replaced by the *maps* $X(T) \longrightarrow Y(T)$, defined by $x \longmapsto \varphi(x) := \varphi \circ x$. The Yoneda Lemma from category theory states that X is determined up to canonical isomorphism by the contravariant functorial assignment $T \longmapsto X(T)$, $(\psi : S \longrightarrow T) \longmapsto (x \longmapsto x \circ \psi)$, called the *functor of points*. Moreover, morphisms $X \longrightarrow Y$ are in bijection with natural transformations $(\varphi_T : X(T) \longrightarrow Y(T))$ of the functors of points.

Another advantage of T-points is that they lead to the notion of *base change*. We will not discuss this in all generality, but instead apply it to extending the notion of supergroup orbits, as we now explain.

5. Isotropies and orbits in families

Let us reconsider the notion of isotropy supergroups through ordinary points. Thus, let $a : G \times X \longrightarrow X$ be an action of a Lie supergroup G on a supermanifold X and let x be an ordinary point. The equation defining the isotropy can be written out in terms of T-points as follows: A T-point $g \in_T G$ is a T-point of the isotropy supergroup if and only if

$$g \cdot x = x.$$

Here, we write $g \cdot x$ for $a(g, x) = a \circ (g, x)$ and x is considered as a T-point of X *via* the composition

$$T \longrightarrow * \xrightarrow{\ x\ } X,$$

where the morphism $T \longrightarrow *$ is unique. Thus, the isotropy supergroup of Kostant and Boyer–Sánchez-Valenzuela is the supergroup G_x, unique up to canonical isomorphism, whose functor of points is

$$G_x(T) = \big\{ g \in_T G \,\big|\, g \cdot x = x \big\}.$$

An equivalent way to state this is that G_x is the fibre product of the point map $x : * \longrightarrow X$ and the orbit map $a_x : G = G \times * \longrightarrow X$, defined by $a_x := a \circ (\mathrm{id}_G \times x)$, *i.e.*, the following diagram is Cartesian:

$$\begin{array}{ccc} G_x & \longrightarrow & G \\ \big\downarrow & & \big\downarrow{\scriptstyle a_x} \\ * & \xrightarrow[\ x\ ]{} & X. \end{array}$$

That is, any pair of morphisms to $*$ and G that lie over the same morphism to X factors uniquely through G_x.

In a similar vein, the orbit $G \cdot x := G/G_x$ is defined by the requirement that the following diagram is a coequaliser:

$$G \times G_x \underset{p_1}{\overset{m}{\rightrightarrows}} G \xrightarrow{\ \pi_x\ } G \cdot x.$$

That is, any morphism defined on G that yields the same morphism on $G \times G_x$ when composed with m and p_1 factors uniquely through π_x.

If now x is a T-point to start with, then the orbit map

$$a \circ (\mathrm{id}_G \times x) : G \times T \longrightarrow X$$

is defined on $G \times T$. (Actually, we prefer to put the T factor first, exchanging factors in the definition.) Thus, the Lie supergroup G gets replaced by a "family" of Lie supergroups $G_T = T \times G$. Formally, this is captured in the following definition.

Definition 6. A *superspace over* T is a morphism $X \longrightarrow T$. A morphism of superspaces over T is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ T & = & T. \end{array}$$

A *supermanifold over* T is a superspace over T locally isomorphic to the model space $\mathbb{A}_T^{p|q} := T \times \mathbb{A}^{p|q}$ with the projection onto T. ("Locally" here means locally in the domain.) A *Lie supergroup over* T is a group object in the category of supermanifolds over T and morphisms over T.

This definition actually makes sense for base superspaces T much more general than supermanifolds, see Ref. [2].

With these notions, the definition of the isotropy supergroup through x is immediate: It is the fibre product of $(p_1, x) : T \longrightarrow X_T = T \times X$ and

$$a_x := (p_1, a \circ (\mathrm{id}_G \times x) \circ (1\,2)) : G_T = T \times G \longrightarrow X_T = T \times X,$$

with $(1\,2)$ denoting the flip. Thus, it makes the following diagram Cartesian:

$$\begin{array}{ccc} G_x & \longrightarrow & G_T \\ \downarrow & & \downarrow{\scriptstyle a_x} \\ T & \xrightarrow[(p_1,x)]{} & X_T. \end{array}$$

In terms of the functor of points, we have

$$G_x(R) := \big\{ (t, g) \in_R G_T \;\big|\; g \cdot x(t) = x(t) \big\}$$

for any supermanifold R over T. Here, recall that $x(t) = x \circ t$.

The notion of isotropies through T-points was defined by Mumford [14] in the context of group schemes. By the Yoneda Lemma, it is clear that G_x is indeed a Lie supergroup over T if only it exists as a supermanifold over T.

A tame example of an action is given by $G = \mathrm{GL}(2, \mathbb{R})$ acting naturally on $X = \mathbb{A}^2$ (where $\mathbb{K} = \mathbb{R}$). For $T = \mathbb{A}^1$ and

$$x(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

we obtain

$$G_x = \left\{ \left(t, \begin{pmatrix} 1 + s \cos t \sin t & s \cos^2 t \\ -s \sin^2 t & 1 - s \cos t \sin t \end{pmatrix} \right) \;\middle|\; s, t \in \mathbb{R} \right\}.$$

In this case, the isotropy supergroup exists and is a Lie group over $\mathbb{A}^1$.

On the other hand, consider the action of $G = \mathbb{A}^{0|1}$ on $X = \mathbb{A}^{1|1}$ defined in Equation (3). Let $T = \mathbb{A}^{0|1}$ with standard coordinate τ and define x by

$$x^\sharp(u) = 0, \quad x^\sharp(\xi) = \tau.$$

In this case, G_x does not exist as a supermanifold over T. However, it does exist as a superspace which is *locally finitely generated* in the sense of Ref. [2]. One computes easily that

$$G_x = \big(*, \mathbb{K}[\theta, \tau]/(\theta\tau)\big).$$

In any case, if G_x exists, then one may define $G \cdot x := G/G_x$ by the requirement that the following diagram is a coequaliser:

$$G_T \times_T G_x \underset{p_1}{\overset{m}{\rightrightarrows}} G_T \xrightarrow{\pi_x} G \cdot x,$$

provided this exists as a supermanifold over T.

In order to understand when the isotropy supergroup exists, we need a piece of data encoding the geometry of the action.

Definition 7. Let $a : G \times X \longrightarrow X$ be an action of a Lie supergroup G on a supermanifold X. For $v \in \mathfrak{g}$, the *fundamental vector field* of v is the unique vector field a_v on X such that

$$(v \otimes 1) \circ a^\sharp = (1 \otimes a_v) \circ a^\sharp.$$

The *fundamental distribution* $\mathscr{A}_\mathfrak{g}$ is the $\mathscr{O}_X$-submodule of the tangent sheaf $\mathscr{T}_X$ spanned by the fundamental vector fields.

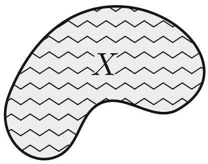

FIGURE 3. The fundamental distribution $\mathscr{A}_\mathfrak{g}$.

The following theorem is proved in Ref. [3].

Theorem 8 (Alldridge–Hilgert–Wurzbacher 2015). *The following are equivalent:*

(i) *The isotropy supergroup G_x exists as a Lie supergroup over T.*
(ii) *The orbit morphism $a_x : G_T \longrightarrow X_T$ has constant rank over T. (That is, its tangent map on the tangent sheaf over T has a locally free cokernel.)*
(iii) *The pullback $x^*\mathscr{A}_\mathfrak{g}$ is locally direct in $x^*\mathscr{T}_X$.*

In this case, $G \cdot x$ is a supermanifold over T and the canonical morphism $G \cdot x \longrightarrow X_T$ is an injective immersion.

We remark that the notion of constant rank morphisms of supermanifolds is much more subtle than for manifolds; in particular, it is not implied by the weaker condition that the rank of the tangent map on the level of tangent spaces is constant.

The theorem subsumes the previous results by Kostant and Boyer–Sánchez-Valenzuela. Moreover, it explains why the isotropy does not exist in the example before Definition 7: the fundamental distribution is spanned by the differential $d = \xi \frac{\partial}{\partial u}$, and its pullback along x is spanned by $\tau \left(x^\sharp \circ \frac{\partial}{\partial u} \right)$, so is not a direct summand.

This phenomenon is not restricted to actions of Lie supergroups. The action on $X = \mathbb{A}^{0|1}$ of the additive group of $\mathbb{A}^1$ that is generated by the even vector field $\xi \frac{\partial}{\partial \xi}$ is also an example where the theorem's assumption fails.

For the particular case of the coadjoint action of G on $\mathfrak{g}^*$, whenever an orbit through a T-point $f \in_T \mathfrak{g}^*$ exists, it carries a symplectic structure (à la Kirillov–Kostant–Souriau), as the following theorem from Ref. [3] shows.

Theorem 9 (Alldridge–Hilgert–Wurzbacher 2015). *Let $f \in_T \mathfrak{g}^*$ be a T-point of $\mathfrak{g}^*$. If the orbit morphism a_f with respect to the coadjoint action $a = \mathrm{Ad}^*$ of G has constant rank, then the coadjoint orbit $G \cdot f$ carries a canonical invariant supersymplectic structure ω_f over T.*

Here, a supersymplectic structure is a non-degenerate super-antisymmetric bilinear form on the tangent sheaf over T (whose sections are vector fields along the fibres of the projection onto T) that is closed for the relative differential $d_{X/T}$.

We emphasize two points: a) The definition of ω_f is the standard one (the precise formulation is somewhat technical since one has to handle the sheaves correctly), and b) in previous attempts by G.M. Tuynman [16] to handle coadjoint orbits through T-points in a more *ad hoc* fashion, it was necessary to consider symplectic forms that where no longer homogeneous with respect to parity. This difficulty disappears in our systematic treatment.

6. Applications to harmonic analysis

We now illustrate in some examples how the point of view introduced in the two previous sections resolves some of the issues around the orbit philosophy for Lie supergroups.

We fix a Lie supergroup G and a T-point f of $\mathfrak{g}^*$. We think of representations of G_T as families over T. For several reasons, the simplest (and most general) way to phrase its representation theory is in terms of contravariant functors on the category $\mathbf{SMan}_T$ of supermanifolds over T. One basic such functor is $\mathcal{O}$, defined by

$$\mathcal{O}(U) := \Gamma(\mathcal{O}_{U,\bar{0}}), \quad \mathcal{O}(f : U \longrightarrow U') := f^\sharp : \mathcal{O}(U') \longrightarrow \mathcal{O}(U).$$

Here, Γ denotes global sections and the subscript $(-)_{\bar{0}}$ the even part. Then $\mathcal{O}$ is a ring object in the category of contravariant functors on $\mathbf{SMan}_T$. The functor of points of G_T is a group object in this category.

Definition 10. A *representation* $(\mathcal{H}, \pi)$ *of* G_T consists of an $\mathcal{O}$-module object $\mathcal{H}$ in the category of contravariant functors on $\mathbf{SMan}_T$ and an $\mathcal{O}$-linear action

$$\pi : G_T \times \mathcal{H} \longrightarrow \mathcal{H}.$$

Let $\mathfrak{h}$ be an $\mathcal{O}_T$-subalgebra of $\mathfrak{g}_T := \mathcal{O}_T \otimes \mathfrak{g}$ (preferably, one that is polarizing in some sense). Then we define a representation $\big(Q(f,\mathfrak{h}), \pi_f^{\mathfrak{h}}\big)$ of G_T as follows. On $(t : U \longrightarrow T) \in \mathbf{SMan}_T$, we define

$$Q(f,\mathfrak{h})(t) := \big\{\psi \in \Gamma(\mathcal{O}_{U\times_T G_T,\bar{0}}) \bigm| \forall v \in \Gamma((t^*\mathfrak{h})_{\bar{0}}) : R_v = -i\langle f(t), v\rangle\psi\big\}.$$

Here, R denotes the right regular representation (by right translation). On morphisms $\varphi : U \longrightarrow U'$ over T, we set

$$Q(f,\mathfrak{h})(\varphi) := (\varphi \times_T \mathrm{id}_{G_T})^\sharp.$$

The action $\pi_f^{\mathfrak{h}}$ is given by restriction of the left regular representation, *viz.*

$$\pi_f^{\mathfrak{h}}(g)\psi := \psi(-, g^{-1}(-)) = \big((\mathrm{id}_U \times_T m) \circ ((\mathrm{id}_U, g^{-1}) \times_T \mathrm{id}_G)\big)^\sharp(\psi)$$

for $\psi \in Q(f,\mathfrak{h})(t)$ and $g \in_U G$. When the coadjoint orbit $G \cdot f$ exists as a supermanifold over T, $Q(f,\mathfrak{h})$ can be seen to define an $\mathcal{O}_T$-module. But it makes sense as a functor in any case.

Let us come back to the most basic example of the Abelian supergroup $G = \mathbb{A}^{0|q}$. Recall from Section 3 that this could not be handled satisfactorily in the traditional approach.

In this case, we have $G_f = G_T$, $G \cdot f = T$, and the Kirillov–Kostant–Souriau form ω_f is zero. Thus, the only reasonable choice for $\mathfrak{h}$ is $\mathfrak{g}_T$. For any $(t : U \longrightarrow T) \in \mathbf{SMan}_T$, the $\mathcal{O}(t)$-module $Q(f,\mathfrak{h})$ is generated by

$$\psi_t = e^{-i\sum_j t_j\xi^j}.$$

Here, t is determined by $t^\sharp(\xi_j) = t_j$, where (ξ_j) is a basis of $\mathfrak{g}$ and (ξ^j) the dual basis. That is, $Q(f,\mathfrak{h})$ comes from a rank $1|0$ locally free $\mathcal{O}_T$-module (or vector bundle on T). The action on ψ_t is given by

$$\pi_f^{\mathfrak{h}}(g)\psi_t = e^{i\langle t,g\rangle}\psi_t.$$

These representations suffice for a decomposition of the regular representation of G. In fact, taking $T = \mathfrak{g}^*$ and $f = \mathrm{id}_T$, one of them is enough.

Proposition 11 (Alldridge–Hilgert–Wurzbacher 2015). *Denote* $\pi_f^{\mathfrak{h}}$ *by* π. *For any superfunction* h *on* G, *we have*

$$\int_T D(\theta)\,\mathrm{str}\,\pi(h) = (-1)^{n(n+1)/2} i^n h_0(0),$$

where $\pi(h)$ *is defined by*

$$\pi(h) := \int_G D(\xi)\,h\pi,$$

and the integrals are Berezin integrals, cf. *Refs.* [7, 12, 13].

This is just the standard inversion formula for the fermionic Fourier transform. In our framework, it acquires an interpretation as the decomposition of the function algebra as an odd direct integral of representations.

We end our discussion by a brief account of the representation theory of the Clifford supergroup G in our framework. Recall that G is the simply-connected Lie supergroup with Lie superalgebra $\mathfrak{g} = \langle x_j, y_j, z | j = 1, \ldots, q\rangle$ where x_j, y_j are odd, z is even, and the bracket is given by

$$[x_j, y_j] = z$$

with all other relations zero. If we choose $T = \mathbb{A}^1 \setminus 0$ and $f = z^*$, where (x^j, y^j, z^*) is the dual basis, then the orbit $G \cdot f$ exists as a supermanifold over T. We choose $\mathfrak{h} = \big\langle x_1, \ldots, x_q, z\big\rangle_{\mathcal{O}_T}$. Then we obtain the following nice characterization of the representation attached to the orbit $G \cdot f$.

Proposition 12 (Alldridge–Hilgert–Wurzbacher 2015). *The representation $\pi = \pi_f^{\mathfrak{h}}$ on $Q(f, \mathfrak{h})$ is the bundle of spinor modules over T of central character $-it$.*

This result can also be reached by other methods, but it is still delightful to see that the spinor module naturally comes out of our construction. Furthermore, this fits nicely together with the following result from Ref. [1].

Theorem 13 (Alldridge–Hilgert–Laubinger 2013). *For any f contained the Schwartz space $\mathscr{S}(G)$, we have the Fourier inversion*

$$f(1) = \frac{(-1)^q}{2\pi} \int_{\mathbb{A}^1} \frac{Dt}{(2t)^{\lfloor (q+1)/2 \rfloor}}\, \tau\big(\pi(f)\big), \ \ \tau = \begin{cases} \mathrm{str} & 2 \mid q, \\ 2^{(1-q)/2} e^{-i\pi/4}\, \mathrm{tr}(\epsilon \cdot) & 2 \nmid q. \end{cases}$$

While a number of issues remain open, these examples may serve as a motivation to study Kirillov's orbit method for Lie supergroups from the more general vantage point that we have suggested here.

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Alexander Alldridge
Universität zu Köln
Mathematisches Institut
Weyertal 86–90
D-50931 Köln, Germany
e-mail: alldridg@math.uni-koeln.de

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Representations of Nilpotent Lie Groups via Measurable Dynamical Systems

Ingrid Beltiţă and Daniel Beltiţă

Abstract. We study unitary representations associated to cocycles of measurable dynamical systems. Our main result establishes conditions on a cocycle, ensuring that ergodicity of the dynamical system under consideration is equivalent to irreducibility of its corresponding unitary representation. This general result is applied to some representations of finite-dimensional nilpotent Lie groups and to some representations of infinite-dimensional Heisenberg groups.

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1. Introduction

A measurable dynamical system is a measure space (X, μ) endowed with a group action on the right $X \times S \to X$, $(x, s) \mapsto x.s$, for which the measure μ is quasi-invariant, hence $\mathrm{d}\mu(x.s) = j(x, s)\mathrm{d}\mu(x)$ for a suitable a.e. defined positive measurable function $j(\cdot, s)$ on X. A scalar cocycle of this measurable dynamical system is a family $\{a(\cdot, s)\}_{s\in S}$ of a.e. defined measurable functions on X with values in the unit circle $\mathbb{T}$, for which the map $\pi_a\colon S \to \mathcal{B}(L^2(X, \mu))$ is a unitary representation, where

$$\pi_a(s)\colon L^2(X, \mu) \to L^2(X, \mu), \quad (\pi_a(s)\varphi)(x) = j(x, s)^{1/2} a(x, s)\varphi(x.s).$$

Our main abstract result is Theorem 7 which establishes conditions on the cocycle a, ensuring that ergodicity of the above dynamical system is equivalent to irreducibility of the unitary representation π_a. The unifying force of this result is

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then illustrated by a variety of applications, including unitary irreducible representations of finite-dimensional nilpotent Lie groups and some representations of infinite-dimensional Heisenberg groups.

Some preliminaries on measure theory

Lemma 1. *Let* (X, μ) *be any measure space and* $\mathcal{H} := L^2(X, \mu)$. *For any* $\psi \in L^\infty(X, \mu)$ *let* $M_\psi \in \mathcal{B}(\mathcal{H})$ *be the multiplication-by-*ψ *operator, and define* $\mathcal{A} := \{M_\psi \mid \psi \in L^\infty(X, \mu)\}$. *If at least one of the following conditions is satisfied:*

1. X *is a locally compact space and* μ *is a Radon measure;*
2. *one has* $\mu(X) < \infty$ *and* $\mathcal{H}$ *is separable;*

then $\mathcal{A}$ *is a maximal abelian self-adjoint subalgebra of* $\mathcal{B}(\mathcal{H})$.

Proof. If the first condition is satisfied then the assertion follows by [5, Ch. I, §7, no. 3, Th. 2]. If the second condition is satisfied, then the constant function $1 \in L^\infty(X, \mu) \subseteq L^2(X, \mu)$ is a cyclic vector for $\mathcal{A}$, hence the conclusion follows by [7, Th. 2.3.4]. □

Definition 2. Let $\alpha \colon G \times X \to X$ be any group action by measurable transformations of a measure space (X, μ). The action α is called *ergodic* if for every measurable set $A \subseteq X$ with $\mu((A \triangle \alpha_g(A)) = 0$ for all $g \in G$, one has either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. (Here, for two sets X and Y, $X \triangle Y$ denotes the symmetric difference $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.)

Remark 3. In the framework of Definition 2 it is straightforward to check that the group action α is ergodic if and only if the equivalence classes of a.e. constant functions in $L^\infty(X, \mu)$ are the only elements $\varphi \in L^\infty(X, \mu)$ with $\varphi \circ \alpha_g = \varphi$ for all $g \in G$. This implies that if α is a transitive action (or more generally, if for every $x \in X$ with its orbit $G.x := \{\alpha_g(x) \mid g \in G\}$ one has $\mu(X \setminus G.x) = 0$), then α is ergodic.

We refer to [8] for the role of ergodic actions in the theory of operator algebras.

2. General results

To begin with, we recall some ideas from [6, Ch. I, Subsect. 1.4].

Definition 4. A *measurable dynamical system* consists of a measure space (X, μ) endowed with a group action on the right

$$\beta \colon X \times S \to X, \quad (x, s) \mapsto \beta_s(x) =: x.s,$$

for which the measure μ is quasi-invariant. Then for every $s \in S$ there is an a.e. defined positive function $j(\cdot, s)$ on X for which $(\beta_s)_*(\mu) = j(\cdot, s)\mu$, where $(\beta_s)_*(\mu)$ denotes the pushforward of the measure μ through the map β_s. Hence for every measurable set $E \subseteq X$ one has

$$\mu(\beta_s(E)) = \int_E j(x, s) \mathrm{d}\mu(x).$$

A *scalar cocycle* of this measurable dynamical system is a family $\{a(\cdot,s)\}_{s\in S}$ consisting of a.e. defined measurable functions on X with values in the unit circle $\mathbb{T}$, satisfying the conditions

$$a(x,s_1s_2)=a(x,s_1)a(x.s_1,s_2) \text{ and } a(x,\mathbf{1})=x$$

for a.e. $x\in X$ and all $s_1,s_2\in S$.

In the above setting we also define $\mathcal{H}:=L^2(X,\mu)$ and for every $s\in S$,

$$\pi_a(s)\colon \mathcal{H}\to\mathcal{H},\quad \pi_a(s)\varphi=j(\cdot,s)^{1/2}a(\cdot,s)(\varphi\circ\beta_s)(\cdot).$$

Remark 5. In Definition 4, since $(\beta_{s_1s_2})_*(\mu)=(\beta_{s_1})_*((\beta_{s_2})_*(\mu))$ for all $s_1,s_2\in S$, it is easily checked that the family $\{a(\cdot,s)\}_{s\in S}$ satisfies the conditions of a scalar cocycle, except that the functions from this family take values in the multiplicative group $(0,\infty)$ instead of the unit circle $\mathbb{T}$.

The following result is a special case of [6, Ch. I, Props. 1.1–1.2] whose proof was not included therein, so we give the sketch of a proof here, for the sake of completeness.

Proposition 6. *Assume the setting of Definition* 4*. Then the following assertions hold:*

(i) *The map $\pi_a\colon S\to\mathcal{B}(\mathcal{H})$ is a unitary representation.*
(ii) *If the representation π_a is irreducible, then the action of S on X is ergodic.*
(iii) *If S is a topological group and one has*

$$\lim_{s\to\mathbf{1}}\mu(E\bigtriangleup(E.s))=\lim_{s\to\mathbf{1}}\int\limits_{E\bigtriangleup(E.s)}|j(\cdot,s)^{1/2}-1|^2\mathrm{d}\mu=\lim_{s\to\mathbf{1}}\int\limits_{E}(a(s,\cdot)-1)\mathrm{d}\mu=0$$

for every measurable set $E\subseteq X$ with $\mu(E)<\infty$, then the representation π_a is continuous.

Proof. Assertion (6) is based on a straightforward computation.

For Assertion (6) note that for every measurable set $E\subseteq X$ which is G-invariant, the multiplication operator $M_{\chi_E}\in\mathcal{B}(\mathcal{H})$ is an orthogonal projection whose image is invariant under $\pi_a(s)$ for all $s\in S$.

For Assertion (6) we use that the values of π_a are unitary operators on $\mathcal{H}$, hence an $(\varepsilon/3)$-argument shows that it suffices to check that $\lim\limits_{s\to\mathbf{1}}\|\pi_a(s)\varphi-\varphi\|=0$ for φ in some subset of $\mathcal{H}$ that spans a dense linear subspace. Using the assumptions, one can check that $\lim\limits_{s\to\mathbf{1}}\|\pi_a(s)\chi_E-\chi_E\|=0$ for every measurable set $E\subseteq X$ with $\mu(E)<\infty$, and this completes the proof. □

For the following theorem we recall that a *multiplicity-free representation* is a unitary representation whose commutant is commutative.

Theorem 7. *Assume the setting of Definition* 4*, where (X,μ) satisfies either of the conditions in Lemma* 1*, and let*

$$S_0:=\{s\in S\mid x.s=x \text{ for a.e. } x\in X\}.$$

If the set $\{a(\cdot,s) \mid s \in S_0\}$ *spans a* w^**-dense linear subspace of* $L^\infty(X,\mu)$*, then* $\pi_a\colon S \to \mathcal{B}(\mathcal{H})$ *is a multiplicity-free representation and moreover the following assertions are equivalent:*

(i) *The action of* S *on* (X,μ) *is ergodic.*
(ii) *The representation* π_a *is irreducible.*

Proof. Recall that $\mathcal{H} = L^2(X,\mu)$ and for any $\psi \in L^\infty(X,\mu)$ we denote by $M_\psi \in \mathcal{B}(\mathcal{H})$ the operator of multiplication by ψ. By Lemma 1, the operator algebra

$$\mathcal{A} := \{M_\psi \mid \psi \in L^\infty(X,\mu)\} \subseteq \mathcal{B}(\mathcal{H})$$

is a maximal self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. To prove that π_a is a multiplicity-free representation, we will show that $\pi_a(S)' \subseteq \mathcal{A}$. Hence we must prove that if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S$, then $T \in \mathcal{A}$. In fact we will prove a stronger fact, namely if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S_0$, then $T \in \mathcal{A}$.

If $s \in S_0$, then it is clear that $j(x,s) = 1$ and $\varphi(x.s) = \varphi(x)$ for a.e. $x \in X$, where $\varphi \in L^2(X,\mu)$ is arbitrary, and it then follows by the definition of π_a that $\pi_a(s)$ is the operator of multiplication by $a(\cdot,s) \in L^\infty(X,\mu)$. Since the set $\{a(\cdot,s) \mid s \in S_0\}$ spans a w^*-dense linear subspace of $L^\infty(X,\mu)$ by hypothesis, it then follows that if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S_0$, then $T \in \mathcal{A}'$. We have seen above that $\mathcal{A}$ is a maximal self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, hence $\mathcal{A}' = \mathcal{A}$, and then $T \in \mathcal{A}$, as claimed above. This completes the proof of the fact that π_a is a multiplicity-free representation.

Moreover, if the representation π_a is irreducible, then the action of S on (X,μ) is ergodic by Proposition 6 (6). Conversely, let us assume that the action of S on (X,μ) is ergodic. In order to prove that the representation π_a is irreducible, we must show that if $T \in \mathcal{B}(\mathcal{H})$ satisfies $T\pi(s) = \pi(s)T$ for all $s \in S$, then T is a scalar multiple of the identity operator on $\mathcal{H}$. In fact, using the condition $T\pi(s) = \pi(s)T$ for all $s \in S_0$, we obtain by the above reasoning that $T = M_\psi$ for some $\psi \in L^\infty(X,\mu)$. Then for all $s \in S$ and $\varphi \in L^\infty(X,\mu)$ one has

$$\begin{aligned}
\psi(x)j(x,s)^{1/2}\varphi(x.s) &= (M_\psi\pi_a(s)\varphi)(x) \\
&= (T\pi_a(s)\varphi)(x) \\
&= (\pi_a(s)T\varphi)(x) \\
&= (\pi_a(s)M_\psi\varphi)(x) \\
&= j(x,s)^{1/2}\psi(x.s)\varphi(x.s)
\end{aligned}$$

for a.e. $x \in X$. This implies that for all $s \in S$ one has $\psi(x) = \psi(x.s)$ for a.e. $x \in X$. Since $\psi \in L^\infty(X,\mu)$ and the action of S on (X,μ) is ergodic, it then follows that ψ is constant a.e. on X, hence the multiplication operator $T = M_\psi$ is a scalar multiplication of the identity operator on $\mathcal{H}$, and this completes the proof. □

Remark 8. As we will see in the examples presented in the following sections of this paper, the group S_0 from Theorem 7 is an abstract version of the Lie subgroup that corresponds to a polarization of a nilpotent Lie algebra. More precisely, one can interpret the representation $\pi_a\colon S \to \mathcal{B}(L^2(X,\mu))$ as the representation induced from the character $\chi_0 \colon S_0 \to \mathbb{T}$, $\chi_0(s) := a(x_0, s)$, for some fixed $x_0 \in X$ (if any) with $x_0.s = x_0$ for all $s \in S_0$. It is worth noting that if there exists such a point $x_0 \in X$, then the above χ_0 is a group homomorphism because of the cocycle properties of a.

3. Applications to group actions on locally compact spaces

The following proposition establishes irreducibility of some unitary representations that play a very significant role in [1] and [2]. See Examples 12–13 below for more specific information in this connection.

Proposition 9. *Let G be a group and (X, μ) be any locally compact space endowed with a Radon measure. Assume that $\alpha \colon G \times X \to X$, $(g, x) \mapsto \alpha_g(x)$, is an action of G on X by measure-preserving transformations. Let $\mathcal{F}$ be any G-invariant vector space of real measurable functions on X, with the corresponding representation $\lambda \colon G \to \mathrm{End}\,(\mathcal{F})$, $\lambda_g(f) := f \circ \alpha_{g^{-1}}$. Assume in addition that the linear span of the set $\{\exp(\mathrm{i}f) \mid f \in \mathcal{F}\}$ is w^*-dense in $L^\infty(X, \mu)$. Then the following conditions are equivalent:*

1. *The action α is ergodic.*
2. *The unitary representation*
$$\pi \colon \mathcal{F} \rtimes_\lambda G \to \mathcal{B}(L^2(X, \mu)), \quad (\pi(f, g)\varphi)(x) = \mathrm{e}^{\mathrm{i}f(x)}\varphi(\alpha_{g^{-1}}(x))$$
is irreducible.

Proof. This is just a special case of Theorem 7, with $S = \mathcal{F} \rtimes_\lambda G$. Indeed in this case $S_0 = (\mathcal{F}, +)$, and it acts trivially on X. The fact that group action of $\mathcal{F} \rtimes_\lambda G$ on (X, μ) is ergodic is equivalent to ergodicity of the action of G on (X, μ), since the action of $\mathcal{F}$ on (X, μ) is trivial. □

Remark 10. In Proposition 9 the ergodicity hypothesis is necessary for the representation π to be irreducible, without imposing any condition on the linear span of the set $\{\exp(\mathrm{i}f) \mid f \in \mathcal{F}\}$. This follows by Proposition 6 (6).

For the transitive group action of a connected simply connected nilpotent Lie group on itself by left translations, the following corollary implies that the unitary representations constructed in [1, Subsect. 2.4] are irreducible. Using suitable global coordinates on coadjoint orbits of nilpotent Lie groups and the transitivity of coadjoint action on its orbits, this corollary also recovers the result of [2, Prop. 5.1(2)]. See Examples 12–13 below for more details in this connection.

Corollary 11. *Let G be a group and X be a finite-dimensional real vector space with a Lebesgue measure μ. Assume that $\alpha \colon G \times X \to X$, $(g, x) \mapsto \alpha_g(x)$, is an action of G on X by measure-preserving transformations. Let $\mathcal{F}$ be any G-invariant vector space of real measurable functions on X, with the corresponding representation $\lambda \colon G \to \mathrm{End}\,(\mathcal{F})$, $\lambda_g(f) := f \circ \alpha_{g^{-1}}$, and define the unitary representation*
$$\pi \colon \mathcal{F} \rtimes_\lambda G \to \mathcal{B}(L^2(X, \mu)), \quad (\pi(f, g)\varphi)(x) = \mathrm{e}^{\mathrm{i}f(x)}\varphi(\alpha_{g^{-1}}(x)).$$

If the linear dual space of X satisfies $X^* \subseteq \mathcal{F}$, then the following conditions are equivalent:

(i) *The action α is ergodic.*
(ii) *The representation π is irreducible.*

Proof. If the representation π is irreducible, then α is ergodic by Remark 10.

Conversely, the result will follow by Proposition 9 as soon as we will have proved that the linear span of the set $\{\exp(\mathrm{i}\xi) \mid \xi \in X^*\}$ is w^*-dense in $L^\infty(X,\mu)$. To check this, recall that the predual of the von Neumann algebra $L^\infty(X,\mu)$ is $L^1(X,\mu)$ and the corresponding duality pairing is

$$L^\infty(X,\mu) \times L^1(X,\mu) \to \mathbb{C}, \quad (\varphi,\psi) \mapsto \langle \varphi,\psi\rangle := \int_X \varphi\psi \mathrm{d}\mu.$$

On the other hand, if $\psi \in L^1(X,\mu)$ and $0 = \langle \exp(\mathrm{i}\xi),\psi\rangle = \int_X \exp(\mathrm{i}\xi)\psi \mathrm{d}\mu$ for all $\xi \in X^*$, then $\psi = 0$ by the injectivity property of the Fourier transform. It then follows by the Hahn–Banach theorem that indeed the linear span of the set $\{\exp(\mathrm{i}\xi) \mid \xi \in X^*\}$ is w^*-dense in $L^\infty(X,\mu)$, and this completes the proof. □

Example 12 ([1, Subsect. 2.4]). Let G be any connected, simply connected, nilpotent Lie group with some fixed left invariant Haar measure, and $\mathcal{F} \subseteq \mathcal{C}^\infty(G)$ be a linear subspace of the space of smooth functions on G. satisfying the following conditions:

1. The linear space $\mathcal{F}$ is invariant under the representation of G by left translations, $\lambda\colon G \to \mathrm{End}\,(\mathcal{C}^\infty(G))$, $(\lambda_g\phi)(x) = \phi(g^{-1}x)$. We denote again by $\lambda\colon G \to \mathrm{End}\,(\mathcal{F})$ the restriction to $\mathcal{F}$ of the above representation λ of G.
2. The mapping $G \times \mathcal{F} \to \mathcal{F}$, $(g,\phi) \mapsto \lambda_g\phi$ is continuous.
3. We have $\mathfrak{g}^* \subseteq \{\phi \circ \exp_G \mid \phi \in \mathcal{F}\}$.

We define $\pi\colon \mathcal{F} \rtimes G \to \mathcal{B}(L^2(G))$ by

$$(\pi(\phi,g)f)(x) = \mathrm{e}^{\mathrm{i}\phi(x)} f(g^{-1}x)$$

for all $\phi \in \mathcal{F}$, $g \in G$, and $f \in L^2(G)$, and almost all $x \in G$.

Hence π is as in Proposition 9. In order to apply that proposition we must check that the linear span of the set $\{\exp(\mathrm{i}\phi) \mid \phi \in \mathcal{F}\}$ is w^*-dense in $L^\infty(G)$, hence that if $\psi \in L^1(G)$ and $\int_G \psi \exp(\mathrm{i}\phi) = 0$ for all $\phi \in \mathcal{F}$, then necessarily $\psi = 0$. To this end, using the above condition for $\phi = \xi \circ \log_G$ with arbitrary $\xi \in \mathfrak{g}^*$ (note that $\phi \in \mathcal{F}$ by the hypothesis 3), we obtain that the Fourier transform of ψ is zero, hence $\psi = 0$. Finally, the right action of $\mathcal{F} \rtimes G$ on G given by

$$G \times (\mathcal{F} \rtimes G) \to G, \quad (x,(g,\phi)) \mapsto g^{-1}x$$

is transitive, hence ergodic (see Remark 3), and then by Proposition 9 the representation π is irreducible.

Let us also note that the above hypotheses on $\mathcal{F}$ ensure that $\mathcal{F}$ is a admissible function space in the sense of [1, Def. 2.8].

Example 13 ([2, Prop. 5.1(2)]). Let G be any connected, simply connected, nilpotent Lie group, with its center Z and the corresponding Lie algebras $\mathfrak{z} \subseteq \mathfrak{g}$. Endow the coadjoint orbit $\mathcal{O}$ with its Liouville measure and define

$$\widetilde{\pi}\colon G \ltimes_{\mathrm{Ad}} \mathfrak{g} \to \mathcal{B}(L^2(\mathcal{O})), \quad (\widetilde{\pi}(g,Y)f)(\xi) = \mathrm{e}^{\mathrm{i}\langle \xi, Y\rangle} f(\mathrm{Ad}_G^*(g^{-1})\xi).$$

Then the following assertions hold:

(i) The group $\widetilde{G} := G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ is nilpotent and its center is $Z \times \mathfrak{z}$.
(ii) $\widetilde{\pi}$ is a unitary irreducible representation of $\widetilde{G}$.

We recall that the multiplication in the semi-direct product group $\widetilde{G}$ is given by

$$(g_1, Y_1) \cdot (g_2, Y_2) = (g_1 g_2, Y_1 + \mathrm{Ad}_G(g_1)Y_2) \tag{1}$$

and the bracket in the corresponding Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$ is defined by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [X_1, Y_2] - [X_2, Y_1]).$$

An inspection of these equations shows that $\widetilde{\mathfrak{g}}$ is a nilpotent Lie algebra with its center $\mathfrak{z} \times \mathfrak{z}$.

To see that $\widetilde{\pi}$ is a representation we need to check that the function

$$a\colon \mathcal{O} \times \widetilde{G} \to \mathbb{T}, \quad a(\xi, (g, Y)) := \mathrm{e}^{\mathrm{i}\langle \xi, Y\rangle}$$

is a cocycle in the sense of Definition 4. In fact, using the right action of $\widetilde{G}$ on $\mathcal{O}$ given by

$$\mathcal{O} \times \widetilde{G} \to \mathcal{O}, \quad (\xi, (g, Y)) \mapsto \xi \circ \mathrm{Ad}_G(g) = \mathrm{Ad}_G^*(g^{-1})\xi, \tag{2}$$

it follows by (1) and the above definition of a that

$$\begin{aligned} a(\xi, (g_1, Y_1)(g_2, Y_2)) &= a(\xi, (g_1 g_2, Y_1 + \mathrm{Ad}_G(g_1)Y_2)) \\ &= \mathrm{e}^{\mathrm{i}\langle \xi, Y_1 + \mathrm{Ad}_G(g_1)Y_2\rangle} \\ &= \mathrm{e}^{\mathrm{i}\langle \xi, Y_1\rangle} \mathrm{e}^{\mathrm{i}\langle \xi \circ \mathrm{Ad}_G(g_1), Y_2\rangle} \\ &= a(\xi, (g_1, Y_1)) a(\xi.(g_1, Y_1), (g_2, Y_2)). \end{aligned}$$

The property $a(\xi, \mathbf{1}) = \xi$ for al $\xi \in \mathcal{O}$ is clear from the definition of a. Also note that the Liouville measure on $\mathcal{O}$ is invariant under the group action (2). It then follows by Proposition 6 that $\widetilde{\pi}$ is a continuous unitary representation.

Moreover, to see that $\widetilde{\pi}$ is irreducible we will use Corollary 11. To this end, first note that the group action (2) is transitive, hence ergodic (see Remark 3). Furthermore, recall that the mapping

$$\mathcal{O} \to \mathfrak{g}_e^*, \quad \xi \to \xi|_{\mathfrak{g}_e}$$

is a global chart which takes the Liouville measure of $\mathcal{O}$ to a Lebesgue measure on $\mathfrak{g}_e^*$, where e is the jump index set of $\mathcal{O}$ with respect to some Jordan–Hölder basis in $\mathfrak{g}$ (see for instance [2]). Then we can use the Fourier transform to see that the linear subspace generated by $\{\mathrm{e}^{\mathrm{i}\langle Y, \cdot\rangle} \mid Y \in \mathfrak{g}\}$ is weak*-dense in $L^\infty(\mathcal{O})$ ($\simeq L^1(\mathcal{O})^*$). Therefore we can use Corollary 11 to obtain that $\widetilde{\pi}$ is irreducible.

In addition to the above properties of $\widetilde{\pi}$ we also recall some additional information on the irreducible representation $\widetilde{\pi}$ that was obtained in [2, Prop. 5.1(2)].

Firstly, the space of smooth vectors for the representation $\widetilde{\pi}$ is $\mathcal{S}(\mathcal{O})$. Moreover, select any Jordan–Hölder basis $X_1, \dots, X_n$ in $\mathfrak{g}$ and define

$$\widetilde{X}_j = \begin{cases} (0, X_j) & \text{for } j = 1, \dots, n, \\ (X_{j-n}, 0) & \text{for } j = n+1, \dots, 2n. \end{cases}$$

Then $\widetilde{X}_1, \dots, \widetilde{X}_{2n}$ is a Jordan–Hölder basis in $\widetilde{\mathfrak{g}}$ and the corresponding predual for the coadjoint orbit $\widetilde{\mathcal{O}} \subseteq \widetilde{\mathfrak{g}}^*$ associated with the representation $\widetilde{\pi}$ is

$$\widetilde{\mathfrak{g}}_{\widetilde{e}} = \mathfrak{g}_e \times \mathfrak{g}_e \subseteq \widetilde{\mathfrak{g}},$$

where $\widetilde{e}$ is the set of jump indices for $\widetilde{\mathcal{O}}$.

4. Application to Gaussian measures on Hilbert spaces

We first recall here a few facts from [3] and [4].

Definition 14. If $\mathcal{V}$ is a real Hilbert space, $A \in \mathcal{B}(\mathcal{V})$ with $(Ax \mid y) = (x \mid Ay)$ for all $x, y \in \mathcal{V}$, and moreover $\operatorname{Ker} A = \{0\}$, then the *Heisenberg algebra* associated with the pair $(\mathcal{V}, A)$ is the real Hilbert space $\mathfrak{h}(\mathcal{V}, A) = \mathcal{V} \dotplus \mathcal{V} \dotplus \mathbb{R}$ endowed with the Lie bracket defined by $[(x_1, y_1, t_1), (x_2, y_2, t_2)] = (0, 0, (Ax_1 \mid y_2) - (Ax_2 \mid y_1))$. The corresponding *Heisenberg group* $\mathbb{H}(\mathcal{V}, A) = (\mathfrak{h}(\mathcal{V}, A), *)$ is the Lie group whose underlying manifold is $\mathfrak{h}(\mathcal{V}, A)$ and whose multiplication is defined by

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + ((Ax_1 \mid y_2) - (Ax_2 \mid y_1))/2)$$

for $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{H}(\mathcal{V}, A)$.

Let $\mathcal{V}_-$ be a real Hilbert space and $(\cdot \mid \cdot)_-$ be its scalar product. For every $a \in \mathcal{V}_-$ and every symmetric, nonnegative, injective, *trace-class* operator K on $\mathcal{V}_-$ there is a unique probability Borel measure γ on $\mathcal{V}_-$ with

$$(\forall x \in \mathcal{V}_-) \qquad \int\limits_{\mathcal{V}_-} \mathrm{e}^{\mathrm{i}(x|y)_-} \mathrm{d}\gamma(y) = \mathrm{e}^{\mathrm{i}(a|x)_- - \frac{1}{2}(Kx|x)_-}$$

and γ is called the *Gaussian measure with the mean a and the variance K*.

Now assume that $a = 0$ and let $\mathcal{V}_+ := \operatorname{Ran} K$ and $\mathcal{V}_0 := \operatorname{Ran} K^{1/2}$ be endowed with the scalar products given by $(Kx \mid Ky)_+ := (x \mid y)_-$ and $(K^{1/2}x \mid K^{1/2}y)_0 := (x \mid y)_-$, respectively, for all $x, y \in \mathcal{V}_-$, which turn the linear bijections $K \colon \mathcal{V}_- \to \mathcal{V}_+$ and $K^{1/2} \colon \mathcal{V}_- \to \mathcal{V}_0$ into isometries. We thus obtain the real Hilbert spaces

$$\mathcal{V}_+ \hookrightarrow \mathcal{V}_0 \hookrightarrow \mathcal{V}_-,$$

where the inclusion maps are Hilbert–Schmidt operators, since $K^{1/2} \in \mathcal{B}(\mathcal{V}_-)$ is a Hilbert–Schmidt operator. Also, the scalar product of $\mathcal{V}_0$ extends to a duality pairing $(\cdot \mid \cdot)_0 \colon \mathcal{V}_- \times \mathcal{V}_+ \to \mathbb{R}$.

We also recall that for every $x \in \mathcal{V}_+$ the translated measure $\mathrm{d}\gamma(-x + \cdot)$ is absolutely continuous with respect to $\mathrm{d}\gamma(\cdot)$ and we have the Cameron–Martin formula

$$\mathrm{d}\gamma(-x + \cdot) = \rho_x(\cdot)\mathrm{d}\gamma(\cdot) \quad \text{with } \rho_x(\cdot) = \mathrm{e}^{(\cdot|x)_0 - \frac{1}{2}(x|x)_0}.$$

Definition 15. Let $\mathcal{V}_+$ be a real Hilbert space with the scalar product denoted by $(x,y) \mapsto (x \mid y)_+$. Also let $A\colon \mathcal{V}_+ \to \mathcal{V}_+$ be a nonnegative, symmetric, injective, trace-class operator. Let $\mathcal{V}_0$ and $\mathcal{V}_-$ be the completions of $\mathcal{V}_+$ with respect to the scalar products

$$(x,y) \mapsto (x \mid y)_0 := (A^{1/2}x \mid A^{1/2}y)_+$$

and

$$(x,y) \mapsto (x \mid y)_- := (Ax \mid Ay)_+,$$

respectively. Then the operator A uniquely extends to a nonnegative, symmetric, injective, trace-class operator $K \in \mathcal{B}(\mathcal{V}_-)$, hence by the above observations one obtains the Gaussian measure γ on $\mathcal{V}_-$ with variance K and mean 0.

One can also construct the Heisenberg group $\mathbb{H}(\mathcal{V}_+, A)$. The *Schrödinger representation* $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-,\gamma))$ is defined by

$$\pi(x,y,t)\phi = \rho_x(\cdot)^{1/2}\mathrm{e}^{\mathrm{i}(t+(\cdot|y)_0+\frac{1}{2}(x|y)_0)}\phi(-x+\cdot)$$

for $(x,y,t) \in \mathbb{H}(\mathcal{V}_+, A)$ and $\phi \in L^2(\mathcal{V}_-,\gamma)$.

Proposition 16. *The representation $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-,\gamma))$ from Definition* 15 *is irreducible.*

Proof. See for instance from [3, Rem. 3.6] or [4]. □

Corollary 17. *In the above setting, the action by translations of $\mathcal{V}_+$ on $(\mathcal{V}_-,\gamma)$ is ergodic.*

Proof. In the present framework, the representation π is the unitary representation associated to the measure space $(\mathcal{V}_-,\gamma)$ acted on by the additive group $(\mathcal{V}_+,+)$ by translations. The cocycle of that measurable dynamical system which gives rise to the representation π is given by

$$a(\cdot,(x,y,t)) = \mathrm{e}^{\mathrm{i}(t+(\cdot|y)_0+\frac{1}{2}(x|y)_0)}$$

for all $(x,y,t) \in \mathbb{H}(\mathcal{V}_+, A)$. The conclusion follows by Propositions 16 and 6 (6) for the right group action

$$\mathcal{V}_- \times \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{V}_-, \quad (v,(x,y,t)) \mapsto -x+v$$

and we are done. □

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Ingrid Beltiţă and Daniel Beltiţă
Institute of Mathematics "Simion Stoilow"
of the Romanian Academy
P.O. Box 1-764
Bucharest, Romania
e-mail: ingrid.beltita@gmail.com
beltita@gmail.com

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Symbolic Interpretation of the Molien Function: Free and Non-free Modules of Covariants

Guillaume Dhont and Boris I. Zhilinskií

Abstract. A mathematical problem originating from molecular physics leads to the exploration of the algebraic structures of sets of multivariate polynomials whose variables are the (x_i, y_i) components of n vectors in a plane with a common origin. The symmetry is assumed to be described by the SO(2) Lie group. The irreducible representations (irreps) of the group are labeled by the integer m. The ring of invariants is the set of polynomials that transform under the action of the SO(2) group according to the $m = 0$ irrep. Such a ring admits a Cohen–Macaulay decomposition. The set of polynomials changing as the (m) irrep, $m \neq 0$, under the elements of the group defines the module of (m)-covariants. The module of (m)-covariants is free when $|m| < n$ and the expression of the Molien function is symbolically interpreted in terms of a standard integrity basis containing one set of denominator polynomials and one set of numerator polynomials. In contrast, the module of (m)-covariants is non-free when $|m| \geq n$ and a generalized integrity basis has to be introduced to throw light on the Molien function. A graphical representation of the algebraic structures of the free and non-free modules is proposed.

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1. Introduction

1.1. Invariant theory and Molien function

It is frequently beneficial to introduce group-theoretical concepts when dealing with a symmetrical physics problem. In particular, group and invariant theory have been introduced to qualitatively understand the dynamics of atoms and molecules [1, 2]. The existence of equivalent atoms, the symmetrical shape of a molecule in its equilibrium configuration or the rotational symmetry in the absence of external fields naturally ask for the introduction of a symmetry group, such as the SO(2) and SO(3) Lie groups [3, 4], a molecular point group [5], or a permutation-inversion group [6].

A typical case in molecular physics is the following: given a symmetry group G and a collection of objects spanning an initial reducible representation Γ_i, it is asked to build from these elements new objects that transform according to a given final irreducible representation (irrep) Γ_f. The objects considered here will be multivariate polynomials whose variables are the (x_i, y_i) components of n vectors in a plane with a common origin.

A common procedure to build symmetry-adapted polynomials in molecular physics uses the projector or Reynolds operator to project to a given irrep [5, 7] together with the shift operator when the dimension of the final irrep is greater than one. An improved method of construction is based on the Clebsch–Gordan coefficients of the considered group. However, neither technique takes into account the global algebraic structure of the problem.

Invariant theory, on the contrary, takes care of the algebraic aspects of the problem [8, 9]. In particular, the invariants, *i.e.*, the objects transforming according to the totally symmetric representation Γ_0 of the group, have a structure of a ring. The objects are otherwise said to be Γ-covariants when they transform according to an irrep Γ different from Γ_0. The Γ-covariants have a structure of a module over a ring of invariants. Sometimes, it is convenient to discuss the ring of invariants as the module of Γ_0-covariants.

Some information about the algebraic structure of the Γ_f-covariants can be encoded in a formal power series. Let c_k denote the dimension of the vector space of Γ_f-covariant multivariate polynomials of total degree k built from the elementary bricks that span the reducible representation Γ_i of group G. The formal power series in the dummy variable λ written as:

$$c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + \cdots = M^G\left(\Gamma_f; \Gamma_i; \lambda\right),$$

defines in the right-hand side the Molien function. One interesting fact is that the Molien function can be directly determined without any knowledge of the c_k numbers [10, 11]:

$$M^G\left(\Gamma_f; \Gamma_i; \lambda\right) = \frac{1}{|G|} \sum_{g \in G} \frac{\bar{\chi}\left(\Gamma_f; g\right)}{\det\left(1_{n\times n} - \lambda D^{\Gamma_i}\left(g\right)\right)}, \tag{1}$$

where the sum in (1) runs over the $|G|$ elements of the group G, $\bar{\chi}\left(\Gamma_f; g\right)$ is the complex conjugate of the character of element g in the final irreducible representation Γ_f, $1_{n\times n}$ is the $n\times n$ unit matrix and $D^{\Gamma_i}\left(g\right)$ is the $n\times n$ matrix representation of element g in the initial reducible representation Γ_i.

The two different symbolic interpretations of the Molien function are illustrated in the next section on the trivial example of the C_i point group with a single vector in the plane. We then deal with the SO(2) group acting on two vectors in the plane and discuss the existence of free and non-free modules.

2. A trivial example

2.1. Point group C_i

Let $C_i = \{e, i\}$ be the two-element group with e the identity operation and i the central inversion which reverses all the coordinates of a point in the plane: $(x, y) \xrightarrow{i} (-x, -y)$. The problem is to find the first terms (ordered by the total degree in x and y) that may appear in the Taylor series of a C_i-invariant function at $(x = 0, y = 0)$: $f_{A_1}(x, y) = f_{A_1}(-x, -y)$.

The character table of the group C_i has only two irreps: the irrep A_1 is the totally symmetric representation and A_2 denotes the unique covariant irrep of the group C_i. The initial reducible irrep spanned by x and y is $\Gamma_i = A_2 \oplus A_2$ and its 2×2 representation matrices are:

$$D^{\Gamma_i}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^{\Gamma_i}(i) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Molien function for the C_i-invariant polynomials built from the x and y variables is easily determined through formula (1):

$$M^{C_i}(A_1; A_2 \oplus A_2; \lambda) = \frac{1 - \lambda^4}{(1 - \lambda^2)^3} = \frac{1 + \lambda^2}{(1 - \lambda^2)^2}. \tag{2}$$

Both rational functions have obviously the same Taylor expansion:

$$M^{C_i}(A_1; A_2 \oplus A_2; \lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots = \lambda^0 + 3\lambda^2 + 5\lambda^4 + 7\lambda^6 + 9\lambda^8 + \cdots,$$

where c_k indicates the number of linearly independent polynomials of degree k or the dimension of the vector space of invariant polynomials of degree k, see Table 1.

TABLE 1. A basis for C_i-invariant polynomials in (x, y) up to degree eight.

degree k	linearly independent polynomials	dimension
0	1	1
2	x^2, xy, y^2	3
4	x^4, , x^3y, x^2y^2, xy^3, y^4	5
6	x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6	7
8	x^8, x^7y, x^6y^2, x^5y^3, x^4y^4, x^3y^5, x^2y^6, xy^7, y^8	9

The next two subsections show that each of the two rational functions in (2) has its own symbolic interpretation.

2.2. Symbolic interpretation à la Hilbert

The rational function

$$\frac{1 - \lambda^4}{(1 - \lambda^2)^3}, \tag{3}$$

allows for a symbolic interpretation à la Hilbert. Its denominator is a product of three terms $(1 - \lambda^2)$, suggesting that the C_i-invariant polynomials may be built as polynomial functions of three polynomial generators of degree two, which are chosen as x^2, y^2 and xy.

However, it is easy to see that some C_i-invariant polynomials may be formed in two different ways from these generators. The polynomial x^2y^2 can, for example, be seen as the square of the generator xy or as the product of the generators x^2 and y^2. In the words of invariant theory, the three generators x^2, y^2 and xy form a syzygy of degree four:

$$\left(x^2\right)\left(y^2\right) = (xy)^2 .$$

Without taking into account the syzygy, the number of linearly independent polynomials of given degree k is overcounted. The $\left(-\lambda^4\right)$ term in the numerator of (3) indicates that enumeration of the C_i-invariant polynomials with the syzygy of degree four in mind gives the correct result.

2.3. Symbolic interpretation of the Molien function in terms of integrity basis

The rational function

$$\frac{1+\lambda^2}{\left(1-\lambda^2\right)^2}, \tag{4}$$

is written as a product of two terms in the denominator and a sum of two terms with positive coefficients in the numerator. A symbolic interpretation can be assigned to (4) if we note that the results in Table 1 hint at the decomposition (5) of the ring of C_i-invariant polynomials in x and y:

$$\mathbb{C}[x,y]^{C_i} = \mathbb{C}[\theta_1,\theta_2]1 \oplus \mathbb{C}[\theta_1,\theta_2]\varphi_0, \tag{5}$$

as a free module over the subring of invariants $\mathbb{C}[\theta_1,\theta_2]$, with $\theta_1(x,y) = x^2$ and $\theta_2(x,y) = y^2$. The polynomials θ_1 and θ_2 can then be found with any positive exponent. They are associated to the two terms of degree two in the denominator of the rational function (4) and are called *denominator* polynomials. The polynomials 1 and $\varphi_0(x,y) = xy$ are the basis of the module. Decomposition (5) indicates that they only appear linearly and are respectively related to the terms of degree zero and two in the numerator of (4). They are named *numerator* polynomials.

The sets of denominator and numerator C_i-invariant polynomials $\left\{\{\theta_1,\theta_2\},\right.$ $\left.\{1,\varphi_1\}\right\}$ form an integrity basis or a homogeneous system of parameters for the ring of C_i-invariants. This decomposition of the ring of invariants is called a Hironaka decomposition and such a ring is an example of a Cohen–Macaulay ring.

In general, for a group G acting on n variables $x_1,\dots,x_n$, the form (6) suggests an integrity basis with D denominator polynomials and N numerator polynomials:

$$M^G\left(\Gamma_f;\Gamma_i;\lambda\right) = \frac{\sum_{k=1}^{k=N}\lambda^{\nu_k}}{\prod_{k=1}^{k=D}\left(1-\lambda^{\delta_k}\right)},\ \nu_k\in\mathbb{N},\ \delta_k\in\mathbb{N}_0=\mathbb{N}\backslash\{0\}. \tag{6}$$

Then any G-invariant polynomial p admits a unique decomposition as a polynomial in denominator and numerator polynomials, reminding the constraint that the numerator polynomials only appear linearly:

$$p\left(x_1,\ldots,x_n\right)=\sum_{k=1}^{N} p_k\left(\theta_1\left(x_1,\ldots,x_n\right),\ldots,\theta_D\left(x_1,\ldots,x_n\right)\right)\times\varphi_k\left(x_1,\ldots,x_n\right),$$

which corresponds to the Cohen–Macaulay decomposition of the ring of invariants:

$$\mathbb{C}\left[x_1,\ldots,x_n\right]^G=\mathbb{C}\left[\theta_1,\ldots,\theta_D\right]\varphi_1+\cdots+\mathbb{C}\left[\theta_1,\ldots,\theta_D\right]\varphi_N.$$

The set of invariants is a free module over the subring $\mathbb{C}[\theta_1,\ldots,\theta_D]$ of denominator invariants.

2.4. Case of covariants

The Molien function for the A_2-covariant polynomials in x and y is computed via formula (1):

$$M^{C_i}\left(A_2;A_2\oplus A_2;\lambda\right)=\frac{2\lambda}{\left(1-\lambda^2\right)^2}.$$

This form can be correlated to an integrity basis: the two C_i-invariant denominator polynomials of degree two are chosen as $\theta_1\left(x,y\right)=x^2$, $\theta_2\left(x,y\right)=y^2$, and the two A_2-covariant numerator polynomials of degree one as $\varphi_1\left(x,y\right)=x$, $\varphi_2\left(x,y\right)=y$. Again, the set of the A_2-covariants has a structure of free module over the subring $\mathbb{C}[\theta_1,\theta_2]$ of invariants. The Cohen–Macaulay decomposition of the module of A_2-covariants reads as:

$$\mathbb{C}[\theta_1,\theta_2]\varphi_1+\mathbb{C}[\theta_1,\theta_2]\varphi_2.$$

This decomposition is general: for a finite group, it is possible to write the ring of invariants or a module of covariants as a module with a Cohen–Macaulay decomposition [8, 9].

3. Invariants and covariants of SO(2)

The elements of the group SO(2) are parametrized by one angle φ, $0\leq\varphi<2\pi$. The sum over the group elements in the Molien formula (1) is replaced by an integral over the angle (Haar measure). It is known that generally the invariant rings of reductive groups are Cohen–Macaulay [12]. The character table of the SO(2) group is given in Table 2. There is an infinite number of irreps: the irrep (0) corresponds to the invariants, while the irrep (m), $m\in\mathbb{Z}$, corresponds to the (m)-covariants.

Table 2. Character table of the group SO(2), $m\in\mathbb{Z}$. The C_φ is the rotation around the z axis by an angle φ.

	e	C_φ
(0)	1	1
(m)	1	$e^{im\varphi}$

Let us consider two vectors of the plane, respectively of coordinates (x_1, y_1) and (x_2, y_2), in a Cartesian basis $(\vec{e}_x, \vec{e}_y)$. They span the initial reducible representation $\Gamma_i = (1) \oplus (-1) \oplus (1) \oplus (-1)$. The problem is to construct integrity bases for the (m)-covariants in the (x_1, y_1, x_2, y_2) coordinates [13, 14].

Table 3 gives the expressions of the Molien functions $M^{\mathrm{SO}(2)}(\Gamma_f; \Gamma_i; \lambda)$ for the final irreducible representation $\Gamma_f = (m)$, $0 \le m \le 3$. All the functions in Table 3 have a symbolic interpretation in terms of integrity basis.

Table 3. Expressions of the Molien functions $M^{\mathrm{SO}(2)}(\Gamma_f; \Gamma_i; \lambda)$ for the final irreducible representation $\Gamma_f = (m)$, $0 \le m \le 3$ and decomposition of the module. The expression of the polynomials are $\theta_1(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2$, $\theta_2(x_1, y_1, x_2, y_2) = x_2^2 + y_2^2$, $\theta_3(x_1, y_1, x_2, y_2) = x_1 x_2 + y_1 y_2$, $\varphi_1(x_1, y_1, x_2, y_2) = 1$, $\varphi_2(x_1, y_1, x_2, y_2) = x_1 y_2 - x_2 y_1$, and $\pi_j = \pi(x_j, y_j) = x_j - i y_j$, $j \in \mathbb{N}$.

Γ_f	$M^{\mathrm{SO}(2)}(\Gamma_f; \Gamma_i; \lambda)$	Module	Decomposition
(0)	$\frac{1+\lambda^2}{(1-\lambda^2)^3}$	free	$\mathbb{C}[\theta_1, \theta_2, \theta_3]\varphi_1 \oplus \mathbb{C}[\theta_1, \theta_2, \theta_3]\varphi_2$
(1)	$\frac{2\lambda}{(1-\lambda^2)^3}$	free	$\mathbb{C}[\theta_1, \theta_2, \theta_3]\pi_1 \oplus \mathbb{C}[\theta_1, \theta_2, \theta_3]\pi_2$
(2)	$\frac{2\lambda^2}{(1-\lambda^2)^3} + \frac{\lambda^2}{(1-\lambda^2)^2}$	non-free	$\mathbb{C}[\theta_1, \theta_2, \theta_3]\pi_1^2 \oplus \mathbb{C}[\theta_1, \theta_2, \theta_3]\pi_1\pi_2$ $\oplus \mathbb{C}[\theta_2, \theta_3]\pi_2^2$
(3)	$\frac{2\lambda^3}{(1-\lambda^2)^3} + \frac{2\lambda^3}{(1-\lambda^2)^2}$	non-free	$\mathbb{C}[\theta_1, \theta_2, \theta_3]\pi_1^3 \oplus \mathbb{C}[\theta_1, \theta_2, \theta_3]\pi_1^2\pi_2$ $\oplus \mathbb{C}[\theta_2, \theta_3]\pi_1\pi_2^2 \oplus \mathbb{C}[\theta_2, \theta_3]\pi_2^3$

3.1. Free modules: invariants and (1)-covariants

3.1.1. Invariants. The expression in Table 3 of the Molien function for invariants ($\Gamma_f = (0)$) suggests an integrity basis for invariants consisting of three denominator polynomials of degree two: θ_1, θ_2, θ_3, and two numerator polynomials: one polynomial of degree zero, φ_1, and one polynomial of degree two: φ_2. This choice of polynomials θ_1, θ_2, θ_3, φ_1, φ_2 is of course not unique. The set of invariants is a free module over the subring of denominator invariants $\mathbb{C}[\theta_1, \theta_2, \theta_3]$.

Any SO(2)-invariant in the (x_1, y_1, x_2, y_2) coordinates is uniquely decomposed as a $\mathbb{C}$-linear combination of $\theta_1^{n_1}\theta_2^{n_2}\theta_3^{n_3}\varphi_i$, $i \in \{1, 2\}$:

$$\sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} c^{(1)}_{n_1,n_2,n_3}\theta_1^{n_1}\theta_2^{n_2}\theta_3^{n_3}\varphi_1 + \sum_{(n_1,n_2,n_3)\in\mathbb{N}^3} c^{(2)}_{n_1,n_2,n_3}\theta_1^{n_1}\theta_2^{n_2}\theta_3^{n_3}\varphi_2. \quad (7)$$

A simple geometric representation of the vector space generated by the $\theta_1^{n_1}\theta_2^{n_2}\theta_3^{n_3}\varphi_i$ polynomials is proposed: each of these polynomials is represented by a point of coordinates (n_1, n_2, n_3) in the three-dimensional lattice associated to either φ_1 or φ_2, see Figure 1. With such a correspondence, the set of all polynomials in (7) can be pictorially described as the two three-dimensional lattices of Figure 2.

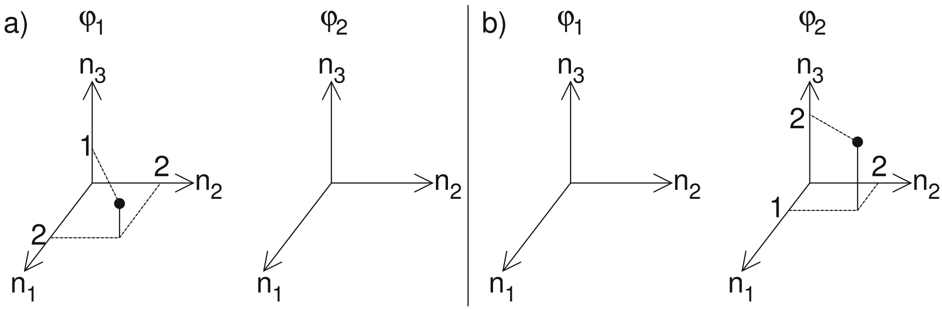

FIGURE 1. a) Polynomial $\theta_1^2\theta_2^2\theta_3\varphi_1$. b) Polynomial $\theta_1\theta_2^2\theta_3^2\varphi_2$.

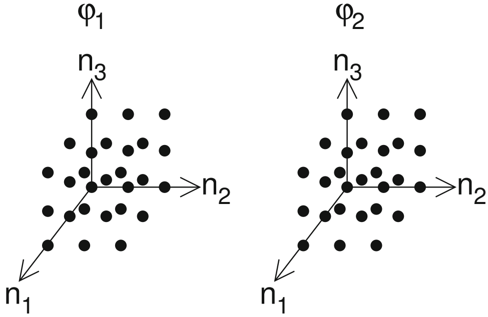

FIGURE 2. Lattice for the $\mathbb{C}$-basis of invariants built from two planar vectors.

3.1.2. (1)-covariants. The integrity basis for (1)-covariants consists of three denominator polynomials of degree two: θ_1, θ_2, θ_3, and two numerator polynomials of degree one: π_1, π_2. The module of (1)-covariants is free and we have a Cohen–Macaulay decomposition of the module, see Table 3. Similar to the invariant case, the graphical representation of the basis of (1)-covariant polynomials is just two copies of a three-dimensional lattice, see Figure 3.

3.2. Non-free modules of (m)-covariants, $m \geq 2$

3.2.1. (2)-covariants. A naive generalization of the results for invariants and (1)-covariants would lead us to decompose the module of (2)-covariants as a module over the subring $\mathbb{C}[\theta_1, \theta_2, \theta_3]$ with π_1^2, $\pi_1\pi_2$ and π_2^2 as a basis. However, a careful inspection shows the relation (8) between (2)-covariants:

$$2\theta_3\pi_1\pi_2 - \theta_2\pi_1^2 - \theta_1\pi_2^2 = 0, \tag{8}$$

which indicates that the module of (2)-covariants is not free. It is impossible to rewrite the Molien function as a unique rational function with a finite product of terms in its denominator and a sum of terms with positive coefficients in its numerator. However, the Molien function for (2)-covariants can be rewritten as

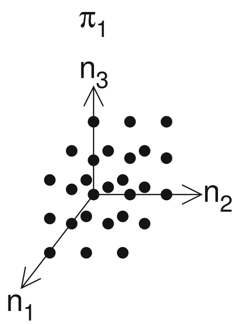

FIGURE 3. Lattice for the $\mathbb{C}$-basis of (1)-covariants built from planar vectors.

a sum of two such rational functions, see Table 3. The second rational function has one term less in the denominator. We associate to the first rational function $\frac{2\lambda^2}{(1-\lambda^2)^3}$ three denominator polynomials of degree two: θ_1, θ_2, θ_3 and two numerators polynomials of degree two: π_1^2, $\pi_1\pi_2$. We associate to the second rational function $\frac{\lambda^2}{(1-\lambda^2)^2}$ only two denominator polynomials of degree two: θ_2, θ_3 and one numerator polynomial of degree two: π_2^2.

The graphical representation of the $\mathbb{C}$-basis of (2)-covariants is made of two three-dimensional lattices and one two-dimensional lattice, see Figure 4.

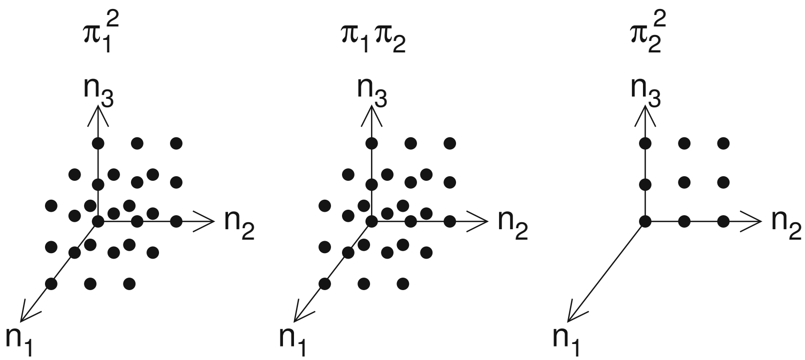

FIGURE 4. Lattice of the $\mathbb{C}$-basis of (2)-covariants built from two planar vectors.

3.2.2. (3)-covariants. The module of (3)-covariants is not free either due to the relations between π_1^3, $\pi_1^2\pi_2$, $\pi_1\pi_2^2$, and π_2^3:

$$2\theta_3\pi_1\pi_2^2 - \theta_1\pi_2^3 - \theta_2\pi_1^2\pi_2 = 0, \quad 2\theta_3\pi_1^2\pi_2 - \theta_1\pi_1\pi_2^2 - \theta_2\pi_1^3 = 0.$$

The first rational function $\frac{2\lambda^3}{(1-\lambda^2)^3}$ suggests θ_1, θ_2, θ_3 as the three denominator polynomials of degree two and the two numerator polynomials of degree two are chosen as π_1^3, $\pi_1^2\pi_2$. As for the (2)-covariants case, the two denominator polynomials of the second rational function are chosen as θ_2 and θ_3. The two numerator polynomials of degree two are selected as $\pi_1\pi_2^2$, π_2^3. The $\mathbb{C}$-basis of (3)-covariants is then graphically represented by two three-dimensional lattices and two two-dimensional lattices, see Figure 5.

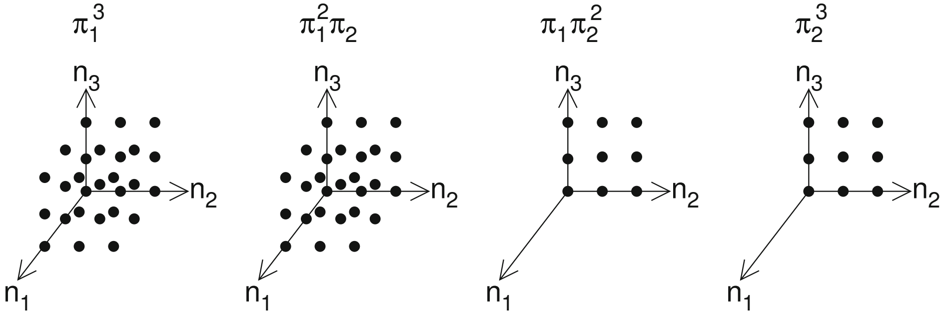

FIGURE 5. Lattice of the $\mathbb{C}$-basis of (3)-covariants built from two planar vectors.

4. Conclusion

The power series expansion of the Molien function gives information on the dimension of the linear vector space of Γ-covariant polynomials of degree k. For finite point groups, the Molien function can be rewritten as a single rational function whose canonical form is different whether a symbolic interpretation à la Hilbert or in terms of integrity basis is looked after.

For the compact Lie group SO(2), we found for the case of $n = 2$ vectors in the plane that a structure of free module exists for the invariants and (1)-covariants: the expression of the Molien function suitable to an interpretation in terms of integrity basis can be written as one rational function and there is a Cohen–Macaulay decomposition over a subring of invariants generated by three denominator invariants.

In contrast, the (m)-covariants, $m \geq 2$ have a non-free module structure: the Molien function has to be written as a sum of two rational functions with a different number of terms in the denominator. The concept of integrity basis has to be generalized and we introduced subrings of invariants generated by only two denominator invariants next to the subrings of invariants generated by three denominator invariants.

This change from a free module structure to a non-free module structure occurs in SO(2) for higher values of n too [13, 14] and in an extension of the problem to SO(3) [15]. Further exploration is required for a better understanding of these algebraic structures induced by physical systems. In particular, the Stanley decomposition of [16] may be useful.

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Guillaume Dhont and Boris I. Zhilinskií
Laboratoire de Physico-Chimie de l'Atmosphère
Université du Littoral Côte d'Opale
59140 Dunkerque, France
e-mail: guillaume.dhont@univ-littoral.fr
zhilin@univ-littoral.fr

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Momentum Maps for Smooth Projective Unitary Representations

Bas Janssens and Karl-Hermann Neeb

Abstract. For a smooth projective unitary representation $(\overline{\rho}, \mathcal{H})$ of a locally convex Lie group G, the projective space $\mathbb{P}(\mathcal{H}^\infty)$ of smooth vectors is a locally convex Kähler manifold. We show that the action of G on $\mathbb{P}(\mathcal{H}^\infty)$ is weakly Hamiltonian, and lifts to a Hamiltonian action of the central $\mathrm{U}(1)$-extension $G^\sharp$ obtained from the projective representation. We identify the non-equivariance cocycles obtained from the weakly Hamiltonian action with those obtained from the projective representation, and give some integrality conditions on the image of the momentum map.

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Keywords. Momentum map, projective representation.

1. Introduction

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$, and let $\overline{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$ be a projective unitary representation of G. It is called *smooth* if the set $\mathbb{P}(\mathcal{H})^\infty$ of smooth rays is dense in $\mathbb{P}(\mathcal{H})$, a ray $[\psi] \in \mathbb{P}(\mathcal{H})$ being called smooth if its orbit map $G \to \mathbb{P}(\mathcal{H})\colon g \mapsto \overline{\rho}(g)[\psi]$ is smooth. For finite-dimensional Lie groups, a projective representation is smooth if and only if it is continuous. For infinite-dimensional Lie groups, smoothness is a natural requirement.

In [3, Theorem 4.3], we showed that for smooth projective unitary representations, the central extension

$$G^\sharp := \{(g, U) \in G \times \mathrm{U}(\mathcal{H})\,;\, \overline{\rho}(g) = [U]\}$$

of G by $\mathrm{U}(1)$ is a central extension of locally convex Lie groups, in the sense that the projection $G^\sharp \to G$ is a homomorphism of Lie groups, as well as a principal $\mathrm{U}(1)$-bundle. Moreover, the projective representation $\overline{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$ of G then lifts to

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a linear representation $\rho\colon G^\sharp \to \mathrm{U}(\mathcal{H})$ of $G^\sharp$, with the property that $\rho(z) = z\mathbf{1}$ for all $z \in \mathrm{U}(1)$. If $\mathcal{H}^\infty \subseteq \mathcal{H}$ is the space of smooth vectors for ρ, then $\mathbb{P}(\mathcal{H}^\infty)$ is equal to $\mathbb{P}(\mathcal{H})^\infty$, the space of smooth rays for $\overline{\rho}$.

The main goal of these notes is to reinterpret this central extension in the context of symplectic geometry of the projective space $\mathbb{P}(\mathcal{H}^\infty)$ and its prequantum line bundle, the tautological bundle $\mathbb{L}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$. In order to equip $\mathbb{P}(\mathcal{H}^\infty)$ with a symplectic structure, we need to consider it as a locally convex manifold. For this, we need a locally convex topology on $\mathcal{H}^\infty$ that is compatible with the $G^\sharp$-action.

Definition 1. The *strong topology* on $\mathcal{H}^\infty$ is the locally convex topology induced by the norm on $\mathcal{H}$ and the seminorms

$$p_B(\psi) := \sup_{\boldsymbol{\xi}\in B} \|\mathtt{d}\rho_k(\boldsymbol{\xi})\psi\|\,,$$

where $B \subseteq (\mathfrak{g}^\sharp)^k$, $k \in \mathbb{N}$, runs over the bounded sets, and the derived representation $\mathtt{d}\rho$ of $\mathfrak{g}^\sharp$ is extended to $(\mathfrak{g}^\sharp)^k$ by $\mathtt{d}\rho_k(\xi_1,\dots,\xi_k) := \mathtt{d}\rho(\xi_1)\cdots\mathtt{d}\rho(\xi_k)$.

We will show that with this topology, $\mathbb{P}(\mathcal{H}^\infty)$ becomes a locally convex Kähler manifold with prequantum line bundle $\mathbb{L}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$. If we identify its tangent space $T_{[\psi]}\mathbb{P}(\mathcal{H}^\infty)$ for any unit vector ψ with $\{\delta v \in \mathcal{H}^\infty\,;\, \langle\psi,\delta v\rangle = 0\}$, then the symplectic form Ω on $\mathbb{P}(\mathcal{H}^\infty)$ is given by

$$\Omega_{[\psi]}(\delta v, \delta w) = 2\mathrm{Im}(\delta v, \delta w)\,.$$

Similarly, the sphere $\mathbb{S}(\mathcal{H}^\infty)$ becomes a locally convex principal $\mathrm{U}(1)$-bundle over $\mathbb{P}(\mathcal{H}^\infty)$, to which the prequantum line bundle $\mathbb{L}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$ is associated along the canonical representation $\mathrm{U}(1) \to \mathrm{GL}(\mathbb{C})$. The connection ∇ on $\mathbb{L}(\mathcal{H}^\infty)$ with curvature $R_\nabla = \Omega$ is associated to the connection 1-form α on $\mathbb{S}(\mathcal{H}^\infty)$, given by

$$\alpha_\psi(\delta v) = -i\langle\psi,\delta v\rangle$$

under the identification $T_\psi\mathbb{S}(\mathcal{H}^\infty) \simeq \{\delta v \in \mathcal{H}^\infty\,;\, \mathrm{Re}\langle\psi,\delta v\rangle = 0\}$.

The group G acts on $\mathbb{P}(\mathcal{H}^\infty)$ by Kähler automorphisms, hence in particular by symplectomorphisms. The action of the central extension $G^\sharp$ lifts to $\mathbb{L}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$, on which it acts by holomorphic quantomorphisms (connection preserving bundle automorphisms). If the projective representation of G is faithful, then the central extension $G^\sharp$ is precisely the group of quantomorphisms of $(\mathbb{L}(\mathcal{H}^\infty),\nabla)$ that cover the G-action on $(\mathbb{P}(\mathcal{H}^\infty),\Omega)$.

$$\begin{array}{ccc} G^\sharp & \circlearrowright & \mathrm{Aut}(\mathbb{L}(\mathcal{H}^\infty),\nabla) \\ \downarrow & & \downarrow \\ G & \circlearrowright & \mathrm{Aut}(\mathbb{P}(\mathcal{H}^\infty),\Omega). \end{array}$$

We show that for any locally convex Lie group G, the action

$$G \times \mathbb{P}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$$

obtained from a smooth projective unitary representation is *separately smooth* in the following sense.

Definition 2. An action $\alpha\colon G \times \mathcal{M} \to \mathcal{M}, (g,m) \mapsto \alpha_g(m)$ of a locally convex Lie group G on a locally convex manifold $\mathcal{M}$ is called *separately smooth* if for every $g \in G$ and $m \in M$, the orbit map $\alpha_m\colon G \to \mathcal{M}, g \mapsto \alpha_g(m)$ and the action maps $\alpha_g\colon \mathcal{M} \to \mathcal{M}$ are smooth.

For Banach Lie groups G, the action is a smooth map $G \times \mathbb{P}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$ by [11, Theorem 4.4], but a certain lack of smoothness is unavoidable as soon as one goes to Fréchet–Lie groups. Indeed, consider the unitary representation of the Fréchet–Lie group $G = \mathbb{R}^{\mathbb{N}}$ on $\mathcal{H} = \ell^2(\mathbb{N}, \mathbb{C})$, defined by $\rho(\phi)\psi = (e^{i\phi_1}\psi_1, e^{i\phi_2}\psi_2, \ldots)$. Then $\mathcal{H}^\infty = \mathbb{C}^{(\mathbb{N})}$ with the direct limit topology [3, Example 3.11], but by [11, Example 4.8], the action of $\mathfrak{g}$ on $\mathbb{C}^{\mathbb{N}}$ is discontinuous for *any* locally convex topology on $\mathcal{H}^\infty$. We therefore propose the following definition of (not necessarily smooth) Hamiltonian actions on locally convex manifolds.

Definition 3. An action $\alpha\colon G \times \mathcal{M} \to \mathcal{M}$ of a locally convex Lie group G on a locally convex, symplectic manifold $(\mathcal{M}, \Omega)$ is called:

- *Symplectic* if it is separately smooth, and $\alpha_g^*\Omega = \Omega$ for all $g \in G$.
- *Weakly Hamiltonian* if it is symplectic, and $i_{X_\xi}\Omega$ is exact for all $\xi \in \mathfrak{g}$, where X_ξ is the fundamental vector field of ξ on $\mathbb{P}(\mathcal{H}^\infty)$.
- *Hamiltonian* if, moreover, $i_{X_\xi}\Omega = d\mu(\xi)$ for a G-equivariant momentum map $\mu\colon \mathcal{M} \to \mathfrak{g}'$ into the continuous dual of $\mathfrak{g}$, which is smooth if $\mathfrak{g}'$ is equipped with the topology of uniform convergence on bounded subsets.

Our main result is that the action of $G^\sharp$ on $\mathbb{P}(\mathcal{H}^\infty)$ is Hamiltonian in the sense of the above definition.

Theorem 4. *The action of $G^\sharp$ on $(\mathbb{P}(\mathcal{H}^\infty), \Omega)$ is Hamiltonian, with momentum map $\mu\colon \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}^{\sharp\prime}$ given by*

$$\mu_{[\psi]}(\xi) = \frac{\langle \psi, i\mathtt{d}\rho(\xi)\psi\rangle}{\langle \psi, \psi\rangle}\,. \tag{1}$$

Since the G-action on $\mathbb{P}(\mathcal{H}^\infty)$ factors through the action of $G^\sharp$, we immediately obtain the following corollary.

Corollary 5. *The action of G on $(\mathbb{P}(\mathcal{H}^\infty), \Omega)$ is weakly Hamiltonian.*

Note that the (classical) momentum associated to $\xi \in \mathfrak{g}^\sharp$ at $[\psi] \in \mathbb{P}(\mathcal{H}^\infty)$ is precisely the corresponding (quantum mechanical) expectation of the self-adjoint operator (observable) $i\mathtt{d}\rho(\xi)$ in the state $[\psi]$.

Sections 2 and 3 of this paper are concerned with the proof of Theorem 4. In Section 2, we show in detail that $\mathbb{P}(\mathcal{H}^\infty)$ is a locally convex, prequantisable Kähler manifold, and in Section 3, we use this to show that the action of $G^\sharp$ on $\mathbb{P}(\mathcal{H}^\infty)$ is Hamiltonian, and lifts to the prequantum line bundle $\mathbb{L}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$.

In the second half of this paper, we give some applications of this symplectic picture to projective representations. In Section 4, we calculate the Kostant–Souriau cocycles associated to the Hamiltonian action, and show that these are precisely the Lie algebra cocycles that one canonically obtains from a projective unitary representation and a smooth ray, cf. [3]. We then prove an integrality result for characters of the stabilizer group that one obtains as the image of the momentum map. Finally, in Section 5, we close with some remarks on smoothness of the action in the context of diffeological spaces.

Momentum maps have been introduced into representation theory by Norman Wildberger [13]. Studying the image of the momentum map has proven to be an extremely powerful tool in the analysis of unitary representations, in particular to obtain information on upper and lower bounds of spectra ([1], [7, 8, 10]). Smoothness properties of the linear Hamiltonian action on the space $\mathcal{H}^\infty$ of smooth vectors of a unitary representation and the corresponding momentum map $\mu_\psi(\xi) := \langle\psi, \mathtt{d}\pi(\xi)\psi\rangle$ have been studied by P. Michor in the context of convenient calculus in [5].

2. The locally convex symplectic space $\mathbb{P}(V)$

In order to equip $\mathbb{P}(\mathcal{H}^\infty)$ with a symplectic structure, we need to consider it as a locally convex manifold (in the sense of [4, Def. 9.1]). Later on, it will be important to choose a locally convex topology on $\mathcal{H}^\infty$ that is compatible with the group action, but for now, it suffices if the scalar product on $\mathcal{H}^\infty \subseteq \mathcal{H}$ is continuous. We will go through the standard constructions of projective geometry, using only a complex, locally convex space V with continuous hermitian scalar product $\langle\,\cdot\,,\,\cdot\,\rangle\colon V \times V \to \mathbb{C}$ (antilinear in the first and linear in the second argument).

Proposition 6. *The projective space $\mathbb{P}(V)$ is a complex manifold modelled on locally convex spaces. The tautological line bundle $\mathbb{L}(V) \to \mathbb{P}(V)$ is a locally convex, holomorphic line bundle over $\mathbb{P}(V)$.*

Proof. We equip $\mathbb{P}(V)$ with the Hausdorff topology induced by the quotient map $V - \{0\} \to \mathbb{P}(V)$. The open neighbourhood

$$U_{[\psi]} := \{[\chi] \in \mathbb{P}(V)\,;\, \langle\psi,\chi\rangle \neq 0\} \tag{2}$$

is then charted by the hyperplane

$$T_{[\psi]} := \{v \in V\,;\, \langle\psi, v\rangle = 0\}\,, \tag{3}$$

and the chart $\kappa_\psi\colon U_{[\psi]} \to T_{[\psi]}$, defined by $\kappa_\psi([\chi]) := \langle\psi,\chi\rangle^{-1}\chi - \psi$ (cf. [9, §V.1]). Note that the map κ_ψ depends on the choice of representative $\psi \in [\psi]$, which we will assume to be of unit length. The inverse chart is $\kappa_\psi^{-1}(v) = [\psi + v]$. We have $\kappa_{z\psi}([\chi]) = z\kappa_\psi([\chi])$ for $z \in \mathrm{U}(1)$. More generally, the transition function $\kappa_{\psi\psi'}\colon \kappa_\psi(U_{[\psi]} \cap U_{[\psi']}) \to \kappa_{\psi'}(U_{[\psi]} \cap U_{[\psi']})$ is given by $v \mapsto \frac{\psi+v}{\langle\psi',\psi+v\rangle} - \psi'$. Since $v \mapsto \langle\psi', \psi + v\rangle$ is continuous and nonzero on $\kappa_\psi(U_{[\psi]} \cap U_{[\psi']})$, the transition functions are holomorphic, making $\mathbb{P}(V)$ into a complex manifold.

For $\mathbb{L}(V) = V - \{0\}$, define the charts $\Lambda_\psi\colon T_{[\psi]} \times \mathbb{C} \to \mathbb{L}(V)$ by $\Lambda_\psi(v, z) = z(\psi + v)$. The transition functions

$$\Lambda_{\psi\psi'}\colon \kappa_\psi(U_{[\psi]} \cap U_{[\psi']}) \times \mathbb{C} \to \kappa_{\psi'}(U_{[\psi]} \cap U_{[\psi']}) \times \mathbb{C}$$

are given by $(v, z) \mapsto (\kappa_{\psi\psi'}(v), \langle \psi', \psi + v\rangle z)$. Since these are holomorphic isomorphisms of trivial locally convex line bundles, the result follows. □

For $\|\psi\| = 1$, we identify the tangent vectors $\delta v, \delta w \in T_{[\psi]}\mathbb{P}(V)$ at the point $[\psi] \in \mathbb{P}(V)$ with their coordinates $\delta v, \delta w \in T_{[\psi]}$ by the tangent map $T_{[\psi]}(\kappa_\psi)$. Accordingly, we define the Hermitean form H on $\mathbb{P}(V)$ by

$$H_{[\psi]}(\delta v, \delta w) := 2\langle \delta v, \delta w\rangle\,. \tag{4}$$

Note that this does not depend on the choice of chart κ_ψ.

Proposition 7. *Equipped with the Hermitean forms $H_{[\psi]}$ of equation* (4), $\mathbb{P}(V)$ *is a Hermitean manifold.*

Proof. As compatibility with the complex structure $J(\delta v) = i\delta v$ is clear, the only thing to show is that H is smooth. Using that the transition map

$$D_v\kappa_{\psi\psi'} \colon T_{[\psi]} \to T_{[\psi']}, \qquad \text{for} \qquad \psi' = \frac{\psi + v}{\|\psi + v\|}$$

is given by

$$D_v\kappa_{\psi,\psi'}(\delta v) = \tfrac{1}{\|\psi+v\|}\left(\delta v - \langle \tfrac{\psi+v}{\|\psi+v\|}, \delta v\rangle \tfrac{\psi+v}{\|\psi+v\|}\right)\,,$$

one sees that in local coordinates for $T^2\mathbb{P}(V)$, the map $T_{[\psi]} \times T_{[\psi]} \times T_{[\psi]} \to \mathbb{C}$ is

$$H_v(\delta v, \delta w) = 2\left(\frac{1}{1 + \|v\|^2}\langle \delta v, \delta w\rangle - \frac{1}{(1 + \|v\|^2)^2}\langle \delta v, v\rangle\langle v, \delta w\rangle\right)\,, \tag{5}$$

which is evidently smooth. □

As the real and imaginary parts of H, we obtain the Fubini–Study metric

$$G_{[\psi]}(\delta v, \delta w) = 2\mathrm{Re}\langle \delta v, \delta w\rangle$$

and the 2-form

$$\Omega_{[\psi]}(\delta v, \delta w) = 2\mathrm{Im}\langle \delta v, \delta w\rangle\,. \tag{6}$$

The 2-form Ω is nondegenerate in the 'weak' sense that $\Omega(\delta v, \delta w) = 0$ for all δw implies $\delta v = 0$. In order to show that Ω is a symplectic form, and hence that $\mathbb{P}(V)$ is Kähler, it thus suffices to prove that it is closed. We will do this by showing that Ω is the curvature of a prequantum bundle.

Proposition 8. *The sphere* $\mathbb{S}(V) = \{\psi \in V\,;\, \|\psi\| = 1\}$ *is a locally convex manifold, and the projection* $\mathbb{S}(V) \to \mathbb{P}(V)$ *is a principal* $\mathrm{U}(1)$*-bundle.*

Proof. The sphere inherits the Hausdorff topology from its inclusion in V. The locally convex space

$$T_\psi := \{v \in V\,;\, \mathrm{Re}\langle \psi, v\rangle = 0\} \subseteq V \tag{7}$$

can be naturally identified with the open neighbourhood

$$U_\psi := \{\chi \in \mathbb{S}(V)\,;\, \mathrm{Re}\langle \psi, \chi\rangle > 0\} \tag{8}$$

of ψ by the chart $\kappa\colon U_\psi \to T_\psi$ with $\kappa_\psi(\chi) = (\mathrm{Re}\langle \psi, \chi\rangle)^{-1}\chi - \psi$, which has inverse $\kappa_\psi^{-1}(v) = \frac{\psi+v}{\|\psi+v\|}$. If ψ and ψ' are not antipodal, then the transition

$U_{\psi\psi'} := U_\psi \cap U_{\psi'}$ is nonempty, and the transition function $\kappa_\psi(U_{\psi\psi'}) \to \kappa_{\psi'}(U_{\psi\psi'})$ is given by $v \mapsto \frac{\psi+v}{\mathrm{Re}\langle\psi',\psi+v\rangle} - \psi'$. This is continuous for the strong topology that T_ψ and $T_{\psi'}$ inherit from V because the scalar product $\langle\cdot\,,\cdot\rangle$ is continuous, and $\mathrm{Re}\langle\psi', \psi+v\rangle$ is nonzero on $\kappa_\psi(U_\psi \cap U_{\psi'})$. In particular, the tangent space $T_\psi\mathbb{S}(V)$ can be canonically identified with T_ψ.

The canonical projection $\mathbb{S}(V) \to \mathbb{P}(V)$ is a smooth principal U(1)-bundle, with local trivialization $\tau_\psi\colon T_{[\psi]} \times \mathrm{U}(1) \to \mathbb{S}(V)$ given by $\tau_\psi(v,z) := z\frac{\psi+v}{\|\psi+v\|}$. (Note that this depends on the representative ψ of $[\psi]$.) For $[\chi]$ in its image $U_\psi \cup U_{-\psi} \cup U_{i\psi} \cup U_{-i\psi}$, we have $z_\psi([\chi]) = \frac{\langle\psi,\chi\rangle}{|\langle\psi,\chi\rangle|}$ and $z_{\psi'}([\chi]) = \frac{\langle\psi',\chi\rangle}{|\langle\psi',\chi\rangle|}$, so the clutching functions $g_{\psi\psi'}\colon U_{[\psi]} \cap U_{[\psi']} \to \mathbb{T}$ are

$$g_{\psi\psi'}([\chi]) = \frac{\langle\psi',\chi\rangle}{\langle\psi,\chi\rangle} \Big/ \left|\frac{\langle\psi',\chi\rangle}{\langle\psi,\chi\rangle}\right| . \qquad \square$$

Identifying $T_\psi\mathbb{S}(V)$ with T_ψ in (7), we define the 1-form α on $\mathbb{S}(V)$ by

$$\alpha_\psi(\delta v) = -i\langle\psi, \delta v\rangle\,. \tag{9}$$

Proposition 9. *The form α is a connection 1-form on $\mathbb{S}(V) \to \mathbb{P}(V)$ with curvature Ω.*

Proof. We start by showing that α is smooth. Using the derivative

$$D_v\kappa_{\psi\psi'}(\delta v) = \frac{1}{\|\psi+v\|}\left(\delta_v - \frac{\mathrm{Re}\langle v,\delta v\rangle}{1+\|v\|^2}(\psi+v)\right) = \frac{\delta v}{\|\psi+v\|} - \frac{\mathrm{Re}\langle v,\delta v\rangle}{\|\psi+v\|^3}(\psi+v)$$

for the transition function with $\psi' = \frac{\psi+v}{\|\psi+v\|}$, one sees that α is represented by the function $T_\psi \times T_\psi \to \mathbb{R}$ given by $(v,\delta v) \mapsto \frac{1}{1+\|v\|^2}\mathrm{Im}\langle v,\delta v\rangle$, which is evidently smooth.

If we identify $T_\psi\mathbb{S}(V)$ with T_ψ and $T_{z\psi}\mathbb{S}(V)$ with $T_{z\psi}$, then the pushforward $R_{z*}\colon T_\psi \to T_{z\psi}$ of the U(1)-action is $R_{z*}(\delta v) = z\delta v$. It follows that α is U(1)-invariant,

$$(R_z^*\alpha_\psi)(\delta v) = \alpha_{z\psi}(z\delta v) = -i\langle z\psi, z\delta v\rangle = -i\langle\psi,\delta v\rangle = \alpha_\psi(\delta v),$$

and since the vector field X_1 generated by the U(1)-action on $\mathbb{S}(V)$ is $X_1(\psi) = \frac{d}{dt}\big|_{t=0} e^{it}\psi = i\psi$, we have $\alpha_\psi(X_1(\psi)) = 1$, so that α is a principal connection 1-form on $\mathbb{S}(V) \to \mathbb{P}(V)$. If we introduce the constant vector fields δv and δw on $U_\psi \subseteq \mathbb{S}(V)$, then at $v = 0$, we have

$$d\alpha_v(\delta v,\delta w) = L_{\delta v}\alpha_v(\delta w) - L_{\delta v}\alpha_v(\delta w) = 2\mathrm{Im}\langle\delta v,\delta w\rangle\,, \tag{10}$$

which agrees with the local expression (5) for $\Omega_{[\psi]}(\delta v,\delta w)$ at $v = 0$, as required. $\square$

In particular, Ω is closed, so $\mathbb{P}(V)$ is a Kähler manifold. Since the tautological line bundle is associated to $\mathbb{S}(V)$ in the sense that $\mathbb{L}(V) := \mathbb{S}(V) \times_{\mathrm{U}(1)} \mathbb{C}$, we have the following result (see also [9]).

Theorem 10. *The projective space* $\mathbb{P}(V)$ *with Hermitean form* H *is a locally convex Kähler manifold. The tautological bundle* $\mathbb{L}(V) \to \mathbb{P}(V)$*, equipped with the connection inherited from the* $\mathrm{U}(1)$*-principal* 1*-form* α*, is a prequantum line bundle for the corresponding symplectic form* Ω*.*

3. Hamiltonian action of $G^\sharp$ on $\mathbb{P}(\mathcal{H}^\infty)$

We return to the situation of a smooth, projective, unitary representation $\overline{\rho}$ of G, and the corresponding unitary representation ρ of $G^\sharp$. In order to obtain a Hamiltonian action of $G^\sharp$ on $\mathbb{P}(\mathcal{H}^\infty)$, we need a locally convex topology on $\mathcal{H}^\infty$ that is compatible with the $G^\sharp$-action. We will equip $\mathcal{H}^\infty$ with the *strong topology* of Definition 1. As the scalar product $\langle \cdot\,, \cdot \rangle \colon \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathbb{C}$ is manifestly continuous, Theorem 10 applies to $\mathbb{P}(\mathcal{H}^\infty)$.

Proposition 11. *The group action* $G^\sharp \times \mathcal{H}^\infty \to \mathcal{H}^\infty$ *is separately smooth for the strong topology.*

Proof. For fixed $g \in G^\sharp$, we show that the linear map $\rho(g) \colon \mathcal{H}^\infty \to \mathcal{H}^\infty$ is continuous. If $B \subseteq (\mathfrak{g}^\sharp)^k$ is bounded, then so is $\mathrm{Ad}_g(B)$, as the action $\mathrm{Ad}_g \colon (\mathfrak{g}^\sharp)^k \to (\mathfrak{g}^\sharp)^k$ of g in the k-fold product of the adjoint representation is a homeomorphism. From

$$\begin{aligned} p_B(\rho(g)\psi) &= \sup_{\boldsymbol{\xi} \in B} \|\mathtt{d}\rho_k(\boldsymbol{\xi})\rho(g)\psi\| \\ &= \sup_{\boldsymbol{\xi} \in B} \|\rho(g)\mathtt{d}\rho_k(\mathrm{Ad}_{g^{-1}}(\boldsymbol{\xi}))\psi\| = p_{\mathrm{Ad}_{g^{-1}}(B)}(\psi)\,, \end{aligned}$$

we then see that $\rho(g)$ is strongly continuous. If we fix $\psi \in \mathcal{H}^\infty$, then the orbit map $g \mapsto \rho(g)\psi$ is smooth in the *norm* topology on $\mathcal{H}^\infty \subseteq \mathcal{H}$ by definition, but we still need to show that it is smooth in the *strong* topology. This follows from [3, Lemma 3.24]. □

Our (somewhat laborious) proof of Theorem 10 now allows us to apply Proposition 11 in local coordinates, yielding the following result.

Proposition 12. *The locally convex Lie group* G *acts separately smoothly on* $\mathbb{P}(\mathcal{H}^\infty)$ *by Kähler automorphisms. This action is covered by a separately smooth action of* $G^\sharp$ *on the prequantum line bundle* $\mathbb{L}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$ *by holomorphic, connection-preserving bundle automorphisms.*

Proof. In local coordinates, the action of $G^\sharp$ looks like $T_{[\psi]} \to T_{\overline{\rho}(g)[\psi]} \colon v \mapsto \rho(g)v$ on $\mathbb{P}(\mathcal{H}^\infty)$, like $T_\psi \to T_{\rho(g)\psi} \colon v \mapsto \rho(g)v$ on $\mathbb{S}(\mathcal{H}^\infty)$, and like

$$T_{[\psi]} \times \mathbb{C} \to T_{\overline{\rho}(g)[\psi]} \times \mathbb{C} \colon v \oplus z \mapsto \rho(g)v \oplus z \text{ on } \mathbb{L}(\mathcal{H}^\infty).$$

It thus follows from Proposition 11 that the group action is separately smooth, and a holomorphic line bundle isomorphism of $\mathbb{L}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$. In local coordinates, the pushforwards $\rho(g)_* \colon T_{[\psi]} \to T_{\overline{\rho}(g)[\psi]}$ and $\rho(g)_* \colon T_\psi \to T_{\rho(g)\psi}$ are simply given by $\delta v \mapsto \rho(g)\delta v$, so $\rho(g)^* H = H$ and $\rho(g)^*\alpha = \alpha$ follow from unitarity of $\rho(g)$ and the definitions (4) and (9). □

For $\xi \in \mathfrak{g}^\sharp$, the fundamental vector field $X_\xi(\psi) = \mathtt{d}\rho(\xi)\psi$ on $\mathbb{S}(\mathcal{H}^\infty)$ is smooth, as it is given in local coordinates $v \in T_\psi$ by

$$X_\xi(u) = \mathtt{d}\rho(\xi)(\psi + v) - \mathrm{Re}\langle\psi, \mathtt{d}\rho(\xi)v\rangle(\psi + v)\,.$$

Since $L_{X_\xi}\alpha = 0$, we have $d(i_{X_\xi}\alpha) + i_{X_\xi}d\alpha = 0$, so since $\Omega = d\alpha$, we find

$$i_{X_\xi}\Omega = d(-i_{X_\xi}\alpha)\,. \tag{11}$$

We therefore find the comomentum map $\mathfrak{g}^\sharp \to C^\infty(\mathbb{P}(\mathcal{H}^\infty)), \xi \mapsto \mu(\xi)$ with

$$\mu_{[\psi]}(\xi) = \alpha_\psi(-X_\xi(\psi))\,. \tag{12}$$

This evaluates to $\langle\psi, i\mathtt{d}\rho(\xi)\psi\rangle$, the expectation in the state $[\psi]$ of the essentially selfadjoint operator $i\mathtt{d}\rho(\xi)$ (cf. Definition 1), which is the observable corresponding to the symmetry generator $\xi \in \mathfrak{g}$. Note that for fixed ξ, the expression $\psi \mapsto \alpha_\psi(-X_\xi(\psi))$ is independent of the unit vector $\psi \in [\psi]$, and smooth because both α and X_ξ are smooth.

We now prove the theorem announced in the introduction (Theorem 4):

Theorem 13. *The action of $G^\sharp$ on $(\mathbb{P}(\mathcal{H}^\infty), \Omega)$ is Hamiltonian, with momentum map $\mu\colon \mathbb{P}(\mathcal{H}^\infty) \to (\mathfrak{g}^\sharp)'$ given by*

$$\mu_{[\psi]}(\xi) = \frac{\langle\psi, i\mathtt{d}\rho(\xi)\psi\rangle}{\langle\psi, \psi\rangle}\,. \tag{13}$$

This is a smooth, $G^\sharp$-equivariant map into the continuous dual $(\mathfrak{g}^\sharp)'$, equipped with the topology of uniform convergence on bounded subsets.

Proof. Since the $G^\sharp$-action preserves α, it preserves Ω, and $i_{X_\xi}\Omega$ is exact by equation (11). Combining (11) and (12), we have $i_{X_\xi}\Omega = d\mu_\xi(\xi)$. The momentum map is equivariant by

$$\mu_{[g\psi]}(\xi) = \langle\rho(g)\psi, i\mathtt{d}\rho(\xi)\rho(g)\psi\rangle = \langle\psi, i\mathtt{d}\rho(\mathrm{Ad}_{g^{-1}}(\xi))\psi\rangle\,.$$

To prove that μ is smooth, consider its pullback to $\mathbb{S}(\mathcal{H}^\infty)$, which is the restriction to $\mathbb{S}(\mathcal{H}^\infty)$ of the map $\widehat{\mu}\colon \mathcal{H}^\infty \to (\mathfrak{g}^\sharp)'$ defined by $\widehat{\mu}_\psi = \langle\psi, \mathtt{d}\rho(\,\cdot\,)\psi\rangle$. Note that the map

$$\mathcal{H}^\infty \to \mathrm{Lin}(\mathfrak{g}^\sharp, \mathcal{H}^\infty), \quad \psi \mapsto \mathtt{d}\rho(\,\cdot\,)\psi$$

is linear, and continuous if $\mathrm{Lin}(\mathfrak{g}^\sharp, \mathcal{H}^\infty)$ is equipped with the topology of uniform convergence on bounded subsets of $\mathfrak{g}$. As the scalar product $\mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathbb{R}$ is continuous, the linear map $\mathcal{H}^\infty \times \mathcal{H}^\infty \to (\mathfrak{g}^\sharp)', (\psi, \chi) \mapsto \langle\psi, \mathtt{d}\rho(\,\cdot\,)\chi\rangle$ is also continuous, and hence smooth. Since $\widehat{\mu}$ is the composition of this map with the (smooth) diagonal map $\mathcal{H}^\infty \to \mathcal{H}^\infty \times \mathcal{H}^\infty$, the result follows. $\square$

4. Cocycles for Hamiltonian actions

A symplectic action of a locally convex Lie group G on a locally convex, symplectic manifold $(\mathcal{M}, \Omega)$ gives rise to *Kostant–Souriau cocycles.*

Proposition 14 (Kostant–Souriau cocycles). *For every $m \in \mathcal{M}$, the map*

$$\omega_m \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

defined by $\omega_m(\xi, \eta) = \Omega_m(X_\xi, X_\eta)$ is a continuous 2-cocycle. If $\mathcal{M}$ is a Kähler manifold, then $\omega_m = \operatorname{Im} h_m$ for a continuous, positive semidefinite, Hermitean form $h_m \colon \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathbb{C}$.

Proof. Since the action is symplectic, $L_{X_\xi}\Omega = 0$, and we have

$$L_{X_\xi}\Omega(X_\eta, X_\zeta) = \Omega([X_\xi, X_\eta], X_\zeta) + \Omega(X_\eta, [X_\xi, X_\zeta])\,.$$

As Ω is closed, it follows that for all $\xi, \eta, \zeta \in \mathfrak{g}$,

$$\begin{aligned} 0 = d\Omega(X_\xi, X_\eta, X_\zeta) &= (L_{X_\xi}\Omega(X_\eta, X_\zeta) + \text{cycl.}) - (\Omega([X_\xi, X_\eta], X_\zeta) + \text{cycl.}) \\ &= \Omega(X_\eta, [X_\xi, X_\zeta]) + \text{cycl.} = \delta\omega(\xi, \eta, \zeta)\,, \end{aligned}$$

and ω_m is a cocycle for every $m \in \mathcal{M}$. Since the orbit map $g \mapsto \alpha_g(m)$ is smooth, the map $\xi \mapsto X_\xi(m)$ is continuous. Since $\Omega_m \colon T_m\mathcal{M} \times T_m\mathcal{M} \to \mathbb{R}$ is smooth, the cocycle ω_m is continuous. If $\mathcal{M}$ is Kähler, then Ω_m is the imaginary part of a positive definite Hermitean form H_m on $T_m\mathcal{M}$. We then have $\omega_m = \operatorname{Im} h_m$ for the pullback $h_m \colon \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathbb{C}$ of H_m along the complexification $D_m\alpha \colon \mathfrak{g}_\mathbb{C} \to T_m\mathcal{M}$ of the derivative of the orbit map. □

4.1. Cocycles for projective unitary representations

For the weakly Hamiltonian action of G on $\mathbb{P}(\mathcal{H}^\infty)$ derived from a smooth projective unitary representation, the Kostant–Souriau cocycles are given by

$$\omega_{[\psi]}(\xi, \eta) = 2\operatorname{Im}\langle \mathtt{d}\rho(\xi^\sharp)\psi, \mathtt{d}\rho(\eta^\sharp)\psi\rangle\,, \tag{14}$$

where $\xi^\sharp, \eta^\sharp \in \mathfrak{g}^\sharp$ are arbitrary lifts of $\xi, \eta \in \mathfrak{g}$. (This does not depend on the choice of lift because $\langle \psi, i\mathtt{d}\rho(\xi^\sharp)\psi\rangle$ is real.)

In particular, we see that the Kostant–Souriau cocycles related to a smooth projective representation arise as the image of the momentum map $\mu \colon \mathbb{P}(\mathcal{H}^\infty) \to (\mathfrak{g}^\sharp)'$, concatenated with the differential $\delta \colon \mathfrak{g}^\sharp \to \mathcal{Z}^2(\mathfrak{g})$ that maps $\lambda \in (\mathfrak{g}^\sharp)'$ to the 2-cocycle

$$(\delta\lambda)(\xi, \eta) := \lambda([\xi^\sharp, \eta^\sharp]),$$

which is again independent of the choice of lift.

Proposition 15. *For the weakly Hamiltonian action of G on $\mathbb{P}(\mathcal{H}^\infty)$ derived from a smooth projective unitary representation, we have $\omega_{[\psi]} = \delta\mu_{[\psi]}$.*

Proof. This is a direct computation. From (13), we obtain

$$\begin{aligned}\delta\mu_{[\psi]}(\xi,\eta) &= \frac{\langle\psi, i\mathtt{d}\rho([\xi^\sharp,\eta^\sharp])\psi\rangle}{\langle\psi,\psi\rangle}\\ &= -i\left(\frac{\langle\mathtt{d}\rho(\xi^\sharp)\psi,\mathtt{d}\rho(\eta^\sharp)\psi\rangle}{\langle\psi,\psi\rangle} - \frac{\langle\mathtt{d}\rho(\eta^\sharp)\psi,\mathtt{d}\rho(\xi^\sharp)\psi\rangle}{\langle\psi,\psi\rangle}\right)\\ &= 2\,\frac{\mathrm{Im}\langle\mathtt{d}\rho(\xi^\sharp)\psi,\mathtt{d}\rho(\eta^\sharp)\psi\rangle}{\langle\psi,\psi\rangle},\end{aligned}$$

which equals $\omega_{[\psi]}(\xi,\eta)$ for $\|\psi\| = 1$ by (14). □

From a smooth projective unitary representation, we thus get not only a *class* $[\omega_{[\psi]}] \in H^2(\mathfrak{g},\mathbb{R})$ in continuous Lie algebra cohomology, but a distinguished set $\mathcal{C} := \{\omega_{[\psi]}\,;\,[\psi]\in\mathbb{P}(\mathcal{H}^\infty)\} \subseteq \mathcal{Z}^2(\mathfrak{g})$ of (cohomologous, cf. [3]) cocycles. As both μ and δ are G-equivariant, this set $\mathcal{C} = \mathrm{Im}(\delta\circ\mu)$ of cocycles is G-invariant, and every $\omega\in\mathcal{C}$ is the imaginary part of a continuous, positive semidefinite, Hermitean form on $\mathfrak{g}_\mathbb{C}$ by Proposition 14. This sheds geometric light on Propositions 6.6, 6.7 and 6.8 of [3].

4.2. Characters of the stabilizer group

The derivative of the momentum map $\mu\colon \mathbb{P}(\mathcal{H}^\infty)\to(\mathfrak{g}^\sharp)'$ is given (for $\|\psi\|=1$) by

$$D_{[\psi]}\mu(\delta v)(\xi) = 2\mathrm{Re}\langle i\mathtt{d}\rho(\xi)\psi,\delta v\rangle\,. \tag{15}$$

This has some interesting consequences. We denote the *real* inner product on $\mathcal{H}^\infty$ by $(v,w)_\mathbb{R} := 2\mathrm{Re}\langle v,w\rangle$, and the orthogonal complement with respect to $(\,\cdot\,,\,\cdot\,)_\mathbb{R}$ by $\perp_\mathbb{R}$. Then the kernel $\mathrm{Ker}(D_{[\psi]}\mu)\subseteq T_{[\psi]} = (\mathbb{C}\psi)^{\perp_\mathbb{R}}$ is precisely the *real* orthogonal complement $(i\mathbb{R}\psi\oplus i\mathtt{d}\rho(\mathfrak{g}^\sharp)\psi)^{\perp_\mathbb{R}}$ in $\mathcal{H}^\infty$.

Proposition 16. *The derivative $D_{[\psi]}\mu\colon T_{[\psi]}\to\mathfrak{g}'$ is injective if and only if $\mathtt{d}\rho(\mathfrak{g}^\sharp)\psi$ spans $\psi^{\perp_\mathbb{R}}\subset\mathcal{H}$ as a real Hilbert space, and zero if and only if the identity component G_0 stabilizes $[\psi]$.*

Proof. The first statement follows immediately from the formula for the kernel. For the second statement, note that $D_{[\psi]}\mu = 0$ is equivalent to $\mathtt{d}\rho(\mathfrak{g}^\sharp)\psi = i\mathbb{R}\psi$. By the Fundamental Theorem of Calculus for locally convex spaces, G_0 stabilizes $[\psi]\in\mathbb{P}(\mathcal{H}^\infty)$ if and only if $\mathfrak{g}$ stabilizes $[\psi]$, which is the case if and only if $\mathtt{d}\rho(\mathfrak{g}^\sharp)\psi\subseteq i\mathbb{R}\psi$. □

We denote the stabilizer of $\lambda\in(\mathfrak{g}^\sharp)'$ under the coadjoint representation by $G^\sharp_\lambda$. Further, we denote by

$$G^\sharp_{[\psi]} := \{g\in G^\sharp\ :\ [\rho(g)\psi] = [\psi]\}$$

the preimage in $G^\sharp$ of the stabilizer $G_{[\psi]}$ of $[\psi]\in\mathbb{P}(\mathcal{H}^\infty)$, and we denote

$$\mathfrak{g}^\sharp_{[\psi]} := \{\xi\in\mathfrak{g}^\sharp\,;\,\mathtt{d}\rho(\xi)\psi\in i\mathbb{R}\psi\}\,.$$

Proposition 17. *For every $[\psi]\in\mathbb{P}(\mathcal{H}^\infty)$, we have $G^\sharp_{[\psi]}\subseteq G^\sharp_{\mu_{[\psi]}}$.*

Proof. Since the momentum map is $G^\sharp$-equivariant, we have $g \in G_{\mu_{[\psi]}}$ if and only if

$$\frac{\langle \rho(g)\psi, i\mathtt{d}\rho(\xi)\rho(g)\psi\rangle}{\langle \rho(g)\psi, \rho(g)\psi\rangle} = \frac{\langle \psi, i\mathtt{d}\rho(\xi)\psi\rangle}{\langle \psi, \psi\rangle} \tag{16}$$

for all $\xi \in \mathfrak{g}$. This is clearly the case if $g \in G^\sharp_{[\psi]}$. □

Proposition 18. *The restriction of* $-i\mu_{[\psi]} \colon \mathfrak{g}^\sharp \to i\mathbb{R}$ *to* $\mathfrak{g}^\sharp_{[\psi]}$ *is a Lie algebra character. It integrates to a group character on any Lie subgroup of* $G^\sharp_{[\psi]}$.

Proof. For $\xi \in \mathfrak{g}^\sharp_{[\psi]}$, we have $\mathtt{d}\rho(\xi)\psi = -i\mu(\xi)\psi$. As

$$\mathtt{d}\rho([\xi,\eta])\psi = [\mathtt{d}\rho(\xi), \mathtt{d}\rho(\eta)]\psi = 0 \quad \text{for} \quad \xi, \eta \in \mathfrak{g}^\sharp_{[\psi]}\,,$$

it follows that $-i\mu$ is an $i\mathbb{R}$-valued character. Similarly, the smooth map

$$F\colon G^\sharp \to \mathbb{C}, \quad F(g) := \frac{\langle \psi, \rho(g)\psi\rangle}{\langle \psi, \psi\rangle}$$

is a $\mathrm{U}(1)$-valued character when restricted to $G^\sharp_{[\psi]}$, as $\rho(g)\psi = F(g)\psi$ on that subgroup. In fact, $F\colon G^\sharp \to \mathbb{C}$ takes values in the unit ball $\Delta \subseteq \mathbb{C}$, and $G^\sharp_{[\psi]}$ is the preimage of the unit circle $\partial\Delta$. The derivative of F at the unit $\mathbf{1} \in G$ is $D_{\mathbf{1}}F = -i\mu_{[\psi]}$, so for any Lie subgroup $H \subseteq G^\sharp_{[\psi]}$, the restriction of F to H is a $\mathrm{U}(1)$-valued smooth character that integrates $-i\mu_{[\psi]}|_{\mathrm{Lie}(H)}$. □

Note that the image of μ is contained in the hyperplane $(\mathfrak{g}^\sharp)'_{-1} \subset (\mathfrak{g}^\sharp)'$ of elements that evaluate to -1 on $1 \in \mathbb{R} = \mathrm{Ker}(\mathfrak{g}^\sharp \to \mathfrak{g})$. Now suppose that the image of $D_{[\psi]}\mu$ is dense in $T_{\mu_{[\psi]}}(\mathfrak{g}^\sharp)'_{-1} = (\mathfrak{g}^\sharp)'_0 \simeq \mathfrak{g}'$. Since $\mathrm{Im}(D_{[\psi]}\mu) \subseteq (\mathfrak{g}^\sharp/\mathfrak{g}^\sharp_\psi)'$, we then have $\mathfrak{g}^\sharp_\psi = \{0\}$, so that $\mathfrak{g}^\sharp_{[\psi]} = \mathbb{R}$. For points $[\psi] \in \mathbb{P}(\mathcal{H}^\infty)$ where the image of $D_{[\psi]}\mu$ is dense, the identity component of any Lie subgroup $H \subseteq G_{[\psi]}$ is therefore $\mathrm{U}(1)$, and since the character on $\mathrm{U}(1) \subseteq G^\sharp_{[\psi]}$ is always the identity, Proposition 18 yields no extra information.

However, Proposition 18 does yield nontrivial integrality requirements if $G^\sharp_{[\psi]}$ is strictly bigger than $\mathrm{U}(1)$, which one expects to be the case for extremal points of the momentum set $\mathrm{Im}\mu$. Compare this to Lemma 2.1 and Theorem 8.1 in [2], where it is shown that for compact Lie groups G, the vertices of the momentum polygon are integral lattice points in the dual $\mathfrak{h}'$ of the Cartan subalgebra.

5. Diffeological Smoothness

As noted in the introduction, the action $G^\sharp \times \mathbb{P}(\mathcal{H}^\infty) \to \mathbb{P}(\mathcal{H}^\infty)$ is separately smooth, but not necessarily smooth. However, if we settle for smoothness in the sense of diffeological spaces, then one can hope for this action to be smooth for the (large) class of *regular* Lie groups modelled on *barrelled* spaces, which includes regular Fréchet and LF Lie groups. Here we prove the infinitesimal version of this, namely that the infinitesimal action $\mathfrak{g}^\sharp \times \mathcal{H}^\infty \to \mathcal{H}^\infty$ is a smooth map of diffeological spaces.

5.1. Infinitesimal action

Let $\overline{\rho}$ be a smooth projective unitary representation of a locally convex Lie group G modelled on a *barrelled* Lie algebra $\mathfrak{g}$.

Lemma 19. *If* $\xi\colon \mathbb{R}^n \to \mathfrak{g}$ *and* $\psi\colon \mathbb{R}^m \to \mathcal{H}^\infty$ *are continuous, then the map* $\mathtt{d}\rho(\xi)\psi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathcal{H}^\infty$ *defined by* $(s,t) \mapsto \mathtt{d}\rho(\xi_t)\psi_s$ *is continuous.*

Proof. Since the Lie algebra action $\mathfrak{g}^\sharp \times \mathcal{H}^\infty \to \mathcal{H}^\infty$ is sequentially continuous by [3, Lemma 3.14], the same holds for its concatenation with the continuous map $(\xi,\psi)\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathfrak{g}^\sharp \times \mathcal{H}^\infty$. Since $\mathbb{R}^n \times \mathbb{R}^m$ is first countable, this implies continuity. □

Lemma 20. *If* $\xi\colon \mathbb{R}^n \to \mathfrak{g}^\sharp$ *and* $\psi\colon \mathbb{R}^m \to \mathcal{H}^\infty$ *are* C^1*, then so is* $\mathtt{d}\rho(\xi)\psi$*, and* $D_{(v_1,v_2)}(\mathtt{d}\rho(\xi)\psi)_{s,t} = \mathtt{d}\rho(\partial_{v_1}\xi_s)\psi_t + \mathtt{d}\rho(\xi_s)\partial_{v_2}\psi_t$.

Proof. For the directional derivative along $(v_1,v_2) \in T_{s,t}(\mathbb{R}^n \times \mathbb{R}^m)$, note that

$$D_{v_1,v_2}(\mathtt{d}\rho(\xi)\psi)_{s,t} = \lim_{\varepsilon\to 0} \mathtt{d}\rho(\Delta\xi_s(\varepsilon))\psi_{t+\varepsilon v_2} + \mathtt{d}\rho(\xi_s)\Delta\psi_t(\varepsilon),$$

with difference quotients $\Delta\xi$ and $\Delta\psi$ defined by $\Delta\xi_s(\varepsilon) := \frac{1}{\varepsilon}(\xi_{s+\varepsilon v_1} - \xi_s)$ and $\Delta\psi_t(\varepsilon) := \frac{1}{\varepsilon}(\psi_{t+\varepsilon v_2} - \psi_t)$ for $\varepsilon \neq 0$, and $\Delta\xi_s(0) := \partial_{v_1}\xi_s$ and $\Delta\psi_t(0) := \partial_{v_2}\psi_t$ for $\varepsilon = 0$. Since $\Delta\xi$ and $\Delta\psi$ are continuous in ε, the formula for $D_{v_1,v_2}(\mathtt{d}\rho(\xi)\psi)_{s,t}$ follows by Lemma 19. Another application of this lemma to $(D\xi,\psi)$ and $(\xi, D\psi)$ shows that the derivative is continuous. □

Proposition 21. *If* $\xi\colon \mathbb{R}^n \to \mathfrak{g}^\sharp$ *and* $\psi\colon \mathbb{R}^m \to \mathcal{H}^\infty$ *are* C^k *for* $k \in \mathbb{N}$ *or* $k = \infty$*, then so is* $\mathtt{d}\rho(\xi)\psi$*.*

Proof. This follows by induction on k, using Lemmas 19 and 20. □

If we equip all locally convex manifolds $\mathcal{M}$ with the diffeology of smooth maps from open subsets of Euclidean space into $\mathcal{M}$, then the following is simply a reformulation of Proposition 21.

Proposition 22. *If* G *is modelled on a barrelled Lie algebra* $\mathfrak{g}$*, then the infinitesimal action* $\mathfrak{g}^\sharp \times \mathcal{H}^\infty \to \mathcal{H}^\infty$ *is a smooth map of diffeological spaces.*

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Bas Janssens
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
NL-3584 CD Utrecht, the Netherlands
e-mail: B.Janssens@uu.nl

Karl-Hermann Neeb
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstrasse 11
D-91058 Erlangen, Germany
e-mail: neeb@math.fau.de

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Canonical Representations for Hyperboloids: an Interaction with an Overalgebra

Vladimir F. Molchanov

Abstract. Canonical representations for the hyperboloid $\mathcal{X} = G/H$ where $G = \mathrm{SO}_0(p,q)$, $H = \mathrm{SO}_0(p,q-1)$, are defined as the restriction to G of maximal degenerate series representations of the overgroup $\widetilde{G} = \mathrm{SL}(n,\mathbb{R})$. We determine explicitly the interaction of Lie operators of $\widetilde{G}$ with operators intertwining canonical representations and representations of G associated with a cone.

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This paper continues the series of our papers [2–5], devoted to the interaction of Poisson and Fourier transforms associated with canonical representations and Lie operators of a larger group (an "overgroup").

This activity was inspired by Neretin's paper [7] (an old Mukunda paper [6] should be also mentioned) for the Lobachevsky plane G/K, where $G = \mathrm{SO}_0(2,1)$, $K = \mathrm{SO}(2)$. In these papers the authors essentially used the Plancherel formula for this manifold.

We use another approach. We use the notions of canonical representations, Poisson and Fourier transforms and "overgroups" and do not need any Plancherel formulae. Nevertheless, even in the framework of our version the computations of explicit formulae is a very difficult analytic problem. Earlier we already studied hyperboloids $\mathcal{X} = G/H$ with $G = \mathrm{SO}_0(p,q)$ and the overgroup $\widetilde{G} = \mathrm{SO}_0(p+1,q)$, see [2–4], and hyperboloids (Lobachevsky spaces) with $G = \mathrm{SO}_0(n-1,1)$ and $\widetilde{G} = \mathrm{SL}(n,\mathbb{R})$, see [5]. Now we consider the hyperboloid $\mathcal{X} = G/H$, where G is the pseudo-orthogonal group $\mathrm{SO}_0(p,q)$, $H = \mathrm{SO}_0(p,q-1)$, and the overgroup is $\widetilde{G} = \mathrm{SL}(n,\mathbb{R})$, $n = p+q$. This case is the most difficult. Notice that expressions of

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the interaction involve differential operators of the fourth, second and zero order (just as in [4]).

One of sources of getting canonical representations consists of the following. Let G be a semi-simple Lie group. We take some group $\widetilde{G}$ (overgroup) containing G such that G is a symmetric subgroup of $\widetilde{G}$, i.e., G is the fixed point subgroup of an involution. Let $\widetilde{P}$ be a maximal parabolic subgroup of $\widetilde{G}$. We take a series of representations $\widetilde{R}_\lambda$ of $\widetilde{G}$ induced by characters of $\widetilde{P}$, they can depend on some discrete parameters, we do not write them. As a rule, representations $\widetilde{R}_\lambda$ are irreducible. They act on functions on some compact manifold Ω (a flag space for $\widetilde{G}$). Denote by R_λ restrictions of $\widetilde{R}_\lambda$ to G:

$$R_\lambda = \widetilde{R}_\lambda \Big|_G .$$

We call these representations R_λ *canonical representations* of the group G. They act on functions on Ω.

Generally speaking, the manifold Ω is not a homogeneous space of the group G, this group has several orbits on Ω. Open G-orbits are semi-simple symmetric spaces G/H_i. Subgroups H_i can be not isomorphic. The manifold Ω is the closure of the union of open G-orbits.

Canonical representations R_λ give rise to *boundary representations*, related to boundaries of G-orbits G/H_i. There are two types of boundary representations. The boundary representations of the first type act on distributions concentrated at the union S of boundaries. The boundary representations of the second type act on jets transversal to S. These two types are dual to each other. Boundary representations are interesting both themselves and as a tool for the decomposition of canonical representations, they glue representations on separate G-orbits.

One can also consider another version of canonical representations: the restriction of these representations above to some open G-orbit in Ω. It is just the case we consider in this paper.

With the canonical representation R_λ, we associate Poisson transforms $P_{\lambda,\sigma}$ and Fourier transforms $F_{\lambda,\sigma}$. They are operators intertwining the representation R_λ with (irreducible) representations T_σ of G occurring in the decomposition of R_λ. Our aim is to find out how the Lie operators of $\widetilde{G}$ in $\widetilde{R}_\lambda$ (i.e., the representation $\widetilde{R}_\lambda$ of the Lie algebra $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$) interact with these transforms.

This problem can be treated as a version of the classical problem on the action of a group (or a Lie algebra) in a basis that is an eigenbasis for some subgroup.

This theory can be considered as a new approach to representation theory of Lie algebras (and Lie groups): in this theory elements of a Lie algebra go to *differential-difference* operators.

In this paper we do not touch the decomposition problem for the canonical and boundary representations, which will be considered elsewhere. Also we do not consider the Fourier transform and do not discuss coefficients of the interactions, since this goes exactly as in [5].

Let us introduce some notation and conventions.

For a character of the group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ we use the following notation:

$$t^{\lambda,\nu} = |t|^\lambda (\operatorname{sgn} t)^\nu, \quad t \in \mathbb{R}^*, \ \lambda \in \mathbb{C}, \ \nu \in \mathbb{Z}.$$

This character depends on ν modulo 2 rather than ν itself.

For a manifold M, $\mathcal{D}(M)$ denotes the space of compactly supported infinitely differentiable complex-valued functions on M, with the usual topology.

For a representation of a Lie group, we retain the same symbol for the corresponding representations of its Lie algebra.

1. Pseudo-orthogonal group and hyperboloid

The group $G = \mathrm{SO}_0(p,q)$ is the connected component of the identity of the group of linear transformations of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$, $n = p+q$, preserving the bilinear form

$$[x,y] = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n.$$

The matrix of this form is $I = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, where

$$\lambda_1 = \cdots = \lambda_p = -1, \lambda_{p+1} = \cdots = \lambda_n = 1.$$

Let K be a subgroup of G consisting of elements g such that $g = IgI$. It is a maximal compact subgroup of G, it is isomorphic to $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

Let us denote by $\langle \cdot \, , \cdot \rangle$ the standard inner products in $\mathbb{R}^p$ and $\mathbb{R}^q$, let us denote by $| \cdot |$ and $\| \cdot \|$ corresponding norms in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. For a point $x \in \mathbb{R}^n$ written as the pair $x = (u,v)$, $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$, we denote $|x| = |u|$ and $\|x\| = \|v\|$ respectively.

We shall consider that G acts on $\mathbb{R}^n$ from the right: $x \mapsto xg$. In accordance with this we write vectors in the row form. Let $\mathcal{X}$ be the hyperboloid defined by equation $[x,x] = 1$, or $-|x|^2 + \|x\|^2 = 1$. The group G acts on it transitively. The stabilizer H of the point $x^0 = (0, \ldots, 0, 1)$ is $\mathrm{SO}_0(p, q-1)$, so that $\mathcal{X}$ is a homogeneous space G/H. It is more convenient for us to use another realization of the hyperboloid $\mathcal{X}$. Let us attach to a point $x \in \mathcal{X}$ the point $y = x/\|x\|$. Then $\mathcal{X}$ becomes a cylinder $\mathcal{Y}$, the direct product of the unit ball $B \subset \mathbb{R}^p$, defined by $|y| < 1$, and the unit sphere $S_2 \subset \mathbb{R}^q$, defined by $\|y\| = 1$.

Let dy be the Euclidean measure on $\mathcal{Y}$, then a G-invariant measure dx on $\mathcal{X}$ is

$$dx = [y,y]^{-n/2} dy.$$

The Lie algebra $\mathfrak{g}$ of the group G consists of matrices $X \in \mathrm{Mat}(n, \mathbb{R})$ satisfying the condition $X' = -XIX$, the prime means matrix transposition. A basis of $\mathfrak{g}$ is formed by matrices $L_{ij} = E_{ij} - \lambda_i \lambda_j E_{ji}$, $i < j$, where E_{ij} is the "matrix unit": it has 1 at the place (i,j) and 0 at other places.

2. Representations of G associated with a cone

Recall [1] some material about representations of the group G associated with a cone (class one representations). We use the "compact picture".

Denote by S the section of the cone $[x,x]=0$ in $\mathbb{R}^n$ by the sphere $x_1^2+\dots+x_n^2=2$. It consists of points s such that $|s|=\|s\|=1$. The section S is the direct product of two unit spheres $S_1\subset\mathbb{R}^p$ and $S_2\subset\mathbb{R}^q$, defined by equations $s_1^2+\dots+s_p^2=1$ and $s_{p+1}^2+\dots+s_n^2=1$ respectively. Let ds be the Euclidean measure on S.

For local coordinates on spheres S_1 and S_2 we take variables s_i omitting one of them for either sphere, say, s_α for S_1 and s_β for S_2. The Laplace–Beltrami operators Δ_1 on S_1 and Δ_2 on S_2 are given respectively by formulae:

$$\Delta_1 = H_1 - D_1^2 - (p-2)D_1\,,$$
$$\Delta_2 = H_2 - D_2^2 - (q-2)D_2\,,$$

where

$$H_1=\sum\frac{\partial^2}{\partial s_i^2}\,,\ D_1=\sum s_i\frac{\partial}{\partial s_i}\,,$$
$$H_2=\sum\frac{\partial^2}{\partial s_j^2}\,,\ D_2=\sum s_j\frac{\partial}{\partial s_j}\,,$$

and derivatives with respect to s_α and s_β have to be omitted.

Let $\sigma\in\mathbb{C}$, $\varepsilon=0,1$. Let us denote by $\mathcal{D}_\varepsilon(S)$ the space of functions $\varphi\in\mathcal{D}(S)$ of parity ε: $\varphi(-s)=(-1)^\varepsilon\varphi(s)$. The representation $T_{\sigma,\varepsilon}$ of the group G acts on $\mathcal{D}_\varepsilon(S)$ by

$$(T_{\sigma,\varepsilon}(g)\varphi)(s)=\varphi\left(\frac{sg}{|sg|}\right)\cdot|sg|^\sigma.$$

If σ is not integer, then $T_{\sigma,\varepsilon}$ is irreducible and equivalent to $T_{2-n-\sigma,\varepsilon}$. For $X\in\mathfrak{g}$, differential operators $T_{\sigma,\varepsilon}(X)$ do not depend on ν, so we omit ε in the notation and write $T_\sigma(X)$.

Here are operators corresponding to basis elements L_{km}:

$$T_\sigma(L_{km})=A_{km},\quad 1\leqslant k<m\leqslant p \text{ or } \ p+1\leqslant k<m\leqslant n,$$
$$T_\sigma(L_{km})=\sigma\, s_k s_m+s_m B_k+s_k B_m,\quad 1\leqslant k\leqslant p<m\leqslant n,$$

where

$$A_{km}=-s_m\frac{\partial}{\partial s_k}+s_k\frac{\partial}{\partial s_m},$$
$$B_k=\frac{\partial}{\partial s_k}-s_kD_1,\quad B_m=\frac{\partial}{\partial s_m}-s_mD_2,$$

as before, derivatives with respect to s_α and s_β have to be omitted.

3. Canonical representations

For an overgroup for the group G, we take the group $\widetilde{G} = \mathrm{SL}(n,\mathbb{R})$. Let $\lambda \in \mathbb{C}$, $\nu = 0,1$. Denote by $\mathcal{D}_{\lambda,\nu}(\mathbb{R}^n)$ the space of functions $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying the following homogeneity condition:

$$f(tx) = t^{\lambda,\nu} f(x), \quad x \in \mathbb{R}^n \setminus \{0\},\ t \in \mathbb{R} \setminus \{0\}.$$

Representations $\widetilde{R}_{\lambda,\nu}$ of $\widetilde{G}$ act on $\mathcal{D}_{-\lambda-n,\nu}(\mathbb{R}^n)$ by translations:

$$\left(\widetilde{R}_{\lambda,\nu}(g)f\right)(x) = f(xg).$$

These representations form a maximal degenerate principal series.

Now let $\widetilde{\mathcal{Y}}$ be a manifold in $\mathbb{R}^n$, defined by $\|y\| = 1$ (it contains $\mathcal{Y}$). Restrict functions in $\mathcal{D}_{-\lambda-n,\nu}(\mathbb{R}^n)$ to $\widetilde{\mathcal{Y}}$. We obtain some space $\mathcal{D}_{-\lambda-n,\nu}(\widetilde{\mathcal{Y}})$ of functions f on $\widetilde{\mathcal{Y}}$ of parity ν:

$$f(-y) = (-1)^\nu f(y), \quad y \in \widetilde{\mathcal{Y}}.$$

In this space the representation $\widetilde{R}_{\lambda,\nu}$ acts as follows:

$$\big(\widetilde{R}_{\lambda,\nu}(g)f\big)(y) = f\left(\frac{yg}{\|yg\|}\right) \|yg\|^{-\lambda-n}, \quad g \in \widetilde{G}.$$

Restrict the representation $\widetilde{R}_{\lambda,\nu}$ of the group $\widetilde{G}$ to its subgroup G. Since G preserves the manifold $\mathcal{Y}$, we also restrict this representation to the space $\mathcal{D}_\nu(\mathcal{Y})$ of functions in $\mathcal{D}(\mathcal{Y})$ of parity ν. Let us call this restriction the *canonical representation*.

Thus, the canonical representation $R_{\lambda,\nu}$, $\lambda \in \mathbb{C}$, $\nu = 0,1$, of the group G acts on the space $\mathcal{D}_\nu(\mathcal{Y})$ by

$$\big(R_{\lambda,\nu}(g)f\big)(y) = f\left(\frac{yg}{\|yg\|}\right) \|yg\|^{-\lambda-n}, \quad g \in G.$$

The Lie algebra $\widetilde{\mathfrak{g}}$ of the group $\widetilde{G}$ consists of matrices $X \in \mathrm{Mat}(n,\mathbb{R})$ with trace zero. It splits into the direct sum $\mathfrak{g} + \mathfrak{m}$ where $\mathfrak{m}$ is the space consisting of matrices X such that $X' = XIX$. Decompose $\mathfrak{m}$ into the direct sum of two subspaces: $\mathfrak{m} = \mathfrak{a} + \mathfrak{n}$ where $\mathfrak{a}$ is the subspace of diagonal matrices and $\mathfrak{n}$ consists of matrices with zero diagonal. A basis of $\mathfrak{n}$ is formed by matrices $M_{ij} = E_{ij} + \lambda_i\lambda_j E_{ji}$, $i < j$, or, more detailed, by matrices $M_{km} = E_{km} + E_{mk}$, $1 \leqslant k < m \leqslant p$ or $p+1 \leqslant k < m \leqslant n$, and $M_{kn} = E_{kn} - E_{nk}$, $1 \leqslant k \leqslant p < m \leqslant n$. The subalgebra $\mathfrak{a}$ is spanned by matrices $Y_{km} = E_{kk} - E_{mm}$, $1 \leqslant k < m \leqslant n$.

For $X \in \widetilde{\mathfrak{g}}$, differential operators $\widetilde{R}_{\lambda,\nu}(X)$ do not depend on ν, so we omit ν in the notation and write $\widetilde{R}_\lambda(X)$.

The centralizer of the group K in $\widetilde{\mathfrak{g}}$ is one-dimensional, a basis is the following matrix in $\mathfrak{a}$:

$$Y_0 = \frac{1}{n}\begin{pmatrix} qE_p & 0 \\ 0 & -pE_q \end{pmatrix}, \tag{1}$$

where E_k is the identity matrix of order k.

In particular, let us write $\widetilde{R}_\lambda(Y_0)$. Write y as a pair $y = (u,v)$, $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$. Then

$$\widetilde{R}_{\lambda,\nu}\left(\mathrm{e}^{tY_0}\right) f(u,v) = f\left(\mathrm{e}^t u, v\right) \mathrm{e}^{t(-\lambda-n)(-p)/n}.$$

Differentiating it with respect to t at the point $t=0$ and passing to polar coordinates: $u = r\omega$, $0 \leqslant r < 1$, $\omega \in S_1$, we obtain

$$\widetilde{R}_\lambda(Y_0) = \sum_{k=1}^{p} u_k \frac{\partial}{\partial u_k} + \frac{p}{n}(\lambda+n) = r\frac{\partial}{\partial r} + \frac{p}{n}(\lambda+n). \tag{2}$$

4. Interaction of the overalgebra with the Poisson transform

The Poisson transform $P_{\lambda,\nu;\sigma}$ is an operator $\mathcal{D}_\nu(S) \to C^\infty(\mathcal{Y})$, defined by

$$(P_{\lambda,\nu,\sigma}\varphi)(y) = a^{(-\lambda-\sigma-n)/2} \int_S [y,s]^{\sigma,\nu}\, \varphi(s)\, ds, \tag{3}$$

where

$$a = [y,y] = 1 - |y|^2.$$

It intertwines $T_{2-n-\sigma,\nu}$ and $R_{\lambda,\nu}$:

$$R_{\lambda,\nu}(g) P_{\lambda,\nu;\sigma} = P_{\lambda,\nu;\sigma} T_{2-n-\sigma,\nu}(g), \quad g \in G. \tag{4}$$

The integral converges absolutely for $\operatorname{Re}\sigma > -1$ and can be continued by analyticity to other λ, σ to a meromorphic function. Considered as a distribution, the function $(P_{\lambda,\nu,\sigma}\varphi)(y)$ has poles in σ (depending on λ) at points

$$\sigma = \lambda - 2k, \quad \sigma = 2 - n - \lambda + 2l,$$

where $k, l \in \{0,1,2,\dots\}$. These poles are simple for generic λ.

We determine explicitly the interaction of the Poisson transform $P_{\lambda,\nu,\sigma}$ with Lie operators of the overgroup $\widetilde{G}$ in the representation $\widetilde{R}_{\lambda,\nu}$, i.e., with the representation $\widetilde{R}_{\lambda,\nu}$ of the Lie algebra $\widetilde{\mathfrak{g}}$ ("overalgebra") of the group $\widetilde{G}$.

We have to write explicitly the compositions $\widetilde{R}_\lambda(X)\, P_{\lambda,\nu,\sigma}$ where $X \in \widetilde{\mathfrak{g}}$. If $X \in \mathfrak{g}$, then, by (4), the answer is simple:

$$\widetilde{R}_\lambda(X)\, P_{\lambda,\nu,\sigma} = P_{\lambda,\nu;\sigma}\, T_{2-n-\sigma}(X).$$

Therefore, it is sufficient to take for X elements in the subspace $\mathfrak{m}$, see Section 3, for example, basis elements M_{km} and Y_{km}.

We write explicitly expressions of $\widetilde{R}_\lambda(X) P_{\lambda,\nu,\sigma}$ for the following elements $X \in \mathfrak{m}$: basis elements M_{km} in $\mathfrak{n}$ and Y_{km} in $\mathfrak{a}$. The crucial step is a computation of the composition $\widetilde{R}_\lambda(Y_0) P_{\lambda,\nu,\sigma}$, where Y_0 is the basis element in the centralizer of the group K, see (11). In order to find expressions for other $X \in \mathfrak{m}$, we use expressions for Y_0 and commutation relations.

Theorem 1. *Let σ be not a pole of the Poisson transform $P_{\lambda,\nu,\sigma}$. The operator $\widetilde{R}_\lambda(X)$, $X \in \mathfrak{m}$, interacts with this transform as follows:*

$$\widetilde{R}_\lambda(X)P_{\lambda,\nu;\sigma} = a(\lambda,\sigma)P_{\lambda,\nu;\sigma+2}\,K_\sigma(X) + b(\lambda,\sigma)P_{\lambda,\nu;\sigma}\,E_\sigma(X) + c(\lambda,\sigma)P_{\lambda,\nu;\sigma-2}\,C(X), \quad (5)$$

where coefficients a, b, c are given by formulae:

$$a(\lambda,\sigma) = \frac{\lambda+\sigma+n}{(\sigma+1)(\sigma+2)(2\sigma+n-2)(2\sigma+n)}, \quad (6)$$

$$b(\lambda,\sigma) = \frac{2\lambda+n}{(2\sigma+n-4)(2\sigma+n)}, \quad (7)$$

$$c(\lambda,\sigma) = \frac{(\lambda-\sigma+2)\sigma(\sigma-1)}{(2\sigma+n-4)(2\sigma+n-2)}, \quad (8)$$

and $K_\sigma(X)$, $E_\sigma(X)$ and $C(X)$ are differential operators on S of order 4, 2 and 0 respectively (the operator $C(X)$ does not depend on σ) linearly depending on $X \in \mathfrak{m}$. In particular, for $X = Y_0$, see (1), we have

$$\begin{aligned} K_\sigma(Y_0) = {} & (\Delta_1-\Delta_2)^2 + (2\sigma^2+2n\sigma+nq+2(p-q))\Delta_1 \\ & + (2\sigma^2+2n\sigma+np-2(p-q))\Delta_2 \\ & + (\sigma+2)(\sigma+p)(\sigma+q)(\sigma+n-2). \end{aligned} \quad (9)$$

$$E_\sigma(Y_0) = \Delta_1 - \Delta_2 + \frac{p-q}{n}\sigma(\sigma+n-2), \quad (10)$$

$$C(Y_0) = 1, \quad (11)$$

Proof. We prove the theorem by straight computations of $\widetilde{R}_\lambda(X)P_{\lambda,\nu,\sigma}$ for basis elements $X \in \mathfrak{m}$.

Let us outline the proof of formulae (5)–(11). Recall (Sections 2 and 3): we write $y \in \mathcal{Y}$ and $s \in S$ as pairs: $s = (\xi,\eta)$ and $y = (u,v)$, so that $[y,s] = -\langle u,\xi\rangle + \langle v,\eta\rangle$. Denote also $A = [y,s]$ and $z = \langle u,\xi\rangle$, $w = \langle v,\eta\rangle$, so that $A = -z+w$. We use polar coordinates $u = r\omega$, $0 \leqslant r < 1$, $\omega \in S_1$.

To simplify the notation, we omit the symbol ν in notation $t^{\lambda,\nu}$, so that, say, A^σ, $A^{\sigma\pm1}$ stand for $A^{\sigma,\nu}$, $A^{\sigma\pm1,\nu\pm1}$, respectively, etc. Then the Poisson transform (3) can be rewritten as

$$P_{\lambda,\nu,\sigma}\varphi = a^\mu \int_S A^\sigma \varphi(s)\,ds, \quad (12)$$

where

$$\mu = \frac{-\lambda-\sigma-n}{2}, \qquad a = 1-r^2. \quad (13)$$

Let us apply to $a^\mu A^\sigma$ the operator $\widetilde{R}_\lambda(Y_0)$, see (2). Using (13), we obtain:

$$\widetilde{R}_\lambda(Y_0)\,(a^\mu A^\sigma) = (\lambda+\sigma+n)a^{\mu-1}A^\sigma - \sigma a^\mu \cdot w \cdot A^{\sigma-1} - q\,\frac{\lambda+n}{n}\,a^\mu A^\sigma, \quad (14)$$

so that, see (12), we get

$$\widetilde{R}_\lambda(Y_0)P_{\lambda,\nu,\sigma}\varphi = (\lambda+\sigma+n)P_{\lambda+2,\sigma}\varphi - \sigma P_{\lambda+1,\sigma-1}(w\cdot\varphi) - q\,\frac{\lambda+n}{n}\,P_{\lambda,\sigma}\varphi.$$

To prove (5)–(11), we have to present function (14) as a linear combination of the following functions:

$$\begin{array}{l} a^{\mu+1}A^{\sigma-2}, \\ a^{\mu}A^{\sigma}, \quad a^{\mu}\Delta_i A^{\sigma}, \\ a^{\mu-1}A^{\sigma+2}, \quad a^{\mu-1}\Delta_i A^{\sigma+2}, \quad a^{\mu-1}\Delta_i\Delta_j A^{\sigma+2}, \end{array}$$

where $i,j\in\{1,2\}$. We do it by rather long computations, we omit them. Note only some relations: we use

$$\begin{aligned} \Delta_1 A^{\sigma} = {} & -\sigma(\sigma+p-2)A^{\sigma} + \sigma(2\sigma+p-3)\cdot w\cdot A^{\sigma-1} \\ & + \sigma(\sigma-1)A^{\sigma-2}(1-a-w^2), \\ \Delta_2 A^{\sigma} = {} & -\sigma(\sigma+q-2)A^{\sigma} - \sigma(2\sigma+q-3)\cdot z\cdot A^{\sigma-1} \\ & + \sigma(\sigma-1)A^{\sigma-2}(1-z^2). \end{aligned}$$

$$\Delta_2\,(w\cdot A^{\sigma}) = w\cdot\Delta_2 A^{\sigma} + \Delta_2 A^{\sigma+1} - A\Delta_2 A^{\sigma}. \qquad \square$$

Now let us go to other elements $X\in\mathfrak{m}$. We use expressions for $X=Y_0$ just found and commutation relations.

Suppose we know (5) with coefficients a,b,c given by (6), (7), (8) for an element $X\in\mathfrak{m}$ and want to find expressions $\widetilde{R}_\lambda(M)P_{\lambda,\nu;\sigma}$ for the element

$$M=[X,L]\in\mathfrak{m}, \tag{15}$$

where L is an element in $\mathfrak{g}$. From (15) we have

$$\widetilde{R}_\lambda(M) = \widetilde{R}_\lambda(X)R_\lambda(L) - R_\lambda(L)\widetilde{R}_\lambda(X).$$

Multiplying this equality by $P_{\lambda,\nu,\sigma}$ from the right and using (4) with L instead of g, we obtain expression (5) for $\widetilde{R}_\lambda(M)$ where

$$\begin{aligned} K_\sigma(M) &= K_\sigma(X)T_{2-n-\sigma}(L) - T_{-n-\sigma}(L)K_\sigma(X), \\ E_\sigma(M) &= E_\sigma(X)T_{2-n-\sigma}(L) - T_{2-n-\sigma}(L)E_\sigma(X), \\ C(M) &= C(X)T_{2-n-\sigma}(L) - T_{4-n-\sigma}(L)C(X). \end{aligned}$$

For example, for $M=M_{kn}$ we take $X=Y_0$, $L=L_{kn}$; for $M=Y_{kn}$ we take $X=(1/2)M_{kn}$ $L=L_{kn}$ and so on. Expanded expressions for operators $K_\sigma(M)$ turn out to be rather cumbersome, we reduce them to products (compositions) of differential operators. Omitting long analytical computations, let us bring the result.

Introduce the following differential operators on S:

$$Z_k(\sigma) = s_k(\Delta_1 - \Delta_2) + (2\sigma + n)B_k - (\sigma + n - 2)(\sigma + p)s_k,$$
$$V_m(\sigma) = s_m(\Delta_1 - \Delta_2) - (2\sigma + n)B_m + (\sigma + n - 2)(\sigma + q)s_m,$$

where $k = 1, \dots, p$, $m = p + 1, \dots, n$.

Then we have

for $X = M_{km}$ $(k \leqslant p < m)$:

$$K_\sigma(M_{km}) = -Z_k(\sigma + 1)V_m(\sigma) - V_m(\sigma + 1)Z_k(\sigma),$$
$$E_\sigma(M_{km}) = -s_m Z_k(\sigma) - s_k V_m(\sigma),$$
$$C(M_{km}) = -2s_k s_m;$$

for $X = Y_{km}$ $(k \leqslant p < m)$:

$$K_\sigma(Y_{km}) = V_m(\sigma)V_m(\sigma + 1) + Z_k(\sigma)Z_k(\sigma + 1),$$
$$E_\sigma(Y_{km}) = s_k Z_k(\sigma) + s_m V_m(\sigma),$$
$$C(Y_{km}) = s_k^2 + s_m^2;$$

for $X = Y_{km}$ $(k < m \leqslant p)$:

$$K_\sigma(Y_{km}) = Z_k(\sigma + 1)Z_k(\sigma) - Z_m(\sigma + 1)Z_m(\sigma),$$
$$E_\sigma(Y_{km}) = s_k Z_k(\sigma) - s_m Z_(\sigma),$$
$$C(Y_{km}) = s_k^2 - s_m^2;$$

for $X = M_{km}$ $(1 \leqslant k < m \leqslant p)$:

$$K_\sigma(M_{km}) = Z_k(\sigma + 1)Z_m(\sigma) + Z_m(\sigma + 1)Z_k(\sigma),$$
$$E_\sigma(M_{km}) = s_k Z_m(\sigma) + s_m Z_k(\sigma),$$
$$C(M_{km}) = 2s_k s_m.$$

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Vladimir F. Molchanov
Derzhavin Tambov State University
Internatsionalnaya, 33
392000 Tambov
Russia
e-mail: v.molchanov@bk.ru

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On p-adic Colligations and 'Rational Maps' of Bruhat–Tits Trees

Yury A. Neretin

Abstract. Consider matrices of order $k+N$ over p-adic field determined up to conjugations by elements of $GL(N)$ over p-adic integers. We define a product of such conjugacy classes and construct the analog of characteristic functions (transfer functions), they are maps from Bruhat–Tits trees to Bruhat–Tits buildings. We also examine categorical quotient for usual operator colligations.

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1. Introduction

1.1. Notation

Denote by $1 = 1_\alpha$ the unit matrix of order α. Below K is an infinite field[1], $\mathbb{K}$ is a locally compact non-Archimedian field, $\mathbb{O} \subset \mathbb{K}$ is the ring of integers. In both cases we keep in mind the p-adic fields. Let $\mathrm{Mat}(n) = \mathrm{Mat}(n, K)$ be the space of matrices of order n over K, $\mathrm{GL}(n, K)$ the group of invertible matrices of order n. We say that an $\infty \times \infty$ matrix g is *finite* if $g - 1$ has finite number of nonzero matrix elements[2]. Denote by $\mathrm{Mat}(\infty) = \mathrm{Mat}(\infty, K)$ the space of finite $\infty \times \infty$ matrices, by $\mathrm{GL}(\infty, K)$ the group of finite invertible finite matrices.

1.2. Colligations

Consider the space $\mathrm{Mat}(\alpha + \infty, K)$ of finite block complex matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of size $(\alpha+\infty)\times(\alpha+\infty)$. Represent the group $\mathrm{GL}(\infty, K)$ as the group of matrices

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[1] We prefer infinite fields, otherwise the rational function (6) is not well defined.

[2] Thus 1_∞ is finite and 0 is not finite.

of the form $\begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}$ of size $\alpha+\infty$. Consider conjugacy classes of $\mathrm{Mat}(\alpha+\infty, K)$ with respect to $\mathrm{GL}(\infty, K)$, i.e., matrices determined up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}^{-1}, \qquad \text{where } u \in \mathrm{GL}(\infty, \mathbb{K}). \tag{1}$$

We call conjugacy classes by *colligations* (another term is '*nodes*'). Denote by

$$\mathrm{Coll}(\alpha) = \mathrm{Coll}(\alpha, K)$$

the set of equivalence classes. There is a natural multiplication on $\mathrm{Coll}(\alpha, K)$, it is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} p & q \\ r & t \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 & q \\ 0 & 1 & 0 \\ r & 0 & t \end{pmatrix} = \begin{pmatrix} ap & b & aq \\ cp & d & cq \\ r & 0 & t \end{pmatrix}. \tag{2}$$

The size of the last matrix is

$$\alpha + \infty + \infty = \alpha + \infty.$$

The following statement is straightforward.

Proposition 1.

a) *The $\circ$-multiplication is a well-defined operation*

$$\mathrm{Coll}(\alpha) \times \mathrm{Coll}(\alpha) \to \mathrm{Coll}(\alpha).$$

b) *The $\circ$-multiplication is associative.*

There is a way to visualize this multiplication. We write the following 'perverse' equation for eigenvalues:

$$\begin{pmatrix} q \\ x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ \lambda x \end{pmatrix}, \tag{3}$$

where $\lambda \in K$. Equivalently,

$$q = ap + \lambda bx; \tag{4}$$

$$x = cp + \lambda dx. \tag{5}$$

We express x from (5),

$$x = (1 - \lambda d)^{-1} cp,$$

substitute it to (4), and get

$$q = \chi_g(\lambda) p,$$

where $\chi_g(\lambda)$

$$\chi_g(\lambda) = a + \lambda b (1 - \lambda d)^{-1} c \tag{6}$$

is a rational function $K \to \mathrm{Mat}(\alpha)$. It is called *characteristic function of g*. The following statement is obvious.

Proposition 2. *If g_1 and g_2 are contained in the same conjugacy class, then their characteristic functions coincide.*

The next statement can be verified by a straightforward calculation (for a more reasonable proof, see below Theorem 20).

Theorem 3. $$\chi_{g\circ h}(\lambda) = \chi_g(\lambda)\chi_h(\lambda).$$

Theorem 4. *Let K be algebraically closed. Then any rational map $K \to \mathrm{Mat}(\alpha, K)$ regular at 0 has the form $\chi_g(\lambda)$ for a certain $g \in \mathrm{Mat}(\infty, K)$.*

See, e.g., [5], Theorem 19.1.

1.3. Origins of the colligations

The colligations and the characteristic functions appeared independently in spectral theory of non-self-adjoint operators (M.S. Livshits, 1946) and in system theory, see, e.g., [3, 5, 8, 9, 13–15, 28, 30]. It seems that in both cases there are no visible reasons to pass to p-adic case.

However, colligations and colligation-like objects arose by independent reasons in representation theory of infinite-dimensional classical groups, see [16, 25].

First, consider a locally compact non-Archimedean field $\mathbb{K}$ and the double cosets
$$M = \mathrm{SL}(2, \mathbb{O}) \setminus \mathrm{SL}(2, \mathbb{K}) / \mathrm{SL}(2, \mathbb{O}).$$
The space of functions on M is a commutative algebra with respect to the convolution on $\mathrm{SL}(2, \mathbb{K})$. This algebra acts in the space of $\mathrm{SL}(2, \mathbb{O})$-fixed vectors of any unitary representation of $\mathrm{SL}(2, \mathbb{K})$. Next (see Ismagilov [10], 1967), let us replace $\mathbb{K}$ by a non-Archimedean non-locally compact field (i.e., the residue field is infinite [10] or the norm group is non-discrete [12]). Then there is no convolution, however double cosets have a natural structure of a semigroup, and this semigroup acts in the space of $\mathrm{SL}(2, \mathbb{O})$-fixed vectors of any unitary representation of $\mathrm{SL}(2, \mathbb{K})$. In particular, this allows to classify all irreducible unitary representations of $\mathrm{SL}(2, \mathbb{K})$ having a non-zero $\mathrm{SL}(2, \mathbb{O})$-fixed vector.

It appeared that these phenomena (semigroup structure on double cosets $L \setminus G/L$ for infinite dimension groups[3,4] and actions of this semigroup in the space of L-fixed vectors) are quite general, see, e.g., [16, 18, 19, 24–26].

In [23] there was proposed a way to construct representations of infinite-dimensional p-adic groups, in particular there appeared semigroups of double cosets and p-adic colligation-like structures. The present work is a simplified parallel of [23]. If we look to the equivalence (1), then a p-adic field is an representative of non-algebraically closed fields. However, [23] suggests another equivalence,
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}^{-1}, \qquad \text{where } u \in \mathrm{GL}(\infty, \mathbb{O}) \tag{7}$$
(we conjugate by the group $\mathrm{GL}(\infty, \mathbb{O})$ of integer matrices). Below we construct analogs of characteristic functions for this equivalence and get 'rational' maps from

[3] There is a elementary explanation initially proposed by Olshanski: such semigroups are limits of Hecke-type algebras at infinity. For more details, see [22].

[4] Conjugacy classes are special cases of double cosets, conjugacy classes G with respect to L are double cosets $L \setminus (G \times L)/L$, where L is embedded to $G \times L$ diagonally, $l \mapsto (l, l)$.

Bruhat–Tits trees to Bruhat–Tits buildings (for $\alpha = 1$ we get maps from trees to trees), the characteristic function (6) is its boundary value on the absolute of the tree.

It is interesting that maps of this type arise in theory of Berkovich rigid analytic spaces[5], see [1, 2, 4]. However I do not understand links between two points of view. For instance, we show that any rational map of a projective line $\mathbb{P}\mathbb{Q}_p^1$ to itself admits a continuation to the Bruhat–Tits tree, and such continuations are enumerated by the set $\mathrm{GL}(\infty, \mathbb{Q}_p)/\,\mathrm{GL}(\infty, \mathbb{O}_p)$. In Berkovich theory continuations of this type are canonical.

1.4. Structure of the paper

In Section 2 we consider characteristic functions over algebraically closed field. We discuss categorical quotient $[\mathrm{Coll}(\alpha)]$ of $\mathrm{Coll}(\alpha)$ with respect to the equivalence (1), the main statement here is Theorem 14. In Section 3 we examine the case $\alpha = 1$. We show that the semigroup $[\mathrm{Coll}(1)]$ is commutative. Also we show that for non-algebraically closed field any rational function $K \to K$ is a characteristic function.

In Section 4 we consider p-adic fields and introduce characteristic functions for conjugacy classes of $\mathrm{GL}(\alpha + \infty, \mathbb{Q}_p)$ by $\mathrm{GL}(\infty, \mathbb{O}_p)$.

In Section 5 we briefly discuss conjugacy classes of $\mathrm{GL}(\alpha + m\infty, \mathbb{Q}_p)$ with respect to $\mathrm{GL}(\infty, \mathbb{O}_p)$.

2. Formalities. Algebraically closed fields

In this section K is an algebraically closed field. For exposition of basic classical theory, see the textbook of Dym [5], Chapter 19. See more in [9, 15, 30, 31]. Our ‘new’ element is the categorical quotient[6] (in a wider generality it was discussed in [21]).

Denote by $\mathbb{P}K^1$ the projective line over K. For an even-dimensional linear space W denote by $\mathrm{Gr}(W)$ the Grassmannian of subspaces of dimension $\frac{1}{2}\dim V$.

2.1. Colligations

Fix $\alpha \geqslant 0$, $N > 0$. Consider the space of matrices $\mathrm{Mat}(\alpha + N, K)$, we write its elements as block matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Consider the group $\mathrm{GL}(N, K)$, we represent its elements as block matrices $\begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}$. Denote by $\mathrm{Coll}_N(\alpha, K)$ the space of conjugacy classes of $\mathrm{Mat}(\alpha + N)$ with respect to $\mathrm{GL}(N, K)$, see (1). Denote by $[\mathrm{Coll}_N(\alpha, K)]$ the categorical quotient (see, e.g., [27]), i.e., the spectrum of the algebra of $\mathrm{GL}(N, K)$-invariant polynomials on $\mathrm{Mat}(\alpha + N)$.

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[5]In Berkovich theory objects are lager than trees and buildings. However, our ‘characteristic functions’ admit extensions to these larger objects.

[6]I have not met discussion of this topic, however sets of ‘nonsingular points’ of $\mathrm{Coll}(\alpha)$ and its completions were discussed in literature, see [9, 30].

2.2. Characteristic function

For an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{Mat}(\alpha + N)$ we assign the characteristic function

$$\chi_g(\lambda) = a + \lambda b(1 - \lambda d)^{-1}c, \qquad \lambda \text{ ranges in } K. \tag{8}$$

If d is invertible, we extend this function to the point $\lambda = \infty$ by setting

$$\chi_g(\infty) = a - bd^{-1}c.$$

Passing to the coordinate $s = \lambda^{-1}$ on $\mathbb{P}K^1$, we get

$$\chi_g(s) = a + b(s - d)^{-1}c.$$

Theorem 5. *Any rational function* $K \to \mathrm{Mat}(\alpha, K)$ *regular at* 0 *is a characteristic function of an operator colligation.*

See, e.g., [5], Theorem 19.1.

2.3. The characteristic function as a map $\mathbb{P}K^1 \to \mathrm{Gr}(K^{2\alpha})$

See [9, 15, 31]. If λ_0 is a regular point of $\chi_g(\lambda)$, we consider its graph $\mathcal{X}_g(\lambda_0)$,

$$\mathcal{X}_g(\lambda_0) \subset K^\alpha \oplus K^\alpha.$$

Singularities of rational maps of $\mathbb{P}K^1$ to a projective variety $\mathrm{Gr}(K^\alpha \oplus K^\alpha)$ are removable. Let us remove a singularity explicitly at a pole $\lambda = \lambda_0$. We can represent $\chi_g(\lambda)$ as

$$A(\lambda - \lambda_0) \begin{pmatrix} \frac{h_1}{(\lambda-\lambda_0)^{m_1}} & 0 & \dots \\ 0 & \frac{h_2}{(\lambda-\lambda_0)^{m_2}} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} B(\lambda - \lambda_0) + S(\lambda - \lambda_0), \tag{9}$$

where $A(\dots)$, $B(\dots)$ are polynomial functions $K \to \mathrm{Mat}(\alpha)$, $A(0)$, $B(0)$ are invertible, the exponents m_i satisfy $m_1 \geqslant m_2 \geqslant \dots$, and $S(\lambda)$ is a rational functions $K \to \mathrm{Mat}(\alpha)$ having zero of any prescribed order $M > 0$ (proof of this is a straightforward repetition of the Gauss elimination procedure). Denote by e_j the standard basis in K^α. Consider the subspace L in $K^\alpha \oplus K^\alpha$ generated by vectors

$$\begin{aligned} e_i \oplus 0, &\qquad \text{for } m_i > 0; \\ h_j e_j \oplus e_j, &\qquad \text{for } m_j = 0; \\ 0 \oplus e_l, &\qquad \text{for } m_l < 0. \end{aligned}$$

Applying the operator $A(0) \oplus B(0)^{-1}$ to L we get $\chi_g(\lambda_0)$.

2.4. An exceptional divisor

A characteristic function is not sufficient for a reconstruction of a colligation. Indeed, consider a block matrix of size $\alpha + k + l$

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

Then the characteristic function is independent of e.

For $g \in \mathrm{Coll}_N(\alpha)$ we define an additional invariant, a divisor[7] $\Xi_g \subset \mathbb{P}K^1$ in the following way: Ξ_g as the divisor of zeros of the polynomial

$$p_g(\lambda) = \det(1 - \lambda d)$$

plus $\lambda = \infty$ with multiplicity $N - \deg p_g$. In the coordinate $s = \lambda^{-1}$ this divisor is simply the set of eigenvalues of d.

Proposition 6.

$$\det \chi_g(\lambda) = \frac{\det \begin{pmatrix} a & -\lambda b \\ c & 1 - \lambda d \end{pmatrix}}{\det(1 - \lambda d)}.$$

Proof. We apply the formula for the determinant of a block matrix. □

Corollary 7. *The divisor Ξ_g contains the divisor of poles of* $\det \chi_g(\lambda)$.

Theorem 8. *For any rational function $\chi K \to \mathrm{Mat}(\alpha)$ regular at 0 there is a colligation g with characteristic function χ such that the divisor Ξ_g coincides with the divisors of poles of* $\det \chi(\lambda)$.

See [5], Theorem 19.8. Such colligations g are called *minimal*.

2.5. Invariants

Theorem 9. *A point $\mathfrak{g}$ of the categorical quotient $[\mathrm{Coll}_N(\alpha)]$ is uniquely determined by the characteristic function $\chi_{\mathfrak{g}}(\lambda)$ and the divisor $\Xi_{\mathfrak{g}}$.*

Proof. Let us describe $\mathrm{GL}(N, K)$-invariants on $\mathrm{Mat}(\alpha+N)$. A point $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(\alpha + N)$ can be regarded as the following collection of data:

a) the matrix d;
b) α vectors (columns $c[j]$ of c);
c) α covectors (rows $b[i]$ of b);
d) scalars a_{ij}.

The algebra of invariants (see [29], Section 11.8.1) is generated by the following polynomials

$$b[i]d^k c[j], \tag{10}$$

$$\mathrm{tr}\, d^k, \tag{11}$$

$$a_{ij}. \tag{12}$$

Expanding the characteristic function in λ,

$$\chi_g(\lambda) = a + \sum_{k=0}^{\infty} \lambda^{k+1} b d^k c$$

[7]i.e., a finite set with multiplicities.

we get in coefficients all the invariants (10), (12). Expanding

$$\ln p_g(\lambda) = \ln\det(1-\lambda d) = -\sum_{j=k}^{\infty} \frac{1}{k}\lambda^k \operatorname{tr} d^k,$$

we get all invariants (11). □

Corollary 10. *Any point of* $[\mathrm{Coll}_N(\alpha)]$ *has a representative of the form*

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix},$$

where e *is diagonal matrix and the colligation* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is minimal.*

2.6. ∘-product

Now we define the operation

$$\mathrm{Coll}_{N_1}(\alpha) \times \mathrm{Coll}_{N_2}(\alpha) \to \mathrm{Coll}_{N_1+N_2}(\alpha)$$

by the formula (2).

Theorem 11.

a) $\chi_{g\circ h}(\lambda) = \chi_g(\lambda)\chi_h(\lambda)$.
b) $\Xi_{g\circ h} = \Xi_g + \Xi_h$.

The statement b) is obvious, a) is well known (see a proof below, Theorem 20).

Corollary 12. *The* ∘*-multiplication is well defined as an operation on categorical quotients,*

$$[\mathrm{Coll}_{N_1}(\alpha)] \times [\mathrm{Coll}_{N_2}(\alpha)] \to [\mathrm{Coll}_{N_1+N_2}(\alpha)]$$

Proof. Indeed, invariants of $g \circ h$ are determined by invariants of g and h. □

2.7. The space $\mathrm{Coll}_\infty(\alpha)$

Consider the natural map

$$I_N := \mathrm{Mat}(\alpha+N) \to \mathrm{Mat}(\alpha+N+1)$$

defined by

$$I_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\chi_{I_N g}(\lambda) = \chi_g(\lambda);$$
$$\Xi_{I_N g} = \Xi_g + \{1\},$$

where $\{1\}$ is the point $1 \in K$.

Lemma 13. *The induced map* $[\mathrm{Coll}_N(\alpha)] \to [\mathrm{Coll}_{N+1}(\alpha)]$ *is an embedding.*

Proof. The restriction of invariants (10–12) defined on $\mathrm{Mat}(\alpha + N + 1)$ to the subspace $\mathrm{Mat}(\alpha + N)$ gives the same expressions for $\mathrm{Mat}(\alpha + N)$. □

Thus, we can define a space $[\mathrm{Coll}(\alpha)] = [\mathrm{Coll}_\infty(\alpha)]$ as an inductive limit

$$[\mathrm{Coll}_\infty(\alpha)] = \lim_{N\to\infty} [\mathrm{Coll}_N(\alpha)].$$

It is equipped with the associative $\circ$-multiplication.

Characteristic function of an element $g \in [\mathrm{Coll}_\infty(\alpha)]$ can be defined in two equivalent ways. The first way, we write the expression (8) for infinite matrix g. The second way. We choose large N such that g has the following representation

$$g = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \begin{matrix} \}\alpha \\ \}N \\ \}\infty \end{matrix} \qquad (13)$$

and write the characteristic function for the upper left block of the size $\alpha + N$.

Next, we define the *exceptional divisor* Ξ_g in $\mathbb{P}K^1$. We represent g in the form (13), write the exceptional divisor for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and add the point $\lambda = 1$ with multiplicity ∞ (in particular, the multiplicity of 1 always is infinity). Thus, we can regard the 'divisor' as a function

$$\xi : \mathbb{P}K^1 \to \mathbb{Z}_+ \cup \infty$$

satisfying the following condition.

a) $\xi(\lambda) = 0$ for all but finite number of λ.
b) $\xi(1) = \infty$, at all other points ξ is finite.

We reformulate statements obtained above in the following form. Denote by Γ_α the semigroup of rational maps $\chi : \mathbb{P}K^1 \to \mathrm{Mat}(\alpha)$ regular at the point $\lambda = 0$. Denote by Δ the set of all divisors in the sense described above. We equip Δ with the operation of addition.

Next, consider the subsemigroup $R_\alpha \subset \Gamma \times \Delta$ consisting of pairs (χ, Ξ) such that divisor of the denominator of $\det \chi(\lambda)$ is contained in the divisor Ξ.

Theorem 14. *The map* $g \mapsto (\chi_g, \Xi_g)$ *is an isomorphism of semigroups*

$$[\mathrm{Coll}_\infty(\alpha)] \quad \textit{and} \quad R_\alpha.$$

Notice, that the semigroup $[\mathrm{Coll}_\infty(\alpha)$ itself is not a product of semigroup of characteristic functions and an Abelian semigroup. A similar object appeared in [16], IX.2.

3. The case $\alpha = 1$

3.1. Commutativity

Theorem 15. *The semigroup* $[\mathrm{Coll}_\infty(1)]$ *is commutative.*

Proof. Indeed, Γ_1 is commutative, therefore $\Gamma_1 \times \Delta$ is commutative. □

REMARK. The semigroup $\mathrm{Coll}_\infty(1)$ is not commutative,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and blocks 'd' on the right-hand side have different Jordan forms. □

3.2. Commutativity – straightforward proof

The proof given below is not necessary in the context of this paper. However, it shows that the commutativity in certain sense is a non-obvious fact (in particular, this proof can be modified for proofs of non-commutativity of $\circ$-products in some cases discussed in [19]).

First, an element of $\mathrm{Coll}_N(1)$ in a general position can be reduced by a conjugation to the form

$$\begin{pmatrix} a & b_1 & b_2 & b_3 & \dots \\ c_1 & \lambda_1 & 0 & 0 & \dots \\ c_2 & 0 & \lambda_2 & 0 & \dots \\ c_3 & 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where λ_j are pairwise distinct. To be short set $N = 2$. Consider two matrices

$$g = \begin{pmatrix} p & b_1 & b_2 \\ c_1 & \lambda_1 & 0 \\ c_2 & 0 & \lambda_2 \end{pmatrix}, \qquad h = \begin{pmatrix} p & q_1 & q_2 \\ r_1 & \mu_1 & 0 \\ r_2 & 0 & \mu_2 \end{pmatrix}$$

with λ_1, λ_2, μ_1, μ_2 being pairwise distinct. We evaluate

$$S = g \circ h = \begin{pmatrix} ap & b_1 & b_2 & aq_1 & aq_2 \\ c_1 p & \lambda_1 & 0 & c_1 q_1 & c_1 q_2 \\ c_2 p & 0 & \lambda_2 & c_2 q_1 & c_2 q_2 \\ r_1 & 0 & 0 & \mu_1 & 0 \\ r_2 & 0 & 0 & 0 & \mu_2 \end{pmatrix}$$

and

$$T = h \circ g = \begin{pmatrix} p & 0 & 0 & q_1 & q_2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ r_1 & 0 & 0 & \mu_1 & 0 \\ r_2 & 0 & 0 & 0 & \mu_2 \end{pmatrix} \begin{pmatrix} a & b_1 & b_2 & 0 & 0 \\ c_1 & \lambda_1 & 0 & 0 & 0 \\ c_2 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} ap & b_1 p & b_2 p & q_1 & q_2 \\ c_1 & \lambda_1 & 0 & 0 & 0 \\ c_2 & 0 & \lambda_2 & 0 & 0 \\ ar_1 & b_1 r_1 & b_2 r_1 & \mu_1 & 0 \\ ar_2 & b_1 r_2 & b_2 r_2 & 0 & \mu_2 \end{pmatrix}.$$

Proposition 16. *In this notation,*

$$T = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}^{-1} S \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix},$$

where

$$U = U_+^{-1} U_d U_-,$$

matrices U_+, U_- are upper (lower) triangular respectively,

$$U_+ = \begin{pmatrix} 1 & 0 & \frac{c_1 q_1}{\lambda_1 - \mu_1} & \frac{c_1 q_2}{\lambda_1 - \mu_2} \\ 0 & 1 & \frac{c_2 q_1}{\lambda_2 - \mu_1} & \frac{c_2 q_2}{\lambda_2 - \mu_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad U_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{b_1 r_1}{\lambda_1 - \mu_1} & \frac{b_2 r_1}{\lambda_2 - \mu_1} & 1 & 0 \\ \frac{b_1 r_2}{\lambda_1 - \mu_2} & \frac{b_2 r_2}{\lambda_2 - \mu_2} & 0 & 1 \end{pmatrix},$$

and U_d is a diagonal matrix with entries

$$p + \frac{q_1 r_1}{\lambda_1 - \mu_1} + \frac{q_2 r_2}{\lambda_1 - \mu_2}, \quad p + \frac{q_1 r_1}{\lambda_2 - \mu_1} + \frac{q_2 r_2}{\lambda_2 - \mu_2},$$
$$\left(a + \frac{b_1 c_1}{\mu_1 - \lambda_1} + \frac{b_2 c_2}{\mu_1 - \lambda_2} \right)^{-1}, \quad \left(a + \frac{b_1 c_1}{\mu_2 - \lambda_1} + \frac{b_2 c_2}{\mu_2 - \lambda_2} \right)^{-1}.$$

Proof. We represent T and S as block matrices,

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

of size $(1+4) \times (1+4)$. We must verify equalities

$$UT_{22} = S_{22} U, \tag{14}$$
$$UT_{21} = S_{21}, \qquad T_{12} = S_{12} U. \tag{15}$$

Represent the first equality in the form

$$U_d (U_- T_{22} U_-^{-1}) = (U_+ S_{22} U_+^{-1}) U_d. \tag{16}$$

The matrices $U_\pm$ are chosen in such a way that

$$U_- T_{22} U_-^{-1} = U_+ S_{22} U_+^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}.$$

Therefore (16) holds for any diagonal matrix U_d. It remains to choose U_d to satisfy (15). □

3.3. Linear-fractional transformations

Proposition 17. *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *be a nondegenerate* 2×2 *matrix. Let* $g \in \mathrm{Coll}_N(\alpha)$. *Let* $\gamma g + \delta$ *be nondegenerate*[8]. *Let* $\chi_g(s)$ *be the characteristic function written in the coordinate* $s = \lambda^{-1}$ *Then the characteristic function of the colligation*

$$h = (\alpha g + \beta)(\gamma g + \delta)^{-1}$$

is

$$\left(\alpha \chi_g\left(\frac{\alpha s + \beta}{\gamma s + \delta}\right) + \beta\right)\left(\gamma \chi_g\left(\frac{\alpha s + \beta}{\gamma s + \delta}\right) + \delta\right)^{-1}.$$

Proof. We represent the equation

$$\begin{pmatrix} q \\ sx \end{pmatrix} = h \begin{pmatrix} p \\ x \end{pmatrix}$$

as

$$\begin{pmatrix} \alpha q + \beta p \\ \alpha s x + \beta x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma q + \delta p \\ \gamma s x + \delta x \end{pmatrix}.$$

Passing to the variable $y = (\gamma s + \delta)x$ we get

$$\begin{pmatrix} \alpha q + \beta p \\ (\alpha s + \beta)(\gamma s + \delta)^{-1} x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma q + \delta p \\ x \end{pmatrix}.$$

This implies the desired statement. □

3.4. Non-algebraically closed fields

Now let K be a non-algebraically closed infinite field.

Proposition 18. *Let*

$$w(\lambda) = \frac{u(\lambda)}{v(\lambda)}$$

be a rational function on $\mathbb{P}K^1$ *such that* $v(0) \neq 0$. *Then it is a characteristic function of a certain element of* $\mathrm{Coll}_\infty(1)$.

---

[8] This is independent on the choice of a representative.

Proof. By induction. Pass to the variable $s = \lambda^{-1}$. We say that degree of $w(s)$ is $\deg v(s)$ (since $s = \infty$ is not a pole of $w(s)$, we have $\deg u(s) \leqslant \deg v(s)$). For functions of degree 1 the statement is correct. Assume that the statement is correct for functions of degree $< n$. Consider a function $w(s)$ of degree n. Take a linear fractional transformation

$$\widetilde{w}(s) := \frac{\alpha w(s) + \beta}{\gamma w(s) + \delta}$$

such that $w(s)$ has a pole at some finite point σ and a zero at some point τ. Then we can decompose $\widetilde{w}(s)$:

$$\widetilde{w}(s) = \frac{s - \tau}{s - \sigma}\, y(s),$$

where $y(s)$ is a rational function of degree $< n$. Both factors are characteristic functions, therefore $\widetilde{w}(s)$ also is a characteristic function. □

4. Maps of Bruhat–Tits trees

Now $\mathbb{K}$ is the p-adic field $\mathbb{Q}_p$ and $\mathbb{O} \subset \mathbb{K}$ is the ring of integers. All considerations below can be automatically extended to arbitrary locally compact non-Archimedian fields (few words must be changed).

4.1. Colligations

Denote by $\mathfrak{Coll}_N(\alpha)$ the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over $\mathbb{K}$ of size $\alpha + N$ defined up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}^{-1}, \qquad \text{where } u \in \mathrm{GL}(N, \mathbb{O}). \tag{17}$$

We define the $\circ$-product

$$\mathfrak{Coll}_{N_1}(\alpha) \times \mathfrak{Coll}_{N_2}(\alpha) \to \mathfrak{Coll}_{N_1+N_2}(\alpha)$$

by the same formula (2).

As above we define $\mathfrak{Coll}_\infty(\alpha)$ and the associative $\circ$-product on $\mathfrak{Coll}_\infty(\alpha)$.

4.2. Bruhat–Tits buildings

Consider a linear space $\mathbb{K}^n$ over $\mathbb{K}$. A *lattice* R in $\mathbb{K}^n$ is a compact $\mathbb{O}$-submodule in $\mathbb{K}^n$ such that $\mathbb{K} \cdot R = \mathbb{K}^n$. In other words (see, e.g., [20, 32]), in a certain basis $e_j \in \mathbb{K}^n$, a submodule R has the form $\oplus_j \mathbb{O} e_j$. The space Lat_n of all lattices is a homogeneous space,

$$\mathrm{Lat}_n \simeq \mathrm{GL}(n, \mathbb{K}) / \mathrm{GL}(n, \mathbb{O}).$$

We intend to construct two simplicial complexes BT_n and BT_n^*.

1. Consider an oriented graph, whose vertices are lattices in $\mathbb{K}^n$. We draw arrow from a vertex R to a vertex T if $T \supset R \supset pT$. If k vertices are pairwise connected by arrows, then we draw a simplex with such vertices. In this way we get a simplicial complex BT_n, all maximal simplices have dimension n. The group $\mathrm{GL}(n, \mathbb{K})$ acts transitively on the set of all maximal simplices (and also on the set of simplices of each given dimension $j = 0, 1, \ldots, n$).
2. Consider a non-oriented graph whose vertices are lattices defined up to a dilatation, $R \sim R'$ if $R = \lambda R'$ for some $\lambda \in \mathbb{K}^\times$. Denote
$$\mathrm{Lat}_n^* := \mathrm{Lat}_n / \mathbb{K}^\times.$$
We connect two vertices $R \not\sim T$ by an edge if for some λ we have $pT \subset \lambda R \subset T$. If k vertices are pairwise connected by edges, then we draw a simplex with such vertices. We get a simplicial complex BT_n^*, dimensions of all maximal simplices are $n-1$. The projective linear group
$$\mathrm{PGL}(n, \mathbb{K}) = \mathrm{GL}(n, \mathbb{K}) / \mathbb{K}^\times$$
acts transitively on the set of all simplices of a given dimension $j = 0, 1, \ldots,$ $n-1$.

We have a natural map
$$\mathrm{BT}_n(\mathbb{K}) \to \mathrm{BT}_n^*(\mathbb{K}),$$
we send a lattice (a vertex) to the corresponding equivalence class, this induces a map of graphs. Moreover, vertices of a k-dimensional simplex fall to vertices of a simplex of dimension $\leqslant k$.

These complexes are called *Bruhat–Tits buildings*, see, e.g., [7, 20]. For $n = 2$ the building $\mathrm{BT}_2(\mathbb{K})$ is an infinite tree, each vertex is an end of $(p+1)$ edges.

4.3. Construction of characteristic functions

Consider the space $\mathbb{K}^2 = \mathbb{K}^1 \oplus \mathbb{K}^1$. For any lattice $R \subset \mathbb{K}^2$ consider the lattice
$$R \otimes \mathbb{O}^N \subset \mathbb{K}^2 \otimes \mathbb{K}^N = \mathbb{K}^N \oplus \mathbb{K}^N.$$
For a colligation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we write the equation
$$\begin{pmatrix} q \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}. \tag{18}$$
Consider the set $\chi_g(R)$ of all $q \oplus p \in \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$ such that there are $y \oplus x \in R \otimes \mathbb{K}^N$ satisfying equation (18).

Proposition 19.

a) *The sets $\chi_g(R)$ are lattices.*
b) *If R, $T \in \mathrm{Lat}_2$ are connected by an arrow, then $\chi_g(R)$ and $\chi_g(T)$ are connected by an arrow or coincide.*
c) *A lattice $\chi_g(R)$ depends on the conjugacy class containing g and not on g itself.*
d) *$\chi_g(\lambda R) = \lambda\, \chi_g(R)$ for $\lambda \in \mathbb{K}^\times$.*

The proof is given in the next subsection.

Thus χ_g is a map

$$\chi_g : \mathrm{BT}_2(\mathbb{K}) \to \mathrm{BT}_n(\mathbb{K}).$$

Since it commutes with multiplications by scalars, we get also a well-defined map

$$\chi_g^* : \mathrm{BT}_2(\mathbb{K}) \to \mathrm{BT}_n^*(\mathbb{K}).$$

4.4. Reformulation of the definition

Consider the space

$$H = \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha \oplus \mathbb{K}^N \oplus \mathbb{K}^N$$

consisting of vectors with coordinates q, p, y, x. Consider the following subspaces and submodules in H:

- $G \subset H$ is the graph of g;
- $U = 0 \oplus 0 \oplus \mathbb{K}^N \oplus \mathbb{K}^N$;
- $V = \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha \oplus 0 \oplus 0$;
- the $\mathbb{O}$-submodule $S = \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha \oplus (R \otimes \mathbb{K}^N)$.

Consider the intersection $G \cap S$ and its projection to V along U. The result is $\chi_g(R)$.

Proof of Proposition 19. The statements a), b), d) follow from new version of the definition, c) follows from $\mathrm{GL}(N, \mathbb{O})$-invariance of $R \otimes \mathbb{O}^N$. □

4.5. Products

Now we wish to obtain an analog of Theorem 11. For this purpose, we need a definition of multiplication of lattices.

Let S, $T \subset \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$ be lattices. We define their *product* ST as the set of all $u \oplus w \in \mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$ such that there is $v \in \mathbb{K}^\alpha$ satisfying $u \oplus v \in S$, $v \oplus w \in T$. This is the usual product of relations (or multi-valued maps), see, e.g., [16].

Theorem 20.

$$\chi_{g\circ h}(S) = \chi_g(S)\chi_h(S).$$

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Let $r = \chi_g(S)q$, $q = \chi_h(S)p$. Then there are z, y, y', x such that

$$y \oplus x \in R \otimes \mathbb{O}^{N_1}, \qquad y' \oplus x' \in R \otimes \mathbb{O}^{N_2}$$

satisfying

$$\begin{pmatrix} r \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ x \end{pmatrix}, \qquad \begin{pmatrix} q \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p \\ x' \end{pmatrix}.$$

Then

$$\begin{pmatrix} r \\ y \\ y' \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ x \\ y' \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & \beta \\ 0 & 1 & 0 \\ \gamma & 0 & \delta \end{pmatrix} \begin{pmatrix} q \\ x \\ x' \end{pmatrix}.$$

This proves the desired statement. □

Consider the natural projection

$$\text{pr} : \mathfrak{Coll}_N(\alpha) \to \text{Coll}_N(\alpha).$$

Formally, we have two characteristic functions of an element of $\mathfrak{Coll}_N(\alpha)$, one is defined on $\mathbb{PK}^1$, another on $\text{BT}_2(\mathbb{K})$. In fact, the second function is the value of the first on the boundary of the building. Now we intend to explain this.

4.6. Convergence of lattices to subspaces

We say that a sequence of lattices $R_j \in \text{Lat}_n$ converges to a subspace $L \subset \mathbb{K}^n$ if

a) For each ε for sufficiently large j a lattice R_j is contained in the ε-neighborhood of L.
b) For each compact set $S \subset L$ we have $R_j \cap L \subset S$ for sufficiently large j.

Proposition 21. *Let a sequence $R^j \in \text{Lat}_n$ converge to a subspace $L \subset \mathbb{O}^n$. Let $M \subset \mathbb{O}^n$ be a subspace. Let $\pi : \mathbb{O}^n \to \mathbb{O}^n/L$ be the natural projection. Then*

a) *$R_j \cap M$ converges to $L \cap M$.*
b) *$\pi(R_j)$ converges to $\pi(M)$.*

The statement is obvious.

We say that a sequence $R_j^* \in \text{Lat}^*$ converges to a subspace L if we have a convergence $R_j \to L$ for some representatives of R_j^*. Notice that a sequence R_j^* can have many limits in this sense[9,10]. However a limit subspace of a given dimension is unique.

4.7. Boundary values

Proposition 22. *Let $g \in \mathfrak{Coll}_N(\alpha)$. Let $\lambda \in \mathbb{PK}^1$ be a nonsingular point of the characteristic function $\chi_{\text{pr}(g)}(\lambda)$ defined on $\mathbb{PK}^1$. Let L be the line in $\mathbb{K}^2$ corresponding λ. Let $R_j \in \text{Lat}_n(\mathbb{K})$ converges to ℓ. Then $\chi_g(R_j)$ converges to $\chi_{\text{pr}(g)}(\lambda)$*

Proof. The statement follows from Subsection 4.4 and Proposition 21. □

4.8. Rational maps of Bruhat–Tits trees

Corollary 23. *Any rational map $\mathbb{PK}^1 \to \mathbb{PK}^1$ can be extended to a continuous map of Bruhat–Tits trees, such that image of a vertex is a vertex and image of an edge is an edge or a vertex.*

Proof. Represent a rational map as a characteristic function of a colligation $q \in \text{Coll}_\infty(1)$. We take a colligation $g \in \mathfrak{Coll}_\infty(1)$ such that $\text{pr}(g) = q$, and take the corresponding map $\text{BT}_2^*(\mathbb{K}) \to \text{BT}_2^*(\mathbb{K})$. □

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[9] Moreover, 0 and $\mathbb{O}^n$ are limits of all sequences according our definition.
[10] See [17].

5. Rational maps of buildings

5.1. m-colligations

Fix $\alpha \geqslant 0$, $m \geqslant 1$. Let $N > 0$. Consider the space $\mathrm{Mat}(\alpha + mN, \mathbb{K})$ of block matrices of size $\alpha + N + \cdots + N$. Denote by $\mathfrak{Coll}_N(\alpha|m)$ the set of such matrices up to the equivalence

$$\begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u \end{pmatrix} \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u \end{pmatrix}^{-1},$$

where $u \in \mathrm{GL}(N, \mathbb{O})$. (19)

We define a multiplication

$$\mathfrak{Coll}_{N_1}(\alpha|m) \times \mathfrak{Coll}_{N_2}(\alpha|m) \to \mathfrak{Coll}_{N_1+N_2}(\alpha|m)$$

by

$$\begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \circ \begin{pmatrix} p & q_1 & \dots & q_m \\ r_1 & t_{11} & \dots & t_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ r_m & t_{m1} & \dots & t_{mm} \end{pmatrix}$$
$$= \begin{pmatrix} a & b_1 & 0 & \dots & b_m & 0 \\ c_1 & d_{11} & 0 & \dots & d_{1m} & 0 \\ 0 & 0 & 1_{N_2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_m & d_{m1} & 0 & \dots & d_{mm} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1_{N_2} \end{pmatrix} \begin{pmatrix} p & 0 & q_1 & \dots & 0 & q_m \\ 0 & 1_{N_1} & 0 & \dots & 0 & 0 \\ r_1 & 0 & t_{11} & \dots & 0 & t_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1_{N_1} & 0 \\ r_m & t_{m1} & 0 & \dots & 0 & t_{mm} \end{pmatrix}$$
$$= \begin{pmatrix} a & b_1 & aq_1 & \dots & b_m & aq_m \\ c_1p & d_{11} & c_1q_1 & \dots & d_{1m} & c_1q_m \\ r_1 & 0 & t_{11} & \dots & 0 & t_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_mp & d_{m1} & c_mq_1 & \dots & d_{mm} & c_mq_m \\ r_m & 0 & t_{m1} & \dots & 0 & t_{mm} \end{pmatrix}.$$

5.2. Characteristic functions

For a lattice $R \in \mathrm{Lat}_{2m}(\mathbb{K})$ consider the lattice

$$R \otimes \mathbb{O}^N \subset \mathbb{K}^{2m} \otimes \mathbb{K}^N = (\mathbb{K}^m \otimes \mathbb{K}^N) \oplus (\mathbb{K}^m \otimes \mathbb{K}^N)$$

For $g \in \mathrm{Mat}(\alpha + km)$ we write the following equation

$$\begin{pmatrix} q \\ y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} p \\ x_1 \\ \vdots \\ x_m \end{pmatrix}, \tag{20}$$

where p, q range in $\mathbb{K}^\alpha$, and $x_j, y_j \in \mathbb{K}^N$. Denote by $\chi_g(R)$ the set of all $q \oplus p \in \mathbb{K}^{2\alpha}$ such that there exists $y \oplus x \in R \otimes \mathbb{O}^N$, for which equality (20) holds.

Theorem 24.

a) *$\chi_g(R)$ is a lattice in $\mathbb{K}^\alpha \oplus \mathbb{K}^\alpha$.*
b) *The characteristic function $\chi_g(R)$ is an invariant of the equivalence* (19).
c) *The map $\chi_g : \mathrm{Lat}_{2m} \to \mathrm{Lat}_{2\alpha}$ induces maps*

$$\mathrm{BT}_{2m} \to \mathrm{BT}_{2\alpha}, \qquad \mathrm{BT}^*_{2m} \to \mathrm{BT}^*_{2\alpha}.$$

d) *For any $g \in \mathfrak{Coll}_{N_1}(\alpha|m)$, $h \in \mathfrak{Coll}_{N_2}(\alpha|m)$, the following identity holds*

$$\chi_{g \circ h}(R) = \chi_g(R)\chi_h(R).$$

Proof. Repeats the proof given above for $m = 1$. See also a more sophisticated object in [23]. □

5.3. Extension to the boundary

Next (see [18, 21]), we extend characteristic functions to the distinguished boundaries of buildings. Let $S \in \mathrm{Mat}(m, \mathbb{K})$. Again write equation (20). We say $q = \chi_g(S)p$ if there exists y such that q, p, y, $x = Sy$ satisfy equation (20). In other words,

$$\chi_g(S) = a + b\widetilde{S}(1 - d\widetilde{S})^{-1}c,$$

where $\widetilde{S} = S \otimes 1_N$,

$$\widetilde{S} = \begin{pmatrix} s_{11} \cdot 1_N & \dots & s_{1m} \cdot 1_N \\ \vdots & \ddots & \vdots \\ s_{m1} \cdot 1_N & \dots & s_{mm} \cdot 1_N \end{pmatrix}.$$

Theorem 25.

a) *For any $g \in \mathfrak{Coll}_{N_1}(\alpha|m)$, $h \in \mathfrak{Coll}_{N_2}(\alpha|m)$,*

$$\chi_{g \circ h}(S) = \chi_g(S)\chi_h(S).$$

b) *If a sequence of lattices $R_j \in \mathrm{Lat}(\mathbb{K}^m \oplus \mathbb{K}^m)$ converges to the graph of S, then $\chi_g(R_j)$ converges to $\chi_g(S)$.*

The proof is the same as above for $m = 1$.

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Yury A. Neretin
Math. Dept., University of Vienna
Nordbergstrasse 15
Vienna, Austria

and

Institute for Theoretical and Experimental Physics
Bolshaya Cheremushkinskaya 25
Moscow 117259, Russia

and

Mech. Math. Dept., Moscow State University
Vorob'evy Gory
Moscow, Russia

and

The Institute of Information Transmission Problems
(Kharkevich Institute)
Moscow, Russia
e-mail: neretin@mccme.ru
URL: www.mat.univie.ac.at/∼neretin
URL: www.th.itep.ru/∼neretin

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Resonances for the Laplacian: the Cases BC_2 and C_2 (except $\mathrm{SO}_0(p, 2)$ with $p > 2$ odd)

J. Hilgert, A. Pasquale and T. Przebinda

Abstract. Let $\mathsf{X} = \mathrm{G}/\mathrm{K}$ be a Riemannian symmetric space of the noncompact type and restricted root system BC_2 or C_2 (except for $\mathrm{G} = \mathrm{SO}_0(p, 2)$ with $p > 2$ odd). The analysis of the meromorphic continuation of the resolvent of the Laplacian of X is reduced from the analysis of the same problem for a direct product of two isomorphic rank-one Riemannian symmetric spaces of the noncompact type which are not isomorphic to real hyperbolic spaces. We prove that the resolvent of the Laplacian of X can be lifted to a meromorphic function on a Riemann surface which is a branched covering of the complex plane. Its poles, that is the resonances of the Laplacian, are explicitly located on this Riemann surface. The residue operators at the resonances have finite rank. Their images are finite direct sums of finite-dimensional irreducible spherical representations of G.

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1. Introduction

The study of resonances has started in quantum mechanics, where they are linked to the metastable states of a system. Mathematically, the resonances appear as poles of the meromorphic continuation of the resolvent $(H - z)^{-1}$ of a Hamiltonian H acting on a space of functions $\mathcal{F}$ on which H is not selfadjoint. In the last thirty years, several articles have considered the case where H is the Laplacian of a Riemannian symmetric space of the noncompact type X and $\mathcal{F}$ is the space $C_c^\infty(\mathsf{X})$ of smooth compactly supported functions on X. The basic problems are the existence, location, counting estimates and geometric interpretation of the resonances. All these problems are nowadays well understood when X is of real rank one, such as the real hyperbolic spaces. The situation is completely different for Riemannian symmetric spaces of higher rank. The pioneering articles proving the analytic continuation of the resolvent of the Laplacian operator across its continuous spectrum are [7] and [8]. However, in these articles, the domains where the continuation was obtained is not sufficiently large to cover the region where the resonances could possibly be found. Indeed, the existence of resonances is linked to the singularities of the Plancherel measure on X. The basic question, whether resonances exist or not for general Riemannian symmetric spaces for which the Plancherel measure is singular, is still open. If the general picture is still unknown, some complete examples in rank 2 have been treated recently: $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ in [5] and the direct products $\mathsf{X}_1 \times \mathsf{X}_2$ of two rank-one Riemannian symmetric spaces of the noncompact type in [6].

The present paper is a natural continuation of [6] and deals with the cases of Riemannian symmetric spaces $\mathsf{X} = \mathrm{G}/\mathrm{K}$ of real rank two and restricted root system BC_2 or C_2 except the case when $\mathrm{G} = \mathrm{SO}_0(p,2)$ with $p > 2$ odd. The reason is that for all the spaces X considered here the analysis of the meromorphic continuation of the resolvent of the Laplacian can be deduced from the same problem on a direct product $\mathsf{X}_1 \times \mathsf{X}_1$ of a Riemannian symmetric space of rank one not isomorphic to the real hyperbolic space.

We prove that for all the spaces X we consider, the resolvent of the Laplacian of X can be lifted to a meromorphic function on a Riemann surface which is a branched covering of $\mathbb{C}$. Its poles, that is the resonances of the Laplacian, are explicitly located on this Riemann surface. If z_0 is a resonance of the Laplacian, then the (resolvent) residue operator at z_0 is the linear operator

$$\mathrm{Res}_{z_0}\widetilde{R} : C_c^\infty(\mathsf{X}) \to C^\infty(\mathsf{X}) \tag{1}$$

defined by

$$\big(\mathrm{Res}_{z_0}\widetilde{R}f\big)(y) = \mathrm{Res}_{z=z_0}[R(z)f](y) \qquad (f \in C_c^\infty(\mathsf{X}),\, y \in \mathsf{X})\,. \tag{2}$$

Since the meromorphic extension takes place on a Riemann surface, the right-hand side of (2) is computed with respect to some coordinate charts and hence determined up to constant multiples. However, the image $\mathrm{Res}_{z_0}\widetilde{R}\big(C_c^\infty(\mathsf{X})\big)$ is a well-defined subspace of $C^\infty(\mathsf{X})$. Its dimension is the rank of the residue operator at z_0. We prove that $\mathrm{Res}_{z_0}\widetilde{R}$ acts on $C_c^\infty(\mathsf{X})$ as a convolution by a finite linear combination of spherical functions of X and is of finite rank. More precisely, write $\mathsf{X} = \mathrm{G}/\mathrm{K}$ for a connected noncompact real semisimple Lie group with finite center G with maximal compact subgroup K. Then the space $\mathrm{Res}_{z_0}\widetilde{R}\big(C_c^\infty(\mathsf{X})\big)$ is a G-module which is a finite direct sum of finite-dimensional irreducible spherical representations of G. The trivial representation of G occurs for the residue operator at the first singularity, associated with the bottom of the spectrum of the Laplacian.

2. Preliminaries

2.1. General notation

We use the standard notation $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$ and $\mathbb{C}^\times$ for the integers, the reals, the positive reals, the complex numbers and the non-zero complex numbers, respectively. For $a \in \mathbb{Z}$, the symbol $\mathbb{Z}_{\geq a}$ denotes the set of integers $\geq a$. We write $[\![a,b]\!] = [a,b] \cap \mathbb{Z}$ for the discrete interval of integers in $[a,b]$. The interior of an interval $I \subseteq \mathbb{R}$ (with respect to the usual topology on the real line) will be indicated by I°. The upper half-plane in $\mathbb{C}$ is $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$; the lower half-plane $-\mathbb{C}^+$ is denoted $\mathbb{C}^-$. If X is a manifold, then $C^\infty(\mathsf{X})$ and $C_c^\infty(\mathsf{X})$ respectively denote the space of smooth functions and the space of smooth compactly supported functions on X.

2.2. Noncompact irreducible Riemannian symmetric spaces of type BC_2 or C_2

Let $\mathsf{X} = \mathrm{G}/\mathrm{K}$ be an irreducible Riemannian symmetric space of the noncompact type and (real) rank 2. Hence G is a connected noncompact semisimple real Lie group with finite center and K is a maximal compact subgroup of G. We can suppose that G is simple and admits a faithful linear representation. Let $\mathfrak{g}$ and $\mathfrak{k}$ be respectively the Lie algebras of G and K, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let us fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The (real) rank 2 condition means that $\mathfrak{a}$ is a 2-dimensional real vector space. We denote by $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$ and by $\mathfrak{a}_\mathbb{C}^*$ the complexification of $\mathfrak{a}^*$. The Killing form of $\mathfrak{g}$ restricts to an inner product on $\mathfrak{a}$. We extend it to $\mathfrak{a}^*$ by duality. The $\mathbb{C}$-bilinear extension of $\langle \cdot, \cdot \rangle$ to $\mathfrak{a}_\mathbb{C}^*$ will be indicated by the same symbol.

Let Σ be the root systems of $(\mathfrak{g}, \mathfrak{a})$. In the following, we suppose that Σ is either of type BC_2 or of type $C_2 = B_2$. The set Σ^+ of positive restricted roots is the form $\Sigma^+ = \Sigma_\mathrm{l}^+ \sqcup \Sigma_\mathrm{m}^+ \sqcup \Sigma_\mathrm{s}^+$, where

$$\Sigma_\mathrm{l}^+ = \{\beta_1, \beta_2\}, \quad \Sigma_\mathrm{m}^+ = \left\{\frac{\beta_2 \pm \beta_1}{2}\right\}, \quad \Sigma_\mathrm{s}^+ = \left\{\frac{\beta_1}{2}, \frac{\beta_2}{2}\right\}$$

with $\Sigma_\mathrm{s}^+ = \emptyset$ in the case $C_2 = B_2$. The two elements of Σ_l^+ form an orthogonal basis of $\mathfrak{a}^*$ and have same norm b. The elements of Σ_m^+ and Σ_s^+ have therefore norm $\frac{\sqrt{2}}{2}b$ and $\frac{b}{2}$, respectively. We define $\mathfrak{a}_+^* = \{\lambda \in \mathfrak{a}^* : \langle \lambda, \beta \rangle > 0 \text{ for all } \beta \in \Sigma^+\}$.

The system of positive unmultipliable roots is $\Sigma_*^+ = \Sigma_\mathrm{l}^+ \sqcup \Sigma_\mathrm{m}^+$. The set Σ_* of unmultipliable roots is a root system. A basis of positive simple roots for Σ_* is $\{\beta_1, \frac{\beta_2 - \beta_1}{2}\}$.

The Weyl group W of Σ acts on the roots by permutations and sign changes. For $a \in \{\mathrm{l}, \mathrm{m}, \mathrm{s}\}$ set $\Sigma_a = \Sigma_a^+ \sqcup (-\Sigma_a^+)$. Then each Σ_a is a Weyl group orbit in Σ. The root multiplicities are therefore triples $m = (m_\mathrm{l}, m_\mathrm{m}, m_\mathrm{s})$ so that m_a is the (constant) value of m on Σ_a for $a \in \{\mathrm{l}, \mathrm{m}, \mathrm{s}\}$. By classification, if $\mathsf{X} = \mathrm{G}/\mathrm{K}$ is Hermitian, then $m_\mathrm{l} = 1$. We adopt the convention that $m_\mathrm{s} = 0$ means that $\Sigma_\mathrm{s}^+ = \emptyset$, i.e., Σ is of type C_2. In this case, if X is Hermitian, then X is said to be of tube type.

The half-sum of positive roots, counted with their multiplicities, is indicated by ρ. Hence

$$2\rho = \sum_{\alpha\in\Sigma^+} m_\alpha \alpha = \Big(m_{\rm l} + \frac{m_{\rm s}}{2}\Big)\beta_1 + \Big(m_{\rm l} + m_{\rm m} + \frac{m_{\rm s}}{2}\Big)\beta_2\,. \tag{3}$$

Table 1 contains the rank-two irreducible Riemannian symmetric spaces G/K with root systems of type BC_2, their root systems, the multiplicities $m = (m_{\rm l}, m_{\rm m}, m_{\rm s})$, and the value of ρ.

Type	AIII	BDI	CII	DIII	EIII
G	$\mathrm{SU}(p,2)$ $(p>2)$	$\mathrm{SO}_0(p,2)$ $(p>2)$	$\mathrm{Sp}(p,2)$ $(p\geq 2)$	$\mathrm{SO}^*(10)$	$E_{6(-14)}$
K	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(2))$	$\mathrm{SO}(p)\times\mathrm{SO}(2)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(2)$	$\mathrm{U}(5)$	$\mathrm{Spin}(10)\times\mathrm{U}(1)$
Hermitian	yes	yes	no	yes	yes
Σ	BC_2	C_2	$p=2$: C_2 $p>2$: BC_2	BC_2	BC_2
$m=(m_{\rm l},m_{\rm m},m_{\rm s})$	$(1,2,2(p-2))$	$(1,p-2,0)$	$(3,4,4(p-2))$	$(1,4,4)$	$(1,6,8)$
2ρ	$(p-1)\beta_1+(p+1)\beta_2$	$\beta_1+(p-1)\beta_2$	$5\beta_1+(5+2(p-2))\beta_2$	$3\beta_1+7\beta_2$	$5\beta_1+8\beta_2$

Table 1. Rank-two irreducible symmetric spaces with root systems BC_2 or C_2

Notice that we are using special low rank isomorphisms (see, e.g., [1, Ch. X, §6, no. 4]), which allow us to omit some cases:

$$\mathrm{SU}(2,2)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2)) \cong \mathrm{SO}_0(4,2)/(\mathrm{SO}(4)\times\mathrm{SO}(2))\,, \tag{4}$$

$$\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(2) \cong \mathrm{SO}_0(3,2)/(\mathrm{SO}(3)\times\mathrm{SO}(2))\,, \tag{5}$$

$$\mathrm{SO}^*(8)/\mathrm{U}(4) \cong \mathrm{SO}_0(6,2)/(\mathrm{SO}(6)\times\mathrm{SO}(2))\,. \tag{6}$$

Observe also that $\mathrm{SO}_0(2,2)/(\mathrm{SO}(2)\times\mathrm{SO}(2)) \cong \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ is not in the list because not irreducible.

Remark 1. Up to isomorphisms, there are four additional irreducible Riemannian symmetric spaces of rank two:

1. $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ (type AI, with root system of type A_2 and one root multiplicity $m=1$; see [5]),
2. $\mathrm{SU}^*(6)/\mathrm{Sp}(3)$ (type AII, with root system of type A_2 and one even root multiplicity $m=4$; see [8]),
3. $E_{6(-26)}/F_4$ (type EIV, with root system of type A_2 and one even root multiplicity $m=8$; see [8]),
4. $G_{2(-14)}/(\mathrm{SU}(2)\times\mathrm{SU}(2))$ (type G, with root system of type G_2 and one root multiplicity $m=1$).

2.3. The Plancherel density of G/K

For $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ and $\beta\in\Sigma$ we shall employ the notation

$$\lambda_\beta = \frac{\langle\lambda,\beta\rangle}{\langle\beta,\beta\rangle}\,. \tag{7}$$

Observe that

$$\lambda_{\beta/2} = 2\lambda_\beta \,, \tag{8}$$

$$\lambda_{(\beta_2 \pm \beta_1)/2} = 2\frac{\langle \lambda, \beta_2 \rangle \pm \langle \lambda, \beta_1 \rangle}{\langle \beta_2, \beta_2 \rangle + \langle \beta_1, \beta_1 \rangle} = \lambda_{\beta_2} \pm \lambda_{\beta_1} \,. \tag{9}$$

For $\beta \in \Sigma_*$, we set

$$\widetilde{\rho}_\beta = \frac{1}{2}\left(m_\beta + \frac{m_{\beta/2}}{2}\right) , \tag{10}$$

where m_β denotes the multiplicity of the root β, and

$$c_\beta(\lambda) = \frac{2^{-2\lambda_\beta}\Gamma(2\lambda_\beta)}{\Gamma\big(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{1}{2}\big)\Gamma\big(\lambda_\beta + \widetilde{\rho}_\beta\big)} \,. \tag{11}$$

Observe that $\widetilde{\rho}_\beta = \rho_\beta = \frac{\langle \rho, \beta \rangle}{\langle \beta, \beta \rangle}$ if β is a simple root in Σ_* (but not in general). In particular, $\widetilde{\rho}_{\beta_1} = \rho_{\beta_1}$.

Harish-Chandra's c-function c_{HC} (written in terms of unmultipliable instead of indivisible roots) is defined by

$$c_{\mathrm{HC}}(\lambda) = c_0 \prod_{\beta \in \Sigma_*^+} c_\beta(\lambda) \,, \tag{12}$$

where c_0 is a normalizing constants so that $c_{\mathrm{HC}}(\rho) = 1$.

In the following we always adopt the convention that empty products are equal to 1. As a consequence of the properties of the gamma function, we have the following explicit expression.

Lemma 1. *The Plancherel density is given by the formula*

$$[c_{\mathrm{HC}}(\lambda)c_{\mathrm{HC}}(-\lambda)]^{-1} = C\Pi(\lambda)P(\lambda)Q(\lambda) \,, \tag{13}$$

where

$$\Pi(\lambda) = \prod_{\beta \in \Sigma_*^+} \lambda_\beta \,, \tag{14}$$

$$P(\lambda) = \prod_{\beta \in \Sigma_*^+} \Bigg(\prod_{k=0}^{(m_{\beta/2})/2-1} \left[\lambda_\beta - \left(\tfrac{m_{\beta/2}}{4} - \tfrac{1}{2}\right) + k\right] \prod_{k=0}^{2\widetilde{\rho}_\beta - 2} [\lambda_\beta - (\widetilde{\rho}_\beta - 1) + k] \Bigg) , \tag{15}$$

$$Q(\lambda) = \prod_{\substack{\beta \in \Sigma_*^+ \\ m_\beta \text{ odd}}} \cot(\pi(\lambda_\beta - \widetilde{\rho}_\beta)) \,, \tag{16}$$

and C is a constant. Consequently, the singularities of the Plancherel density $[c_{\mathrm{HC}}(\lambda)c_{\mathrm{HC}}(-\lambda)]^{-1}$ are at most simple poles located along the hyperplanes of the equation

$$\pm\lambda_\beta = \widetilde{\rho}_\beta + k$$

where $\beta \in \Sigma_*^+$ has odd multiplicity m_β, and $k \in \mathbb{Z}_{\geq 0}$.

Proof. The singularities of $[c_{\mathrm{HC}}(\lambda)c_{\mathrm{HC}}(-\lambda)]^{-1}$ are those of $\cot(\pi(\lambda_\beta - \widetilde{\rho}_\beta))$, for $\beta \in \Sigma_*^+$ with m_β odd, which are not killed by zeros of the polynomial $\Pi(\lambda)P(\lambda)$. $\square$

The following corollary will allow us to establish a region of holomorphic extension of the resolvent.

Corollary 2. *Set*

$$L = \min\{\widetilde{\rho}_\beta |\beta| : \beta \in \Sigma_*^+,\ m_\beta \ \textit{odd}\}\,. \tag{17}$$

Then, for every fixed $\omega \in \mathfrak{a}^*$ *with* $|\omega| = 1$*, the function*

$$r \mapsto [c_{\rm HC}(r\omega)c_{\rm HC}(-r\omega)]^{-1}$$

is holomorphic on $\mathbb{C} \setminus \big(]-\infty, -L] \cup [L, +\infty[\big)$.

The values of $\widetilde{\rho}_\beta$ for the roots in Σ_*^+, as well as the value of L, are given in Table 2. Recall that $b = \langle \beta_1, \beta_1 \rangle = \langle \beta_2, \beta_2 \rangle$.

G	$\mathrm{SU}(p,2)$ $(p>2)$	$\mathrm{SO}_0(p,2)$ $(p>2)$	$\mathrm{Sp}(p,2)$ $(p \geq 2)$	$\mathrm{SO}^*(10)$	$E_{6(-14)}$
$\widetilde{\rho}_{\beta_j} = \frac{1}{2}\big(m_{\rm l} + \frac{m_{\rm s}}{2}\big)$	$(p-1)/2$	$1/2$	$p-1/2$	$3/2$	$5/2$
$\widetilde{\rho}_{(\beta_2\pm\beta_1)/2} = \frac{m_{\rm m}}{2}$	1	$(p-2)/2$	2	2	3
$\Sigma_{*,{\rm odd}}^+ = \{\beta \in \Sigma_*^+ : m_\beta \text{ odd}\}$	$\{\beta_1, \beta_2\}$	p even: $\{\beta_1,\beta_2\}$ p odd: $\{\beta_1,\beta_2,\frac{\beta_2\pm\beta_1}{2}\}$	$\{\beta_1, \beta_2\}$	$\{\beta_1, \beta_2\}$	$\{\beta_1, \beta_2\}$
$L = \min\{\widetilde{\rho}_\beta|\beta| : \beta \in \Sigma_{*,{\rm odd}}^+\}$	$\frac{\sqrt{p-1}}{2}b$	$p=3$: $\frac{\sqrt{2}}{4}b$ $p>3$: $\frac{b}{2}$	$\big(\frac{3}{2} + 2(p-2)\big)b$	$\frac{3}{2}b$	$\frac{5}{2}b$

TABLE 2. The values of $\widetilde{\rho}_\beta$ for $\beta \in \Sigma_*^+$ and of L

A computation using the values in the tables together with [6, §2] yields the following corollary.

Corollary 3. *If* $\mathrm{G} \neq \mathrm{SO}_0(p,2)$ *with* p *odd, then* $\{\beta \in \Sigma_*^+ : m_\beta \textit{ is odd}\}$ *is equal to* $\{\beta_1, \beta_2\}$. *Hence*

$$[c_{\rm HC}(\lambda)c_{\rm HC}(-\lambda)]^{-1} = \Pi_0(\lambda)P_0(\lambda)[c_{\rm HC}^\times(\lambda)c_{\rm HC}^\times(-\lambda)]^{-1}\,, \tag{18}$$

where

$$\Pi_0(\lambda) = \lambda_{(\beta_2-\beta_1)/2}\lambda_{(\beta_2+\beta_1)/2} = \lambda_{\beta_2}^2 - \lambda_{\beta_1}^2\,, \tag{19}$$

$$P_0(\lambda) = \prod_{\beta=(\beta_2\pm\beta_1)/2} \ \prod_{k=0}^{2\widetilde{\rho}_\beta-2} [\lambda_\beta - (\widetilde{\rho}_\beta - 1) + k]\,, \tag{20}$$

and $[c_{\rm HC}^\times(\lambda)c_{\rm HC}^\times(-\lambda)]^{-1}$ *is the Plancherel density of the product* $\mathsf{X}_1 \times \mathsf{X}_1$ *of two isomorphic rank-one Riemannian symmetric spaces with root systems of type* BC_1 *(or* A_1*) and multiplicities* $(m_{\beta_j}, m_{\beta_j/2}) = (m_{\rm l}, m_{\rm s})$.

If $\mathrm{G} = \mathrm{SO}_0(p,2)$ *with* $p \geq 3$ *odd, then* $\Sigma_*^+ = \Sigma^+$ *and*

$$\begin{aligned}
\Pi(\lambda) &= \lambda_{\beta_1}\lambda_{\beta_2}(\lambda_{\beta_2}^2 - \lambda_{\beta_1}^2) \\
P(\lambda) &= \prod_{k=0}^{p-2}\Big(\lambda_{(\beta_2-\beta_1)/2} - \Big(\frac{p-2}{2} - 1\Big) + k\Big)\Big(\lambda_{(\beta_2+\beta_1)/2} - \Big(\frac{p-2}{2} - 1\Big) + k\Big) \\
&= \prod_{k=0}^{p-2}\Big(\lambda_{\beta_2} - \lambda_{\beta_1} - \Big(\frac{p-2}{2} - 1\Big) + k\Big)\Big(\lambda_{\beta_2} + \lambda_{\beta_1} - \Big(\frac{p-2}{2} - 1\Big) + k\Big) \\
Q(\lambda) &= \cot\big(\pi\big(\lambda_{\beta_1} - \tfrac{1}{2}\big)\big)\cot\big(\pi\big(\lambda_{\beta_2} - \tfrac{1}{2}\big)\big)\cot\big(\pi\big(\lambda_{\beta_2} - \lambda_{\beta_1} - \tfrac{p}{2} + 1\big)\big) \\
&\qquad \times \cot\big(\pi\big(\lambda_{\beta_2} + \lambda_{\beta_1} - \tfrac{p}{2} + 1\big)\big).
\end{aligned}$$

2.4. The resolvent of Δ

Endow the Euclidean space $\mathfrak{a}^*$ with the Lebesgue measure normalized so that the unit cube has volume 1. On the Furstenberg boundary $\mathrm{B} = \mathrm{K}/\mathrm{M}$ of X, where M is the centralizer of $\mathfrak{a}$ in K, we consider the K-invariant measure db normalized so that the volume of B is equal to 1. Let X be equipped with its (suitably normalized) natural G-invariant Riemannian measure and let Δ denote the corresponding (positive) Laplacian. As in the cases treated in [5] and [6], it will be convenient to identify $\mathfrak{a}^*$ with $\mathbb{C}$ as vector spaces over $\mathbb{R}$. More precisely, we want to view $\mathfrak{a}_1^*$ and $\mathfrak{a}_2^*$ as the real and the purely imaginary axes, respectively. To distinguish the resulting complex structure in $\mathfrak{a}^*$ from the natural complex structure of $\mathfrak{a}_{\mathbb{C}}^*$, we shall indicate the complex units in $\mathfrak{a}^* \equiv \mathbb{C}$ and $\mathfrak{a}_{\mathbb{C}}^*$ by i and $\mathbf{i}$, respectively. So $\mathfrak{a}^* \equiv \mathbb{C} = \mathbb{R} + i\mathbb{R}$, whereas $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* + \mathbf{i}\mathfrak{a}^*$. For $r, s \in \mathbb{R}$ and $\lambda, \nu \in \mathfrak{a}^*$ we have $(r + is)(\lambda + \mathbf{i}\nu) = (r\lambda - s\nu) + \mathbf{i}(r\nu + s\lambda) \in \mathfrak{a}_{\mathbb{C}}^*$.

By the Plancherel Theorem [3, Ch. III, §1, no. 2], the Helgason–Fourier transform $\mathcal{F}$ is a unitary equivalence of Δ acting on $L^2(\mathsf{X})$ with the multiplication operator M on $L^2(\mathfrak{a}_+^* \times B, [c_{\mathrm{HC}}(\mathbf{i}\lambda)c_{\mathrm{HC}}(-\mathbf{i}\lambda)]^{-1}\, d\lambda\, db)$ given by

$$MF(\lambda, b) = \Gamma(\Delta)(\mathbf{i}\lambda)F(\lambda, b) = (\langle\rho,\rho\rangle + \langle\lambda,\lambda\rangle)F(\lambda, b) \qquad ((\lambda, b) \in \mathfrak{a}^* \times B)\,. \quad (21)$$

It follows, in particular, that the spectrum of Δ is the half-line $[\rho_{\mathsf{X}}^2, +\infty[$, where $\rho_{\mathsf{X}}^2 = \langle\rho,\rho\rangle$. By the Paley–Wiener theorem for $\mathcal{F}$, see, e.g., [3, Ch. III, §5], for every $u \in \mathbb{C} \setminus [\rho_{\mathsf{X}}^2, +\infty[$ the resolvent of Δ at u maps $C_c^\infty(\mathsf{X})$ into $C^\infty(\mathsf{X})$.

Recall that for sufficiently regular functions $f_1, f_2 : \mathsf{X} \to \mathbb{C}$, the convolution $f_1 \times f_2$ is the function on X defined by $(f_1 \times f_2) \circ \pi = (f_1 \circ \pi) * (f_2 \circ \pi)$. Here $\pi : \mathrm{G} \to \mathsf{X} = \mathrm{G}/\mathrm{K}$ is the natural projection and $*$ denotes the convolution product of functions on G.

The Plancherel formula yields the following explicit expression for the image of $f \in C_c^\infty(\mathsf{X})$ under the resolvent operator $R(z) = (\Delta - \rho_{\mathsf{X}}^2 - z^2)^{-1}$ of the shifted Laplacian $\Delta - \rho_{\mathsf{X}}^2$:

$$[R(z)f](y) = \int_{\mathfrak{a}^*} \frac{1}{\langle\lambda,\lambda\rangle - z^2}\,(f \times \varphi_{\mathbf{i}\lambda})(y)\,\frac{d\lambda}{c_{\mathrm{HC}}(\mathbf{i}\lambda)c_{\mathrm{HC}}(-\mathbf{i}\lambda)}, \quad (z \in \mathbb{C}^+\,, y \in \mathsf{X})\,. \quad (22)$$

See [4, formula (14)]. Here and in the following, resolvent equalities as (22) are given up to non-zero constant multiples.

3. Meromorphic extension in the case $G \neq SO_0(2,p)$, $p > 2$ odd

3.1. The resolvent kernel in polar coordinates

We write $\mathfrak{a}_1^* = \mathbb{R}\beta_1$ and $\mathfrak{a}_2^* = \mathbb{R}\beta_2$, so that $\mathfrak{a}^* = \mathfrak{a}_1^* \oplus \mathfrak{a}_2^*$ and $\lambda = \lambda_1 + \lambda_2 = x_1\beta_1 + x_2\beta_2 \in \mathfrak{a}^*$. Introduce the coordinates

$$\mathbb{R}^2 \ni (x_1, x_2) \to x_1\beta_1 + x_2\beta_2 \in \mathfrak{a}_1^* \oplus \mathfrak{a}_2^* = \mathfrak{a}^*. \tag{23}$$

Hence $x_j = \lambda_{\beta_j}$ if $\lambda = x_1\beta_1 + x_2\beta_2$.

In view of Table 2, the functions Π_0 and P_0 from (19) and (20) can be rewritten in these coordinates, as

$$\Pi_0(\lambda) = \Pi_0(x_1\beta_1 + x_2\beta_2) \;=\; x_2^2 - x_1^2\,, \tag{24}$$

$$\begin{aligned} P_0(\lambda) = P_0(x_1\beta_1 + x_2\beta_2) \;&=\; \prod_{\beta=(\beta_2\pm\beta_1)/2}\ \prod_{k=0}^{m_{\mathrm{m}}-2}\left[\lambda_\beta - \left(\frac{m_{\mathrm{m}}}{2} - 1\right) + k\right] \\ &= \prod_{k=1}^{m_{\mathrm{m}}-1}\left[(x_2 + x_1) - \frac{m_{\mathrm{m}}}{2} + k\right]\left[(x_2 - x_1) - \frac{m_{\mathrm{m}}}{2} + k\right] \\ &= \prod_{k=1}^{m_{\mathrm{m}}-1}\left[\left(x_2 - \frac{m_{\mathrm{m}}}{2} + k\right)^2 - x_1^2\right] \end{aligned} \tag{25}$$

since

$$(\beta_j)_{(\beta_2\pm\beta_1)/2} = (\beta_j)_{\beta_2} \pm (\beta_j)_{\beta_1} = \begin{cases} \pm 1 & \text{for } j = 1, \\ 1 & \text{for } j = 2. \end{cases}$$

We write

$$\vartheta_0(x_1, x_2) = \Pi_0(\lambda)P_0(\lambda) = (x_2^2 - x_1^2)\prod_{k=1}^{m_{\mathrm{m}}-1}\left[\left(x_2 - \frac{m_{\mathrm{m}}}{2} + k\right)^2 - x_1^2\right]. \tag{26}$$

Further we write

$$[c_{\mathrm{HC}}^{\mathsf{X}_1}(\mathbf{i}\lambda)c_{\mathrm{HC}}^{\mathsf{X}_1}(-\mathbf{i}\lambda)]^{-1} = C_1\Pi_1(\mathbf{i}\lambda)P_1(\mathbf{i}\lambda)Q_1(\mathbf{i}\lambda) \tag{27}$$

for the Plancherel density of the space X_1 in Corollary 3, so that

$$[c_{\mathrm{HC}}^{\times}(\mathbf{i}\lambda)c_{\mathrm{HC}}^{\times}(-\mathbf{i}\lambda)]^{-1} = C_1^2\Pi_1(ix_1)\Pi_1(ix_2)P_1(ix_1)P_1(ix_2)Q_1(ix_1)Q_1(ix_2). \tag{28}$$

See [6, §1 and §2]. Using (18) and omitting non-zero constant multiples, we can therefore rewrite (22) as

$$\begin{aligned} R(z)f(y) &= \int_{\mathfrak{a}^*}\frac{1}{\langle\lambda,\lambda\rangle - z^2}(f\times\varphi_{\mathbf{i}\lambda})(y)\frac{1}{c_{\mathrm{HC}}(\mathbf{i}\lambda)c_{\mathrm{HC}}(-\mathbf{i}\lambda)}\,d\lambda \\ &= \int_{\mathbb{R}^2}\frac{(f\times\varphi_{\mathbf{i}x_1\beta_1+\mathbf{i}x_2\beta_2})(y)}{x_1^2b^2 + x_2^2b^2 - z^2}\vartheta_0(ix_1, ix_2)x_1x_2P_1(ix_1)P_1(ix_2) \\ &\quad \times Q_1(ix_1)Q_1(ix_2)\,dx_1\,dx_2\,. \end{aligned}$$

Introduce the polar coordinates

$$x_1 = \frac{r}{b}\cos\theta, \quad x_2 = \frac{r}{b}\sin\theta \qquad (0 < r\,,\ 0 \le \theta < 2\pi)$$

on $\mathbb{R}^2$ and set

$$p_1(x) = P_1\big(i\tfrac{x}{b}\big) \qquad \text{and} \qquad q_1(x) = Q_1\big(i\tfrac{x}{b}\big). \tag{29}$$

In these terms (up to a non-zero constant multiple)

$$R(z)f(y) = \int_0^\infty \frac{1}{r^2 - z^2} F(r)\, r\, dr,$$

where

$$\begin{aligned} F(r) = \int_0^{2\pi} &(f \times \varphi_{\mathbf{i}\frac{r}{b}\cos\theta\,\beta_1 + \mathbf{i}\frac{r}{b}\sin\theta\,\beta_2})(y)\, \vartheta_{0,\mathrm{pol}}(r,\theta) r^2 \cos\theta \sin\theta \\ &\times\ p_1(r\cos\theta) q_1(r\cos\theta) p_1(r\sin\theta) q_1(r\sin\theta)\, d\theta\,, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \vartheta_{0,\mathrm{pol}}(r,\theta) = \vartheta_0(i\,x_1, i\,x_2) = -\frac{r^2}{b^2}(\sin^2\theta - \cos^2\theta) \\ \times \prod_{k=1}^{m_{\mathrm{m}}-1} \left[\Big(\frac{r}{b} i \sin\theta - \frac{m_{\mathrm{m}}}{2} + k \Big)^2 + \frac{r^2}{b^2}\cos^2\theta \right]. \end{aligned} \tag{31}$$

Here and in the following, we omit from the notation the dependence of F on the function $f \in C_c^\infty(\mathsf{X})$ and on $y \in \mathsf{X}$.

Recall the functions

$$\mathrm{c}(w) = \frac{w + w^{-1}}{2}\,, \qquad \mathrm{s}(w) = \frac{w - w^{-1}}{2} = i\mathrm{c}(-iw) \qquad (w \in \mathbb{C}^\times) \tag{32}$$

from [6, (20)] and notice that

$$\cos\theta = \mathrm{c}(e^{i\theta})\,, \qquad \sin\theta = \frac{\mathrm{s}(e^{i\theta})}{i} = \mathrm{c}(-ie^{i\theta})\,, \qquad d\theta = \frac{de^{i\theta}}{ie^{i\theta}}.$$

For $z \in \mathbb{C}$ and $w \in \mathbb{C}^\times$ define

$$\psi_z(w) = (f \times \varphi_{\mathbf{i}\frac{z}{b}\mathrm{c}(w)\,\beta_1 + \mathbf{i}\frac{z}{b}\mathrm{c}(-iw)\,\beta_2})(y) \tag{33}$$

$$\phi_z(w) = -z^2 \mathrm{c}(w) \frac{\mathrm{s}(w)}{w} p_1\big(z\mathrm{c}(w)\big) q_1\big(z\mathrm{c}(w)\big) p_1\big(z\mathrm{c}(-iw)\big) q_1\big(z\mathrm{c}(-iw)\big), \tag{34}$$

as in [6, (32) and (33)] together with

$$\vartheta_z(w) = \frac{z^2}{b^2}\big(\mathrm{c}(w)^2 - \mathrm{c}(-iw)^2\big) \prod_{k=1}^{m_{\mathrm{m}}-1} \left[\Big(\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + k\Big)^2 + \frac{z^2}{b^2}\mathrm{c}(w)^2 \right], \tag{35}$$

which is a polynomial function of z. Then

$$F(r) = \int_{|w|=1} \vartheta_r(w) \psi_r(w) \phi_r(w)\, dw\,. \tag{36}$$

Lemma 4. *Let $z \in \mathbb{C}$ and $w \in \mathbb{C}^\times$. Then:*

$$\begin{aligned} &\psi_{-z}(w) = \psi_z(w), \qquad &&\psi_z(-w) = \psi_z(w), \qquad &&\psi_z(iw) = \psi_z(w), \\ &\phi_{-z}(w) = \phi_z(w), \qquad &&\phi_z(-w) = -\phi_z(w), \qquad &&\phi_z(iw) = -i\phi_z(w), \\ &\vartheta_{-z}(w) = \vartheta_z(w), \qquad &&\vartheta_z(-w) = \vartheta_z(w), \qquad &&\vartheta_z(iw) = \vartheta_z(w). \end{aligned}$$

Proof. Set $\mu(z,w) = \mathbf{i}\frac{z}{b}\mathrm{c}(w)\beta_1 + \mathbf{i}\frac{z}{b}\mathrm{c}(-iw)\beta_2$, so that $\psi_z(w) = f \times \varphi_{\mu(z,w)}(y)$. Then $\mu(-z,w)$, $\mu(z,-w)$ and $\mu(z,iw)$ are transformed into $\mu(z,w)$ by sign changes and transposition of β_1 and of β_2. The equalities for $\psi_z(w)$ then follow because the spherical function φ_λ is W-invariant in the parameter λ.

The equalities for $\phi_z(w)$ are an immediate consequence of (32) and the fact that the functions c, s and $p_1 q_1$ are odd.

To prove the relations for $\vartheta_z(w)$, notice that $-\frac{m_{\mathrm{m}}}{2} + k = \frac{m_{\mathrm{m}}}{2} - h$ where $h = m_{\mathrm{m}} - k \in \{1, \dots, m_{\mathrm{m}} - 1\}$ when $k \in \{1, \dots, m_{\mathrm{m}} - 1\}$. Hence

$\prod_{k=1}^{m_{\mathrm{m}}-1} \left[\left(-\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + k\right)^2 + \frac{z^2}{b^2}\mathrm{c}(w)^2\right] = \prod_{h=1}^{m_{\mathrm{m}}-1} \left[\left(\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + h\right)^2 + \frac{z^2}{b^2}\mathrm{c}(w)^2\right].$

This proves the first two equalities for $\vartheta_z(w)$ since $\mathrm{s}(-w) = -\mathrm{s}(w)$. For the last equality, notice that

$$\begin{aligned}
&\left(\frac{z}{b}\mathrm{s}(iw) - \frac{m_{\mathrm{m}}}{2} + k\right)^2 + \frac{z^2}{b^2}\mathrm{c}(iw)^2 \\
&\quad = \left[\frac{z}{b}i\mathrm{c}(w) - \frac{m_{\mathrm{m}}}{2} + k + i\frac{z}{b}i\mathrm{s}(w)\right]\left[\frac{z}{b}i\mathrm{c}(w) - \frac{m_{\mathrm{m}}}{2} + k - i\frac{z}{b}i\mathrm{s}(w)\right] \\
&\quad = \left[-\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + k + i\frac{z}{b}\mathrm{c}(w)\right]\left[\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + k - i\frac{z}{b}\mathrm{c}(w)\right].
\end{aligned}$$

Hence, since m_{m} is even,

$$\begin{aligned}
&\prod_{k=1}^{m_{\mathrm{m}}-1} \left[\left(\frac{z}{b}\mathrm{s}(iw) - \frac{m_{\mathrm{m}}}{2} + k\right)^2 + \frac{z^2}{b^2}\mathrm{c}(iw)^2\right] \\
&\quad = \prod_{k=1}^{m_{\mathrm{m}}-1} \left[\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + k + i\frac{z}{b}\mathrm{c}(w)\right] \cdot (-1)^{m_{\mathrm{m}}-1} \\
&\qquad \times \prod_{h=1}^{m_{\mathrm{m}}-1} \left[\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + h - i\frac{z}{b}\mathrm{c}(w)\right] \\
&\quad = -\prod_{k=1}^{m_{\mathrm{m}}-1} \left[\left(\frac{z}{b}\mathrm{s}(w) - \frac{m_{\mathrm{m}}}{2} + k\right)^2 + \frac{z^2}{b^2}\mathrm{c}(w)^2\right].
\end{aligned}$$

This proves the claim because $\mathrm{c}(w)^2 - \mathrm{c}(-iw)^2$ changes sign under the transformation $w \to iw$. □

Thus [6, Lemma 3] generalizes as follows.

Lemma 5. *The function $F(r)$, (36), extends holomorphically to*

$$F(z) = \int_{|w|=1} \vartheta_z(w)\psi_z(w)\phi_z(w)\, dw\,, \tag{37}$$

where

$$z \in \mathbb{C} \setminus i((-\infty, -L] \cup [L, +\infty))$$

and L is the constant defined in (17). The function $F(z)$ is even and $F(z)z^{-2}$ is bounded near $z = 0$.

The following proposition, giving an initial holomorphic extension of the resolvent across the spectrum of the Laplacian, has been independently proven by Mazzeo and Vasy [7, Theorem 1.3] and by Strohmaier [8, Proposition 4.3] for general Riemannian symmetric spaces of the noncompact type and even rank. It shows that all possible resonances of the resolvent are located along the half-line $i(-\infty,-L]$. According to our conventions, we will omit f and y from the notation and write $R(z)$ instead of $[R(z)f](y)$.

Proposition 6. *The resolvent $R(z) = [R(z)f](y)$ extends holomorphically from $\mathbb{C} \setminus \big((-\infty,0] \cup i(-\infty,-L]\big)$ to a logarithmic Riemann surface branched along $(-\infty,0]$, with the preimages of $i\big((-\infty,-L] \cup [L,+\infty)\big)$ removed and, in terms of monodromy, it satisfies the following equation*

$$R(ze^{2i\pi}) = R(z) + 2i\pi\, F(z) \qquad \big(z \in \mathbb{C} \setminus \big((-\infty,0] \cup i(-\infty,-L] \cup i[L,+\infty)\big)\big).$$

The starting point for studying the meromorphic extension of R across $i(-\infty,-L]$ is the Proposition 7 below. It says that this meromorphic extension is equivalent to that of function F. This proposition is analogous to [6, Proposition 4] and its proof is omitted.

Proposition 7. *Fix $x_0 > 0$ and $y_0 > 0$. Let*

$$Q = \{z \in \mathbb{C}; \Re z > x_0,\ y_0 > \Im z \geq 0\}$$
$$U = Q \cup \{z \in \mathbb{C}; \Im z < 0\}.$$

Then there is a holomorphic function $H : U \to \mathbb{C}$ (depending on $f \in C_c^\infty(\mathsf{X})$ and $y \in \mathsf{X}$, which are omitted from the notation) such that

$$R(z) = H(z) + \pi i\, F(z) \qquad (z \in Q). \tag{38}$$

As a consequence, the resolvent $R(z) = [R(z)f](y)$ extends holomorphically from $\mathbb{C}^+$ to $\mathbb{C} \setminus \big((-\infty,0] \cup i(-\infty,-L]\big)$.

3.2. Meromorphic extension and residue computations

This section is devoted to the meromorphic extension of the function F (and hence of the resolvent) across the half-line $i(-\infty,-L]$. We set

$$\psi_z^\vartheta(w) = \vartheta_z(w)\psi_z(w) \tag{39}$$

and follow the stepwise extension procedure for F from [6, §2 and §3] with $\psi_z(w)$ replaced by $\psi_z^\vartheta(w)$. Some formulas are simplified by the fact that we are only dealing with the special case of $\mathsf{X}_1 = \mathsf{X}_2$ with β_1 and β_2 of equal norms $b_1 = b_2 = b$ and equal odd multiplicities $m_{\beta_1} = m_{\beta_2}$. Notice also that in this paper, studying the singularities of the Plancherel density, we are replacing the elements ρ_{β_1} and ρ_{β_2} used in [6] with $\widetilde{\rho}_{\beta_1}$ and $\widetilde{\rho}_{\beta_2}$, which are equal and have value $\frac{1}{2}\big(m_{\mathrm{l}} + \frac{m_{\mathrm{s}}}{2}\big)$. Indeed, in the case of direct product of rank one symmetric spaces treated in [6], there was no need of introducing multiple notation by distinguishing between ρ_β and $\widetilde{\rho}_\beta$ for $\beta \in \Sigma_*$. The distinction is now necessary since $\rho_{\beta_1} = \widetilde{\rho}_{\beta_1} = \widetilde{\rho}_{\beta_2} \neq \rho_{\beta_2}$. Furthermore, we omit the index j from the notation used in [6] when it only refers to which of the two factors one considers. So, for instance [6, (38)] yields, for the set of singularities of the product $p_1 q_1$ from (29), the set

$$S = S_+ \cup (-S_+)\,, \tag{40}$$

where

$$S_+ = ib(\widetilde{\rho}_{\beta_1} + \mathbb{Z}_{\geq 0}) = ib\Big(\frac{1}{2}\left(m_\mathrm{l} + \frac{m_\mathrm{s}}{2}\right) + \mathbb{Z}_{\geq 0}\Big). \tag{41}$$

For $r > 0$ and $c, d \in \mathbb{R} \setminus \{0\}$ recall the sets

$$\mathrm{D}_r = \{z \in \mathbb{C};\ |z| < r\}\,,$$
$$\mathrm{E}_{c,d} = \left\{\xi + i\eta \in \mathbb{C};\ \left(\tfrac{\xi}{c}\right)^2 + \left(\tfrac{\eta}{d}\right)^2 < 1\right\}\,,$$

and the role they play in [6, §1.4] for the functions s and c introduced in (32). Then [6, Prop. 6] translates in the following proposition.

Proposition 8. *Suppose $z \in \mathbb{C} \setminus i((-\infty, -L] \cup [L, \infty))$ and $r > 0$ are such that*

$$S \cap z\partial E_{\mathrm{c}(r),\mathrm{s}(r)} = \emptyset. \tag{42}$$

Then

$$F(z) = F_r(z) + 2\pi i\, F_{r,\mathrm{res}}(z), \tag{43}$$

where

$$F_r(z) = \int_{\partial D_r} \psi_z^\vartheta(w)\phi_z(w)\,dw,$$
$$F_{r,\mathrm{res}}(z) = {\sum_{w_0}}' \psi_z^\vartheta(w_0) \operatorname*{Res}_{w=w_0} \phi_z(w),$$

and $\sum_{w_0}'$ denotes the sum over all the w_0 such that

$$z\mathrm{c}(w_0) \in S \cap z(\mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)} \setminus [-1, 1]) \tag{44}$$

or

$$z\mathrm{c}(-iw_0) \in S \cap z(\mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)} \setminus [-1, 1]). \tag{45}$$

Both F_r and $F_{r,\mathrm{res}}$ are holomorphic functions on the open subset of $\mathbb{C} \setminus i((-\infty, -L] \cup [L, \infty))$ where the condition (42) holds. Furthermore, F_r extends to a holomorphic function on the open subset of $\mathbb{C}$ where the condition (42) holds.

To make the function $F_{r,\mathrm{res}}(z)$ explicit, we proceed as in [6, §3.1]. The present situation is in fact simpler, because only the case $L_{1,\ell} = L_{2,\ell}$ occurs. We denote this common value by L_ℓ, i.e., we define for $\ell \in \mathbb{Z}_{\geq 0}$

$$L_\ell = b(\widetilde{\rho}_{\beta_1} + \ell) = b\big(\tfrac{m_\mathrm{l}}{2} + \tfrac{m_\mathrm{s}}{4} + \ell\big)\,. \tag{46}$$

So $S_+ = \{iL_\ell; \ell \in \mathbb{Z}_{\geq 0}\}$.

If $0 \neq z \in \mathbb{C} \setminus i\big((-\infty, -L_\ell] \cup [L_\ell, +\infty)\big)$, then $\frac{iL_\ell}{z} \in \mathbb{C} \setminus [-1, 1]$ and we can uniquely define $w_1^\pm \in \mathrm{D}_1 \setminus \{0\}$ satisfying

$$z\mathrm{c}(w_1^\pm) \quad = \quad \pm iL_\ell\,. \tag{47}$$

Since $c(-w) = -c(w)$, we obtain that $w_1^- = -w_1^+$. Moreover, w_1 satisfies (44) if and only if $w_2 = iw_1$ satisfies (45) because $z(\mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)} \setminus [-1,1])$ is symmetric with respect to the origin $0 \in \mathbb{C}$. Hence

$$F_{r,\mathrm{res}}(z) = \sum_{w_1^+}{}' \Big[\psi_z^\vartheta(w_1^+) \operatorname*{Res}_{w=w_1^+} \phi_z(w) + \psi_z^\vartheta(w_1^-) \operatorname*{Res}_{w=w_1^-} \phi_z(w) + \psi_z^\vartheta(iw_1^+) \operatorname*{Res}_{w=iw_1^+} \phi_z(w) + \psi_z^\vartheta(iw_1^-) \operatorname*{Res}_{w=iw_1^-} \phi_z(w)\Big], \quad (48)$$

where $\sum'_{w_1^+}$ denotes the sum over all the w_1^+ such that $zc(w_1^+) \in S_+ \cap z(\mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)} \setminus [-1,1])$ and $w_1^- = -w_1^+$.

Then, using Lemma 4, we obtain the following analogue of [6, Lemma 9].

Lemma 9. *For $\ell \in \mathbb{Z}_{\geq 0}$ and $0 \neq z \in \mathbb{C} \setminus i\big((-\infty, -L_\ell] \cup [L_\ell, +\infty)\big)$, let $w_1^\pm$ be defined by* (47)*. Then*

$$\psi_z^\vartheta(w_1^+) = \psi_z^\vartheta(w_1^-) = \psi_z^\vartheta(iw_1^+) = \psi_z^\vartheta(iw_1^-), \quad (49)$$

$$\operatorname*{Res}_{w=w_1^+} \phi_z(w) = \operatorname*{Res}_{w=w_1^-} \phi_z(w) = \operatorname*{Res}_{w=iw_1^+} \phi_z(w) = \operatorname*{Res}_{w=-iw_1^-} \phi_z(w). \quad (50)$$

$$(51)$$

Explicitly,

$$\psi_z^\vartheta(w_1^+) = \psi_z^\vartheta\Big(\mathrm{c}^{-1}\big(\tfrac{iL_\ell}{z}\big)\Big),$$

$$\operatorname*{Res}_{w=w_1^+} \phi_z(w) = -C_\ell\, p_1\Big(iz(\mathrm{s} \circ \mathrm{c}^{-1})\big(\tfrac{iL_\ell}{z}\big)\Big)\, q_1\Big(iz(\mathrm{s} \circ \mathrm{c}^{-1})\big(\tfrac{iL_\ell}{z}\big)\Big),$$

where

$$C_\ell = \frac{b}{\pi} L_\ell\, p_1(iL_\ell) \neq 0. \quad (52)$$

Corollary 10. *Let $\ell \in \mathbb{Z}_{\geq 0}$ and $0 \neq z \in \mathbb{C} \setminus i\big((-\infty, -L_\ell] \cup [L_\ell, +\infty)\big)$. Set*

$$G_\ell(z) = -C_\ell \psi_z^\vartheta\Big(\mathrm{c}^{-1}\big(\tfrac{iL_\ell}{z}\big)\Big) p_1\Big(iz(\mathrm{s} \circ \mathrm{c}^{-1})\big(\tfrac{iL_\ell}{z}\big)\Big)\, q_1\Big(iz(\mathrm{s} \circ \mathrm{c}^{-1})\big(\tfrac{iL_\ell}{z}\big)\Big), \quad (53)$$

$$\mathrm{S}_{r,z,+} = \{\ell \in \mathbb{Z}_{\geq 0} : iL_\ell \in z\big(\mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)} \setminus [-1,1]\big)\}. \quad (54)$$

Then the function $F_{r,\mathrm{res}}(z)$ on the right-hand side of (43) *is given by*

$$F_{r,\mathrm{res}}(z) = 4 \sum_{\ell \in \mathrm{S}_{r,z,+}} G_\ell(z). \quad (55)$$

The following proposition is analogous to [6, Proposition 10].

Proposition 11. *For $0 < r < 1$ and $0 \neq z \in \mathbb{C} \setminus i((-\infty, -L] \cup [L, +\infty))$, let $\mathrm{S}_{r,z,+}$ be as in* (54)*. Moreover, let $W \subseteq \mathbb{C}$ be a connected open set such that*

$$S \cap W \partial \mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)} = \emptyset \quad (56)$$

and set

$$S_{r,W,+} = \{\ell \in \mathbb{Z}_{\geq 0} : iL_\ell \in W\mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)}\} \qquad (z \in W \setminus i\mathbb{R}). \quad (57)$$

Then $S_{r,z,+} = S_{r,W,+}$. In particular, $S_{r,z,+}$ does not depend on $z \in W \setminus i\mathbb{R}$.

Proceeding now as in [6, Corollaries 11, 13 and Lemma 12], we obtain the following result for points on $i\mathbb{R}$.

Corollary 12. *For every $iv \in i\mathbb{R}$ and for every r with $0 < r < 1$ and $vc(r) \notin iS$ there is a connected open neighborhood W_v of iv in $\mathbb{C}$ satisfying the following conditions.*

1. $S \cap W_v \partial \mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)} = \emptyset$.
2. $S_{r,W_v,+} = \{\ell \in \mathbb{Z}_{\geq 0} : iL_\ell \in iv\mathrm{E}_{\mathrm{c}(r),\mathrm{s}(r)}\} = [\![0, N_v]\!]$ *for some* $N_v \in \mathbb{Z}_{\geq 0}$.
3. *For* $n \in \mathbb{Z}_{\geq 0}$, *set*
$$I_n = b\widetilde{\rho}_{\beta_1} + b[n, n+1) = [L_n, L_{n+1})\,. \tag{58}$$
If $v \in I_n$ then $N_v = n$. Hence
$$F_{r,\mathrm{res}}(z) = 4\sum_{\ell=1}^{n} G_\ell(z) \qquad (z \in W_v \setminus i\mathbb{R})\,. \tag{59}$$

We recall the relevant Riemann surfaces from [6, (76)]. Fix $\ell \in \mathbb{Z}_{\geq 0}$. Then
$$\mathrm{M}_\ell = \left\{(z,\zeta) \in \mathbb{C}^\times \times (\mathbb{C} \setminus \{i,-i\}) \ : \ \zeta^2 = \left(\frac{iL_\ell}{z}\right)^2 - 1\right\} \tag{60}$$
is a Riemann surface above $\mathbb{C}^\times$, with projection map $\pi_\ell : \mathrm{M}_\ell \ni (z,\zeta) \to z \in \mathbb{C}^\times$. The fiber of π_ℓ above $z \in \mathbb{C}^\times$ is $\{(z,\zeta),(z,-\zeta)\}$. In particular, the restriction of π_ℓ to $\mathrm{M}_\ell \setminus \{(\pm iL_\ell, 0)\}$ is a double cover of $\mathbb{C}^\times \setminus \{\pm iL_\ell\}$.

Now [6, Lemma 15] has the following analogue. The difference is that we have replaced $\psi_z(w)$ by $\psi_z^\vartheta(w)$. So we have to look for possible cancellations of singularities arising from the additional polynomial factor ϑ_z.

Lemma 13. *In the above notation,*
$$\widetilde{G}_\ell : \mathrm{M}_\ell \ni (z,\zeta) \to \frac{b}{\pi} L_\ell\, p_1(iL_\ell)\psi_z^\vartheta\left(\frac{iL_\ell}{z} - \zeta\right) p_1(iz\zeta)q_1(iz\zeta) \in \mathbb{C} \tag{61}$$
is the meromorphic extension to M_ℓ of a lift of G_ℓ.

The function $\widetilde{G}_\ell$ has simple poles at all points $(z,\zeta) \in \mathrm{M}_\ell$ such that
$$z = \pm i\sqrt{L_\ell^2 + L_m^2}, \tag{62}$$
where $m \in \mathbb{Z}_{\geq 0} \setminus \left[\!\left[\ell - \left(\frac{m_{\mathrm{m}}}{2} - 1\right), \ell + \left(\frac{m_{\mathrm{m}}}{2} - 1\right)\right]\!\right]$.

Proof. Formula (61) is obtained using the lifts of c^{-1} and soc^{-1}, as in [6, Lemma 15].

The poles of $\widetilde{G}_\ell$ are the points $(z,\zeta) \in \mathrm{M}_\ell$ for which the function $\vartheta_z\big(\frac{iL_\ell}{z} - \zeta\big)p_1(iz\zeta)q_1(iz\zeta)$ is singular, i.e., the points for which $p_1(iz\zeta)q_1(iz\zeta)$ is singular and $\vartheta_z\big(\frac{iL_\ell}{z} - \zeta\big) \neq 0$. By construction, $p_1(iz\zeta)q_1(iz\zeta)$ is singular if and only if $iz\zeta \in S$, see (40). In this case, there exist $\epsilon \in \{\pm 1\}$ and $m \in \mathbb{Z}$ so that $\zeta = \frac{\epsilon L_m}{z}$. Hence $\zeta^2 = \frac{L_m^2}{z^2}$. Since $(z,\zeta) \in \mathrm{M}_\ell$, we also have $\zeta^2 = -\frac{L_\ell^2}{z^2} - 1$. Thus $z = \pm i\sqrt{L_\ell^2 + L_m^2}$.

We now compute $\vartheta_z\big(\frac{iL_\ell}{z}-\zeta\big)$ for such (z,ζ). Set

$$w=\frac{iL_\ell}{z}-\zeta=\frac{iL_\ell}{z}-\frac{\epsilon L_m}{z}=\frac{iL_\ell-\epsilon L_m}{\pm i\sqrt{L_\ell^2+L_m^2}}\,.$$

Then

$$w^{-1}=\frac{\pm i\sqrt{L_\ell^2+L_m^2}}{iL_\ell-\epsilon L_m}=\frac{iL_\ell+\epsilon L_m}{\pm i\sqrt{L_\ell^2+L_m^2}}\,.$$

So,

$$\mathrm{c}(w)=\frac{w+w^{-1}}{2}=\frac{L_\ell}{\pm\sqrt{L_\ell^2+L_m^2}}\,,$$

$$\mathrm{c}(-iw)=\frac{w-w^{-1}}{2i}=\frac{\epsilon L_m}{\pm\sqrt{L_\ell^2+L_m^2}}\,.$$

Hence

$$z\mathrm{c}(w)=iL_\ell\,,\qquad z\mathrm{c}(-iw)=i\epsilon L_m\,,\qquad z\mathrm{s}(w)=-\epsilon L_m\,.$$

Substituting in (35), we obtain

$$\vartheta_z(w)=\frac{\big(L_m^2-L_\ell^2\big)}{b^2}\prod_{k=1}^{m_{\mathrm{m}}-1}\Big[\Big(\frac{-\epsilon L_m}{b}-\frac{m_{\mathrm{m}}}{2}+k\Big)^2-\frac{L_\ell^2}{b^2}\Big]\,. \tag{63}$$

The same argument used in the proof of Lemma 4 shows that the right-hand side of this equation is independent of $\epsilon\in\{\pm1\}$. Using (46), we therefore obtain

$$\begin{aligned}\vartheta_z(w)&=\big((\widetilde{\rho}_{\beta_1}+m)^2-(\widetilde{\rho}_{\beta_1}+\ell)^2\big)\prod_{k=1}^{m_{\mathrm{m}}-1}\Big[\big(\widetilde{\rho}_{\beta_1}+m-\frac{m_{\mathrm{m}}}{2}+k\big)^2-(\widetilde{\rho}_{\beta_1}+\ell)^2\Big]\\&=(m-\ell)(m+\ell+2\widetilde{\rho}_{\beta_2})\prod_{k=1}^{m_{\mathrm{m}}-1}\Big(m-\ell-\frac{m_{\mathrm{m}}}{2}+k\Big)\Big(m+\ell+2\widetilde{\rho}_{\beta_1}-\frac{m_{\mathrm{m}}}{2}+k\Big)\\&=(m-\ell)(m+\ell+2\widetilde{\rho}_{\beta_2})\prod_{h=-(\frac{m_{\mathrm{m}}}{2}-1)}^{\frac{m_{\mathrm{m}}}{2}-1}\Big(m-\ell+h\Big)\Big(m+\ell+2\widetilde{\rho}_{\beta_1}+h\Big)\,.\end{aligned}$$

The values of $m\in\mathbb{Z}_{\geq0}$ making this polynomial vanish are:

$$m=\ell\,, \tag{64}$$

$$m\in\mathbb{Z}_{\geq0}\cap\big[\!\big[\ell-\big(\tfrac{m_{\mathrm{m}}}{2}-1\big),\ell+\big(\tfrac{m_{\mathrm{m}}}{2}-1\big)\big]\!\big]\,, \tag{65}$$

$$m\in\mathbb{Z}_{\geq0}\cap\big[\!\big[-\ell-2\widetilde{\rho}_{\beta_1}-\big(\tfrac{m_{\mathrm{m}}}{2}-1\big),-\ell-2\widetilde{\rho}_{\beta_1}+\big(\tfrac{m_{\mathrm{m}}}{2}-1\big)\big]\!\big]\,. \tag{66}$$

Observe that $-\ell-2\widetilde{\rho}_{\beta_1}+\big(\frac{m_{\mathrm{m}}}{2}-1\big)\geq0$ if and only if $(0\leq)\ell\leq-2\widetilde{\rho}_{\beta_1}+\big(\frac{m_{\mathrm{m}}}{2}-1\big)$. Looking at the first two rows of Table 2, we see that this can happen if and only if $\mathrm{G}=\mathrm{SO}_0(p,2)$ with even $p\geq6$. In this case,

$$\begin{aligned}\big[\!\big[\ell-\big(\tfrac{m_{\mathrm{m}}}{2}-1\big),\ell+\big(\tfrac{m_{\mathrm{m}}}{2}-1\big)\big]\!\big]&=\big[\!\big[\ell+2-\tfrac{p}{2},\ell-2+\tfrac{p}{2}\big]\!\big]\\\big[\!\big[-\ell-2\widetilde{\rho}_{\beta_1}-\big(\tfrac{m_{\mathrm{m}}}{2}-1\big),-\ell-2\widetilde{\rho}_{\beta_1}+\big(\tfrac{m_{\mathrm{m}}}{2}-1\big)\big]\!\big]&=\big[\!\big[-\ell+1-\tfrac{p}{2},-\ell-3+\tfrac{p}{2}\big]\!\big].\end{aligned}$$

Hence

$$\mathbb{Z}_{\geq 0} \cap \llbracket -\ell+1-\tfrac{p}{2}, -\ell-3+\tfrac{p}{2}\rrbracket = \llbracket 0, -\ell-3+\tfrac{p}{2}\rrbracket$$

does not add zeros to those in (65). In fact, $-\ell-3+\frac{p}{2} \leq \ell-2+\frac{p}{2}$ and, if $-\ell-3+\frac{p}{2} \geq 0$, i.e., $\ell \leq \frac{p}{2}-3$, then $\ell+2-\frac{p}{2} \leq 0$. □

For $\ell, m \in \mathbb{Z}_{\geq 0}$, set

$$z_{\ell,m} = i\sqrt{L_\ell^2 + L_m^2} \tag{67}$$

and

$$\zeta_{\ell,m} = i\sqrt{\frac{L_m^2}{L_\ell^2 + L_m^2}}\,. \tag{68}$$

Let $\epsilon \in \{\pm 1\}$. Then all points $(\pm z_{\ell,m}, \epsilon\zeta_{\ell,m})$ are in M_ℓ. Open neighborhoods in M_ℓ of these points are the sets

$$\mathrm{U}_{\ell,\pm} = \{(z,\zeta) \in \mathrm{M}_\ell\ ;\ \pm\Im z > 0\}\,, \tag{69}$$

and local charts on them are

$$\kappa_{\ell,\pm} : \mathrm{U}_{\ell,\pm} \ni (z,\zeta) \to \zeta \in \mathbb{C} \setminus i\big((-\infty,-1] \cup [1,+\infty)\big)\,, \tag{70}$$

inverted by setting $z = \pm i\,\frac{L_\ell}{\sqrt{\zeta^2+1}}$.

Lemma 14. *The local expressions for $\widetilde{G}_\ell$ in terms of the charts (70) are*

$$\big(\widetilde{G}_\ell \circ \kappa_{\ell,\pm}^{-1}\big)(\zeta) = \pm\frac{b}{\pi}\,L_\ell\, p_1(iL_\ell)p_2\Big(\tfrac{L_\ell\zeta}{\sqrt{\zeta^2+1}}\Big)q_1\Big(\tfrac{L_\ell\zeta}{\sqrt{\zeta^2+1}}\Big)\psi^{\vartheta}_{i\frac{L_\ell}{\sqrt{\zeta^2+1}}}\big(\sqrt{\zeta^2+1} \mp \zeta\big)\,. \tag{71}$$

Suppose $m \in \mathbb{Z}_{\geq 0} \setminus \big\llbracket \ell - \big(\frac{m_{\mathrm{m}}}{2}-1\big), \ell + \big(\frac{m_{\mathrm{m}}}{2}-1\big)\big\rrbracket$. Then the residue of the local expression of $\widetilde{G}_\ell$ at a point $(z,\zeta) \in \mathrm{M}_\ell$ with $z = \pm z_{\ell,m}$ is

$$\operatorname*{Res}_{\zeta=\pm\zeta_{\ell,m}} (\widetilde{G}_\ell \circ \kappa_{\ell,\pm}^{-1})(\zeta) = \pm\frac{1}{i\pi^2}C_{\ell,m}(f \times \varphi_{\frac{L_\ell\beta_1+L_m\beta_2}{b}})(y)\,. \tag{72}$$

In (72),

$$C_{\ell,m} = bL_\ell\, p_1(iL_\ell)p_2(iL_m)\vartheta_0\Big(\frac{L_\ell}{b}, \frac{L_m}{b}\Big)\,, \tag{73}$$

where ϑ_0 is as in (26)*, is a positive constant.*

Proof. The computation of the residues is as in [6, Lemma 16]. The constant $\vartheta_0\big(\frac{L_\ell}{b}, \frac{L_m}{b}\big)$ agrees with (63) with $(z,\zeta) = (z_{\ell,m}, \epsilon\zeta_{\ell,m})$, and we only need to prove that it is positive. Recall that (63) is independent of ϵ. Hence

$$\begin{aligned}\vartheta_0\Big(\frac{L_\ell}{b}, \frac{L_m}{b}\Big) &= \frac{\big(L_m^2 - L_\ell^2\big)}{b^2}\\ &\quad\times \prod_{k=1}^{m_{\mathrm{m}}-1}\Big(\frac{L_m}{b} - \Big(\frac{m_{\mathrm{m}}}{2}-k\Big) - \frac{L_\ell}{b}\Big)\Big(\frac{L_m}{b} - \Big(\frac{m_{\mathrm{m}}}{2}-k\Big) + \frac{L_\ell}{b}\Big)\,.\end{aligned} \tag{74}$$

If $m > \ell + \big(\frac{m_{\mathrm{m}}}{2}-1\big) \geq \ell$, then all factors appearing in the above product are positive. If $m < \ell - \big(\frac{m_{\mathrm{m}}}{2}-1\big) \leq \ell$, then all factors $\frac{L_m}{b} - \big(\frac{m_{\mathrm{m}}}{2}-k\big) + \frac{L_\ell}{b}$ are positive, whereas $L_m^2 - L_\ell^2$ as well as the $m_{\mathrm{m}} - 1$ factors $\frac{L_m}{b} - \left(\frac{m_{\mathrm{m}}}{2} - k\right) - \frac{L_\ell}{b}$ are negative. Since m_{m} is even, we conclude that $\vartheta_0\left(\frac{L_\ell}{b}, \frac{L_m}{b}\right) > 0$ in all cases. □

A different parametrization of the singularities of $\widetilde{G}_\ell$ will turn out to be more convenient. Observe first that, by (3) and (10),

$$\widetilde{\rho}_{\beta_1} = \rho_{\beta_1} = \rho_{\beta_2} - \frac{m_{\mathrm{m}}}{2}\,.$$

We will use the following notation for $(\ell_1, \ell_2) \in \mathbb{Z}^2_{\geq 0}$:

$$\lambda(\ell_1, \ell_2) = (\rho_{\beta_1} + \ell_1)\beta_1 + (\rho_{\beta_2} + \ell_2)\beta_2 = \frac{1}{b}\left(L_{\ell_1}\beta_1 + L_{\ell_2 + \frac{m_{\mathrm{m}}}{2}}\beta_2\right). \tag{75}$$

Corollary 15. *Keep the notation of Lemma 14. If $\ell \in [\![0, \frac{m_{\mathrm{m}}}{2} - 1]\!]$, then $\widetilde{G}_\ell$ has simple poles at the points $(z, \zeta) \in \mathrm{M}_\ell$ with $z = \pm i|z|$ and*

$$b^{-2}|z|^2 = (\rho_{\beta_1} + \ell)^2 + (\rho_{\beta_2} + \ell + k)^2 \qquad (k \in \mathbb{Z}_{\geq 0})\,. \tag{76}$$

If $\ell \in \frac{m_{\mathrm{m}}}{2} + \mathbb{Z}_{\geq 0}$, then $\widetilde{G}_\ell$ has simple poles at the points $(z, \zeta) \in \mathrm{M}_\ell$ with $z = \pm i|z|$ and satisfying either (76) *or*

$$b^{-2}|z|^2 = (\rho_{\beta_1} + m)^2 + (\rho_{\beta_2} + \ell_0)^2 \qquad (m \in [\![0, \ell_0]\!])\,, \tag{77}$$

where $\ell_0 = \ell - \frac{m_{\mathrm{m}}}{2}$.

The residue of the local expression of $\widetilde{G}_\ell$ at a point $(z, \zeta) \in \mathrm{M}_\ell$ with $z = \pm i|z|$ satisfying (76) *is*

$$\operatorname*{Res}_{\zeta = \pm\zeta_{\ell, \ell + \frac{m_{\mathrm{m}}}{2} + k}} (\widetilde{G}_\ell \circ \kappa_{\ell,\pm}^{-1})(\zeta) = \pm\frac{1}{i\pi^2} C_{\ell, \ell + \frac{m_{\mathrm{m}}}{2} + k}\left(f \times \varphi_{\lambda(\ell, \ell+k)}\right)(y)\,. \tag{78}$$

The residue of the local expression of $\widetilde{G}_\ell$ at a point $(z, \zeta) \in \mathrm{M}_\ell$ with $z = \pm i|z|$ satisfying (77) *is*

$$\operatorname*{Res}_{\zeta = \pm\zeta_{\ell, m}} (\widetilde{G}_\ell \circ \kappa_{\ell,\pm}^{-1})(\zeta) = \pm\frac{1}{i\pi^2} C_{\ell, m}\left(f \times \varphi_{\lambda(m, \ell_0)}\right)(y)\,. \tag{79}$$

Proof. We have $\ell \in [\![0, \frac{m_{\mathrm{m}}}{2} - 1]\!]$ if and only if $0 \in [\![\ell - \left(\frac{m_{\mathrm{m}}}{2} - 1\right), \ell + \left(\frac{m_{\mathrm{m}}}{2} - 1\right)]\!]$. In this case, $m \in \mathbb{Z}_{\geq 0} \setminus [\![\ell - \left(\frac{m_{\mathrm{m}}}{2} - 1\right), \ell + \left(\frac{m_{\mathrm{m}}}{2} - 1\right)]\!] = \ell + \frac{m_{\mathrm{m}}}{2} + \mathbb{Z}_{\geq 0}$ is of the form $m = \ell + \frac{m_{\mathrm{m}}}{2} + k$ with $k \in \mathbb{Z}_{\geq 0}$. Hence $\frac{L_\ell}{b} = \rho_{\beta_1} + \ell$ and $\frac{L_m}{b} = \widetilde{\rho}_{\beta_1} + \frac{m_{\mathrm{m}}}{2} + \ell + k = \rho_{\beta_2} + \ell + k$.

On the other hand, if $\ell \in \frac{m_{\mathrm{m}}}{2} + \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0} \setminus [\![\ell - \left(\frac{m_{\mathrm{m}}}{2} - 1\right), \ell + \left(\frac{m_{\mathrm{m}}}{2} - 1\right)]\!]$, then either $m \in \ell + \frac{m_{\mathrm{m}}}{2} + \mathbb{Z}_{\geq 0}$ (and the above applies), or $m \in [\![0, \ell_0]\!]$. In the latter case, $\frac{L_\ell}{b} = \widetilde{\rho}_{\beta_1} + \frac{m_{\mathrm{m}}}{2} + \ell_0 = \rho_{\beta_2} + \ell_0$ and $\frac{L_m}{b} = \rho_{\beta_1} + m$. Observe also that $\varphi_{\lambda(\ell_0, m)} = \varphi_{\lambda(m, \ell_0)}$ by W-invariance. □

We now proceed with the piecewise extension of F along the negative imaginary half-line $-i[L, +\infty)$. Recall from Corollary 12 that for $v \in I_n = [L_n, L_{n+1})$

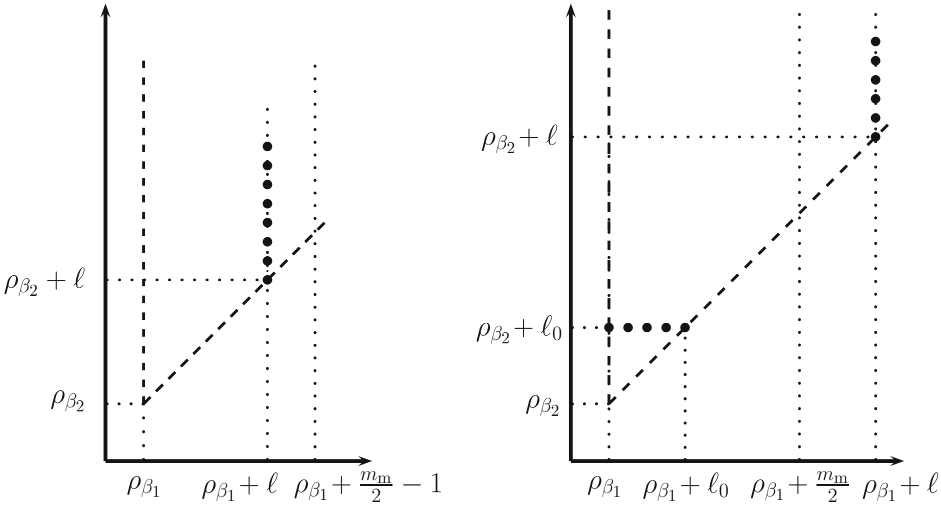

FIGURE 1. On the left: $\lambda(\ell,\ell+k)$ for $\ell\in[\![0,\frac{m_{\rm m}}{2}]\!]$. On the right: $\lambda(\ell,\ell+k)$ and $\lambda(m,\ell_0)$ for $\ell\geq\frac{m_{\rm m}}{2}$

with $n\in\mathbb{Z}_{\geq 0}$ there exists $0<r_v<1$ and an open neighborhood W_v of $-iv$ in $\mathbb{C}$ such that

$$F(z)=F_{r_v}(z)+4\sum_{\ell=0}^{n}G_\ell(z)\qquad(z\in W_v\setminus i\mathbb{R}),\tag{80}$$

where the function F_{r_v} is holomorphic in W_v. This equality extends then to $I_{-1}=(0,L)$ by allowing empty sums. By possibly shrinking W_v, we may also assume that W_v is an open disk around $-iv$ such that

$$W_v\cap i\mathbb{R}\subseteq\begin{cases}-iI_n & \text{for } v\in I_n^\circ,\\ -i(I_n-\frac{b}{2}) & \text{for } v=L_n\,.\end{cases}$$

In addition, for $0<v<L$ we define W_v to be an open ball around $-iv$ in $\mathbb{C}$ such that $W_v\cap i\mathbb{R}\subset(0,L)$. If $v\in I_n$, $v'\geq L$ and $W_v\cap W_{v'}\neq\emptyset$, then we obtain for $z\in W_v\cap W_{v'}$

$$F_{r_{v'}}(z)=F_{r_v}(z)+\begin{cases}0 & \text{if } v'\in I_n,\\ 4\,G_n(z) & \text{if } v'\in I_{n-1}.\end{cases}$$

Now we set

$$W_{(-1)}=\bigcup_{v\in I_{-1}}W_v\qquad\text{and}\qquad W_{(n)}=\bigcup_{v\in I_n}W_v\qquad(n\in\mathbb{Z}_{\geq 0})\,.$$

For $n\in\mathbb{Z}_{\geq 1}$ we define a holomorphic function $F_{(n)}:W_{(n)}\to\mathbb{C}$ by

$$F_{(n)}(z)=\begin{cases}F_{r_v}(z) & \text{if } n\in\mathbb{Z}_{\geq 0},\ v\in I_n \text{ and } z\in W_v\\ F(z) & \text{if } n=-1 \text{ and } z\in W_{(-1)}\,.\end{cases}$$

We therefore obtain the following analogue of [6, Proposition 18].

Proposition 16. *For every integer $n \in \mathbb{Z}_{\geq -1}$ we have*

$$F(z) = F_{(n)}(z) + 4\sum_{\ell=0}^{n} G_\ell(z) \qquad (z \in W_{(n)} \setminus i\mathbb{R}), \tag{81}$$

where $F_{(n)}$ is holomorphic in $W_{(n)}$, the G_ℓ are as in (53)*, and empty sums are defined to be equal to* 0*.*

We can continue F across $-i(0, +\infty)$ inductively, as in the case of the direct product of two rank one symmetric spaces in [6]. Our specific case $\mathsf{X}_1 = \mathsf{X}_2$ is slightly easier, as for instance one gets just one regularly spaced sequence of branching points L_ℓ. Since the procedure does not involve new steps, we will limit ourself to overview the different parts and state the final result, referring the reader to [6] for the details.

For a fixed positive integer N, we construct a Riemann surface $\mathrm{M}_{(N)}$ by "pasting together" the Riemann surfaces M_ℓ to which all functions G_ℓ, with $\ell = 0, 1, \dots, N$, admit meromorphic extension. Namely, we set

$$\mathrm{M}_{(N)} = \big\{(z, \zeta) \in \mathbb{C}^- \times \mathbb{C}^{N+1}; \zeta = (\zeta_0, \dots, \zeta_N),\ (z, \zeta_\ell) \in \mathrm{M}_\ell,\ \ell \in \mathbb{Z}_{\geq 0},\ 0 \leq \ell \leq N\big\}. \tag{82}$$

Then $\mathrm{M}_{(N)}$ is a Riemann surface, and the map

$$\pi_{(N)} : \mathrm{M}_{(N)} \ni (z, \zeta) \to z \in \mathbb{C}^- \tag{83}$$

is a holomorphic 2^{N+1}-to-1 cover, except when $z = -iL_\ell$ for some $\ell \in \mathbb{Z}_{\geq 0}$ with $0 \leq \ell \leq N$. The fiber above each of these elements $-iL_\ell$ consists of 2^N branching points of $\mathrm{M}_{(N)}$. A choice of square root function $\zeta_\ell^+(z)$, see [6, (81)], for every coordinate function ζ_ℓ on $\mathrm{M}_{(N)}$ yields a section

$$\sigma_{(N)}^+ : z \to (z, \zeta_0^+(z), \dots, \zeta_N^+(z))$$

of the projection $\pi_{(N)}$. All possible sections of $\pi_{(N)}$ are obtained by choosing a sign $\pm\zeta_\ell^+$ for each coordinate function. We obtain in this way a parametrization of all sections of $\pi_{(N)}$ by means of elements $\varepsilon = (\varepsilon_0, \dots, \varepsilon_N) \in \{\pm 1\}^{N+1}$.

For $0 \leq \ell \leq N$ consider the holomorphic projection

$$\pi_{(N,\ell)} : \mathrm{M}_{(N)} \ni (z, \zeta) \to (z, \zeta_\ell) \in \mathrm{M}_\ell. \tag{84}$$

Then the meromorphic function

$$\widetilde{G}_{(N,\ell)} = \widetilde{G}_\ell \circ \pi_{(N,\ell)} : \mathrm{M}_{(N)} \to \mathbb{C} \tag{85}$$

is holomorphic on $(\pi_{(N)})^{-1}(\mathbb{C}^- \setminus i\mathbb{R})$. Moreover, on $\mathbb{C}^- \setminus i\mathbb{R}$,

$$\widetilde{G}_{(N,\ell)} \circ \sigma_{(N)}^+ = G_\ell\,.$$

So, $\widetilde{G}_{(N,\ell)}$ is the meromorphic extension of a lift of G_ℓ to $\mathrm{M}_{(N)}$. Using the right-hand side of (81) with $F_{(n)}$ constant on the z-fibers, we obtain a lift of F to $\pi_{(N)}^{-1}(W_{(n)} \setminus i\mathbb{R})$.

The next step is to "glue together" all these local meromorphic extensions of F, moving from branching point to branching point, to get a meromorphic extension of F along the branched curve γ_N in $\mathrm{M}_{(N)}$ covering the interval $-i(0, L_{N+1})$. Define, as in [6, section 4.3], the open sets $U_{n,\varepsilon}$, $U_{\varepsilon(n^\vee)}$ (with $n \in \mathbb{Z}_{\geq 0}$, $\varepsilon \in \{\pm 1\}^{N+1}$) and the open neighborhood M_{γ_N} of γ_N in $\mathrm{M}_{(N)}$. Every open set $U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$ is a homeomorphic lift to $\mathrm{M}_{(N)}$ of the neighborhood $W_{(n)}$ of $-[L_n, L_{n+1})$. Then we have the following analogue of [6, Theorem 19].

Theorem 17. *For $n \in \{-1, 0, \dots, N\}$, $\varepsilon \in \{\pm 1\}^{N+1}$ and $(z, \zeta) \in U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$ define*

$$\widetilde{F}(z,\zeta) = F_{(n)}(z) + 4\sum_{\ell=0}^{n} \widetilde{G}_{(N,\ell)}(z,\zeta) + 4 \sum_{\substack{n<\ell\leq N \\ \text{with } \varepsilon_\ell = -1}} \left[\widetilde{G}_{(N,\ell)}(z,\zeta) - \widetilde{G}_{(N,\ell)}(z,-\zeta)\right], \quad (86)$$

where the first sum is equal to 0 if $\ell = -1$ and the second sum is 0 if $\varepsilon_\ell = 1$ for all $\ell > n$. Then $\widetilde{F}$ is the meromorphic extension of a lift of F to the open neighborhood M_{γ_N} of the branched curve γ_N lifting $-i(0, L_{N+1})$ in $\mathrm{M}_{(N)}$.

Order the singularities according to their distance from the origin $0 \in \mathbb{C}$, and let $\{z_{(h)}\}_{h\in\mathbb{Z}_{\geq 0}}$ be the resulting ordered sequence. For a fixed $h \in \mathbb{Z}_{\geq 0}$ set

$$S_h = \{\ell \in \mathbb{Z}_{\geq 0};\ \exists k \in \mathbb{Z}_{\geq 0} \text{ so that } b^{-2}|z_{(h)}|^2 = (\rho_{\beta_1} + \ell)^2 + (\rho_{\beta_2} + \ell + k)^2\}. \quad (87)$$

Notice that if $\ell \in S_h$, then the corresponding element k is uniquely determined. Let $N \in \mathbb{Z}_{\geq 0}$ be such that $|z_{(h)}| < L_{N+1}$ and $n \in [\![0, N]\!]$ such that $|z_{(h)}| \in [L_n, L_{n+1})$. Then the possible singularities of $\widetilde{F}$ at points of $\mathrm{M}_{(N)}$ above $z_{(h)}$ are those of

$$\sum_{\ell=0}^{n} \widetilde{G}_{(N,\ell)}(z,\zeta) = \sum_{\ell=0}^{n} \widetilde{G}_{\ell}(z,\zeta_\ell).$$

Indeed, the singularities of $\widetilde{G}_{(N,\ell)}(z,\zeta) = \widetilde{G}_\ell(z,\zeta_\ell)$ occur at points $(z,\zeta) \in \mathrm{M}_{(N)}$ with $|z|^2 = L_\ell^2 + L_m^2 > L_\ell^2$. Hence the second sum on the right-hand side of (86) is holomorphic on $U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$.

The singular points of $\widetilde{F}$ above $z_{(h)}$ are parametrized by $\varepsilon \in \{\pm 1\}^{N+1}$. We denote by $(z_{(h)}, \zeta^{(h,\varepsilon)})$ the one in $U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$. The local expression of $\widetilde{F}$ on $U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$ is computed in terms of the chart $\kappa_{n,\varepsilon}$ defined for $(z,\zeta) \in U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$ by $\kappa_{n,\varepsilon}(z,\zeta) = \zeta_n$.

Suppose $\widetilde{G}_{(N,\ell)}(z,\zeta)$ is singular at $(z_{(h)}, \zeta^{(h,\varepsilon)})$. Then, by [6, Proposition 21],

$$\operatorname*{Res}_{\zeta_n=\zeta_n^{(h,\varepsilon)}} \left(\widetilde{G}_{(N,\ell)} \circ \kappa_{n,\varepsilon}^{-1}\right)(\zeta_n) = \varepsilon_\ell\varepsilon_n \frac{L_n^2}{L_\ell^2} \frac{\sqrt{|z_{(h)}|^2 - L_\ell^2}}{\sqrt{|z_{(k)}|^2 - L_n^2}} \operatorname*{Res}_{\zeta_\ell=\zeta_\ell^{(h,\varepsilon)}} \left(\widetilde{G}_\ell \circ \kappa_{\ell,-}^{-1}\right)(\zeta_\ell). \quad (88)$$

If ℓ satisfies (76) with $z = z_{(h)}$ for some $k \in \mathbb{Z}_{\geq 0}$, then $|z_{(h)}|^2 - L_\ell^2 = b^2(\rho_{\beta_2} + \ell + k)^2 = L^2_{\ell+\frac{m_\mathrm{m}}{2}+k}$. If $\ell \geq \frac{m_\mathrm{m}}{2}$ satisfies (77) with $z = z_{(h)}$ for some $m \in [\![0, \ell_0]\!]$ and $\ell_0 = \ell - \frac{m_\mathrm{m}}{2}$, then $|z_{(h)}|^2 - L_\ell^2 = b^2(\rho_{\beta_1} + m) = L_m^2$.

In the first case, by (78), the right-hand side of (88) is equal to

$$\frac{\varepsilon_n L_n^2}{\sqrt{|z_{(h)}|^2 - L_n^2}} \frac{L_{\ell+\frac{m_\mathrm{m}}{2}+k}}{L_\ell^2} \operatorname*{Res}_{\zeta_\ell = -\zeta_{\ell,\ell+\frac{m_\mathrm{m}}{2}+k}} \big(\widetilde{G}_\ell \circ \kappa_{\ell,-}^{-1}\big)(\zeta_\ell)$$
$$= \frac{i}{\pi^2} \frac{\varepsilon_n L_n^2}{\sqrt{|z_{(h)}|^2 - L_n^2}} \frac{L_{\ell+\frac{m_\mathrm{m}}{2}+k}}{L_\ell^2} C_{\ell,\ell+\frac{m_\mathrm{m}}{2}+k}\big(f \times \varphi_{\lambda(\ell,\ell+k)}\big)(y).$$

In the second case, by (79), the right-hand side of (88) is equal to

$$\frac{\varepsilon_n L_n^2}{\sqrt{|z_{(h)}|^2 - L_n^2}} \frac{L_m}{L_\ell^2} \operatorname*{Res}_{\zeta_\ell = -\zeta_{\ell,m}} \big(\widetilde{G}_\ell \circ \kappa_{\ell,-}^{-1}\big)(\zeta_\ell)$$
$$= \frac{i}{\pi^2} \frac{\varepsilon_n L_n^2}{\sqrt{|z_{(h)}|^2 - L_n^2}} \frac{L_m}{L_\ell^2} C_{\ell,m}\big(f \times \varphi_{\lambda(m,\ell_0)}\big)(y).$$

Observe that in both cases, the constants appearing are i times a positive constant. Observe also that if $\ell \geq \frac{m_\mathrm{m}}{2}$ and $\widetilde{G}_{(N,\ell)}$ is singular at $(z_{(h)}, \zeta^{(h,\varepsilon)})$ with ℓ satisfying (77) with $z = z_{(h)}$, some $m \in [\![0, \ell_0]\!]$ and $\ell_0 = \ell - \frac{m_\mathrm{m}}{2}$, then $(z_{(h)}, \zeta^{(h,\varepsilon)})$ is also a singularity of $\widetilde{G}_{(N,m)}$ and m satisfies (76) with $z = z_{(h)}$ and $k = \ell_0 - m \in \mathbb{Z}_{\geq 0}$. Of course, $\varphi_{\lambda(m,\ell_0)} = \varphi_{\lambda(m,m+k)}$ in this case. It follows that the set S_h is sufficient to parametrize the residues of $\widetilde{F}$ at $(z_{(h)}, \zeta^{(h,\varepsilon)})$.

It follows that

$$\operatorname*{Res}_{\zeta_n = \zeta_n^{(h,\varepsilon)}} \big(\widetilde{F} \circ \kappa_{n,\varepsilon}^{-1}\big)(\zeta_n) = \frac{i\varepsilon_n L_n^2}{\sqrt{|z_{(h)}|^2 - L_n^2}} \sum_{\ell \in S_h} c_\ell \big(f \times \varphi_{\lambda(\ell,\ell+k)}\big)(y), \tag{89}$$

where $k \in \mathbb{Z}_{\geq 0}$ is associated with ℓ as in the definition of S_h and c_ℓ is a positive constant depending only on ℓ.

By Proposition 7, the meromorphic extensions on the half-line $i(-\infty, -L]$ of F and of the resolvent R of the Laplacian are equivalent. Thus the resolvent R can be lifted and meromorphically extended along the curve γ_N in M_{γ_N}. Its singularities (i.e., the resonances of the Laplacian) are those of the meromorphic extension $\widetilde{F}$ of F and are located at the points of M_{γ_N} above the elements $z_{(h)}$. They are simple poles. The precise description is given by the following theorem.

Theorem 18. *Let $f \in C_c^\infty(\mathsf{X})$ and $y \in \mathsf{X}$ be fixed. Let $N \in \mathbb{N}$ and let γ_N be the curve lifting the interval $-i(0, N+1)$ in $\mathrm{M}_{(N)}$. Then the resolvent $R(z) = [R(z)f](y)$ lifts as a meromorphic function to the neighborhood M_{γ_N} of the curve γ_N in $\mathrm{M}_{(N)}$. We denote the lifted meromorphic function by $\widetilde{R}_{(N)}(z,\zeta) = \big[\widetilde{R}_{(N)}(z,\zeta)f\big](y)$.*

The singularities of $\widetilde{R}_{(N)}$ are at most simple poles at the points $(z_{(h)}, \zeta^{(h,\varepsilon)}) \in \mathrm{M}_{(N)}$ with $h \in \mathbb{Z}_{\geq 0}$ so that $|z_{(h)}| < L_{N+1}$ and $\varepsilon \in \{\pm 1\}^{N+1}$. Explicitly, for

$(n,\varepsilon) \in [\![0,N]\!] \times \{\pm 1\}^{N+1}$,

$$\widetilde{R}_{(N)}(z,\zeta) = \widetilde{H}_{(N,m,\varepsilon)}(z,\zeta) + 2\pi i \sum_{\ell=0}^{m} \widetilde{G}_{(N,\ell)}(z,\zeta) \quad \left((z,\zeta) \in U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}\right), \tag{90}$$

where $\widetilde{H}_{(N,m,\varepsilon)}$ *is holomorphic and* $\widetilde{G}_{(N,\ell)}(z,\zeta)$ *is in fact independent of* N *and* ε *(but dependent on* f *and* y*, which are omitted from the notation). The singularities of* $\widetilde{R}_{(N)}(z,\zeta)$ *in* $U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$ *are simple poles at the points* $(z_{(h)}, \zeta^{(h,\varepsilon)})$ *belonging to* $U_{\varepsilon(n^\vee)} \cup U_{n,\varepsilon}$*. The residue of the local expression of* $\widetilde{R}_{(N)}$ *at one such point is* $i\pi$ *times the right-hand side of* (89).

4. The resolvent operators

Recall the notation $\lambda(\ell_1,\ell_2) = (\rho_{\beta_1} + \ell_1)\beta_1 + (\rho_{\beta_2} + \ell_2)\beta_2$ introduced in (75). For a fixed $h \in \mathbb{Z}_{\geq 0}$, the sum over S_h appearing on the right-hand side of (89) is independent either of N or n. It can be used to define the residue operator $\mathrm{Res}_{z_{(h)}} \widetilde{R}$ of the meromorphically extended resolvent at $z_{(h)}$. Explicitly,

$$\mathrm{Res}_{z_{(h)}} \widetilde{R} = \sum_{\ell \in S_h} c_\ell R_{\lambda(\ell,\ell+k_\ell)} \tag{91}$$

where, c_ℓ are non-zero constants and, as in [5, (57)], $R_\lambda : C_c^\infty(\mathsf{X}) \to C^\infty(\mathsf{X})$ is defined by $R_\lambda f = f \times \varphi_\lambda$. We know from [2, Chapter IV, Theorem 4.5] that $R_\lambda(C_c^\infty(\mathsf{X}))$ is an irreducible representation of G. Furthermore, two such representations are equivalent if and only if the spectral parameters λ are in the same Weyl group orbit. Since, in our case, the Weyl group acts by transposition and sign changes, the element $\lambda(\ell_1,\ell_2)$ is dominant with respect to the fixed choice of positive roots if and only if

$$\rho_{\beta_2} + \ell_2 \geq \rho_{\beta_1} + \ell_1 \geq 0\,, \quad \text{i.e.,} \quad \ell_2 + \frac{m_{\mathrm{m}}}{2} \geq \ell_1\,.$$

In particular, all $\lambda(\ell, \ell + k_\ell)$ are distinct and dominant. Hence, as a G-module,

$$\mathrm{Res}_{z_{(h)}} \widetilde{R}(C_c^\infty(\mathsf{X})) = \bigoplus_{\ell \in S_{(h)}} R_{\lambda(\ell,\ell+k_\ell)}(C_c^\infty(\mathsf{X}))\,. \tag{92}$$

Theorem 19. *If* $\ell, k \in \mathbb{Z}^2_{\geq 0}$*, then* $\dim R_{\lambda(\ell,\ell+k)}(C_c^\infty(\mathsf{X})) < \infty$. *Thus*

$$\mathrm{Res}_{z_{(h)}} \widetilde{R}(C_c^\infty(\mathsf{X}))$$

is a finite-dimensional G*-module.*

The G*-module* $\mathrm{Res}_{z_{(h)}} \widetilde{R}(C_c^\infty(\mathsf{X}))$ *is bounded (and unitary) if and only it is the trivial representation, which occurs for* $h = 0$*, i.e., when* $z_{(0)} = -i\sqrt{\langle \rho, \rho \rangle}$.

Proof. By [3, Ch II, §4, Theorem 4.16], $\dim R_{\lambda(\ell_1,\ell_2)}(C_c^\infty(\mathsf{X})) < \infty$ if and only if there is $w \in W$ such that

$$(w\lambda(\ell_1,\ell_2) - \rho)_\beta \in \mathbb{Z}_{\geq 0} \qquad (\beta \in \Sigma_*^+, \beta \text{ simple})\,. \tag{93}$$

Recall that the simple roots in Σ_*^+ are β_1 and $\frac{\beta_2-\beta_1}{2}$. Moreover, $\mu_{\frac{\beta_2-\beta_1}{2}} = \mu_{\beta_2} - \mu_{\beta_1}$ for $\mu \in \mathfrak{a}_{\mathbb{C}}^*$. Since

$$\lambda(\ell, \ell+k) - \rho = (\rho_{\beta_1} + \ell)\beta_1 + (\rho_{\beta_2} + \ell + k)\beta_2 - (\rho_{\beta_1}\beta_1 + \rho_{\beta_2}\beta_2) = \ell\beta_1 + (\ell+k)\beta_2 \,,$$

we conclude that

$$(\lambda(\ell, \ell+k) - \rho)_{\beta_1} = \ell \in \mathbb{Z}_{\geq 0} \,,$$
$$(\lambda(\ell, \ell+k) - \rho)_{(\beta_2-\beta_1)/2} = k \in \mathbb{Z}_{\geq 0} \,,$$

which satisfies (93) with $w = \mathrm{id}$. □

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J. Hilgert
Department of Mathematics
Paderborn University
Warburger Str. 100
D-33098 Paderborn, Germany
e-mail: hilgert@math.uni-paderborn.de

A. Pasquale
Université de Lorraine
Institut Elie Cartan de Lorraine
UMR CNRS 7502
F-57045 Metz, France
e-mail: angela.pasquale@univ-lorraine.fr

T. Przebinda
Department of Mathematics
University of Oklahoma
Norman, OK 73019, USA
e-mail: tprzebinda@ou.edu

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Howe's Correspondence and Characters

Tomasz Przebinda

Abstract. The purpose of this note is to explain how is Howe's correspondence used to construct irreducible unitary representations of low Gel'fand–Kirillov dimension and to recall and motivate a conjecture concerning the distribution characters of the representations involved.

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1. Introduction

In this note we would like to shed some light at the wide open problem of understanding the distribution character Θ_Π, [8], of an irreducible unitary representation Π of a real reductive group G. The representations of low Gel'fand–Kirillov dimension are of special interest.

The notion of the Gel'fand–Kirillov dimension $GK \dim \Pi$ of an irreducible admissible representation Π of G (or rather of the corresponding Harish-Chandra module X_Π) was introduced in [21]. It is equal to one half times the Gel'fand–Kirillov dimension of the algebra $\mathcal{U}(\mathfrak{g})/\operatorname{Ann} X_\Pi$, a concept defined earlier in [7]. (See also [4] for more explanation.) Here $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G and $\operatorname{Ann} X$ is the annihilator of X_Π.

We explain why Howe's correspondence, [15], is a suitable tool for constructing irreducible unitary representations of low Gel'fand–Kirillov dimension and recall a conjecture concerning the distribution characters of the representations occurring in the correspondence [3].

2. The Weil representation

Let W be a vector space of finite dimension $2n$ over $\mathbb{R}$ with a non-degenerate symplectic form $\langle\cdot,\cdot\rangle$. Denote by $\mathrm{Sp} \subseteq \mathrm{GL(W)}$ the corresponding symplectic group.

Denote by $\mathfrak{sp}$ the Lie algebra of Sp. Fix a compatible positive complex structure J on W. Hence $J \in \mathfrak{sp}$ is such that $J^2 = -1$ (minus the identity in End(W)) and the symmetric bilinear form $\langle J\cdot, \cdot\rangle$ is positive definite on W.

For an element $g \in \mathrm{Sp}$, let $J_g = J^{-1}(g-1)$. Then its adjoint with respect to the form $\langle J\cdot, \cdot\rangle$ is $J_g^* = Jg^{-1}(1-g)$. In particular J_g and J_g^* have the same kernel. Hence the image of J_g is

$$J_g\mathrm{W} = (\mathrm{Ker}\, J_g^*)^\perp = (\mathrm{Ker}\, J_g)^\perp$$

where $\perp$ denotes the orthogonal complement with respect to $\langle J\cdot, \cdot\rangle$. Therefore, the restriction of J_g to $J_g\mathrm{W}$ defines an invertible element. Thus it makes sense to consider $\det(J_g)^{-1}_{J_g\mathrm{W}}$, the reciprocal of the determinant of the restriction of J_g to $J_g\mathrm{W}$. Let

$$\widetilde{\mathrm{Sp}} = \{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C}, \quad \xi^2 = i^{\dim(g-1)\mathrm{W}} \det(J_g)^{-1}_{J_g\mathrm{W}}\}\,. \tag{1}$$

Then there exists a 2-cocycle $C : \mathrm{Sp} \times \mathrm{Sp} \to \mathbb{C}$, such that $\widetilde{\mathrm{Sp}}$ is a group with respect to the multiplication

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2))\,. \tag{2}$$

In fact, by [1, Lemma 52],

$$|C(g_1, g_2)| = \sqrt{\left|\frac{\det(J_{g_1})_{J_{g_1}\mathrm{W}} \det(J_{g_2})_{J_{g_2}\mathrm{W}}}{\det(J_{g_1 g_2})_{J_{g_1 g_2}\mathrm{W}}}\right|} \tag{3}$$

and by [1, Proposition 46 and formula (109)],

$$\frac{C(g_1, g_2)}{|C(g_1, g_2)|} = \chi(\frac{1}{8}\,\mathrm{sgn}(q_{g_1, g_2})), \tag{4}$$

where $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$, is a fixed unitary character of the additive group $\mathbb{R}$ and $\mathrm{sgn}(q_{g_1,g_2})$ is the signature of the symmetric form

$$\begin{aligned} q_{g_1,g_2}(u', u'') = {} & \frac{1}{2}\langle (g_1+1)(g_1-1)^{-1}u', u''\rangle \\ & + \frac{1}{2}\langle (g_2+1)(g_2-1)^{-1}u', u''\rangle, \quad u', u'' \in (g_1-1)\mathrm{W} \cap (g_2-1)\mathrm{W}. \end{aligned} \tag{5}$$

By the signature of a (possibly degenerate) symmetric form we understand the difference between the maximal dimension of a subspace where the form is positive definite and the maximal dimension of a subspace where the form is negative definite. The group $\widetilde{\mathrm{Sp}}$ is known as the metaplectic group.

Let dw be the Lebesgue measure on W such that the volume of the unit cube with respect to this form is 1. (Since all positive complex structures are conjugate by elements of Sp, this normalization does not depend on the particular choice of J.) Let $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ be a complete polarization. We normalize the Lebesgue measures on X and on Y similarly.

Each element $K \in \mathcal{S}^*(\mathrm{X}\times\mathrm{X})$ defines an operator

$$\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$$

by

$$\mathrm{Op}(K)v(x) = \int_{\mathrm{X}} K(x,x')v(x')\,dx'. \tag{6}$$

Here $\mathcal{S}(\mathrm{V})$ and $\mathcal{S}^*(\mathrm{V})$ denote the Schwartz space on the vector space V and the space of tempered distributions on V, respectively. The map $\mathrm{Op} : \mathcal{S}^*(\mathrm{X}\times\mathrm{X}) \to \mathrm{Hom}(\mathcal{S}(\mathrm{X}), \mathcal{S}^*(\mathrm{X}))$ is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [12, Theorem 5.2.1].

Fix the unitary character $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$, and recall the Weyl transform

$$\begin{aligned} &\mathcal{K} : \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\mathrm{X}\times\mathrm{X}) \\ &\mathcal{K}(f)(x,x') = \int_{\mathrm{Y}} f(x-x'+y)\chi\left(\frac{1}{2}\langle y, x+x'\rangle\right)dy \qquad (f \in \mathcal{S}(\mathrm{W})). \end{aligned} \tag{7}$$

Let

$$\chi_{c(g)}(u) = \chi\left(\frac{1}{4}\langle (g+1)(g-1)^{-1}u, u\rangle\right) \qquad (u = (g-1)w,\ w \in \mathrm{W}). \tag{8}$$

(In particular, if $g-1$ is invertible on W, then $\chi_{c(g)}(u) = \chi(\frac{1}{4}\langle c(g)u, u\rangle$ where $c(g) = (g+1)(g-1)^{-1}$ is the usual Cayley transform.) For $\tilde{g} = (g,\xi) \in \widetilde{\mathrm{Sp}}$ define

$$\Theta(\tilde{g}) = \xi, \qquad T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathrm{W}}, \qquad \omega(\tilde{g}) = \mathrm{Op}\circ\mathcal{K}\circ T(\tilde{g}), \tag{9}$$

where $\mu_{(g-1)\mathrm{W}}$ is the Lebesgue measure on the subspace $(g-1)\mathrm{W}$ normalized so that the volume of the unit cube with respect to the form $\langle J\cdot,\cdot\rangle$ is 1. In these terms, $(\omega, L^2(\mathrm{X}))$ is the Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ.

3. Dual pairs

A real reductive dual pair is a pair of subgroups $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathsf{W})$ which act reductively on the symplectic space W, G' is the centralizer of G in Sp and G is the centralizer of G' in Sp, [13]. We shall be concerned with the irreducible pairs in the sense that there is no non-trivial direct sum decomposition of W preserved by G and G'. For brevity we shall simply call them dual pairs. They are listed in Table 1.

4. Howe's correspondence

For a member G of a dual pair, let $\mathcal{R}(\widetilde{\mathrm{G}}, \omega) \subseteq \mathcal{R}(\widetilde{\mathrm{G}})$ denote the subset of the representations which may be realized as quotients of $\mathcal{S}(X)$ by closed $\widetilde{\mathrm{G}}$-invariant subspaces. Let us fix a representation Π in $\mathcal{R}(\widetilde{\mathrm{G}}, \omega)$ and let $N_\Pi \subseteq \mathcal{S}(X)$ be the intersection of all the closed G-invariant subspaces $N \subseteq \mathcal{S}(X)$ such that Π is

Dual pair	$\mathbb{D}$	ι	$(\ ,\ )$	$(\ ,\ )'$	dim W	stable range
$\mathrm{GL}_m(\mathbb{D}),\ \mathrm{GL}_n(\mathbb{D})$	$\mathbb{R},\ \mathbb{C},\ \mathbb{H}$				$2nm\ \dim_{\mathbb{R}}(\mathbb{D})$	$m \leq \frac{n}{2}$
$\mathrm{O}_{p,q},\ \mathrm{Sp}_{2n}(\mathbb{R})$	$\mathbb{R}$	$\iota = 1$	+	-	$2n(p+q)$	$p+q \leq n$
$\mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{O}_{p,q}$	$\mathbb{R}$	$\iota = 1$	-	+	$2n(p+q)$	$2n \leq \min\{p,q\}$
$\mathrm{O}_p(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$	$\mathbb{C}$	$\iota = 1$	+	-	$4np$	$p \leq n$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_p(\mathbb{C})$	$\mathbb{C}$	$\iota = 1$	-	+	$4np$	$2n \leq \frac{p}{2}$
$\mathrm{U}_{p,q},\ \mathrm{U}_{r,s}$	$\mathbb{C}$	$\iota \neq 1$	+	-	$2(p+q)(r+s)$	$p+q \leq \min\{r,s\}$
$\mathrm{Sp}_{p,q},\ \mathrm{O}^*_{2n}$	$\mathbb{H}$	$\iota \neq 1$	+	-	$8n(p+q)$	$p+q \leq n$
$\mathrm{O}^*_{2n},\ \mathrm{Sp}_{p,q}$	$\mathbb{H}$	$\iota \neq 1$	-	+	$8n(p+q)$	$2n \leq \min\{p,q\}$

TABLE 1. Dual pairs

infinitesimally equivalent to $\mathcal{S}(X)/N$. This is a representation of both $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$. As such, it is infinitesimally isomorphic to

$$\Pi \otimes \Pi_1', \tag{10}$$

for some representation Π_1' of $\widetilde{\mathrm{G}}'$. Howe proved, [15, Theorem 1A], that Π_1' is a finitely generated admissible quasisimple representation of $\widetilde{\mathrm{G}}'$, which has a unique irreducible quotient $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$. Conversely, starting with $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$ and applying the above procedure with the roles of G and G′ reversed, we arrive at the representation $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$. The resulting bijection

$$\mathcal{R}(\widetilde{\mathrm{G}}, \omega) \ni \Pi \to \Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega) \tag{11}$$

is called Howe's correspondence, or local θ correspondence, for the pair (G, G′).

Recall the unnormalized moment map

$$\tau' : \mathsf{W} \to \mathfrak{g}'^*, \quad \tau'(w)(X) = \langle X(w), w\rangle \qquad (X \in \mathfrak{g}', w \in \mathsf{W}), \tag{12}$$

and the notion of the wave front set $WF(\Pi)$ of an irreducible admissible representation Π of a real reductive group G, [14], [20, Theorem 3.4]. Then, in terms of (11),

$$WF(\Pi') \subseteq \tau'(\mathsf{W}), \tag{13}$$

see [17, Corollary 2.8]. Since the wave front set is contained in the nilpotent cone $\mathcal{N}' \subseteq \mathfrak{g}'^*$, see [14, Proposition 1.2] and [20, Theorem 3.4], we actually have

$$WF(\Pi') \subseteq \tau'(\mathsf{W}) \cap \mathcal{N}' \tag{14}$$

Recall that, by [21, Theorem 1.1], [2, Theorem 4.1] and [20, Theorem C],

$$GK \dim(\Pi') = \frac{1}{2} \dim WF(\Pi'). \tag{15}$$

Hence we get a bound for the Gel'fand–Kirillov dimension of Π',

$$GK\dim(\Pi') \leq \frac{1}{2}\dim(\tau'(\mathsf{W}) \cap \mathcal{N}'). \tag{16}$$

One may realize the symplectic space as the tensor product of the defining modules for the groups G and G′. For example if $\mathrm{G} = \mathrm{O}_{p,q}$ and $\mathrm{G}' = \mathrm{Sp}_{2n}(\mathbb{R})$, then $\mathsf{W} = \mathbb{R}^{p+q} \otimes \mathbb{R}^{2n}$. Hence, roughly, the smaller the dimension of the defining module for the group G, the smaller the right-hand side of (16). One may compute this number for each dual pair using [5, Corollary 6.1.4] and [6, Table 3, page 456], but the formulas are not illuminating. We provide a sample in Table 2 below.

On the other hand, as shown in [16, Theorem A], for dual pairs in the stable range, with G-the smaller member (see table 1), if Π is unitary then so is Π'. (The case $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2n,2n}, \mathrm{Sp}_{2n})$ and Π trivial is excluded.) Later this fact was generalized beyond the stable range in [17, Theorem 3.1] and in [10, Theorem 1.1]. Thus Howe's correspondence provides a method for understanding irreducible unitary representations of classical groups of low Gel'fand–Kirillov dimension. What remains is to understand their characters and we propose an approach in the next section.

Dual pair G, G'	$\dim \tau'(\mathsf{W}) \cap \mathcal{N}'$
$\mathrm{GL}_m(\mathbb{D}),\ \mathrm{GL}_n(\mathbb{D})$, $m \leq n$, $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$	$\dim \mathbb{D}\,(2mn - m - m^2)$
$\mathrm{O}_p,\ \mathrm{Sp}_{2n}(\mathbb{R})$, $p \leq n$	$2np - p^2 + p$
$\mathrm{O}_p,\ \mathrm{Sp}_{2n}(\mathbb{R})$, $p > n$	$n(n+1)$
$\mathrm{O}_{2p}(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$, $p \leq n$	$2(2n^2 - 2(n-p)^2 - 2(n-p))$
$\mathrm{O}_{2p+1}(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$, $p \leq n$	$2(2n^2 - 2(n-p)^2 + 2(n-p))$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_{2p}(\mathbb{C})$, $p \leq n$	$2(4pn - 2n - 2n^2)$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_{2p+1}(\mathbb{C})$, $p \leq n$	$2(4pn - 2n^2)$

TABLE 2. Examples of $\dim \tau'(\mathsf{W}) \cap \mathcal{N}'$

5. The Cauchy Harish-Chandra integral

The wave front set of the character Θ of the Weil representation is given by

$$\mathrm{WF}(\Theta) = \{(g, \xi) \in \widetilde{\mathrm{Sp}} \times \mathfrak{sp}^*;\ \xi \in \mathrm{WF}_1(\Theta),\ Ad(g)^*(\xi) = \xi\}, \tag{17}$$

where the fiber over the identity, $\mathrm{WF}_1(\Theta)$ is the closure of O_{min}, one of the two minimal non-zero nilpotent coadjoint orbits in $\mathfrak{sp}^*$. (The closure of the other minimal nilpotent orbit is in the wave front set of the contragredient Weil representation.) The formula is a key to a construction of an operator from the space of the invariant eigen-distributions on $\widetilde{\mathrm{G}}$ to the space of the invariant eigen-distributions on $\widetilde{\mathrm{G}}'$, [19], [3], assuming that the rank of G is less or equal to the rank of G′. We recall it below.

A maximal compact subgroup $\mathrm{K} \subseteq \mathrm{G}$ consists of the points fixed by a Cartan involution $\theta : \mathrm{G} \to \mathrm{G}$. Let $\mathrm{P} \subseteq \mathrm{G}$ be the subset of the elements $g \in \mathrm{G}$ such that $\theta(g) = g^{-1}$. Then $\mathrm{G} = \mathrm{KP}$. Any Cartan subgroup $\mathrm{H} \subseteq \mathrm{G}$ is conjugate to one which is invariant under θ. Thus let H be a θ-stable Cartan subgroup of G. Set $\mathrm{A} = \mathrm{H} \cap \mathrm{P}$. This is called the vector part of H, [22].

Denote by $\mathrm{A}' \subseteq \mathrm{Sp}$ the centralizer of A and let $\mathrm{A}'' \subseteq \mathrm{Sp}$ be the centralizer of A'. There is a measure $d\dot{w}$ on the quotient space $\mathrm{A}''\backslash\mathsf{W}$ defined by

$$\int_{\mathsf{W}} \phi(w)\, dw = \int_{\mathrm{A}''\backslash\mathsf{W}} \int_{\mathrm{A}''} \phi(aw)\, da\, d\dot{w}. \tag{18}$$

Let $\widetilde{\mathrm{A}'}$ be the preimage of A' in the metaplectic group. Recall, (9), the embedding

$$T : \widetilde{\mathrm{Sp}} \to S^*(\mathsf{W})\,.$$

The formula

$$Chc(f) = \int_{\mathrm{A}''\backslash\mathsf{W}} \int_{\widetilde{\mathrm{A}'}} f(g)T(g)(w)\, dg\, d\dot{w} \qquad (f \in C_c^\infty(\widetilde{\mathrm{A}'})), \tag{19}$$

where each consecutive integral is absolutely convergent, defines a distribution on $\widetilde{\mathrm{A}'}$, [19, Lemma 2.9]. Fix a regular element $h \in \mathrm{H}^{reg}$. Let $\widetilde{h}$ be an element in the preimage of h in the metaplectic group. The intersection of the wave front set of the distribution (19) with the conormal bundle of the embedding

$$\widetilde{\mathrm{G}'} \ni \widetilde{g} \to \widetilde{h}\widetilde{g'} \in \widetilde{\mathrm{A}''} \tag{20}$$

is empty (i.e., contained in the zero section), [19, Proposition 2.10]. Hence there is a unique restriction of the distribution (19) to $\widetilde{\mathrm{G}}$, denoted $Chc_{\widetilde{h}}$.

Harish-Chandra's Regularity Theorem, [9, Theorem 2], implies that the character of an irreducible representation coincides with a function multiplied by the Haar measure. Thus for $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}})$ we may consider the following integral

$$\int_{\widetilde{\mathrm{H}^{reg}}} \Theta_\Pi(\widetilde{h}^{-1})|\det(Ad(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}}|Chc_{\widetilde{h}}(f)\, d\widetilde{h} \qquad (f \in C_c^\infty(\widetilde{\mathrm{G}'})). \tag{21}$$

In fact, this integral is absolutely convergent, [19, Theorem 2.14].

Recall the Weyl–Harish-Chandra integration formula

$$\int_{\widetilde{\mathrm{G}}} f(g)\, dg = \sum \frac{1}{|\mathcal{W}(\mathrm{H},\mathrm{G})|} \int_{\widetilde{\mathrm{H}^{reg}}} |\det(Ad(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}}| \int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{H}}} f(g\widetilde{h}g^{-1})\, d\dot{g}\, d\widetilde{h}, \tag{22}$$

where $\mathcal{W}(\mathrm{H},\mathrm{G})$ is the Weyl group of H in G and the summation is over a maximal family of mutually non-conjugate (θ-stable) Cartan subgroups $\widetilde{\mathrm{G}}$. In terms of (22), set

$$\Theta'_\Pi(f) = C_\Pi \sum \frac{1}{|\mathcal{W}(\mathrm{H},\mathrm{G})|} \int_{\widetilde{\mathrm{H}^{reg}}} \Theta_\Pi(\widetilde{h}^{-1})|\det(Ad(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}}|Chc_{\widetilde{h}}(f)\, d\widetilde{h}, \tag{23}$$

where C_Π is a non-zero constant. This is an invariant distribution on $\widetilde{\mathrm{G}'}$. Hence a finite linear combination of irreducible characters, see [11, page 52]. In fact, with the appropriate normalization of all the measures involved, [3, Theorem 4], Θ'_Π is an invariant eigen-distribution whose infinitesimal character is equal to the one obtained from the infinitesimal character of Θ_Π by ([18, Theorem 1.19]). There are reasons to believe that (for an appropriate constant C_Π) Θ'_Π coincides with the character of the representation Π'_1, (10). Since quite often, $\Pi'_1 = \Pi'$, the above construction could explain Howe's correspondence on the level of characters in the sense that knowing the character of the representation of the small group gives a formula for the character of the representation of the large group. Though the conjecture holds in many cases, see for example [19], there is no proof of the equality $\Theta'_\Pi = \Theta_{\Pi'_1}$ in general. In the next section we recall how is our conjecture related to the classical Cauchy Determinantal Identity.

6. The pair $\mathrm{G} = \mathrm{U}_p$, $\mathrm{G}' = \mathrm{U}_s$

In this case $\Pi'_1 = \Pi'$, $\Theta'_\Pi = \Theta_{\Pi'}$ and the formula (23) coincides with the following equality

$$\int_{\widetilde{\mathrm{G}}} \int_{\widetilde{\mathrm{G}'}} f(g')\Theta(g'g)\Theta_\Pi(g^{-1})\, dg'\, dg = \int_{\widetilde{\mathrm{G}'}} \Theta_{\Pi'}(g')f(g')\, dg',$$

where each consecutive integral is absolutely convergent. This is an explicit version of the following equality of distributions

$$\int_{\widetilde{\mathrm{G}}} \Theta(g'g)\Theta_\Pi(g^{-1})\, dg = \Theta_{\Pi'}(g'), \tag{24}$$

which is equivalent to the First Fundamental Theorem of Classical Invariant Theory. By restricting to the maximal tori one sees that, for $r = s$, (24) is equivalent to the Cauchy Determinantal Identity:

$$\det\left(\frac{1}{1 - h_i h'_j}\right) = \frac{\prod_{i<j}(h_i - h_j) \cdot \prod_{i<j}(h'_i - h'_j)}{\prod_{i<j}(1 - h_i h'_j)}. \tag{25}$$

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Tomasz Przebinda
Department of Mathematics
University of Oklahoma
Norman, OK 73019, USA
e-mail: tprzebinda@gmail.com

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Local Inverse Scattering

A.V. Domrin

Abstract. We develop a local version of the inverse scattering method for studying soliton equations of parabolic type (this includes, for example, Korteweg–de Vries, nonlinear Schrödinger, and Boussinesq equations, but not sine-Gordon). The potentials are germs of holomorphic matrix-valued functions, without any boundary conditions. The scattering data are matrix-valued formal power series in the spectral parameter. We give a precise description of all possible scattering data and exact criteria for solubility of the local holomorphic Cauchy problem for a soliton equation of parabolic type in terms of the scattering data of the initial conditions. As an application, we prove the strongest possible version of the Painlevé property for such equations: every local holomorphic solution admits a global meromorphic extension with respect to the space variable.

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1. Introduction

The first general result in the theory of partial differential equations was the Cauchy–Kowalevsky theorem [1, 2]. We need only the following simple version of it. *Let P be a holomorphic function of x, t in a neighborhood of a given point $(x_0, t_0) \in \mathbb{C}^2$ and a polynomial in the other variables. Then the Cauchy problem*

$$\partial_t^m u = P(x, t, \{\partial_x^k \partial_t^l u\}_{k+l \leq m, (k,l) \neq (0,m)});$$
$$\partial_t^j u(x, t_0) = \varphi_j(x), \quad 0 \leq j \leq m-1,$$

has a unique local holomorphic solution $u(x, t)$ in a neighborhood of (x_0, t_0) for all holomorphic germs $\varphi_0, \varphi_1, \ldots, \varphi_{m-1} \in \mathcal{O}(x_0)$. To explain the necessity of the conditions $k + l \leq m$ and $(k, l) \neq (0, m)$, Kowalevsky [2] established the following

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theorem on forced analytic extension for solutions of the heat equation. *Each solution* $u \in \mathcal{O}(D)$ *of the equation* $u_t = u_{xx}$ *on an arbitrary bidisk* $D = \{(x,t) \in \mathbb{C}^2 \mid |x - x_0| < \varepsilon_1,\ |t - t_0| < \varepsilon_2\}$ *admits an analytic continuation to a solution* $\widetilde{u} \in \mathcal{O}(S)$ *of the same equation on the strip* $S = \{(x,t) \in \mathbb{C}^2 \mid |t - t_0| < \varepsilon_2\}$. In other words, every local holomorphic solution $u(x,t)$ extends to an entire function of x for each admissible value of t. Subsequent works of Salekhov [3], Kiselman [4] and Zerner [5] showed that the same assertion holds for all local holomorphic solutions $u(x,t)$ of all equations in the following larger classes:

$$\partial_t^p u = \partial_x^m u + \sum_{j=0}^{m-1} c_j \partial_x^j u, \tag{1}$$

$$\partial_x^m u = \sum_{k+l<m} c_{kl} \partial_x^k \partial_t^l u, \tag{2}$$

$$\partial_x^m u = \sum_{k+l<m} a_{kl}(x,t) \partial_x^k \partial_t^l u, \tag{3}$$

where $m \geq 2$ and p, $1 \leq p < m$ are integers, $c_j, c_{kl} \in \mathbb{C}$ are constant coefficients, and the functions $a_{kl} \in \mathcal{O}(\{(x,t) \in \mathbb{C}^2 \mid |t - t_0| < \delta\})$ are assumed to be entire functions of x and holomorphic germs in t at the point $t_0 \in \mathbb{C}$. Modern exposition of the results about analytic extension of holomorphic solutions of *linear* partial differential equations is given in Hörmander's well-known monograph [6, § 9.4], and papers of Henkin [7] and Rigat [8].

Up to now, all attempts to generalize these results and approaches to *nonlinear* equations and systems led only to partial and sporadic results (see, for example, [9, 10] and references therein). One can mention the extensive recent studies of the dissipative smoothing phenomenon (the regularizing effect of dispersive evolutionary equations of mathematical physics), which produce results looking very similar to the forced analytic extension (see various approaches in [11, 12] and references therein). However, the solution $u(x,t)$ in these results must always satisfy certain global restrictions as a function of x for $t = t_0 \in \mathbb{R}$, and the conclusion about analytic extension to a neighborhood of the real axis $\mathbb{R}_x^1 \subset \mathbb{C}_x^1$ is derived only for real $t > t_0$. There are several exceptions from this rule [13, 14], but neither of them gives any information about analytic extension of arbitrary local solutions that are holomorphic in x and t.

It was a long-standing challenge to obtain such information at least in the case of soliton equations[1], where it is referred to as "rigorous Painlevé analysis". As Kruskal *et al.* put it [15, p. 195], "To date there is no proof that the Korteweg–de Vries equation possesses the Painlevé property. The main problem lies in a lack of methods for obtaining the global analytic description of a locally defined solution in the space of several complex variables." In what follows we present such a method (which was suggested in [16, 17]) and use it to give a definitive answer to the question of analytic continuation of all local solutions (see Theorem 1 below).

[1] Of parabolic type since the hyperbolic case is trivial by the Cauchy–Kowalevsky theorem.

Before doing this, we briefly explain the notion of a soliton equation starting with the most popular examples:

$$u_t = au_{xxx} + buu_x, \qquad a, b \in \mathbb{C} \setminus \{0\}, \tag{4}$$

$$u_{tt} = au_{xxxx} + buu_{xx} + bu_x^2, \qquad a, b \in \mathbb{C} \setminus \{0\}, \tag{5}$$

$$iu_t = au_{xx} + bu|u|^2, \qquad a, b \in \mathbb{R} \setminus \{0\}. \tag{6}$$

In (6), $|u|^2$ is understood as $u(x,t)\overline{u(\overline{x},\overline{t})}$. The inverse scattering method first appeared as a tool for solving the Korteweg–de Vries equation (4) (with real a, b), which describes long waves on shallow water. It was noted in the pioneering paper of Gardner, Green, Kruskal and Miura [18] that if the potential $u(x,t)$ evolves according to (4), then the evolution of its scattering data (certain spectral characteristics of the operator $L = \partial_x^2 + u(x,t)$ on the Hilbert space $L^2(\mathbb{R}_x^1)$) turns out to be linear and "explicitly integrable", which enables one to construct examples of solutions and study the properties of all solutions in certain classes. An explanation of the unexpected success of this approach was given by Lax [19], who showed that the equation (4) (to be definite, with $a = 1/4$, $b = 3/2$) is a necessary and sufficient condition for solubility of the *auxiliary linear problem*

$$L\psi = \lambda\psi, \qquad \psi_t = P\psi \tag{7}$$

for the operators $L := \partial_x^2 + u$ and $P := \partial_x^3 + (3/2)u\partial_x + (3/4)u_x$. In other words, (4) may be written in the form $L_t = [P, L]$. Since P is skew-Hermitian, it follows that the evolution of L consists in its conjugation by a t-dependent unitary operator on $L^2(\mathbb{R}_x^1)$. This conjugation clearly preserves the spectrum, and then it is no surprise that the more refined spectral characteristics (scattering data) also evolve in a simple and tractable way.

The nonlinear Schrödinger equation (6) describes the evolution of a slowly varying dispersive wave envelope in nonlinear media and arises in optics, hydrodynamics and plasma physics. It was first studied in terms of the inverse scattering method by Zakharov and Shabat [20], who modified (7) replacing the scalar second-order differential equation $L\psi = \lambda\psi$ by a matrix first-order 2×2-system of differential equations with subsequent reduction (that is, a choice of matrices of special algebraic structure: in the present case, skew-Hermitian). The auxiliary linear problem takes the form

$$E_x = UE, \qquad E_t = VE \tag{8}$$

for some matrix-valued polynomials $U(x,t,z)$ and $V(x,t,z)$ of degrees 1 and 2, respectively in the spectral parameter $z \in \mathbb{C}$ (which is related to the parameter λ in (7) by the formula $\lambda = z^n$ in case of $n \times n$-matrices). Hence the equation (6) turned out to be written, although implicitly, as a reduction of the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{9}$$

which plays a fundamental role in our approach. The first explicit presentation of soliton equations as reductions of zero curvature equations and first corollaries of this presentation are due to Novikov [21].

Finally, the Boussinesq equation (5) describes water waves (like the Korteweg–de Vries equation) but admits wave motion in any direction (unlike the Korteweg–de Vries equation). It was studied in terms of the inverse scattering method by Zakharov (1973) and turned out to be the first physically relevant example where the 2×2-matrices in (8) or second-order operators L in (7) should be replaced by 3×3-matrices or third-order operators.

Thus all equations (4)–(6) are reductions of (9), where U and V are polynomials in z, the degree of U is equal to 1 and the degree of V is equal to $m \geq 2$. Taking this property for the definition[2] of a *soliton equation of parabolic type*, we shall give a complete answer to the question about analytic continuation of local holomorphic solutions of such equations. The situation appears to be almost the same as for the linear equations (1)–(3) with the only difference: the solutions now extend to globally meromorphic (not necessarily entire) functions of x.

Theorem 1. *For each of equations* (4)–(6)*, every local holomorphic solution* $u(x,t)$ *in a bidisk* $D = \{(x,t) \in \mathbb{C}^2 \,||x - x_0| < \varepsilon_1,\ |t - t_0| < \varepsilon_2\}$ (*with real centre* $(x_0, t_0) \in \mathbb{R}^2$ *in case* (6)) *admits an analytic continuation to a meromorphic function* $\widetilde{u}(x,t)$ *in the strip* $S = \{(x,t) \in \mathbb{C}^2 \,|\, |t - t_0| < \varepsilon_2\}$.

It follows from the Cauchy–Kowalevsky theorem (with x as a time variable) that Theorem 1 is unimprovable: for each of equations (4)–(6) one can find a solution u whose extension $\widetilde{u}$ admits no further extension (holomorphic or meromorphic) beyond the strip S. In other words, the envelope of meromorphy of any local holomorphic (or meromorphic) solution covers the whole complex line in the x-direction and may be arbitrary (any prescribed Riemann surface over the t-axis) in the t-direction.

To prove Theorem 1, we develop a local version of the inverse scattering method for soliton equations of parabolic type (this method was previously used only for equations of hyperbolic type, where the results and techniques are quite different; see [22, Ch. I] or [23, Part II, Ch. I, §§ 6–8]). The potentials are holomorphic germs without any boundary conditions. When one additionally imposes rapidly decaying or quasiperiodic (finite-gap) boundary conditions, the local version becomes naturally isomorphic to the corresponding standard version of the inverse scattering method. This may be regarded as a step towards solving another old puzzle: give a unified treatment of finite-gap solutions and rapidly decaying solutions (in the words of Bennequin [24, pp. 35–36], "...comment marier les solutions géometriques, attachées aux courbes algébriques [...] avec les diffusions qui viennent du scattering-inverse (solutions L^2 de KdV par exemple)?"). Many other applications of the local inverse scattering approach are yet to be developed.

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[2]This definition will be sharpened in § 2.

Structure of the paper. In § 2 we recall the construction of some holomorphic solutions of zero curvature equations using a very simple version of the Riemann problem. It serves to motivate our approach and describe the algebraic structure of zero curvature equations. Then § 3 introduces the main definitions and results of the local inverse scattering method. We give only the barest sketches of proofs (along with references to their full versions) but preserve all motivations and accurate statements in the hope to present the logical structure of the theory clearly and comprehensively. In § 4 we formally deduce Theorem 1 from the results of § 3. Thus our exposition is organized as a proof of Theorem 1 and all other results should be regarded as lemmas. However, we call some of them theorems in view of their importance.

2. The Riemann problem and zero curvature equations

We start with the *zero curvature equations* (9), where $U(x,t,z)$, $V(x,t,z)$ are $\mathrm{gl}(n,\mathbb{C})$-valued[3] rational functions of an auxiliary parameter z with coefficients depending on the space and time variables x,t. Here the poles of U,V must be fixed in advance and independent of x,t, and the coefficients of a rational function are defined as the coefficients of its partial fraction expansion or, equivalently, as the coefficients of the principal parts of its Laurent expansions at all poles. A *holomorphic solution of* (9) *on a domain* $\Omega \subset \mathbb{C}^2_{xt}$ is a pair of rational $\mathrm{gl}(n,\mathbb{C})$-valued functions U,V of z with prescribed positions and multiplicities of poles such that all coefficients of U,V are defined and holomorphic on Ω and all coefficients of the rational function $U_t - V_x + [U,V]$ of z are identically equal to 0 on Ω. Equation (9) with a fixed z (different from the poles of U and V) holds on a simply connected domain $\Omega \subset \mathbb{C}^2_{xt}$ if and only if the auxiliary linear system (8) with the same value of z has a holomorphic solution $E : \Omega \to \mathrm{GL}(n,\mathbb{C})$. Note that this solution is unique up to right multiplication by an invertible matrix (possibly depending on z).

Our approach uses the Riemann problem (see, for example, in [25, Ch. III] or [23, Part I, Ch. II and Part II, Ch. I, §§ 6–8]) on factorization of matrix-valued functions on a circle, or rather a generalization of this problem to the case of divergent series of Gevrey type (see the next section). Let D_+, D_- be disjoint open disks whose closures cover the whole extended complex plane $\overline{D}_+ \cup \overline{D}_- = \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. A continuous function $\gamma : \Gamma \to \mathrm{GL}(n,\mathbb{C})$ on the circle $\Gamma := \overline{D}_+ \cap \overline{D}_-$ is said to be *left-factorable* (resp. *right-factorable*) if there are continuous functions $\gamma_\pm : \overline{D}_\pm \to \mathrm{GL}(n,\mathbb{C})$ that are holomorphic on $D_\pm$ and satisfy $\gamma = \gamma_+\gamma_-^{-1}$ on Γ (resp. $\gamma = \gamma_-^{-1}\gamma_+$ on Γ). We regard the function γ as *data* of the Riemann problem and the pair (γ_+, γ_-) as a *solution*. If a solution exists, it is unique up to

[3] Throughout the paper $\mathrm{gl}(n,\mathbb{C})$ stands for the set of all $n \times n$-matrices with complex entries, $\mathrm{GL}(n,\mathbb{C})$ is the group of all invertible matrices in $\mathrm{gl}(n,\mathbb{C})$, and $[A,B] = AB - BA$ is the commutator of matrices $A, B \in \mathrm{gl}(n,\mathbb{C})$.

right (resp. left) multiplication of both elements of the pair by the same invertible constant matrix.

We now describe a holomorphic version of the Zakharov–Shabat *dressing method* [26]. Let (U_0, V_0) be a holomorphic solution (which may, for example, be identically equal to zero) of equation (9) on a domain $\Omega \subset \mathbb{C}^2$, and let E_0 be the corresponding solution of the auxiliary linear problem (8) normalized by the condition $E_0(x_0, t_0, z) = I$ (the identity matrix) for all z, where $(x_0, t_0) \in \Omega$ is a fixed point. Consider any covering of the extended complex plane $\overline{\mathbb{C}}$ by the disks D_+, D_- such that the circle $\Gamma = \overline{D}_+ \cap \overline{D}_-$ contains no poles of the rational functions U_0, V_0, and take any right-factorable continuous function $g : \Gamma \to \mathrm{GL}(n, \mathbb{C})$. For every $(x, t) \in \Omega$ pose the Riemann problem of finding invertible continuous functions $\theta_\pm : \overline{D}_\pm \to \mathrm{GL}(n, \mathbb{C})$ that are holomorphic on $D_\pm$ and satisfy

$$E_0(x, t, z) g(z) E_0^{-1}(x, t, z) = \theta_-^{-1}(x, t, z) \theta_+(x, t, z) \quad \text{for} \quad z \in \Gamma. \tag{10}$$

To make the solution $\theta_\pm$ unique, we fix a point $z_0 \in D_-$ and impose the additional condition $\theta_-(x, t, z_0) = I$ for all x, t. By a theorem of Malgrange [27, § 4], the set Ω_g of all points $(x, t) \in \Omega$ for which the Riemann problem (10) is soluble, is either the whole domain Ω, or the complement to a complex curve $C_g \subset \Omega$ that does not pass through the point (x_0, t_0), and the matrix-valued functions $\theta_\pm(x, t, z)$ are meromorphic on $\Omega \times D_\pm$ with at most a pole in (x, t) along this curve for every fixed $z \in D_\pm$. We put

$$U_1(x, t, z) = \begin{cases} (\theta_+)_x \theta_+^{-1} + \theta_+ U_0 \theta_+^{-1} & \text{for} \quad z \in \overline{D}_+, \\ (\theta_-)_x \theta_-^{-1} + \theta_- U_0 \theta_-^{-1} & \text{for} \quad z \in \overline{D}_- \end{cases} \tag{11}$$

and define $V_1(x, t, z)$ by the same formula with U_0 replaced by V_0 and the derivatives in x replaced by the derivatives in t. Then the pair $(U_1(x, t, z), V_1(x, t, z))$ is a holomorphic solution of (9) on the domain $\Omega_g \subset \mathbb{C}^2_{xt}$ (or a meromorphic solution on Ω) *with the same positions and multiplicities of poles* of the rational functions U_1, V_1 as they were for the rational functions U_0, V_0. We say that this solution is obtained by *dressing the solution* U_0, V_0 *by means of the function* g.

In what follows we always assume that the divisors of poles of the rational functions U, V are equal to ∞ and $m\infty$ for some integer $m \geq 2$, that is, U is a polynomial of degree 1 in z, and V is a polynomial of degree $m \geq 2$ in z (see the definition of soliton equations of parabolic type in the Introduction). Then it is natural to consider a limiting case of the dressing method when the disk D_- contracts to the point ∞ and the disk D_+ expands to the whole plane $\mathbb{C}$. (An analogue of this construction was studied by Krichever [22, Ch. I] in the hyperbolic case when the sets of poles of U and V are disjoint[4].) For the dressing function $g(z)$ we take the germ at ∞ of an arbitrary holomorphic invertible matrix-valued map,

[4]This enabled him to present all local holomorphic solutions of (9) with disjoint sets of poles of U and V as a result of dressing of "trivial" solutions and write any local holomorphic solution as a non-linear superposition of two waves running along the characteristics similarly to the d'Alembert formula for solutions of the wave equation. Clearly, none of these results holds in the case of parabolic equations, which we study here.

and for the contour Γ we take any circle of large radius lying in the domain of that germ. To state the limiting version of the dressing method, we digress on the algebraic structure of zero-curvature equations (9) for polynomials U, V of the form specified above. It may be assumed from the very beginning that

$$\begin{cases} U(x,t,z) = az + q(x,t) \ , \\ V(x,t,z) = bz^m + r_1(x,t)z^{m-1} + \cdots + r_m(x,t) \end{cases} \tag{12}$$

for some *diagonal* matrices $a, b \in \mathrm{gl}(n, \mathbb{C})$ and holomorphic matrix-valued functions $q, r_1, \ldots, r_m : \Omega \to \mathrm{gl}(n, \mathbb{C})$ on a given domain $\Omega \subset \mathbb{C}^2$. Then (9) is a system of $m+1$ matrix equations for $m+1$ unknown matrix functions $q, r_1, \ldots, r_m$. Assume for non-degeneracy that the matrix a has *simple spectrum* (that is, all of its eigenvalues are distinct) and the matrix-valued function $q(x,t)$ is *off-diagonal* (that is, $q_{ii}(x,t) \equiv 0$ for $i = 1, \ldots, n$). Then the first m equations of the system and the diagonal part of the last $(m+1)$th equation are soluble in a purely algebraic way. Hence the system can be reduced to one off-diagonal matrix equation for one off-diagonal unknown matrix-valued function $q(x,t)$.

To state this more precisely, we fix an arbitrary point $x_0 \in \mathbb{C}$ and introduce the set $\mathcal{R}(x_0)$ of all germs of holomorphic $\mathrm{gl}(n, \mathbb{C})$-valued maps at x_0 and the set $\mathcal{R}(x_0)^{\mathrm{od}}$ of all off-diagonal germs $q \in \mathcal{R}(x_0)$, that is, the germs with $q_{ii}(x) \equiv 0$ for $i = 1, \ldots, n$. A map $F : \mathcal{R}(x_0) \to \mathcal{R}(x_0)$ is called a *differential polynomial* if each entry of the matrix-valued function $F(\kappa)$ is an ordinary polynomial (the same for all κ) in the entries of κ and their derivatives (of any order) with respect to x. We need the following assertion ([28, Lemma 1]) whose content and proof must be regarded as well known.

Lemma 1. *Let $a, b, c_1, c_2, \ldots \in \mathrm{gl}(n, \mathbb{C})$ be diagonal matrices such that a has simple spectrum. Then there is a unique sequence of differential polynomials $F_j : \mathcal{R}(x_0) \to \mathcal{R}(x_0)$ $(j = 0, 1, 2, \ldots)$ with the following properties:*

(a) $F_0(\kappa) \equiv b$,
(b) $F_j(0) \equiv c_j$ *for all* $j = 1, 2, \ldots$,
(c) *the formal Laurent series* $F(\kappa, z) := \sum_{j=0}^{\infty} F_j(\kappa) z^{-j}$ *satisfies the differential equation* $\partial_x F(\kappa, z) = [az + \kappa, F(\kappa, z)]$ *identically with respect to x and z for all* $\kappa \in \mathcal{R}(x_0)^{\mathrm{od}}$.

Arguing as in the proof of [28, Theorem 1], we see that a pair of polynomials $U(x,t,z)$, $V(x,t,z)$ of the form (12) with diagonal matrices $a, b \in \mathrm{gl}(n, \mathbb{C})$ (where a has simple spectrum) and off-diagonal function $q(x,t)$ is a holomorphic solution of (9) in a domain $\Omega \subset \mathbb{C}^2$ if and only if the following two conditions hold. First, the coefficients $r_1, \ldots, r_m : \Omega \to \mathrm{gl}(n, \mathbb{C})$ of the polynomial V must be expressed in terms of the off-diagonal function $q : \Omega \to \mathrm{gl}(n, \mathbb{C})$ by the formulae

$$r_1 = F_1(q), \quad \ldots, \quad r_m = F_m(q)$$

for some diagonal matrices $c_1(t), \ldots, c_m(t) \in \mathrm{gl}(n, \mathbb{C})$ that depend holomorphically on t in the domain equal to the projection of Ω to the coordinate axis $\mathbb{C}^1_t$. Second, the holomorphic off-diagonal function $q : \Omega \to \mathrm{gl}(n, \mathbb{C})$ must satisfy the equation

$$q_t = [a, F_{m+1}(q)] \tag{13}$$

on Ω for the same choice of the diagonal matrices $c_1(t), \dots, c_m(t)$ as in the first condition and for an arbitrary diagonal matrix $c_{m+1} \in \mathrm{gl}(n, \mathbb{C})$ (the right-hand side of (13) actually does not depend on the choice of c_{m+1}).

Among all solutions U, V of the form (12) of the zero curvature equations (9), we are interested only in those that correspond to solutions of (13) for some t-*independent* diagonal matrices $c_1, \dots, c_j$. We also assume for non-degeneracy that both matrices a, b have simple spectrum. Then we call (13) the *soliton equation of parabolic type defined by the matrices* $a, b, c_1, \dots, c_m$. This equation is equivalent to the zero curvature equation $U_t - V_x + [U, V] = 0$ for the polynomials

$$U(x, t, z) = az + q(x, t), \qquad V(x, t, z) = \sum_{j=0}^{m} F_{m-j}(q)(x, t)z^j, \tag{14}$$

where F_0, F_1, …, F_m are the differential polynomials determined by the sequence of matrices $a, b, c_1, \dots, c_m$ according to Lemma 1. Examples of reductions of soliton equations of parabolic type are the linear equations of the form $u_t = P(\partial_x)u$ for an arbitrary polynomial P of degree ≥ 2, the Korteweg–de Vries equation (4), the nonlinear Schrödinger equation (5) and others (see, for example, [28, end of § 2]).

We now state the limiting version of the dressing method for constructing holomorphic solutions of the equations studied. The identically zero solution U_0, V_0 will be dressed by means of any germ $g = f^{-1} \in \mathcal{D}$, where $\mathcal{D}$ is the set of all holomorphic $\mathrm{GL}(n, \mathbb{C})$-valued functions f on $\{z \in \mathbb{C} \mid |z| > R_0\} \cup \{\infty\}$ (R_0 depends on f) with $f(\infty) = I$. In other words, $\mathcal{D}$ consists of the germs of holomorphic $\mathrm{GL}(n, \mathbb{C})$-valued functions f at ∞ with $f(\infty) = I$. The following assertion ([28, Theorem 1]) must be regarded as well known, although it was not explicitly stated and completely proved anywhere in the literature. Versions of it are contained in [29, Proposition 2.7], [30, Theorem 3.2.6] and [31, Proposition 2.9].

Lemma 2. *Let $a, b, c_1, c_2, \ldots \in \mathrm{gl}(n, \mathbb{C})$ be diagonal matrices such that a has simple spectrum. We fix an integer $m \geq 2$ and a point $(x_0, t_0) \in \mathbb{C}^2$. For every function $f \in \mathcal{D}$ let $\Omega(f)$ be the set of all $(x, t) \in \mathbb{C}^2$ such that the function*

$$\gamma(x, t, z) := \exp\{az(x - x_0) + (bz^m + c_1 z^{m-1} + \cdots + c_m)(t - t_0)\}f^{-1}(z)$$

is right-factorable on some (and then on any) circle $\{|z|=R\}$, $R_0 < R < +\infty$. Then the set $\Omega(f) \subset \mathbb{C}^2$ is either the whole of $\mathbb{C}^2$ or the complement to an entire complex curve (the set of zeros of an entire function) not passing through (x_0, t_0). For every point $(x, t) \in \Omega(f)$ let $(\gamma_+(x, t, z), \gamma_-(x, t, z))$ be the solution of the Riemann problem

$$\gamma(x, t, z) = \gamma_-^{-1}(x, t, z)\gamma_+(x, t, z) \quad \textit{for} \quad R_0 < |z| < +\infty, \tag{15}$$

normalized by the condition $\gamma_-(x, t, \infty) = I$. We put

$$q_f(x, t) := \lim_{z \to \infty} z[\gamma_-(x, t, z) - I, a]. \tag{16}$$

Then the function $q_f : \Omega(f) \to \mathrm{gl}(n, \mathbb{C})$ *is an off-diagonal holomorphic solution on* $\Omega(f)$ *of the soliton equation of parabolic type* (13) *determined by the matrices* $a, b, c_1, c_2, \dots$.

We note that the Riemann problem (15) coincides with (10) up to the notation $f = g^{-1}$, $\gamma_- = \theta_-$, $\gamma_+ = \theta_+ E_0$, and the definition (16) of the solution constructed in Lemma 2 is obtained by equating the coefficients of z^0 in second equation (11).

The class of solutions $q_f(x,t)$ constructed in Lemma 2 contains all *finite-gap* solutions (they correspond to those matrices $f \in \mathcal{D}$ whose columns are eigenvectors of some non-degenerate[5] rational $\mathrm{gl}(n, \mathbb{C})$-valued function $G(z)$; see [32, Theorem 5]) and many *rapidly decreasing* solutions (as described in [32, § 5]). In both cases the construction of Lemma 2 coincides with the corresponding version of the inverse scattering method if we understand the germ $f \in \mathcal{D}$ as the *scattering data* of a matrix-valued potential $q_f(x, t_0) \in \mathcal{R}(x_0)$. This is explained at length in [32, §§ 4, 5]. Note that our "potentials" determine their "scattering data" not uniquely, but only up to right multiplication by any diagonal germ in $\mathcal{D}$. This can be expressed in the following form (see [32, Theorem 4(A)] and its proof).

Lemma 3. *Two functions* $f, g \in \mathcal{D}$ *determine the same solution* $q_f(x,t) = q_g(x,t)$ *of equation* (13) *in a neighborhood of the point* $(x_0, t_0) \in \mathbb{C}^2$ *if and only if the function* $g^{-1}f \in \mathcal{D}$ *is diagonal. This condition is also necessary and sufficient for the equality* $q_f(x, t_0) = q_g(x, t_0)$ *in a neighbourhood of the point* $x_0 \in \mathbb{C}$.

3. The main definitions and results

The construction of solutions described in Lemma 2 is far from giving *all local holomorphic solutions* of (13) in a neighborhood of the given point $(x_0, t_0) \in \mathbb{C}^2$. (For example, all solutions constructed in Lemma 2 extend meromorphically to $\mathbb{C}^2$, while the Cauchy–Kowalevsky theorem stated in the introduction enables us to construct local solutions with any prescribed singularity in t.) We now present a natural modification of this construction which is free from this disadvantage as well as from the non-uniqueness (described in Lemma 3) of the correspondence between potentials and their scattering data[6]. Recall that the set $\mathcal{D}$ of "scattering

[5] Here non-degeneracy means that the complex curve $C_G := \{(z,w) \in \mathbb{C}^2 \mid \det(G(z) - wI) = 0\}$ splits into n distinct holomorphic branches over a punctured neighbourhood $\{|z| > R\}$ of the point $z = \infty$. This automatically holds if the matrix $G(\infty)$ has simple spectrum. The algebraic curve C_G is known as a *spectral curve* and plays an important role in the theory of finite-gap solutions. Replacing "rational" by "holomorphic at ∞" in the definition of G gives an equivalent description of the set of all solutions constructed in Lemma 2.

[6] Note, however, that this non-uniqueness is sometimes an asset: it provides a flexible and natural language in some important constructions. For example, adding a soliton to a given solution q_f is very conveniently described in the notation of Lemma 2 as multiplication of f by a Blaschke factor (see, for example, [25, Ch. III, § 2] or [31, Proposition 4.2]), but this description becomes cumbersome if we insist on using the normalized scattering data, which are introduced below.

data” consists of all convergent (in a neighborhood of the point $z=\infty$) series of the form

$$f(z) = I + \frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \cdots,$$

where $\varphi_0, \varphi_1, \ldots \in \mathrm{gl}(n,\mathbb{C})$. We now want to replace it by the set $\mathcal{D}_{1/m}$ of all formal power series of the same form with off-diagonal matrices $\varphi_k \in \mathrm{gl}(n,\mathbb{C})$ (this makes the correspondence between the germs of solutions and their scattering data one-to-one in contrast to Lemma 2) and with

$$\sum_{k=0}^{\infty} \frac{|\varphi_k|}{k!^{1/m}} A^k < \infty \quad \text{for some } A > 0,$$

where $m \geq 2$ is the number of the equation (13) in its hierarchy. The class $\mathcal{D}_{1/m}$ is natural because its elements are precisely those formal power series for which the left-hand side of (15) (that is, the data of the Riemann problem) is well defined as a formal Laurent series in z for all (x,t) in a neighborhood of the point $(x_0,t_0) \in \mathbb{C}^2$. (This follows from Lemma 4 below.) In the case when $m=1$ (or, equivalently, $b=c_1=c_2=\cdots=0$), equation (13) takes a trivial form $q_t=0$, but its “solutions” (that is, all germs $q(x)$ of holomorphic off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued functions at the point $x_0 \in \mathbb{C}$) are also described by their scattering data. This is an important part of the whole method (see Theorem 3 below).

Let us describe appropriate Banach spaces of formal power series. For every $\alpha \geq 0$ we introduce the set Gev_α (referred to as *Gevrey class* α) of all formal power series of the form $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-(k+1)}$ $\varphi_k \in \mathrm{gl}(n,\mathbb{C})$ such that the series $\sum_{k=0}^{\infty} (k!)^{-\alpha} |\varphi_k| x^k$ has a non-zero radius of convergence. Here $|\cdot|$ is any fixed norm on $\mathrm{gl}(n,\mathbb{C})$ with the property $|AB| \leq |A||B|$. The vector space Gev_α is the union of an increasing family of Banach spaces isometrically isomorphic to l_1. Namely, $\mathrm{Gev}_\alpha = \bigcup_{A>0} G_\alpha(A)$, where $G_\alpha(A)$ is the set of all formal power series $\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-(k+1)}$ with $\varphi_k \in \mathrm{gl}(n,\mathbb{C})$ such that $\|\varphi\|_{\alpha,A} := \sum_{k=0}^{\infty} (k!)^{-\alpha} |\varphi_k| A^k < \infty$.

In the same vein, for every $m \geq 1$ we write the vector space Ent_m of all $\mathrm{gl}(n,\mathbb{C})$-valued entire functions of order $\leq m$ and finite type (for order exactly m) in the form $\mathrm{Ent}_m = \bigcup_{B>0} E_m(B)$, where $E_m(B)$ is the set of all formal power series $\varepsilon(z) = \sum_{l=0}^{\infty} \varepsilon_l z^l$ with $\varepsilon_l \in \mathrm{gl}(n,\mathbb{C})$ such that $\|\varepsilon\|_{m,B} := \sup_{l\geq 0} |\varepsilon_l| (l!)^{1/m} B^{-l} < \infty$ (this condition guarantees that the series converges for all $z \in \mathbb{C}$). Clearly, each $E_m(B)$ is a Banach space isometrically isomorphic to l_∞.

An important property of the Banach spaces $G_\alpha(A)$ and $E_m(B)$ is the possibility to multiply their elements for $\alpha m \leq 1$ and $B < A$ (and, generally speaking, these inequalities are unimprovable). This fact is expressed by the following assertion [17, Lemma 1], where $\{\cdot\}_+$ and $\{\cdot\}_-$ stand respectively for the positive and negative parts of a Laurent series: $\{\sum_{k\in\mathbb{Z}} a_k z^k\}_+ = \sum_{k\geq 0} a_k z^k$, $\{\sum_{k\in\mathbb{Z}} a_k z^k\}_- = \sum_{k\leq -1} a_k z^k$.

Lemma 4. *Suppose that $A > B > 0$, $m \geq 1$ and $0 \leq \alpha \leq 1/m$. Then the product of elements of $G_\alpha(A)$ and $E_m(B)$ in any order is a well-defined formal Laurent series belonging to the direct sum $G_\alpha(A-B) + E_m(B)$. The maps $(\varphi,\varepsilon) \mapsto \{\varphi\varepsilon\}_\pm$*

and $(\varphi, \varepsilon) \mapsto \{\varepsilon\varphi\}_\pm$ *are continuous bilinear forms on* $G_\alpha(A) \times E_m(B)$ *with values in* $G_\alpha(A-B)$ *and* $E_m(B)$.

We can now state the main result (a slightly extended version of [17, Theorem 3] with basically the same proof) on the solubility of the Riemann problem (15) in the context of divergent power series and on the analytic properties of its solutions as functions of parameters. We actually need only two very special cases: first, when Ω is $\mathbb{C}^2_{xt}$ and the polynomial $P(x,t,z)$ is of the form $a(x-x_0)z + (bz^m + c_1 z^{m-1} + \cdots + c_m)(t-t_0)$ for some integer $m \geq 2$ with the same diagonal matrices $a, b, c_1, c_2, \ldots \in \mathrm{gl}(n,\mathbb{C})$ as in Lemma 2 and, second, when Ω is $\mathbb{C}^1_x$, the polynomial $P(x,z)$ is equal to $a(x-x_0)z$, and $m = 1$. In part (B) we use the notation $\mathrm{Gev}_{\alpha-0} := \bigcup_{0\leq s<\alpha} \mathrm{Gev}_s$.

Theorem 2.

(A) *Let* Ω *be a complex manifold,* $m \geq 1$ *an integer,* $p_0, p_1, \ldots, p_m : \Omega \to \mathrm{gl}(n,\mathbb{C})$ *holomorphic maps, and* $\xi_0 \in \Omega$ *a point with* $p_k(\xi_0) = 0$ *for* $k = 0, 1, \ldots, m$. *Put* $P(\xi, z) := \sum_{k=0}^m p_k(\xi) z^k$ *for all* $\xi \in \Omega$, $z \in \mathbb{C}$. *Then for every series* $f \in I + \mathrm{Gev}_{1/m}$ *one can find a neighborhood* $\Omega(f)$ *of the point* ξ_0 *in* Ω, *numbers* $A, B > 0$ *and holomorphic maps* $\gamma_- : \Omega(f) \to I + G_{1/m}(A)$ *and* $\gamma_+ : \Omega(f) \to E_m(B)$ *such that the following equality of formal Laurent series holds for all* $\xi \in \Omega(f)$:

$$e^{P(\xi,z)} f^{-1}(z) = \gamma_-^{-1}(\xi, z)\gamma_+(\xi, z) \tag{17}$$

and all values of the entire function $z \mapsto \gamma_+(\xi, z)$ *belong to the group* $\mathrm{GL}(n,\mathbb{C})$ *of invertible complex* $n \times n$*-matrices and satisfy the equality*

$$\det \gamma_+(\xi, z) = e^{\mathrm{tr}\, P(\xi,z)} \quad \text{for all} \quad z \in \mathbb{C}. \tag{18}$$

(B) *Under the hypotheses of part* (A), *if we additionally know that* $f \in I + \mathrm{Gev}_{(1/m)-0}$ *and* Ω *is a Stein manifold*[7] *with* $H^2(\Omega, \mathbb{Z}) = 0$, *then there is a holomorphic non-vanishing at* ξ_0 *function* $\tau_f \in \mathcal{O}(\Omega)$ *with the following properties.*

  (a) *The germs of the holomorphic maps* $\xi \mapsto \tau_f(\xi)(\gamma_-(\xi, z) - I)$ *and* $\xi \mapsto \tau_f(\xi)(\gamma_-^{-1}(\xi, z) - I)$ *at the point* ξ_0 *admit an analytic continuation to holomorphic maps* $\Omega \to G_{1/m}(A)$ *for every* $A > 0$.

  (b) *For every exhaustion* $\{\xi_0\} = K_0 \subset K_1 \subset \cdots$ *of the manifold* Ω *by holomorphically convex compact sets* $K_j \subset \mathrm{int}\, K_{j+1}$ *with* $H^2(\mathrm{int}\, K_j, \mathbb{Z}) = 0$ *there is a sequence of numbers* $B_j > 0$ *such that the germs* $\xi \mapsto \tau_f(\xi)\gamma_+(\xi, z)$ *and* $\xi \mapsto \tau_f(\xi)\gamma_+^{-1}(\xi, z)$ *admit an analytic continuation to holomorphic maps* $\mathrm{int}\, K_j \to E_m(B_j)$ *for every* $j = 1, 2, \ldots$.

  (c) *The equalities* (17) *and* (18) *hold for all* $\xi \in \Omega$ *with* $\tau_f(\xi) \neq 0$.

[7] A Stein manifold may be defined as a closed complex submanifold of $\mathbb{C}^N$. The additional requirements on Ω in part (B) guarantee the solubility of the second Cousin problem on Ω (see, for example, [33, subsections 41 and 49]). All the hypotheses of part (B) automatically hold in our cases when Ω is either $\mathbb{C}^2$ or $\mathbb{C}^1$.

To say this in simpler words, if a formal $\mathrm{gl}(n,\mathbb{C})$-valued power series $f(z) = I+\varphi_0 z^{-1}+\varphi_1 z^{-2}+\cdots$ belongs to a Gevrey class such that the left-hand side of (17) is well defined in a neighborhood of ξ_0 by Lemma 4, then the Riemann problem (17) is soluble in some neighborhood of ξ_0, and its solution $\gamma_\pm(\xi,z)$ is holomorphic with respect to ξ in this neighborhood. This fact further supports the idea of natural appearance of the Gevrey classes in our approach. But if we additionally assume (as in part (B)) that the series $f(z)$ belongs to a strictly smaller Gevrey class than in part (A), then the Riemann problem (17) becomes soluble everywhere on Ω except possibly for a complex hypersurface $\{\xi\in\Omega \,|\, \tau_f(\xi)=0\}$ that does not pass through ξ_0, and the solution $\gamma_\pm(\xi,z)$ is globally meromorphic with respect to ξ in Ω with at most poles along this hypersurface. The hypotheses of part (B) certainly hold (for any m) when $f(z)$ is an ordinary convergent series in a neighborhood of $z=\infty$ (this situation was described in Lemma 2), and we thus recover the (needed part of the) result of Malgrange [27, which was mentioned in § 1].

To prove part (A) of Theorem 2, we reduce the Riemann problem (17) to a linear inhomogeneous equation of the form $X(\xi)\varphi = u(\xi)$ on the Banach space $E = G_\alpha(A)$ for appropriate values of $\alpha \le 1/m$ and $A>0$, where $\varphi=\varphi(\xi)\in E$ is the unknown vector, $X(\xi): E\to E$ is a known linear operator (a slightly modified version of the Toeplitz operator on the Hardy space) and $u(\xi)\in E$ is a known vector. Here $X(\xi)$ and $u(\xi)$ depend holomorphically on ξ in a neighborhood of ξ_0 and $X(\xi_0)=I$ is the identity operator. Once this is done, it is clear that the solution $\varphi(\xi) = X(\xi)^{-1}u(\xi)$ exists, is unique and depends holomorphically on ξ in a neighborhood of ξ_0. The details are given in [17, § 5].

To prove part (B) of Theorem 2, we note that under the hypotheses of part (B) the operator $X(\xi)$ and the vector $u(\xi)$ are defined and holomorphic with respect to ξ on the whole parameter space Ω and, moreover, the operator $Y(\xi) := X(\xi)-I$ is compact for every $\xi\in\Omega$. Therefore the desired conclusion follows from the "meromorphic Fredholm alternative" contained in the following lemma[8], which can be found along with a proof in [17, Lemma 8].

Lemma 5. *Let Ω be a Stein manifold with $H^2(\Omega,\mathbb{Z})=0$ and let $Y:\Omega\to\mathcal{B}(E)$ be a holomorphic map from Ω to the Banach space $\mathcal{B}(E)$ of all linear continuous operators on a Banach space E. Suppose that the operators $Y(\xi)$ are compact for all $\xi\in\Omega$ and the operator $I+Y(\xi_0)$ is invertible for some point $\xi_0\in\Omega$. Then there is a holomorphic function $\tau\in\mathcal{O}(\Omega)$ with $\tau(\xi_0)=1$ such that the following assertions hold.*

(i) *The operator $I+Y(\xi)$ is invertible for those and only those $\xi\in\Omega$ that satisfy $\tau(\xi)\neq 0$.*

(ii) *$\xi\mapsto\tau(\xi)(I+Y(\xi))^{-1}$ is a holomorphic map $\Omega\to\mathcal{B}(E)$.*

[8]This result seems to be first stated at the needed level of generality (that is, for operators on general Banach spaces and not only on the Hilbert space) by Gokhberg (1953). Then it was rediscovered many times by various authors. Amazingly, references to the possible authorship of this result in the well-known monographs of Kato (1966), Lang (1975), Reed and Simon (1978) and Yafaev (1993) give us four *mutually disjoint* sets of authors.

Theorem 2(A) enables us to define the inverse scattering transform for all $f \in I+\mathrm{Gev}_1$ (recall that Lemma 2 did so only for $f \in I+\mathrm{Gev}_0$) by the formula (16) with $t = t_0$ (or, equivalently, with $b = c_1 = c_2 = \cdots = 0$). We now describe this definition in more detail. Fix a diagonal matrix $a \in \mathrm{gl}(n, \mathbb{C})$ with simple spectrum and an arbitrary point $x_0 \in \mathbb{C}$. For every formal power series $\varphi \in \mathrm{Gev}_1$ consider the solution $\gamma_\pm(x, z)$ of the Riemann problem (17) with $P(x, z) = a(x - x_0)z$ and $f(z) = I + \varphi(z)$. Let $\mathcal{R}(x_0)$ be the set of all germs of holomorphic $\mathrm{gl}(n, \mathbb{C})$-valued functions at x_0, and let $\mathcal{R}(x_0)^{\mathrm{od}}$ be the set of all off-diagonal germs $q \in \mathcal{R}(x_0)$ (that is, those with $q_{ll}(x) \equiv 0$ for $l = 1, \ldots, n$). Then all coefficients $g_k(x)$ of the expansion $\gamma_-(x, z) = I + \sum_{k=0}^{\infty} g_k(x) z^{-(k+1)}$ belong to $\mathcal{R}(x_0)$, and the formula (basically (16) with $t = t_0$)

$$B\varphi(x) := [g_0(x), a] \quad \text{for} \quad x - x_0 \in \Omega(f) \tag{19}$$

determines a map $B : \mathrm{Gev}_1 \to \mathcal{R}(x_0)^{\mathrm{od}}$. We call this map the *inverse scattering transform*. The notation $B\varphi$ is chosen in honor of the classical Borel transform, to which (19) reduces for upper-triangular $\mathrm{gl}(2, \mathbb{C})$-valued convergent series $\varphi \in \mathrm{Gev}_0$ (as explained in [28, § 6], or [17, § 2]).

The *direct scattering transform* $L : \mathcal{R}(x_0)^{\mathrm{od}} \to \mathrm{Gev}_1$ is defined by the formula

$$Lq(z) := \mu(x_0, z) - I, \tag{20}$$

where $\mu(x, z) = I + \sum_{k=0}^{\infty} m_k(x) z^{-(k+1)}$ is a unique solution of the differential equation $\mu_x = (az + q(x))\mu - \mu az$ in the class of formal power series of the form indicated with $m_k \in \mathcal{R}(x_0)$, $k = 0, 1, 2, \ldots$, such that all coefficients of the series $\mu(x_0, z) - I$ are off-diagonal (the existence and uniqueness of this solution are proved in [17], § 6, the paragraph before Lemma 10). The notation Lq is chosen in honor of the classical Laplace transform

$$Lu(z) = \int_0^\infty u(x) e^{-xz}\, dx,$$

to which (20) reduces in case of upper-triangular $\mathrm{gl}(2, \mathbb{C})$-valued potentials $q(x)$ that are entire functions of exponential type (see [28, § 6], [17, § 2]). The definition of Lq may seem strange (where does the differential equation $\mu_x = (az + q(x))\mu - \mu az$ come from?), but it is natural in view of the following observation (which is rather standard in the Riemann-problem approach to integrable systems). Consider the Riemann problem (17) with $\xi = x$ and $P(\xi, z) = a(x - x_0)z$, differentiate it with respect to x (the Leibniz rule for the derivative of a product still holds because of the last assertion of Lemma 4) and separate the positive and negative powers of z in the resulting Laurent series. This yields that the components of the solution of the Riemann problem satisfy the differential equations

$$\partial_x \gamma_+ = (az + q(x))\gamma_+, \qquad \partial_x \gamma_- = (az + q(x))\gamma_- - \gamma_- az \tag{21}$$

with initial conditions $\gamma_+(x_0, z) = I$, $\gamma_-(x_0, z) = f(z)$, where $q(x) := Bf(x)$ is defined in (19). Thus we see that the differential equation for $\mu(x, z)$ just selects the candidates for the role of $\gamma_-(x, z)$, and the initial condition restores $f(z)$ by the formula (20). This observation also motivates the first part of the following theorem (an extended version of [17, Theorem 1]) which says that the maps L and B are indeed inverse to each other if we restrict ourselves by only off-diagonal series in Gev_1 (as already mentioned in the definition of $\mathcal{D}_{1/m}$ above, this restriction removes the non-uniqueness described in Lemma 3). The second part of Theorem 3 follows from the first part and Theorem 2(B). It says that all potentials whose scattering data belong to strictly smaller Gevrey classes than necessary for the Riemann problem (17) to be well defined, are globally meromorphic in x.

Theorem 3.

(A) *The map $q \mapsto Lq$ is a bijection of the set $\mathcal{R}(x_0)^{\mathrm{od}}$ onto the set $\mathrm{Gev}_1^{\mathrm{od}}$ of all off-diagonal series in Gev_1. The inverse map is the restriction to $\mathrm{Gev}_1^{\mathrm{od}}$ of the map $B : \mathrm{Gev}_1 \to \mathcal{R}(x_0)^{\mathrm{od}}$ defined in* (19).

(B) *If $q \in \mathcal{R}(x_0)^{\mathrm{od}}$ and $Lq \in \mathrm{Gev}_{1-0}$, then the germ $q(x)$ admits an analytic continuation to a globally meromorphic off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued function on $\mathbb{C}^1_x$ (denoted again by $q(x)$) such that for every $z \in \mathbb{C}$ the auxiliary linear system $E_x = (az + q(x))E$ has a globally meromorphic fundamental system of solutions.*

A key role in the proof of part (A) of Theorem 3 is played by the following particular case of a theorem of Sibuya on formal solutions of singularly perturbed ordinary differential equations (see [34, Theorem A.5.4.1 on pp. 254–256] or [35, Theorem XII-5-2]). *Let $m, \nu \geq 1$ be integers, $A : \mathbb{C}^\nu \to \mathbb{C}^\nu$ an invertible linear operator, and $y(x,z) = \sum_{k=0}^\infty a_k(x) z^{-k}$ a formal power series whose coefficients $a_k(x)$ are $\mathbb{C}^\nu$-valued holomorphic germs at $x_0 \in \mathbb{C}$. Suppose that*

$$\frac{dy}{dx} = z^m A y + \sum_{j=0}^{m-1} z^j B_j(x,y) \tag{22}$$

for some $\mathbb{C}^\nu$-valued polynomials $B_j(x,y)$ in the components of the vector y with coefficients in $\mathcal{O}(x_0)$. Then the series $y(x_0,z)$ belongs to $\mathrm{Gev}_{1/m}$.

In our applications of this result, $\mathbb{C}^\nu$ is the vector space of all off-diagonal matrices $X \in \mathrm{gl}(n,\mathbb{C})$ and the operator $AX := [C,X]$ sends every such matrix to its commutator with a given diagonal matrix $C \in \mathrm{gl}(n,\mathbb{C})$. The role of C is played by a in the proof of Theorem 3 and b in the proof of Theorem 4 below. Since the operator A is invertible if and only if the matrix C has simple spectrum, this explains our non-degeneracy assumptions (made in the definition of soliton equations of parabolic type in § 2) that the matrices a and b have simple spectrum.

The detailed proof of part (A) of Theorem 3 is given in [17, §§6, 7] and we shall not repeat it here. Once part (A) (or rather the equality $q = BLq$ for all $q \in \mathcal{R}(x_0)^{\mathrm{od}}$) is proved, part (B) follows easily. Indeed, if $Lq \in \mathrm{Gev}_{1-0}$, then the germ $BLq(x)$ admits a global meromorphic extension from a neighborhood of x_0 to the whole of $\mathbb{C}^1_x$ by Theorem 2(B). Since $BLq = q$, this proves the first assertion of Theorem 3(B). To prove the second assertion, note that the component $\gamma_+(x,z)$ of the solution of the Riemann problem (17) with $P(x,z) = a(x-x_0)z$ satisfies the auxiliary linear system $E_x = (az + q(x))E$ for all $z \in \mathbb{C}$ (this follows from the first equality in (21)) and its columns are linearly independent by (18). Hence its columns form a fundamental system of solutions. On the other hand, it follows from Theorem 2(B), assertion (b), that $\gamma_+(x, z)$ is a globally meromorphic function on $\mathbb{C}_x^1$ with denominator $\tau_f(x)$ for every fixed z. This proves the second assertion of Theorem 3(B), which is also known as the *trivial-monodromy property.*

We can now state a criterion for solubility of the local holomorphic Cauchy problem for soliton equations of parabolic type. Consider any system of evolution equations of the form (13), where $q(x, t)$ is an unknown off-diagonal $\mathrm{gl}(n, \mathbb{C})$-valued function, $m \geq 2$ is a given integer and $F_0, F_1, F_2, \ldots$ is the sequence of differential polynomials in x corresponding to a given sequence of diagonal matrices $a, b, c_1, c_2, \ldots \in \mathrm{gl}(n, \mathbb{C})$ according to Lemma 1. We always assume that the nondegeneracy condition holds: *the matrices a, b have simple spectrum.* Let $\mathcal{R}(x_0, t_0)$ be the set of all germs of holomorphic $\mathrm{gl}(n, \mathbb{C})$-valued maps at the point $(x_0, t_0) \in \mathbb{C}^2$, and let $\mathcal{R}(x_0, t_0)^{\mathrm{od}}$ be the set of all off-diagonal germs in $\mathcal{R}(x_0, t_0)$. The *local holomorphic Cauchy problem* for (22) is posed as follows. Given an off-diagonal holomorphic germ $q_0 \in \mathcal{R}(x_0)^{\mathrm{od}}$, it is required to find a germ $q \in \mathcal{R}(x_0, t_0)^{\mathrm{od}}$ that satisfies equation (13) and the initial condition $q(x, t_0) = q_0(x)$. The following theorem is an extended version of [17, Theorem 2].

Theorem 4.

(A) *The Cauchy problem $q(x, t_0) = q_0(x)$ for equation* (13) *admits a local holomorphic solution at the point $(x_0, t_0) \in \mathbb{C}^2$ if and only if $Lq_0 \in \mathrm{Gev}_{1/m}$. If such a solution $q(x, t)$ exists, it is unique.*

(B) *Every local holomorphic solution $q(x, t)$ of equation* (13) *in an arbitrary bidisk $D := \{(x, t) \in \mathbb{C}^2 \mid |x - x_0| < \delta_1,\ |t - t_0| < \delta_2\}$ admits an analytic continuation to a meromorphic function in the strip $S := \{(x, t) \in \mathbb{C}^2 \mid |t - t_0| < \delta_2\}$ possessing the trivial-monodromy property with respect to x (in the sense of Theorem* 3(B)) *for every fixed t. On the other hand, one can find a holomorphic solution $q_0(x, t)$ of* (13) *in D that admits no further analytic extension beyond the strip S.*

(C) *The envelope of meromorphy of any local holomorphic solution $q \in \mathcal{R}(x_0, t_0)^{\mathrm{od}}$ of equation* (22) *can be written in the form $\mathbb{C}_x^1 \times X$, where X is a Riemannian domain over $\mathbb{C}_t^1$. Conversely, for every Riemannian domain $\pi : X \to \mathbb{C}_t^1$ over $\mathbb{C}_t^1$ and every point $(x_0, t_0) \in \mathbb{C} \times \pi(X)$ one can find a local holomorphic solution $q \in \mathcal{R}(x_0, t_0)^{\mathrm{od}}$ of equation* (22) *whose envelope of meromorphy is equal to $\mathbb{C}_x^1 \times X$.*

(D) *In the notation of part* (A), *if the germ $q_0(x) := q(x, t_0)$ satisfies $Lq_0 \in \mathrm{Gev}_{(1/m)-0}$, then the solution $q(x, t)$ of the corresponding Cauchy problem admits an analytic continuation to a meromorphic off-diagonal* $\mathrm{gl}(n, \mathbb{C})$*-valued function on $\mathbb{C}^2$ possessing the following trivial-monodromy property with respect to x and t. For every $z \in \mathbb{C}$ the auxiliary linear system $E_x = (az + q(x, t))E$, $E_t = (bz^m + \sum_{j=1}^m F_j(q)(x, t) z^{m-j})E$ (which is defined by the formulae* (8) *on account of* (14)) *has a globally meromorphic fundamental system of solutions on $\mathbb{C}_{xt}^2$.*

In connection with the terminology in part (C) of the theorem we recall that a *Riemannian domain over* $\mathbb{C}^N$ is a complex manifold X together with a holomorphic locally invertible map $\pi : X \to \mathbb{C}^N$ (see [33, Subsection 22]), and the *envelope of meromorphy* of an arbitrary family of germs of holomorphic functions at a point $\zeta_0 \in \mathbb{C}^N$ is defined as the largest holomorphically separable Riemannian domain over $\mathbb{C}^N$ such that all the germs in this family can be analytically continued to meromorphic functions on this Riemannian domain (see [33, subsection 41]). This domain over $\mathbb{C}^N$ admits a more constructive description as the union of the results of all possible analytic extensions along chains of polydisks, similarly to the definition of a complete analytic function in the sense of Weierstrass ([33, Russian page 276]). By the envelope of meromorphy of a $\mathrm{gl}(n,\mathbb{C})$-valued germ (or a family of such germs) we understand the envelope of meromorphy of all entries of these germs.

To prove the necessity of the condition $Lq_0 \in \mathrm{Gev}_{1/m}$ for the existence of a local holomorphic solution $q(x,t)$ of the Cauchy problem, one should reduce the ordinary differential equation for $\mu(x_0,t,z)$ (where $\mu(x,t,z)$ is the formal series from the definition (20) of the scattering data $Lq(t,z)$) to the form (22) with x replaced by t and then apply Sibuya's theorem mentioned above (using the assumption that the matrix b has simple spectrum). This part of the argument is done in [17] by a reference to [28], but the exposition of this proof in [28, § 5] contains an inaccuracy that will be corrected now. Contrary to the last paragraph of [28, § 5], one cannot in general remove all terms with negative powers of z from the formula (5.1) of [28] by making the transformation indicated there. However, there is no actual need to remove them. Just replace the last paragraph of [28, § 5] by the following paragraph (which uses our current notation $\mu(x,t,z)$ for what was denoted by $m(x,t,z)$ in [28]; the other notation is from [28]).

By the definition of the series $\widetilde{V}$ in [28], the off-diagonal series $N(t,z) := \mu(x_0,t,z) - I$ satisfies the differential equation $N_t = V(I+N) - (I+N)\widetilde{V}$, where V is defined by (14). Taking the diagonal parts of both sides of this equation, we have $0 = V_{\mathrm{d}} + (V_{\mathrm{od}}N)_{\mathrm{d}} - \widetilde{V}$. Now, substituting $\widetilde{V} = V_{\mathrm{d}} + (V_{\mathrm{od}}N)_{\mathrm{d}}$ into the equality of the off-diagonal parts, we obtain the following equation of the form (22) for $N(t,z)$:

$$N_t = VN - NV_{\mathrm{d}} + V_{\mathrm{od}} - (I+N)(V_{\mathrm{od}}N)_{\mathrm{d}},$$

where the subscripts "d" and "od" denote the diagonal and off-diagonal part respectively. To verify that this equation is indeed of the form (22) (with the variable t instead of x and after rearranging all entries of the matrix N into one vector $y \in \mathbb{C}^\nu$, $\nu = n(n-1)$), we note the following. First, V_{od} and the difference between $VN - NV_{\mathrm{d}}$ and $[bz^m, N]$ are polynomials of degree at most $m-1$ in z whose coefficients depend holomorphically on t and polynomially on N. Second, the linear operator $N \mapsto [b,N]$ is invertible[9] on the space of all off-diagonal matrices. Therefore all the hypotheses of Sibuya's theorem hold, and we arrive at the desired

[9]This is the only place where we use the assumption that b has a simple spectrum.

conclusion: the formal series $Lq_0(z) = N(t_0, z)$ belongs to $\mathrm{Gev}_{1/m}$. This proves the necessity in part (A).

To prove the sufficiency of the condition $Lq_0 \in \mathrm{Gev}_{1/m}$ for the existence of the local holomorphic solution of the Cauchy problem, we consider the Riemann problem (17) with parameter $\xi = (x, t) \in \mathbb{C}^2$, the polynomial $P(\xi, z) = az(x - x_0) + (bz^m + c_1 z^{m-1} + \cdots + c_m)(t - t_0)$, and the formal series $f(z) = I + Lq_0(z)$. By Theorem 2(A), the solution $\gamma_\pm(x, t, z)$ of this problem exists in a neighborhood of the point $(x_0, t_0) \in \mathbb{C}^2$ and depends holomorphically on x, t. Putting $q(x, t) := [g_0(x, t), a]$, where $g_0(x, t)$ is the coefficient at z^{-1} in the expansion $\gamma_-(x, t, z) = I + \sum_{k=0}^{\infty} g_k(x, t) z^{-(k+1)}$, we claim that the holomorphic off-diagonal $\mathrm{gl}(n, \mathbb{C})$-valued function $q(x, t)$ satisfies equation (13) in a neighborhood of (x_0, t_0) along with the initial condition $q(x, t_0) = q_0(x)$. Indeed, the initial condition $q(x, t_0) = q_0(x)$ follows from the equality $BLq_0 = q_0$, which holds by Theorem 2(A). Furthermore, the first equality (21) shows that in a neighborhood of (x_0, t_0) we have $E_x = UE$, where $E(x, t, z) := \gamma_+(x, t, z)$ and $U(x, t, z) := az + q(x, t)$. Repeating verbatim the proof of Lemma 2 (which is legitimate in our case because of Lemma 4), we obtain that $E_t = VE$, where $V(x, t, z)$ is given by the formula (14) with the same differential polynomials $F_j : \mathcal{R}(x_0) \to \mathcal{R}(x_0)$ as in (13). The resulting equations $E_x = UE$ and $E_t = VE$ form the auxiliary linear system (8) whose solubility (with invertible E) implies that we have the zero curvature condition (9): $U_t - V_x + [U, V] = 0$, which is equivalent to the equation (13). This completes the proof of part (A) of Theorem 4.

Once part (A) is proved, part (B) follows easily from it and Theorem 3(B) since we always have $1/m < 1$ for all $m \geq 2$. Examples mentioned in the last assertion of part (B) can be constructed in abundance using the Cauchy–Kowalevsky theorem (this was done in [36, § 4], for all equations appearing in Theorem 1). The rest of Theorem 4 can also be easily obtained from part (A) and Theorems 2, 3, but we omit the details since these results have no direct use in the proof of Theorem 1.

4. Proof of Theorem 1

We start by showing that every local holomorphic solution $u(x, t)$ of any of equations (4)–(6) induces a local holomorphic solution $q(x, t)$ of an appropriate system (13). Indeed, if u satisfies (4), then a rescaling of x and t yields that $u_t = u_{xxx} - 6uu_x$, which is equivalent to (13) for $m = 3$, $a = b = \mathrm{diag}(1/2, -1/2)$, $c_1 = c_2 = c_3 = 0$ and $q(x, t) = \left(\begin{smallmatrix} 0 & u(x,t) \\ 1 & 0 \end{smallmatrix}\right)$. If u satisfies (6), then a rescaling of x and t by real factors yields that $iu_t + u_{xx} = \pm u|u|^2$, which is equivalent to (13) for $m = 2$, $a = b = \mathrm{diag}(-i/2, i/2)$, $c_1 = c_2 = 0$ and $q(x, t) = \begin{pmatrix} 0 & u(x,t) \\ \pm\overline{u(\overline{x}, \overline{t})} & 0 \end{pmatrix}$. If u satisfies (5), then the reduction is more complicated. It is described, for example, in [36] and follows Drinfeld and Sokolov [37]. First, a rescaling of x and t yields that $u_{tt} = -1/3 u_{xxxx} - 4/3 (uu_x)_x$, which is the condition for solubility of the system

$$\varphi_x = u_t, \quad \varphi_t = -1/3 u_{xxx} - 4/3 uu_x$$

in the bidisk D. This enables us to write the rescaled equation (6) in the form $L_t = [P, L]$ (see (7)), where $L := \partial_x^3 + u\partial_x + 1/2(\varphi + u_x)$ and $P := \partial_x^2 + 2/3u$. Second, writing $L = (\partial_x - v_3)(\partial_x - v_2)(\partial_x - v_1)$ for some $v_1, v_2, v_3 \in \mathcal{O}(D)$, we define an off-diagonal (because $v_1 + v_2 + v_3 = 0$) matrix-valued function $q(x,t) := K^{-1}\operatorname{diag}(v_1(x,t), v_2(x,t), v_3(x,t))K$ with $K \in \mathrm{GL}(3,\mathbb{C})$ being the matrix with entries $K_{ij} = (\alpha_j)^{i-1}$, where $\alpha_1, \alpha_2, \alpha_3$ are the cubic roots of 1 written in an arbitrary order. Then the rescaled equation (6) is equivalent to (13) for $m = 2$, $a = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$, $b = a^2$ and $c_1 = c_2 = 0$.

Now, to prove Theorem 1, we apply Theorem 4(B) to $q(x,t)$ and conclude that $q(x,t)$ extends to a global meromorphic function of x for every fixed t. Recovering $u(x,t)$ from $q(x,t)$ by the formulae above, we see that the same conclusion holds for $u(x,t)$, as required. □

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A.V. Domrin
Moscow State University
Faculty of Mechanics and Mathematics
119991, GSP-1, Leninskie Gory 1, Moscow, Russia
e-mail: domrin@mi.ras.ru

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Painlevé Equations and Supersymmetric Quantum mechanics

David J. Fernández C.

Abstract. An algorithm to generate solutions to the Painlevé IV and V equations is presented, based on supersymmetric quantum mechanics applied to the harmonic and radial oscillators, respectively. These solutions are expressed in terms of confluent hypergeometric functions, leading to a classification in solution hierarchies, according to the specific special functions they involve.

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1. Introduction

The generation of explicit solutions of nonlinear differential equations is a subject of mathematical physics worth studying. This applies, in particular, to the six nonlinear ordinary second-order differential equations, discovered by purely mathematical considerations by Paul Painlevé at the beginning of twentieth century, which nowadays are called Painlevé equations [1–4].

Another topic concerns the polynomial deformations of the Heisenberg–Weyl algebra (polynomial Heisenberg algebras, by short) such that the commutator between the annihilation and creation operators is substituted by a polynomial of the Hamiltonian. It turned out that such deformations can be realized straightforwardly by the supersymmetric (SUSY) partners of the harmonic and radial oscillators [5–8]. Furthermore, a connection can be established between second- (third-) order polynomial Heisenberg algebras and Painlevé IV (Painlevé V) equations, where the annihilation and creation operators are of third (fourth) order [9–20]. In particular, the existence can be shown of SUSY partner Hamiltonians for the harmonic (radial) oscillator possessing also third- (fourth-) order ladder operators, and thus being connected with Painlevé IV (Painlevé V) equation. Consequently,

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an algorithm to generate solutions of the Painlevé IV (Painlevé V) equation can be designed, by identifying the so-called extremal states of the system (for a review of this subject see, e.g., [20]).

In this paper, based on the harmonic and radial oscillators, we will try to combine the three subjects previously mentioned, namely, polynomial Heisenberg algebras (PHA), Painlevé equations and supersymmetric quantum mechanics (for reviews on the last subject, see, e.g., [21–26]). In order to achieve this goal, we have organized this article as follows.

In the next section the polynomial Heisenberg algebras will be introduced, paying special attention to those of second and third order. In Section 3 the SUSY partners of the harmonic oscillator will be constructed, as well as the corresponding solutions to the PIV equation. In Section 4 the same treatment will be applied to the SUSY partners of the radial oscillator and the PV equation. Our conclusions are contained in Section 5.

2. Polynomial Heisenberg algebras

The $(m-1)$th-order polynomial Heisenberg algebras are deformations of the Heisenberg–Weyl algebra with three generators $\{H, \mathcal{L}_m^+, \mathcal{L}_m^-\}$ satisfying [7]:

$$[H, \mathcal{L}_m^{\pm}] = \pm\mathcal{L}_m^{\pm}, \tag{1}$$

$$[\mathcal{L}_m^-, \mathcal{L}_m^+] \equiv N_m(H+1) - N_m(H) \equiv P_{m-1}(H). \tag{2}$$

The analogue of the number operator, $N_m(H) \equiv \mathcal{L}_m^+\mathcal{L}_m^-$, is a polynomial of degree m of the Hamiltonian H, which can be factorized as:

$$N_m(H) = \prod_{i=1}^{m} (H - \mathcal{E}_i)\,. \tag{3}$$

A straightforward representation of such a structure is a differential one, where H acquires the standard Schrödinger form

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \tag{4}$$

while $\mathcal{L}_m^{\pm}$ are mth-order differential ladder operators.

Note that there are special states of the system, the so-called *extremal states* of H, such that $\mathcal{L}_m^-\psi_{\mathcal{E}_i} = 0$ and $H\psi_{\mathcal{E}_i} = \mathcal{E}_i\psi_{\mathcal{E}_i}$, $i = 1,\dots,m$. They supply all information we can get about the spectrum of H, Sp(H). In fact:

(a) Suppose that s extremal states $\{\psi_{\mathcal{E}_i}, i = 1,\dots,s\}$ satisfy the right boundary conditions. Then, Sp(H) will consist of s infinite energy ladders, arising from the iterated action of $\mathcal{L}_m^+$ onto $\{\psi_{\mathcal{E}_i}, i = 1,\dots,s\}$ (Figure 1, left).

(b) However, if for the jth extremal state it turns out that $(\mathcal{L}_m^+)^{n-1}\psi_{\mathcal{E}_j} \neq 0$ and $(\mathcal{L}_m^+)^n\psi_{\mathcal{E}_j} = 0$, then the jth energy ladder instead of being infinite will end up after n steps, while the $s-1$ remaining ladders will stay infinite (Figure 1, right).

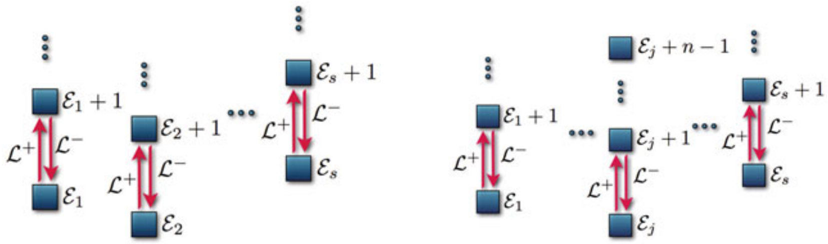

FIGURE 1. Spectrum of H when s extremal states obey the boundary conditions (left) and when the jth one satisfies also $(\mathcal{L}_m^+)^{n-1}\,\psi_{\mathcal{E}_j}\neq 0$ and $(\mathcal{L}_m^+)^{n}\,\psi_{\mathcal{E}_j}=0$ (right).

Since they are connected with Painlevé equations, let us present a brief overview of the polynomial Heisenberg algebras of second and third order.

2.1. Second-order PHA

Let $\mathcal{L}_3^{\pm}$ be the following third-order differential operators

$$\mathcal{L}_3^+ = A_3^+A_2^+A_1^+, \qquad \mathcal{L}_3^- = A_1^-A_2^-A_3^-, \tag{5}$$

$$A_j^{\pm} = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} - f_j\right), \tag{6}$$

such that

$$H_{j+1}A_j^+ = A_j^+H_j, \quad H_jA_j^- = A_j^-H_{j+1}, \quad j=1,2,3. \tag{7}$$

In addition, let us suppose that the closure relation is satisfied,

$$H_4 = H_1 - 1 \equiv H - 1, \tag{8}$$

which ensures that the commutator between H and $\mathcal{L}_3^{\pm}$ in Eq. (1) is valid. A diagram representing the chain of intertwining relations of Eq. (7) as well as the global one is shown in Figure 2.

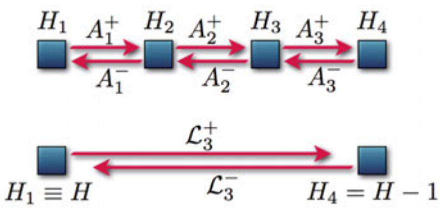

FIGURE 2. Representation of the intertwining relations of Eq. (7) and the global one leading to Eq. (1).

Equations (5–8) lead to a system of coupled differential equations for the unknown functions f_j and the potential V. After a long but straightforward calculation [20], it turns out that all these unknowns become expressed in terms of just one function g:

$$f_1 = -\frac{g}{2} + \frac{g'}{2g} + \frac{\epsilon_1 - \epsilon_2}{g}, \tag{9}$$

$$f_2 = -\frac{g}{2} - \frac{g'}{2g} - \frac{\epsilon_1 - \epsilon_2}{g}, \tag{10}$$

$$f_3 = x + g, \tag{11}$$

$$V(x) = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \epsilon_3 + \frac{1}{2}, \tag{12}$$

which satisfies the Painlevé IV equation:

$$gg'' = \frac{1}{2}{g'}^2 + \frac{3}{2}g^4 + 4xg^3 + 2\left(x^2 - a\right)g^2 + b, \tag{13}$$

with parameters $a = \epsilon_1 + \epsilon_2 - 2\epsilon_3 - 1,\ b = -2(\epsilon_1 - \epsilon_2)^2$. Moreover, the generalized number operator

$$N_3(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}_3),$$

has three roots $\mathcal{E}_i = \epsilon_i + 1,\ i = 1, 2, 3$, which are associated to the three extremal states:

$$\psi_{\mathcal{E}_1} \propto \left(\frac{g'}{2g} - \frac{g}{2} - \frac{1}{g}\sqrt{-\frac{b}{2}} - x\right) \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} - \frac{1}{g}\sqrt{-\frac{b}{2}}\right) \mathrm{d}x\right], \tag{14}$$

$$\psi_{\mathcal{E}_2} \propto \left(\frac{g'}{2g} - \frac{g}{2} + \frac{1}{g}\sqrt{-\frac{b}{2}} - x\right) \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{1}{g}\sqrt{-\frac{b}{2}}\right) \mathrm{d}x\right], \tag{15}$$

$$\psi_{\mathcal{E}_3} \propto \exp\left(-\frac{x^2}{2} - \int g \, \mathrm{d}x\right). \tag{16}$$

Thus, once we get a solution g of the PIV equation the system we are dealing with becomes determined, since all relevant quantities are expressed in terms of such a g. This is the so-called direct approach to the problem.

On the other hand, in the inverse approach the extremal states of a system ruled by a second-order PHA determine the solutions to the PIV equation. In fact, from Eq. (16) it is obtained:

$$g(x) = -x - \{\ln[\psi_{\mathcal{E}_3}(x)]\}'. \tag{17}$$

In this way we can generate three solutions of the PIV equation, by selecting as $\psi_{\mathcal{E}_3}$ any of the three extremal states of the system.

2.2. Third-order PHA

Now, let $\mathcal{L}_4^{\pm}$ be the fourth-order operators

$$\mathcal{L}_4^+ = A_4^+ A_3^+ A_2^+ A_1^+, \qquad \mathcal{L}_4^- = A_1^- A_2^- A_3^- A_4^-, \tag{18}$$

$$A_j^{\pm} = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} - f_j\right), \tag{19}$$

such that

$$H_{j+1}A_j^+ = A_j^+ H_j, \quad H_j A_j^- = A_j^- H_{j+1}, \quad j = 1, 2, 3, 4. \tag{20}$$

In order to ensure that Eq. (1) is satisfied, the following closure relation has to be fulfilled:

$$H_5 = H_1 - 1 \equiv H - 1. \tag{21}$$

A representation of the chain of intertwining relations of Eq. (20) as well as the global one is shown in Figure 3.

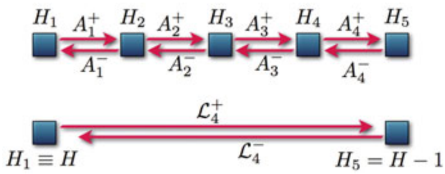

FIGURE 3. Representation of the intertwining relations of Eq. (20) and the global one leading to Eq. (1).

Equations (18–21) produce a system of coupled differential equations for the functions f_j and the potential V. After an even longer calculation than in the previous case, it turns out that all of them depend on just one function $w(z)$ which satisfies now the Painlevé V equation:

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right) w'^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(aw + \frac{b}{w}\right) + c\frac{w}{z} + d\frac{w(w+1)}{w-1}, \tag{22}$$

where $z = x^2$ and the PV parameters are given by

$$a = \frac{(\epsilon_1 - \epsilon_2)^2}{2}, \; b = -\frac{(\epsilon_3 - \epsilon_4)^2}{2}, \; c = \frac{\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - 1}{2}, \; d = -\frac{1}{8}. \tag{23}$$

Two functions are going to be quite important for generating solutions of the PV equation later on:

$$g(x) = \frac{x}{w(x^2) - 1}, \quad h(x) = -x - g(x). \tag{24}$$

Note that the number operator $N_4(H)$ is now a polynomial of fourth degree in H:

$$N_4(H) = (H - \mathcal{E}_1)(H - \mathcal{E}_2)(H - \mathcal{E}_3)(H - \mathcal{E}_4),$$

with roots $\mathcal{E}_j = \epsilon_j + 1,\ j = 1, 2, 3, 4$ and associated extremal states given by:

$$\psi_{\mathcal{E}_1} \propto \left[\frac{h}{2}\left(\frac{g'}{2g} - \frac{h'}{2h} - \frac{x}{2} + \frac{\mathcal{E}_2 - \mathcal{E}_1}{g}\right) - \mathcal{E}_1 + \frac{\mathcal{E}_3 + \mathcal{E}_4}{2}\right] \times \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{\mathcal{E}_2 - \mathcal{E}_1}{g}\right) \mathrm{d}x\right], \tag{25}$$

$$\psi_{\mathcal{E}_2} \propto \left[\frac{h}{2}\left(\frac{g'}{2g} - \frac{h'}{2h} - \frac{x}{2} + \frac{\mathcal{E}_1 - \mathcal{E}_2}{g}\right) - \mathcal{E}_2 + \frac{\mathcal{E}_3 + \mathcal{E}_4}{2}\right] \times \exp\left[\int \left(\frac{g'}{2g} + \frac{g}{2} + \frac{\mathcal{E}_1 - \mathcal{E}_2}{g}\right) \mathrm{d}x\right], \tag{26}$$

$$\psi_{\mathcal{E}_3} \propto e^{\int\left(\frac{h'}{2h} + \frac{h}{2} + \frac{\mathcal{E}_4 - \mathcal{E}_3}{h}\right)\mathrm{d}x}, \qquad \psi_{\mathcal{E}_4} \propto e^{\int\left(\frac{h'}{2h} + \frac{h}{2} + \frac{\mathcal{E}_3 - \mathcal{E}_4}{h}\right)\mathrm{d}x}. \tag{27}$$

As in the previous case, once $w(z)$ is found all the relevant functions of the system can be determined, in particular the potential $V(x)$. Once again, this is nothing but the direct approach to the problem.

On the other hand, in the inverse approach one uses the expressions for the two extremal states of Eq. (27) in order to find $h(x)$, which in turn is related with the solution $w(z)$ to the PV equation as follows:

$$h(x) = \frac{2(\mathcal{E}_3 - \mathcal{E}_4)}{[\ln(\psi_{\mathcal{E}_4}) - \ln(\psi_{\mathcal{E}_3})]'} = \{\ln\left[W(\psi_{\mathcal{E}_3}, \psi_{\mathcal{E}_4})\right]\}', \tag{28}$$

$$w(z) = \frac{h(\sqrt{z})}{\sqrt{z} + h(\sqrt{z})}. \tag{29}$$

Thus, if we choose as $\psi_{\mathcal{E}_3}$ and $\psi_{\mathcal{E}_4}$ any pair between the four extremal states for systems ruled by third-order PHA we can build several solutions of the PV equation.

Let us explore next the possibilities offered by these techniques to generate solutions to the PIV and PV equations by constructing the SUSY partners of the harmonic and radial oscillators.

3. Harmonic oscillator SUSY partners

The kth-order SUSY transformation applied to the harmonic oscillator employs k Schrödinger seed solutions for $V_0(x) = x^2/2$ of the form [5]:

$$u = e^{-\frac{x^2}{2}}\left[{}_1F_1\left(\frac{1-2\epsilon}{4}, \frac{1}{2}; x^2\right) + 2x\nu\frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\,{}_1F_1\left(\frac{3-2\epsilon}{4}, \frac{3}{2}; x^2\right)\right]. \tag{30}$$

If the transformation creates k new levels with the following ordering:

$$\epsilon_k < \epsilon_{k-1} < \cdots < \epsilon_1 < E_0 = \frac{1}{2}, \tag{31}$$

then the constants $\nu_i, i = 1, \ldots, k$, must satisfy

$$|\nu_i| < 1 \text{ for odd } i, \quad |\nu_i| > 1 \text{ for even } i. \tag{32}$$

The new potential is given by

$$V_k(x) = \frac{x^2}{2} - \{\ln[W(u_1,\ldots,u_k)]\}'', \quad k \geq 1, \tag{33}$$

while the eigenfunctions of H_k read (the corresponding eigenvalues are written to their right):

$$\psi_n^{(k)} = \frac{B_k^+\psi_n}{[(E_n-\epsilon_1)\ldots(E_n-\epsilon_k)]^{1/2}}, \qquad E_n, \tag{34}$$

$$\psi_{\epsilon_j}^{(k)} \propto \frac{W(u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_k)}{W(u_1,\ldots,u_k)}, \quad \epsilon_j. \tag{35}$$

Thus, the spectrum of H_k turns out to be:

$$\mathrm{Sp}(H_k) = \left\{\epsilon_j,\ E_n = n+\frac{1}{2},\ j=1,\ldots,k,\ n=0,1,\ldots\right\}. \tag{36}$$

There is a pair of intertwining relations for H_0 and H_k:

$$H_kB_k^+ = B_k^+H_0, \qquad H_0B_k = B_kH_k. \tag{37}$$

Thus, the following factorized expressions appear:

$$B_k^+B_k = (H_k-\epsilon_1)\ldots(H_k-\epsilon_k), \tag{38}$$

$$B_kB_k^+ = (H_0-\epsilon_1)\ldots(H_0-\epsilon_k). \tag{39}$$

As a consequence, there is a natural pair of ladder operators for H_k:

$$L_k^- = B_k^+aB_k, \tag{40}$$

$$L_k^+ = B_k^+a^+B_k, \tag{41}$$

which are $(2k+1)$th-order differential operators such that:

$$[H_k, L_k^\pm] = \pm L_k^\pm. \tag{42}$$

Moreover, the analogue of the number operator becomes a polynomial of degree $2k+1$ in H_k:

$$N(H_k) \equiv L_k^+L_k^- = \left(H_k-\frac{1}{2}\right)\prod_{i=1}^{k}(H_k-\epsilon_i-1)(H_k-\epsilon_i). \tag{43}$$

Hence, the operators $\{L_k^-,\ L_k^+,\ H_k\}$ generate a $(2k)$th-order polynomial Heisenberg algebra, since

$$[L_k^-, L_k^+] = N(H_k+1) - N(H_k). \tag{44}$$

A diagram representing the action of the operators L_k^-, L_k^+, B_k, B_k^+ is given in Figure 4.

Note that for $k=1$ the natural ladder operators $L_1^\pm$ are of third order, so there is a straightforward link with the PIV equation. Furthermore, for $k>1$ it can happen that some SUSY partner Hamiltonians H_k have as well third-order ladder operators, thus connecting also with the PIV equation. The requirements

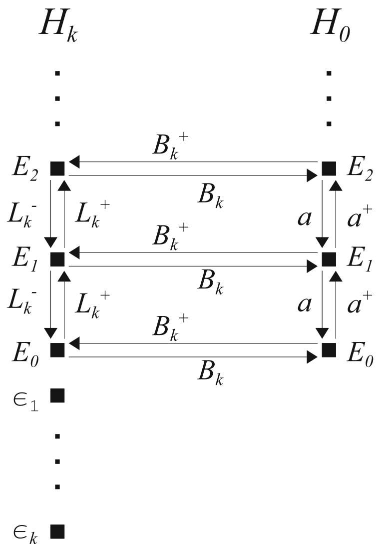

FIGURE 4. Diagram representing the action of the operators L_k^-, L_k^+, B_k, B_k^+.

to produce such an *order reduction* are contained in the following theorem, which has been proven elsewhere [8, 18].

Theorem 1 (Reduction Theorem). *Suppose that H_k is generated from the harmonic oscillator Hamiltonian H_0 by k connected seed solutions*

$$u_j = a^{j-1}u_1, \quad \epsilon_j = \epsilon_1 - (j-1), \quad j = 1, \ldots, k, \tag{45}$$

u_1 being a nodeless seed solution given by Eq. (30) *for $\epsilon_1 < 1/2$ and $|\nu_1| < 1$. Thus,*

$$L_k^+ = P_{k-1}(H_k) l_k^+, \tag{46}$$

$$P_{k-1}(H_k) = (H_k - \epsilon_1)\ldots(H_k - \epsilon_{k-1}), \tag{47}$$

where l_k^+ is a third-order differential ladder operator such that

$$[H_k, l_k^+] = l_k^+, \tag{48}$$

$$l_k^+ l_k^- = (H_k - \epsilon_k)\left(H_k - \frac{1}{2}\right)(H_k - \epsilon_1 - 1). \tag{49}$$

An immediate consequence of this theorem is that H_k, l_k^-, l_k^+ generate a second-order polynomial Heisenberg algebra. Thus, solutions of the PIV equation can be obtained departing from the extremal states of the system (the corresponding energies appear to the right):

$$\psi_{\mathcal{E}_1} \propto B_k^+ e^{-x^2/2}, \qquad \mathcal{E}_1 = \frac{1}{2}, \tag{50}$$

$$\psi_{\mathcal{E}_2} \propto B_k^+ a^+ u_1, \qquad \mathcal{E}_2 = \epsilon_1 + 1, \tag{51}$$

$$\psi_{\mathcal{E}_3} \propto \frac{W(u_1, \dots, u_{k-1})}{W(u_1, \dots, u_k)}, \qquad \mathcal{E}_3 = \epsilon_k. \tag{52}$$

Indeed, by using Eq. (17) the corresponding PIV solution turns out to be

$$g_k(x) = -x - \{\ln[\psi_{\mathcal{E}_3}(x)]\}' = -x - \left\{\ln\left[\frac{W(u_1, \dots, u_{k-1})}{W(u_1, \dots, u_k)}\right]\right\}'. \tag{53}$$

The parameters associated to this solution are given by:

$$a = -\epsilon_1 + 2k - \frac{3}{2}, \quad b = -2\left(\epsilon_1 + \frac{1}{2}\right)^2. \tag{54}$$

Now, by permuting the indices for the extremal states of Eqs. (50–52) we obtain two additional PIV solutions for different parameters a, b:

$$g_k(x) = -x - \left\{\ln\left[B_k^+ e^{-x^2/2}\right]\right\}', \quad a = 2\epsilon_1 - k, \quad b = -2k^2, \tag{55}$$

$$g_k(x) = -x - \left\{\ln\left[B_k^+ a^+ u_1\right]\right\}', \; a = -\epsilon_1 - k - \frac{3}{2}, \; b = -2\left(\epsilon_1 - k + \frac{1}{2}\right)^2. \tag{56}$$

As an illustration, in Figure 5 we have shown some PIV solutions generated through first-order SUSY ($k = 1$), while in Figure 6 we are plotting the ones generated with second-order SUSY ($k = 2$).

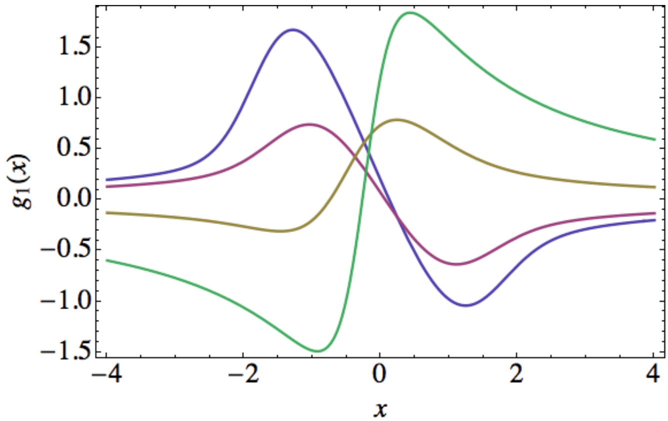

FIGURE 5. Solutions of the PIV equation generated through first-order SUSY for $(\epsilon_1, \nu_1) = (0.25, 0.99)$ (blue), $(\epsilon_1, \nu_1) = (0, 0.1)$ (magenta), $(\epsilon_1, \nu_1) = (-1, 0.5)$ (yellow), and $(\epsilon_1, \nu_1) = (-4, 0.5)$ (green).

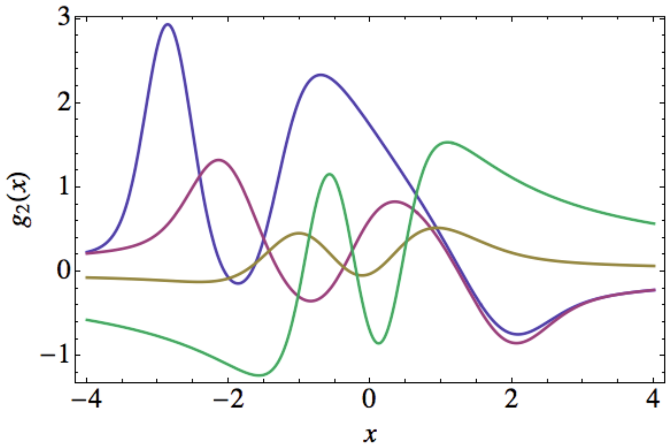

FIGURE 6. Solutions of the PIV equation generated from second-order SUSY for $(\epsilon_1, \nu_1) = (0.25, 0.99)$ (blue), $\epsilon_1 = \{0.25$ (magenta), -0.75 (yellow), -2.75 (green)$\}$ and $\nu_1 = 0.5$.

It is important to notice that our previous formulas for generating solutions to the PIV equation, and the reduction theorem, remain valid even if the seed solutions are complex [27–30]. In particular, for real ϵ_1 we can take,

$$u_1(x) = e^{-x^2/2}\left[{}_1F_1\left(\frac{1-2\epsilon_1}{4}, \frac{1}{2}; x^2\right) + \Lambda\, x\, {}_1F_1\left(\frac{3-2\epsilon_1}{4}, \frac{3}{2}; x^2\right)\right],$$

where $\Lambda = \lambda + i\kappa$ ($\lambda, \kappa, \epsilon_1 \in \mathbb{R}$). These complex seed solutions give place to complex solutions of the PIV equation; some examples are given in Figures 7 and 8, for $k = 1$ and $k = 2$ respectively.

Since the general seed solution $u(x)$ is given in terms of confluent hypergeometric functions, it is said that the PIV solutions belong to the confluent hypergeometric function hierarchy, as at the end they are expressed in terms of these special functions. In addition, more specific hierarchies appear for particular values of ϵ_1 [8, 31, 32], e.g., the error function hierarchy arises for

$$\epsilon_1 \in \left\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots, -\frac{(2m+1)}{2}, \dots\right\}. \tag{57}$$

For example, PIV solutions belonging to the error function hierarchy are:

$$g_1(x, -1/2) = \frac{2\nu_1}{\varphi_{\nu_1}(x)}, \tag{58}$$

$$g_1(x, -3/2) = \frac{\varphi_{\nu_1}(x)}{1 + x\varphi_{\nu_1}(x)}, \tag{59}$$

$$g_1(x, -5/2) = \frac{4[\nu_1 + \varphi_{\nu_1}(x)]}{2\nu_1 x + (1 + 2x^2)\varphi_{\nu_1}(x)}, \tag{60}$$

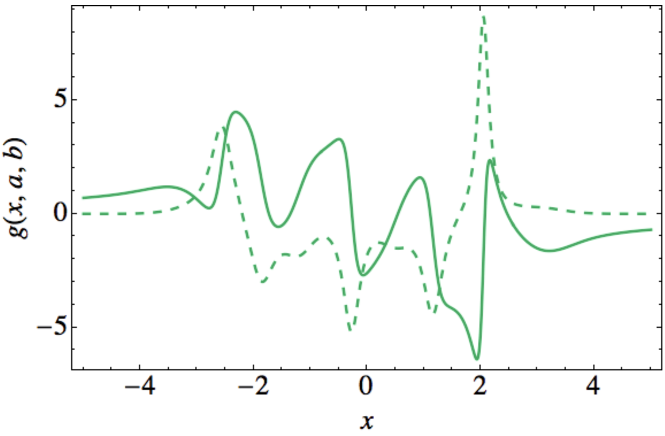

FIGURE 7. Real (solid) and imaginary (dashed) parts of a complex PIV solution for $a_{iii} = -5$, $b_{iii} = -8$ ($k = 1$, $\epsilon_1 = 5/2$, $\lambda = \kappa = 1$).

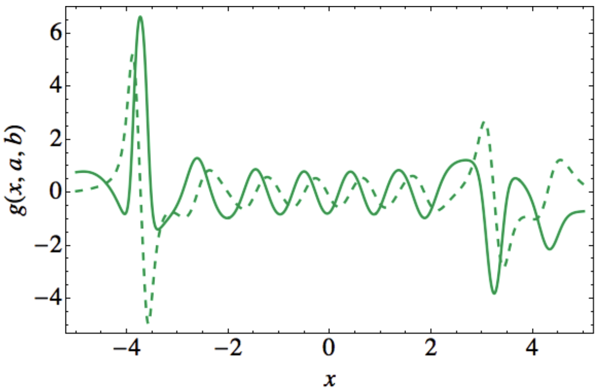

FIGURE 8. Real (solid) and imaginary (dashed) parts of the complex PIV solution for $a_{ii} = 12$, $b_{ii} = -8$ ($k = 2$, $\epsilon_1 = 7$, $\lambda = \kappa = 1$).

$$g_2(x, -1/2) = \frac{4\nu_1[\nu_1 + 6\varphi_{\nu_1}(x)]}{\varphi_{\nu_1}(x)[\varphi^2_{\nu_1}(x) - 2\nu_1 x\varphi_{\nu_1}(x) - 2\nu_1^2]}, \tag{61}$$

where $\varphi_{\nu_1}(x) \equiv \sqrt{\pi}e^{x^2}[1 + \nu_1 \operatorname{erf}(x)]$. There appears also the rational hierarchy for $\nu_1 = 0$ and

$$\epsilon_1 \in \left\{-\frac{1}{2}, -\frac{5}{2}, \dots, -\frac{(4m+1)}{2}, \dots\right\}. \tag{62}$$

Explicit expressions for elements of this hierarchy are given by:

$$g_1(x,-5/2) = \frac{4x}{1+2x^2}, \tag{63}$$

$$g_1(x,-9/2) = \frac{8(3x+2x^3)}{3+12x^2+4x^4}, \tag{64}$$

$$g_1(x,-13/2) = \frac{12(15x+20x^3+4x^5)}{15+90x^2+60x^4+8x^6}, \tag{65}$$

$$g_2(x,-5/2) = -\frac{4x}{1+2x^2}+\frac{16x^3}{3+4x^4}, \tag{66}$$

$$g_2(x,-9/2) = -\frac{8(3x+2x^3)}{3+12x^2+4x^4}+\frac{32(15x^3+12x^5+4x^7)}{45+120x^4+64x^6+16x^8}, \tag{67}$$

$$\begin{aligned} g_2(x,-13/2) = & -\frac{12(15x+20x^3+4x^5)}{15+90x^2+60x^4+8x^6} \\ & +\frac{48(525x^3+840x^5+600x^7+160x^9+16x^{11})}{1575+6300x^4+6720x^6+3600x^8+768x^{10}+64x^{12}}, \end{aligned} \tag{68}$$

$$g_3(x,-5/2) = \frac{4x(27-72x^2+16x^8)}{27+54x^2+96x^6-48x^8+32x^{10}}, \tag{69}$$

$$\begin{aligned} g_3(x,-9/2) = & -\frac{32(15x^3+12x^5+4x^7)}{45+120x^4+64x^6+16x^8} \\ & +\frac{24(225x-150x^3+120x^5+240x^7+80x^9+32x^{11})}{675+2700x^2-900x^4+480x^6+720x^8+192x^{10}+64x^{12}}. \end{aligned} \tag{70}$$

On the other hand, the imaginary error function hierarchy arises for $\epsilon_1 \geq E_0$, for example:

$$g_{iii}(x;5/2) = \frac{4\Lambda(1-x^2)+2x(-3+2x^2)\phi_\Lambda^i(x)}{2\Lambda x+(1-2x^2)\phi_\Lambda^i(x)}, \tag{71}$$

where $\phi_\Lambda^i = \mathrm{e}^{-x^2}[4+\Lambda\pi^{1/2}\mathrm{erfi}(x)]$. Moreover, the first kind modified Bessel function hierarchy appears for $\epsilon_1 = 0, k = 1, \Lambda = i$:

$$g_i(x;0) = \frac{\Gamma\left(\frac{3}{4}\right)\left[I_{-\frac{5}{4}}\left(\frac{x^2}{2}\right)+(1-x^2)I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)\right]+2ix^2\Gamma\left(\frac{5}{4}\right)\left[I_{-\frac{3}{4}}\left(\frac{x^2}{2}\right)-I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)\right]}{x\Gamma\left(\frac{3}{4}\right)I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)+2ix\Gamma\left(\frac{5}{4}\right)I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)}. \tag{72}$$

It is important to explore a similar link, arising now between the SUSY partners of the radial oscillator and PV equation, which is the subject of the next Section.

4. Radial oscillator SUSY partners

The radial oscillator Hamiltonian reads

$$H_\ell = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\ell(x) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{8} + \frac{\ell(\ell+1)}{2x^2}, \tag{73}$$

where $\ell \geq 0$. It has second-order differential ladder operators given by:

$$b_\ell^\pm = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} \mp x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x^2}{4} - \frac{\ell(\ell+1)}{x^2} \mp \frac{1}{2}\right), \tag{74}$$

which generate a first-order polynomial Heisenberg algebra [33]:

$$[H_\ell, b_\ell^\pm] = \pm b_\ell^\pm, \tag{75}$$

$$[b_\ell^-, b_\ell^+] = 2H_\ell. \tag{76}$$

The analogue of the number operator takes the form:

$$b_\ell^+ b_\ell^- = (H_\ell - \mathcal{E}_1)(H_\ell - \mathcal{E}_2) = \left(H_\ell - \frac{\ell}{2} - \frac{3}{4}\right)\left(H_\ell + \frac{\ell}{2} - \frac{1}{4}\right). \tag{77}$$

The associated extremal states, which are formal eigenfunctions of H_ℓ annihilated also by b_ℓ^-, become:

$$\psi_{\mathcal{E}_1} \propto x^{\ell+1}\exp(-x^2/4), \qquad \mathcal{E}_1 = \frac{\ell}{2} + \frac{3}{4} \equiv E_{0\ell}, \tag{78}$$

$$\psi_{\mathcal{E}_2} \propto x^{-\ell}\exp(-x^2/4), \qquad \mathcal{E}_2 = -\frac{\ell}{2} + \frac{1}{4} = -E_{0\ell} + 1. \tag{79}$$

Since the first one fulfills vanishing boundary conditions for $\ell \geq 0$, it leads to a ladder of physical eigenfunctions of H_ℓ. The spectrum of the radial oscillator is therefore:

$$\mathrm{Sp}(H_\ell) = \left\{E_{n\ell} = n + \frac{\ell}{2} + \frac{3}{4}, n = 0, 1, \dots\right\}. \tag{80}$$

Let us apply now a kth-order SUSY transformation to H_ℓ by using k seed solutions of the form [7]:

$$u(x,\epsilon) = x^{-\ell}\mathrm{e}^{-\frac{x^2}{4}}\left[{}_1F_1\left(\frac{1-2\ell-4\epsilon}{4}, \frac{1-2\ell}{2}; \frac{x^2}{2}\right)\right.$$
$$\left. + \nu\frac{\Gamma\left(\frac{3+2\ell-4\epsilon}{4}\right)}{\Gamma\left(\frac{3+2\ell}{2}\right)}\left(\frac{x^2}{2}\right)^{\ell+1/2} {}_1F_1\left(\frac{3+2\ell-4\epsilon}{4}, \frac{3+2\ell}{2}; \frac{x^2}{2}\right)\right], \tag{81}$$

which creates k new levels below $E_{0\ell}$ ordered as follows:

$$\epsilon_k < \epsilon_{k-1} < \cdots < \epsilon_1 < E_{0\ell}. \tag{82}$$

In order to avoid singularities in the new potential for $x > 0$ it must be fulfilled that $\nu_i \geq -\Gamma\left(\frac{1-2\ell}{2}\right)/\Gamma\left(\frac{1-2\ell-4\epsilon}{4}\right)$ for i odd and $\nu_i \leq -\Gamma\left(\frac{1-2\ell}{2}\right)/\Gamma\left(\frac{1-2\ell-4\epsilon}{4}\right)$ for i

even. Thus, the new potential and associated spectrum become

$$V_k(x) = \frac{x^2}{8} + \frac{\ell(\ell+1)}{2x^2} - \{\ln[W(u_1,\dots,u_k)]\}'', \tag{83}$$

$$\mathrm{Sp}(H_k) = \{\epsilon_k,\dots,\epsilon_1, E_{0\ell}, E_{1\ell},\dots\}. \tag{84}$$

The natural ladder operators for H_k are now of $(2k+2)$th order,

$$L_k^{\pm} = B_k^+ b_\ell^{\pm} B_k^-, \tag{85}$$

and generate a $(2k+1)$th-order polynomial Heisenberg algebra since:

$$[H_k, L_k^{\pm}] = \pm L_k^{\pm}, \tag{86}$$

$$[L_k^-, L_k^+] = N(H_k+1) - N(H_k), \tag{87}$$

where the number operator is a polynomial of degree $2k+2$ in H_k:

$$N(H_k) = \left(H_k - \frac{\ell}{2} - \frac{3}{4}\right)\left(H_k + \frac{\ell}{2} - \frac{1}{4}\right)\prod_{j=1}^{k}(H_k-\epsilon_j)(H_k-\epsilon_j-1). \tag{88}$$

For $k=1$ the natural ladder operators $L_1^{\pm}$ are of fourth order, then the corresponding system is linked directly with the PV equation. Moreover, for $k>1$ some SUSY partner Hamiltonians H_k have as well fourth-order ladder operators, thus they are also connected with the PV equation. The requirements for producing such an order reduction are formulated in the next theorem, which proof is given in references [8, 34].

Theorem 2 (Reduction Theorem). *Let the SUSY partner Hamiltonian H_k of the radial oscillator be generated by k connected seed solutions*

$$u_i = (b_\ell^-)^{i-1}u_1, \quad \epsilon_i = \epsilon_1 - (i-1), \quad i = 1,\dots,k, \tag{89}$$

where u_1 is a nodeless seed solution given in Eq. (81) *for $\epsilon_1 < E_0 = \frac{\ell}{2}+\frac{3}{4}$, $\nu_1 \geq -\frac{\Gamma\left(\frac{1-2\ell}{2}\right)}{\Gamma\left(\frac{1-2\ell-4\epsilon_1}{4}\right)}$. Thus:*

$$L_k^+ = P_{k-1}(H_k)\mathbf{l}_k^+, \tag{90}$$

$$P_{k-1}(H_k) = (H_k-\epsilon_1)\dots(H_k-\epsilon_{k-1}), \tag{91}$$

$\mathbf{l}_k^+$ being a fourth-order differential ladder operator such that

$$[H_k, \mathbf{l}_k^+] = \mathbf{l}_k^+, \tag{92}$$

$$\mathbf{l}_k^+\mathbf{l}_k^- = (H_k - E_0)(H_k + E_0 - 1)(H_k-\epsilon_k)(H_k-\epsilon_1-1). \tag{93}$$

This theorem ensures that the operators $\{H_k, \mathbf{l}_k^-, \mathbf{l}_k^+\}$ satisfy a third-order polynomial Heisenberg algebra and thus they can give place to solutions of the

PV equation, departing from the associated extremal states

$$\psi_{\mathcal{E}_1} \propto B_k^+ b^+ u_1, \qquad \mathcal{E}_1 = \epsilon_1 + 1, \tag{94}$$

$$\psi_{\mathcal{E}_2} \propto B_k^+ \left[x^{-\ell} \exp(-x^2/4)\right], \qquad \mathcal{E}_2 = -E_0 + 1, \tag{95}$$

$$\psi_{\mathcal{E}_3} \propto \frac{W(u_1, \dots, u_{k-1})}{W(u_1, \dots, u_k)}, \qquad \mathcal{E}_3 = \epsilon_k, \tag{96}$$

$$\psi_{\mathcal{E}_4} \propto B_k^+ \left[x^{\ell+1} \exp(-x^2/4)\right], \qquad \mathcal{E}_4 = E_0. \tag{97}$$

If we remember that

$$h(x) = \{\ln [W(\psi_{\mathcal{E}_3}, \psi_{\mathcal{E}_4})]\}', \tag{98}$$

$$w(z) = \frac{h(\sqrt{z})}{\sqrt{z} + h(\sqrt{z})}, \tag{99}$$

using first-order SUSY we have produced the PV solutions of Figure 9, while those of Figure 10 were generated with the second-order SUSY.

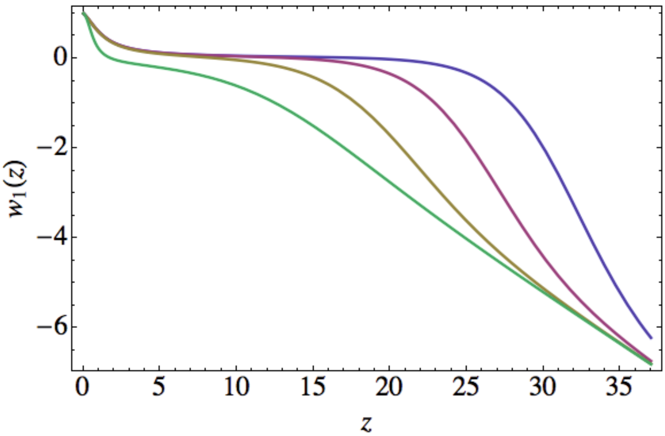

FIGURE 9. Real PV solutions generated through first-order SUSY for $\ell = 1$, $\epsilon_1 = 1$, $\nu_1 = \{0.905$ (blue), 0.913 (magenta), 1 (yellow), 10 (green)$\}$.

We can employ the same formulas for generating complex solutions of the PV equation, by taking complex transformation functions. One of them is shown in Figure 11, where we have plotted a complex PV solution by using a complex transformation function for real ϵ_1, which makes the PV parameters a, b, c, d to be real.

In general, the solutions we have derived belong to the so-called confluent hypergeometric function hierarchy, since the involved seed solution u_1 contains such a special function. In addition, for particular values of the factorization energy ϵ_1 and the parameter ν_1 of the seed solution more specific hierarchies will be

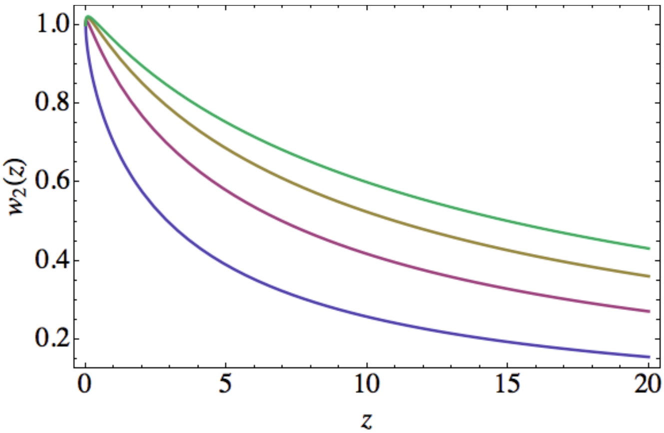

FIGURE 10. Real PV solutions generated from second-order SUSY for $\ell = 0$, $\nu_1 = 0$, $\epsilon_1 = \{1/4$ (blue), $-3/4$ (magenta), $-7/4$ (yellow), $-11/4$ (green)$\}$.

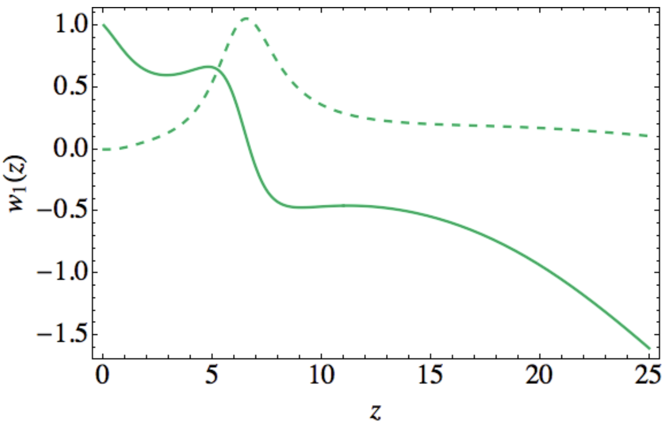

FIGURE 11. Complex PV solution generated through first-order SUSY for $\ell = 2$, $\epsilon_1 = 2$, $\nu_1 = i$.

obtained, e.g., below we are showing two elements of the Laguerre polynomial hierarchy:

$$w_1(z) = 1 - z^{-1/2}, \tag{100}$$

$$w_1(z) = 1 - \frac{z^{3/2} L_1^{(\alpha)}(z^2/2)}{2L_1^{(\alpha)}(z^2/2) - 2\alpha - 1}, \tag{101}$$

where $\alpha = -(2\ell + 1)/2$. Another hierarchy of solutions is the Hermite polynomial hierarchy, two of its member being:

$$w_1(z) = 1 - \frac{z^{3/2} H_{2n}(z)}{(z^2+1)H_{2n}(z) - 4nzH_{2n-1}(z)}, \tag{102}$$

$$w_1(z) = 1 + \frac{z^{1/2} H_{2n}(z)}{4nH_{2n-1}(z) - zH_{2n}(z)}. \tag{103}$$

It is clear that more hierarchies can be identified; a deeper and more detailed study of them can be found in [34].

5. Conclusions

We have shown that systems ruled by polynomial Heisenberg algebras are interesting from a physical as well as from a mathematical point of view. The ones having a second-order polynomial Heisenberg algebra as their intrinsic algebraic structure are connected with the PIV equation, while those with a third-order polynomial Heisenberg algebra are linked with the PV equation. In addition, by applying supersymmetric quantum mechanics to the harmonic and radial oscillators we have found the simplest non-trivial examples realizing such algebras. Moreover, we have introduced a recipe for generating solutions to the PIV and PV equations, which employs the SUSY partners of the harmonic and radial oscillators, respectively.

As a final remark, let us point out that we have achieved our initial goal of combining successfully polynomial Heisenberg algebras, supersymmetric quantum mechanics and Painlevé equations, three subjects of mathematical physics which from now on cannot be seen as independent anymore.

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David J. Fernández C.
Physics Department, Cinvestav
P.O. Box 14-740
07000 México D.F., Mexico
e-mail: david@fis.cinvestav.mx

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Change in Energy Eigenvalues Against Parameters

Toshihiro Iwai and Boris Zhilinskii

Abstract. A topological characterization of energy-band rearrangements against parameters for molecular problems with slow/fast variables comes around to a study of a Dirac equation with a parameter. In this article, the Dirac equation of space-dimension two is studied under both the APS (an abbreviation of Atiyah–Patodi–Singer) and the chiral bag boundary conditions, where the mass is viewed as a parameter ranging over all real numbers. The APS boundary condition requires that eigenstates evaluated on the boundary should belong to the subspace of eigenstates associated with positive or negative eigenvalues for a boundary operator, and the chiral bag boundary condition requires that eigenstates evaluated on the boundary have chiral components related by a unitary operator. The spectral flow for a one-parameter family of operators is the net number of eigenvalues passing through zeros in the positive direction as the parameter runs. It is shown that the spectral flow for the Dirac equation with the APS boundary condition is ± 1, depending on the sign of the total angular momentum eigenvalue. A counterpart of the spectral flow in the case of the chiral bag boundary condition is treated as an extension of spectral flow. In addition, discrete symmetry is discussed to explain the pattern of eigenvalues as functions of the parameter.

Mathematics Subject Classification (2010). 81Qxx, 81Q05, 35Q41.

Keywords. Energy band, Dirac equation, APS boundary condition, chiral bag boundary condition, spectral flow, winding number.

1. Setting up

There is a class of band structures of molecular spectra, in which energy excitations can be separated into low and high ones. For example, the energy of typical rotational excitation is much smaller than the typical vibrational excitation (see Fig. 1). The low and the high excited levels form high density states and a small number of isolated states, respectively. Accordingly, the whole dynamical variables

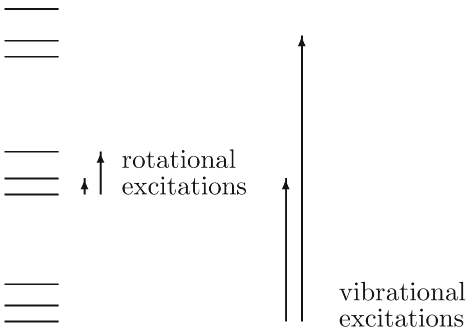

FIGURE 1. A characteristic pattern of energy levels for a molecular problem with one slow (rotational) and one fast (vibrational) degrees of freedom.

are separated into slow and fast variables in such a way that the slow variables are for describing high density states and the fast ones for a small number of isolated states.

We give a simple model Hamiltonian consisting of slow and fast variables of different nature. Let J_k and S_k be generators of $SU(2)$, which are taken as describing the orbital and the spin variables, respectively. We define a one-parameter family of Hamiltonian operators to be

$$\hat{H}_\tau = (1-\tau)\mathbb{1} \otimes (-S_3) + \tau \sum_{k=1}^{3} J_k \otimes S_k, \quad 0 \le \tau \le 1. \tag{1}$$

For the representation parameters $j = 1, s = \frac{1}{2}$, the $\hat{H}_\tau$ takes the form of 6×6 Hermitian matrix. The eigenvalues are easily found as functions of the parameter τ, which exhibit the band rearrangement against τ, as is shown in Figure 2.

If j is sufficiently large, the J_k can be taken as slow variables and treated as classical ones but the fast variables S_k remain to be quantum ones. The treatment of slow and fast variables as classical and quantum variables, respectively, is called a semi-quantum model.

On the assumption that the J_k can be treated as classical variables [2], the operator $\hat{H}_\tau$ with J_k replaced by x_k is converted into

$$H_\tau(\boldsymbol{x}) = \frac{1-\tau}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\tau}{2}\begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \quad \boldsymbol{x} \in S^2 \subset \mathbb{R}^3, \tag{2}$$

where $\boldsymbol{x} = (x_k)$ has been restricted to the unit sphere S^2 by the normalization J_k/J due to the conservation of the angular momentum. The eigenvalues of $H_\tau(\boldsymbol{x})$ are

$$\lambda^{\pm}(\tau, \boldsymbol{x}) = \pm\sqrt{-\frac{1}{4} + \frac{\tau(1-\tau)}{2}(1+x_3)}, \quad |\boldsymbol{x}| = 1, \tag{3}$$

which are degenerate if and only if $x_3 = 1$ (or $\boldsymbol{x} = \boldsymbol{e}_3 = (0,0,1)$) and $\tau = \frac{1}{2}$.

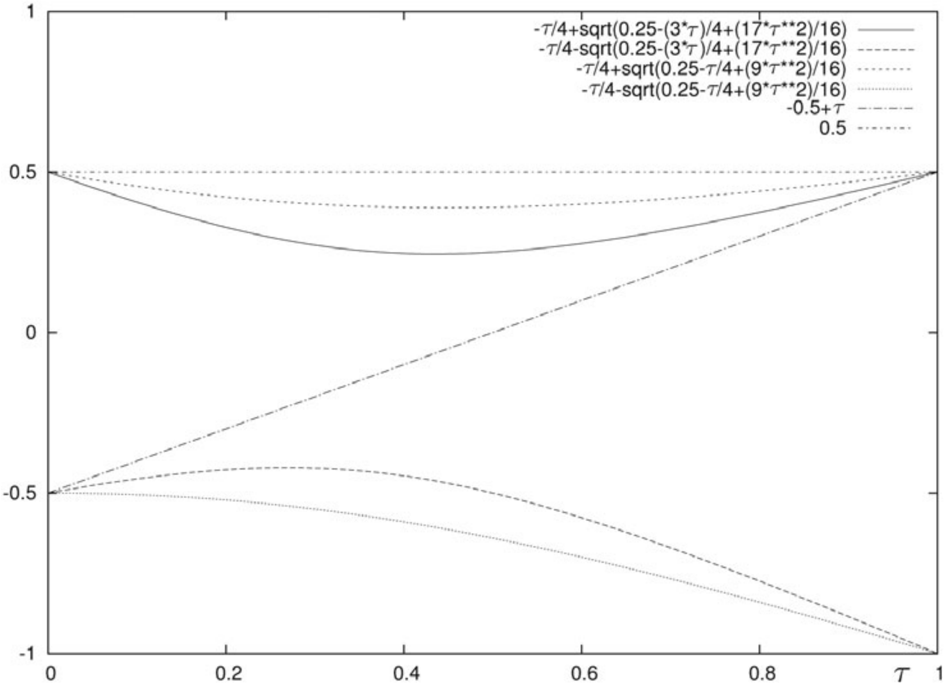

FIGURE 2. A redistribution of eigenvalues for $\hat{H}_\tau$ against the parameter τ.

For each of eigenvalues $\lambda^{\pm}(\tau, \boldsymbol{x})$, the associated eigenspace is attached at $\boldsymbol{x} \in S^2$, and the totality of such eigenspaces forms a complex line bundle over S^2, which we denote by $L^{\pm}(\tau)$, respectively, and call the eigen-line bundles. As long as the eigenvalues are not degenerate, we have the direct sum of eigen-line bundles, $L^{+}(\tau) \oplus L^{-}(\tau)$. When $\tau = 0$, both of the eigen-line bundles are trivial; $L^{\pm}(0) = S^2 \times \mathbb{C}$. When the parameter passes the value $\tau = \frac{1}{2}$, the direct sum of the eigen-line bundles fails, since the eigenvalues are degenerate at $\boldsymbol{x} = \boldsymbol{e}_3$ for $\tau = \frac{1}{2}$. This means that accompanying the variation in the parameter τ, the eigen-line bundles topologically change. This change can be detected by using the first Chern number assigned to each of $L^{\pm}(\tau)$. For $\tau = 1$, the Hamiltonian is expressed as

$$H_1(\boldsymbol{x}) = \frac{1}{2}\begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \quad \boldsymbol{x} \in S^2 \subset \mathbb{R}^3, \tag{4}$$

and the first Chern numbers of $L^{\pm}(1)$ are easily calculated as

$$c_1(L^{\pm}(1)) = \frac{i}{2\pi}\int_{S^2} F^{\pm} = \mp 1, \tag{5}$$

where $F^{\pm}$ denote the curvature forms assigned to $L^{\pm}(1)$, respectively. Since the Chern number is integer-valued and depends continuously on the parameter τ, it is constant in τ except for $\tau = \frac{1}{2}$. Thus, the modification of band structure against the parameter τ shown in Figure 2 finds a counterpart in the corresponding semi-quantum model, in the form of a piece-wise constant behavior of Chern numbers shown in Figure 3.

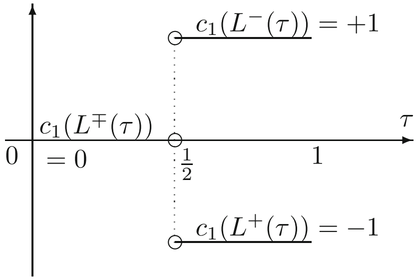

FIGURE 3. Change in the Chern numbers against τ for the semi-quantum model (2).

We give here, after [3], a more complicated semi-quantum Hamiltonian than (2),

$$H(\boldsymbol{x}) = \begin{pmatrix} X & Y - iZ \\ Y + iZ & -X \end{pmatrix}, \quad \boldsymbol{x} \in S^2 \subset \mathbb{R}^3, \tag{6}$$

where

$$X(\boldsymbol{x}) = b_1(y^2 - x^2) + b_2 zy, \tag{7a}$$

$$Y(\boldsymbol{x}) = 2b_1 yx - b_2 zx, \tag{7b}$$

$$Z(\boldsymbol{x}) = a_1 z + a_2 y(y^2 - 3x^2), \tag{7c}$$

and (a_1, a_2, b_1, b_2) are real constants with the assumption that $(a_1, a_2) \neq (0, 0)$ and $(b_1, b_2) \neq (0, 0)$. We note that this Hamiltonian admits D_3 symmetry,

$$D^E(g)H(\boldsymbol{x})D^E(g)^{-1} = H(D^{E\oplus A_2}(g)\boldsymbol{x}), \quad g \in D_3, \tag{8}$$

where D_3 is a discrete subgroup of $SO(3)$ and where E and A_2 denote a two-dimensional and one-dimensional not totally-symmetric representations of D_3, respectively. The action of D_3 on the sphere, denoted by the symbol $D^{E\oplus A_2}$, is illustrated in Figure 4. The z-axis is the C_3 symmetry axis. Three C_2 symmetry axes belong to the xy-plane. Two intersection points of the C_3 symmetry axis and the sphere form the two-point orbit with C_3 stabilizer. Six points of intersection of three C_2 symmetry axes with the sphere form two three-point orbits with stabilizer C_2.

After [3], we describe the Chern numbers of the eigen-line bundles for the Hamiltonian (6). Owing to the invariance of the Chern numbers with respect to the scaling of the parameters (a_1, a_2, b_1, b_2), the parameter space $(\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 -$

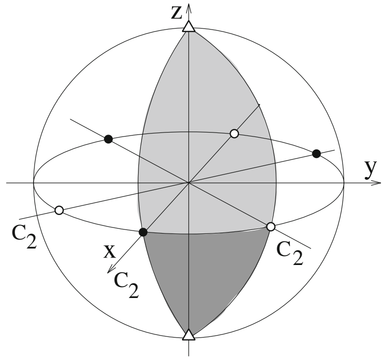

FIGURE 4. D_3 group and its action on the sphere.

$\{0\}$) reduces to the two-torus T^2 described as $a_1 = \cos\phi_1, a_2 = \sin\phi_1$ and $b_1 = \cos\phi_2, b_2 = \sin\phi_2$. The reduced parameter space is divided into a certain number of connected regions to which respective fixed Chern numbers are assigned, and such regions are called iso-Chern domains. The parameter space with such partition and Chern numbers is called the iso-Chern diagram. The iso-Chern diagram for the eigen-line bundle associated with the positive eigenvalue is shown in Figure 5. The red and blue lines ($\phi_1 = \pm\frac{\pi}{2}, \phi_2 = \pm\frac{\pi}{2}$) and black curves ($\cos\phi_1\cos\phi_2 = \sin\phi_1\sin^3\phi_2$) are the sets of degeneracy points in the reduced parameter space T^2.

The iso-Chern diagram for the eigen-line bundle associated with negative eigenvalue is obtained by opposing the sign of the Chern number assigned to each iso-Chern domain.

In view of Figure 5, we observe that when we move from an iso-Chern domain to an adjacent one, passing the boundary between them, the change in the Chern

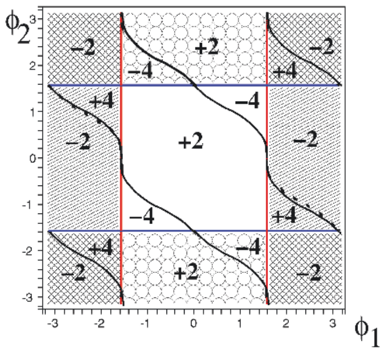

FIGURE 5. The iso-Chern diagram for the eigen-line bundle associated with the positive eigenvalue of the D_3 invariant Hamiltonian (6)

number, which we call a delta-Chern, is one of the four values, $\pm 2, \pm 6$. The numbers 2 and 6 are the orders of D_3 orbits with stabilizers C_3 and C_2, respectively. In fact, we can show that degeneracy points on S^2 form D_3 orbits and that the delta-Chern is given by "± 1 times the order of the orbit in question [5]". The number $+1$ or -1 originally comes from a winding number, and is called a local delta-Chern, which has already appeared in Figure 3. In the case of Hamiltonian (2), the number of degeneracy points is one and the initial bundle $L^{\pm}(\tau)$ with $\tau < \frac{1}{2}$ is trivial, so that the delta-Chern and the local delta-Chern coincide.

The local delta-Chern can be evaluated through a linearization method at the degeneracy point in question. Let $(\tau_0, \boldsymbol{x}_0)$ be a degeneracy point, where τ_0 is a parameter value at which a path in the parameter space crosses the boundary between adjacent iso-Chern domains and where $\boldsymbol{x}_0$ is a degeneracy point on the sphere at which the two eigenvalues are degenerate for τ_0. Then, the Hamiltonian can be homotopically deformed to the linearized Hamiltonian $H_{\mathrm{loc}}(t, p; \tau_0, \boldsymbol{x}_0)$ in the neighborhood of $(\tau_0, \boldsymbol{x}_0)$ by means of deleting higher-order terms in (t, p_1, p_2), and hence the winding number attached to the degeneracy point can be evaluated by using $H_{\mathrm{loc}}(t, p; \tau_0, \boldsymbol{x}_0)$ to obtain the local delta-Chern. This idea is mentioned not in [3] but in [6]. For the semi-quantum Hamiltonian (2), the linearized Hamiltonian at $(\tau, \boldsymbol{x}) = (\frac{1}{2}, \boldsymbol{x}_0)$ is given by

$$\begin{aligned} H_{\mathrm{loc}}(t, q; \frac{1}{2}, \boldsymbol{x}_0) &= t\dot{H}_{\frac{1}{2}}(\boldsymbol{x}_0) + p_1 \nabla H_{\frac{1}{2}}(\boldsymbol{x}_0) \cdot \boldsymbol{e}_1 + p_2 \nabla H_{\frac{1}{2}}(\boldsymbol{x}_0) \cdot \boldsymbol{e}_2 \\ &= \frac{1}{4} \begin{pmatrix} 4t & p_1 - ip_2 \\ p_1 + ip_2 & -4t \end{pmatrix}, \end{aligned} \tag{9}$$

where $\boldsymbol{e}_k$ are the standard basis vectors with $\boldsymbol{x}_0 = \boldsymbol{e}_3$, and where $\boldsymbol{e}_1, \boldsymbol{e}_2$ are viewed as tangent vectors to S^2 at $\boldsymbol{x}_0$.

We are interested in what corresponds to the delta-Chern, in full quantum description. To this end, we consider a full quantum Hamiltonian corresponding to a linearized semi-quantum Hamiltonian. For notational simplicity, we take up the simple semi-quantum Hamiltonian, in place of (9),

$$H(t, p) = \begin{pmatrix} t & p_1 - ip_2 \\ p_1 + ip_2 & -t \end{pmatrix}. \tag{10}$$

Replacing p_k by $-i\partial/\partial q_k$, we obtain the corresponding full quantum Hamiltonian expressed as

$$\hat{H}_t = \begin{pmatrix} t & -i\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2} \\ -i\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} & -t \end{pmatrix} = -i \sum_{k=1}^{2} \sigma_k \frac{\partial}{\partial q_k} + t\sigma_3, \tag{11}$$

where σ_k are the Pauli matrices. Thus we come to a Dirac operator $\hat{H}_t$.

2. The Dirac equation on a bounded domain

A Dirac operator on $\mathbb{R}^d$ is given by

$$H = -i \sum_{k=1}^{d} \gamma_k \nabla_k + \mu \gamma_{d+1}, \quad \nabla_k = \partial/\partial x_k, \tag{12}$$

where μ is a mass parameter which is assumed to take all real values in this article, and where γ_k are the gamma matrices satisfying

$$\begin{aligned} \gamma_k \gamma_j + \gamma_j \gamma_k &= 2\delta_{jk} I, \quad j,k = 1, \ldots, d, \\ \gamma_k \gamma_{d+1} + \gamma_{d+1} \gamma_k &= 0, \\ (\gamma_{d+1})^2 &= I, \\ (\gamma_\nu)^\dagger &= \gamma_\nu, \qquad \nu = 1, \ldots, d, d+1, \end{aligned}$$

with I denoting the identity matrix of the same size as γ_k. In the present article, our interest centers on the case of $d = 2$, and the gamma matrices are realized as the Pauli matrices, $\gamma_\nu = \sigma_\nu$, $\nu = 1, 2, 3$ (see (11)).

To pose a boundary condition, we need Green's formula [1]. Let V and S denote a bounded domain in $\mathbb{R}^d$ and its boundary, respectively. Green's formula for the Dirac operator H is given by

$$\langle \Phi, H\Psi \rangle_V - \langle H\Phi, \Psi \rangle_V = -i \langle \phi, \vec{\gamma} \cdot \vec{n}\, \psi \rangle_S, \tag{13}$$

where $\phi = \Phi|_S, \psi = \Psi|_S$ and $\vec{\gamma} \cdot \vec{n} = \sum \gamma_j n_j$ with $\vec{n}$ being the outward unit normal to S.

Any boundary condition for the Dirac equation $H\Phi = E\Phi$ should require the vanishing of the right-hand side of the above equation. If such a boundary condition is adopted, the operator H becomes a symmetric operator. Furthermore, with some Sobolev conditions, it becomes self-adjoint.

In what follows, we give two boundary conditions, the APS and the chiral bag boundary conditions. The APS boundary condition is given as follows: If we can find a self-adjoint boundary operator B on S such that B has no zero eigenvalue, we obtain the decomposition of the Hilbert space $\mathcal{H}(S)$ into

$$\mathcal{H}(S) = \mathcal{H}^{(+)}(S) \oplus \mathcal{H}^{(-)}(S), \tag{14}$$

where $\mathcal{H}^{(\pm)}(S)$ are subspaces such that $B|_{\mathcal{H}^{(+)}(S)} > 0$ and $B|_{\mathcal{H}^{(-)}(S)} < 0$. We assume further that

$$(\vec{\gamma} \cdot \vec{n}) \mathcal{H}^{(\pm)}(S) = \mathcal{H}^{(\mp)}(S). \tag{15}$$

The APS boundary condition requires that eigenstates evaluated on the boundary should belong to $\mathcal{H}^{(+)}(S)$ or $\mathcal{H}^{(-)}(S)$.

To describe the chiral bag boundary condition, we decompose spinors into the sum of chiral components,

$$\Phi = \Phi_+ + \Phi_-, \quad \Phi_\pm := \frac{1}{2}(I \pm \vec{\gamma} \cdot \vec{n})\Phi. \tag{16}$$

The components $\Phi_\pm$ belong to the eigenspaces associated with the eigenvalues ± 1 of $\vec{\gamma}\cdot\vec{n}$, respectively, and those eigenspaces are orthogonal to each other, so that any chiral components, $\Phi_\pm$ and $\Psi_\pm$, satisfy

$$\vec{\gamma}\cdot\vec{n}\,\Phi_+ = \Phi_+, \quad \vec{\gamma}\cdot\vec{n}\,\Phi_- = -\Phi_-, \quad \langle \Psi_+, \Phi_- \rangle = 0. \tag{17}$$

Then, the right-hand side of Green's formula is brought into

$$-i\langle \phi, \vec{\gamma}\cdot\vec{n}\,\psi\rangle_S = -i\langle \phi_+, \psi_+\rangle_S + i\langle \phi_-, \psi_-\rangle_S. \tag{18}$$

If the chiral components $\psi_\pm$ of $\psi = \Psi|_S$ are related by

$$\psi_- = U\gamma_{d+1}\psi_+, \tag{19}$$

where U is any unitary operator acting on spinors defined on the boundary and further commutes with $\vec{\gamma}\cdot\vec{n}$, then those components satisfy $\langle \phi_-, \psi_-\rangle_S = \langle \phi_+, \psi_+\rangle_S$, so that the boundary integral vanishes. The above equation is called the chiral bag boundary condition.

From a physical point of view, we have to consider currents on the boundary. The continuity equation of the current and the density is described as

$$\frac{\partial}{\partial \tau}(\Psi^\dagger\Psi) + \sum_{k=1}^{d} \frac{\partial}{\partial x_k}(\Psi^\dagger\gamma_k\Psi) = 0, \tag{20}$$

where τ denotes the time parameter in this equation only. The transverse component of the current vector $\vec{J} = (\Psi^\dagger\gamma_k\Psi)$, which is given by

$$\Psi^\dagger(\vec{\gamma}\cdot\vec{n})\Psi, \tag{21}$$

should vanish on the boundary S in time-independent models.

3. Feasible solutions to the 2D Dirac equation

Before solving the Dirac equation for the Hamiltonian $\hat{H}_t$ given in (11), we have to mention the $U(1)$ symmetry of $\hat{H}_t$. Let

$$D(e^{i\tau}) := \begin{pmatrix} e^{-i\tau/2} & 0 \\ 0 & e^{i\tau/2} \end{pmatrix}, \quad R(\tau) := \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}, \quad \tau \in \mathbb{R}. \tag{22}$$

Then, the $U(1)$ action U_τ on the two-component spinor Φ on $\mathbb{R}^2$ is defined to be

$$U_\tau\Phi = D(e^{i\tau})\Phi \circ R(-\tau). \tag{23}$$

As is straightforwardly verified, the $\hat{H}_t$ admits the $U(1)$ symmetry,

$$U_\tau \hat{H}_t U_\tau^{-1} = \hat{H}_t. \tag{24}$$

The infinitesimal generator $\hat{J}$ of U_τ, which is defined through $U_\tau = \exp(-i\tau\hat{J})$, is called the (spin-orbital) angular momentum operator. By differentiation of U_τ with respect to τ at $\tau = 0$, we obtain

$$\hat{J} = \frac{1}{2}\sigma_3 + i\mathbb{1}\Big(q_2\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial q_2}\Big) = \frac{1}{2}\sigma_3 - i\mathbb{1}\frac{\partial}{\partial \theta}, \tag{25}$$

where (r, θ) are the polar coordinates. The differentiation of (24) with respect to τ at $\tau = 0$ yields

$$[\hat{J}, \hat{H}_t] = 0. \tag{26}$$

The Hamiltonian (11) is expressed in the polar coordinates as

$$\hat{H}_t = -i\sigma_r \frac{\partial}{\partial r} - \frac{i}{r}\sigma_\theta \frac{\partial}{\partial \theta} + t\sigma_3, \tag{27}$$

where

$$\sigma_r = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \quad \sigma_\theta = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}. \tag{28}$$

We now apply the separation of variables method in the polar coordinates. We start with the eigenvalue equation $J\Phi = j\Phi$, which is solved by

$$\Phi_j(r, \theta) = \begin{pmatrix} e^{i(j-\frac{1}{2})\theta}\phi_j^{(-)}(r) \\ e^{i(j+\frac{1}{2})\theta}\phi_j^{(+)}(r) \end{pmatrix}, \quad j \in \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \dots\right\}, \tag{29}$$

where $\phi_j^{(\pm)}(r)$ are unknown radial functions. The Dirac equation $\hat{H}_t\Phi = E\Phi$ then reduces to $\hat{H}_t\Phi_j = E_j\Phi_j$, which gives for radial functions $\phi_j^{(\pm)}(r)$

$$-i\frac{d\phi_j^{(+)}}{dr} - \frac{i}{r}(j + \frac{1}{2})\phi_j^{(+)} + t\phi_j^{(-)} = E_j\phi_j^{(-)}, \tag{30a}$$

$$-i\frac{d\phi_j^{(-)}}{dr} + \frac{i}{r}(j - \frac{1}{2})\phi_j^{(-)} - t\phi_j^{(+)} = E_j\phi_j^{(+)}. \tag{30b}$$

These equations are put together to give rise to a second-order differential equation. According as $|E_j| > |t|$ or $|E_j| < |t|$, the differential equation in question is the Bessel equation or the modified Bessel equation.

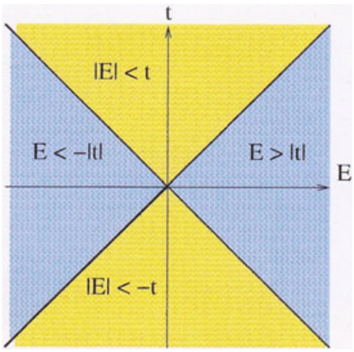

FIGURE 6. The (E, t)-parameter space is divided into four regions with different solutions to (30).

To each of four regions shown in Figure 6, assigned is a type of feasible solution:

(i) Feasible solutions with $|E_j| > |t|$:

$$\Phi_j(r,\theta) = c\begin{pmatrix} \sqrt{E_j + t}e^{i(j-\frac{1}{2})\theta}J_{j-\frac{1}{2}}(\beta_j r) \\ i\sqrt{E_j - t}e^{i(j+\frac{1}{2})\theta}J_{j+\frac{1}{2}}(\beta_j r) \end{pmatrix} \quad \text{for} \quad E_j > 0, \tag{31a}$$

$$\Phi_j(r,\theta) = c'\begin{pmatrix} \sqrt{|E_j + t|}e^{i(j-\frac{1}{2})\theta}J_{j-\frac{1}{2}}(\beta_j r) \\ -i\sqrt{|E_j - t|}e^{i(j+\frac{1}{2})\theta}J_{j+\frac{1}{2}}(\beta_j r) \end{pmatrix} \quad \text{for} \quad E_j < 0, \tag{31b}$$

where $\beta_j = \sqrt{E_j^2 - t^2}$ and where c and c' are complex constants.

(ii) Feasible solutions with $|E_j| < |t|$:

$$\Phi_j(r,\theta) = c\begin{pmatrix} \sqrt{t + E_j}e^{i(j-\frac{1}{2})\theta}I_{j-\frac{1}{2}}(\varepsilon_j r) \\ -i\sqrt{t - E_j}e^{i(j+\frac{1}{2})\theta}I_{j+\frac{1}{2}}(\varepsilon_j r) \end{pmatrix} \quad \text{for} \quad t > 0, \tag{32a}$$

$$\Phi_j(r,\theta) = c'\begin{pmatrix} \sqrt{|t + E_j|}e^{i(j-\frac{1}{2})\theta}I_{j-\frac{1}{2}}(\varepsilon_j r) \\ i\sqrt{|t - E_j|}e^{i(j+\frac{1}{2})\theta}I_{j+\frac{1}{2}}(\varepsilon_j r) \end{pmatrix} \quad \text{for} \quad t < 0, \tag{32b}$$

where $\varepsilon_j = \sqrt{t^2 - E_j^2}$ and where c and c' are complex constants.

In the limit as $t \to 0$ within the constraint $|E_j| < |t|$, one has $E_j = t = 0$.

(iii) Feasible solutions with $E_j = t = 0$:

$$\Phi_j(r,\theta) = c\begin{pmatrix} 0 \\ e^{i(j+\frac{1}{2})\theta}r^{-(j+\frac{1}{2})} \end{pmatrix} \quad \text{for} \quad j < 0, \tag{33a}$$

$$\Phi_j(r,\theta) = c'\begin{pmatrix} e^{i(j-\frac{1}{2})\theta}r^{j-\frac{1}{2}} \\ 0 \end{pmatrix} \quad \text{for} \quad j > 0. \tag{33b}$$

In terms of $z = re^{i\theta}$, these solutions are expressed as

$$c\begin{pmatrix} 0 \\ \overline{z}^{|j|-\frac{1}{2}} \end{pmatrix} \quad \text{for} \quad j < 0, \quad \text{and} \quad c'\begin{pmatrix} z^{j-\frac{1}{2}} \\ 0 \end{pmatrix} \quad \text{for} \quad j > 0, \tag{34}$$

respectively, where $|j| - \frac{1}{2}$ and $j - \frac{1}{2}$ are non-negative integers.

4. The APS boundary condition

Let A_t be the restriction of $\hat{H}_t$ to the circle $r = R$. The boundary operator B_t is then defined to be and expressed as

$$B_t = i\sigma_r A_t = \begin{pmatrix} \frac{i}{R}\frac{\partial}{\partial\theta} & -ite^{-i\theta} \\ ite^{i\theta} & -\frac{i}{R}\frac{\partial}{\partial\theta} \end{pmatrix}, \tag{35}$$

where $t \neq 0$. The case of $t = 0$ will be treated separately. Further, we note that

$$\sigma_r B_t + B_t \sigma_r = \frac{1}{R}\sigma_r. \tag{36}$$

Eigenvalues and associated eigenstates of B_t are easily obtained as follows:

$$\phi_j^{(-)}(\theta) = c'_j \begin{pmatrix} -it\, e^{i(j-\frac{1}{2})\theta} \\ (\frac{j}{R} + \lambda_j^-) e^{i(j+\frac{1}{2})\theta} \end{pmatrix} \quad \text{for} \quad \kappa_j^- := \frac{1}{2R} + \lambda_j^- < 0, \tag{37a}$$

$$\phi_j^{(+)}(\theta) = c_j \begin{pmatrix} -it\, e^{i(j-\frac{1}{2})\theta} \\ (\frac{j}{R} + \lambda_j^+) e^{i(j+\frac{1}{2})\theta} \end{pmatrix} \quad \text{for} \quad \kappa_j^+ := \frac{1}{2R} + \lambda_j^+ > 0, \tag{37b}$$

where

$$\lambda_j^{\pm} = \pm\sqrt{\frac{j^2}{R^2} + t^2}, \quad t \neq 0. \tag{38}$$

Let D_R^2 and ∂D_R^2 denote the 2-disk of radius R and its boundary, respectively. Define

$$\mathcal{H}^{(\pm)}(\partial D_R^2) = \text{span}\Big\{\phi_j^{(\pm)},\, j \in \Big\{\pm\frac{1}{2}, \pm\frac{2}{3}, \dots\Big\}\Big\}. \tag{39}$$

Then, the Hilbert space $\mathcal{H}(\partial D_R^2)$ attached to ∂D_R^2 is decomposed into

$$\mathcal{H}(\partial D_R^2) = \mathcal{H}^{(+)}(\partial D_R^2) \oplus \mathcal{H}^{(-)}(\partial D_R^2), \tag{40}$$

where

$$\mathcal{H}^{(+)}(\partial D_R^2) \perp \mathcal{H}^{(-)}(\partial D_R^2), \quad \sigma_r \mathcal{H}^{(\mp)}(\partial D_R^2) = \mathcal{H}^{(\pm)}(\partial D_R^2). \tag{41}$$

The APS boundary condition for $t \neq 0$ is now described as

$$\Phi_j(R,\theta) \in \mathcal{H}^{(-)}(\partial D_R^2) \quad \text{or} \quad \Phi_j(R,\theta) \in \mathcal{H}^{(+)}(\partial D_R^2). \tag{42}$$

In what follows, we list functional equations to determine eigenvalues [6].

(i) Edge state eigenvalues with $\Phi_j(R,\theta) \in \mathcal{H}^{(-)}(\partial D_R^2)$ are determined by the functional equations

$$t\sqrt{\frac{t+E_j}{t-E_j}} I_{j-\frac{1}{2}}(\varepsilon_j R) = \Big(\frac{j}{R} + \sqrt{\frac{j^2}{R^2} + t^2}\Big) I_{j+\frac{1}{2}}(\varepsilon_j R), \quad \text{for} \quad t > 0, \tag{43a}$$

$$|t|\sqrt{\frac{|t+E_j|}{|t-E_j|}} I_{j-\frac{1}{2}}(\varepsilon_j R) = \Big(\frac{j}{R} + \sqrt{\frac{j^2}{R^2} + t^2}\Big) I_{j+\frac{1}{2}}(\varepsilon_j R), \quad \text{for} \quad t < 0. \tag{43b}$$

These equations can be solved numerically to provide edge state eigenvalues as functions of t, as is shown in Figure 7.

(ii) There exist no edge state eigenvalues with $\Phi_j(R,\theta) \in \mathcal{H}^{(+)}(\partial D_R^2)$.

(iii) Regular state eigenvalues with $\Phi_j(R,\theta) \in \mathcal{H}^{(-)}(\partial D_R^2)$ are determined by the functional equations

$$-t\sqrt{\frac{E_j+t}{E_j-t}} J_{j-\frac{1}{2}}(\beta_j R) = \Big(\frac{j}{R} + \sqrt{\frac{j^2}{R^2} + t^2}\Big) J_{j+\frac{1}{2}}(\beta_j R) \quad \text{for} \quad E_j > 0, \tag{44a}$$

$$t\sqrt{\frac{|E_j+t|}{|E_j-t|}} J_{j-\frac{1}{2}}(\beta_j R) = \Big(\frac{j}{R} + \sqrt{\frac{j^2}{R^2} + t^2}\Big) J_{j+\frac{1}{2}}(\beta_j R) \quad \text{for} \quad E_j < 0. \tag{44b}$$

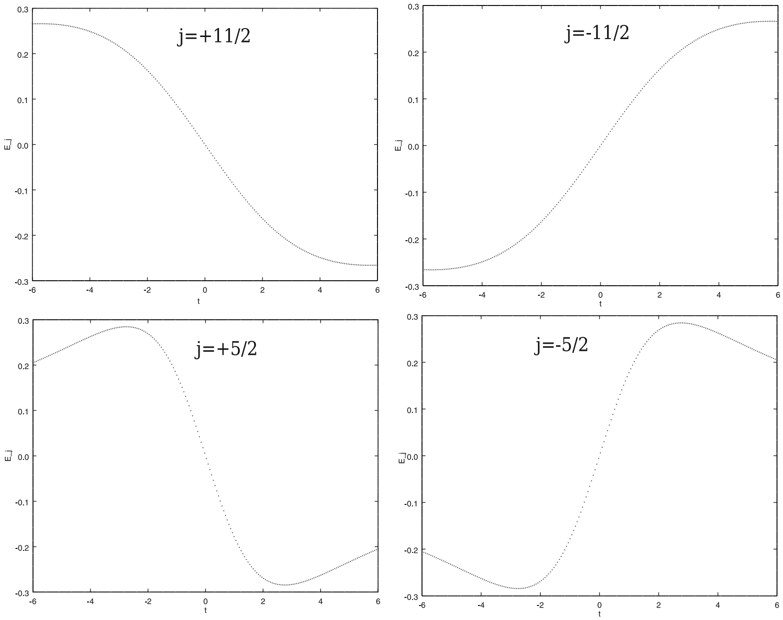

FIGURE 7. Edge state eigenvalues as functions of t. Left panels are for $j = 11/2$, $j = 5/2$, and right panels for $j = -11/2$ and $j = -5/2$.

(iv) Regular state eigenvalues with $\Phi_j(R,\theta) \in \mathcal{H}^{(+)}(\partial D_R^2)$ are determined by the functional equations

$$-t\sqrt{\frac{|E_j+t|}{|E_j-t|}}J_{j-\frac{1}{2}}(\beta_j R) = \Big(\frac{j}{R} - \sqrt{\frac{j^2}{R^2}+t^2}\Big)J_{j+\frac{1}{2}}(\beta_j R) \quad \text{for} \quad E_j > 0, \tag{45a}$$

$$t\sqrt{\frac{|E_j+t|}{|E_j-t|}}J_{j-\frac{1}{2}}(\beta_j R) = \Big(\frac{j}{R} - \sqrt{\frac{j^2}{R^2}+t^2}\Big)J_{j+\frac{1}{2}}(\beta_j R) \quad \text{for} \quad E_j < 0. \tag{45b}$$

Equations (44) and (45) are numerically solved to give regular and edge state eigenvalues as functions of t, respectively, as is shown in Figure 8.

We turn to the case of $t = 0$. To pose the APS boundary condition for $t = 0$, we find eigenstates of the boundary operator B_0. From (35) with $t = 0$, the eigenvalues and associated eigenstates for B_0 prove to be given by

$$B_0\phi_j^{(0,+)} = \frac{1}{R}(j+\frac{1}{2})\phi_j^{(0,+)}, \qquad \phi_j^{(0,+)} = \begin{pmatrix} 0 \\ a_j e^{i(j+\frac{1}{2})\theta} \end{pmatrix}, \tag{46a}$$

$$B_0\phi_j^{(0,-)} = -\frac{1}{R}(j-\frac{1}{2})\phi_j^{(0,-)}, \quad \phi_j^{(0,-)} = \begin{pmatrix} b_j e^{i(j-\frac{1}{2})\theta} \\ 0 \end{pmatrix}. \tag{46b}$$

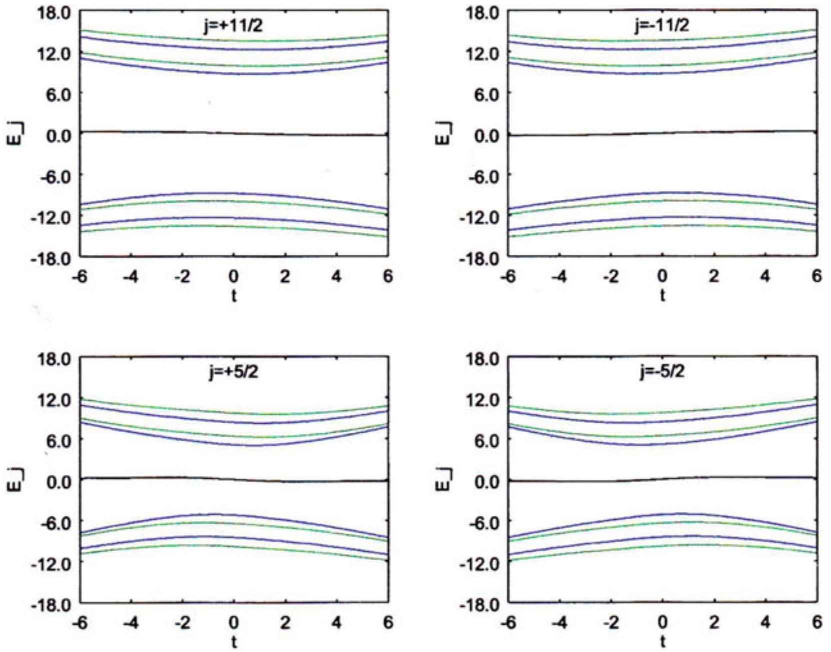

FIGURE 8. Regular state eigenvalues as functions of t under the APS boundary condition. Green lines are eigenvalues with $\Phi_j(R,\theta) \in \mathcal{H}^{(-)}(\partial D_R^2)$. Blue lines are eigenvalues with $\Phi_j(R,\theta) \in \mathcal{H}^{(+)}(\partial D_R^2)$. Black lines are edge state eigenvalues.

Now we are in a position to state the APS boundary condition for $t = 0$. Define

$$\mathcal{H}_0^{(\pm)}(\partial D_R^2) = \text{span}\left\{\phi_j^{(0,\pm)};\, j = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots\right\}. \tag{47}$$

Then, we obtain the decomposition

$$\mathcal{H}(\partial D_R^2) = \mathcal{H}_0^{(+)}(\partial D_R^2) \oplus \mathcal{H}_0^{(-)}(\partial D_R^2), \tag{48}$$

where

$$\mathcal{H}_0^{(+)}(\partial D_R^2) \perp \mathcal{H}_0^{(-)}(\partial D_R^2), \quad \sigma_r \mathcal{H}_0^{(\pm)}(\partial D_R^2) = \mathcal{H}_0^{(\mp)}(\partial D_R^2). \tag{49}$$

In spite of the superscripts $(\pm)$, both $\mathcal{H}_0^{(\pm)}(\partial D_R^2)$ have eigenstates associated with negative, zero, and positive eigenvalues of B_0.

The APS boundary condition for $t = 0$ is expressed as

$$\Phi_j(R,\theta) \in \mathcal{H}_0^{(-)}(\partial D_R^2) \quad \text{or} \quad \Phi_j(R,\theta) \in \mathcal{H}_0^{(+)}(\partial D_R^2). \tag{50}$$

The solutions given in (33) are shown to satisfy the APS boundary condition

$$\Phi_j(R,\theta) \in \mathcal{H}_0^{(+)}(\partial D_R^2) \quad \text{for} \quad j<0, \tag{51a}$$

$$\Phi_j(R,\theta) \in \mathcal{H}_0^{(-)}(\partial D_R^2) \quad \text{for} \quad j>0. \tag{51b}$$

Eigenstates associated with zero eigenvalue are called zero modes.

We now show that the zero modes are indeed linked with edge eigenstates when the parameter t reaches the zero value. To this end, we introduce the power series $I_\nu^P(z)$ through

$$I_\nu(z) = \Big(\frac{z}{2}\Big)^\nu I_\nu^P(z), \quad I_\nu^P(z) = \sum_{n=0}^\infty \frac{1}{n!\Gamma(\nu+n+1)}\Big(\frac{z}{2}\Big)^{2n}. \tag{52}$$

Using $I^P_{j+\frac{1}{2}}(z)$ with $j>0$ and choosing suitable constant factors, we can modify (32) with prescribed ε_j into the edge eigenstates of the form

$$\tilde{\Phi}^{(+)}_{\text{edg}} = \begin{pmatrix} e^{i(j-\frac{1}{2})\theta} r^{j-\frac{1}{2}} I^P_{j+\frac{1}{2}}(\varepsilon_j r) \\ -i\frac{t-E_j}{2} r^{j+\frac{1}{2}} e^{i(j+\frac{1}{2})\theta} I^P_{j+\frac{1}{2}}(\varepsilon_j r) \end{pmatrix} \quad \text{for} \quad t>0, \tag{53a}$$

$$\tilde{\Phi}^{(-)}_{\text{edg}} = \begin{pmatrix} e^{i(j-\frac{1}{2})\theta} r^{j-\frac{1}{2}} I^P_{j+\frac{1}{2}}(\varepsilon_j r) \\ i\frac{|t-E_j|}{2} r^{j+\frac{1}{2}} e^{i(j+\frac{1}{2})\theta} I^P_{j+\frac{1}{2}}(\varepsilon_j r) \end{pmatrix} \quad \text{for} \quad t<0. \tag{53b}$$

Then, as t tends to zero, the both edge states $\tilde{\Phi}^{(\pm)}_{\text{edg}}$ prove to reach the same limit,

$$\tilde{\Phi}^{(-)}_{\text{edg}} \longrightarrow \frac{1}{\Gamma(j+\frac{1}{2})} \begin{pmatrix} r^{j-\frac{1}{2}} e^{i(j-\frac{1}{2})\theta} \\ 0 \end{pmatrix} \longleftarrow \tilde{\Phi}^{(+)}_{\text{edg}}, \quad \text{as} \quad E_j \to 0. \tag{54}$$

In a similar manner, for the eigenstates defined for $j<0$ to be

$$\tilde{\Psi}^{(+)}_{\text{edg}} = \begin{pmatrix} \frac{t+E_j}{2} e^{-i(|j|+\frac{1}{2})\theta} r^{|j|+\frac{1}{2}} I^P_{|j|+\frac{1}{2}}(\varepsilon_j r) \\ -ie^{-i(|j|-\frac{1}{2})\theta} r^{|j|-\frac{1}{2}} I^P_{|j|-\frac{1}{2}}(\varepsilon_j r) \end{pmatrix} \quad \text{for} \quad t>0, \tag{55a}$$

$$\tilde{\Psi}^{(-)}_{\text{edg}} = \begin{pmatrix} -\frac{|t+E_j|}{2} e^{-i(|j|+\frac{1}{2})\theta} r^{|j|+\frac{1}{2}} I^P_{|j|+\frac{1}{2}}(\varepsilon_j r) \\ -ie^{-i(|j|-\frac{1}{2})\theta} r^{|j|-\frac{1}{2}} I^P_{|j|-\frac{1}{2}}(\varepsilon_j r) \end{pmatrix} \quad \text{for} \quad t<0, \tag{55b}$$

we find that

$$\tilde{\Psi}^{(-)}_{\text{edg}} \longrightarrow \frac{1}{\Gamma(|j|+\frac{1}{2})} \begin{pmatrix} 0 \\ -ie^{-i(|j|-\frac{1}{2})\theta} r^{|j|-\frac{1}{2}} \end{pmatrix} \longleftarrow \tilde{\Psi}^{(+)}_{\text{edg}}, \quad \text{as} \quad E_j \to 0. \tag{56}$$

In the rest of this section, we discuss discrete symmetry and currents on the boundary. As is easily verified, the operators H_t, J, and B_t defining the Dirac equation with the APS boundary condition satisfy

$$\sigma_1 \overline{H}_t \sigma_1 = -H_t, \tag{57a}$$

$$\sigma_1 \overline{J} \sigma_1 = -J, \tag{57b}$$

$$\sigma_1 \overline{B}_t \sigma_1 = B_t. \tag{57c}$$

These equations imply that if E_j is a regular (resp. edge) state eigenvalue with the angular momentum j then $-E_j$ is a regular (resp. edge) state eigenvalue with the angular momentum $-j$. This fact explains the pattern of eigenvalues shown in Figures 7 and 8. If the graph of one of left panels is reflected with respect to the t-axis (the horizontal axis with $E = 0$), then the resultant graph coincides with the graph of the adjacent right panel.

We turn to another discrete symmetry. In a similar manner to the above, we verify that

$$i\sigma_2\overline{H}_t(-i\sigma_2) = H_{-t}, \tag{58a}$$

$$i\sigma_2\overline{J}(-i\sigma_2) = -J, \tag{58b}$$

$$i\sigma_2\overline{B}_t(-i\sigma_2) = B_{-t}. \tag{58c}$$

It then follows that if E_j is a regular (resp. edge) state eigenvalue with the angular momentum j for t, then E_j is a regular (resp. edge) state eigenvalue with the angular momentum $-j$ for $-t$. This fact explains that the pattern of eigenvalues shown in Figures 7 and 8 is of t-reflection along with j-inversion.

We proceed to currents on the boundary. We recall that the boundary values of both edge and regular eigenstates are proportional to eigenstates of the boundary operator B_t. Then, we can easily verify that the radial and the tangential components of the current for $\phi_j^{(\pm)}$ given in (37) are evaluated as

$$(\phi_j^{(\pm)})^\dagger\sigma_r\phi_j^{(\pm)} = 0, \quad (\phi_j^{(\pm)})^\dagger\sigma_\theta\phi_j^{(\pm)} = 2t|c|^2\Big(\frac{j}{R} \pm \sqrt{\frac{j^2}{R^2} + t^2}\Big), \tag{59}$$

respectively, where c is a constant. While the radial component vanishes, the tangential component alternates the sign, according as $t < 0$ or $t > 0$.

5. The chiral bag boundary condition

If the unitary operator U in (19) is chosen as

$$U = e^{2i\arctan e^\lambda}\mathbb{1}, \tag{60}$$

the chiral bag boundary condition is brought into

$$\sigma_r\psi = -ie^{\lambda\sigma_3}\sigma_3\psi. \tag{61}$$

With this boundary condition applied to feasible solutions, the functional equations for determining edge and regular state eigenvalues are found to be given as follows [7]: (i) For $|E_j| < |t|$, those functional equations are

$$\sqrt{\frac{t+E_j}{t-E_j}}I_{j-\frac{1}{2}}(\varepsilon_j R) = e^{-\lambda}I_{j+\frac{1}{2}}(\varepsilon_j R) \quad \text{for} \quad t > 0, \tag{62a}$$

$$-\sqrt{\frac{|t+E_j|}{|t-E_j|}}I_{j-\frac{1}{2}}(\varepsilon_j R) = e^{-\lambda}I_{j+\frac{1}{2}}(\varepsilon_j R) \quad \text{for} \quad t < 0, \tag{62b}$$

and (ii) for $|E_j| > |t|$, they are

$$\sqrt{\frac{E_j+t}{E_j-t}}J_{j-\frac{1}{2}}(\beta_j R) = -e^{-\lambda}J_{j+\frac{1}{2}}(\beta_j R) \quad \text{for} \quad E_j > 0, \tag{63a}$$

$$\sqrt{\frac{|E_j+t|}{|E_j-t|}}J_{j-\frac{1}{2}}(\beta R) = e^{-\lambda}J_{j+\frac{1}{2}}(\beta_j R) \quad \text{for} \quad E_j < 0. \tag{63b}$$

Though Eq. (62b) has no solution, the other functional equations for regular and edge state eigenvalues are numerically solved to provide the eigenvalues as functions of t, as is shown in Figure 9.

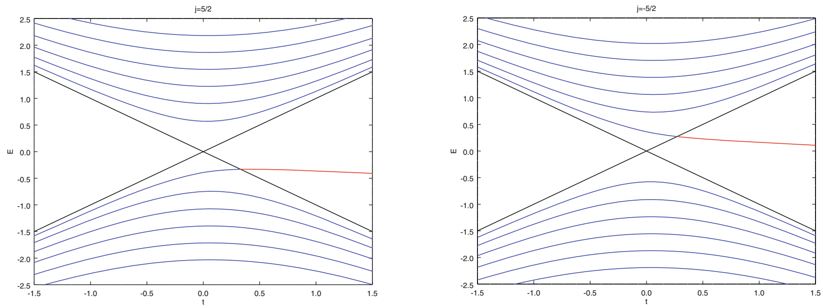

FIGURE 9. Eigenvalues of regular (blue) and edge (red) eigenstates with $R = 10, \lambda = 0.1$ for $j = 5/2$ (left panel) and for $j = -5/2$ (right panel).

A remarkable property observed in Figure 9 is that one of regular state eigenvalues is connected with an edge state eigenvalue. We refer to the state as a critical state, which corresponds to the eigenvalue as a limit of both the regular and the edge state eigenvalues. Since the critical states are characterized by the conditions that $E = \pm t$, we can easily solve Eq. (30) with $E = \pm t$ to find the critical states within constant multiples, along with the eigenvalues,

$$\Phi = \begin{pmatrix} ir^{j-\frac{1}{2}}e^{i(j-\frac{1}{2})\theta} \\ \frac{e^{\lambda}}{R}r^{j+\frac{1}{2}}e^{i(j+\frac{1}{2})\theta} \end{pmatrix}, \qquad E_j^{\text{cri}} = -\frac{e^{\lambda}(j+\frac{1}{2})}{R}, \text{ for } j > 0, \tag{64a}$$

$$\Phi = \begin{pmatrix} -\frac{e^{-\lambda}}{R}r^{-(j-\frac{1}{2})}e^{i(j-\frac{1}{2})\theta} \\ ir^{-(j+\frac{1}{2})}e^{i(j+\frac{1}{2})\theta} \end{pmatrix}, \quad E_j^{\text{cri}} = \frac{e^{-\lambda}(|j|+\frac{1}{2})}{R}, \text{ for } j < 0. \tag{64b}$$

Like (34), these critical states are also described in $z = re^{i\theta}$ as

$$\Phi = \begin{pmatrix} iz^{j-\frac{1}{2}} \\ \frac{e^{\lambda}}{R}z^{j+\frac{1}{2}} \end{pmatrix}, \qquad j > 0, \tag{65a}$$

$$\Phi = \begin{pmatrix} -\frac{e^{-\lambda}}{R}\overline{z}^{|j|+\frac{1}{2}} \\ i\overline{z}^{|j|-\frac{1}{2}} \end{pmatrix}, \quad j < 0. \tag{65b}$$

We can verify that the transition indeed occurs from a regular eigenstate to an edge eigenstate. Like (53), choosing a suitable scaling factor, we can introduce a regular and an edge eigenstates for $j > 0$ of the form

$$\tilde{\Phi}_{\rm reg} = \begin{pmatrix} r^{j-\frac{1}{2}} e^{i(j-\frac{1}{2})\theta} J^P_{j-\frac{1}{2}}(\beta_j r) \\ -i\frac{|E_j - t|}{2} r^{j+\frac{1}{2}} e^{(j+\frac{1}{2})\theta} J^P_{j+\frac{1}{2}}(\beta_j r) \end{pmatrix}, \tag{66a}$$

$$\tilde{\Phi}_{\rm edg} = \begin{pmatrix} e^{i(j-\frac{1}{2})\theta} r^{j-\frac{1}{2}} I^P_{j+\frac{1}{2}}(\varepsilon_j r) \\ -i\frac{t-E_j}{2} r^{j+\frac{1}{2}} e^{i(j+\frac{1}{2})\theta} I^P_{j+\frac{1}{2}}(\varepsilon_j r) \end{pmatrix}, \tag{66b}$$

respectively, where J^P_ν is a power series defined through

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu J^P_\nu(z), \quad J^P_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n}. \tag{67}$$

It is easily shown that as $E_j(t) \to -t$, there occurs the transition

$$\tilde{\Phi}_{\rm reg} \longrightarrow \frac{1}{\Gamma(j+\frac{1}{2})} \begin{pmatrix} r^{j-\frac{1}{2}} e^{i(j-\frac{1}{2})\theta} \\ -i\frac{e^\lambda}{R} r^{j+\frac{1}{2}} e^{i(j+\frac{1}{2})\theta} \end{pmatrix} \longleftarrow \tilde{\Phi}_{\rm edg}. \tag{68}$$

For $j < 0$, we take

$$\tilde{\Psi}_{\rm reg} = \begin{pmatrix} \frac{E_j+t}{2} e^{-i(|j|+\frac{1}{2})\theta} r^{|j|+\frac{1}{2}} J^P_{|j|+\frac{1}{2}}(\beta_j r) \\ -ie^{-i(|j|-\frac{1}{2})\theta} r^{|j|-\frac{1}{2}} J^P_{|j|-\frac{1}{2}}(\beta_j r) \end{pmatrix}, \tag{69a}$$

$$\tilde{\Psi}_{\rm egd} = \begin{pmatrix} \frac{t+E_j}{2} e^{-i(|j|+\frac{1}{2})\theta} r^{|j|+\frac{1}{2}} I^P_{|j|+\frac{1}{2}}(\varepsilon_j r) \\ -ie^{-i(|j|-\frac{1}{2})\theta} r^{|j|-\frac{1}{2}} I^P_{|j|-\frac{1}{2}}(\varepsilon_j r) \end{pmatrix}. \tag{69b}$$

Then, a straightforward calculation shows that as $E_j(t) \to t$, there occurs the transition

$$\tilde{\Psi}_{\rm reg} \longrightarrow \frac{1}{\Gamma(|j|+\frac{1}{2})} \begin{pmatrix} \frac{e^{-\lambda}}{R} e^{-i(|j|+\frac{1}{2})\theta} r^{|j|+\frac{1}{2}} \\ -ie^{-i(|j|-\frac{1}{2})\theta} r^{|j|-\frac{1}{2}} \end{pmatrix} \longleftarrow \tilde{\Psi}_{\rm edg}. \tag{70}$$

In the rest of this section, we mention discrete symmetry and boundary currents. Like (57), we verify that

$$\sigma_1 \overline{H}_t \sigma_1 = -H_t, \tag{71a}$$

$$\sigma_1 \overline{J} \sigma_1 = -J, \tag{71b}$$

$$\sigma_r \sigma_1 \overline{\psi} = -ie^{-\lambda\sigma_3} \sigma_3 \sigma_1 \overline{\psi}. \tag{71c}$$

In contrast to (57c), the chiral bag boundary condition is not invariant under the $\sigma_1 K$, where K denotes the complex conjugation. In fact, in the right-hand side of (71c), the exponent λ of the boundary condition (61) is replaced by $-\lambda$. However, if the λ is viewed as a real parameter, the inversion $\lambda \to -\lambda$ is acceptable as a transformation, so that we may view the above equations as representing a pseudo-symmetry of the family of the Dirac equations with the chiral bag boundary condition depending on λ. It then turns out that if E_j is a regular (resp. edge) state eigenvalue with the angular momentum j then $-E_j$ is a regular (resp. edge) state eigenvalue with the angular momentum $-j$ under the boundary condition with the parameter value $-\lambda$. This symmetry is observed in the pattern of eigenvalues shown in Figure 10.

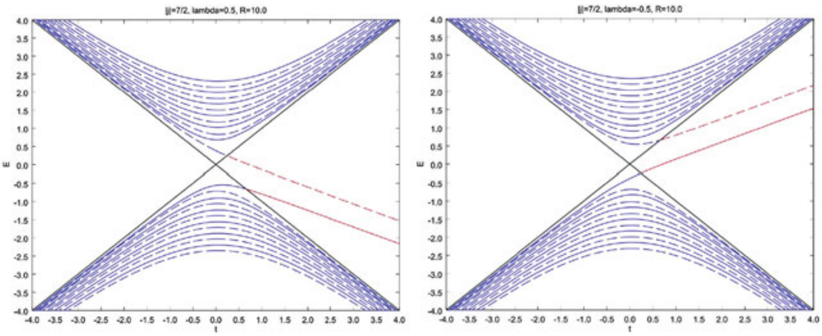

FIGURE 10. Eigenvalues of regular (blue) and edge (red) eigenstates with $|j| = \frac{7}{2}$, $R = 10$ and with $\lambda = -0.5$ (left panel) and $\lambda = 0.5$ (right panel). The solid and dashed curves are for $j > 0$ and for $j < 0$, respectively. Black lines are auxiliary lines $E = \pm t$ separating the regions referred to in Figure 6.

In contrast to (58), the operator $i\sigma_2 K$ cannot be a symmetry operator for the eigenvalue problem with the chiral bag boundary condition. In fact, we obtain the following equations in correspondence with (58),

$$i\sigma_2 \overline{H}_t(-i\sigma_2) = H_{-t}, \tag{72a}$$

$$i\sigma_2 \overline{J}(-i\sigma_2) = -J, \tag{72b}$$

$$\sigma_r(i\sigma_2)\overline{\psi} = ie^{-\lambda\sigma_3}\sigma_3(i\sigma_2)\overline{\psi}. \tag{72c}$$

As is seen in (72c), the boundary condition is not (pseudo-)invariant under the action of $i\sigma_2 K$. In the right-hand side of the (72c), the factor $-i$ of the condition (61) is replaced by i. This gives a reason why the pattern of eigenvalues shown in Figure 10 is not of t-reflection along with j-inversion.

Currents on the boundary for edge states (32) with $r = R$ and E_j specified are given, within constant multiples, by

$$\psi^\dagger \sigma_r \psi = 0, \tag{73a}$$

$$\psi^\dagger \sigma_\theta \psi = -2(t + E_j)e^\lambda I_{j-\frac{1}{2}}(\varepsilon_j R)^2 \quad \text{for} \quad t > 0. \tag{73b}$$

For regular states (31) with $r = R$ and E_j specified, one has, within constant multiples,

$$\psi^\dagger \sigma_r \psi = 0, \tag{74a}$$

$$\psi^\dagger \sigma_\theta \psi = \begin{cases} -2(t + E_j)e^\lambda J_{j-\frac{1}{2}}(\beta_j R)^2 & \text{for} \quad E_j > 0, \\ -2|t + E_j|e^\lambda J_{j-\frac{1}{2}}(\beta_j R)^2 & \text{for} \quad E_j < 0. \end{cases} \tag{74b}$$

6. Comparison between the APS and the chiral bag boundary conditions

We are interested in transition states both for the APS and the chiral bag boundary conditions. As is well known, the spectral flow for a one-parameter family of

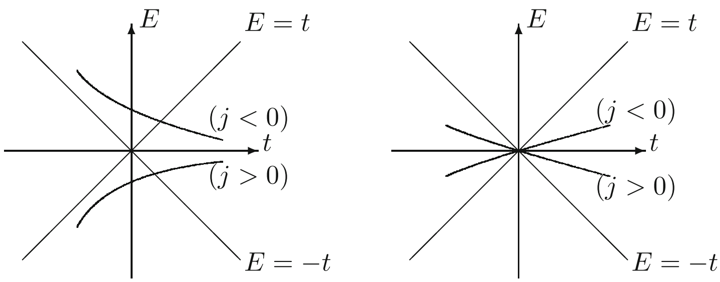

FIGURE 11. A schematic view of transient eigenvalue curves. The left and the right panels are for the chiral bag and the APS boundary conditions, respectively. In the left panel, the parameter is chosen as $\lambda = 0$ for simplicity.

operators is the net number of eigenvalues passing through zero in the positive direction as the parameter runs. This notion works well for characterizing the band rearrangement under the APS boundary condition. In fact, the spectral flow in question is given by $-\mathrm{sgn}(j)$. However, it does not serve as a characteristic quantity under the chiral bag boundary condition, since the zero eigenvalue does not carry a special meaning.

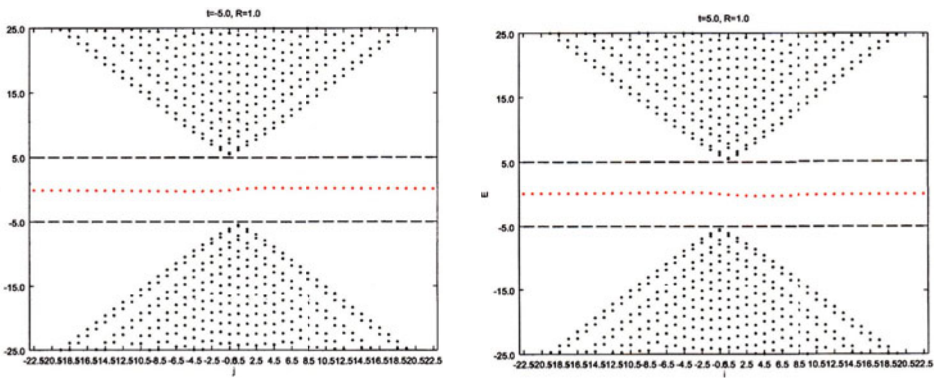

FIGURE 12. Eigenvalues of regular (black) and edge (red) eigenstates against j under the APS boundary condition with $R = 1.0$ for $t = -5.0$ (left panel) and for $t = 5.0$ (right panel). The dashed horizontal lines in the left and the right panels correspond to the lines $E = \pm t$ with $t = -5.0$ and $t = 5.0$, respectively.

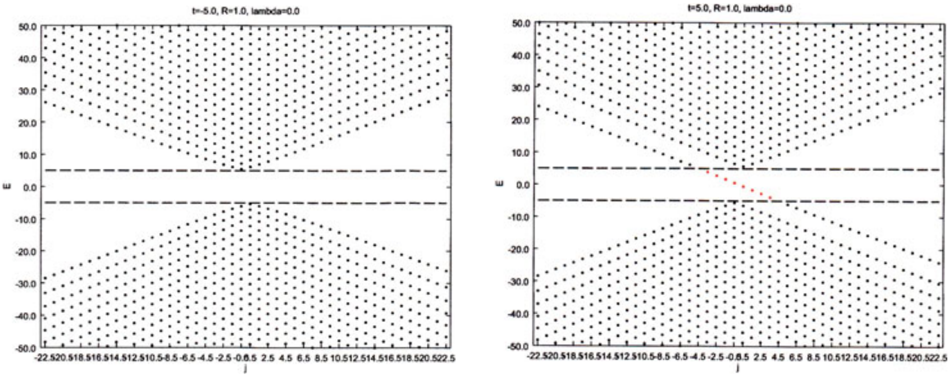

FIGURE 13. Eigenvalues of regular (black) and edge (red) eigenstates against j under the chiral bag boundary condition with $R = 1.0, \lambda = 0.0$ for $t = -5.0$ (left panel) and for $t = 5.0$ (right panel). The dashed horizontal lines in the left and the right panels correspond to the lines $E = \pm t$ with $t = -5.0$ and $t = 5.0$, respectively.

We need an extended notion of the spectral flow to characterize the band rearrangement under the chiral bag boundary condition. In the case of the chiral bag boundary condition, there exists a transient eigenvalue curve which crosses one of the boundary lines $E = \pm t$, depending on whether $j > 0$ or $j < 0$ (see Fig. 11, the left and the right panels of which are abstracted from Figure 9 and Figure 7, respectively). If we assign -1 and $+1$ to the crossing of the boundary lines $E = -t$ and $E = t$, respectively, the extended spectral flow for the chiral bag boundary condition is given by $-\mathrm{sgn}(j)$. Then, band rearrangement is characterized by $-\mathrm{sgn}(j)$ in both cases of the APS and the chiral bag boundary conditions.

In conclusion, we compare two boundary conditions from another point of view. If we plot energy eigenvalues against j under the chiral bag boundary condition with the parameter t fixed at a value, we obtain Figures 12 and 13 for the APS and the chiral bag boundary conditions, respectively. Difference is distinctively observed in the pattern of edge state eigenvalues (red dots).

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Toshihiro Iwai
Kyoto University
Kyoto, Japan
e-mail: iwai.toshihiro.63u@st.kyoto-u.ac.jp

Boris Zhilinskii
Université du Littoral Côte d'Opale
Dunkerque, France
e-mail: zhilin@univ-littoral.fr

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Time-dependent Pais–Uhlenbeck Oscillator and Its Decomposition

Hirosuke Kuwabara, Tsukasa Yumibayashi and Hiromitsu Harada

Abstract. The Pais–Uhlenbeck(PU) oscillator is the simplest model with higher time derivatives, and its properties has been studied for a long time. In this paper, we extend the 4th-order free PU oscillator to a non-trivial case, dubbed the 4th-order time-dependent PU (tdPU) oscillator, which has time-dependent frequencies. We show that this model cannot be decomposed into two harmonic oscillators in contrast to the original PU oscillator by a linear coordinate canonical transformation derived by Smilga. As a result of sustaining canonicality of this transformation for the tdPU oscillator, an interaction is added.

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Keywords. Pais–Uhlenbeck oscillator, higher-order theories, canonical transformations.

1. Introduction

The higher derivative theories [1–5] are worth investigating and the Pais–Uhlenbeck (PU) oscillator [6] is a good theoretical laboratory for studying classical and quantum models with higher derivatives. The key property of the PU oscillator is that it can be decomposed into two standard harmonic oscillators by a canonical transformation shown by Smilga[7]. In the following section, we review this fact. In higher derivative theories, their Hamiltonian needed for quantization and examining stability can be obtained by Ostrogradski's method [8, 9]. The Hamiltonian of the PU oscillator given by this method seems a little complicated, but it can be rewritten by a canonical transformation in a very simple form in terms of several independent harmonic oscillators. For example, as regards the 4th-order PU oscillator, whose equation of motion(EOM) has up to the 4th-order time derivative, its Hamiltonian H_{PU} can be separated into two independent harmonic oscillator Hamiltonians, $H_{\mathrm{PU}} = H_1 - H_2$, by Smilga's canonical transformation [7]. The important point is that Smilga's transformation is canonical, which means that it preserves the EOM of the 4th-order PU oscillator. When one applies a transformation for some purposes in the Hamiltonian formalism, for instance, to simplify calculation or to make physical meaning clearer, it is necessary to restrict it to canonical transformation not to change physics. In the decomposed form, in fact, quantization is quite simple, just by quantizing each of harmonic oscillators.

One may wonder, whether the decomposition still applies to an interacting PU oscillator. In this paper we demonstrate that it is not the case for the 4th-order time-dependent PU(tdPU) oscillator, which has time-dependent frequencies, and it is needed to add interaction terms when one uses the time-dependent Smilga transformation. When we "naively" use the time-dependent Smilga transformation in the tdPU oscillator, a problem arises, that is, the transformation ceases to be canonical. Here "naively" means to take an assumption that the tdPU Hamiltonian is invariant under the transformation. If one imposes this assumption, equations of motion before and after the transformation are not equivalent, which shows that this transformation is not canonical in our model. Therefore, for making the time-dependent Smilga transformation canonical, we need to add some correction terms to the "naively" transformed Hamiltonian. We find those correction terms to obtain a canonical transformation in two steps, (i) a heuristic method: a comparison of the differential equations, and (ii) a generating function method: we confirm to reproduce the same result obtained in (i). Finally, we show that the separability of our model is surely deformed because the corrections become the interaction terms of two time-dependent harmonic oscillators.

2. 4th-order free PU oscillator

In [6, 7], the 4th-order PU oscillator is defined as

$$L_{\mathrm{PU}}(x, \dot{x}, \ddot{x}) = \frac{1}{2}\ddot{x}^2 - \frac{1}{2}\Omega_1 \dot{x}^2 + \frac{1}{2}\Omega_2 x^2, \tag{1}$$

and its EOM is

$$\ddddot{x} + \Omega_1 \ddot{x} + \Omega_2 x = 0, \tag{2}$$

where $\Omega_1 := {\omega_1}^2 + {\omega_2}^2, \Omega_2 := {\omega_1}^2 {\omega_2}^2$. Let us introduce the PU Hamiltonian by applying Ostrogradski's method [8] to (1). The PU Hamiltonian H_{PU} is

$$\begin{aligned} H_{\mathrm{PU}}(q_1, q_2, p_1, p_2) &= p_1 \dot{q_1} + p_2 \dot{q_2} - L_{\mathrm{PU}} \\ &= p_1 q_2 + \frac{1}{2}{p_2}^2 + \frac{1}{2}\Omega_1 {q_2}^2 - \frac{1}{2}\Omega_2 {q_1}^2, \end{aligned} \tag{3}$$

where q_i, p_i $(i = 1, 2)$ are canonical coordinates,

$$\begin{cases} q_1 = x, \\ p_1 = \dfrac{\partial L_{\mathrm{PU}}}{\partial \dot{x}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L_{\mathrm{PU}}}{\partial \ddot{x}} = -\Omega_1 \dot{x} - \dddot{x}, \\ q_2 = \dot{x}, \\ p_2 = \dfrac{\partial L_{\mathrm{PU}}}{\partial \ddot{x}} = \ddot{x}. \end{cases} \tag{4}$$

In [7], Smilga gave the transformation that separates $H_{\rm PU}$ as follows:

$$\begin{cases} Q_1 = \dfrac{\gamma}{\omega_1}({\omega_1}^2 q_2 + p_1), \\ Q_2 = \gamma({\omega_1}^2 q_1 + p_2), \\ P_1 = \omega_1\gamma(p_2 + {\omega_2}^2 q_1), \\ P_2 = \gamma(p_1 + {\omega_2}^2 q_2). \end{cases} \tag{5}$$

In this paper, we call it Sm and where $\gamma := 1/\sqrt{{\omega_1}^2 - {\omega_2}^2}$ We also assume $\omega_1 > \omega_2$ for simplicity, and this transformation is a canonical transformation. According to (5), $H_{\rm PU}$ can be rewritten to

$$H_{\rm PU}(Q_1, Q_2, P_1, P_2) = H_1(Q_1, P_1) - H_2(Q_2, P_2), \tag{6}$$

where $H_i(Q_i, P_i) := \frac{1}{2}{P_i}^2 + \frac{1}{2}{\omega_i}^2{Q_i}^2$. Thus we see that the PU oscillator can be separated into two harmonic oscillators. Smilga used this property to quantize the PU oscillator and investigate its quantum stability.

3. Time-dependent PU oscillator

Our Lagrangian of the time-dependent PU(tdPU) oscillator is

$$L_{\rm tdPU}(x, \dot{x}, \ddot{x}) = \frac{1}{2}\ddot{x}^2 - \frac{1}{2}\Omega_1(t)\dot{x}^2 + \frac{1}{2}\Omega_2(t)x^2, \tag{7}$$

where $\Omega_1(t) := {\omega_1}^2(t) + {\omega_2}^2(t), \Omega_2(t) := {\omega_1}^2(t){\omega_2}^2(t)$. This Lagrangian can be obtained by replacing $\omega_i \to \omega_i(t)$ $(i = 1, 2)$ in (1). Its EOM is

$$\ddot{\ddot{x}} + \Omega_1(t)\ddot{x} + \dot{\Omega}_1(t)\dot{x} + \Omega_2(t)x = 0. \tag{8}$$

The Hamiltonian and its EOM are

$$\begin{aligned} H_{\rm tdPU}(q_1, q_2, p_1, p_2) &= p_1\dot{q}_1 + p_2\dot{q}_2 - L_{\rm tdPU} \\ &= p_1 q_2 + \frac{1}{2}{p_2}^2 + \frac{1}{2}\Omega_1(t){q_2}^2 - \frac{1}{2}\Omega_2(t){q_1}^2, \end{aligned} \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(q_1, q_2, p_1, p_2)^T = A\ (q_1, q_2, p_1, p_2)^T, \tag{10}$$

where

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_2(t) & 0 & 0 & 0 \\ 0 & -\Omega_1(t) & -1 & 0 \end{pmatrix}. \tag{11}$$

Here ^{T} denotes transpose of a matrix, p_i, q_i $(i = 1, 2)$ are defined by (4) with time-dependent frequencies, (8) and (10) are equivalent in our case.

By using (5) with time-dependent frequencies, (9) can be written as

$$H_{\rm tdPU}(Q_1, Q_2, P_1, P_2) = (H_{\rm td})_1(Q_1, P_1) - (H_{\rm td})_2(Q_2, P_2), \tag{12}$$

where $(H_{\mathrm{td}})_i(Q_i, P_i) = \frac{1}{2}{P_i}^2 + \frac{1}{2}{\omega_i}^2(t){Q_i}^2$. Its EOM is

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q_1, Q_2, P_1, P_2)^T = B\ (Q_1, Q_2, P_1, P_2)^T. \tag{13}$$

where

$$B := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -{\omega_1}^2(t) & 0 & 0 & 0 \\ 0 & {\omega_2}^2(t) & 0 & 0 \end{pmatrix}, \tag{14}$$

It is straightforward to check that (9) is equal to (12) under the Smilga transformation. However, the time-dependent Smilga transformation is not a canonical transformation. In other words, it doesn't reproduce the EOM (8) or, equivalently, (10) does not follow from (13). In the next two sections, we show that some correction terms are needed to correct the situation, and they become interaction terms of the harmonic oscillators.

4. Comparison of EOMs

Here we find the correction term by a heuristical method, a comparison of the EOMs (10) and (13). Let us introduce some notation first.

$$X := (q_1, q_2, p_1, p_2)^T, \tag{15}$$

$$Y := (Q_1, Q_2, P_1, P_2)^T, \tag{16}$$

X, Y and A, B are related by

$$Y = MX, \tag{17}$$

$$MA = BM, \tag{18}$$

where M is the coefficient matrix of the Smilga transformation (5). With this notation, the EOM (10) and (13) can be written as

$$\dot{X} = AX, \tag{19}$$

$$\dot{Y} = BY + \alpha. \tag{20}$$

Here α represents a correction to (13). Differentiating the first relation (17) with respect to t and substituting into the left-hand side of (20), we find

$$\dot{Y} = M\dot{X} + \dot{M}X = MAX + \dot{M}X = BY + \alpha = BMX + \alpha.$$

Then,

$$\alpha = \dot{M}X = \dot{M}M^{-1}Y.$$

Therefore, $\alpha \neq 0$, unless the model is time-independent. More explicitly, α is

$$\alpha = \gamma^2(t) \begin{pmatrix} C_1 & 0 & 0 & E_1 \\ 0 & C_2 & E_1 & 0 \\ 0 & E_2 & C_3 & 0 \\ E_2 & 0 & 0 & C_4 \end{pmatrix} Y,$$

where

$$C_1 := \frac{\dot\gamma(t)}{\gamma^3(t)} + \frac{\dot\omega_1(t)}{\omega_1(t)}\Omega_1(t), \qquad C_2 := \frac{\dot\gamma(t)}{\gamma^3(t)} + 2\omega_1(t)\dot\omega_1(t),$$
$$C_3 := \frac{\dot\gamma(t)}{\gamma^3(t)} + \frac{\dot\omega_1(t)}{\omega_1(t)\gamma^2(t)} - 2\omega_2(t)\dot\omega_2(t), \quad C_4 := \frac{\dot\gamma(t)}{\gamma^3(t)} - 2\omega_2(t)\dot\omega_2(t),$$
$$E_1 := -2\dot\omega_1(t),\ E_2 := 2\omega_1(t)\omega_2(t)\dot\omega_2(t), \quad \gamma(t) := 1/\sqrt{{\omega_1}^2(t) - {\omega_2}^2(t)}.$$

5. Generating function method

In this part, we show that the correction term can be also obtained by using a canonical Hamiltonian H'_{tdPU}. The latter can be found by the generating function method, via solving the equations

$$p_i = \frac{\partial W(q,Q,t)}{\partial q_i}, \quad P_i = -\frac{\partial W(q,Q,t)}{\partial Q_i},$$
$$H_{\mathrm{tdPU}}(q,p) - H'_{\mathrm{tdPU}}(Q,P) = -\frac{\partial W(q,Q,t)}{\partial t},$$

where $q := (q_1, q_2), p := (p_1, p_2), Q := (Q_1, Q_2), P := (P_1, P_2)$ and $W(q,Q,t)$ is a generating function of a canonical transformation. Then we find

$$H'_{\mathrm{tdPU}}(Q,P) = H_{\mathrm{tdPU}}(Q,P) + \frac{\partial W}{\partial t} = H_{\mathrm{tdPU}}(Q,P) + H_{\mathrm{int}}, \tag{21}$$

where

$$\begin{aligned} H_{\mathrm{int}} := & -2\gamma^2(t)\omega_1(t)\omega_2(t)\dot\omega_2(t)Q_1Q_2 - 2\gamma^2(t)\dot\omega_1(t)P_1P_2 \\ & + \frac{\gamma^2(t)\omega_2(t)}{\omega_1(t)}(\omega_1(t)\dot\omega_2(t) + \dot\omega_1(t)\omega_2(t))P_1Q_1 + \gamma^2(t)(\omega_1(t)\dot\omega_1(t) \\ & + \dot\omega_2(t)\omega_2(t))P_2Q_2 + \dot g(t), \end{aligned} \tag{22}$$

and $g(t)$ is an arbitrary time-dependent function. Its EOM is

$$\dot Y = BY + \gamma^2(t) \begin{pmatrix} C_1 & 0 & 0 & E_1 \\ 0 & C_2 & E_1 & 0 \\ 0 & E_2 & C_3 & 0 \\ E_2 & 0 & 0 & C_4 \end{pmatrix} Y,$$

It is easily confirmed that the last term is equal to α. Hence, the interacting Hamiltonian (21) is canonical and the correction α of the EOM (13) is equivalent to the interaction terms in the canonical Hamiltonian (21).

6. Conclusion

In this paper, we extended the free 4th-order PU oscillator to the non-trivial case, the 4th-order tdPU oscillator, whose frequencies depend on time. We found that our model cannot be written down in the form of two harmonic oscillators. This is because the Smilga transformation is not canonical in our extended model, so that it was necessary to add some correction terms to make it canonical. We obtained those corrections in two steps: (i) a comparison of the differential equations, and (ii) the generating function method. We showed that the correction terms in the tdPU EOM can be written as interaction terms in the canonical tdPU Hamiltonian.

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Hirosuke Kuwabara, Tsukasa Yumibayashi and Hiromitsu Harada
Department of Physics, Tokyo Metropolitan University
Hachioji-shi 1-1, Tokyo 192-0397, Japan
e-mail: kuwabara-hirosuke@ed.tmu.ac.jp
yumibayashi-tsukasa@ed.tmu.ac.jp
harada-hiromitsu@ed.tmu.ac.jp

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Quantum Walks in Low Dimension

Tatsuya Tate

Abstract. Discrete-time quantum walks are defined as a non-commutative analogue of the usual random walks on standard lattices and have been formulated in computer sciences. They are new objects in mathematics and are investigated in various areas, such as computer sciences, quantum physics, probability theory, and discrete geometric analysis. In this article, recent works on point-wise asymptotic behavior and an effective formula for nth power of the discrete-time quantum walks in one dimension are surveyed. The idea to obtain the formula for the nth power in one dimension is applied in this paper to compute the nth power of certain two-dimensional quantum walk, called the Grover walk to obtain a new formula for the two-dimensional Grover walk. The formula for nth power in one dimension has been used to prove a weak limit theorem. In this paper, the large deviation asymptotics, in one dimension, is deduced by using this formula which is a new proof of a previously obtained result.

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1. Introduction

The notion of discrete-time quantum walks is discovered in computer sciences ([1, 2, 12]), and recently it is investigated in various areas such as computer sciences, quantum physics, probability theory and discrete geometric analysis. A historical background to this area can be found in [8] and some of results introduced here are surveyed in [11]. It is defined as a non-commutative analogue of usual random walks and it is sometimes discussed in comparison with usual random walks. It seems that this notion is going to have appropriate positions in mathematics. However, compared with usual random walks, the investigations to this area seems to be far from being enough.

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One of main purposes of the present paper is to review some of known results, mainly obtained by the author and a collaborator, in order that the notion itself, results on it and ideas behind them can be shared among researchers in various different areas.

We mainly consider here one-dimensional discrete-time quantum walks with "two interior degrees of freedom". This is a simplest one among many other models. Let us give its definition here. The Hilbert space on which the transition operators of quantum walks are defined is the ℓ^2-space, $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, consisting of all square summable functions on the set of integers, $\mathbb{Z}$, taking values in the two-dimensional complex vector space $\mathbb{C}^2$ with the standard Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$. We choose a two-by-two unitary matrix C, which is called a constant *coin matrix*. For simplicity of notation, the unitary matrix C is assumed to have determinant one. Then the matrix C is written in the form

$$C = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \quad (a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1), \tag{1}$$

which is decomposed as

$$C = P_C + Q_C, \quad P_C = \begin{pmatrix} a & 0 \\ -\overline{b} & 0 \end{pmatrix}, \quad Q_C = \begin{pmatrix} 0 & b \\ 0 & \overline{a} \end{pmatrix}.$$

Now the operator $U(C)$ defined by

$$(U(C)f)(x) = P_C(f(x-1)) + Q_C(f(x+1)) \quad (f \in \ell^2(\mathbb{Z}, \mathbb{C}^2),\ x \in \mathbb{Z}) \tag{2}$$

is what we call a *discrete-time quantum walk* with a constant coin matrix C. This is a unitary operator on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, and the transition probability is defined by the formula

$$p_n(x) = p_n(\phi; x) := \|U(C)^n(\delta_o \otimes \phi)(x)\|^2_{\mathbb{C}^2}, \tag{3}$$

where the initial state, which is a function in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, can be taken arbitrarily but here only the initial state of the form $\delta_z \otimes \phi$ with $z \in \mathbb{Z}$ and $\phi \in \mathbb{C}^2$ with $\|\phi\|_{\mathbb{C}^2} = 1$ defined by

$$(\delta_z \otimes \phi)(y) = \begin{cases} 1 & (y = z), \\ 0 & (y \neq z), \end{cases}$$

is considered. One of the main issue for the quantum walks is, as in the theory of random walks, to obtain various aspects of asymptotic behavior of transition probabilities in the long-time limit. As in the case of random walks the transition probability is defined in terms of the nth power of the transition operator, $U(C)$. Therefore, the computation of the nth power would be necessary to analyze its asymptotic behavior. A usual method is to compute the eigenvalues of the Fourier transform of the unitary operator $U(C)$, which is in this case two-by-two unitary matrix-valued function on the unit circle, and to diagonalize it. However, the computation of powers of higher-dimensional quantum walks by using this method would become complicated because the size of the matrix becomes large. Thus it would be reasonable to find an algebraic structure behind it and to use it to compute nth powers. In Section 4, an effective formula for nth power for one-dimension case is reviewed. The original idea for this formula is a decomposition of $U(C)$ into two other quantum walks, and which leads us to the regular representation of the infinite dihedral group. In this section the formula is easily obtained from the de Moivre formula for unitary operators. However this idea is applicable to some other types of quantum walks. In particular, this idea is applied to the two-dimensional *Grover walk* in the section. It seems that this formula for the two-dimensional Grover walk is new.

Section 3 is devoted to review some results on the asymptotics of transition probability of one-dimensional quantum walks defined above. After describing the weak limit theorem due to Konno ([9]), we explain point-wise asymptotics of the transition probability. There are three types of asymptotic behaviors. One is for the case that the points accumulate a point inside the wall of the weak limit measure, and in this region the transition probability has oscillation. The behavior is asymptotically expressed by the Airy function when the points are close to the wall. Outside the wall is the region for the large deviation asymptotics. These are originally obtained by using the concrete expression of eigenvalues and hence it would be rather hard to generalize it to higher-dimensional case. However, it turns out that the nth power formula can be also applicable to obtain the asymptotic behavior inside the wall and in the large deviation region. In particular, a new computation for the rate function in large deviation asymptotics is given here. Hopefully the nth power formula for the two-dimensional Grover walk could be applicable to get its rate function as well.

2. An algebraic structure

2.1. De Moivre formula

As in Introduction, it is necessary to compute or to analyze, in an appropriate sense, the nth power of $U(C)$, because the transition probability is defined by $U(C)^n$. We start with the de Moivre (or de Moivre–Euler) formula for the nth power of the complex number $e^{i\theta} = \cos\theta + i\sin\theta$ of modulus one. By introducing the Chebyshev polynomials

$$T_n(x) = \cos n\theta, \quad U_{n-1}(x) = \frac{\sin n\theta}{\sin\theta} \quad (x = \cos\theta),$$

the de Moivre–Euler formula is written in the form

$$(e^{i\theta})^n = T_n(x) + iyU_{n-1}(x) \quad (x = \cos\theta,\ y = \sin\theta). \tag{4}$$

The same formula holds for any unitary operator on any Hilbert space. Indeed, let U be a unitary operator on a Hilbert space $\mathcal{H}$ and we set $X = (U + U^*)/2$, $Y = (U - U^*)/(2i)$ so that $U = X + iY$. These operators are self-adjoint and commute with each other. Since U is unitary, we see $X^2 + Y^2 = I$. Simple application of the binomial expansion shows

$$\begin{aligned} U^n &= (X+iY)^n \qquad (5) \\ &= \sum_{k=0}^{[n/2]} \binom{n}{2k} X^{n-2k}(X^2-I)^k + iY \sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} X^{n-2k-1}(X^2-I)^k, \end{aligned}$$

where $[n/2]$ denotes the integer not greater than $n/2$. Since the sums in the above coincide, as is easily shown, with $T_n(X)$ and $U_{n-1}(X)$, respectively, we have obtained the following.

Lemma 1. *In the above notation, the following formula holds.*

$$U^n = T_n(X) + iYU_{n-1}(X). \qquad (6)$$

2.2. One-dimensional quantum walks and the dihedral group

For the one-dimensional quantum walk $U(C)$, the formula (6) is extremely useful. One of reasons for the usefulness of (6) in one dimension comes from the fact that X is an operator acting on scalar functions. However, for higher-dimensional quantum walks, the real part X has rather complicated structure. One of a strategies to analyze the nth power is, in general, to utilize an algebraic structure behind the operators. In the one-dimensional case an algebraic structure is brought to us by the *infinite dihedral group* Γ. This is not quite necessary, in one dimension, for the asymptotic analysis of the transition probability. However, it is still worth mentioning, because there is an algebraic structure for certain two-dimensional quantum walks which is similar to the one-dimensional case.

One way to define the infinite dihedral group Γ is to specify a set of generators and relations as

$$\Gamma = \langle\, \epsilon, \nu \mid \nu\epsilon = \epsilon\nu^{-1},\, \epsilon^2 = 1 \,\rangle,$$

where 1 is the unit. The group Γ is isomorphic to the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2$ defined by an obvious action of $\mathbb{Z}_2 = \{\pm 1\}$ on $\mathbb{Z}$. Suppose we are given a unitary representation ρ of Γ on a Hilbert space $\mathcal{H}$. We set

$$V = \rho(\nu), \quad W = i\rho(\epsilon) \qquad (7)$$

to get two unitary operators V, W satisfying the relations

$$VW = WV^{-1}, \quad W^2 = -I. \qquad (8)$$

The above relations make an operator of the form

$$U = sV + tW \quad (s, t \in \mathbb{R},\, s^2 + t^2 = 1) \qquad (9)$$

unitary. We set

$$x = \frac{s}{2}(V + V^*), \quad v = \frac{s}{2i}(V - V^*), \quad w = tW. \qquad (10)$$

The operators x, v and w satisfy the relations $xv = vx$, $xw = wx$, $vw + wv = 0$. The imaginary part y of the unitary operator U defined by (9) is written as $y = v - iw$. Hence by the de Moivre formula (6) we have the following.

Lemma 2. *For the unitary operator U defined by* (9) *we have*

$$U^n = T_n(x) + i(v - iw)U_{n-1}(x).$$

To relate the above with the one-dimensional quantum walks, we only need to construct a unitary representation $(\rho, \mathcal{H})$ of the dihedral group Γ. For a fixed coin matrix C given in (1), we suppose that $ab \neq 0$ and set $\alpha = a/|a|$, $\beta = b/|b|$. Then, the two quantum walks

$$V = U(V_o), \quad W = U(W_o) \tag{11}$$

with coin matrices

$$V_o = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}, \quad W_o = \begin{pmatrix} 0 & \beta \\ -\overline{\beta} & 0 \end{pmatrix}$$

satisfy the relation (8) and hence they define, through (7), a unitary representation of Γ. The following theorem is proved in [15] and also easily shown.

Theorem 3. *The unitary representation $(\rho, \ell^2(\mathbb{Z}, \mathbb{C}^2))$ of the infinite dihedral group Γ defined by* (11), (7) *is unitarily equivalent to the regular representation $(R, \ell^2(\Gamma))$.*

2.3. Two-dimensional Grover walk

Almost the same discussion can be applied to a certain two-dimensional quantum walk, called Grover walk which was used for an improvement ([3]) of Grover's quantum search algorithm ([6]).

2.3.1. Grover walk and a semi-direct product. This time, the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$ is replaced by the semi-direct product $\mathbb{Z}^2 \rtimes K$ where K is the Klein four group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We set $K = \{1, p, q, pq\}$ with the relation $p^2 = q^2 = 1$, $pq = qp$. Then, K acts linearly on $\mathbb{Z}^2$ by the following way:

$$pe_1 = -e_2, \; pe_2 = -e_1, \; qe_1 = e_2, \; qe_2 = e_1,$$

where $\{e_1, e_2\}$ denotes the standard basis of $\mathbb{Z}^2$. The group $\mathbb{Z}^2 \rtimes K$ has the presentation

$$\mathbb{Z}^2 \rtimes K = \langle\, d_1, d_2, w \mid w^2 = (d_1 d_2)^2 = (d_1 d_2^{-1})^2 = (wd_1)^2 = (wd_2)^2 = 1 \,\rangle,$$

which can be proved by the correspondence

$$d_1 = (e_1, p), \quad d_2 = (-e_1, q), \quad w = (e_2, pq). \tag{12}$$

Therefore, any unitary representation of the group $\mathbb{Z}^2 \rtimes K$ gives three unitary operators D_1, D_2 and W satisfying the relation

$$W^2 = (D_1 D_2)^2 = (D_1 D_2^{-1})^2 = (WD_1)^2 = (WD_2)^2 = I. \tag{13}$$

To relate this with quantum walk, we need to introduce the Grover walk in two dimensions. Let G_o be a 4×4 unitary matrix, which is sometimes called the Grover matrix, defined by

$$G_o = \frac{1}{2} J - I,$$

where J is the matrix whose entries are all 1 and I is the identity matrix. This simple matrix is self-adjoint and unitary so that we have $G_o^* = G_o^{-1} = G_o$. Let

$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ denote the standard basis of $\mathbb{C}^4$, and let P_i with $i = 1, \ldots, 4$ denote the orthogonal projection onto the one-dimensional subspace spanned by $\boldsymbol{e}_i$. The two-dimensional Grover walk is then defined as a unitary operator on $\ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ by the formula

$$G = U(G_o) := G_o P_1 \tau_1 + G_o P_2 \tau_1^{-1} + G_o P_3 \tau_2 + G_o P_4 \tau_2^{-1},$$

where τ_i denotes the shift operator along e_i-axis defined by

$$(\tau_i f)(x) = f(x - e_i) \quad (f \in \ell^2(\mathbb{Z}^2, \mathbb{C}^4),\ x \in \mathbb{Z}^2).$$

As in the one-dimensional case, it would be reasonable to express the Grover walk G as a linear combination of other quantum walks. To obtain useful decomposition, we note the following decomposition of G_o itself:

$$G_o = \frac{1}{2}(-I + W_o + (D_1)_o + (D_2)_o) \tag{14}$$

where I is the identity matrix and other matrices W_o, $(D_1)_o$, $(D_2)_o$ are defined by

$$W_o = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (D_1)_o = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (D_2)_o = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The corresponding quantum walks

$$A = U(I), \quad W = U(W_o), \quad D_1 = U((D_1)_o), \quad D_2 = U((D_2)_o) \tag{15}$$

satisfy the relations (13) and $A = D_1 W D_2$, and hence we obtain the representation $(\rho, \ell^2(\mathbb{Z}^2, \mathbb{C}^4))$. We have the following

Theorem 4. *The unitary representation $(\rho, \ell^2(\mathbb{Z}^2, \mathbb{C}^4))$ of $\mathbb{Z}^2 \rtimes K$ defined by the Grover walk as above is unitarily equivalent to the right regular representation $(R, \ell^2(\mathbb{Z}^2 \rtimes K))$.*

Proof. The unitary operator $u : \ell^2(\mathbb{Z}^2 \rtimes K) \to \ell^2(\mathbb{Z}^2, \mathbb{C}^4)$ defined by

$$u\delta_{(\alpha,1)} = \delta_\alpha \otimes \boldsymbol{e}_3,\ u\delta_{(\alpha,p)} = \delta_\alpha \otimes \boldsymbol{e}_2,\ u\delta_{(\alpha,q)} = \delta_\alpha \otimes \boldsymbol{e}_1,\ u\delta_{(\alpha,pq)} = \delta_\alpha \otimes \boldsymbol{e}_4$$

intertwines the two representations. □

2.3.2. *n*th power of the Grover walk. As in the above, the two-dimensional Grover walk G can be written in the form

$$G = \frac{1}{2}(-A + W + D_1 + D_2).$$

The real part X of the two-dimensional Grover walk G does not have quite easy form. To simplify the computation, it would be better to work in the group $\mathbb{Z}^2 \rtimes K$. The element in $\mathbb{Z}^2 \rtimes K$ corresponding to the operator A given in (15) under the isomorphism (12) is $a = d_1 w d_2 = (-e_2, 1)$. We set $b = w d_1 d_2 = (e_1, 1)$. The two elements a, b generate a subgroup isomorphic to $\mathbb{Z}^2$ and the corresponding operators $A = D_1WD_2$, $B = WD_1D_2$ are easy to handle. Thus, it would be rather natural to consider the sum of their real parts

$$Z = \frac{1}{4}(A + A^{-1} + B + B^{-1}),$$

which, in the representation defined by using the Grover walk, is given by

$$Z = \frac{1}{4}(\tau_1 + \tau_1^{-1} + \tau_2 + \tau_2^{-1})I_4.$$

Thus, the self-adjoint operator Z is essentially the transition operator of the usual random walk on $\mathbb{Z}^2$ acting on scalar functions. Here is a lemma.

Lemma 5. *We have the following.*

$$\begin{aligned}(G - G^{-1})(G + G^{-1} + 2Z) &= 0,\\ (X + Z)^2 &= (W + Z)(X + Z).\end{aligned} \tag{16}$$

These can be easily deduced for the representation of $\mathbb{Z}^2 \rtimes K$ defined by using the Grover walk as above. In particular the first formula is proved by the Hamilton–Cayley theorem. However, it can be shown directly by using only the relation (13), although the computation becomes a little bit long. The first formula in equation (16) is equivalent to

$$Y(X + Z) = 0 \tag{17}$$

and it is easy to show that Z commutes with X and Y, where Y is the imaginary part of G. Thus we have $YU_{n-1}(X) = YU_{n-1}(-Z)$. Somewhat complicated is the term $T_n(X)$ in (6). By using (5), relations (17) and $X^2 - I = -Y^2$, we see

$$T_n(X) = X^n - (-Z)^n + T_n(-Z).$$

From this and the second formula in (16), it is not hard to show the following.

Theorem 6. *The nth power of the two-dimensional Grover walk G is given by the following formula.*

$$G^n = \frac{(X + Z)(W^n - (-Z)^n)}{W + Z} + T_n(-Z) + iYU_{n-1}(-Z).$$

According to the above formula, the nth power G^n is computable by W^k and Z^k. Since Z is essentially an operator acting on scalar functions and $W^2 = I$, the above formula gives us a computable formula for G^n.

3. Asymptotics for one-dimensional quantum walks

3.1. Weak limit theorem and its dynamical structure

First of all, let us review a weak limit theorem of the transition probability of one-dimensional quantum walks. The following is due to Konno ([9]).

Theorem 7 (Konno, [9]). *Let $\phi = {}^t(\phi_1, \phi_2)$ be a unit vector in $\mathbb{C}^2$. Suppose that $ab \neq 0$, where a, b are the components of the coin matrix C given in* (1)*. Then, we have the following weak limit formula:*

$$\underset{n\to\infty}{\text{w-lim}} \sum_{x\in\mathbb{Z}} p_n(\phi; x)\delta_{x/n} = \chi_{(-|a|,|a|)}(\xi)\frac{|b|(1-\lambda_C(\phi)\xi)}{\pi(1-\xi^2)\sqrt{|a|^2-\xi^2}}\, d\xi,$$

where $\delta_{x/n}$ denotes the Dirac measure at x/n, $\chi_{(-|a|,|a|)}(y)$ is the characteristic function of the interval $(-|a|,|a|)$ and the constant $\lambda_C(\phi)$ is given by

$$\lambda_C(\phi) = |\phi_1|^2 - |\phi_2|^2 - \frac{1}{|a|^2}(ab\overline{\phi}_1\phi_2 + \overline{a}\overline{b}\phi_1\overline{\phi}_2).$$

Konno uses a set of relations satisfied by the matrix units of size two and an asymptotic properties of the Jacobi polynomials. But there is another method, see for example, [5]. The nth power formula in Lemma 2 is also used to prove Theorem 7 in a systematic way ([15]). It would be worth mentioning that the constant $\lambda_C(\phi)$ can be expressed in terms of the moment map of S^1-action on S^2 and the Hopf fibration $S^3 \to S^2$.

In Theorem 7 the reason why the case $a = 0$ or $b = 0$ are excluded is obvious; one can compute thoroughly in these cases. However it is interesting to note these cases, where the corresponding weak limit formulas become

$$\begin{aligned} &\underset{n\to\infty}{\text{w-lim}} \sum_{x\in\mathbb{Z}} p_n(\phi; x)\delta_{x/n} = |\phi_1|^2\delta_1 + |\phi_2|^2\delta_{-1} \quad (b = 0), \\ &\underset{n\to\infty}{\text{w-lim}} \sum_{x\in\mathbb{Z}} p_n(\phi; x)\delta_{x/n} = \delta_0 \qquad\qquad\qquad\quad (a = 0). \end{aligned} \tag{18}$$

Equation (18) shows that when $b = 0$ the quantum walker is repelled from the initial position and when $a = 0$ the quantum walker is attracted to the initial position. This dynamical behavior is expressed in the algebraic relation (8), and the decomposition (9) for $U = U(C)$ describes a competition of these two dynamically different terms.

3.2. Pointwise asymptotics

In the decomposition (8) of $U = U(C)$, we have $W^2 = -I$. Since the transition probability is defined by the norm square, the minus sign here is not important. The nth power W^n could be regarded as giving a fluctuation for the transition probability. The following version of the pointwise asymptotic behavior indeed shows such a fluctuation.

Theorem 8 ([13]). *Suppose that a sequence of integers x_n satisfies that*

$$\xi_n = \xi + O(1/n), \quad \xi_n = x_n/n \tag{19}$$

for some $\xi \in (-|a|,|a|)$. Then we have the following:

$$p_n(\phi; x_n) = \frac{(1+(-1)^{n+x_n})|b|}{\pi n(1-\xi^2)\sqrt{|a|^2-\xi^2}}\left[1 - \lambda_C(\phi)\xi + \mathrm{OSC}_n(\xi_n) + O(1/n)\right] \tag{20}$$

as $n \to \infty$ uniformly in x satisfying (19)*. Here* $\mathrm{OSC}_n(\xi)$ *is a function of the form*

$$A(\xi)\cos(n\theta(\xi)) + B(\xi)\sin(n\theta(\xi)).$$

We remark that the term $(1+(-1)^{n+x_n})$ in (20) reflects the fact that the left-hand side vanishes if $n+x_n$ is odd. According to Theorems 7 and 8 it would be rather easy to think that the transition probability will achieve its maximum around $x \sim \pm|a|$ because the leading term diverges at $\xi = \pm|a|$. Indeed we have the following

Theorem 9 ([13]). *Suppose that a sequence of integers x_n satisfies that*

$$x_n = \pm n|a| + d_n, \quad d_n = O(n^{1/3}). \tag{21}$$

Then we have

$$\begin{aligned} p_n(\phi; x_n) &= (1+(-1)^{n+x_n})\alpha^2 n^{-2/3} \\ &\qquad \times \left|\mathrm{Ai}\left(\pm\alpha n^{-1/3} d_n\right)\right|^2 (1 \mp |a|\lambda_C(\phi)) + O(1/n), \end{aligned} \tag{22}$$

where $\alpha = (2/|a||b|^2)^{1/3}$ and $\mathrm{Ai}(x)$ *is the Airy function.*

In the formula (22), the argument $n^{-1/3}d_n$ of the Airy function is bounded and hence the transition probability is, in this region close to the wall $x = \pm|a|$, of order $O(n^{-2/3})$. In the region inside the wall, $|x| < |a|$, Theorem 8 tells us that it is of order $O(n^{-1})$, and hence the transition probability grows around the wall.

The transition probability can be still positive even if the point x is outside the wall. However, if the wall $x = \pm|a|$ could be regarded as an analogue of potential wall, the transition probability outside the wall should be exponentially decaying. An asymptotic formula for this region corresponds in probability theory to something which is called the large deviation asymptotics. The following is the large deviation asymptotics for one-dimensional quantum walks.

Theorem 10 ([13]). *Let $\xi \in \mathbb{R}$ satisfy $|a| < |\xi| < 1$. Suppose that a sequence of integers x_n satisfies* (19)*. Then we have*

$$p_n(\phi; x_n) = \frac{(1+(-1)^{n+x_n})|b|}{\pi n(1-\xi^2)\sqrt{\xi^2-|a|^2}} e^{-nH_Q(\xi_n)}(G(\xi) + O(1/n)), \tag{23}$$

where $\xi_n = x_n/n$, $G(\xi)$ is a smooth non-negative function in $|a| < |\xi| < 1$, and the function $H_Q(\xi)$, which is positive and convex in this region, is given by

$$\begin{aligned} H_Q(\xi) = 2|\xi| \log\left(|b||\xi| + \sqrt{\xi^2-|a|^2}\right) - 2\log\left(|b| + \sqrt{\xi^2-|a|^2}\right) \\ + (1-|\xi|)\log\left(1-\xi^2\right) - 2|\xi|\log|a|. \end{aligned} \tag{24}$$

The function $H_Q(\xi)$ given in (24) is what we call *the rate function* in large deviation asymptotics, and this function expresses the exponential decay of the transition probability. Indeed, the following is directly deduced from Theorem 10.

Corollary 11. *Suppose that a sequence of integers x_n satisfies* (19). *If $p_n(\phi; x_n)$ is not zero for every sufficiently large n, then we have the following:*

$$\lim_{n\to\infty} \frac{1}{n} \log p_n(\phi; x_n) = -H_Q(\xi). \tag{25}$$

Note that, in equation (23), the function $G(\xi)$ can be computed concretely in terms of the components of the coin matrix C and the initial state ϕ. It is also interesting to note that the rate function $H_Q(\xi)$ does not depend on the initial state, ϕ.

Theorems 8, 9 and 10 tell us the whole picture of the behavior of the one-dimensional quantum walks. There is a wall at $\xi = \pm|a|$ where the weak limit distribution diverges. In finite time, the transition probability penetrates this wall. However the probability penetrating the wall is so small that, as time tends to infinity, it decays exponentially. This exponential decay rate is given by the rate function $H_Q(\xi)$. The probability reaches its maximum around the wall. Inside of the wall, the probability has a fluctuation.

Strangely enough the point-wise behavior of the Hermite function is very similar to this picture. It can be deduced from the Plancherel–Rotach formula. The readers who are interested in this formula may be referred to [14]. But it is interesting to note that the weak limit distribution for rescaled Hermite functions is quite similar to that of one-dimensional quantum walks given in Theorem 7. Indeed, let $\varphi_n(x)$ denote the Hermite function, which is an eigenfunction of the harmonic oscillator $-(d/dx)^2 + x^2$, normalized so that its L^2-norm on $\mathbb{R}$ is one. The eigenvalue corresponding to φ_n is λ_n^2 with $\lambda_n = \sqrt{2n+1}$. We rescale the Hermite function as

$$\rho_n(x) = \lambda_n |\varphi_n(\lambda_n x)|^2,$$

whose L^2-norm is still one. Hence we can regard it as a probability distribution on $\mathbb{R}$, and it is not hard to show that

$$\operatorname*{w-lim}_{n\to\infty} \rho_n(x)\, dx = \frac{dx}{\pi\sqrt{1-x^2}}. \tag{26}$$

This weak limit distribution has the wall at $x = \pm 1$, and it would be very natural to think that this wall is created by an effect of the potential term x^2 of the harmonic oscillator. Therefore, the wall for one-dimensional quantum walks might be regarded as a ‘virtual potential wall’. However, there is no information about such a potential term in the definition of the one-dimensional quantum walks. It is also interesting to note that the weak limit distribution for *continuous-time* one-dimensional quantum walk ([4]), which is the unitary evolution of the adjacency operator, is precisely equal to (26) ([10]). Here, again there is no information about such a potential term because the adjacency operator is essentially the graph Laplacian. Hence the discretization of the time might not be so crucial for the creation of the wall. It is not clear what kind of dynamical structure of both of discrete- and continuous-time quantum walks creates such a wall.

4. *n*th power formula and the large deviation

In this section, we give a new approach to the large deviation asymptotics given in Theorem 10. The same analysis also proves Theorem 8, although the detail is omitted. In this section, the sequence of integers $x = x_n$ is assumed to satisfy the condition (19). Furthermore, for simplicity we assume that

$$s < |\xi| < \frac{1}{\sqrt{1+t^2}}. \tag{27}$$

When this condition does not hold it is necessary to use some other analytic continuation of a function used below.

The transition probability $p_n(x)$ $(x \in \mathbb{Z})$ is defined by

$$p_n(x) = \|U(C)^n(\delta_0 \otimes \phi)(x)\|^2$$

where $\phi \in \mathbb{C}^2$ with $\|\phi\| = 1$. The quantum walk $U(C)$ is written in the form $U(C) = x + iv + w$ where x, v and w are self adjoint operators defined in (10), (11). These operators are given explicitly as follows:

$$x = \frac{s}{2}(\alpha\tau + \alpha^{-1}\tau^{-1})I, \quad v = \frac{s}{2i}(\alpha\tau - \alpha^{-1}\tau^{-1})J,$$
$$w = \begin{pmatrix} 0 & 0 \\ -t\overline{\beta} & 0 \end{pmatrix}\tau + \begin{pmatrix} 0 & t\beta \\ 0 & 0 \end{pmatrix}\tau^{-1},$$

where $\alpha = a/|a|$, $\beta = b/|b|$, $s = |a|$, $t = |b|$, I is the identity matrix of size two and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Denoting by $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ the standard basis of $\mathbb{C}^2$, we have

$$\begin{aligned}
x(\delta_m \otimes \boldsymbol{e}_i) &= \frac{s}{2}[(\alpha\tau + \alpha^{-1}\tau^{-1})\delta_m] \otimes \boldsymbol{e}_i \quad (i = 1, 2),\\
v(\delta_m \otimes \boldsymbol{e}_1) &= \frac{s}{2i}[(\alpha\tau - \alpha^{-1}\tau^{-1})\delta_m] \otimes \boldsymbol{e}_1,\\
v(\delta_m \otimes \boldsymbol{e}_2) &= -\frac{s}{2i}[(\alpha\tau - \alpha^{-1}\tau^{-1})\delta_m] \otimes \boldsymbol{e}_2,\\
w(\delta_m \otimes \boldsymbol{e}_1) &= -t\overline{\beta}\tau\delta_m \otimes \boldsymbol{e}_2,\\
w(\delta_m \otimes \boldsymbol{e}_2) &= t\beta\tau^{-1}\delta_m \otimes \boldsymbol{e}_1.
\end{aligned}$$

We define the polynomials $p_{ij}^n(z)$ in $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by

$$\begin{aligned}
p_{11}^n(z) &= T_n(s(z+z^{-1})/2) + \frac{s}{2}(z - z^{-1})U_{n-1}(s(z+z^{-1})/2),\\
p_{21}^n(z) &= -tzU_{n-1}(s(z+z^{-1})/2),\\
p_{12}^n(z) &= tz^{-1}U_{n-1}(s(z+z^{-1})/2),\\
p_{22}^n(z) &= T_n(s(z+z^{-1})/2) - \frac{s}{2}(z - z^{-1})U_{n-1}(s(z+z^{-1})/2),
\end{aligned}$$

so that we have

$$\begin{aligned}
U(C)^n(\delta_0 \otimes \boldsymbol{e}_1) &= p_{11}^n(\alpha\tau)\delta_0 \otimes \boldsymbol{e}_1 + \overline{\alpha}\overline{\beta}p_{21}^n(\alpha\tau)\delta_0 \otimes \boldsymbol{e}_2,\\
U(C)^n(\delta_0 \otimes \boldsymbol{e}_2) &= \alpha\beta p_{12}^n(\alpha\tau)\delta_0 \otimes \boldsymbol{e}_1 + p_{22}^n(\alpha\tau)\delta_0 \otimes \boldsymbol{e}_2.
\end{aligned}$$

For any Laurent polynomial $q(z) = \sum_{y\in\mathbb{Z}} \mathrm{c}_y(q)z^y$ with coefficients $\mathrm{c}_y(q) \in \mathbb{C}$, we have $[q(\tau)\delta_0 \otimes \phi](x) = \mathrm{c}_x(q)\phi$. Therefore, since each $\mathrm{c}_x(p_{ij}^n)$ is real, we obtain the following:

$$\begin{aligned} U(C)^n(\delta_0 \otimes \phi)(x) &= \alpha^x \left[(\phi_1 \mathrm{c}_x(p_{11}^n) + \phi_2\alpha\beta \mathrm{c}_x(p_{12}^n))\, \boldsymbol{e}_1 \right. \\ &\qquad \left. + \left(\phi_1\overline{\alpha}\overline{\beta}\mathrm{c}_x(p_{21}^n) + \phi_2 \mathrm{c}_x(p_{22}^n)\right) \boldsymbol{e}_2\right], \\ p_n(x) &= |\phi_1 \mathrm{c}_x(p_{11}^n) + \phi_2\alpha\beta \mathrm{c}_x(p_{12}^n)|^2 + \left|\phi_1\overline{\alpha}\overline{\beta}\mathrm{c}_x(p_{21}^n) + \phi_2 \mathrm{c}_x(p_{22}^n)\right|^2 \\ &= \left(\mathrm{c}_x(p_{11}^n)^2 + \mathrm{c}_x(p_{21}^n)^2\right)|\phi_1|^2 + \left(\mathrm{c}_x(p_{12}^n)^2 + \mathrm{c}_x(p_{22}^n)^2\right)|\phi_2|^2 \\ &\qquad + 2\left(\mathrm{c}_x(p_{11}^n)\mathrm{c}_x(p_{12}^n) + \mathrm{c}_x(p_{21}^n)\mathrm{c}_x(p_{22}^n)\right) \mathrm{Re}\,(\overline{\phi}_1\phi_2\alpha\beta). \end{aligned}$$

Using the relations $\overline{p_{22}^n(z)} = p_{11}^n(z)$ and $\overline{p_{21}^n(z)} = -p_{12}^n(z)$ for $z \in U(1)$ we arrive at the following

$$\begin{aligned} p_n(x) &= \left(\mathrm{c}_x(p_{11}^n)^2 + \mathrm{c}_{-x}(p_{12}^n)^2\right)|\phi_1|^2 + \left(\mathrm{c}_{-x}(p_{11}^n)^2 + \mathrm{c}_x(p_{12}^n)^2\right)|\phi_2|^2 \\ &\qquad + 2\left(\mathrm{c}_x(p_{11}^n)\mathrm{c}_x(p_{12}^n) - \mathrm{c}_{-x}(p_{11}^n)\mathrm{c}_{-x}(p_{12}^n)\right) \mathrm{Re}\,(\overline{\phi}_1\phi_2\alpha\beta). \end{aligned} \tag{28}$$

Therefore, it is only necessary to find asymptotic formula for $\mathrm{c}_x(p_{11}^n)$ and $\mathrm{c}_x(p_{12}^n)$. What we are going to prove the following about these coefficients.

Proposition 12. *Suppose that $x = x_n \in \mathbb{Z}$ satisfies* (19) *with $\xi \in \mathbb{R}$ satisfying* (27). *We set $x = n\xi + \alpha_n$. Then we have*

$$\begin{aligned} \mathrm{c}_x(p_{11}^n) &= (1 + (-1)^{x+n})n^{-1/2}A(\xi)e^{-nH_Q(\xi)/2} \\ &\qquad \times \left((1+\xi)e^{-i\pi\alpha_n/2}R(\xi)^{\alpha_n} + O(n^{-1})\right), \\ \mathrm{c}_x(p_{12}^n) &= (1 + (-1)^{x+n})n^{-1/2}A(\xi)e^{-nH_Q(\xi)/2} \\ &\qquad \times \left(-\sqrt{1-\xi^2}e^{-i\pi\alpha_n/2}R(\xi)^{\alpha_n+1} + O(n^{-1})\right) \end{aligned}$$

where $H_Q(\xi)$ is defined in (24), *and $A(\xi)$ and $R(\xi)$ are positive functions given by*

$$A(\xi) = \sqrt{\frac{t}{2\pi}}\frac{1}{\sqrt{(1-\xi^2)\sqrt{\xi^2 - s^2}}}, \quad R(\xi) = \frac{t|\xi| - \sqrt{\xi^2 - s^2}}{s\sqrt{1-\xi^2}}.$$

According to Proposition 12 and equation (28), transition probability decays exponentially and its exponential decay rate is given by $-H_Q(\xi)$.

4.1. Asymptotics of $\mathrm{c}_x(p_{11}^n)$

For any $x \in \mathbb{Z}$, we have

$$\mathrm{c}_x(p_{11}^n) = \int_{|z|=1} z^{-x}p_{11}^n(z)\,đz = I_n(x) + J_n(x), \quad đz = \frac{dz}{2\pi i z},$$

where the measure $đz$ is the normalized Lebesgue measure on the unit circle and the integrals $I_n(x)$ and $J_n(x)$ are given by

$$I_n(x) = \int_{|z|=1} z^{-x} T_n(s(z+z^{-1})/2)\,đz = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\theta} T_n(s\cos\theta)\,d\theta,$$
$$J_n(x) = \int_{|z|=1} z^{-x}\frac{s}{2}(z-z^{-1})U_{n-1}(s(z+z^{-1})/2)\,đz$$
$$= \frac{is}{2\pi}\int_{-\pi}^{\pi} e^{-ix\theta} s\sin\theta U_{n-1}(s\cos\theta)\,d\theta.$$

From this we have

$$I_n(-x) = \overline{I_n(x)} = I_n(x), \quad J_n(-x) = -\overline{J_n(x)} = -J_n(x).$$

Then, the asymptotic formula for $c_x(p_{11}^n)$ can be deduced from the following lemma.

Lemma 13. *Under the same assumptions as in Proposition* 12*, we have*

$$I_n(x) = (1+(-1)^{x+n})n^{-1/2}A(\xi)e^{-nH_Q(\xi)/2} \times \left(e^{-i\pi\alpha_n/2}R(\xi)^{\alpha_n} + O(n^{-1})\right),$$
$$J_n(x) = (1+(-1)^{x+n})n^{-1/2}A(\xi)e^{-nH_Q(\xi)/2} \times \left(|\xi|e^{-i\pi\alpha_n/2}R(\xi)^{\alpha_n} + O(n^{-1})\right).$$

It is interesting to note that the integral $I_n(x)$ can be written in the following form:

$$I_n(x) = \frac{1}{\pi}\int_0^1 T_x(u)T_n(su)\frac{du}{\sqrt{1-u^2}}.$$

Thus, we have obtained asymptotic expansion of the integral of this form.

4.1.1. Computation of $I_n(x)$. First let us simplify the integral $I_n(x)$. Using the identity $T_n(-x) = (-1)^n T_n(x)$, we see

$$I_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ix\theta}T_n(s\cos\theta)\,d\theta = \frac{1+(-1)^{x+n}}{2\pi}\int_0^{\pi} e^{-ix\theta}T_n(s\cos\theta)\,d\theta,$$

and hence $I_n(x) = 0$ when $x+n$ is odd. Thus we assume that $x+n$ is even. Then, changing the integral contour for the integral $I_n(x)$ leads us to

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{-ix\theta}T_n(s\cos\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} e^{-ix(\theta+i\rho)}T_n(s\cos(\theta+i\rho))\,d\theta \qquad (29)$$

for every $\rho \in \mathbb{R}$, since the integrals on the vertical lines for the rectangle are identical by the assumption that $x+n$ is even. Let Log and $\sqrt{z}$ be the principal branch of the logarithm and the square root, respectively. Then a branch of the arccosine function is given by

$$\arccos(w) = -i\mathrm{Log}\left(w + i\sqrt{1-w^2}\right)$$

which coincides with the inverse function of $\cos\theta$ for $\theta \in [0,\pi]$, when $w \in [-1,1]$. We define a function $h(\zeta)$ by

$$\begin{aligned} h(\zeta) &= i\arccos(s\cos\zeta) \\ &= \operatorname{Log}\left(s\cos(\theta+i\rho) + i\sqrt{1-s^2\cos^2(\theta+i\rho)}\right), \quad \zeta = \theta + i\rho. \end{aligned}$$

Since $\cos(\theta+i\rho) = \cos\theta\cosh\rho - i\sin\theta\sinh\rho$ the function $h(\zeta)$ is holomorphic on the region $|\sinh\rho| < t/s$. Thus we choose ρ in (29) so that $|\sinh\rho| < t/s$. Now we write $x = n\xi + \alpha_n$ with $|\alpha_n| \le C$. For $0 \ne \rho \in \mathbb{R}$ with $|\sinh\rho| < t/s$, we set

$$\Phi_\rho(\theta) = i(\theta+i\rho)\xi - h(\theta+i\rho), \quad \Psi_\rho(\theta) = i(\theta+i\rho)\xi + h(\theta+i\rho)$$

so that

$$I_n(x) = \frac{1}{2\pi}\left(\int_0^\pi e^{-n\Phi_\rho(\theta)}e^{-i(\theta+i\rho)\alpha_n}\,d\theta + \int_0^\pi e^{-n\Psi_\rho(\theta)}e^{-i(\theta+i\rho)\alpha_n}\,d\theta\right).$$

4.1.2. Critical points of the phase functions. In what follows, we assume for simplicity $\xi > 0$. To use the method of stationary phase, we need to examine the phase functions Φ_ρ, Ψ_ρ. We note that

$$\begin{aligned} \Phi'_\rho(\theta) &= i\left(\xi - \frac{s\sin(\theta+i\rho)}{\sqrt{1-s^2\cos^2(\theta+i\rho)}}\right), \\ \Psi'_\rho(\theta) &= i\left(\xi + \frac{s\sin(\theta+i\rho)}{\sqrt{1-s^2\cos^2(\theta+i\rho)}}\right). \end{aligned}$$

To find a critical point of Φ_ρ and Ψ_ρ, we need to examine the equation

$$\xi^2 = \frac{s^2\sin^2(\theta+i\rho)}{1-s^2\cos^2(\theta+i\rho)} = \frac{s^2 - s^2\cos^2(\theta+i\rho)}{1-s^2\cos^2(\theta+i\rho)},$$

which is solved as

$$\cos^2(\theta+i\rho) = \frac{1}{s^2}\frac{s^2-\xi^2}{1-\xi^2} = -\frac{1}{s^2}\frac{\xi^2-s^2}{1-\xi^2}.$$

Since we have assumed $s < \xi < 1$, we have $\cos^2(\theta+i\rho) < 0$ and hence $\cos\theta = 0$. Since $0 \le \theta \le \pi$, we find that the critical point occurs only at $\theta = \pi/2$. We see that

$$\Phi'_\rho(\pi/2) = i\left(\xi - \frac{s\cosh\rho}{\sqrt{1+s^2\sinh^2\rho}}\right), \quad \Psi'_\rho(\pi/2) = i\left(\xi + \frac{s\cosh\rho}{\sqrt{1+s^2\sinh^2\rho}}\right).$$

Therefore, when $s < \xi < 1$, Ψ_ρ has no critical points, and Φ_ρ has only critical points at $\theta = \pi/2$. We set

$$\rho(\xi) := -\sinh^{-1}\left(\frac{1}{s}\sqrt{\frac{\xi^2-s^2}{1-\xi^2}}\right) \qquad (s < \xi < 1/\sqrt{1+t^2}).$$

Then, we see that $\rho(\xi) < 0$, $|\sinh(\rho(\xi))| < t/s$ and for $\rho = \rho(\xi)$, Φ_ρ has the critical point $\theta = \pi/2$ and there are no other critical points. In what follows, we put $\rho = \rho(\xi)$. By our assumption that $s < \xi < 1/\sqrt{1+t^2}$, we see

$$\sinh\rho = -\frac{1}{s}\sqrt{\frac{\xi^2 - s^2}{1-\xi^2}}, \quad \cosh\rho = \frac{1}{s}\frac{t\xi}{\sqrt{1-\xi^2}},$$
$$\cos(\theta + i\rho) = \frac{t\xi}{s\sqrt{1-\xi^2}}\cos\theta + \frac{i}{s}\sqrt{\frac{\xi^2 - s^2}{1-\xi^2}}\sin\theta,$$
$$\sin(\theta + i\rho) = \frac{t\xi}{s\sqrt{1-\xi^2}}\sin\theta - \frac{i}{s}\sqrt{\frac{\xi^2 - s^2}{1-\xi^2}}\cos\theta.$$

A direct computation shows

$$\Phi_\rho''(\theta) = -i\frac{st^2\cos(\theta + i\rho)}{(1 - s^2\cos^2(\rho + i\theta))^{3/2}}, \quad \Phi_\rho''(\pi/2) = \frac{1}{t}(1-\xi^2)\sqrt{\xi^2 - s^2}$$

and hence the critical point $\theta = \pi/2$ is non-degenerate.

4.1.3. Real parts and asymptotics. Next we consider real parts

$$\operatorname{Re}\Phi_\rho = -\rho\xi - \operatorname{Re}h, \quad \operatorname{Re}\Psi_\rho = -\rho\xi + \operatorname{Re}h,$$
$$\operatorname{Re}h = \log\left|s\cos(\theta + i\rho) + i\sqrt{1 - s^2\cos^2(\theta + i\rho)}\right|$$

of the phase functions. A direct computation shows

$$\frac{d}{d\theta}(\operatorname{Re}h(\zeta)) = \operatorname{Re}h'(\zeta) = -\operatorname{Im}\left(\frac{s\sin(\theta + i\rho)}{\overline{\sqrt{1 - s^2\cos^2(\theta + i\rho)}}}\right).$$

Thus, we see

$$(\operatorname{Re}\Phi_\rho)_\theta = -(\operatorname{Re}h)_\theta = \operatorname{Im}\left(\frac{s\sin(\theta + i\rho)}{\overline{\sqrt{1 - s^2\cos^2(\theta + i\rho)}}}\right) = -(\operatorname{Re}\Psi_\rho)_\theta.$$

Hence, if $(\operatorname{Re}h)_\theta = 0$, we see that

$$\frac{s\sin(\theta + i\rho)}{\overline{\sqrt{1 - s^2\cos^2(\theta + i\rho)}}}$$

is real. Therefore, if $(\operatorname{Re}h)_\theta = 0$, we see that $\cos^2(\theta + i\rho)$ is real. This means that $\cos\theta\sin\theta = 0$. When $\cos\theta = 0$, that is $\theta = \pi/2$, $1 - s^2\cos^2(\pi/2 + i\rho) = t^2/(1-\xi^2) > 0$ and $\sin(\pi/2 + i\rho) = t\xi/s\sqrt{1-\xi^2}$ are real. When $\sin\theta = 0$, still we have $1 - s^2\cos^2(\theta + i\rho) = \frac{1-(1+t^2)\xi^2}{1-\xi^2} > 0$ with $\theta = 0, \pi$. But $\sin(\theta + i\rho)$ with $\theta = 0, \pi$ is pure imaginary. Thus, we see that $(\operatorname{Re}\Phi)_\theta = 0$ or $(\operatorname{Re}\Psi)_\theta = 0$ if and only if $\theta = \pi/2$. The values $\mathrm{Re}\,(\Phi(i\rho))$, $\mathrm{Re}\,(\Phi(\pi + i\rho))$, $\mathrm{Re}\,(\Phi(\pi/2 + i\rho))$ are given as

$$\begin{aligned}
&\mathrm{Re}\,(\Phi(i\rho)) = \mathrm{Re}\,(\Phi(\pi + i\rho)) = -\xi\rho = \mathrm{Re}\,(\Psi(i\rho)) = \mathrm{Re}\,(\Psi(\pi + i\rho)) \\
&\mathrm{Re}\,(\Phi(\pi/2 + i\rho)) = -\xi\rho + \frac{1}{2}\log(1-\xi^2) - \log(t + \sqrt{\xi^2 - s^2}), \\
&\mathrm{Re}\,(\Psi(\pi/2 + i\rho)) = -\xi\rho - \frac{1}{2}\log(1-\xi^2) + \log(t + \sqrt{\xi^2 - s^2}).
\end{aligned}$$

Now it is easy to show that

$$\frac{1}{2}\log(1-\xi^2) - \log(t + \sqrt{\xi^2 - s^2}) < 0.$$

From this it follows that $\mathrm{Re}\,(\Phi_\rho(\theta))$ attains its minimum at $\theta = \pi/2$ and $\mathrm{Re}\,(\Psi_\rho(\theta))$ attains its minimum at $\theta = 0$ or $\theta = \pi$ with minimum $-\xi\rho(\xi) > 0$. We then define the function $H_Q(\xi)$ by (twice of) the minimum value of $\mathrm{Re}\,(\Phi_{\rho(\xi)}(\theta))$, namely

$$\begin{aligned}
H_Q(\xi) &:= 2\mathrm{Re}\,(\Phi_{\rho(\xi)}(\pi/2)) \\
&= -2\xi\rho(\xi) + \log(1-\xi^2) - 2\log\left(t + \sqrt{\xi^2 - s^2}\right) \\
&= 2\xi\log\left(t\xi + \sqrt{\xi^2 - s^2}\right) + (1-\xi)\log(1-\xi^2) \\
&\quad - 2\log\left(t + \sqrt{\xi^2 - s^2}\right) - 2\xi\log s.
\end{aligned} \tag{30}$$

Then we see that

$$\mathrm{Re}\,(\Psi_{\rho(\xi)}(\theta)) - H_Q(\xi)/2 = -\frac{1}{2}\log(1-\xi^2) + \log\left(t + \sqrt{\xi^2 - s^2}\right) > 0,$$

and hence we obtain

$$\int_0^\pi e^{-n\Psi_{\rho(\xi)}(\theta)} e^{-i\alpha_n(\theta + i\rho)}\, d\theta = e^{-nH_Q(\xi)/2} O(e^{-nc(\xi)}),$$

where $c(\xi)$ is a constant depending on ξ. The integral with the phase function $\Phi_{\rho(\xi)}$ is localized around the critical point $\theta = \pi/2$ with an exponentially decaying error term. Therefore, a method of stationary phase ([7]) tells us that

$$\begin{aligned}
&\int_0^\pi e^{-n\Phi_{\rho(\xi)}(\theta)} e^{-i\alpha_n(\theta + i\rho)}\, d\theta \\
&\quad = \sqrt{\frac{2\pi}{n}} \frac{e^{-n\Phi_{\rho(\xi)}(\pi/2)}}{\sqrt{\Phi''_{\rho(\xi)}(\pi/2)}} \left(e^{-i\alpha_n(\pi/2 + i\rho(\xi))} + O(n^{-1})\right) \\
&\quad = \frac{\sqrt{2\pi t}}{\sqrt{n(1-\xi^2)\sqrt{\xi^2 - s^2}}} e^{-nH_Q(\xi)/2}(e^{-i\alpha_n(\pi/2 + i\rho(\xi))} + O(n^{-1})).
\end{aligned}$$

Since $R(\xi) = e^{\rho(\xi)}$, we have obtain the formula for $I_n(x)$ given in Lemma 13.

4.1.4. Asymptotics of $J_n(x)$. We have

$$J_n(x) = \frac{is(1+(-1)^{x+n})}{2\pi} \int_0^\pi e^{-ix\theta} \sin\theta U_{n-1}(s\cos\theta)\, d\theta.$$

From this we see $J_n(x) = 0$ if $x+n$ is odd. Thus, we may suppose that $x+n$ is even, and in this case we see

$$J_n(x) = \frac{is}{\pi} \int_0^\pi e^{-ix(\theta+i\rho)} \sin(\theta+i\rho) \frac{\sin(-inh(\theta+i\rho))}{\sin(-ih(\theta+i\rho))}\, d\theta.$$

Set

$$f_\rho(\theta) = e^{-i\alpha_n(\theta+i\rho)} \frac{\sin(\theta+i\rho)}{\sinh(h(\theta+i\rho))}.$$

Then we have

$$J_n(x) = \frac{s}{2\pi}\left(\int_0^\pi e^{-n\Phi_\rho(\theta)} f_\rho(\theta)\, d\theta - \int_0^\pi e^{-n\Psi_\rho(\theta)} f_\rho(\theta)\, d\theta\right).$$

It is easy to see that $f_{\rho(\xi)}(\pi/2) = \frac{\xi}{s} e^{-i\alpha_n(\pi/2+i\rho(\xi))}$ and hence we have obtained Lemma 13.

We note that when $\xi < 0$, we take the different sign in the definition of $\rho(\xi)$ and then the main term in the asymptotics is now given by the integral whose phase function is $\Psi_{\rho(\xi)}(\theta)$.

4.2. Asymptotics of $\mathrm{c}_x(p_{12}^n)$

Finally we analyze the coefficient of the Laurent polynomial p_{12}^n. A computation similar to that for $\mathrm{c}_x(p_{11}^n)$ shows that it is zero when $x+n$ is odd, and when $x+n$ is even we have

$$\mathrm{c}_x(p_{12}^n) = \frac{t}{\pi} \int_0^\pi e^{-ix\theta} e^{-i\theta} U_{n-1}(s\cos\theta)\, d\theta.$$

The same computation as before shows

$$\mathrm{c}_x(p_{12}^n) = \frac{t}{2\pi}\left(\int_0^\pi e^{-n\Phi_\rho(\theta)} g_\rho(\theta)\, d\theta - \int_0^\pi e^{-n\Psi_\rho(\theta)} g_\rho(\theta)\, d\theta\right)$$

where $g_\rho(\theta)$ is given by

$$g_\rho(\theta) = e^{-i(\theta+i\rho)\alpha_n} \frac{e^{-i(\theta+i\rho)}}{\sinh(h(\theta+i\rho)).}$$

Therefore, the same computation shows Proposition 12.

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Tatsuya Tate
Mathematical Institute, Graduate School of Sciences
Tohoku University
Aoba, Sendai 980-8578, Japan
e-mail: tate@math.tohoku.ac.jp

Part IV

Algebraic Structures

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Center-symmetric Algebras and Bialgebras: Relevant Properties and Consequences

Mahouton Norbert Hounkonnou and Mafoya Landry Dassoundo

Abstract. Lie admissible algebra structures, called center-symmetric algebras, are defined. Their main properties and algebraic consequences are derived and discussed. Bimodules are given and used to build a center-symmetric algebra on the direct sum of the underlying vector space and a finite-dimensional vector space. Then, the matched pair of center-symmetric algebras is established and related to the matched pair of sub-adjacent Lie algebras. Besides, Manin triples of center-symmetric algebras are defined and linked with their associated matched pairs. Further, center-symmetric bialgebras of center-symmetric algebras are investigated and discussed. Finally, a theorem yielding the equivalence between Manin triples of center-symmetric algebras, matched pairs of Lie algebras and center-symmetric bialgebras is provided.

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Keywords. Lie-admissible algebra; Lie algebra; center-symmetric algebra; matched pair; Manin triple; bialgebra; cocycle.

1. Introduction

Consider an algebra $(\mathcal{A}, \mu)$, i.e., a $\mathbb{K}$ vector space $\mathcal{A}$ endowed with a binary operation or law (bilinear homomorphism) μ defined as:

$$\begin{aligned} \mu : \mathcal{A} \times \mathcal{A} &\longrightarrow \mathcal{A} \\ (x, y) &\longmapsto \mu(x, y). \end{aligned}$$

Define the *associator of the binary product* by a trilinear homomorphism on $\mathcal{A}$ as follows [4]:

$$\begin{aligned} \mathrm{ass}_\mu : \mathcal{A} \times \mathcal{A} \times \mathcal{A} &\longrightarrow \mathcal{A} \\ (x, y, z) &\longmapsto \mu(\mu(x, y), z) - \mu(x, \mu(y, z)). \end{aligned}$$

Let $\sigma \in \Sigma_3$ (here Σ_n is the symmetric group of degree n ($n \in \mathbb{N}$)), acting on the associator as:

$$\sigma(x_1, x_2, x_3) = \left(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}\right).$$

Definition 1 ([9]). The algebra $\mathcal{A} = (\mathcal{A}, \mu)$ is called Lie admissible if the commutator of μ, denoted by $[,]_\mu$, defines on $\mathcal{A}$ a Lie algebra structure, i.e., $[x, y]_\mu = \mu(x, y) - \mu(y, x)$ (bilinear and skew-symmetric) and $[[x, y]_\mu, z]_\mu + [[z, x]_\mu, y]_\mu + [[y, z]_\mu, x]_\mu = 0$ (Jacobi identity).

Definition 2 ([4]). The algebra $(\mathcal{A}, \mu)$ is called Lie-admissible if and only if μ satisfies:

$$\sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} \operatorname{ass}_\mu \circ \sigma = 0, \tag{1}$$

where ε is the signature of σ.

Definition 3 ([8]). Let G be a subgroup of Σ_3. We say that the algebra law is G-associative if

$$\sum_{\sigma \in \mathrm{G}} (-1)^{\varepsilon(\sigma)} \operatorname{ass}_\mu \circ \sigma = 0. \tag{2}$$

The subgroups of Σ_3 are well known. We have: $\mathrm{G}_1 = \{\mathrm{id}\}$, $\mathrm{G}_2 = \{\mathrm{id}, \tau_{12}\}$, $\mathrm{G}_3 = \{\mathrm{id}, \tau_{23}\}$, $\mathrm{G}_4 = \{\mathrm{id}, \tau_{13}\}$, $\mathrm{G}_5 = \{\mathrm{A}_3\}$ (the alternating group) and $\mathrm{G}_6 = \Sigma_3$. Here τ_{ij} is the transposition between i and j, i.e., explicitly:

$$\tau_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \tau_{13} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \tau_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \mathrm{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

We deduce the following types of Lie admissible algebras:

1. If μ is G_1-associative, then μ is an associative law.
2. If μ is G_2-associative, then μ is a law of a Vinberg algebra [12]. If $\mathcal{A}$ is finite-dimensional, then the associated Lie admissible algebra is provided with an affine structure.
3. If μ is G_3-associative, then μ is a law of a pre-Lie algebra (also called left-symmetric algebra).
4. If μ is G_4-associative, then μ satisfies

$$(xy)z - x(yz) = (zy)x - z(yx), \qquad \forall x, y, z \in \mathcal{A}. \tag{3}$$

   We called the corresponding algebra *center-symmetric algebra*.

5. If μ is G_5-associative, then μ satisfies the generalized Jacobi condition, i.e.,

$$(xy)z + (yz)x + (zx)y = x(yz) + y(zx) + z(xy). \tag{4}$$

   Moreover, if the law is antisymmetric, then it is a law of a Lie algebra.

6. If μ is G_6-associative, then μ is a Lie admissible law.

This work aims at studying G_4-associative structures, called center-symmetric algebras. Their algebraic properties are investigated. Related bimodule and matched pairs are given. The associated Manin triples look like the Manin triples of Lie algebras [2]. Besides, from the symmetry role of matched pairs, equivalent relations are established in the framework of center-symmetric bialgebras making some bridges with the Lie-bialgebra construction by Drinfeld [3].

Throughout this work, we consider $\mathcal{A}$, a finite-dimensional vector space over the field $\mathbb{K}$ of characteristic zero (0) together with a bilinear product $\cdot$ defined as $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $(x, y) \mapsto x \cdot y$.

2. Basic properties: main definitions and algebraic consequences

In this section, we give the definition of a center-symmetric algebra, provide their basic properties and deduce relevant algebraic consequences, similarly to the known framework of left-symmetric algebras [1].

Definition 4. $(\mathcal{A}, \cdot)$, (or simply $\mathcal{A}$), is said to be a center-symmetric algebra if $\forall\, x, y, z \in \mathcal{A}$, the associator of the bilinear product $\cdot$, defined by $(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$, is symmetric in x and z, i.e.,

$$(x, y, z) = (z, y, x). \tag{5}$$

To simplify the notation, we will denote $x \cdot y$ by xy if there is no danger of confusion.

Remark 5. Any associative algebra is a center-symmetric algebra.

Proposition 6. *The bilinear product (commutator)* $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, $(x, y) \longmapsto [x, y] = x \cdot y - y \cdot x$ *gives a Lie bracket structure on* $\mathcal{A}$, *known as the sub-adjacent Lie algebra* $\mathcal{G}(\mathcal{A}) := (\mathcal{A}, [\cdot, \cdot])$ *of* $(\mathcal{A}, \cdot)$.

Proof. By definition of the commutator, $[\cdot, \cdot]$ is bilinear and skew symmetric. The Jacoby identity comes from a straightforward computation. □

Thus, as in the case of left-symmetric algebras, $(\mathcal{A}, \cdot)$ can be called the compatible center-symmetric algebra structure of the Lie algebra $\mathcal{G}(\mathcal{A})$.

Considering the representations of the left L and right R multiplication operations:

$$L : \mathcal{A} \longrightarrow \mathfrak{gl}(\mathcal{A})$$

$$x \longmapsto L_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & x \cdot y, \end{array} \tag{6}$$

$$R : \mathcal{A} \longrightarrow \mathfrak{gl}(\mathcal{A})$$

$$x \longmapsto R_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & y \cdot x, \end{array} \tag{7}$$

we infer the adjoint representation $\mathrm{ad} := L - R$ of the sub-adjacent Lie algebra $\mathcal{G}(\mathcal{A})$ of a center-symmetric algebra $\mathcal{A}$ as follows:

$$\mathrm{ad} : \mathcal{A} \longrightarrow \mathfrak{gl}(\mathcal{A})$$

$$x \longmapsto \mathrm{ad}_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & [x, y], \end{array} \tag{8}$$

such that $\forall\, x, y \in \mathcal{A}, \mathrm{ad}_x(y) := (L_x - R_x)(y)$.

Proposition 7. *Let $(\mathcal{A}, \cdot)$ be a center-symmetric algebra, and L, (resp. R), be the linear representation of the left, (resp. right), multiplication operator. Then,*

1. *For all $x, y \in \mathcal{A}$ we have: $[L_x, R_y] = [L_y, R_x]$ and $L_{x\cdot y} - L_x L_y = R_x R_y - R_{y\cdot x}$.*
2. $\mathrm{ad} = L - R$ *is a linear representation of the sub-adjacent Lie algebra $\mathcal{G}(\mathcal{A})$ of $(\mathcal{A}, \cdot)$, i.e.,* $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$, $\forall\, x, y \in \mathcal{A}$.

Proof. It is immediate from the definitions of considered operators. □

3. Bimodules and matched pairs

Definition 8. Let $\mathcal{A}$ be a center-symmetric algebra, V be a vector space. Suppose $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ be two linear maps satisfying: For all $x, y \in \mathcal{A}$,

$$[l_x, r_y] = [l_y, r_x] \tag{9}$$

$$l_{xy} - l_x l_y = r_x r_y - r_{yx}. \tag{10}$$

Then, (l, r, V) (or simply (l, r)) is called bimodule of the center-symmetric algebra $\mathcal{A}$.

In this case, the following statement can be proved by a direct computation.

Proposition 9. *Let $(\mathcal{A}, \cdot)$ be a center-symmetric algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$. Then, (l, r, V) is a bimodule of $\mathcal{A}$ if and only if, the semi-direct sum $\mathcal{A} \oplus V$ of vector spaces is turned into a center-symmetric algebra by defining the multiplication in $\mathcal{A} \oplus V$ by $(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l_{x_1} v_2 + r_{x_2} v_1)$, $\forall x_1, x_2 \in \mathcal{A}$, $v_1, v_2 \in V$. We denote it by $\mathcal{A} \ltimes_{l,r} V$ or simply $\mathcal{A} \ltimes V$.*

Furthermore, we derive the next result.

Proposition 10. *Let $\mathcal{A}$ be a center-symmetric algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$, such that (l, r, V) is a bimodule of $\mathcal{A}$. Then, the map: $l - r : \mathcal{A} \longrightarrow \mathfrak{gl}(V)\ x \longmapsto l_x - r_x$, is a linear representation of the sub-adjacent Lie algebra of $\mathcal{A}$.*

Example. According to Proposition 7, one can deduce that (L, R, A) is a bimodule of the center-symmetric algebra $\mathcal{A}$, where L and R are the left and right multiplication operator representations, respectively.

Definition 11 ([6]). Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie algebras and let $\mu : \mathcal{H} \to \mathfrak{gl}(\mathcal{G})$ and $\rho : \mathcal{G} \to \mathfrak{gl}(\mathcal{H})$ be two Lie algebra representations satisfying: For all $x, y \in \mathcal{G}$, $a, b \in \mathcal{H}$,

$$\rho(x)\,[a, b] - [\rho(x)a, b] - [a, \rho(x)b] + \rho(\mu(a)x)b - \rho(\mu(b)x)a = 0, \tag{11}$$

$$\mu(a)\,[x, y] - [\mu(a)x, y] - [x, \mu(a)y] + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0. \tag{12}$$

Then, $(\mathcal{G}, \mathcal{H}, \rho, \mu)$ is called a matched pair of the Lie algebras $\mathcal{G}$ and $\mathcal{H}$, denoted by $\mathcal{H} \bowtie^{\rho}_{\mu} \mathcal{G}$. In this case, $(\mathcal{G} \oplus \mathcal{H}, *)$ defines a Lie algebra with respect to the product $*$ satisfying:

$$(x + a) * (y + b) = [x, y] + \mu(a)y - \mu(b)x + [a, b] + \rho(x)b - \rho(y)a.$$

Theorem 12. *Let* $(\mathcal{A},\cdot)$ *and* $(\mathcal{B},\circ)$ *be two center-symmetric algebras. Suppose that* $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ *and* $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ *are bimodules of* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively, obeying the relations:*

$$\begin{aligned}-r_{\mathcal{A}}(x)(a\circ b)+r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a+a\circ(r_{\mathcal{A}}(x)b)+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a&\\ +(l_{\mathcal{A}}(x)b)\circ a-l_{\mathcal{A}}(x)(b\circ a)=0,&\end{aligned}\tag{13}$$

$$\begin{aligned}-r_{\mathcal{B}}(a)(x\cdot y)+r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x+x\cdot(r_{\mathcal{B}}(a)y)+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x&\\ +(l_{\mathcal{B}}(a)y)\cdot x-l_{\mathcal{B}}(a)(y\cdot x)=0,&\end{aligned}\tag{14}$$

$$\begin{aligned}a\circ(l_{\mathcal{A}}(x)b)+(r_{\mathcal{A}}(x)b)\circ a-(r_{\mathcal{A}}(x)a)\circ b-l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b+r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a&\\ +l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a-b\circ(l_{\mathcal{A}}(x)a)-r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b=0,&\end{aligned}\tag{15}$$

$$\begin{aligned}x\cdot(l_{\mathcal{B}}(a)y)+(r_{\mathcal{B}}(a)y)\cdot x-(r_{\mathcal{B}}(a)x)\cdot y-l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y+r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x&\\ +l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x-y\cdot(l_{\mathcal{B}}(a)x)-r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y=0,&\end{aligned}\tag{16}$$

for any $x, y \in A$ *and* $a, b \in \mathcal{B}$*. Then, there is a center-symmetric algebra structure on* $\mathcal{A}\oplus\mathcal{B}$ *given by:* $(x+a)*(y+b) = (x\cdot y+l_{\mathcal{B}}(a)y+r_{\mathcal{B}}(b)x)+(a\circ b+l_{\mathcal{A}}(x)b+r_{\mathcal{A}}(y)a)$*. We denote this center-symmetric algebra by* $\mathcal{A} \bowtie^{l_{\mathcal{A}},r_{\mathcal{A}}}_{l_{\mathcal{B}},r_{\mathcal{B}}} \mathcal{B}$*, or simply by* $\mathcal{A} \bowtie \mathcal{B}$*. Then* $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ *satisfying the above conditions is called matched pair of the center-symmetric algebras* $\mathcal{A}$ *and* $\mathcal{B}$*.*

Proof. Consider $x, y \in \mathcal{A}$ and $a, b \in \mathcal{B}$. We have:

$$(x+a)*(y+b) = (xy+l_{\mathcal{B}}(a)y+r_{\mathcal{B}}(b)x)+(a\circ b+l_{\mathcal{A}}(x)b+r_{\mathcal{A}}(y)a),$$

and the associator takes the form:

$$\begin{aligned}(x+a,y+b,z+c)=&\,(x,y,z)+(a,b,c)+\{r_{\mathcal{B}}(c)(x\cdot y)+l_{\mathcal{A}}(x\cdot y)c\\ &-x\cdot(r_{\mathcal{B}}(c)y)-l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c)-r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x\}\\ &+\{r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c-r_{\mathcal{B}}(b\circ c)x\\ &+(l_{\mathcal{A}}(x)b)\circ c-l_{\mathcal{A}}(x)(b\circ c)\}+\{(r_{\mathcal{B}}(b)x)\cdot z\\ &+l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z+r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b)-x\cdot(l_{\mathcal{B}}(b)z)\\ &-r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x-l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b)\}+\{(l_{\mathcal{B}}(a)y)\cdot z\\ &+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z+r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a)-l_{\mathcal{B}}(a)(y\cdot z)\\ &-r_{\mathcal{A}}(y\cdot z)a\}+\{r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)+(r_{\mathcal{A}}(y)a)\circ c\\ &+l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c-l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)-a\circ(l_{\mathcal{A}}(y)c)\\ &-r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a\}+\{l_{\mathcal{B}}(a\circ b)z+r_{\mathcal{A}}(z)(a\circ b)\\ &-l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z)-a\circ(r_{\mathcal{A}}(z)b)-r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a\},\end{aligned}$$

which can also be re-expressed as:

$$\begin{aligned}(x+a,y+b,z+c)=&\,(x,y,z)+(x,y,c)+(x,b,z)+(x,b,c)\\ &+(a,y,z)+(a,y,c)+(a,b,z)+(a,b,c).\end{aligned}\tag{17}$$

Similarly,

$$(z+c, y+c, x+a) = (z,y,x) + (z,y,a) + (z,b,x) + (z,b,a) \\ + (c,y,x) + (c,b,a) + (c,y,a) + (c,b,x). \quad (18)$$

Using the fact that $(l_{\mathcal{A}}, r_{\mathcal{A}})$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}})$ is a bimodule of $\mathcal{B}$, one arrives at the following result:

$$(x+a, y+b, z+c) = (z+c, y+b, x+a) \Longleftrightarrow \begin{cases} (x,y,z) = (z,y,x) \\ (x,y,c) = (c,y,x) \\ (x,b,z) = (z,b,x) \\ (x,b,c) = (c,b,x) \\ (a,y,z) = (z,y,a) \\ (a,y,c) = (c,y,a) \\ (a,b,z) = (z,b,a) \\ (a,b,c) = (c,b,a) \end{cases}$$

$$\Longleftrightarrow \begin{cases} (l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B}), (l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A}) \\ \text{are bimodules of} \\ \mathcal{A} \text{ and } \mathcal{B}, \text{ respectively,} \\ (x,y,c) = (c,y,x) \ (14) \\ (x,b,z) = (z,b,x) \ (16) \\ (x,b,c) = (c,b,x) \ (13) \\ (a,y,c) = (c,y,a).(15) \end{cases}.$$

This last relation ends the proof. □

Moreover, every center-symmetric algebra which is a direct sum of the underlying spaces of two center-symmetric sub-algebras can be obtained in the above way.

Corollary 13. *Let* $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ *be a matched pair of center-symmetric algebras. Then,* $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{B}), l_{\mathcal{A}} - r_{\mathcal{A}}, l_{\mathcal{B}} - r_{\mathcal{B}})$ *is a matched pair of sub-adjacent Lie algebras* $\mathcal{G}(\mathcal{A})$ *and* $\mathcal{G}(\mathcal{B})$.

Proof. By using Proposition 10, the bimodules $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$, we have: $\mathrm{ad}_{\mathcal{A}} := l_{\mathcal{A}} - r_{\mathcal{A}}$ and $\mathrm{ad}_{\mathcal{B}} := l_{\mathcal{B}} - r_{\mathcal{B}}$ are the linear representations of the sub-adjacent Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ of the center-symmetric algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, the statement that $\mathcal{G}(\mathcal{A}) \bowtie^{\mathrm{ad}_{\mathcal{A}}}_{\mathrm{ad}_{\mathcal{B}}} \mathcal{G}(\mathcal{B})$ is a matched pair of the Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ follows from Theorem 12. Hence, ($\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{B}), \mathrm{ad}_{\mathcal{A}}, \mathrm{ad}_{\mathcal{B}}$) is a matched pair of sub-adjacent Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$. □

Definition 14. Let (l, r, V) be a bimodule of a center-symmetric algebra $\mathcal{A}$, where V is a finite-dimensional vector space. The dual maps l^*, r^* of the linear maps l, r, are defined, respectively, as: $l^*, r^* : \mathcal{A} \to \mathfrak{gl}(V^*)$ such that:

$$l^* : \mathcal{A} \longrightarrow \mathfrak{gl}(V^*)$$
$$x \longmapsto l^*_x : \begin{array}{ccc} V^* & \longrightarrow & V^* \\ u^* & \longmapsto & l^*_x u^* : \begin{array}{ccc} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle l^*_x u^*, v\rangle := \langle u^*, l_x v\rangle \end{array} \end{array}, \tag{19}$$

$$r^* : \mathcal{A} \longrightarrow \mathfrak{gl}(V^*)$$
$$x \longmapsto r^*_x : \begin{array}{ccc} V^* & \longrightarrow & V^* \\ u^* & \longmapsto & r^*_x u^* : \begin{array}{ccc} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle r^*_x u^*, v\rangle := \langle u^*, r_x v\rangle \end{array} \end{array}, \tag{20}$$

for all $x \in \mathcal{A}, u^* \in V^*, v \in V$.

Proposition 15. *Let $\mathcal{A}$ be a center-symmetric algebra and $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ be two linear maps, where V is a finite-dimensional vector space. The following conditions are equivalent:*

1. *(l, r, V) is a bimodule of $\mathcal{A}$.*
2. *(r^*, l^*, V^*) is a bimodule of $\mathcal{A}$.*

Proof. It stems from Definition 14. □

Theorem 16. *Let $(\mathcal{A}, \cdot)$ be a center-symmetric algebra. Suppose that there exists a center-symmetric algebra structure "$\circ$" on its dual space $\mathcal{A}^*$. Then,*

$$(\mathcal{A}, \mathcal{A}^*, R^*_\cdot, L^*_\cdot, R^*_\circ, L^*_\circ)$$

is a matched pair of center-symmetric algebras $\mathcal{A}$ and $\mathcal{A}^*$ if and only if

$$(\ \mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), -\operatorname{ad}^*_\cdot, -\operatorname{ad}^*_\circ\ )$$

is a matched pair of Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$.

Proof. By considering Theorem 12, setting $l_{\mathcal{A}} := R^*_\cdot, r_{\mathcal{A}} := L^*_\cdot, l_{\mathcal{B}} := R^*_\circ, r_{\mathcal{B}} := L^*_\circ$, and exploiting Definition 11 with $\mathcal{G} := \mathcal{G}(\mathcal{A}), \mathcal{H} := \mathcal{G}(\mathcal{A}^*)$, $\rho := R^*_\cdot - L^*_\cdot, \mu := R^*_\circ - L^*_\circ$, and the relations (19) and (20), we get the equivalences. □

Proposition 17. *Let $\mathcal{G}$ be a Lie algebra. Suppose $\rho : \mathcal{G} \to \mathfrak{gl}(V)$ and $\mu : \mathcal{G} \to \mathfrak{gl}(W)$ be two linear representations of $\mathcal{G}$, where V and W are two vector spaces. Then, the linear map $\rho \otimes 1 + 1 \otimes \mu : \mathcal{G} \to \mathfrak{gl}(V \otimes W)$ given by $(\rho \otimes 1 + 1 \otimes \mu)(v, w) := \rho(x)v \otimes w + v \otimes \mu(x)v$ is also a representation of $\mathcal{G}$.*

Proof. It comes from a straightforward computation. □

Theorem 18. *Let $\mathcal{A}$ be a center-symmetric algebra with the product given by the linear map $\beta^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Suppose there is a center-symmetric algebra structure "$\circ$" on the dual space $\mathcal{A}^*$ provided by a linear map $\alpha^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$. Then, $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), -\operatorname{ad}^*_\cdot, -\operatorname{ad}^*_\circ)$ is a matched pair of Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$ if and only if $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a 1-cocycle of $\mathcal{G}(\mathcal{A})$ associated to $(-\operatorname{ad}_\cdot) \otimes 1 + 1 \otimes (-\operatorname{ad}_\cdot)$ and $\beta : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ is a 1-cocycle of $\mathcal{G}(\mathcal{A}^*)$ associated to $(-\operatorname{ad}_\circ) \otimes 1 + 1 \otimes (-\operatorname{ad}_\circ)$.*

Proof. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of $\mathcal{A}$ and $\{e_1^*, e_2^*, \cdots, e_n^*\}$ its dual basis. Consider $e_i \cdot e_j = \sum_{k=1}^{n} c_{ij}^k e_k$ and $e_i^* \circ e_j^* = \sum_{k=1}^{n} f_{ij}^k e_k^*$, where $c_{ij}^k, f_{ij}^k \in \mathbb{K}$ are structure constants associated to $\cdot$ and $\circ$, respectively. Then, it follows that

$$\alpha(e_k) = \sum_{i,j=1}^{n} f_{ij}^k e_i \otimes e_j, \quad \beta(e_k^*) = \sum_{i,j=1}^{n} c_{ij}^k e_i^* \otimes e_j^*,$$

and

$$\alpha([e_i, e_j]) = \sum_{m,l=1}^{n} \sum_{k=1}^{n} \{(c_{ij}^k - c_{ji}^k) f_{ml}^k\} \, e_m \otimes e_l, \tag{21}$$

$$\beta([e_i^*, e_j^*]) = \sum_{m,l=1}^{n} \sum_{k=1}^{n} \{(f_{ij}^k - f_{ji}^k) c_{ml}^k\} \, e_m^* \otimes e_l^*, \tag{22}$$

and we get:

$$\begin{aligned} &\{(-\operatorname{ad}.)(e_i) \otimes 1 + 1 \otimes (-\operatorname{ad}.)(e_i)\}\alpha(e_j) \\ &\qquad - \{(-\operatorname{ad}.)(e_j) \otimes 1 + 1 \otimes (-\operatorname{ad}.)(e_j)\}\alpha(e_i) \\ &= \sum_{m,l=1}^{n} \sum_{k=1}^{n} \big\{ - f_{kl}^j (c_{ik}^m - c_{ki}^m) + f_{kl}^i (c_{jk}^m - c_{kj}^m) \\ &\qquad\qquad - f_{mk}^j (c_{ik}^l - c_{ki}^l) + f_{mk}^i (c_{jk}^l - c_{kj}^l) \big\} e_m \otimes e_l. \end{aligned} \tag{23}$$

Taking into account the fact that α is a 1-cocycle of $\mathcal{G}(\mathcal{A})$ associated to $(-\operatorname{ad}.) \otimes 1 + 1 \otimes (-\operatorname{ad}.)$, and using the relations (21) and (23) yield:

$$\begin{aligned} &\sum_{k=1}^{n} (c_{ij}^k - c_{ji}^k) f_{ml}^k \\ &\quad = \sum_{k=1}^{n} \big\{ f_{kl}^i (c_{jk}^m - c_{kj}^m) - f_{kl}^j (c_{ik}^m - c_{ki}^m) + f_{mk}^i (c_{jk}^l - c_{kj}^l) - f_{mk}^j (c_{ik}^l - c_{ki}^l) \big\}. \end{aligned} \tag{24}$$

Besides, we obtain:

$$\begin{aligned} &\{(-\operatorname{ad}_\circ)(e_i^*) \otimes 1 + 1 \otimes (-\operatorname{ad}_\circ)(e_i^*)\}\beta(e_j^*) \\ &\qquad - \{(-\operatorname{ad}_\circ)(e_j^*) \otimes 1 + 1 \otimes (-\operatorname{ad}_\circ)(e_j^*)\}\beta(e_i^*) \\ &= \sum_{m,l=1}^{n} \sum_{k=1}^{n} \big\{ - c_{kl}^j (f_{ik}^m - f_{ki}^m) + c_{kl}^i (f_{jk}^m - f_{kj}^m) \\ &\qquad\qquad - c_{mk}^j (f_{ik}^l - f_{ki}^l) + c_{mk}^i (f_{jk}^l - f_{kj}^l) \big\} (e_m^* \otimes e_l^*). \end{aligned} \tag{25}$$

As β is the 1-cocycle issued from $(-\operatorname{ad}_\circ)\otimes 1 + 1\otimes(-\operatorname{ad}_\circ)$ and using the relations (22) and (25), we obtain:

$$\begin{aligned}&\sum_{k=1}^{n}(f_{ij}^{k}-f_{ji}^{k})c_{ml}^{k} \\ &\quad=\sum_{k=1}^{n}\left\{c_{kl}^{i}(f_{jk}^{m}-f_{kj}^{m})-c_{kl}^{j}(f_{ik}^{m}-f_{ki}^{m})+c_{mk}^{i}(f_{jk}^{l}-f_{kj}^{l})c_{mk}^{j}(f_{ik}^{l}-f_{ki}^{l})\right\}.\end{aligned}\tag{26}$$

Now, let us find the relations associated to equations (11), (12) of the matched pair of Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$. We have $\forall\, i,j,k$:

$$\left\langle(-\operatorname{ad}^*_\cdot)(e_i)e_j^*, e_k\right\rangle = -\left\langle\sum_{k=1}^{m}(c_{ik}^{j}-c_{ki}^{j})e_k^*, e_k\right\rangle,$$

providing

$$(-\operatorname{ad}^*_\cdot)(e_i)e_j^* = -\sum_{k=1}^{n}(c_{ik}^{j}-c_{ki}^{j})e_k^*.\tag{27}$$

Similarly,

$$(-\operatorname{ad}^*_\circ)(e_i^*)e_j = -\sum_{k=1}^{n}(f_{ik}^{j}-f_{ki}^{j})e_k,\tag{28}$$

$$\begin{aligned}(-\operatorname{ad}^*_\circ)(e_m^*)[e_i,e_j] &= \sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})(-\operatorname{ad}^*_\circ)(e_m^*)e_k \\ &\qquad -\sum_{l=1}^{n}\sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})(f_{ml}^{k}-f_{lm}^{k})e_l.\end{aligned}$$

Then,

$$\begin{aligned}&(-\operatorname{ad}^*_\circ)(e_m^*)[e_i,e_j] \\ &\quad= -\sum_{l=1}^{n}\sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})(f_{ml}^{k}-f_{lm}^{k})e_l, -\operatorname{ad}^*_\circ(\operatorname{ad}^*_\cdot(e_i)e_m^*)e_j \\ &\qquad\qquad -[e_i,\operatorname{ad}^*_\circ(e_m^*)e_j]+\operatorname{ad}^*_\circ(\operatorname{ad}^*_\cdot(e_j)e_m^*)-[\operatorname{ad}^*_\circ(e_m^*)e_i,e_j] \\ &\quad=\sum_{l=1}^{n}\sum_{k=1}^{n}\{-(c_{ik}^{m}-c_{ki}^{m})(f_{kl}^{j}-f_{lk}^{j})-f_{mk}^{j}(c_{ik}^{l}-c_{ki}^{l}) \\ &\qquad\qquad +f_{km}^{j}(c_{ik}^{l}-c_{ki}^{l})(c_{jk}^{m}-c_{kj}^{m})(f_{kl}^{i}-f_{lk}^{i})-(f_{mk}^{i}-f_{km}^{i})(c_{kj}^{l}-c_{jk}^{l})\}e_l.\end{aligned}\tag{29}$$

Using the fact that $(\mathcal{G}(\mathcal{A}),\mathcal{G}(\mathcal{A}^*),\operatorname{ad}^*_\cdot,\operatorname{ad}^*_\circ)$ is a bimodule of Lie algebras, we have

$$\begin{aligned}&\sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})(f_{ml}^{k}-f_{lm}^{k}) \\ &\quad=\sum_{k=1}^{n}-(c_{ik}^{m}-c_{ki}^{m})(f_{kl}^{j}-f_{lk}^{j})-f_{mk}^{j}(c_{ik}^{l}-c_{ki}^{l})+f_{km}^{j}(c_{ik}^{l}-c_{ki}^{l}) \\ &\qquad\quad +(c_{jk}^{m}-c_{kj}^{m})(f_{kl}^{i}-f_{lk}^{i})+(f_{mk}^{i}-f_{km}^{i})(c_{jk}^{l}-c_{kj}^{l}),\end{aligned}\tag{30}$$

that is,

$$\begin{aligned}\sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})f_{ml}^{k}+\sum_{k=1}^{n}&(c_{ik}^{m}-c_{ki}^{m})f_{kl}^{j}+(c_{ik}^{l}-c_{ki}^{l})f_{mk}^{j}\\ &-(c_{jk}^{m}-c_{kj}^{m})f_{kl}^{i}-(c_{jk}^{l}-c_{kj}^{l})f_{mk}^{i}\\ =\sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})f_{lm}^{k}+\sum_{k=1}^{n}&(c_{ik}^{m}-c_{ki}^{m})f_{lk}^{j}+(c_{ik}^{l}-c_{ki}^{l})f_{km}^{j}\\ &-(c_{jk}^{m}-c_{kj}^{m})f_{lk}^{i}-(c_{jk}^{l}-c_{kj}^{l})f_{km}^{i}.\end{aligned}$$

Replacing l (resp. m) by m (resp. l) in the right-hand side of the equality leads to:

$$\begin{aligned}\sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})f_{ml}^{k}=\sum_{k=1}^{n}\big\{&-(c_{ik}^{m}-c_{ki}^{m})f_{kl}^{j}-(c_{ik}^{l}-c_{ki}^{l})f_{mk}^{j}\\ &+(c_{jk}^{m}-c_{kj}^{m})f_{kl}^{i}+(c_{jk}^{l}-c_{kj}^{l})f_{mk}^{i}\big\},\end{aligned}\tag{31}$$

which is identical to equation (24). Besides,

$$\begin{aligned}&(-\operatorname{ad}^*_\cdot)(e_m)[e_i^*,e_j^*]\\ &\quad=-\sum_{l=1}^{n}\sum_{k=1}^{n}\big\{(f_{ij}^{k}-f_{ji}^{k})(c_{ml}^{k}-c_{lm}^{k})\big\}\,e_l^*-\operatorname{ad}^*_\cdot(\operatorname{ad}^*_\circ(e_i^*)e_m)e_j^*\\ &\qquad-[e_i^*,\operatorname{ad}^*_\cdot(e_m)e_j^*]+\operatorname{ad}^*_\cdot(\operatorname{ad}^*_\circ(e_j^*)e_m)e_m^*-[\operatorname{ad}^*_\cdot(e_m)e_i^*,e_j^*]\\ &\quad=\sum_{l=1}^{n}\sum_{k=1}^{n}\{-(f_{ik}^{m}-f_{ki}^{m})(c_{kl}^{j}-c_{lk}^{j})-c_{mk}^{j}(f_{ik}^{l}-f_{ki}^{l})\\ &\qquad+c_{km}^{j}(f_{ik}^{l}-f_{ki}^{l})+(f_{jk}^{m}-f_{kj}^{m})(c_{kl}^{i}-c_{lk}^{i})-(c_{mk}^{i}-c_{km}^{i})(f_{kj}^{l}-f_{jk}^{l})\}e_l^*.\end{aligned}\tag{32}$$

Then, with $\mathcal{G}(\mathcal{A})\bowtie^{-\operatorname{ad}^*_\cdot}_{-\operatorname{ad}^*_\circ}\mathcal{G}(\mathcal{A}^*)$ and the relation (32), we obtain

$$\begin{aligned}&\sum_{k=1}^{n}(f_{ij}^{k}-f_{ji}^{k})(c_{ml}^{k}-c_{lm}^{k})\\ &\quad=\sum_{k=1}^{n}-(f_{ik}^{m}-f_{ki}^{m})(c_{kl}^{j}-c_{lk}^{j})-c_{mk}^{j}(f_{ik}^{l}-f_{ki}^{l})\\ &\qquad\quad+c_{km}^{j}(f_{ik}^{l}-f_{ki}^{l})+(f_{jk}^{m}-f_{kj}^{m})(c_{kl}^{i}-c_{lk}^{i})+(c_{mk}^{i}-c_{km}^{i})(f_{jk}^{l}-f_{kj}^{l}),\end{aligned}$$

i.e.,

$$\begin{aligned}\sum_{k=1}^{n}(f_{ij}^{k}-f_{ji}^{k})c_{ml}^{k}+\sum_{k=1}^{n}&c_{kl}^{j}(f_{ik}^{m}-f_{ki}^{m})+c_{kl}^{i}(f_{jk}^{m}-f_{kj}^{m})\\ &-c_{mk}^{j}(f_{ik}^{l}-f_{ki}^{l})-c_{mk}^{i}(f_{jk}^{l}-f_{kj}^{l})\\ =\sum_{k=1}^{n}(f_{ij}^{k}-f_{ji}^{k})c_{lm}^{k}+\sum_{k=1}^{n}&c_{lk}^{j}(f_{ik}^{m}-f_{ki}^{m})+c_{lk}^{i}(f_{jk}^{m}-f_{kj}^{m})\\ &-c_{km}^{j}(f_{ik}^{l}-f_{ki}^{l})-c_{km}^{i}(f_{jk}^{l}-f_{kj}^{l}),\end{aligned}$$

Replacing l, (resp. m), by m,(resp. l), in the right-hand side of the equality leads to

$$\sum_{k=1}^{n}(f_{ij}^{k}-f_{ji}^{k})c_{ml}^{k}=\sum_{k=1}^{n}-c_{kl}^{j}(f_{ik}^{m}-f_{ki}^{m})+c_{kl}^{i}(f_{jk}^{m}-f_{kj}^{m}) \\ -c_{mk}^{j}(f_{ik}^{l}-f_{ki}^{l})+c_{mk}^{i}(f_{kj}^{l}-f_{jk}^{l}), \tag{33}$$

which is identical to equation (26). □

4. Manin triples and center-symmetric bialgebras

In this section, similarly to the notion of a Manin triple of Lie algebras introduced in [2], we first give the definition of a Manin triple of a center-symmetric algebra and investigate its associated bialgebra structure. Then, we provide the basic definition and properties of center-symmetric bialgebras.

Definition 19. A Manin triple of center-symmetric algebras is a triple $(\mathcal{A},\mathcal{A}^{+},\mathcal{A}^{-})$ together with a non degenerate symmetric bilinear form $\mathfrak{B}\ (\ ;\ )$ on $\mathcal{A}$ which is invariant, i.e., $\forall x,y,z\in\mathcal{A}$, $\mathfrak{B}(x*y,z)=\mathfrak{B}(x,y*z)$, satisfying:

1. $\mathcal{A}=\mathcal{A}^{+}\oplus\mathcal{A}^{-}$ as $\mathbb{K}$-vector space;
2. $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$ are center-symmetric subalgebras of $\mathcal{A}$;
3. $\mathcal{A}^{+}$ and $\mathcal{A}^{-}$ are isotropic with respect to $\mathfrak{B}(;)$.

Two Manin triples $(\mathcal{A}_1,\mathcal{A}_1^{+},\mathcal{A}_1^{-},\mathfrak{B}_1)$ and $(\mathcal{A}_2,\mathcal{A}_2^{+},\mathcal{A}_2^{-},\mathfrak{B}_2)$ of center-symmetric algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are homomorphic (isomorphic) if there is a homomorphism (isomorphism) $\varphi:\mathcal{A}_1\to\mathcal{A}_2$ such that: $\varphi(\mathcal{A}_1^{+})\subset\mathcal{A}_2^{+}$, $\varphi(\mathcal{A}_1^{-})\subset\mathcal{A}_2^{-}$, $\mathfrak{B}_1(x,y)=\varphi^{*}\mathfrak{B}_2(\varphi(x),\varphi(y))=\mathfrak{B}_2(\varphi(x),\varphi(y))$. In particular, if $(\mathcal{A},\cdot)$ is a center-symmetric algebra, and if there exists a center-symmetric algebra structure on its dual space $\mathcal{A}^{*}$ denoted $(\mathcal{A}^{*},\circ)$, then there is a center-symmetric algebra structure on the direct sum of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^{*}$ (see Theorem 12) such that $(\mathcal{A}\oplus\mathcal{A}^{*},\mathcal{A},\mathcal{A}^{*})$ is the associated Manin triple with the invariant bilinear symmetric form given by $\mathfrak{B}_{\mathcal{A}}(x+a^{*},y+b^{*})=<x,b^{*}>+<y,a^{*}>$, $\forall x,y\in\mathcal{A};a^{*},b^{*}\in\mathcal{A}^{*}$, called the standard Manin triple of the center-symmetric algebra $\mathcal{A}$.

Theorem 20. *Let $(\mathcal{A},\cdot)$ and $(\mathcal{A}^{*},\circ)$ be two center-symmetric algebras. Then, the six-tuple $(\mathcal{A},\mathcal{A}^{*},R_{\cdot}^{*},L_{\cdot}^{*};R_{\circ}^{*},L_{\circ}^{*})$ is a matched pair of center-symmetric algebras $\mathcal{A}$ and $\mathcal{A}^{*}$ if and only if $(\mathcal{A}\oplus\mathcal{A}^{*},\mathcal{A},\mathcal{A}^{*})$ is their standard Manin triple.*

Proof. By considering that $(\mathcal{A},\mathcal{A}^{*},R_{\cdot}^{*},L_{\cdot}^{*};R_{\circ}^{*},L_{\circ}^{*})$ is a matched pair of center-symmetric algebras, it follows that the bilinear product $*$ defined in Theorem 12 is center-symmetric on the direct sum of the underlying vectors spaces, $\mathcal{A}\oplus\mathcal{A}^{*}$. Computing and comparing the relations, we get:
$\mathfrak{B}_{\mathcal{A}}\left((x+a)*(y+b),(z+c)\right)=\mathfrak{B}_{\mathcal{A}}\left((x+a),(y+b)*(z+c)\right)\ \forall x,y,z\in\mathcal{A};$
$a,b,c\in\mathcal{A}^{*}$, which expresses the invariance of the standard bilinear form on $\mathcal{A}\oplus\mathcal{A}^{*}$. Therefore, $(A\oplus\mathcal{A}^{*},\mathcal{A},\mathcal{A}^{*})$ is the standard Manin triple of the center-symmetric algebras $\mathcal{A}$ and $\mathcal{A}^{*}$. □

Definition 21. Let $\mathcal{A}$ be a vector space. A center-symmetric bialgebra structure on $\mathcal{A}$ is a pair of linear maps (α, β) such that $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\beta : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ satisfying:

1. $\alpha^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ is a center-symmetric algebra structure on $\mathcal{A}^*$,
2. $\beta^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a center-symmetric algebra structure on $\mathcal{A}$,
3. α is a 1-cocycle of $\mathcal{G}(\mathcal{A})$ associated to $(-\operatorname{ad}_.) \otimes 1 + 1 \otimes (-\operatorname{ad}_.)$,
4. β is 1-cocycle of $\mathcal{G}(\mathcal{A}^*)$ associated to $(-\operatorname{ad}_\circ) \otimes 1 + 1 \otimes (-\operatorname{ad}_\circ)$.

We also denote this center-symmetric bialgebra by $(\mathcal{A}, \mathcal{A}^*, \alpha, \beta)$ or simply $(\mathcal{A}, \mathcal{A}^*)$.

Proposition 22. *Let $(\mathcal{A}, \cdot)$ be a center-symmetric algebra and $(\mathcal{A}^*, \circ)$ be a center-symmetric algebra structure on its dual space $\mathcal{A}^*$. Then the following conditions are equivalent:*

1. $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ *is the standard Manin triple of considered center-symmetric algebras;*
2. $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), -\operatorname{ad}^*_., -\operatorname{ad}^*_\circ)$ *is a matched pair of sub-adjacent Lie algebras;*
3. $(\mathcal{A}, \mathcal{A}^*, R^*_., L^*_., R^*_\circ, L^*_\circ)$ *is a matched pair of center-symmetric algebras;*
4. $(\mathcal{A}, \mathcal{A}^*)$ *is a center-symmetric bialgebra.*

Proof. From Theorem 16, (2) $\Longleftrightarrow$ (3), while from Theorem 18, (2) $\Longleftrightarrow$ (4). Theorem 20 shows that (1) $\Longleftrightarrow$ (3). □

5. Concluding remarks

In this work, we have defined Lie admissible algebra structures, called center-symmetric algebras for which the main properties and algebraic consequences have been derived and discussed. Bimodules have been given and used to build a center-symmetric algebra on the direct sum of a center-symmetric algebra and a vector space. Then, we have established the matched pairs of center-symmetric algebras, which have been related to the matched pairs of sub-adjacent Lie algebras. Besides, we have defined the Manin triples of center-symmetric algebras and linked it with their associated matched pairs. Further, we have investigated and discussed center-symmetric bialgebras of center-symmetric algebras. Finally, we have provided a theorem yielding the equivalence between Manin triples of center-symmetric algebras, matched pairs of Lie algebras and center-symmetric algebras, and center-symmetric bialgebra.

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Mahouton Norbert Hounkonnou and Mafoya Landry Dassoundo
University of Abomey-Calavi
International Chair in Mathematical
Physics and Applications
ICMPA-UNESCO Chair, P.O. 072 Box 50
Cotonou, Rep. of Benin
e-mail: norbert.hounkonnou@cipma.uac.bj
with copy to hounkonnou@yahoo.fr
mafoya.dassoundo@cipma.uac.bj

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N-point Virasoro Algebras Considered as Krichever–Novikov Type Algebras

Martin Schlichenmaier

Abstract. We explain how the recently again discussed N-point Witt, Virasoro, and affine Lie algebras are genus zero examples of the multi-point versions of Krichever–Novikov type algebras as introduced and studied by Schlichenmaier. Using this more general point of view, useful structural insights and an easier access to calculations can be obtained. As example, explicit expressions for the three-point situation are given.

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1. Introduction

In the context of genus zero conformal field theory (CFT) the Witt algebra and its universal central extension, the Virasoro algebra, play an important role by encoding conformal symmetry [1]. Krichever–Novikov algebras are higher genus and multi-point analogs of them. For higher genus, but still only for two points where poles are allowed, some of the algebras were generalized in 1986 by Krichever and Novikov [18–20]. In 1990 the author [22, 24, 25] extended this approach further to the general multi-point case. These extensions were not straightforward generalizations. The crucial point was to introduce a replacement of the graded algebra structure present in the "classical" case. Krichever and Novikov found that an almost-grading, see Definition 1, will be enough to allow constructions in representation theory, like triangular decompositions, highest weight modules, Verma modules and many more things. In [24, 25] it was realized that a splitting of the set of points A where poles are allowed into two disjoint non-empty subsets

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$A = I \cup O$ is crucial for introducing an almost-grading. For every such splitting the corresponding almost-grading was given. Essentially different splittings (not just corresponding to interchanging I and O) will yield essentially different almost-gradings. For the general theory (including the classical case) see the recent monograph [31].

The genus zero but more than two point case was also addressed by Bremner [3–5]. Recently, there was again a revived interest in the genus zero, multi-point situation. See, e.g., work by Cox, Jurisisch, Martins, and collaborators [6–9, 15]. In particular, this interest comes from representation theory and its interpretation in the context of quantization of (conformal) field theory. In some of these articles the vector field algebras were called N-Virasoro algebras, affine algebras N-point affine algebras, etc. Here we like to stress the fact, that these algebras are also examples of genus zero Krichever–Novikov (KN) type algebras in their multi-point version as introduced by the current author.

In a recent manuscript [32] I showed this in detail. Furthermore, I give a common treatment of all these kinds of algebras. Taking this interpretation seriously, gives a better understanding of the situation and an easier approach to calculations. Furthermore, it explains certain properties remarked by the authors of [7, 9, 15].

In this write-up of a talk presented at the Bialowieza meeting in 2015, I will report on the results obtained in [32] and add some additional comments. For all proofs and calculations I refer to [32]. For general background information on Krichever–Novikov type algebras see [31], or the review [30].

Here we will recall the geometric setup for KN type algebras, introduce the algebras including their almost-grading and triangular decomposition. Then we determine "all" their central extensions.

The outcome will be that all cocycle classes for the vector field algebra and the differential operator algebras are geometric and that their universal central extensions can be explicitly given. The same is done for the current algebra. In this way multi-point affine algebras are obtained. The Heisenberg algebra will be obtained from the function algebra by cocycles which are multiplicative or equivalently $\mathcal{L}$-invariant, see the definitions below. The presentation allows an easy access to calculations of structure constants and cocycle values for these algebras. As an illustration we give explicit results for the three point genus zero situation.

2. Classical algebras

In purely algebraic terms the *Virasoro algebra* $\mathcal{V}$ can be defined in terms of generators $\{e_n (n \in \mathbb{Z}), t\}$ and (Lie algebra) relations [1]

$$[e_n, e_m] = (m-n)e_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m} \cdot t, \quad [t, e_n] = 0. \tag{1}$$

The element t is called the central element.

[1] δ_n^m is the Kronecker delta, which is 1 if $n = m$, otherwise 0.

Without term coming with the central element the algebra is called the *Witt algebra* $\mathcal{W}$. With respect to $\mathcal{W}$, the algebra $\mathcal{V}$ is its universal central extension.

There are other algebras which are relatives of the Virasoro algebra. We only recall the definition of the affine algebras [16, 21]. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and β the Cartan–Killing form. For $\overline{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ we take the Lie algebra structure

$$[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}, \quad x, y \in \mathfrak{g}, \quad n, m \in \mathbb{Z}. \tag{2}$$

Now we set $\widehat{\mathfrak{g}} = \mathbb{C} \otimes \overline{\mathfrak{g}}$ as vector space, denote by $\widehat{x \otimes z^n} := (0, x \otimes z^n)$ and $t := (1, 0)$, and take as Lie structure on $\widehat{\mathfrak{g}}$

$$[\widehat{x \otimes z^n}, \widehat{y \otimes z^m}] := \widehat{[x, y] \otimes z^{n+m}} - \beta(x, y) \cdot n\, \delta_m^{-n} \cdot t, \quad [t, \widehat{\mathfrak{g}}] = 0. \tag{3}$$

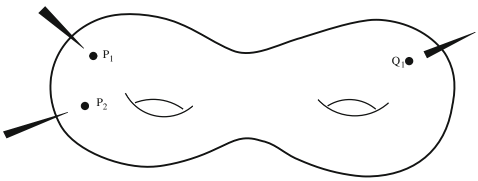
Indeed, this is a Lie algebra $\widehat{\mathfrak{g}}$. It is called the affine Lie algebra associated to $\mathfrak{g}$. Without central term, the algebra is called current or loop algebra.

We remark that all these Lie algebras are infinite-dimensional graded Lie algebras. The grading is given by defining

$$\deg(e_n) = n, \quad \deg(x \otimes z^n) = n, \quad \deg(t) = 0. \tag{4}$$

3. Geometric set-up

Even if the results which we present here are dealing with genus zero, for a deeper understanding of the structure it will be helpful to consider Riemann surfaces of arbitrary genus. Hence, let Σ_g be a compact Riemann surface of genus $g = g(\Sigma_g)$ and A be a finite set of points of Σ_g (also called marked points). Let furthermore A be decomposed into $A = I \cup O$, $I = \{P_1, \ldots, P_K\}$ (in-points) and $O = \{Q_1, \ldots, Q_M\}$ (out-points), both non-empty and disjoint. All points should be pairwise distinct.



For the case of genus zero with $A = \{P_1, P_2, \ldots, P_N\}$, by a fractional linear transformation (i.e., a complex automorphism of the Riemann sphere), the point P_N can be brought to ∞. We obtain

$$P_i = a_i, \quad a_i \in \mathbb{C}, \ i = 1, \ldots, N-1, \qquad P_N = \infty, \tag{5}$$

with the local coordinates $z - a_i$, $i = 1, \ldots, N-1$, $w = 1/z$, at the marked points. The *classical situation* is given by

$$\Sigma_0 = S^2, \qquad I = \{0\}, \quad O = \{\infty\}\ . \tag{6}$$

4. Geometric realizations of the Krichever–Novikov type algebras

Let $\mathcal{K}$ be the canonical line bundle, i.e., the line bundle over Σ whose local sections are the local holomorphic differentials. We consider the tensor power

$$\mathcal{K}^\lambda := \mathcal{K}^{\otimes \lambda} \quad \text{for } \lambda \in \mathbb{Z}\,. \tag{7}$$

Its sections are the forms of weight λ. For example, for $\lambda = -1$ we obtain the local holomorphic vector fields and $\lambda = 0$ yields the functions. After fixing a square root L of $\mathcal{K}$ (also called theta characteristics, or spin structure) we can even consider half-integer λ powers. For higher genus g we have a finite number of choices. But for $g = 0$ there is only one square-root, the tautological bundle U. In this presentation we ignore the half-forms (e.g., the supercase).

Next we set

$$\mathcal{F}^\lambda := \mathcal{F}^\lambda(A) := \{f \text{ is a global meromorphic section of } K^\lambda \mid \text{such that } f \text{ is holomorphic over } \Sigma \setminus A\}. \tag{8}$$

These are infinite-dimensional vector spaces, their elements are called meromorphic forms of weight λ. We sum over all $\lambda \in \mathbb{Z}$ (respectively $\in 1/2\mathbb{Z}$)

$$\mathcal{F} := \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{F}^\lambda. \tag{9}$$

An associative structure

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{10}$$

is defined in terms of local representing meromorphic functions

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto s\,dz^\lambda \cdot t\,dz^\nu = s \cdot t\; dz^{\lambda+\nu}. \tag{11}$$

This turns $\mathcal{F}$ to an associative and commutative graded algebra. Note that $\mathcal{A} := \mathcal{F}^0$ is a subalgebra and that the $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.

Next we define a Lie algebra structure

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (s,t) \mapsto [s,t], \tag{12}$$

in local representatives of the sections as

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto [s\,dz^\lambda, t\,dz^\nu] := \left((-\lambda) s \frac{dt}{dz} + \nu\, t \frac{ds}{dz}\right) dz^{\lambda+\nu+1}, \tag{13}$$

The space $\mathcal{F}$ with $[.,.]$ is a Lie algebra and with respect to $\cdot$ and $[.,.]$ it is a Poisson algebra. Obviously, $\mathcal{L} := \mathcal{F}^{-1}$ is a Lie subalgebra (the algebra of vector fields), and the $\mathcal{F}^\lambda$s are Lie modules over $\mathcal{L}$.

The subspace $\mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L} =: \mathcal{D}^1$ is also a Lie subalgebra of $\mathcal{F}$. It is the Lie algebra of differential operators of degree ≤ 1.

Finally, we define the (generalized) current algebra as follows. We fix an arbitrary finite-dimensional complex Lie algebra $\mathfrak{g}$. The generalized current algebra is defined as $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with the Lie product

$$[x \otimes f, y \otimes g] = [x,y] \otimes f \cdot g, \qquad x,y \in \mathfrak{g}, \quad f,g \in \mathcal{A}. \tag{14}$$

All the above algebras consist of meromorphic objects defined over compact Riemann surfaces. We call them *Krichever–Novikov* (*KN*) *type algebras*. The classical algebras of Section 2 are obtained for the classical geometric situation (6).

5. Almost-graded structure

In the classical situation the introduced algebras are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [18] there is a weaker concept, an almost-grading, which to a large extent is a valuable replacement of a honest grading. As shown in [24] such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O.

Definition 1. Let $\mathcal{L}$ be a Lie or an associative algebra such that

$$\mathcal{L} = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n \tag{15}$$

is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,

(ii) There exist constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

The elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called *homogeneous subspace* of degree n.

In a similar manner almost-graded modules over almost-graded algebras are defined. In [24], see also [31], an almost-grading for $\mathcal{F}^\lambda$ is introduced by exhibiting certain elements $f^\lambda_{n,p} \in \mathcal{F}^\lambda$, $p = 1, \ldots, K$ which constitute a basis of a subspace $\mathcal{F}^\lambda_n$ of homogeneous elements of degree n.

For the current presentation the details are not of importance. We only note that the basis element $f^\lambda_{n,p}$ of degree n fulfills

$$\operatorname{ord}_{P_i}(f^\lambda_{n,p}) = (n + 1 - \lambda) - \delta^p_i, \quad P_i \in I,\ i = 1, \ldots, K\,, \tag{16}$$

and that there are certain prescriptions at the points in O such that the element $f^\lambda_{n,p}$ is essentially unique. In the next section we will give the elements for genus zero explicitly.

But here a warning is in order: The decomposition (and hence the almost-grading) depends on the splitting of A into $I \cup O$.

6. Genus zero – standard splitting

Now we return to the genus zero case. We take the standard splitting:

$$I = \{P_1, P_2, \ldots, P_{N-1}\}, \qquad O = \{\infty\}, \tag{17}$$

and have $K = N - 1$. It is enough to construct a basis $\{A_{n,p}\}$ of $\mathcal{A}$, as then $\mathcal{F}_n^\lambda = \mathcal{A}_{n-\lambda} dz^\lambda$, with $f_{n,p}^\lambda = A_{n-\lambda,p} dz^\lambda$. We set for $n \in \mathbb{Z}$

$$A_{n,p}(z) := (z - a_p)^n \cdot \prod_{\substack{i=1 \\ i \neq p}}^{K} (z - a_i)^{n+1} \cdot \alpha(p)^{n+1}, \quad p = 1, \ldots, K\,. \tag{18}$$

Here $\alpha(p)$ is a normalization factor such that in the local coordinate $(z - a_p)$

$$A_{n,p}(z) = (z - a_p)^n (1 + O(z - a_p))\,. \tag{19}$$

The order at ∞ is automatically fixed as $-(Kn + K - 1)$. For the vector fields we take

$$e_{n,p} := f_{n,p}^{-1} = A_{n+1,p} \frac{d}{dz}, \quad p = 1, \ldots, K\,. \tag{20}$$

The above algebras are almost-graded algebras. In fact,

$$\mathcal{F}^\lambda = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}_m^\lambda, \qquad \text{with} \quad \dim \mathcal{F}_m^\lambda = K, \tag{21}$$

and there exist R_1, R_2 (independent of n and m) such that

$$\mathcal{A}_n \cdot A_m \subseteq \bigoplus_{h=n+m}^{n+m+R_1} \mathcal{A}_h\ , \qquad [\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{L}_h\ . \tag{22}$$

For genus zero and the standard splitting we have

$$R_1 = \begin{cases} 0, & N = 2, \\ 1, & N > 2, \end{cases} \qquad R_2 = \begin{cases} 0, & N = 2, \\ 1, & N = 3, \\ 2, & N > 3\,. \end{cases} \tag{23}$$

An important consequence of the almost-grading (not only in genus zero) is the existence of a triangular decomposition $\mathcal{U} = \mathcal{U}_{[-]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[+]}$ with

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m. \tag{24}$$

Here $\mathcal{U}_{[+]}$ and $\mathcal{U}_{[-]}$ are subalgebras, whereas $\mathcal{U}_{[0]}$ is only a subspace. Such a triangular decomposition is of relevance for the construction of highest weight representation.

Another basis. Our algebra $\mathcal{A}$ can also be given as the algebra

$$\mathcal{A} = \mathbb{C}[(z - a_1), (z - a_1)^{-1}, (z - a_2)^{-1}, \ldots, (z - a_{N-1})^{-1}], \tag{25}$$

with the obvious relations. If we set $A_n^{(i)} := (z - a_i)^n$, then

$$A_n^{(i)}, \quad n \in \mathbb{Z},\ i = 1, \ldots, N - 1 \tag{26}$$

is a generating set of $\mathcal{A}$. A basis is given, e.g., by

$$A_n^{(1)},\ n \in \mathbb{Z}, \quad A_{-n}^{(i)},\ n \in \mathbb{N},\ i = 2, \dots, N-1. \tag{27}$$

But this basis does not define an almost-graded structure.

7. Central extensions

Next we want to introduce central extensions of our algebras. The following is also valid in arbitrary genus.

Let C_i be positively oriented (deformed) circles on the Riemann surface Σ_g around the points P_i in I, $i = 1, \dots, K$, and C_j^* positively oriented circles around the points Q_j in O, $j = 1, \dots, M$. A cycle C_S on Σ_g is called a separating cycle if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points.

In the following we will integrate over closed curves C meromorphic differentials on Σ_g without poles in $\Sigma_g \setminus A$. In this context integration over C and C' gives the same value if $[C] = [C']$ in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{Z})$. Moreover,

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*]. \tag{28}$$

Given such a separating cycle C_S (respectively cycle class) we define the linear form

$$\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega\,. \tag{29}$$

This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega \;=\; \sum_{i=1}^{K} \mathrm{res}_{P_i}(\omega) \;=\; -\sum_{l=1}^{M} \mathrm{res}_{Q_l}(\omega). \tag{30}$$

A central extension of a Lie algebra $\mathcal{U}$ is defined on the vector space direct sum $\widehat{\mathcal{U}} = \mathbb{C} \oplus U$ ($\hat{x} := (0, x)$, $t := (1, 0)$)

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \Phi(x, y) \cdot t, \quad [t, \widehat{U}] = 0, \quad x, y \in U, \tag{31}$$

with a bilinear form Φ on $\mathcal{U}$. Recall that $\widehat{U}$ will be a Lie algebra, if and only if Φ is antisymmetric and fulfills the Lie algebra 2-cocycle condition for all $x, y, z \in \mathcal{U}$

$$0 = d_2\Phi(x, y, z) := \Phi([x, y], z) + \Phi([y, z], x) + \Phi([z, x], y). \tag{32}$$

A 2-cocycles Φ is a coboundary if there exists a $\phi : \mathcal{U} \to \mathbb{C}$ such that

$$\Phi(x, y) = d_1\phi(x, y) = \phi([x, y]). \tag{33}$$

It is well known that the second Lie algebra cohomology $\mathrm{H}^2(\mathcal{U}, \mathbb{C})$ of $\mathcal{U}$ with values in the trivial module $\mathbb{C}$ classifies equivalence classes of central extensions.

A Lie algebra $\mathcal{U}$ is called perfect if $[\mathcal{U}, \mathcal{U}] = \mathcal{U}$. Perfect Lie algebras admit universal central extensions.

8. Local and bounded cocycles

In the previous section we considered all central extensions. Now we are heading towards central extensions which are "compatible" with the almost-grading.

Definition 2.

(a) Let γ be a 2-cocycle for the almost-graded Lie algebra $\mathcal{U}$, then γ is called a *local cocycle* if $\exists T_1, T_2$ such that

$$\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies T_2 \leq n + m \leq T_1. \tag{34}$$

(b) A 2-cocycle γ is called *bounded* (from above) if $\exists T_1$ such that

$$\gamma(\mathcal{U}_n, \mathcal{U}_m) \neq 0 \implies n + m \leq T_1. \tag{35}$$

(c) A cocycle class $[\gamma]$ is called a *local* (*bounded*) *cohomology class* if and only if it admits a representing cocycle which is local (respectively bounded).

Note that, e.g., in a local cocycle class not all representing cocycles are local. Obviously, the set of local (or bounded) cocycles is a subspace of all cocycles. Moreover, the set $\mathrm{H}^2_{\mathrm{loc}}(\mathcal{U}, \mathbb{C})$ (respectively $\mathrm{H}^2_{\mathrm{b}}(\mathcal{U}, \mathbb{C})$) of local (respectively bounded) cohomology classes is a subspace of the full cohomology space.

In [27] and [28] I classified all bounded and local cocycles for the KN type algebras.

A cocycle $\gamma : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$ is called a *geometric cocycle* if there is a bilinear map $\widehat{\gamma} : \mathcal{U} \times \mathcal{U} \to \mathcal{F}^1$, such that γ is the composition of $\widehat{\gamma}$ with an integration, i.e.,

$$\gamma = \gamma_C := \frac{1}{2\pi \mathrm{i}} \int_C \widehat{\gamma} \tag{36}$$

with C a curve on $\Sigma_g \setminus A$.

Given $\widehat{\gamma}$ only the class of C in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C})$ plays a role. Recall that

$$\dim \mathrm{H}_1(\Sigma_g \setminus A, \mathbb{C}) = \begin{cases} 2g, & \#A = N = 0, 1, \\ 2g + (N-1), & \#A = N \geq 2\,. \end{cases} \tag{37}$$

In particular, for genus zero and $N \geq 1$ we have

$$\dim \mathrm{H}_1(\Sigma_0 \setminus A, \mathbb{C}) = (N-1). \tag{38}$$

In this case a basis is, e.g., given by circles C_i around the points P_i, where we leave out one of them. For example $[C_i]$, $i = 1, \ldots, N-1$ will do. But there might be a more convenient choice, e.g., for the standard splitting we take $[C_S] = -[C_\infty]$ and $[C_i],\ i = 1, \ldots, N-2$.

9. Main results

The results presented in this section are valid for genus zero and the multi-point situation. In this situation the algebras are sometimes called N-Virasoro algebra or N-point $\mathfrak{g}$-algebras.

The results presented here (and some more) are obtained in [32]. There also the proofs can be found. Here I only give the results and the basic strategy employed.

1. We show that all cocycle classes are bounded cocycle classes with respect to the almost-grading induced by the standard splitting.
2. Next, the classification result of bounded cocycle classes [27, 28] of the author is used which gives a complete classification and explicit expressions by integrals over curves.
3. In particular, in genus zero our cocycles classes are geometric cocycles classes with respect to certain explicitly given one-forms.
4. In genus zero the geometric cocycles can be obtained via integration over circles around the points in I, or alternatively around ∞ and hence can be calculate via residues.
5. In case that the Lie algebra is perfect the universal central extension can directly be given.

9.1. Function algebra – Heisenberg algebra

The function algebra is abelian, hence there are too many Lie algebra cocycles. For the above classification we have to restrict ourselves to the following naturally given cocycle classes:

- A cocycle γ is called $\mathcal{L}$-invariant if and only if

$$\gamma(e\,.\,f,g)+\gamma(f,e\,.\,g)=0,\qquad f,g\in\mathcal{A},\quad e\in\mathcal{L}. \tag{39}$$

- A cocycle γ is called multiplicative if

$$\gamma(fg,h)+\gamma(gh,f)+\gamma(hf,g)=0,\qquad f,g,h\in\mathcal{A}. \tag{40}$$

Theorem 3 ([32]**).** *If one of the above properties is fulfilled then it is a geometric cocycle. It is linear combination of*

$$\gamma_i^{\mathcal{A}}(f,g)=\frac{1}{2\pi \mathrm{i}}\int_{C_i} f dg=\mathrm{res}_{a_i}(f dg),\quad i=1,\ldots,N-1. \tag{41}$$

The cocycle is bounded from above with respect to the almost-grading given by the standard splitting.

As one can show that the cocycles of the type (41) are both $\mathcal{L}$-invariant and multiplicative, we obtain that every $\mathcal{L}$-invariant cocycle is multiplicative and vice versa.

The unique cocycle (up to scaling) of this type which is local with respect to the standard splitting is obtained as the sum of the $\gamma_i^{\mathcal{A}}$, $i=1,\ldots,N-1$, or alternatively as $\gamma_\infty^{\mathcal{A}}$ [27].

In the two point situation we obtain

$$\gamma(A_n, A_m) = \alpha \cdot (-n) \cdot \delta_m^{-n}. \tag{42}$$

The Heisenberg algebra is a central extension of the function algebra obtained via such a cocycle. This could be either the local one or the "full" one (depending on the convention one is using). For the full one the center is $(N-1)$-dimensional. Of course, the function algebra does not have a universal central extension, but the full Heisenberg algebra might be some kind of substitute.

9.2. Vector field algebra

Theorem 4 ([32]). *Every cocycle class is geometric and given by*

$$\gamma^{\mathcal{L}}_{C,R}(e,f) = \frac{1}{2\pi \mathrm{i}} \int_C \left(\frac{1}{2}(ef''' - e'''f) - R(ef' - e'f)\right)dz, \tag{43}$$

where R is a projective connection.

We do no repeat the definition of a projective connection here, as for our coordinates we can take $R = 0$. The strategy explained above yields that $\mathrm{H}^2(\mathcal{L}, \mathbb{C})$ is $(N-1)$-dimensional and is generated by integrating (43) over the C_i. Furthermore, these cocycles generate the universal central extension.

By different techniques Skryabin [33] has shown the existence of a universal central extension for arbitrary genus.

9.3. Differential operator algebra

Also here the main result is that all cocycle classes are geometric. The $\mathcal{L}$-invariant cocycles for $\mathcal{A}$ and arbitrary cocycles for $\mathcal{L}$ define two cocycle types for $\mathcal{D}^1$. But there is a another type, called mixing cocycles

$$\gamma^{(m)}_{C,T}(e,g) := \frac{1}{2\pi \mathrm{i}} \int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A}, \tag{44}$$

Here T is an affine connection. As it can be taken to be zero on the affine part we do not repeat its definition here.

Theorem 5 ([32]). *All cocycle classes are geometric and are linear combinations of the introduced three types. The Lie algebra $\mathcal{D}^1$ is perfect and the universal central extension has a $3 \cdot (N-1)$-dimensional center.*

9.4. Current algebra

Recall that the current algebra $\overline{\mathfrak{g}}$ is defined with respect to a finite-dimensional Lie algebra $\mathfrak{g}$. For the classification results we assume that $\mathfrak{g}$ is simple. Let β be the Cartan–Killing form, then we show [32] that all cocycles are geometric and cohomologous to (with C an arbitrary curve)

$$\gamma^{\overline{\mathfrak{g}}}_{\beta,C}(x \otimes f, y \otimes g) = \beta(x,y) \cdot \gamma^{\mathcal{A}}_C(f,g) = \beta(x,y) \cdot \frac{1}{2\pi \mathrm{i}} \int_C f dg\,. \tag{45}$$

As $\overline{\mathfrak{g}}$ is perfect, it admits a universal central extension which has a $(N-1)$-dimensional center which can be explicitly given. If we consider only central extensions which admit an extension of the almost-grading (e.g., with respect to the standard splitting) we obtain that this central extension class is unique [28].

The author has also corresponding results for the general reductive case. Furthermore, the superalgebra can be treated in the same manner [29].

10. Three-point algebras

The case of only three points where poles are allowed is to a certain extent special as we have additional symmetries. These symmetries can be used to simplify the calculations of structure constants even further. Additionally, the three-point case plays a role in quite a number of applications. See, e.g., the tetrahedron algebra appearing in statistical mechanics, in particular the work of Terwilliger and collaborators [2, 13, 14]. See also Kazhdan and Lusztig [17]. For applications to deformations of Lie algebras see also [10–12, 26].

By a fractional linear transformation, respectively by a PGL(2) action, the three points can be brought to the points $0, 1$ and ∞. After having fixed $A = \{0, 1, \infty\}$, by applying an automorphism from the remaining symmetry group, we obtain the situation

$$A = I \cup O, \qquad I := \{0, 1\}, \quad \text{and} \quad O := \{\infty\}. \tag{46}$$

This we will consider here. We will "symmetrize" and "anti-symmetrize" our basis elements (18)

$$A_n(z) := z^n(z-1)^n, \quad B_n(z) := z^n(z-1)^n(2z-1)\,. \tag{47}$$

The structure equations read as

$$\begin{aligned} A_n \cdot A_m &= A_{n+m}, \\ A_n \cdot B_m &= B_{n+m}, \\ B_n \cdot B_m &= A_{n+m} + 4A_{n+m+1}. \end{aligned} \tag{48}$$

The space of cocycles is two-dimensional. First we take the residues around ∞ and get for the cocycle values

$$\begin{aligned} \gamma_\infty^{\mathcal{A}}(A_n, A_m) &= 2n\,\delta_m^{-n}, \\ \gamma_\infty^{\mathcal{A}}(A_n, B_m) &= 0, \\ \gamma_\infty^{\mathcal{A}}(B_n, B_m) &= 2n\delta_m^{-n} + 4(2n+1)\,\delta_m^{-n-1}\,. \end{aligned} \tag{49}$$

Then around 0 and get

$$\begin{aligned} \gamma_0^{\mathcal{A}}(A_n, A_m) &= -\,n\,\delta_m^{-n}, \\ \gamma_0^{\mathcal{A}}(A_n, B_m) &= n\,\delta_m^{-n} + 2n\,\delta_m^{-n-1} + \sum_{k=2}^{\infty} n\,(-1)^{k-1} 2^k \frac{(2k-3)!!}{k!}\delta_m^{-n-k}\,, \\ \gamma_0^{\mathcal{A}}(B_n, B_m) &= -\,n\delta_m^{-n} - 2(2n+1)\,\delta_m^{-n-1}\,. \end{aligned} \tag{50}$$

We see that the cocycle $\gamma_\infty^{\mathcal{A}}$ is local but $\gamma_0^{\mathcal{A}}$ is not. This is in accordance to the uniqueness result of [27].

Next we come to the vector field algebra. The basis is given by

$$e_n := A_{n+1}\frac{d}{dz}, \qquad f_n := B_{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}. \tag{51}$$

This yields the structure equations

$$\begin{aligned} [e_n, e_m] &= (m-n)\, f_{m+n}, \\ [e_n, f_m] &= (m-n)\, e_{m+n} + (4(m-n)+2)\, e_{n+m+1}, \\ [f_n, f_m] &= (m-n)\, f_{m+n} + 4(m-n)\, f_{n+m+1}. \end{aligned} \tag{52}$$

Its universal central extension is two-dimensional, and as above obtained by calculating residues of (43) at ∞ and 0:

$$\begin{aligned} \gamma_0^{\mathcal{L}}(e,f) &= 1/2\,\mathrm{res}_0(e\cdot f''' - f\cdot e''')dz \\ \gamma_\infty^{\mathcal{L}}(e,f) &= 1/2\,\mathrm{res}_\infty(e\cdot f''' - f\cdot e''')\,. \end{aligned} \tag{53}$$

We get at ∞:

$$\begin{aligned} \gamma_\infty^{\mathcal{L}}(e_n, e_m) &= 2(n^3-n)\,\delta_m^{-n} + 4n(n+1)(2n+1)\delta_m^{-n-1} \\ \gamma_\infty^{\mathcal{L}}(e_n, f_m) &= 0, \\ \gamma_\infty^{\mathcal{L}}(f_n, f_m) &= 2(n^3-n)\,\delta_m^{-n} + 8n(n+1)(2n+1)\delta_m^{-n-1} \\ &\qquad + 8(n+1)(2n+1)(2n+3)\delta_m^{-n-2}\,, \end{aligned} \tag{54}$$

and at 0:

$$\begin{aligned} \gamma_0^{\mathcal{L}}(e_n, e_m) &= \; -(n^3-n)\,\delta_n^{-m} - 2n(n+1)(2n+1)\delta_m^{-n-1} \\ \gamma_0^{\mathcal{L}}(e_n, f_m) &= (n^3-n)\,\delta_m^{-n} + 6n^2(n+1)\delta_m^{-n-1} + 6n(n+1)^2\delta_m^{-n-2} \\ &\qquad + \sum_{k\geq 3} n(n+1)(n+k-1)(-1)^k 2^k\cdot 3\cdot \frac{(2k-5)!!}{k!}\delta_m^{-n-k} \\ \gamma_0^{\mathcal{L}}(f_n, f_m) &= \; -(n^3-n)\,\delta_m^{-n} - 4n(n+1)(2n+1)\delta_m^{-n-1} \\ &\qquad - 4(n+1)(2n+1)(2n+3)\delta_m^{-n-2}\,. \end{aligned} \tag{55}$$

In accordance with the results in [27] $\gamma_\infty^{\mathcal{L}}$ is local, but $\gamma_0^{\mathcal{L}}$ is not.

The principal picture should be clear now. For the corresponding results for the differential operator algebra and the Lie superalgebra I refer to [32]. Also there (in Appendix B), the universal central extension for the $\mathfrak{sl}(2,\mathbb{C})$ current algebra is given.

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Martin Schlichenmaier
University of Luxembourg
Mathematics Research Unit, FSTC
Campus Kirchberg
6, rue Coudenhove-Kalergi
L-1359 Luxembourg-Kirchberg, Luxembourg
e-mail: martin.schlichenmaier@uni.lu

Part V

Field Theory and Quantization

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Star Products on Graded Manifolds and α'-corrections to Double Field Theory

Andreas Deser

Abstract. Originally proposed as an $O(d,d)$-invariant formulation of classical closed string theory, double field theory (DFT) offers a rich source of mathematical structures. Most prominently, its gauge algebra is determined by the so-called C-bracket, a generalization of the Courant bracket of generalized geometry, in the sense that it reduces to the latter by restricting the theory to solutions of a "strong constraint". Recently, infinitesimal deformations of these structures in the string sigma model coupling α' were found. In this short contribution, we review constructing the Drinfel'd double of a Lie bialgebroid and offer how this can be applied to reproduce the C-bracket of DFT in terms of Poisson brackets. As a consequence, we are able to explain the α'-deformations via a graded version of the Moyal–Weyl product in a class of examples. We conclude with comments on the relation between B- and β-transformations in generalized geometry and the Atiyah algebra on the Drinfel'd double.

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1. Introduction: Brackets and deformations in DFT

Due to the extended nature of closed strings moving in a background, the field theory describing its classical dynamics is different from that of a point particle. In particular, the string can "wind" around compact cycles of the background manifold. This gives rise to two sets of parameters (or zero modes) characterizing the solutions to the classical equations of motion. One of them is associated to the center of mass momentum p_i of the closed string and the corresponding configuration space coordinates x^i span the phase space of the center of mass treated as a point particle. The second set $\tilde{p}^i$ is associated to the winding and gives rise to a second set of coordinates $\tilde{x}_i$. DFT is a field theory on this "doubled configuration space" which can be reduced to ordinary configuration space by using the strong constraint

$$\partial_i\phi(x,\tilde{x})\,\tilde{\partial}^i\psi(x,\tilde{x}) + \tilde{\partial}^i\phi(x,\tilde{x})\,\partial_i\psi(x,\tilde{x}) = 0, \tag{1}$$

for functions ϕ, ψ on the doubled configuration space. This constraint has its origin in the level matching condition for physical fields in string theory and restores the right amount of coordinates of a physical configuration space. We refer to the reader especially to [2] and the lecture notes [1] for an introduction to DFT.

1.1. C-bracket and bilinear form

In [3, 4], a Lagrangian action for DFT was formulated and gauge symmetries were identified. Due to a lack of space, we only present results that are important for the rest of the presentation. To state the gauge symmetries, we use notation conventions of *generalized geometry*. On a d-dimensional manifold M, generalized vector fields V are locally given by sections of $TM \oplus T^*M$, i.e., $V = V^i\partial_i + V_i dx^i$. To state local expressions in DFT, the components are allowed to depend on the doubled configuration space with coordinates $(x^i, \tilde{x}_i)$. Furthermore one uses a capital index to denote objects transforming in the fundamental representation of $O(d,d)$, i.e., $V^M = (V^i(x,\tilde{x}), V_i(x,\tilde{x}))$, where $A \in O(d,d)$ obeys

$$A\eta A^t = \eta\,, \qquad \eta = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix}\,, \tag{2}$$

and id is the d-dimensional identity matrix. We will denote the bilinear form represented by η by $\langle\cdot,\cdot\rangle$. Capital indices are raised and lowered by the latter, so for generalized vectors V, W we have

$$\langle V, W\rangle = V^P W^Q \eta_{PQ} = V^i W_i + V_i W^i\,. \tag{3}$$

The gauge symmetries of DFT are given by the action of a generalized Lie derivative, acting on functions ϕ by[1] $\mathcal{L}_V\phi = V^K\partial_K\phi$ and generalized vectors W according to

$$\begin{aligned}(\mathcal{L}_V W)_K &= V^P\partial_P W_K + (\partial_K V^P - \partial^P V_K)W_P\,,\\ (\mathcal{L}_V W)^K &= V^P\partial_P W^K - (\partial_P V^K - \partial^K V_P)W^P\,.\end{aligned} \tag{4}$$

Finally, the commutator of two generalized Lie derivatives gives the generalized Lie derivative with respect to the *C-bracket* of two generalized vectors, which is given in components by

$$\Big([V,W]_C\Big)^P = V^K\partial_K W^P - W^K\partial_K V^P - \frac{1}{2}\Big(V^K\partial^P W_K - W^K\partial^P V_K\Big)\,. \tag{5}$$

Note that for the specific solution $\tilde{\partial}^i = 0$, this bracket reduces to the well-known Courant bracket of generalized geometry. In the following subsection, we will present a deformation of the bilinear form η and the C-bracket found in double field theory.

[1]We use the notation ∂_M for the pair $(\partial_i, \tilde{\partial}^i)$, so expressions like $V^M\partial_M$ are expanded as $V^i\partial_i + V_i\tilde{\partial}^i$.

1.2. α'-deformations

Classical closed string theory is described by a two-dimensional sigma model. Perturbative expansions are formal power series in the coupling constant $\alpha' = l_s^2$, where l_s is the fundamental string length. Recently, corrections to the bilinear form and C-bracket up to first order in α' were given in [5, 6]. For the correction of the bilinear form, we introduce the notation $\langle V, W\rangle_{\alpha'} := \langle V, W\rangle - \alpha'\langle\langle V, W\rangle\rangle$, where the component expression for the correction is

$$\langle\langle V, W\rangle\rangle = \partial_P V^Q \partial_Q W^P \,. \tag{6}$$

Similarly, for the correction to the C-bracket, we introduce the short notation $[V, W]_{\alpha'} := [V, W]_C - \alpha'[[V, W]]$, where the correction is given by

$$[[V, W]]^K = \frac{1}{2}\Big(\partial^K \partial_Q V^P \partial_P W^Q - V \leftrightarrow W\Big) \,. \tag{7}$$

Note that this expression has a form part and a vector part. As an example, we expand the vector part in terms of partial derivatives:

$$\begin{aligned} [[V, W]]_i = \frac{1}{2}\Big(&\partial_i\partial_m V^n \partial_n W^m + \partial_i\partial_m V_n \tilde{\partial}^n W^m + \partial_i\tilde{\partial}^m V^n \partial_n W_m \\ &+ \partial_i\tilde{\partial}^m V_n \tilde{\partial}^n W_m - V \leftrightarrow W\Big) \,. \end{aligned} \tag{8}$$

The goal of this work is to get a systematic explanation of the derivative expansions (6) and (8). In the following section, we are going to set up a mathematical formalism to rewrite the bilinear form and the C-bracket in terms of Poisson brackets. This will allow us finally to identify the deformation using a Moyal–Weyl star product on a specific symplectic supermanifold.

2. Lie bialgebroids and double fields

For finite-dimensional vector spaces $\mathcal{V}$, it is a standard exercise to show the isomorphism between the exterior algebra and the algebra of polynomials in the parity reversed version $\Pi\mathcal{V}$:

$$\wedge^\bullet \mathcal{V}^* \simeq \mathrm{Pol}^\bullet(\Pi\mathcal{V}) \,. \tag{9}$$

For a finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1$, parity reversion Π acts according to $(\Pi\mathcal{W})_0 = \mathcal{W}_1$ and $(\Pi\mathcal{W})_1 = \mathcal{W}_0$. In (9), elements of $\mathcal{V}$ have degree 0 and elements of $\Pi\mathcal{V}$ have degree 1. In the case of vector bundles, differentials are derivations of the exterior algebra, which get mapped to derivations on functions, i.e., vector fields. Squaring to zero means that the vector fields are actually *homological*. These statements are summarized by the structure of a Lie algebroid:

Definition 1. A *Lie algebroid* is a vector bundle $A \to M$ together with a homological vector field d_A of degree 1 on the supermanifold ΠA.

A pair (A, A^*) of a Lie algebroid and its linear dual has the structure of a *Lie bialgebroid* if the differentials respect the brackets on the dual spaces. This will be the basic structure used in the following sections.

2.1. Lie bialgebroids and the Drinfel'd double

Let (A, A^*) be a pair of dual Lie algebroids over a manifold M. The homological vector field d_A can be lifted to a function on the cotangent bundle $T^*\Pi A \xrightarrow{\mathfrak{p}} M$. Similarly, the corresponding operator d_{A^*} for the dual can be lifted to $T^*\Pi A^* \xrightarrow{\bar{\mathfrak{p}}} M$. Similarly to the case of standard phase spaces, there is a Legendre transform $L : T^*\Pi A \to T^*\Pi A^*$, which can be used to pull back functions. Thus we have the situation

$$\begin{array}{ccc} T^*\Pi A & \xrightarrow{L} & T^*\Pi A^* \\ \big\downarrow \mathfrak{p} & & \big\downarrow \bar{\mathfrak{p}} \\ \Pi A & & \Pi A^* \end{array} \ . \tag{10}$$

For local formulas we use coordinates (x^i, ξ^a) on ΠA, where x^i are coordinates on the base manifold and ξ^a denote the (Grassmann odd) fibre coordinates. On its cotangent bundle, we have in addition the canonical conjugate momenta, i.e., $(x^i, \xi^a, x^*_i, \xi^*_a)$. As in the purely even case, there is a canonical Poisson bracket on $T^*\Pi A$, given by the relations

$$\{x^i, x^*_j\} = \delta^i_j \ , \qquad \{\xi^a, \xi^*_b\} = \delta^a_b \ . \tag{11}$$

Using this Poisson structure and the "lifted" vector field

$$\theta := h_{d_A} + L^* h_{d_{A^*}} \ , \tag{12}$$

it is possible to write down the following concise characterization of (A, A^*) being a Lie bialgebroid:

Theorem 2. *The pair (A, A^*) is a Lie bialgebroid if and only if $\{\theta, \theta\} = 0$.*

We refer to [7] for a proof and further details on the mathematical structures introduced in the present work. Theorem 2 is the motivation for the following definition:

Definition 3. For a Lie bialgebroid as above, the bundle $T^*\Pi A$, equipped with the homological vector field $\{\theta, \cdot\}$ is called the *Drinfel'd double* of (A, A^*).

We refer to [9, 10] for the original work on the Drinfel'd double in this context. The essential ingredient for the homological vector field is the function θ in (12).

2.2. C-bracket in terms of Poisson brackets

Let M be a Poisson manifold. Then the standard example of a Lie bialgebroid is $(A, A^*) = (TM, T^*M)$. The respective brackets are the Lie bracket and Koszul bracket[2], giving rise to the de Rham and Poisson–Lichnerowicz differential, respectively. We use their lifts to functions on the Drinfel'd double to define two sets of momentum variables $p_i, \tilde{p}^i$:

$$\begin{aligned} h_{d_A} &= a_i^j(x)x_j^*\xi^i - \tfrac{1}{2}f_{ij}^k(x)\xi^i\xi^j\xi_k^* &=:\ \xi^i p_i\ , \\ h_{d_{A^*}} &= a^{ij}(x)x_i^*\xi_j^* + \tfrac{1}{2}Q_k^{ij}(x)\xi^k\xi_i^*\xi_j^* &=:\ \xi_i^*\tilde{p}^i\ , \end{aligned} \tag{13}$$

where we denote the anchor maps by a_i^j and a^{ij}, and f and Q are determined by the brackets on A and A^*, respectively[3]. We consider the momenta p_i and $\tilde{p}^i$ to act on functions on $T^*\Pi A$ by using the Poisson bracket, e.g., $\{p_i, \cdot\}$. In particular, lifting functions $\phi \in \mathcal{C}^\infty(M)$ to $T^*\Pi A$ (we use the same letter ϕ for the lift), we define the following two differential operators:

$$\partial_i\phi := \{p_i, \phi\}\ , \qquad \tilde{\partial}^i\phi := \{\tilde{p}^i, \phi\}\ . \tag{14}$$

Lifting furthermore generalized vectors to $T^*\Pi A$, i.e., if locally $X^i\partial_i + \omega_i dx^i \in \Gamma(TM \oplus T^*M)$, we define $V := X^i\xi_i^* + \omega_i\xi^i \in T^*\Pi A$, we are able to show the following result by rewriting the proof done in [7] for Courant brackets, but using ∂_i and $\tilde{\partial}^i$ here:

Theorem 4. *For vanishing f and Q, let V, W be lifts of generalized vectors to $T^*\Pi A$. Furthermore, define the Dorfmann product $\circ$ by*

$$V \circ W := \Big\{\{\xi^i p_i + \xi_i^*\tilde{p}^i, V\}, W\Big\}\ . \tag{15}$$

Then the C-bracket of V, W (lifted to $T^*\Pi A$) is given by

$$[V, W]_C = \frac{1}{2}\Big(V \circ W - W \circ V\Big)\ . \tag{16}$$

The proof is an easy evaluation in local coordinates of $T^*\Pi A$, and comparison with (5), see [11]. The generalization for non-vanishing f and Q would give a version of the C-bracket containing "fluxes", which, as far as we know, has not been done so far in the physics literature. As a final remark for this subsection, we observe that the bilinear form $\langle V, W\rangle$ is given by evaluating the Poisson bracket $\{V, W\}$ of the lifted quantities to $T^*\Pi A$. These observations will be used in the following sections to suggest a way to understand the deformations (6) and (8) of the bilinear form and C-bracket encountered in DFT.

[2]The Koszul bracket of forms $\omega_1, \omega_2 \in \Gamma(T^*M)$ is given by

$$[\omega_1, \omega_2]_K = \mathcal{L}_{\pi^\sharp(\omega_1)}\omega_2 - \iota_{\pi^\sharp(\omega_2)}d\omega_1\ ,$$

where $\mathcal{L}$ is the Lie derivative and $\pi^\sharp$ is the anchor determined by the Poisson structure.

[3]The notation f and Q is common in the physics literature, where these objects play a role in flux compactifications of string theory.

3. Deformation of the metric and C-bracket

The result of Theorem 4 immediately suggests the interpretation of α'-corrections such as (6) and (8) in terms of deformation theory. Given a formal star product on the algebra of smooth functions on a Poisson manifold[4], the star-commutator reproduces the Poisson bracket in the first non-trivial order:

$$\{f,g\} = \lim_{t\to\infty}\frac{1}{t}\Big(f\star g - g\star f\Big)\,. \tag{17}$$

Thus, higher orders lead to deformations of the Poisson bracket and as a consequence of Theorem 4 of the metric and C-bracket. In the following, we will define an appropriate notion of star-commutator taking into account the Koszul signs on the graded manifold $T^*\Pi A$. Furthermore, we will give a (constant) Poisson structure on $T^*\Pi A$ such that the corrections of DFT are reproduced by taking star-commutators w.r.t. the corresponding Moyal–Weyl product.

3.1. Star-commutator and Poisson structure

For the Moyal–Weyl case, let $I = i_1\cdots i_k,\ J = j_1\cdots j_k$, with $\partial_I = \partial_{x^{i_1}}\cdots\partial_{x^{i_k}}$, then the star commutator for purely even manifolds has the standard form

$$\{f,g\}^* = \sum_{k=1}^{\infty} t^k\Big(\sum_{IJ} m_k^{IJ}(\partial_I f\partial_J g - \partial_I g\partial_J f)\Big)\,. \tag{18}$$

In the case of the symplectic supermanifold $T^*\Pi A$, we will replace this by the following expression:

$$\{f,g\}^* = \sum_{k=1}^{\infty}\Big(\sum_{IJ}(\partial_I f\partial_J g - (-1)^{\epsilon}\partial_I g\partial_J f)\Big)\,. \tag{19}$$

The sign $(-1)^\epsilon$ takes care of the $\mathbb{Z}_2$-grading and is given by

$$\epsilon = |f||g| + |x^J|(|f|-1) + |x^I|(|g|-1)\,, \tag{20}$$

where $|f|$ denotes the $\mathbb{Z}_2$-degree of a function and the shorthand notation $|x^I| := |x^{i_1}|+\cdots+|x^{i_k}|$ is used. We remark that in contrast to the Moyal–Weyl case where the odd powers of the deformation don't contribute due to the antisymmetry of the Poisson tensor, in the graded case there are such contributions due to the different sign rule. In our case this will open the possibility to get the appropriate α'-correction.

Finally, we have to choose a Poisson structure on $T^*\Pi A$ which correctly reproduces both, the correction to the bilinear form $\langle\cdot,\cdot\rangle$ and the C-bracket. Furthermore, the corresponding Poisson brackets, i.e., the first-order star commutators still have to give the result of Theorem 4. It turns out that this is indeed possible. To avoid long calculations we choose a setup which is as simple as possible, but still shows the essential features. Let M be a symplectic manifold with Poisson

[4]More precisely on formal power series in a deformation parameter t, usually denoted by $\mathcal{C}^\infty(M)[[t]]$. We refer to [12–15] for recent applications of deformation theory in closed string theory and to [16–18] for star products on graded manifolds.

tensor π. In this case (TM, T^*M) is a Lie bialgebroid. In the expressions for the α'-corrections, there are no f- and Q-fluxes. We can achieve the latter by taking the standard basis of vector fields on the tangent bundle. As a consequence, we get

$$h_{d_A} = \xi^m x_m^* \,, \qquad L^* h_{d_{A^*}} = \xi_m^* \pi^{mn} x_n^* \,. \tag{21}$$

This is a special solution of the strong constraint of double field theory, with $\tilde{\partial}^i f = \{\tilde{p}^i, f\} = \pi^{ij}\partial_j f$. We choose the following Poisson structure on the Drinfel'd double:

$$\pi_{T^*\Pi A} = \frac{\partial}{\partial x_i^*} \wedge \frac{\partial}{\partial x^i} + \frac{\partial}{\partial \xi_i^*} \wedge \frac{\partial}{\partial \xi^i} + \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \xi_i^*} - \pi^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \xi^j} \,. \tag{22}$$

We will give our results for the deformation for this situation. In the general case, we have a differential operator $\tilde{\partial}^i = \{p^i, \cdot\}$, whose action on functions depends on the chosen Lie bialgebroid. If it is possible to associate a vector field $\frac{\partial}{\partial \tilde{x}_i}$ to this operator, the corresponding Poisson structure would be

$$\pi_{T^*\Pi A} = \frac{\partial}{\partial x_i^*} \wedge \frac{\partial}{\partial x^i} + \frac{\partial}{\partial \xi_i^*} \wedge \frac{\partial}{\partial \xi^i} + \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \xi_i^*} + \frac{\partial}{\partial \tilde{x}_i} \wedge \frac{\partial}{\partial \xi^i} \,. \tag{23}$$

We will leave the investigation of existence and properties of such a Poisson structure and its relation to double field theory for future work and give our deformation results for the Poisson tensor (22) in the following.

3.2. Deformation of the metric

Due to the various terms of the graded Poisson structure (22), computing higher orders of the graded Moyal–Weyl product is lengthy, but straightforward. We therefore refer the reader to [19] for computational details and only give the results. We will use the notation $\tilde{\partial}^i$ for $\{p^i, \cdot\}$. Furthermore, we use the following notation for star-commutators:

$$\{f, g\} = \sum_{k=1}^{\infty} t^k \{f, g\}_{(k)} \,. \tag{24}$$

Taking $V = V^m(x)\xi_m^* + V_m(x)\xi^m$ and $W = W^m(x)\xi_m^* + W_m(x)\xi^m$ to be lifts of generalized vectors to $T^*\Pi A$, we get the following results for the first two orders in the deformation parameter:

$$\begin{aligned} \{V, W\}_{(1)} &= (V^i W_i + V_i W^i) = \langle V, W \rangle \,, \\ \{V, W\}_{(2)} &= -\partial_i V^j \partial_j W^i - \partial_i V_j \tilde{\partial}^j W^i - \tilde{\partial}^i V^j \partial_j W_i - \tilde{\partial}^i V_j \tilde{\partial}^j W_i \,. \end{aligned} \tag{25}$$

Comparing the latter expressions with the formulas from DFT (6), we get the following statement:

Theorem 5. *Let $V = V^i\xi_i^* + V_i\xi^i$ and $W = W^i\xi_i^* + W_i\xi^i$ be two generalized vectors, lifted to $T^*\Pi A$. Then we have*

$$\frac{1}{t}\{V, W\}^* = \langle V, W \rangle - t\langle\langle V, W \rangle\rangle + \mathcal{O}(t^2) \,, \tag{26}$$

i.e., the graded star-commutator gives the deformation of the inner product $\langle \cdot, \cdot \rangle$ up to second order.

For convenience of notation, we always denote the generalized vectors V, W and their lifts to $T^*\Pi A$ by the same letters. It is clear from the context which objects are used.

3.3. Deformation of the C-bracket

Using Theorem 4, we are now able to compute corrections to the C-bracket. First, it is easy to see that the Poisson structure (22) together with the sign rule given in (19) correctly reproduce the Dorfmann product $\circ$:

$$V \circ W = \Big\{\{\theta, V\}_{(1)}, W\Big\}_{(1)} . \tag{27}$$

To see which Poisson brackets contribute to the first non-trivial corrections to $V \circ W$, we expand the double Poisson bracket up to order t^4:

$$\begin{aligned} \Big\{\{\theta, V\}, W\Big\}^* = &\ t^2\, V \circ W + t^3 \Big\{\{\theta, V\}_{(2)}, W\Big\}_{(1)} \\ &+ t^3 \Big\{\{\theta, V\}_{(1)}, W\Big\}_{(2)} + \mathcal{O}(t^4) . \end{aligned} \tag{28}$$

A short calculation shows the vanishing of $\{\theta, V\}_{(2)}$ for the chosen setup (π constant) and we have

$$\begin{aligned} \{\theta, V\}_{(1)} = \xi^m \xi^n \partial_m V_n + \xi^*_k \xi^m \pi^{kn} \partial_n V_m + \xi^*_k \xi^*_m \pi^{kn} \partial_n V^m \\ + V_n \pi^{nm} x^*_m + x^*_n V^n . \end{aligned}$$

Inserting this expression into $\Big\{\{\theta, V\}_{(1)}, W\Big\}_{(2)}$ gives exactly the contribution which was encountered for this setup in DFT, see equation (8). Thus we state the following result:

Theorem 6. *Let $V = V^i \xi^*_i + V_i \xi^i$ and $W = W^i \xi^*_i + W_i \xi^i$ be two generalized vectors lifted to $T^*\Pi A$, then we have*

$$\frac{1}{2t^2}\Big(\Big\{\{\theta, V\}^*, W\Big\}^* - \Big\{\{\theta, W\}^*, V\Big\}^*\Big) = [V, W]_C + t[[V, W]]_C + \mathcal{O}(t^2) , \tag{29}$$

i.e., the two-fold star commutator coincides with the α'-corrected C-bracket of DFT up to second order in the deformation parameter $t = \alpha'$.

The proof is a straightforward but lengthy evaluation in local coordinates. We refer the reader to the original article [19] for details, especially concerning the Koszul signs. To sum up, in the framework chosen above, it is possible to explain α'-corrections to the bilinear pairing and C-bracket encountered in string theory via a star commutator with respect to a graded version of the Moyal–Weyl product.

4. Outlook: B-, β-transformations and the Atiyah algebra

In the final section we want to give additional evidence for the relevance of the introduced mathematical framework in physics, especially to the structures arising in DFT. First we recall that a *B-transform* of a generalized vector (X, ω) is defined by

$$(X, \omega) \mapsto (X, \omega + \iota_X B) \ , \quad B \in \Gamma(\wedge^2 T^* M) \ . \tag{30}$$

Furthermore, a β-transform is given in an analogous way by

$$(X, \omega) \mapsto (X + \iota_\omega \beta, \omega) \ , \quad \beta \in \Gamma(\wedge^2 TM) \ . \tag{31}$$

Finally a linear transformation is given by the following definition

$$(X, \omega) \mapsto (X + C(X), \omega + C^{-t}(\omega)) \ , \quad C \in \Gamma(TM \otimes T^* M) \ , \tag{32}$$

where A^{-t} means the inverse transpose of the invertible matrix C. The idea to lift these transformations to $T^*\Pi A$ lies at hand, thus introducing the lifts

$$B = \tfrac{1}{2} B_{ij} \xi^i \xi^j \ , \quad \beta = \tfrac{1}{2} \beta^{ij} \xi^*_i \xi^*_j \ , \quad C = C^j_i \xi^*_j \xi^i \ , \tag{33}$$

it is a straightforward exercise to show that the action of B-, β- and linear transformations on the lift $\Sigma = X^i \xi^*_i + \omega_i \xi^i$ of a generalized vector (X, ω) is given by

$$\begin{gathered} \Sigma \mapsto \Sigma + \{\Sigma, B\} \ , \quad \Sigma \mapsto \Sigma + \{\Sigma, \beta\} \\ \Sigma \mapsto \Sigma + \{\Sigma, C\} \ . \end{gathered} \tag{34}$$

Comparing with [20], we see that the transformations (34) are the lifts to $T^*\Pi A$ of the generators of the *Atiyah algebra* of infinitesimal bundle transformations of $A \oplus A^*$, preserving the bilinear form η. With this very convenient rewriting of the transformations used frequently in the generalized geometry applications to string theory, an immediate open question is about the deformation of these transformations. The tools established in this work will be helpful to investigate this further. In addition to that, the inclusion of fluxes as "fibre translations" in the sense of [20] could be performed conveniently as suggested in [19].

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Andreas Deser
Institute of Theoretical Physics
University of Hannover
Appelstr. 2
D-30167 Hannover, Germany
e-mail: andreas.deser@itp.uni-hannover.de

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Adiabatic Limit in Ginzburg–Landau and Seiberg–Witten Equations

Armen Sergeev

Abstract. We study the adiabatic limit procedure in the $(1+2)$-dimensional Ginzburg–Landau equations and 4-dimensional Seiberg–Witten equations.

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The hyperbolic Ginzburg–Landau equations arise in gauge field theory as the Euler–Lagrange equations for the $(1+2)$-dimensional Abelian Higgs model describing the interaction between the electromagnetic and scalar fields. They are nonlinear hyperbolic equations containing, in particular, a nonlinear covariant D'Alembertian. The main problem is to describe the moduli space of solutions of these equations. This problem for static solutions, called otherwise the Ginzburg–Landau vortices, was successfully solved by Taubes but the corresponding problem for general, or dynamic Ginzburg–Landau solutions is far from being completed.

Manton proposed to study dynamical solutions with small kinetic energy with the help of adiabatic limit procedure by introducing the "slow time" parameter on solution trajectories. In this limit dynamical solutions converge to geodesics on the space of vortices with respect to the metric, generated by the kinetic energy functional. Thus, the original equations reduce to ordinary Euler geodesic equations so that by solving the latter equations we can describe the behavior of slowly moving dynamical solutions.

It turns out that this procedure has a 4-dimensional analogue. Namely, for Seiberg–Witten equations on 4-dimensional symplectic manifolds it is possible to introduce an analogue of the adiabatic limit. In this limit solutions of Seiberg–

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Witten equations reduce to families of vortices in normal planes to pseudoholomorphic curves which may be considered as complex analogues of geodesics parameterized by the "complex time". This case may be considered as a $(2+2)$-dimensional analogue of Abelian Higgs model.

1. Static Ginzburg–Landau equations

1.1. Ginzburg–Landau Lagrangian

The Ginzburg–Landau Lagrangian, defined on the plane $\mathbb{R}^2_{(x_1,x_2)}$ with coordinates (x_1, x_2), has the form

$$\mathcal{L}(A,\Phi) = |F_A|^2 + |d_A\Phi|^2 + \frac{1}{4}(1-|\Phi|^2)^2.$$

From the physical point of view the variable A represents the electromagnetic vector-potential while mathematically it is a U(1)-connection on $\mathbb{R}^2_{(x_1,x_2)}$, given by the 1-form

$$A = A_1 dx_1 + A_2 dx_2$$

with smooth pure imaginary coefficients.

The curvature F_A of this connection, given by the 2-form

$$F_A = dA = \sum_{i,j=1}^{2} F_{ij} dx_i \wedge dx_j = 2F_{12} dx_1 \wedge dx_2$$

with coefficients

$$F_{ij} = \partial_i A_j - \partial_j A_i, \quad \partial_j := \partial/\partial x_j,$$

is physically interpreted as the electromagnetic field strength so that the term $|F_A|^2$ may be identified with the Maxwell Lagrangian.

The variable Φ is the Higgs field, given by a smooth complex-valued function $\Phi = \Phi_1 + i\Phi_2$ on $\mathbb{R}^2_{(x_1,x_2)}$. From the physical point of view it is a scalar field, interacting with the electromagnetic field A. In superconductivity Φ describes the superconducting current so that $\Phi = 0$ means that the superconductivity is absent while $|\Phi| = 1$ relates to pure superconductivity.

The covariant exterior derivative $d_A\Phi$ in the second term of the Ginzburg–Landau Lagrangian is given by the formula

$$d_A\Phi = d\Phi + A\Phi = \sum_{i=1}^{2} (\partial_i + A_i)\Phi \, dx_i,$$

and the term $|d_A\Phi|^2$ is responsible for the interaction of the electromagnetic field with the Higgs field Φ.

The last term $\frac{1}{4}(1-|\Phi|^2)^2$ is the most important ingredient in the Ginzburg–Landau Lagrangian. It is responsible for the nonlinear character of the "self-interaction" of the field Φ. We require that $|\Phi| \to 1$ for $|x| \to \infty$ which means, physically, that we have pure superconductivity at infinity. The zeros of the function $\Phi = \rho e^{i\theta}$ correspond to the points where the superconductivity is absent. In a neighborhood of a zero the vector field $\vec{v} = \nabla\theta$ behaves like a hydrodynamical vortex. By this reason solutions of the considered model are also called vortices.

1.2. Vortices

We define the potential energy of our model as the integral of the Ginzburg–Landau Lagrangian:

$$U(A, \Phi) = \frac{1}{2} \int \mathcal{L}(A, \Phi)\, d^2x.$$

The condition $|\Phi| \to 1$ implies that the considered problem has a topological invariant, given by the rotation number of the map Φ, sending circles of sufficiently large radius to topological circles. This invariant is called the vortex number and has integer values.

We give now the mathematical definition of vortices. These are the pairs (A, Φ), on which the minimum of the potential energy $U(A, \Phi) < \infty$ is realized in the given topological class, fixed by the value of the vortex number d. If $d > 0$ then such pairs are called d-vortices while for $d < 0$ they are called the $|d|$-antivortices.

One of the most important features of our model is the invariance of the potential energy $U(A, \Phi)$ under the infinite-dimensional group of gauge transforms, given by the formula

$$A \longmapsto A + id\chi, \quad \Phi \longmapsto e^{-i\chi}\Phi$$

where χ is an arbitrary smooth real-valued function on $\mathbb{R}^2_{(x_1,x_2)}$.

We are interested in the moduli space of d-vortices, defined as the quotient

$$\mathcal{M}_d = \frac{\{d\text{-vortices } (A, \Phi)\}}{\{\text{gauge transforms}\}} .$$

This space is described by the following theorem of Taubes.

Introduce the complex coordinate $z = x_1 + ix_2$ in the plane $\mathbb{R}^2_{(x_1,x_2)}$, identifying $\mathbb{R}^2_{(x_1,x_2)}$ with the complex plane $\mathbb{C}_z$ and suppose that $d > 0$.

Theorem 1 (Taubes, [1],[12]). *For a given integer $d > 0$ and any unordered collection $z_1, \ldots, z_k$ of different points on the complex plane $\mathbb{C}$ taken with multiplicities $d_1, \ldots, d_k$ such that $\sum_{j=1}^k d_j = d$ there exists a unique (up to gauge transforms) d-vortex (A, Φ) such that the map Φ vanishes precisely at the points $z_1, \ldots, z_k$ with given multiplicities $d_1, \ldots, d_k$.*

Moreover, Taubes has proved (cf. [1]) that any critical point (A, Φ) of the functional $U(A, \Phi) < \infty$ with vortex number $d > 0$ is gauge equivalent to some d-vortex. In other words, all solutions of the Euler–Lagrange equations for the functional $U(A, \Phi)$ with finite energy are stable and have minimal energy in their topological class.

The Taubes theorem implies that the moduli space $\mathcal{M}_d$ of d-vortices may be identified with the vector space $\mathbb{C}^d$ by associating with the collection $z_1, \ldots, z_k$ with multiplicities $d_1, \ldots, d_k$ such that $\sum_{j=1}^k d_j = d$ the monic polynomial, having its zeros precisely at the points $z_1, \ldots, z_k$ with given multiplicities $d_1, \ldots, d_k$. The antivortices with $d < 0$ admit an analogous description.

This result has the following physical interpretation. Solutions of the Euler–Lagrange equations for the functional $U(A,\Phi)$ consist either of vortices, or of antivortices. Our model cannot contain simultaneously both vortices and antivortices – such bound states should "annihilate" before the system is transformed into the static state.

2. Dynamical Ginzburg–Landau equations

2.1. Ginzburg–Landau action

Let us switch on the time in our model by adding the variable $x_0 = t$. In this case the Higgs field $\Phi = \Phi(t, x_1, x_2)$ will be given by a smooth complex-valued function on the space $\mathbb{R}^3_{(t,x_1,x_2)}$ while the 1-form A will be replaced by the form

$$\mathcal{A} = A_0 dt + A_1 dx_1 + A_2 dx_2$$

with the coefficients $A_\mu = A_\mu(t, x_1, x_2)$, $\mu = 0, 1, 2$, being smooth functions with pure imaginary values on the space $\mathbb{R}^3_{(t,x_1,x_2)}$. Denote by $A^0 = A_0 dt$ the time component of the form $\mathcal{A}$ and by $A = A_1 dx_1 + A_2 dx_2$ its space component.

The potential energy of the system is given by the same formula, as before, i.e., $U(\mathcal{A}, \Phi) = U(A, \Phi)$ while the kinetic energy is given by

$$T(\mathcal{A}, \Phi) = \frac{1}{2} \int \left\{ |F_{01}|^2 + |F_{02}|^2 + |d_{A^0}\Phi|^2 \right\} dx_1 dx_2$$

where F_{0j}, $j = 1, 2$, are defined in the same way, as before, i.e.,

$$F_{0j} = \partial_0 A_j - \partial_j A_0,$$

and $d_{A^0}\Phi = d\Phi + A_0\, dt$. In other words, the formula for the kinetic energy contains the same terms, as those entering the formula for the potential energy, but containing the derivative in time.

The described dynamical model is governed by the Ginzburg–Landau action functional:

$$S(\mathcal{A}, \Phi) = \int_0^{T_0} \left(T(\mathcal{A}, \Phi) - U(\mathcal{A}, \Phi) \right) dt.$$

2.2. Ginzburg–Landau equations

The Euler–Lagrange equations for the Ginzburg–Landau action functional, called also the Ginzburg–Landau equations, have the form

$$\begin{cases} \partial_1 F_{01} + \partial_2 F_{02} = -i\, \mathrm{Im}(\bar{\Phi} \nabla_{A,0} \Phi) \\ \partial_0 F_{0j} + \sum_{k=1}^{2} \varepsilon_{jk} \partial_k F_{12} = -i\, \mathrm{Im}(\bar{\Phi} \nabla_{A,j} \Phi), \; j = 1, 2 \\ (\nabla^2_{A,0} - \nabla^2_{A,1} - \nabla^2_{A,2})\Phi = \frac{1}{2}\Phi(1 - |\Phi|^2), \end{cases}$$

where

$$\nabla_{A,\mu} = \partial_\mu + A_\mu, \; \mu = 0, 1, 2; \quad \varepsilon_{12} = -\varepsilon_{21} = 1, \; \varepsilon_{11} = \varepsilon_{22} = 0.$$

The first of these equations is of the constraint type meaning that it is satisfied for any t if it is fulfilled at the initial moment of time. The last equation, containing the covariant D'Alembertian in its left-hand side, has the form of a nonlinear wave equation.

The Ginzburg–Landau equations, as well as the action $S(\mathcal{A}, \Phi)$, are invariant under the dynamical gauge transforms, given by the same formula, as in the static case

$$A_\mu \longmapsto A_\mu + i\partial_\mu \chi, \quad \Phi \longmapsto e^{-i\chi}\Phi, \quad \mu = 0, 1, 2,$$

but with χ being now a smooth real-valued function on $\mathbb{R}^3_{(t,x_1,x_2)}$.

Our main goal is to describe the solutions of the above Ginzburg–Landau equations up to dynamical gauge transforms. We call solutions of these equations, for brevity, the dynamical solutions (in contrast with static solutions, considered before). The quotient of the space of dynamical solutions modulo gauge transforms is called the moduli space of dynamical solutions.

3. Adiabatic limit in Ginzburg–Landau equations

3.1. Temporal gauge

For the analysis of dynamical solutions it is convenient to choose the gauge function χ so that the time component of the potential vanishes, i.e., $A_0 = 0$. Such a choice of χ is called the temporal gauge. (Note that after imposing this condition on the gauge function χ we are still left with the gauge freedom with respect to static gauge transforms.)

In the temporal gauge a dynamical solution of the Ginzburg–Landau equations may be considered as a trajectory of the form

$$\gamma : t \longmapsto [A(t), \Phi(t)]$$

where $[A, \Phi]$ denotes the gauge class of the pair (A, Φ) with respect to static gauge transforms. This trajectory lies in the configuration space

$$\mathcal{N}_d = \frac{\{(A, \Phi) \text{ with } U(A, \Phi) < \infty \text{ and vortex number } d\}}{\{\text{static gauge transforms}\}}$$

which contains, in particular, the moduli space of d-vortices $\mathcal{M}_d$.

The configuration space $\mathcal{N}_d$ may be thought of as a canyon with the bottom coinciding with the moduli space $\mathcal{M}_d$ of d-vortex solutions, having the minimal energy in $\mathcal{N}_d$. We can also think of a dynamical solution as the trajectory $\gamma(t)$ of a small ball rolling along the walls of the canyon. The lower is the kinetic energy of the ball, the closer is its trajectory to the bottom. Our ball may even hit the bottom but cannot stop there since, having a non-zero kinetic energy, it should assent the canyon wall again.

3.2. Adiabatic limit

Consider a family of dynamical solutions γ_ϵ of Ginzburg–Landau equations, depending on a parameter $\epsilon > 0$, with trajectories

$$\gamma_\epsilon : t \longmapsto [A_\epsilon(t), \Phi_\epsilon(t)].$$

Suppose that the kinetic energy of these trajectories

$$T(\gamma_\epsilon) := \int_0^{T_0} T(\gamma_\epsilon(t))dt \approx \epsilon$$

tends to zero for $\epsilon \to 0$, proportional to ϵ. Then in the limit $\epsilon \to 0$ the trajectory γ_ϵ converts into a static solution, i.e., a point of $\mathcal{M}_d$.

However, if we introduce on γ_ϵ the "slow time" $\tau = \epsilon t$ and consider the limit of the "rescaled" trajectories $\gamma_\epsilon(\tau)$ for $\epsilon \to 0$ then in this limit we shall obtain not a point, but a trajectory γ_0, lying in $\mathcal{M}_d$. Of course, such a trajectory cannot be a solution of the original dynamical equations since any of its points is a static solution. However, these trajectories describe approximately dynamical solutions with small kinetic energy.

The described procedure is called the adiabatic limit. In this limit the original dynamical equations reduce to the adiabatic equations whose solutions are called the adiabatic trajectories.

3.3. Adiabatic principle

The adiabatic trajectories admit the following intrinsic description in terms of the space $\mathcal{M}_d$.

Theorem 2 ([9],[10]). *The kinetic energy functional generates a Riemannian metric on the space $\mathcal{M}_d$, called the kinetic or T-metric. The adiabatic trajectories γ_0 are the geodesics of this metric.*

The idea of the approximate description of "slow" dynamical solutions in terms of the moduli space of static solutions was proposed on a heuristic level by Manton [2] who postulated the following *adiabatic principle*: for any geodesic trajectory γ_0 on the moduli space of d-vortices $\mathcal{M}_d$ it should exist a sequence $\{\gamma_\epsilon\}$ of dynamical solutions, converging to γ_0 in the adiabatic limit.

A rigorous mathematical formulation and the proof of this principle were given recently by Roman Palvelev [4] (cf. also [5]).

So we have the following correspondence established by the adiabatic limit procedure:

$$\left\{\begin{matrix}\text{solutions of Ginzburg–}\\ \text{Landau equations}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{geodesics on the vortex moduli}\\ \text{space in } T\text{-metric}\end{matrix}\right\}$$

and

$$\left\{\begin{matrix}\text{Ginzburg–Landau}\\ \text{equations}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Euler geodesic equations on the}\\ \text{vortex moduli space}\end{matrix}\right\} .$$

3.4. Scattering of vortices

The adiabatic principle reduces the problem of scattering of vortices in our model to the description of geodesics on the moduli space of d-vortices $\mathcal{M}_d$ in the kinetic metric, i.e., to the solution of ordinary Euler geodesic equation on the space $\mathcal{M}_d$ provided with T-metric. Unfortunately, apart from the case $d = 2$, no explicit formulas for this metric are known. (In the case $d = 2$ such a formula was obtained in a joint paper with Sergey Chechin [11].)

The reason is that the Taubes theorem itself does not provide any explicit formula for the d-vortex solution with zeros in the prescribed points of the complex plane. This theorem only states the existence of such a vortex in a neighborhood of the linearized solution with given zeros.

Despite the absence of explicit formulas in the general case we can establish some useful properties of the kinetic metric. For example, it can be proved that it is smooth in the coordinates, given by the symmetric functions of zeros of Φ.

Recall that, according to Taubes' theorem, the moduli space of d-vortices can be identified with $\mathbb{C}^d$ by assigning to a given collection of k points $z_1, \dots, z_k$ on the complex plane with multiplicities $d_1, \dots, d_k$ such that $\sum_{j=1}^k d_j = d$ the monic polynomial with zeros in these points with given multiplicities $d_1, \dots, d_k$. The coefficients of this polynomial, being symmetric functions of zeros, define in a natural way coordinates on the moduli space $\mathcal{M}_d$. In these coordinates the kinetic metric is proved to be smooth (this result was also obtained by Roman Palvelev [3]).

Using this property, Palvelev has proved [3] that in the process of central-symmetric head-on collision of d vortices their trajectories rotate by the angle π/d. In particular, two vortices after their collision scatter by a right angle. This effect is well known to physicists and can be observed in physical experiments.

4. Adiabatic limit in Seiberg–Witten equations

4.1. Seiberg–Witten equations on Riemannian 4-manifolds

The Ginzburg–Landau equations, as it became clear recently, are closely related to the Seiberg–Witten equations. The Seiberg–Witten equations, along with the Yang–Mills equations, are the limiting cases of a more general supersymmetric Yang–Mills theory (cf. [7], [8]). Note that, opposite to the conformally invariant Yang–Mills equations, the Seiberg–Witten equations are not invariant under the change of the scale which is one of their main features. So to draw "useful information" from these equations one should introduce the scale parameter λ into them and take the limit $\lambda \to \infty$.

Let us recall the definition of Seiberg–Witten equations (a detailed discussion of these equations may be found in [6] or [10]). Let (X, g) be an oriented compact Riemannian 4-manifold. We assume that X is provided with a Spin^c-structure given by the spinor bundle $W = W^+ \oplus W^-$ of rank 4 provided with a spinor connection given by the covariant derivative ∇_A generated by a connection form A on the characteristic line bundle $L \to X$ of the structure.

The Seiberg–Witten action functional is defined by the expression

$$S(A, \Phi) = \frac{1}{2} \int_X \left\{ |F_A|^2 + |\nabla_A \Phi|^2 + \frac{|\Phi|^2}{4}(s + |\Phi|^2) \right\} d\,\mathrm{vol}$$

where Φ is a section of the positive spinor bundle W^+, $s = s(g)$ is the scalar curvature of (X, g) and $d\,\mathrm{vol}$ is the volume element on (X, g). Note that $S(A, \Phi)$ may take negative values if the curvature s is negative.

The equations for the local minima of this functional may be written in the form

$$\begin{cases} D_A\Phi = 0 \\ F_A^+ = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \cdot \mathrm{Id} \end{cases}$$

where D_A is the covariant Dirac operator associated with the connection ∇_A and F_A^+ is the selfdual component of the curvature F_A.

4.2. Seiberg–Witten equations on symplectic 4-manifolds

If the manifold X is symplectic, i.e., provided with a symplectic form ω compatible with the Riemannian metric g, then the pair (ω, g) uniquely determines an almost complex structure J compatible with both structures.

The almost complex structure J determines a canonical Spin^c-structure $W_{\mathrm{can}} = W_{\mathrm{can}}^+ \oplus W_{\mathrm{can}}^-$ and canonical spinor connection ∇_{can} with

$$W_{\mathrm{can}}^+ = \Lambda^0(X) \oplus \Lambda^{0,2}(X), \quad W_{\mathrm{can}}^- = \Lambda^{0,1}(X)$$

and characteristic line bundle $L_{\mathrm{can}} = \Lambda^{0,2}(X)$.

Having a Hermitian line bundle $E \to X$ provided with a Hermitian connection B we can construct the corresponding Spin^c-structure on X. For this structure the spinor bundle W_E is the tensor product $W_{\mathrm{can}} \otimes E$ and the spinor connection coincides with the tensor product of the canonical connection A_{can} on L_{can} and B.

In other words,

$$W_E^+ = W_{\mathrm{can}}^+ \otimes E = \Lambda^0(X, E) \oplus \Lambda^{0,2}(X, E), \quad W_E^- = W_{\mathrm{can}}^- \otimes E = \Lambda^{0,1}(X, E)$$

with the characteristic bundle

$$L_{\mathrm{can}} = L_{\mathrm{can}} \otimes E = \Lambda^{0,2}(X, E)$$

and spinor connection $\nabla_A = \nabla_{\mathrm{can}} + B$.

In this case the Dirac operator D_A will coincide with

$$D_A = \bar{\partial}_B + \bar{\partial}_B^*$$

where $\bar{\partial}_B^*$ is the operator L^2-adjoint to $\bar{\partial}_B$, and Φ is a section of the positive spinor bundle W_E^+ having the form

$$\Phi = (\varphi_0, \varphi_2) \in \Omega^0(X, E) \oplus \Omega^{0,2}(X, E).$$

In this case the Seiberg–Witten equations with scale parameter λ will have the form

$$\begin{cases} \bar{\partial}_{B_\lambda}\alpha_\lambda + \bar{\partial}_{B_\lambda}^*\beta_\lambda = 0 \\ \frac{2}{\lambda}F_{B_\lambda}^{0,2} = \bar{\alpha_\lambda}\beta_\lambda, \\ \frac{4i}{\lambda}F_{B_\lambda}^\omega = 4\pi + |\beta_\lambda|^2 - |\alpha_\lambda|^2 \end{cases}$$

where $\alpha_\lambda = \frac{\varphi_0}{\sqrt{\lambda}}$, $\beta_\lambda = \frac{\varphi_2}{\sqrt{\lambda}}$ are the normalized forms and $F_{B_\lambda}^\omega$ is the component of the curvature F_{B_λ} parallel to the form ω.

We assume that the necessary solvability conditions, having the topological nature (cf. [6], [13]), are satisfied. Then these equations will have solutions for all sufficiently large λ.

4.3. Adiabatic limit in Seiberg–Witten equations

Taubes in [13] has shown that these solutions have the following behavior for $\lambda \to \infty$:

1) $|\alpha_\lambda| \to 1$ for $\lambda \to \infty$ everywhere outside its zeros;
2) $|\beta_\lambda| \to 0$ for $\lambda \to \infty$ everywhere with its first derivatives.

Denote by $C_\lambda := \alpha_\lambda^{-1}(0)$ the zero set of α_λ. Then (cf. [13]) C_λ converge (in the sense of currents) to some pseudoholomorphic divisor, i.e., a chain composed of pseudoholomorphic curves C_k taken with multiplicities m_k. In the same limit the original Seiberg–Witten equations will reduce to a family of vortex equations defined in the complex planes normal to the curves C_k. The chain $\sum m_k C_k$ may be considered as a complex analogue of adiabatic geodesics in $(1+2)$-dimensional case.

Conversely, in order to reconstruct the solution of Seiberg–Witten equations from the chain $\sum m_k C_k$, the family of vortex solutions in normal planes should satisfy a nonlinear $\bar{\partial}$-equation which may be considered as a complex analogue of the Euler equation for adiabatic geodesics with "complex time" (cf. [10]).

So we have the following correspondence established by the adiabatic limit procedure:

$$\left\{\begin{matrix}\text{solutions of Seiberg–}\\ \text{Witten equations}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{families of vortices in normal planes to}\\ \text{pseudoholomorphic divisors}\end{matrix}\right\}$$

and

$$\left\{\begin{matrix}\text{Seiberg–Witten}\\ \text{equations}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{families of vortex equations in normal}\\ \text{planes to pseudoholomorphic divisors}\end{matrix}\right\} .$$

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Armen Sergeev
Steklov Mathematical Institute
Gubkina 8
119991 Moscow, Russia
e-mail: sergeev@mi.ras.ru

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Variational Tricomplex and BRST Theory

A.A. Sharapov

Abstract. By making use of the variational tricomplex, a covariant procedure is proposed for deriving the classical BRST charge of the BFV formalism from a given BV master action.

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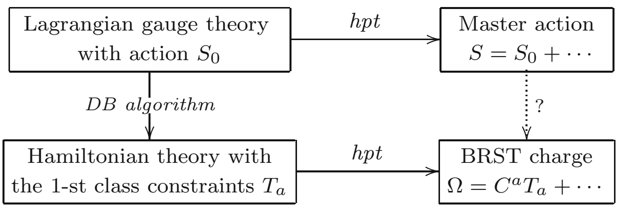
1. Introduction

The BRST theory provides the most powerful approach to the quantization of gauge systems [1]. It includes the Batalin–Vilkovisky (BV) formalism for Lagrangian gauge systems and its Hamiltonian counterpart known as the Batalin–Fradkin–Vilkovisky (BFV) formalism. Usually, the two formalisms are developed in parallel starting, respectively, from the classical action or the first-class constraints on the phase space of the system. In either case one applies the homological perturbation theory (hpt) to obtain the master action or the classical BRST charge at the output. A relationship between both the pictures of gauge dynamics is established through the Dirac–Bergmann (DB) algorithm, which allows one to generate the complete set of first-class constraints by the classical action. All these can be displayed diagrammatically as follows:



Looking at this picture it is natural to ask about the dotted arrow making the diagram commute. The arrow symbolizes a hypothetical map or construction connecting the BV and BFV formalisms at the level of generating functionals.

As we show below such a map really exists. By making use of the variational tricomplex [10], we propose a direct construction of the classical BRST charge from the BV master action. The construction is explicitly covariant (even though we pass to the Hamiltonian picture) and generates the full spectrum of BFV ghosts immediately from that of the BV theory. We also derive a covariant Poisson bracket on the extended phase space of the theory, with respect to which the classical BRST charge obeys the master equation. The construction of the covariant Poisson bracket is similar to that presented in [5], except that our Poisson bracket is defined off shell.

Finally, it should be noted that the first variational tricomplex for gauge systems was introduced in [6] as the Koszul–Tate resolution of the usual variational bicomplex for partial differential equations. Using this tricomplex, the authors of [6] were able to relate various Lie algebras associated with the global symmetries and conservation laws of a classical gauge system. Our tricomplex is similar in nature but involves the full BRST differential, and not its Koszul–Tate part.

2. Variational tricomplex of a local gauge system

In modern language the classical fields are just the sections of a locally trivial, fiber bundle $\pi : E \to M$ over an n-dimensional space-time manifold M. The typical fiber F of E is called the *target space of fields*. In case the bundle is trivial, i.e., $E = M \times F$, the fields are merely the smooth mappings from M to F. For the sake of simplicity, we restrict ourselves to fields associated with vector bundles. In this case the space of fields $\Gamma(E)$ has the structure of a real vector space.

Bearing in mind gauge theories as well as field theories with fermions, we assume $\pi : E \to M$ to be a $\mathbb{Z}$-graded supervector bundle over the ordinary (non-graded) smooth manifold M. The Grassmann parity and the $\mathbb{Z}$-grading of a homogeneous object A will be denoted by $\epsilon(A)$ and $\deg A$, respectively. It should be emphasized that in the presence of fermionic fields there is no natural correlation between the Grassmann parity and the $\mathbb{Z}$-grading. Since throughout the paper we work exclusively in the category of $\mathbb{Z}$-graded supermanifolds, we omit the boring prefixes "super" and "graded" whenever possible. For a quick introduction to the graded differential geometry and some of its applications we refer the reader to [7–9].

In the local field theory, the dynamics of fields are governed by partial differential equations. The best way to account for the local structure of fields is to introduce the variational bicomplex $\Lambda^{*,*}(J^\infty E; d, \delta)$ on the infinite jet bundle $J^\infty E$ associated with the vector bundle $\pi : E \to M$. Here d and δ denote the horizontal and vertical differentials in the bigraded space $\Lambda^{*,*}(J^\infty E) = \bigoplus \Lambda^{p,q}(J^\infty E)$ of differential forms on $J^\infty E$, where p and q refer to the vertical and horizontal degrees, respectively. A brief account of the concept of a variational bicomplex can be found in [4, 5].

The free variational bicomplex represents thus a natural kinematical basis for defining local field theories. In order to specify dynamics two more geometrical ingredients are needed. These are the classical BRST differential and the BRST-invariant (pre)symplectic structure on $J^\infty E$. Let us give the corresponding definitions.

2.1. Presymplectic structure

By a *presymplectic* $(2,m)$-form on $J^\infty E$ we understand an element $\omega \in \widetilde{\Lambda}^{2,m}(J^\infty E)$ satisfying [1]

$$\delta\omega \simeq 0\,. \tag{1}$$

The form ω is assumed to be homogeneous, so that we can speak of an odd or even presymplectic structure of definite $\mathbb{Z}$-degree. The triviality of the relative "δ modulo d" cohomology in positive vertical degree (see [5, Sec. 19.3.9]) implies that any presymplectic $(2,m)$-form is exact, namely, there exists a homogeneous $(1,m)$-form θ such that $\omega \simeq \delta\theta$. The form θ is called the *presymplectic potential* for ω. Clearly, the presymplectic potential is not unique. If θ_0 is one of the presymplectic potentials for ω, then setting $\omega_0 = \delta\theta_0$ we get

$$\delta\omega_0 = 0\,, \qquad \omega_0 \simeq \omega\,.$$

In other words, any presymplectic form has a δ-closed representative.

Denote by $\ker\omega$ the space of all evolutionary vector fields X on $J^\infty E$ that fulfill the relation[2]

$$i_X\omega \simeq 0\,.$$

A presymplectic form ω is called non-degenerate if $\ker\omega = 0$, in which case we refer to it as a *symplectic form*.

An evolutionary vector field X is called *Hamiltonian* with respect to ω if it preserves the presymplectic form, that is,

$$L_X\omega \simeq 0\,. \tag{2}$$

Obviously, the Hamiltonian vector fields form a subalgebra in the Lie algebra of all evolutionary vector fields. Eq. (2) is equivalent to

$$\delta i_X\omega \simeq 0\,.$$

Again, because of the triviality of the relative δ-cohomology, we can write

$$i_X\omega \simeq \delta H \tag{3}$$

for some $H \in \widetilde{\Lambda}^{0,m}(J^\infty E)$. We refer to H as a *Hamiltonian form* (or *Hamiltonian*) associated with X. Sometimes, to indicate the relationship between the Hamiltonian vector fields and forms, we will write X_H for X. In general, the relationship is far from being one-to-one.

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[1] By abuse of notation, we denote by ω an element of the quotient space $\widetilde{\Lambda}^{2,m} = \Lambda^{2,m}/d\Lambda^{2,m-1}$ and its representative in $\Lambda^{2,m}$. The sign $\simeq$ means equality modulo $d\Lambda^{*,*}$.

[2] Recall that a vertical vector field X is called *evolutionary* if $i_X d + (-1)^{\epsilon(X)} d i_X = 0$, where i_X is the operation of contraction of X with differential forms.

The space $\Lambda^{0,m}_{\omega}(J^\infty E)$ of all Hamiltonian m-forms can be endowed with the structure of a Lie algebra. The corresponding Lie bracket is defined as follows: If X_A and X_B are two Hamiltonian vector fields associated with the Hamiltonian forms A and B, then

$$\{A,B\} = (-1)^{\epsilon(X_A)} i_{X_A} i_{X_B}\omega\,. \tag{4}$$

The next proposition shows that the bracket is well defined and possesses all the required properties.

Proposition 1 ([10]). *The bracket* (4) *is bilinear over reals, maps the Hamiltonian forms to Hamiltonian ones, enjoys the symmetry property*

$$\{A,B\} \simeq -(-1)^{(\epsilon(A)+\epsilon(\omega))(\epsilon(B)+\epsilon(\omega))}\{B,A\}\,, \tag{5}$$

and obeys the Jacobi identity

$$\{C,\{A,B\}\} \simeq \{\{C,A\},B\} + (-1)^{(\epsilon(C)+\epsilon(\omega))(\epsilon(A)+\epsilon(\omega))}\{A,\{C,B\}\}\,. \tag{6}$$

2.2. Classical BRST differential

An odd evolutionary vector field Q on $J^\infty E$ is called *homological* if

$$[Q,Q] = 2Q^2 = 0\,, \qquad \deg Q = 1\,. \tag{7}$$

The Lie derivative along the homological vector field Q will be denoted by δ_Q. It follows from the definition that $\delta_Q^2 = 0$. Hence, δ_Q is a differential of the algebra $\Lambda^{*,*}(J^\infty E)$ increasing the $\mathbb{Z}$-degree by 1. Moreover, the operator δ_Q anticommutes with the coboundary operators d and δ:

$$\delta_Q d + d\delta_Q = 0\,, \qquad \delta_Q\delta + \delta\delta_Q = 0\,.$$

This allows us to speak of the tricomplex $\Lambda^{*,*,*}(J^\infty E; d,\delta,\delta_Q)$, where

$$\delta_Q : \Lambda^{p,q,r}(J^\infty E) \to \Lambda^{p,q,r+1}(J^\infty E)\,.$$

In the physical literature the homological vector field Q is known as the *classical BRST differential* and the $\mathbb{Z}$-grading is called the *ghost number*. These are the two main ingredients of all modern approaches to the covariant quantization of gauge theories. In the BV formalism, for example, the BRST differential carries all the information about equations of motions, their gauge symmetries and identities, and the space of physical observables is naturally identified with the group $H^{0,n,0}(J^\infty E; \delta_Q/d)$ of "δ_Q modulo d" cohomology in ghost number zero. For general non-Lagrangian gauge theories the classical BRST differential was systematically defined in [2, 3].

The equations of motion of a gauge theory can be recovered by considering the zero locus of the homological vector field Q. In terms of adapted coordinates (x^i, ϕ^a_I) on $J^\infty E$ the vector field Q, being evolutionary, assumes the form[3]

$$Q = \partial_I Q^a \left(\partial/\,\partial\phi^a_I\right)\,.$$

[3]We use the multi-index notation according to which the multi-index $I = i_1 i_2\cdots i_k$ represents the set of symmetric covariant indices and $\partial_I = \partial_{i_1}\cdots\partial_{i_k}$. The *order* of the multi-index is given by $|I| = k$.

Then there exists an integer l such that the equations

$$\partial_I Q^a = 0\,, \qquad |I| = k\,,$$

define a submanifold $\Sigma^k \subset J^{l+k}E$. The standard regularity condition implies that Σ^{k+1} fibers over Σ^k for each k. This gives the infinite sequence of projections

$$\cdots \longrightarrow \Sigma^{l+3} \longrightarrow \Sigma^{l+2} \longrightarrow \Sigma^{l+1} \longrightarrow \Sigma^l \to M\,,$$

which enables us to define the zero locus of Q as the inverse limit

$$\Sigma^\infty = \lim_{\longleftarrow} \Sigma^k\,.$$

In physics, the submanifold $\Sigma^\infty \subset J^\infty E$ is usually referred to as the *shell*. The terminology is justified by the fact that the classical field equations as well as their differential consequences can be written as

$$(j^\infty \phi)^* (\partial_I Q^a) = 0\,.$$

In other words, the field $\phi \in \Gamma(E)$ satisfies the classical equations of motion iff $j^\infty \phi \in \Sigma^\infty$.

2.3. *Q*-invariant presymplectic structure and its descendants

By a *gauge system* on $J^\infty E$ we will mean a pair (Q, ω) consisting of a homological vector field Q and a Q-invariant presymplectic $(2,m)$-form ω. In other words, the vector field Q is supposed to be Hamiltonian with respect to ω, so that $\delta_Q \omega \simeq 0$. The last relation implies the existence of forms ω_1, H, and θ_1 such that

$$\delta_Q \omega = d\omega_1\,, \qquad i_Q \omega = \delta H + d\theta_1\,. \tag{8}$$

As was mentioned in Section 2.1, we can always assume that $\omega = \delta\theta$ for some presymplectic potential θ, so that $\delta\omega = 0$. Then applying δ to the second equality in (8) and using the first one, we find $d(\omega_1 - \delta\theta_1) = 0$. On account of the exactness of the variational bicomplex, the last relation is equivalent to

$$\omega_1 \simeq \delta\theta_1\,.$$

Thus, ω_1 is a presymplectic $(2, m-1)$-form on $J^\infty E$ coming from the presymplectic potential θ_1. Furthermore, the form ω_1 is Q-invariant as one can easily see by applying δ_Q to the first equality in (8) and using once again the fact of exactness of the variational bicomplex. Let H_1 denote the Hamiltonian for Q with respect to ω_1, i.e.,

$$i_Q \omega_1 \simeq \delta H_1\,, \qquad H_1 \in \widetilde{\Lambda}^{0,m-1}(J^\infty E)\,.$$

Given the pair (Q, ω), we call ω_1 the *descendent presymplectic structure* on $J^\infty E$ and refer to (Q, ω_1) as the *descendent gauge system*.

The next proposition provides an alternative definition for the descendent Hamiltonian of the homological vector field.

Proposition 2 ([10]). *Let ω be a δ-closed representative of a presymplectic $(2,m)$-form on $J^\infty E$ and* $\deg H_1 \neq 0$*, then*

$$dH_1 = -\frac{1}{2}\{H,H\}\,. \tag{9}$$

Corollary 3. *H is a Maurer–Cartan element of the Lie algebra $\Lambda^{0,m}_\omega(J^\infty E)$, that is,*

$$\{H,H\} \simeq 0\,.$$

Corollary 4. *The Hamiltonian form H_1 is d-closed on-shell. In particular, for $m=n$ it defines a conservation law.*

Proposition 5 ([10]). *Suppose that the Q-invariant presymplectic form ω of top horizontal degree has the structure*

$$\omega = P_{ab}\wedge\delta\phi^a\wedge\delta\phi^b\,, \qquad P_{ab}\in\Lambda^{0,n}(J^\infty E)\,, \tag{10}$$

and H is the Hamiltonian of Q with respect to ω. Then the presymplectic potential for the descendent presymplectic $(2,n-1)$-form $\omega_1\simeq\delta\theta_1$ is defined by the equation

$$\delta H = \delta\phi^a\wedge\frac{\delta H}{\delta\phi^a} - d\theta_1\,. \tag{11}$$

The above construction of the descendent gauge system (Q,ω_1) can be iterated producing a sequence of gauge systems (Q,ω_k), where the kth presymlectic form $\omega_k\in\Lambda^{2,m-k}(J^\infty E)$ is the descendant of ω_{k-1}. The minimal k for which $\omega_k\simeq 0$ gives a numerical invariant of the original gauge system (Q,ω).

3. BFV from BV

In this section, we apply the construction of the variational tricomplex for establishing a direct correspondence between the BV formalism of Lagrangian gauge systems and its Hamiltonian counterpart known as the BFV formalism. We start from a very brief account of both the formalisms in a form suitable for our purposes. For a systematic exposition of the subject we refer the reader to [1].

3.1. BV formalism

The starting point of the BV formalism is an infinite-dimensional manifold $\mathcal{M}_0$ of gauge fields that live on an n-dimensional space-time M. Depending on a particular structure of gauge symmetry the manifold $\mathcal{M}_0$ is extended to an $\mathbb{N}$-graded manifold $\mathcal{M}$ containing $\mathcal{M}_0$ as its body. The new fields of positive $\mathbb{N}$-degree are called the *ghosts* and the $\mathbb{N}$-grading is referred to as the *ghost number*. Let us collectively denote all the original fields and ghosts by Φ^A and refer to them as fields. At the next step the space of fields $\mathcal{M}$ is further extended by introducing the odd cotangent bundle $\Pi T^*[-1]\mathcal{M}$. The fiber coordinates, called *antifields*, are denoted by Φ^*_A. These are assigned with the following ghost numbers and Grassmann parities:

$$\mathrm{gh}(\Phi^*_A) = -\mathrm{gh}(\Phi^A)-1\,, \qquad \epsilon(\Phi^*_A)=\epsilon(\Phi^A)+1 \pmod 2\,.$$

Thus, the total space of the odd cotangent bundle $\Pi T^*[-1]\mathcal{M}$ becomes a $\mathbb{Z}$-graded supermanifold. The canonical Poisson structure on $\Pi T^*[-1]\mathcal{M}$ is determined by the following odd Poisson bracket in the space of functionals of Φ and Φ^*:

$$(A,B)=\int_M\left(\frac{\delta_r A}{\delta\Phi^A}\frac{\delta_l B}{\delta\Phi^*_A}-\frac{\delta_r A}{\delta\Phi^*_A}\frac{\delta_l B}{\delta\Phi^A}\right)d^nx\,. \tag{12}$$

Here d^nx is a volume form on M and the subscripts l and r refer to the standard left and right functional derivatives. In the physical literature the above bracket is usually called the *antibracket* or the *BV bracket.*

The functionals of the form

$$A=\int_M(j^\infty\phi)^*(a)\,,$$

where $\phi=(\Phi,\Phi^*)$ and $a\in\widetilde{\Lambda}^{0,n}(J^\infty E)$, are called *local.* Under suitable boundary conditions for ϕs the map $a\mapsto A$ defines an isomorphism of vector spaces, which gives rise to a pulled-back Poisson bracket on $\widetilde{\Lambda}^{0,n}(J^\infty E)$. This last bracket is determined by the symplectic structure

$$\omega=\delta\Phi^*_A\wedge\delta\Phi^A\wedge d^nx \tag{13}$$

according to (4). By definition, $\mathrm{gh}(\omega)=-1$ and $\epsilon(\omega)=1$.

The central goal of the BV formalism is the construction of a *master action* S on the space of fields and antifields. This is defined as a proper solution to the *classical master equation*

$$(S,S)=0\,. \tag{14}$$

The local functional S is required to be of ghost number zero and start with the action S_0 of the original fields to which one couples vertices involving antifields. All these vertices can be found systematically from the master equation (14) by means of the so-called *homological perturbation theory* [1].

The classical BRST differential on the space of fields and antifields is canonically generated by the master action through the antibracket:

$$Q=(S\,,\,\cdot\,)\,. \tag{15}$$

Because of the master equation for S and the Jacobi identity for the antibracket (12), the operator Q squares to zero in the space of smooth functionals. The physical quantities are then identified with the cohomology classes of Q in ghost number zero. When restricted to the subspace of local functionals the classical BRST differential (15) induces a homological vector field on the total space of the jet bundle $J^\infty E$.

3.2. BFV formalism

The Hamiltonian formulation of the same gauge dynamics implies a prior splitting $M=N\times\mathbb{R}$ of the original space-time into space and time; the factor N can be viewed as the physical space at a given instant of time. The initial values of the original fields are then considered to form an infinite-dimensional manifold $\mathcal{N}_0$. To allow for possible constraints on the initial data of fields the manifold $\mathcal{N}_0$ is extended to an $\mathbb{N}$-graded supermanifold $\mathcal{N}$ by adding new fields, called ghosts, of positive $\mathbb{N}$-degree. Then the space of original fields and ghosts is doubled by introducing the cotangent bundle $T^*\mathcal{N}$ endowed with the canonical symplectic structure. If we denote the local coordinates on $\mathcal{N}$ by Φ^a and the linear coordinates in the cotangent spaces by $\bar{\Phi}_a$, then the canonical Poisson bracket in the space of functionals of Φ^a and $\bar{\Phi}_a$ reads

$$\{A,B\} = \int_N \left(\frac{\delta_r A}{\delta \Phi^a} \frac{\delta_l B}{\delta \bar{\Phi}_a} - (-1)^{\epsilon(\Phi_a)} \frac{\delta_r A}{\delta \bar{\Phi}_a} \frac{\delta_l B}{\delta \Phi^a} \right) d^{n-1}x \,. \tag{16}$$

Here $d^{n-1}x$ stands for a volume form on N. By the definition of the cotangent bundle of a graded manifold

$$\mathrm{gh}(\bar{\Phi}_a) = -\mathrm{gh}(\Phi^a)\,, \qquad \epsilon(\bar{\Phi}_a) = \epsilon(\Phi^a)\,.$$

Again, the space of local functionals, i.e., functionals of the form

$$B = \int_N j^\infty(\phi)^*(b)\,, \qquad \phi = (\Phi, \bar{\Phi})\,, \qquad b \in \tilde{\Lambda}^{0,n-1}(J^\infty E)\,,$$

appears to be closed w.r.t. the even Poisson bracket (16) and the map $b \mapsto B$ induces an even Poisson bracket on $\widetilde{\Lambda}^{0,n-1}(J^\infty E)$. The latter is determined by the even symplectic form

$$\omega_1 = \delta \bar{\Phi}_a \wedge \delta \Phi^a \wedge d^{n-1}x$$

of ghost number zero.

The gauge structure of the original dynamics is encoded by the *classical BRST charge* Ω. This is given by an odd, local functional of ghost number 1 satisfying the classical master equation

$$\{\Omega, \Omega\} = 0\,.$$

The classical BRST differential in the extended space of fields and momenta is given now by the Hamiltonian action of the BRST charge:

$$Q = \{\Omega\,, \cdot\,\}\,. \tag{17}$$

It is clear that $Q^2 = 0$. The group of Q-cohomology in ghost number zero is then naturally identified with the space of physical observables. Upon restriction to the space of local functionals the variational vector field (17) induces a homological vector field on the total space of the infinite jet bundle.

3.3. From BV to BFV

It must be clear from the discussion above that any gauge system in the BFV formalism may be viewed as the descendant of the same system in the BV formalism. More precisely, we can define the even presymplectic structure ω_1 on the phase space of a gauge theory as the descendant of the odd symplectic structure (13):

$$d\omega_1 = \delta_Q(\delta\Phi^*_A \wedge \delta\Phi^A \wedge d^n x) = \delta \left(\delta\Phi^A \wedge \frac{\delta S}{\delta \Phi^A} + \delta \Phi^*_A \wedge \frac{\delta S}{\delta \Phi^*_A} \right)\,.$$

The corresponding classical BRST charge is given by

$$\Omega_N = \int_N (j^\infty\phi)^*(J)\,,$$

where $N \subset M$ is a space-like Cauchy hypersurface and $J \in \Lambda^{0,n-1}_{\omega_1}(J^\infty E)$ is the Hamiltonian of the classical BRST differential $Q = (S,\,\cdot\,)$ w.r.t. the descendent presymplectic form ω_1, i.e.,

$$\delta J \simeq i_Q\omega_1\,. \tag{18}$$

It is clear that $\mathrm{gh}(\Omega) = 1$. In virtue of Corollary 3, the functional Ω obeys the classical master equation $\{\Omega,\Omega\} = 0$ with respect to the even Poisson bracket associated with ω_1. According to Corollary 4 the form J represents a conserved current, the BRST current. Formally, this means that the "value" of the odd charge Ω_N does not depend on the choice of N provided that $j^\infty\phi \in \Sigma^\infty$.

Since the canonical symplectic structure (13) on the space of fields and antifields is δ-exact, we can give an equivalent definition for J in terms of the antibracket (12). For this end, consider the dynamics of fields in a domain $D \subset M$ bounded by two Cauchy hypersurfaces N_1 and N_2. The fields and antifields are assumed to vanish on space infinity together with their derivatives. By Proposition 2,

$$-\frac{1}{2}(S,S) = \int_D (j^\infty\phi)^*(dJ) = \int_D d[(j^\infty\phi)^*(J)] = \Omega_{N_2} - \Omega_{N_1}\,.$$

Let us illustrate the general construction by a particular example of gauge theory.

3.4. Maxwell's electrodynamics

In the BV formalism, the free electromagnetic field in a 4-dimensional Minkowski space is described by the master action

$$S = \int L\,, \qquad L = -\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + C\partial^\mu A^*_\mu\Big)d^4x\,. \tag{19}$$

Here

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the strength tensor of the electromagnetic field, A^*_μ is the antifield to the electromagnetic potential A_μ, and C is the ghost field associated with the standard gauge transformation $\delta_\varepsilon A_\mu = \partial_\mu\varepsilon$.

Since the gauge symmetry is abelian, the master action (19) does not involve the ghost antifield C^*. The odd symplectic structure (13) on the space of fields and antifields assumes the form

$$\omega = (\delta A^*_\mu \wedge \delta A^\mu + \delta C^* \wedge \delta C)\wedge d^4x\,, \qquad d^4x = dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,,$$

and the action of the classical BRST differential is given by

$$\delta_Q A_\mu = \partial_\mu C\,, \qquad \delta_Q A^*_\mu = \partial^\nu F_{\nu\mu}\,, \qquad \delta_Q C = 0\,, \qquad \delta_Q C^* = \partial^\mu A^*_\mu\,. \tag{20}$$

The variation of the Lagrangian density reads

$$\delta L = (\partial^\mu F_{\mu\nu}\delta A^\nu + \partial^\mu A^*_\mu \delta C + \partial^\mu C \delta A^*_\mu - \partial^\mu \theta_\mu) \wedge d^4x\,, \qquad \theta_\mu = F_{\mu\nu}\delta A^\nu + C\delta A^*_\mu\,.$$

One can easily check that $i_Q\omega \simeq \delta L$. By Proposition 5 the form

$$\theta_1 = -\theta_\mu \wedge d^3x^\mu\,, \qquad d^3x^\mu = \eta^{\mu\nu} i_{\frac{\partial}{\partial x^\nu}} d^4x\,,$$

defines the potential for the descendent presymplectic form

$$\omega_1 = \delta\theta_1 = -(\delta F_{\nu\mu} \wedge \delta A^\mu + \delta C \wedge \delta A^*_\nu) \wedge d^3x^\nu\,. \tag{21}$$

(Of course, one could arrive at this expression by considering the BRST variation $\delta_Q\omega = d\omega_1$ of the original symplectic structure.)

Applying the BRST differential to the form ω_1 yields one more descendent presymplectic form

$$\omega_2 = \delta C \wedge \delta F_{\mu\nu} \wedge d^2x^{\mu\nu}\,, \qquad d^2x^{\mu\nu} = \eta^{\mu\alpha} i_{\frac{\partial}{\partial x^\alpha}} d^3x^\nu\,.$$

This last form, being “absolutely” invariant under the BRST transformations (20), leaves no further descendants.

The 3-form of the conserved BRST current J associated to the BRST symmetry transformations (20) is determined by Eq. (18). We find

$$J = J_\nu d^3x^\nu \simeq -C\partial^\mu F_{\mu\nu} d^3x^\nu\,.$$

Once we identify x^0 with time in the Hamiltonian formalism, the antifield A^*_0 plays the role of ghost momentum canonically conjugate to C with respect to the presymplectic structure (21). The on-shell conservation of the corresponding BRST charge $\Omega = \int_{\mathbb{R}^3} J_0 d^3x$ expresses nothing but the Gauss law $\partial^i F_{i0} = 0$.

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A.A. Sharapov
Physics Faculty
Tomsk State University
Tomsk 634050, Russia
e-mail: sharapov@phys.tsu.ru

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Quantization of Hitchin's Moduli Space of a Non-orientable Surface

Siye Wu

Abstract. We review the geometry of the moduli space of flat connections and Hitchin's moduli space for an orientable or non-orientable surface, and study various line bundles over the moduli spaces. After a survey of the background materials, we consider the quantization of Hitchin's moduli space for a non-orientable surface by branes and mirror symmetry.

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1. Introduction

The aim of quantization is to obtain, from a classical system, a quantum Hilbert space and an algebra of quantum operators acting on it. Geometrically, a classical system is a symplectic manifold and, though outstanding questions still exist, geometric quantization [23, 32] is a well-developed procedure for achieving these goals. On the other hand, deformation quantization [4] produces a deformation of the algebras of functions on a symplectic or Poisson manifold in a formal power series, beginning with the Poisson bracket in the first order. There are also problems such as the convergence of the formal series. A few years ago, a new method called quantization via branes was proposed [13] using a suitable topological A-model on a complexification of the symplectic manifold to be quantized. Both the algebra of quantum observables and the space of quantum states are realized as the spaces of open strings ending on branes. A dual point of view via mirror symmetry was subsequently studied [12] using a topological B-model. In this paper, we explore the quantization of Hitchin's moduli spaces for a non-orientable surface.

On an orientable surface, Hitchin's equations were introduced in [15] as a reduction of the self-dual Yang–Mills equation in four dimensions. The moduli space of solutions (up to gauge equivalence) is hyper-Kähler. When the gauge group is replaced by its Langlands dual [10, 25], Hitchin's moduli space turns to its mirror manifold [8, 14]. In [21] (see [37] for recent development), a twisted version of the $N = 4$ gauge theory in four dimensions was compactified in two directions to obtain, at low energies, a sigma-model whose target space is Hitchin's moduli space. The electric-magnetic duality (or S-duality) in four dimensions reduces to the mirror symmetry in two dimensions and explains the geometric Langlands correspondence [21]. Hitchin's moduli space for a non-orientable surface was studied in [19], building upon earlier works [17, 18] on the moduli space of flat connections on a non-orientable surface. The mathematical structures were further explored in [39] from the point of view of four-dimensional gauge theory, using especially the discrete electric and magnetic fluxes [34] and their duality.

The present paper is structured as follows. In §2, we review the construction in [30] of the prequantum line bundle over the moduli space of flat connections on an orientable surface, beginning with a finite-dimensional model for clarity. In §3, we recall Hitchin's moduli space for an orientable surface and study various line bundles over it by applications of the finite-dimensional setting. We give a survey of branes in sigma-models using Hitchin's moduli space as an example. In §4, we explain the geometry of the moduli space of flat connections and Hitchin's moduli space for a non-orientable surface and study the line bundles over them. In §5, we review the general theory of quantization via branes and mirror symmetry. We then apply it to the quantization of Hitchin's moduli space for a non-orientable surface.

2. Line bundle on moduli space of flat connections

2.1. Symplectic quotient and prequantum line bundle

Let (M, ω) be a symplectic manifold and let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Suppose G acts on (M, ω) on the left and the action is Hamiltonian with a moment map $\mu\colon M \to \mathfrak{g}^*$. This means that if X_a is the vector field on M corresponding to $a \in \mathfrak{g}$, then $\iota_{X_a}\omega = d\mu_a$, where $\mu_a = \langle \mu, a\rangle$, and $\omega(X_a, X_b) = -X_a\mu_b = X_b\mu_a = \{\mu_a, \mu_b\} = \mu_{[a,b]}$ for all $a, b \in \mathfrak{g}$. If 0 is a regular value and G acts freely (or with a constant stabilizer) on $\mu^{-1}(0)$, then the symplectic quotient $M^0 := \mu^{-1}(0)/G$ is a smooth manifold with an induced symplectic form ω^0 [26, 27]. For simplicity, we will make this assumption throughout our discussion. In general, M^0 is a stratified symplectic space [31], but we will then consider only the smooth part of M^0.

A symplectic manifold (M, ω) is (pre)quantisable if there is a Hermitian line bundle $L \to M$ with a unitary connection ∇ with curvature $\omega/\sqrt{-1}$. We assume that the Hamiltonian G-action on M can be lifted to L preserving the connection ∇. Let Z be the vector field on the principal $U(1)$-bundle P or on $L^\times$ (the total space L with the zero section deleted) that generates the standard $U(1)$-action. Let $\bar{X}$ be the horizontal lift to P or $L^\times$ of a vector field X on M. Then $[\bar{X}, \bar{Y}] =$

$\overline{[X,Y]} + \omega(X,Y)Z$ for all vector fields X, Y on M. For $a \in \mathfrak{g}$, let $\hat{X}_a$ be the vector field of the lifted action on L. Then $\hat{X}_a = \bar{X}_a + \mu_a Z$ on P or on $L^\times$ [23]. The restriction of L to $\mu^{-1}(0)$ descends to a Hermitian line bundle $L^0 \to M^0$ with an induced connection ∇^0 with curvature $\omega^0/\sqrt{-1}$, i.e., L^0 is a prequantum line bundle over (M^0, ω^0) [11].

2.2. A finite-dimensional model

Now suppose the symplectic form ω on M is exact, i.e., $\omega = d\theta$, where θ is a 1-form on M. In this case, we can take $L = M \times \mathbb{C}$ with a (unitary) trivialization under which the gauge potential of the connection ∇ is $\theta/\sqrt{-1}$. Suppose the lifted G-action on L is $g \cdot (x,z) = (gx, \gamma(g,x)z)$, where $g \in G$, $x \in M$, $z \in \mathbb{C}$. Then $\gamma\colon G \times M \to U(1)$ satisfies the cocycle condition

$$\gamma(gh,x) = \gamma(h,x)\gamma(g,hx) \tag{1}$$

for all $g, h \in G$, $x \in M$. Writing $\gamma_g = \gamma(g,\cdot)\colon M \to U(1)$ for each $g \in G$ as a $U(1)$-valued function on M, the condition (1) is equivalent to $\gamma_{gh} = \gamma_h\, h^*\gamma_g$. Since ∇ is G-invariant, $g \in G$ pulls back the gauge potential to a gauge transformation (determined by γ_g) of itself, i.e.,

$$g^*\theta - \theta = -\sqrt{-1}\gamma_g^{-1} d\gamma_g. \tag{2}$$

We study the function $\gamma_a := -\sqrt{-1}\frac{d}{dt}\big|_{t=0}\gamma(e^{ta},\cdot)$ on M, where $a \in \mathfrak{g}$. It is clearly linear in $a \in \mathfrak{g}$. The vector field of the lifted action on $P = M \times U(1)$ or on $L^\times = M \times \mathbb{C}^\times$ is $\hat{X}_a = X_a + \gamma_a Z$. We have the identity

$$X_a\gamma_b - X_b\gamma_a = -\gamma_{[a,b]} \tag{3}$$

for all $a, b \in \mathfrak{g}$. This follows either from differentiating (1) with respect to g, h at $e \in G$ or from $[\hat{X}_a, \hat{X}_b] = -\hat{X}_{[a,b]}$. On the other hand, the horizontal lift of any vector field X on M is $\bar{X} = X + \iota_X\theta\, Z$, and since $\hat{X}_a = \bar{X}_a + \mu_a Z$, we get, for any $a \in \mathfrak{g}$,

$$\gamma_a = \mu_a + \iota_{X_a}\theta. \tag{4}$$

In fact, this identity almost follows from (1) and (2): by differentiating (2) with respect to g, we get $\gamma_a = \mu_a + \iota_{X_a}\theta + \kappa_a$, where κ_a is a constant which is also linear in $a \in \mathfrak{g}$ (hence $\kappa \in \mathfrak{g}^*$). Using (3), we can prove that κ is zero on $[\mathfrak{g},\mathfrak{g}]$. So $\kappa = 0$ if $\mathfrak{g}$ is semi-simple; in general, κ can be made zero by shifting either the G-action or the moment map.

We show that $\gamma\colon G \times M \to U(1)$ and $\theta \in \Omega^1(M)$ satisfying (1) and (2) (as well as $\kappa = 0$) are sufficient to determine the prequantum line bundle L^0 over M^0 and its connection ∇^0. First, the total space $L^0 = \mu^{-1}(0) \times_\gamma G$ is the quotient of $\mu^{-1}(0) \times \mathbb{C}$ by G. So a section ψ of L^0 can be identified with a complex-valued function $\tilde{\psi}$ on $\mu^{-1}(0)$ satisfying $\tilde{\psi}(gx) = \gamma(g,x)\tilde{\psi}(x)$ for all $g \in G$ and $x \in \mu^{-1}(0)$, or $g^*\tilde{\psi} = \gamma_g\tilde{\psi}$. Second, the connection ∇^0 on L^0 is given by $\widetilde{\nabla^0\psi} = d\tilde{\psi} - \sqrt{-1}\theta\tilde{\psi}$. We can check that it is covariant under G, i.e., $g^*\widetilde{\nabla^0\psi} = \gamma_g\widetilde{\nabla^0\psi}$, and horizontal, i.e., $\iota_{X_a}\widetilde{\nabla^0\psi} = 0$; the latter follows from (4) and from $\mu_a = 0$

on $\mu^{-1}(0)$. Alternatively, consider the connection 1-form $\alpha := u^{-1}du - \sqrt{-1}\theta$ on $P = M \times U(1)$ or on $L^\times = M \times \mathbb{C}^\times$ satisfying $\iota_Z \alpha = \sqrt{-1}$. The same identity (4) shows that α on $\mu^{-1}(0) \times U(1)$ or on $\mu^{-1}(0) \times \mathbb{C}^\times$ is basic with respect to the G-action and thus descends to a connection 1-form α^0 on $P^0 = \mu^{-1}(0) \times_\gamma U(1)$ or on $(L^0)^\times = \mu^{-1}(0) \times_\gamma \mathbb{C}^\times$.

If we take a different (unitary) trivialization of L related to the previous one by $\beta\colon M \to U(1)$, then we obtain a cocycle γ' cohomologous to γ, i.e.,

$$\gamma'(g,x) = \beta(gx)\gamma(g,x)\beta(x)^{-1}, \tag{5}$$

for all $g \in G$, $x \in M$, or equivalently, $\gamma'_g = g^*\beta\, \gamma_g\, \beta^{-1}$. Also, $\omega = d\theta'$, where

$$\theta' := \theta - \sqrt{-1}\beta^{-1}d\beta. \tag{6}$$

Then θ' and $\gamma'_a := -\sqrt{-1}\frac{d}{dt}\big|_{t=0}\gamma'(e^{ta},\cdot)$ (for $a \in \mathfrak{g}$) satisfy the analogs of (2), (4). Conversely, given γ', θ' that are related to γ, θ by (5), (6), they define a prequantum line bundle $(L^0)'$ over M^0 that is isomorphic to L^0. In fact there is a bundle isomorphism $\varrho_\beta : L^0 \to (L^0)'$, $[(x,z)] \mapsto [(x,\beta(x)z)]$, where $x \in \mu^{-1}(0)$, $z \in \mathbb{C}$. We show that it respects the connections on L^0 and $(L^0)'$. If $\psi \in \varGamma(L^0)$ and $\psi' := \varrho_\beta \circ \psi \in \varGamma((L^0)')$, they define functions $\tilde\psi$, $\tilde\psi'$ on $\mu^{-1}(0)$ that are related by $\tilde\psi' = \beta\tilde\psi$. It is easy to check that $\widetilde{(\nabla^0)'\psi'} = d\tilde\psi' - \sqrt{-1}\theta'\tilde\psi' = \beta\, \widetilde{\nabla^0\psi}$. Alternatively, the new connection 1-form $\alpha' = u^{-1}du - \sqrt{-1}\theta'$ on $\mu^{-1}(0) \times U(1)$ is related to α by a gauge transformation, $\alpha' = \alpha - \sqrt{-1}\beta^{-1}d\beta$, and it descends to a connection 1-form $(\alpha^0)'$ on $(P^0)' := \mu^{-1}(0) \times_{\gamma'} U(1)$ that is related to α^0 by $\varrho_\beta^*(\alpha^0)' = \alpha^0$.

On the other hand, if $\theta = \theta' + \lambda$ in $\omega = d\theta$, and λ is a G-invariant 1-form on M, then a cocycle γ satisfying $g^*\theta' - \theta' = -\sqrt{-1}\gamma_g^{-1}d\gamma_g$ also satisfies (2). Let $\omega' = d\theta'$, which is not necessarily symplectic on M. Suppose $\mu'\colon M \to \mathfrak{g}^*$ (or $\mu'_a := \langle \mu', a\rangle$, $a \in \mathfrak{g}$) satisfies $\iota_{X_a}\omega' = d\mu'_a$, $\omega'(X_a, X_b) = -X_a\mu'_b = X_b\mu'_a = \mu'_{[a,b]}$ and $\gamma_a = \mu'_a + \iota_{X_a}\theta'$ for all $a, b \in \mathfrak{g}$. Then $\mu\colon M \to \mathfrak{g}^*$ given by $\mu_a := \mu'_a - \iota_{X_a}\lambda$ satisfies $\iota_{X_a}\omega = d\mu_a$ and

$$\{\mu_a, \mu_b\} = -X_a(\mu'_b - \iota_{X_b}\lambda) = \mu'_{[a,b]} + \iota_{[X_a,X_b]}\lambda = \mu_{[a,b]}.$$

Therefore μ is the moment map of the Hamiltonian G-action on (M,ω). Since μ_a also satisfies (4), we obtain a prequantum line bundle L^0 on $M^0 = \mu^{-1}(0)/G$ (with curvature $\omega^0/\sqrt{-1}$). In particular, if $\theta' = 0$ and $\theta = \lambda$ itself is G-invariant, then we can choose $\gamma = 1$ and μ given by $\mu_a = -\iota_{X_a}\lambda$ ($a \in \mathfrak{g}$). In this case, the bundle $L^0 = M^0 \times \mathbb{C}$ is topologically trivial and λ on $\mu^{-1}(0)$ descends to a 1-form λ^0 on M^0 such that the covariant derivative of L^0 is $\nabla^0 = d - \sqrt{-1}\lambda^0$.

2.3. The Ramadas–Singer–Weitsman construction

We specialize to the infinite-dimensional setting of [30]. Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$ and let C be a closed orientable surface of genus $g(C) \geq 2$. Given a principal G-bundle P over C, let $\mathcal{A}(P)$ be set of connections on P and let $\mathcal{G}(P)$ be the group of gauge transformations. When G is a matrix group, the gauge transformation of $A \in \mathcal{A}(P)$ by $g \in \mathcal{G}(P)$ is

$g \cdot A = gAg^{-1} - dg\, g^{-1}$. The Lie algebra of $\mathcal{G}(P)$ is $\mathrm{Lie}(\mathcal{G}(P)) \cong \Omega^0(C, \mathrm{ad}P)$. Its dual has an identification $\mathrm{Lie}(\mathcal{G}(P))^* \cong \Omega^2(C, \mathrm{ad}P)$ by the pairing

$$\langle F, \epsilon \rangle = -\frac{1}{2\pi} \int_C \mathrm{tr}(\epsilon F)$$

between $\epsilon \in \Omega^0(C, \mathrm{ad}P)$ and $F \in \Omega^2(C, \mathrm{ad}P)$. Here tr is the standard trace on matrices when $G = SU(n)$; in general, $-\,\mathrm{tr}$ denotes the inner product on $\mathfrak{g}$ such that the long roots of each simple factor of G are of length $\sqrt{2}$.

Note that $\mathcal{A}(P)$ is an affine space and the tangent space at any $A \in \mathcal{A}(P)$ is $T_A\mathcal{A}(P) \cong \Omega^1(C, \mathrm{ad}P)$. There is a symplectic form $\omega^\sharp$ on $\mathcal{A}(P)$ given by

$$\omega^\sharp(\alpha, \beta) = -\frac{1}{2\pi} \int_C \mathrm{tr}(\alpha \wedge \beta), \tag{7}$$

where $\alpha, \beta \in \Omega^1(C, \mathrm{ad}P)$. With respect to $\omega^\sharp$, the action of $\mathcal{G}(P)$ is Hamiltonian and the moment map is $\mu\colon \mathcal{A}(P) \to \mathrm{Lie}(\mathcal{G}(P))^*$, $A \mapsto F_A$ (the curvature of A) [2]. The symplectic quotient $\mathcal{M}(P) := \mu^{-1}(0)/\mathcal{G}(P)$ is the moduli space of flat connections up to gauge equivalence. In fact, we will consider the subset of irreducible flat connections on P in a smooth part of $\mathcal{M}(P)$, which we denote by the same notation. It has a natural symplectic form ω and is of dimension $2(g(C) - 1)\dim G$ [2].

To construct the prequantum line bundle over $\mathcal{M}(P)$, we recall the Chern–Simons functional

$$\mathrm{CS}(A) = \frac{1}{4\pi} \int_B \mathrm{tr}\, A \wedge \left(dA + \frac{1}{3}[A, A] \right)$$

of a connection A on a 3-manifold B. Now choose a 3-manifold B whose boundary is C. Then the bundle P over C and all $g \in \mathcal{G}(P)$, $A \in \mathcal{A}(P)$ extend to B [30] (which we denote by the same notations). We define

$$\gamma(g, A) = \exp\left[\sqrt{-1}(\mathrm{CS}(g \cdot A) - \mathrm{CS}(A)) \right]. \tag{8}$$

It can be shown that $\gamma(g, A)$ does not depend on the choice of B and the extensions of g and A, and satisfies the cocycle condition in (1) [30]. Curiously, the exponent in (8) is equal to the Chern–Simons functional on a closed 3-manifold. In fact, let $\hat{B} = B \cup_C (-B)$ be the double of B by gluing B and $-B$ (B with the opposite orientation) along C, $\hat{P}_g$ be the gluing of the same bundle P over B and $-B$ along C by the gauge transformation g. Then the connections $g \cdot A$ on B and A on $-B$ defines a connection $\hat{A}_g$ on $\hat{P}_g$ over the closed 3-manifold $\hat{B}$ such that $\mathrm{CS}(g \cdot A) - \mathrm{CS}(A) = \mathrm{CS}(\hat{A}_g)$.

We will explore the fact that the symplectic form (7) is exact. Explicitly, we have $\omega^\sharp = d\theta$, where θ is a 1-form on $\mathcal{A}(P)$ defined by[1]

$$\theta_A(\alpha) = -\frac{1}{4\pi} \int_C \mathrm{tr}(A \wedge \alpha), \tag{9}$$

[1]Strictly speaking, A on the right-hand side should be $A - A_0 \in \Omega^1(C, \mathrm{ad}P)$, where A_0 is a fixed reference connection on P. On the oriented double cover of a non-orientable surface (cf. §4), A_0 should be invariant under the involution.

for all $A \in \mathcal{A}(P)$, $\alpha \in \Omega^1(C, \mathrm{ad}P)$. On the other hand, the differential of γ at $(g, A) \in \mathcal{G}(P) \times \mathcal{A}(P)$ is given by [30]

$$-\sqrt{-1}\gamma^{-1} d\gamma_{(g,A)}(\epsilon, \alpha) = \frac{1}{4\pi} \int_C \mathrm{tr}\left(dg\, g^{-1} \wedge g \cdot \alpha - 2\epsilon F_{g\cdot A} - d_{g\cdot A}\, \epsilon \wedge g \cdot A\right),$$

where $\epsilon \in \Omega^0(C, \mathrm{ad}P)$, $\alpha \in \Omega^1(C, \mathrm{ad}P)$. Here $dg\, g^{-1}$ is the (pull-back of) the (right invariant) Maurer–Cartan form on G and $g \cdot \alpha = \mathrm{Ad}_g \alpha$ (which is $g\alpha g^{-1}$ when G is a matrix group). This implies that γ and θ satisfy (2), (4). Therefore following the construction in §2.2, we obtain a Hermitian line bundle $\mathcal{L} \to \mathcal{M}(P)$ with a unitary connection whose curvature is $\omega/\sqrt{-1}$. It is also the determinant line bundle of a family of $\bar{\partial}$ operators [30].

If we use, instead of γ, the cocycle

$$\gamma_k(g, A) := \gamma(g, A)^k = \exp\left[\sqrt{-1}k(\mathrm{CS}(g\cdot A) - \mathrm{CS}(A))\right],$$

where $k \in \mathbb{Z}$, then we get the line bundle $\mathcal{L}^{\otimes k}$ over $\mathcal{M}(P)$ whose curvature is $k\omega/\sqrt{-1}$. The bundle is positive if and only if $k > 0$.

We can combine the moduli spaces $\mathcal{M}(P)$ with various topological types of P which are labeled by $w_1(P) \in H^2(G, \pi_1(G)) \cong \pi_1(G)$. The resulting (usually disconnected) space $\mathcal{M}(C, G)$ can be identified with the representation variety $\mathrm{Hom}(\pi_1(C), G)/G$. The symplectic form and prequantum line bundle on $\mathcal{M}(C, G)$ are denoted by the same notations, ω and $\mathcal{L}$, respectively.

3. Line bundles and branes on Hitchin's moduli space

3.1. Hitchin's moduli space as a hyper-Kähler quotient

As above, C is a closed orientable surface of genus $g(C) \geq 2$, G is a compact semisimple Lie group, and P is a principal G-bundle over C. The group $\mathcal{G}(P)$ of gauge transformations acts on the space $\mathcal{A}(P)$ of connections as before and on $\Omega^1(C, \mathrm{ad}P)$ by $g \colon \Phi \mapsto g\Phi = \mathrm{Ad}_g \Phi$. We consider the space $\mathcal{A}(P) \times \Omega^1(C, \mathrm{ad}P)$ and we denote a tangent vector at (A, Φ) by (α, ξ), where $\alpha, \xi \in \Omega^1(C, \mathrm{ad}P)$. Choosing a complex structure on C (which defines the Hodge star $*$ on the 1-forms on C), the space $\mathcal{A}(P) \times \Omega^1(C, \mathrm{ad}P)$ is hyper-Kähler [15], and the three complex structures $I^\sharp$, $J^\sharp$, $K^\sharp$ are given by (in the convention of [21])

$$I^\sharp(\alpha, \xi) = (*\alpha, -*\xi), \quad J^\sharp(\alpha, \xi) = (\xi, -\alpha), \quad K^\sharp(\alpha, \xi) = (*\xi, *\alpha).$$

The metric and the three corresponding symplectic forms are, respectively,

$$\begin{aligned} \mathsf{g}^\sharp\big((\alpha, \xi), (\beta, \eta)\big) &= -\frac{1}{2\pi} \int_C \mathrm{tr}(\alpha \wedge *\beta + \xi \wedge *\eta), \\ \omega_I^\sharp\big((\alpha, \xi), (\beta, \eta)\big) &= -\frac{1}{2\pi} \int_C \mathrm{tr}(\alpha \wedge \beta - \xi \wedge \eta), \end{aligned}$$

$$\omega_J^\sharp\big((\alpha,\xi),(\beta,\eta)\big) = -\frac{1}{2\pi}\int_C \mathrm{tr}(-\alpha\wedge *\eta + \xi\wedge *\beta),$$
$$\omega_K^\sharp\big((\alpha,\xi),(\beta,\eta)\big) = -\frac{1}{2\pi}\int_C \mathrm{tr}(\alpha\wedge \eta + \xi\wedge \beta),$$

where $(\alpha,\xi),(\beta,\eta)\in T_{(A,\Phi)}(\mathcal{A}(P)\times\Omega^1(C,\mathrm{ad}P))\cong\Omega^1(C,\mathrm{ad}P)^{\oplus 2}$. Moreover,

$$\omega_I^\sharp = \omega_I'^\sharp + d\lambda_I^\sharp, \quad \omega_J^\sharp = d\lambda_J^\sharp, \quad \omega_K^\sharp = d\lambda_K^\sharp,$$

where $\omega_I'^\sharp = d\theta_I'$, with (see for example [21])

$$\theta_I'\big((\alpha,\xi)\big) = -\frac{1}{4\pi}\int_C \mathrm{tr}(A\wedge\alpha), \qquad \lambda_I^\sharp\big((\alpha,\xi)\big) = \frac{1}{4\pi}\int_C \mathrm{tr}(\Phi\wedge\xi),$$
$$\lambda_J^\sharp\big((\alpha,\xi)\big) = -\frac{1}{2\pi}\int_C \mathrm{tr}(\Phi\wedge *\alpha), \quad \lambda_K^\sharp\big((\alpha,\xi)\big) = -\frac{1}{2\pi}\int_C \mathrm{tr}(\Phi\wedge\alpha).$$

The action of $\mathcal{G}(P)$ on $\mathcal{A}(P)\times\Omega^1(C,\mathrm{ad}P)$ is Hamiltonian with respect to all three symplectic forms. For $\epsilon\in\mathrm{Lie}(\mathcal{G}(P))\cong\Omega^0(C,\mathrm{ad}P)$, the vector field $\mathcal{V}_\epsilon$ at (A,Φ) is $(-d_A\epsilon,[\epsilon,\Phi])$. The three moment maps are [15]

$$\mu_I(A,\Phi) = F_A - \tfrac{1}{2}[\Phi,\Phi], \quad \mu_J(A,\Phi) = -d_A*\Phi, \quad \mu_K(A,\Phi) = d_A\Phi, \qquad (10)$$

all valued in $\mathrm{Lie}(\mathcal{G}(P))^*\cong\Omega^2(C,\mathrm{ad}P)$. The 1-forms $\lambda_I^\sharp,\lambda_J^\sharp,\lambda_K^\sharp$ are invariant under $\mathcal{G}(P)$ and are related to the moment maps by (cf. §2.2)

$$\langle\mu_I,\epsilon\rangle = \langle\mu_I',\epsilon\rangle - \iota_{\mathcal{V}_\epsilon}\lambda_I^\sharp, \quad \langle\mu_J,\epsilon\rangle = -\iota_{\mathcal{V}_\epsilon}\lambda_J^\sharp, \quad \langle\mu_K,\epsilon\rangle = -\iota_{\mathcal{V}_\epsilon}\lambda_K^\sharp,$$

where $\mu_I' = F_A$ satisfies $\iota_{\mathcal{V}_\epsilon}\omega_I'^\sharp = d\langle\mu_I',\epsilon\rangle$ as in §2.3. Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(P) := \mu_I^{-1}(0)\cap\mu_J^{-1}(0)\cap\mu_K^{-1}(0)/\mathcal{G}(P)$ is the space of gauge equivalence classes of solutions to Hitchin's equations

$$\mu_I(A,\Phi) = \mu_J(A,\Phi) = \mu_K(A,\Phi) = 0. \qquad (11)$$

It is the hyper-Kähler quotient [16] of $\mathcal{A}(P)\times\Omega^1(C,\mathrm{ad}P)$ by $\mathcal{G}(P)$ at level 0, and is of real dimension $4(g(C)-1)\dim G$. We will only consider the smooth part of the moduli space, which is also denoted by $\mathcal{M}_{\mathrm{H}}(P)$. We denote by I, J, K the three complex structures and by ω_I, ω_J, ω_K the three symplectic forms on $\mathcal{M}_{\mathrm{H}}(P)$. The forms $\omega_I'^\sharp$, $\lambda_I^\sharp$, $\lambda_J^\sharp$, $\lambda_K^\sharp$ descend, respectively, to ω_I', λ_I, λ_J, λ_K on $\mathcal{M}_{\mathrm{H}}(P)$, where we have the equations

$$\omega_I = \omega_I' + d\lambda_I, \quad \omega_J = d\lambda_J, \quad \omega_K = d\lambda_K.$$

There is an involution $\sigma^\sharp\colon (A,\Phi)\mapsto(A,-\Phi)$ on $\mathcal{A}(P)\times\Omega^1(C,\mathrm{ad}P)$, which is holomorphic with respect to $I^\sharp$ but anti-holomorphic with respect to $J^\sharp$ and $K^\sharp$. The action of $\sigma^\sharp$ on the tangent spaces is $\sigma_*^\sharp\colon(\alpha,\xi)\to(\alpha,-\xi)$. Thus we obtain $(\sigma^\sharp)^*\lambda_I^\sharp = \lambda_I^\sharp$, $(\sigma^\sharp)^*\lambda_J^\sharp = -\lambda_J^\sharp$, $(\sigma^\sharp)^*\lambda_K^\sharp = -\lambda_K^\sharp$, and hence $(\sigma^\sharp)^*\omega_I^\sharp = \omega_I^\sharp$, $(\sigma^\sharp)^*\omega_J^\sharp = -\omega_J^\sharp$, $(\sigma^\sharp)^*\omega_K^\sharp = -\omega_K^\sharp$. Since $\sigma^\sharp$ does not act on $\mathcal{G}(P)$ and commutes with the action of $\mathcal{G}(P)$ on $\mathcal{A}(P)\times\Omega^1(C,\mathrm{ad}P)$, it descends to an involution σ on the hyper-Kähler quotient $\mathcal{M}_{\mathrm{H}}(P)$, which is holomorphic in I but anti-holomorphic in J, K, and which satisfies $\sigma^*\lambda_I = \lambda_I$, $\sigma^*\lambda_J = -\lambda_J$, $\sigma^*\lambda_K = -\lambda_K$ and $\sigma^*\omega_I = \omega_I$, $\sigma^*\omega_J = -\omega_J$, $\sigma^*\omega_K = -\omega_K$. Therefore the fixed-point set $\mathcal{M}_{\mathrm{H}}(P)^\sigma$ is a complex submanifold of $\mathcal{M}_{\mathrm{H}}(P)$ with respect to I but a Lagrangian submanifold with respect to ω_J and ω_K. It contains $\mathcal{M}(P)$ as a connected component [15].

Given a pair $(A,\Phi) \in \mathcal{A}(P) \times \Omega^1(C,\mathrm{ad}P)$, the combination $A - \sqrt{-1}\Phi$ is a connection on the $G^{\mathbb{C}}$-bundle $P^{\mathbb{C}} := P \times_G G^{\mathbb{C}}$. Two of Hitchin's equations (11), $\mu_I = \mu_K = 0$, are equivalent to the flatness of $A - \sqrt{-1}\Phi$, and moreover, Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(P)$ can be identified with the moduli space $\mathcal{M}(P^{\mathbb{C}})$ of (reductive) flat $G^{\mathbb{C}}$-connections on $P^{\mathbb{C}}$ up to complex gauge transformations [6, 9]. In this way, $\mathcal{M}_{\mathrm{H}}(P)$ naturally inherits the complex structure J, under which σ is anti-holomorphic.

Finally, Hitchin's moduli spaces $\mathcal{M}_{\mathrm{H}}(P)$ with various topological types of P also combine to a (usually disconnected) moduli space $\mathcal{M}_{\mathrm{H}}(C,G)$, and the complex structures, differential forms, involution on each $\mathcal{M}_{\mathrm{H}}(P)$ define those on $\mathcal{M}_{\mathrm{H}}(C,G)$, denoted by the same notations. Also, $\mathcal{M}_{\mathrm{H}}(C,G)$ can be identified with the representation variety $\mathrm{Hom}(\pi_1(C),G^{\mathbb{C}})/\!\!/G^{\mathbb{C}}$.

3.2. Line bundles and complex Chern–Simons gauge theory

We begin with a simple observation in the finite-dimensional setting. Let M be a (finite-dimensional) hyper-Kähler manifold with an action of a compact Lie group G that is Hamiltonian with respect to all three symplectic structures ω_I, ω_J, ω_K, and let $\mu_I,\mu_J,\mu_K\colon M \to \mathfrak{g}^*$ be the respective moment maps. Let ω_I^0, ω_J^0, ω_K^0 be the induced symplectic forms on the hyper-Kähler quotient $M^0 := \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)/G$. Assuming all spaces concerned are smooth, we can regard M^0 as either the symplectic quotient of the symplectic manifold $(\mu_J^{-1}(0)\cap\mu_K^{-1}(0),\omega_I)$ by G or a symplectic submanifold (with respect to ω_I^0) of the symplectic quotient $\mu_I^{-1}(0)/G$. Therefore the method in §2.2 applies to hyper-Kähler quotients. Suppose L_I is a prequantum line bundle of (M,ω_I) and the G-action on M lifts to L_I preserving the connection. As in §2.2, suppose $\omega_I = d\theta_I$ and let $\gamma_I\colon G \times M \to U(1)$ be the cocycle defined by the G-action on $L_I = M \times \mathbb{C}$. Then θ_I and γ_I satisfying (1), (2) and (4) define a prequantum line bundle L_I^0 of (M^0,ω_I^0). Similarly, we can construct prequantum line bundles L_J^0 of (M^0,ω_J^0) and L_K^0 of (M^0,ω_K^0).

Returning to Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(P)$, where P is a principal G-bundle over C, we consider the 1-forms $\theta_I = \theta_I' + \lambda_I^\sharp$, $\theta_J = \lambda_J^\sharp$, $\theta_K = \lambda_K^\sharp$ on $\mathcal{A}(P) \times \Omega^1(C,\mathrm{ad}P)$ given in §3.1. Since $\lambda_I^\sharp$, $\lambda_J^\sharp$, $\lambda_K^\sharp$ are invariant under $\mathcal{G}(P)$, the cocycles are $\gamma_I(g,A,\Phi) = \gamma(g,A)$ as in (8) and $\gamma_J(g,A,\Phi) = \gamma_K(g,A,\Phi) = 1$, where $g \in \mathcal{G}(P)$ and $(A,\Phi) \in \mathcal{A}(P) \times \Omega^1(C,\mathrm{ad}P)$. We then obtain prequantum line bundles $\mathcal{L}_I$, $\mathcal{L}_J$, $\mathcal{L}_K$ over $\mathcal{M}_{\mathrm{H}}(P)$ with respect to ω_I, ω_J, ω_K. (See [7] for an approach using determinant line bundles.) To relate $\mathcal{L}_I$ to the bundle $\mathcal{L} \to \mathcal{M}(P)$ in §2.3, we recall that (the total space of) the cotangent bundle $T^*\mathcal{M}(P)$ is contained in $\mathcal{M}_{\mathrm{H}}(P)$ as an open dense subset [15]. Topologically, the restriction of $\mathcal{L}_I$ to $T^*\mathcal{M}(P)$ is the pull-back of the bundle $\mathcal{L} \to \mathcal{M}(P)$, but the connection is modified by $\lambda_I/\sqrt{-1}$, which has a non-zero contribution to the curvature along the fibres of $T^*\mathcal{M}(P)$, so that the total curvature of $\mathcal{L}_I$ is $\omega_I/\sqrt{-1}$. The line bundles $\mathcal{L}_J$ and

$\mathcal{L}_K$ are topologically trivial, but their connections $d-\sqrt{-1}\lambda_J$ and $d-\sqrt{-1}\lambda_K$ are non-trivial and their curvatures are $\omega_J/\sqrt{-1}$ and $\omega_K/\sqrt{-1}$, respectively.

Under the involution $\sigma^\sharp$ on $\mathcal{A}(P)\times\Omega^0(C,\mathrm{ad}P)$, the cocycle γ_I is invariant, hence the induced involution σ on $\mathcal{M}_{\mathrm{H}}(P)$ lifts to $\mathcal{L}_I$. Moreover, since $(\sigma^\sharp)^*\theta_I=\theta_I$, the connection on $\mathcal{L}_I$ is preserved by σ and the restriction of $\mathcal{L}_I$ to $\mathcal{M}(P)\subset\mathcal{M}_{\mathrm{H}}(P)^\sigma$ is the line bundle $\mathcal{L}\to\mathcal{M}(C,G)$ constructed in §2.3. On the other hand, since $(\sigma^\sharp)^*\lambda_J^\sharp=-\lambda_J^\sharp$ and $(\sigma^\sharp)^*\lambda_K^\sharp=-\lambda_K^\sharp$, there are bundle isomorphisms $\sigma^*\mathcal{L}_J\cong\mathcal{L}_J^{-1}$, $\sigma^*\mathcal{L}_K\cong\mathcal{L}_K^{-1}$ respecting the connections. In particular, the restrictions of $\mathcal{L}_J$, $\mathcal{L}_K$ to $\mathcal{M}(P)^\sigma$, or to $\mathcal{M}(P)$, together with their connections, are trivial. These are the refinements of the fact that $\mathcal{M}(P)$ or $\mathcal{M}_{\mathrm{H}}(P)^\sigma$ is symplectic in ω_I but Lagrangian in ω_J, ω_K.

The line bundles constructed above are related to the Chern–Simons gauge theory with a complex gauge group. For an oriented 3-manifold B and a principal G-bundle P over B, if A is a connection on P and $\Phi\in\Omega^1(B,\mathrm{ad}P)$, then $A-\sqrt{-1}\Phi$ is a connection of the $G^{\mathbb{C}}$-bundle $P^{\mathbb{C}}:=P\times_G G^{\mathbb{C}}$. The action

$$\begin{aligned}S_t(A,\Phi)=&\frac{t}{2}\,\mathrm{CS}(A-\sqrt{-1}\Phi)+\frac{\bar{t}}{2}\,\mathrm{CS}(A+\sqrt{-1}\Phi)\\=&\frac{k}{4\pi}\int_B\mathrm{tr}\left(A\wedge\left(dA+\tfrac{1}{3}[A,A]\right)-\Phi\wedge d_A\Phi\right)\\&-\frac{s}{4\pi}\Big(\int_B\mathrm{tr}\left(2\Phi\wedge F_A+\tfrac{1}{3}\Phi\wedge[\Phi,\Phi]\right)+\int_{\partial B}\mathrm{tr}(A\wedge\Phi)\Big),\end{aligned}$$

where $t=k+\sqrt{-1}s\in\mathbb{C}$ (with $k\in\mathbb{Z}$ and $s\in\mathbb{R}$) is a complex parameter, is invariant under the group $\mathcal{G}(P^{\mathbb{C}})$ of complex gauge transformations [35].

In fact, Hitchin's moduli space, equipped with the symplectic form $\omega_t:=k\omega_I+s\omega_K$ ($t=k+\sqrt{-1}s$), is the phase space of the above three-dimensional complex Chern–Simons gauge theory. On the other hand, the symplectic form ω_J appears when we regard Hitchin's moduli space as the total space of a cotangent bundle (up to a compliment of high codimensions).

Returning to the G-bundle P over a closed orientable surface C, given $A\in\mathcal{A}(P)$, $\Phi\in\Omega^1(C,\mathrm{ad}P)$ and $g\in\mathcal{G}(P)$, we choose a 3-manifold B with boundary $\partial B=C$ and we extend P, A, Φ, g to B. Let

$$\gamma_t(g,A,\Phi):=\exp[\sqrt{-1}(S_t(g\cdot A,g\cdot\Phi)-S_t(A,\Phi))].\tag{12}$$

Since $k\in\mathbb{Z}$, $\gamma_t(g,A,\Phi)$ is independent of the choice of B and the extensions.

A simple calculation shows that

$$\gamma_t(g,A,\Phi)=\beta_s(g\cdot A,g\cdot\Phi)\gamma_k(g,A)\beta_s(A,\Phi)^{-1},$$

where $\gamma_k(g,A)=\gamma(g,A)^k$ as in §2.3 and β_s is a $U(1)$-valued function on $\mathcal{A}(P)\times\Omega^1(C,\mathrm{ad}P)$ defined by

$$\beta_s(A,\Phi)=\exp\Big[-\frac{\sqrt{-1}s}{4\pi}\int_C\mathrm{tr}(A\wedge\Phi)\Big].$$

By differentiating (12), we verify that the cocycle γ_t and the 1-form

$$\theta_t := k(\theta'_I + \lambda_I^\sharp) + s\lambda_K^\sharp - \sqrt{-1}\beta_s^{-1}d\beta_s$$

satisfy the analogs of (2), (4).

As explained in §2.2, we can replace γ_t and θ_t by $\gamma'_t(g, A, \Phi) := \gamma_k(g, A)$ and $\theta'_t := k(\theta'_I + \lambda_I^\sharp) + s\lambda_K^\sharp$, respectively, as the difference in the cocycles is a coboundary. The resulting Hermitian line bundle (with a unitary connection) on $\mathcal{M}_{\mathrm{H}}(P)$ is thus $\mathcal{L}_t = \mathcal{L}_I^{\otimes k} \otimes \mathcal{L}_K^{\otimes s}$, where $\mathcal{L}_K^{\otimes s}$ means the topologically trivial line bundle with connection $d - \sqrt{-1}s\lambda_K$ and curvature $s\omega_K/\sqrt{-1}$. (Compare [7], where a complex-valued Chern–Simons term $\mathrm{CS}(A - \sqrt{-1}\Phi)$ was used to produce a non-Hermitian line bundle over $\mathcal{M}_{\mathrm{H}}(P)$.)

Finally, the line bundles $\mathcal{L}_I$, $\mathcal{L}_J$, $\mathcal{L}_K$, $\mathcal{L}_t$ on $\mathcal{M}_{\mathrm{H}}(P)$ with various topological types of P form line bundles over $\mathcal{M}_{\mathrm{H}}(C, G)$ denoted by the same notations and with the same properties.

3.3. Branes on Hitchin's moduli space

In a two-dimensional sigma-model whose target space is Y, a brane $\mathcal{B}$ is a pair (Z, E), where Z is a submanifold of Y and $E \to Z$ is a complex vector bundle with a unitary connection. If the worldsheet Σ has a boundary $\partial\Sigma$, then Z is where $\partial\Sigma$ maps to while E gives extra degrees of freedom on the boundary. (See [1] for a comprehensive survey of branes in mathematics and physics.) A brane $\mathcal{B} = (Z, E)$ is called space-filling if $Z = Y$.

A-model and B-model are topological sigma-models depending on, respectively, a symplectic structure and a complex structure on the target Y [36]. If (Y, ω) is symplectic, an A-brane requires a boundary condition that is compatible with the supersymmetry of the A-model. Typically, an A-brane consists of a Lagrangian submanifold Z with a flat bundle E. There are exceptional A-branes supported on coisotropic submanifolds [20]. In particular, let ℓ be a Hermitian line bundle over Y with a unitary connection whose curvature is F. Then the condition for the space-filling brane $\mathcal{B}_{\mathrm{cc}} = (Y, \ell)$ to be an A-brane is that $\omega^{-1}(\sqrt{-1}F)$ is an integrable complex structure on Y. More generally, an A-brane can be an object in the (extended) Fukaya category of (Y, ω). On the other hand, when Y is a complex manifold, a B-brane, consistent with the supersymmetry of the B-model, is of the form $\mathcal{B} = (Z, E)$, where Z is a holomorphic submanifold of Y and E is a holomorphic vector bundle over Z. More generally, a B-brane can be an object in the (derived) category of coherent sheaves on Y.

Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(C, G)$ is the target space of a sigma-model on an (orientable) two-dimensional worldsheet Σ, as the low energy theory of the $N = 4$ gauge theory on a four-dimensional manifold $\Sigma \times C$, compactified along C [21]. Here C remains a closed orientable surface of genus $g(C) > 2$. Recall that (the smooth part of) $\mathcal{M}_{\mathrm{H}}(C, G)$ is a hyper-Kähler manifold of real dimension $4(g(C) - 1)\dim G$, with complex structures I, J, K and symplectic forms $\omega_I, \omega_J, \omega_K$. Consider the Hitchin fibration $h\colon \mathcal{M}_{\mathrm{H}}(C, G) \to \boldsymbol{B}$, where $\boldsymbol{B}$ is a vector space of real dimension $2(g(C) - 1)\dim G$ [15]. For a generic $b \in \boldsymbol{B}$, the fibre $F_b := h^{-1}(b)$ is a union of tori of real dimension $2(g(C)-1)\dim G$. It is Lagrangian in $\omega^\sharp_J$ and $\omega^\sharp_K$ but holomorphic in I. Given a flat line bundle ℓ' on F_b, we have a brane $\mathcal{B}_{F_b}=(F_b,\ell')$ of type (B,A,A) [21]. Another brane $\mathcal{B}'$ of type (B,A,A) is from the most degenerate fibre $h^{-1}(0)=\mathcal{M}(C,G)$ with a flat line bundle over it. $\mathcal{M}(C,G)$ is only part of $\mathcal{M}_{\mathrm{H}}(C,G)^\sigma$; other components also support branes of type (B,A,A).

Being hyper-Kähler, Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(C,G)$ also carries a number of space-filling branes. First, $\mathcal{B}_I:=(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_I)$ is clearly a B-brane with respect to I. Since the curvature of $\mathcal{L}_I$ is $\omega_I/\sqrt{-1}$ and $\omega_J^{-1}\omega_I=-K$, $\omega_K^{-1}\omega_I=J$, they are coisotropic A-branes with respect to the symplectic forms ω_J, ω_K (or any combination $\omega_J\cos\alpha+\omega_K\sin\alpha$). So $\mathcal{B}_I$ is a brane of type (B,A,A). Further, the brane $\mathcal{B}_I^{\otimes k}:=(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_I^{\otimes k})$ $(k\in\mathbb{Z}\backslash\{0\})$ is also of type (B,A,A) in the sense that it is a B-brane with respect to I and an A-brane with respect to $k\omega_J$, $k\omega_K$. Second, $\mathcal{B}_J:=(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_J)$ is likewise a brane of type (A,B,A), and so is $\mathcal{B}_J^{\otimes s}:=(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_J^{\otimes s})$ $(s\in\mathbb{R}\backslash\{0\})$, which is a B-brane with respect to J and an A-brane with respect to $s\omega_I$, $s\omega_K$. When $s>0$ takes a particular value determined by the gauge coupling in four dimensions, $\mathcal{B}_J^{\otimes s}$ is the canonical coisotropic brane $\mathcal{B}_{\mathrm{cc}}$ in [21]. Third, $\mathcal{B}_K:=(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_K)$ and $\mathcal{B}_K^{\otimes s}:=(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_K^{\otimes s})$ $(s\in\mathbb{R}\backslash\{0\})$ are branes of type (A,A,B) for the same reason. Finally, if $t=k+\sqrt{-1}s\neq 0$ $(k\in\mathbb{Z},\ s\in\mathbb{R})$, then $\mathcal{B}_t:=(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_t)$ is a B-brane for the complex structure $(kI+sK)/|t|$ and is an A-brane for the symplectic form $|t|\omega_J$.

4. Line bundles on moduli spaces from a non-orientable surface

4.1. Moduli spaces from a non-orientable surface

Now let C be a closed non-orientable surface, while G remains a compact semisimple Lie group. Following the work [17] on the moduli space $\mathcal{M}(C,G)$ of flat connections on C, Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(C,G)$ was studied in [19] using the orientation double cover $\tilde{C}$ of C. As in [17] for $\mathcal{M}(C,G)$, it is related to part of the space $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ that is invariant under an involution,

Let $\pi\colon\tilde{C}\to C$ be the orientation double cover of C. Here $\tilde{C}$ is a connected closed orientable surface on which the non-trivial deck transformation τ acts as an orientation-reversing involution without fixed points. Given a principal G-bundle $P\to C$, let $\tilde{P}=\pi^*P$ be the pull back. Then τ acts naturally on $\tilde{P}$ as a G-equivariant involution (cf. [18]), and hence also on the set of differential forms $\Omega^\bullet(\tilde{C},\mathrm{ad}\tilde{P})$ on $\tilde{C}$ (valued in $\mathrm{ad}\tilde{P}$), the space $\mathcal{A}(\tilde{P})$ of connections on $\tilde{P}$ and the group $\mathcal{G}(\tilde{P})$ of gauge transformations. The fixed-point sets in these spaces are the corresponding spaces from C, i.e., $\Omega^\bullet(\tilde{C},\mathrm{ad}\tilde{P})^\tau=\Omega^\bullet(C,\mathrm{ad}P)$, $\mathcal{A}(\tilde{P})^\tau=\mathcal{A}(P)$ and $\mathcal{G}(\tilde{P})^\tau=\mathcal{G}(P)$ [17, 18].

The moduli space $\mathcal{M}(\tilde{P})$ of flat connections on $\tilde{P}$ over the orientable surface $\tilde{C}$ has a symplectic form $\tilde{\omega}$ and a complex structure $\tilde{J}$ (if we choose a complex structure on $\tilde{C}$ so that τ is anti-holomorphic). On $\mathcal{M}(\tilde{P})$, the action of τ is anti-symplectic, i.e., $\tau^*\tilde{\omega} = -\tilde{\omega}$, and anti-holomorphic. Therefore the fixed-point set $\mathcal{M}(\tilde{P})^\tau$ in $\mathcal{M}(\tilde{P})$ is Lagrangian and totally real [18]. The moduli space $\mathcal{M}(\tilde{C}, G)$ of flat G-connections on $\tilde{C}$ on bundles of all topological types is identifiable with the character variety, and there is an anti-symplectic involution τ on it so that $\mathcal{M}(\tilde{C}, G)^\tau$ is Lagrangian and totally real.

The moduli space $\mathcal{M}(C, G)$ for topological types of bundles over C is a regular cover of a subset $\mathcal{M}(\tilde{C}, G)^\tau_0$ in $\mathcal{M}(\tilde{C}, G)^\tau$ via the pull-back map π^* with $Z(G)_{[2]}$ as the group of deck transformations [17].[2] Here the centre $Z(G)$ of G is a finite Abelian group, and $Z(G)_{[2]}$ is the subgroup of 2-torsion elements. In particular, if $Z(G)$ is of odd order, then $\mathcal{M}(C, G)$ coincides with $\mathcal{M}(\tilde{C}, G)^\tau$ [17], but in general, $\pi^*\colon \mathcal{M}(C, G) \to \mathcal{M}(\tilde{C}, G)^\tau$ is a local diffeomorphism which is neither surjective nor injective. What covers other parts of $\mathcal{M}(\tilde{C}, G)^\tau$ are the moduli spaces of flat connections twisted by flat torsion gerbes (or B-fields) on C, or twisted character varieties [38].

To formulate Hitchin's equations on a non-orientable surface C, we choose a conformal structure on C; then the Hodge star $*$ on the 1-forms on C is defined up to a sign. Therefore Hitchin's equations (11) on $(A, \Phi) \in \mathcal{A}(P) \times \Omega^1(C, \mathrm{ad}P)$ by setting the right-hand sides of μ_I, μ_J, μ_K in (10) to zero still make sense even though C is not orientable [19]. Let $\mathcal{M}_{\mathrm{H}}(P)$ be the moduli space of the solutions (A, Φ) to Hitchin's equations on C up to gauge equivalence. Although C is non-orientable, there is a complex structure $J^\sharp$ and a symplectic form $\omega^\sharp_J$ on $\mathcal{A}(P) \times \Omega^1(C, \mathrm{ad}P)$ which descend respectively to J and ω_J on $\mathcal{M}_{\mathrm{H}}(P)$. The complex structure J can also be understood from the moduli space of flat $G^{\mathbb{C}}$-connections on C [19]. Let $\mathcal{M}_{\mathrm{H}}(C, G)$ be the union of $\mathcal{M}_{\mathrm{H}}(P)$ with all topological types of P.

On the oriented double cover $\tilde{C}$, there is a unique complex structure which is compatible with the conformal structure on C and the orientation on $\tilde{C}$. So Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(\tilde{C}, G)$ for $\tilde{C}$ is hyper-Kähler, with three complex structures $\tilde{I}$, $\tilde{J}$, $\tilde{K}$ and three symplectic forms $\tilde{\omega}_I$, $\tilde{\omega}_J$, $\tilde{\omega}_K$. The involution τ acts on $\mathcal{M}_{\mathrm{H}}(\tilde{C}, G)$, and it is holomorphic and symplectic in $\tilde{J}$ and $\tilde{\omega}_J$, but anti-holomorphic and anti-symplectic in $\tilde{I}$ and $\tilde{\omega}_I$, $\tilde{K}$ and $\tilde{\omega}_K$. Consequently, the fixed-point set $\mathcal{M}_{\mathrm{H}}(\tilde{C}, G)^\tau$ is holomorphic and symplectic in $\mathcal{M}_{\mathrm{H}}(\tilde{C}, G)$ with respect to $\tilde{J}$ and $\tilde{\omega}_J$ but totally real and Lagrangian with respect to $\tilde{I}$ and $\tilde{\omega}_I$, $\tilde{K}$ and $\tilde{\omega}_K$ [19].[3]

More interestingly, when restricted to appropriate smooth part, Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(C, G)$ for the non-orientable surface C is a regular cover via

[2]It was stated in [17] that $\mathcal{M}(C, G)$ is a $|Z(G)/2Z(G)|$-sheeted cover of $\mathcal{M}_{\mathrm{H}}(\tilde{C}, G)^\tau_0$. This is compatible with the slightly more refined statement (cf. [19, 38]) here because $Z_{[2]}$ is always of the same order as $Z/2Z$ if Z is a finite Abelian group.

[3]In fact, similar results hold for Hitchin's equations on higher-dimensional non-orientable manifolds [19]. See [3] for anti-holomorphic involutions on surfaces with possible fixed points and [5] for involutions that also acts on the structure group.

the pull-back map π^* of a subset $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)_0^\tau$ of $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$, with the 2-torsion subgroup $Z(G)_{[2]}$ of the centre $Z(G)$ as the group of deck transformations [19] (compare the same pattern [17] for $\mathcal{M}(C,G)$ explained above). The covering map is Kähler with respect to J, ω_J on $\mathcal{M}(C,G)$ and the restrictions of $\tilde{J}$, $\tilde{\omega}_J$ to $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)_0^\tau$. There are similar coverings of other parts of $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$ by the moduli spaces of solutions to Hitchin's equations for twisted G-bundles over C, or twisted character varieties of the group $G^{\mathbb{C}}$ [38].

4.2. Line bundle on moduli space of flat connections (non-orientable case)

We study line bundles over $\mathcal{M}(C,G)$ and $\mathcal{M}(\tilde{C},G)^\tau$, where C is a closed non-orientable surface, $\pi\colon \tilde{C}\to C$ is its orientation double covering, τ is the non-trivial deck transformation on $\tilde{C}$, and G is a simple, connected and simply connected compact Lie group.

Let $P\to C$ be a principal G-bundle. Then the moduli space $\mathcal{M}(\tilde{C},G)$ of flat connections on $\tilde{P}:=\pi^*P$ over $\tilde{C}$ has a symplectic structure $\tilde{\omega}$ and a Hermitian line bundle $\tilde{\mathcal{L}}$ over $\mathcal{M}(\tilde{C},G)$ with a unitary connection whose curvature is $\tilde{\omega}/\sqrt{-1}$. In the construction of $\tilde{\mathcal{L}}$ (cf. §2.3), we need a cocycle given by (8) ($g\in\mathcal{G}(\tilde{P})$, $A\in\mathcal{A}(\tilde{P})$ in our context), as well as a 1-form $\tilde{\theta}$ on $\mathcal{A}(\tilde{P})$ defined by (9) (but replacing C by $\tilde{C}$), satisfying the analogs of (1), (2). Recall that τ acts as an anti-symplectic involution on $\mathcal{M}(\tilde{C},G)$. In fact, since τ reverses the orientation of $\tilde{C}$, we have a stronger statement $\tau^*\tilde{\theta}=-\tilde{\theta}$. We claim that for all $g\in\mathcal{G}(\tilde{P})$, $A\in\mathcal{A}(\tilde{P})$, we have

$$\gamma(\tau^*g,\tau^*A)=\gamma(g,A)^{-1}. \tag{13}$$

To prove (13), recall that in (8), we need to choose an oriented 3-manifold B whose boundary is $\tilde{C}$ and extend $\tilde{P}$, g, A to B (denoted by the same notations). Likewise, choose another oriented 3-manifold B' with the same boundary $\tilde{C}$ and extend $\tilde{P}$, τ^*g, τ^*A to $\tilde{P}'$, g', A' on B'. Consider a closed 3-manifold $\hat{B}_\tau$ obtained by gluing B and B' along the boundary $\tilde{C}$ via τ. We note that since τ reverses the orientation on $\tilde{C}$, there is no need to reverse the orientation of B or B' to obtain a consistent orientation on $\hat{B}_\tau$. Moreover, the bundles $P\to B$ and $P'\to B'$ glue along $\tilde{C}$ via the lifted action of τ on $\tilde{P}$ to form a G-bundle $\hat{P}_\tau$ over $\hat{B}_\tau$. Then the connections A on $\tilde{P}$ and A' on $\tilde{P}'$ determine a connection $\hat{A}_\tau$ on $\hat{P}_\tau$, and the gauge transformations g of $\tilde{P}$ and g' of $\tilde{P}'$ determine a gauge transformation $\hat{g}_\tau$ of $\hat{P}_\tau$. Therefore, (13) follows easily from the equalities

$$\mathrm{CS}(\hat{A}_\tau)=\mathrm{CS}(A)+\mathrm{CS}(A'),\quad \mathrm{CS}(\hat{g}_\tau\cdot\hat{A}_\tau)=\mathrm{CS}(g\cdot A)+\mathrm{CS}(g'\cdot A')$$

and the fact that $\mathrm{CS}(\hat{g}_\tau\cdot\hat{A}_\tau)-\mathrm{CS}(\hat{A}_\tau)\in 2\pi\mathbb{Z}$.

(13) shows that the involution τ on $\mathcal{M}(\tilde{C},G)$ does not lift to an action on $\tilde{\mathcal{L}}$, but there is a bundle isomorphism $\tau^*\tilde{\mathcal{L}}\to\tilde{\mathcal{L}}^{-1}$ preserving the respective connections. Since the curvature of the connection on $\tilde{\mathcal{L}}^{-1}$ is $-\tilde{\omega}/\sqrt{-1}$, this statement is a refinement of $\tau^*\tilde{\omega}=-\tilde{\omega}$ that we saw earlier. In particular, restricting to the fixed-point set $\mathcal{M}(\tilde{C},G)^\tau$ (on which τ is the identity map), the bundles $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^{-1}$ are isomorphic. That is, the restriction of $\tilde{\mathcal{L}}$ to $\mathcal{M}(\tilde{C},G)^\tau$ is flat and $\tilde{\mathcal{L}}^{\otimes 2}$ (with

its connection) is trivial on $\mathcal{M}(\tilde{C},G)^\tau$. This is a refinement of the statement that $\mathcal{M}(\tilde{C},G)^\tau$ is Lagrangian in $(\mathcal{M}(\tilde{C},G),\tilde{\omega})$.

Let $i_0\colon \mathcal{M}(\tilde{C},G)^\tau_0 \to \mathcal{M}(\tilde{C},G)$ be the inclusion map and $p_0\colon \mathcal{M}(C,G) \to \mathcal{M}(\tilde{C},G)^\tau_0$ be the $|Z(G)_{[2]}|$-sheeted covering map [17, 38] explained in §4.1. Then the bundles $\mathcal{L} := p_0^* i_0^* \tilde{\mathcal{L}}$ over $\mathcal{M}(C,G)$ is clearly flat and 2-torsion. We argue that it is in fact trivial. Using $\mathcal{A}(P) = \mathcal{A}(\tilde{P})^\tau$, $\mathcal{G}(P) = \mathcal{G}(\tilde{P})^\tau$, the space

$$\mathcal{M}(C,G) = (\mathcal{A}(P) \cap \mu^{-1}(0))/\mathcal{G}(P) = \mu^{-1}(0)^\tau/\mathcal{G}(\tilde{P})^\tau$$

is a Lagrangian quotient [29], and the bundle $\mathcal{L}$ can be identified with the quotient of $\mathcal{A}(P) \cap \mu^{-1}(0) \times \mathbb{C}$ by the $\mathcal{G}(P)$-action $g\colon (A,z) \mapsto (g \cdot A, \gamma_C(g,A)z)$, where $\gamma_C(g,A) := \gamma(\pi^* g, \pi^* A)$ for all $g \in \mathcal{G}(P)$, $A \in \mathcal{A}(P)$. It suffice to show that $\gamma_C \equiv 1$. Indeed, we extend the bundle $\tilde{P}$, the connection $\pi^* A$ and the gauge transformation π^* to a 3-manifold B whose boundary is $\tilde{C}$. Then they descend to a closed orientable 3-manifold obtained from B by collapsing the boundary $\tilde{C}$ to C. Thus the exponent in (8) is an integer multiple of $2\pi\sqrt{-1}$, and the result follows.

Finally, the regular coverings $p_0\colon \mathcal{M}(C,G) \to \mathcal{M}(\tilde{C},G)^\tau_0$ itself induces flat line bundles over $\mathcal{M}(\tilde{C},G)^\tau_0$. Let e be a character of the group $Z(G)_{[2]}$, i.e., $e \in (Z(G)_{[2]})^\vee := \mathrm{Hom}(Z(G)_{[2]}, U(1))$, the Pontryagin dual of $Z(G)_{[2]}$. Then for each $e \in (Z(G)_{[2]})^\vee$, we have a flat line bundle $\ell_e := \mathcal{M}(C,G) \times_e \mathbb{C}$ over $\mathcal{M}(\tilde{C},G)^\tau_0$. The trivial line bundle over $\mathcal{M}(C,G)$ pushes down via p_0 to the vector bundle $\bigoplus_{e \in (Z(G)_{[2]})^\vee} \ell_e$ over $\mathcal{M}(\tilde{C},G)^\tau_0$.

4.3. Line bundles on Hitchin's moduli space (non-orientable case)

Our goal is to study the properties of the line bundles $\tilde{\mathcal{L}}_I$, $\tilde{\mathcal{L}}_J$, $\tilde{\mathcal{L}}_K$ over $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ under the action of τ, the restriction of these bundles to the fixed-point set $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$, and subsequently the pull-back of these restrictions to $\mathcal{M}_{\mathrm{H}}(C,G)$. We will obtain refinements of the statement that τ on $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ is symplectic with respect to $\tilde{\omega}_J$ but anti-symplectic with respect to $\tilde{\omega}_I$, $\tilde{\omega}_K$.

First, recall that the bundle $\tilde{\mathcal{L}}_I$ is defined by the cocycle $\gamma_I(g,A,\Phi) = \gamma(g,A)$ which satisfies (13). The arguments in §4.2 show that $\tau^*\tilde{\mathcal{L}}_I \cong \tilde{\mathcal{L}}_I^{-1}$ as bundles. Moreover, the 1-form $\tilde{\theta}_I$ on $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ satisfies $\tau^*\tilde{\theta}_I = -\tilde{\theta}_I$, hence the above bundle isomorphism respects the connections. Therefore, the restriction of $\tilde{\mathcal{L}}_I$ to $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$ is flat and 2-torsion, i.e., $\tilde{\mathcal{L}}_I^{\otimes 2}$ (together with its connection) is trivial on $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$. Let $i_0\colon \mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau_0 \to \mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ be the inclusion map and let $p_0\colon \mathcal{M}_{\mathrm{H}}(C,G) \to \mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau_0$ be the $|Z(G)_{[2]}|$-sheeted covering map. Then the bundle $\mathcal{L}_I := p_0^* i_0^* \tilde{\mathcal{L}}_I$ on $\mathcal{M}_{\mathrm{H}}(C,G)$ is not only flat and 2-torsion, but is in fact trivial as in §4.2.

Second, the bundle $\tilde{\mathcal{L}}_J$ is quite different. Since $\gamma_J(g,A,\Phi) = 1$, $\tilde{\mathcal{L}}_J$ is a product bundle topologically. However, the connection on $\tilde{\mathcal{L}}_J$ is $d - \sqrt{-1}\tilde{\lambda}_J$, where $\tilde{\lambda}_J$ is the 1-form on $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ defined in §3.1 such that $\tilde{\omega}_J = d\tilde{\lambda}_J$. Since $\tau^*\tilde{\lambda}_J = \tilde{\lambda}_J$, there is an isomorphism $\tau^*\tilde{\mathcal{L}}_J \cong \tilde{\mathcal{L}}_J$ respecting the connections. That is, the action of τ on $\mathcal{M}_{\rm H}(\tilde{C},G)$ lifts to $\tilde{\mathcal{L}}_J$ preserving the connection, a fact that will be very crucial in §5. Consequently, the restriction of $\tilde{\mathcal{L}}_J$ to $\mathcal{M}(\tilde{C},G)_0^\tau$ is a prequantum line bundle, and its pull back $\mathcal{L}_J := p_0^* i_0^* \tilde{\mathcal{L}}_J$ to $\mathcal{M}_{\rm H}(C,G)$ is also a prequantum line bundle (with respect to ω_J).

Third, since $\gamma_K(g,A,\Phi)=1$, $\tilde{\mathcal{L}}_K$ is topologically trivial again. The connection on $\tilde{\mathcal{L}}_K$ is $d-\sqrt{-1}\tilde{\lambda}_K$, where $\tilde{\lambda}_K$ is a 1-form such that $\tilde{\omega}_K = d\tilde{\lambda}_K$. Since $\tau^*\tilde{\lambda}_K = -\tilde{\lambda}_K$ in this case, there is a bundle isomorphism $\tau^*\tilde{\mathcal{L}}_K \cong \tilde{\mathcal{L}}_K^{-1}$ respecting the connection. The restriction of $\tilde{\mathcal{L}}_K$ to $\mathcal{M}_{\rm H}(\tilde{C},G)^\tau$, with its connection, is trivial, and so is the pull-back $\mathcal{L}_K := p_0^* i_0^* \tilde{\mathcal{L}}_K$ to $\mathcal{M}_{\rm H}(C,G)$.

To summarize, if $\tilde{\mathcal{L}}_I$, $\tilde{\mathcal{L}}_J$, $\tilde{\mathcal{L}}_K$ are the prequantum line bundles on $\mathcal{M}_{\rm H}(\tilde{C},G)$ with respect to $\tilde{\omega}_I$, $\tilde{\omega}_J$, $\tilde{\omega}_K$ and if $\mathcal{L}_I$, $\mathcal{L}_J$, $\mathcal{L}_K$ are the pull-back to $\mathcal{M}_{\rm H}(C,G)$ of the restriction of these bundles to $\mathcal{M}_{\rm H}(\tilde{C},G)_0^\tau$, then $\mathcal{L}_I$, $\mathcal{L}_K$ are trivial while $\mathcal{L}_J$ is the prequantum line bundle of $(\mathcal{M}(C,G),\omega_J)$. In particular, the line bundle $\tilde{\mathcal{L}}_t = \tilde{\mathcal{L}}_I^{\otimes k} \otimes \tilde{\mathcal{L}}_K^{\otimes s}$ from complex Chern–Simons gauge theory with parameter $t = k+\sqrt{-1}s$ ($k\in\mathbb{Z}$, $s\in\mathbb{R}$) is flat and 2-torsion when restricted to $\mathcal{M}_{\rm H}(\tilde{C},G)^\tau$, and its pull back $\mathcal{L}_t \to \mathcal{M}_{\rm H}(C,G)$ is trivial with a trivial connection.

Finally, we construct the flat line bundles induced by the regular covering map $p_0\colon \mathcal{M}_{\rm H}(C,G) \to \mathcal{M}_{\rm H}(\tilde{C},G)_0^\tau$. As in §4.2, we get a flat line bundle ℓ'_e over $\mathcal{M}_{\rm H}(\tilde{C},G)_0^\tau$ for each $e\in (Z(G)_{[2]})^\vee$, and hence a brane $\mathcal{B}'_e := (\mathcal{M}_{\rm H}(\tilde{C},G)_0^\tau, \ell')$ of type (A,B,A). Furthermore, we have

$$(p_0)_*\mathcal{L}_J = \bigoplus_{e\in (Z(G)_{[2]})^\vee} \tilde{\mathcal{L}}_J \otimes \ell'_e \tag{14}$$

as vector bundles over $\mathcal{M}_{\rm H}(\tilde{C},G)_0^\tau$.

5. Quantization via branes and mirror symmetry

5.1. General theory and example

We begin with an outline of quantization via branes and mirror symmetry [12, 13]. Let (M,ω) be a symplectic manifold to be quantized. Assume that there is a complexification of M, i.e., a complex manifold Y (whose complex structure is denoted by I) with an anti-holomorphic involution τ such that M is one of the components of the fixed-point set. Assume also that there is a holomorphic symplectic $(2,0)$-form $\Omega_I = \omega_J + \sqrt{-1}\omega_K$ on Y such that $\tau^*\Omega_I = \bar{\Omega}_I$ and ω_J restricts to ω on M. In several important examples, Y is hyper-Kähler, and we will use the notations suggested by hyper-Kähler geometry. Assume further that there is a Hermitian line bundle ℓ over Y with a unitary connection whose curvature is $\mathrm{Re}\,\Omega_I/\sqrt{-1} = \omega_J/\sqrt{-1}$ and that the action of τ can be lifted to ℓ, acting trivially over M and preserving its connection. Then the restriction of ℓ to M is a prequantum line bundle.

Consider the A-model of the symplectic manifold (Y,ω_K). Though ultimately the A-model depends on ω_K only, to define (pseudo-)holomorphic curves, one needs to choose an (almost) complex structure K on Y compatible with ω_K. Given two branes $\mathcal{B}_1$ and $\mathcal{B}_2$ in the A-model, let $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ be the space of states of open strings whose boundary conditions at the two ends are given by $\mathcal{B}_1$, $\mathcal{B}_2$. If $\mathcal{B}_1$, $\mathcal{B}_2$ are Lagrangian A-branes, then $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ is the symplectic Floer homology. If one of them is coisotropic, the result is more complicated. For example, if $Y = \mathcal{M}_{\mathrm{H}}(C, G)$, where C is orientable, $\mathcal{B}_{\mathrm{cc}} = \mathcal{B}_J^{\otimes s}$ is the canonical coisotropic brane and $\mathcal{B}'$ is supported on a fibre of the Hitchin fibration (cf. §3.3), then $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}')$ gives rise to a sheaf of $\mathcal{D}$-module on $\mathcal{M}(C, G)$ related to the geometric Langlands programme [21].

For quantization, we use two A-branes: a canonical coisotropic brane $\mathcal{B}_{\mathrm{cc}} := (Y, \ell)$ and a Lagrangian brane $\mathcal{B}' := (M, \ell')$, where ℓ' is a flat line bundle on M. According to [13], $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ is the space of quantum observables and $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}')$ is the space of quantum states when quantising the symplectic manifold (M, ω) with the prequantum line bundle $L := (\ell|_M) \otimes (\ell')^{-1}$. In fact, $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ is a deformation of the algebra of classical observables (which are analytic functions on Y), whereas $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}')$ is the space $\mathcal{H} = H^0(M, L \otimes K_M^{1/2})$ of quantum states normally constructed by geometric quantization. In this way, $\mathcal{H}$ is naturally a $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$-module. The algebra $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ remains the same if we quantize a submanifold of Y fixed by another anti-holomorphic involution.

The involution τ defines a Hermitian structure on $\mathcal{H}$, and analytic functions f on Y satisfying $\tau^* f = \bar{f}$ act as self-adjoint operators on $\mathcal{H}$. To obtain enough quantum operators acting on $\mathcal{H}$, we need many analytic functions on Y. This is best achieved by choosing Y as the minimal complexification of M, which is homotopic to M and Stein [24].

In the dual approach, we consider the B-model on the mirror manifold $Y^\vee$ of Y in the complex structure $J^\vee$. For two B-branes $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$, the space of string states in the B-model is simply $\mathrm{Ext}(\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2)$, whose (virtual) dimension is given by the Grothendieck–Riemann–Roch formula

$$\sum_k (-1)^k \dim \mathrm{Ext}^k(\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2) = \int_{Y^\vee} \mathrm{ch}(\tilde{\mathcal{B}}_1)^* \wedge \mathrm{ch}(\tilde{\mathcal{B}}_2) \wedge \mathrm{td}(Y^\vee),$$

where $\alpha^* := (-1)^{k+1}\alpha$ if α is a $2k$-form. Kontsevich's homological mirror symmetry [22] maps an A-brane $\mathcal{B}$ in Y to a B-brane $\mathcal{B}^\vee$ in $Y^\vee$, and for two A-branes $\mathcal{B}_1$, $\mathcal{B}_2$, there is an isomorphism $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2) \cong \mathrm{Ext}(\mathcal{B}_1^\vee, \mathcal{B}_2^\vee)$. Therefore, by mirror symmetry, quantization of (M, ω), which is a problem in the A-model of (Y, ω_K), is transformed to an easier problem of finding $\mathrm{Ext}(\mathcal{B}_{\mathrm{cc}}^\vee, \mathcal{B}_{\mathrm{cc}}^\vee)$ and $\mathrm{Ext}(\mathcal{B}_{\mathrm{cc}}^\vee, (\mathcal{B}')^\vee)$ in the B-model of $(Y^\vee, J^\vee)$, provided that $Y^\vee$, $\mathcal{B}_{\mathrm{cc}}^\vee$, $(\mathcal{B}')^\vee$ are known explicitly [12].

When there is a fibration $p\colon Y \to \boldsymbol{B}$ by (special) Lagrangian tori, the mirror manifold $Y^\vee$ is the dual torus fibration $p^\vee\colon Y^\vee \to \boldsymbol{B}$ [33]. In this case, each fibre $p^{-1}(b)$ ($b \in \boldsymbol{B}$) in Y together with a flat line bundle on it is an A-brane, and its mirror is the B-brane supported on the point in the dual torus $(p^\vee)^{-1}(b)$ representing this flat line bundle on $p^{-1}(b)$. If the fibration of Y is holomorphic in another complex structure J and if $\mathcal{B}$ is a B-brane in Y with respect to J, then the mirror $\mathcal{B}^\vee$ of $\mathcal{B}$ can be calculated by a Fourier–Mukai transform, i.e., $\mathcal{B}^\vee = R(h^\vee)_*(h^*\mathcal{B} \otimes \mathcal{P})$, where $\mathcal{P}$ is the Poincaré line bundle over the fibre product $Y \times_{\boldsymbol{B}} Y^\vee$. Hence

$$\operatorname{ch}(\mathcal{B}^\vee) = \int_{(Y \times_{\boldsymbol{B}} Y^\vee)/Y^\vee} p^* \operatorname{ch}(\mathcal{B}) \wedge \operatorname{ch}(\mathcal{P}).$$

In particular, if $\mathcal{B} = (Y, \ell)$ is a space-filling brane and the restriction of the line bundle ℓ to each fibre $h^{-1}(b)$ $(b \in \boldsymbol{B})$ is flat, then the mirror $\mathcal{B}^\vee$ is supported on the subset in $Y^\vee$ consisting of elements $p \in Y^\vee$ representing the flat line bundle $\ell|_{h^{-1}(h^\vee(p))}$.

An archetypical example in mirror symmetry comes from Hitchin's moduli spaces. Let $G^\vee$ be the Langlands dual [10, 25] of a compact semisimple Lie group G. If C is a closed orientable surface, then the mirror of $\mathcal{M}_{\mathrm{H}}(C, G)$ with the symplectic structure ω_K is $\mathcal{M}_{\mathrm{H}}(C, G^\vee)$ with the complex structure $J^\vee$ [8, 14, 21]. In fact, consider the Hitchin fibrations $h\colon \mathcal{M}_{\mathrm{H}}(C, G) \to \boldsymbol{B}$ and $h^\vee\colon \mathcal{M}_{\mathrm{H}}(C, G^\vee) \to \boldsymbol{B}^\vee$. There is a natural identification $\varXi\colon \boldsymbol{B} \to \boldsymbol{B}^\vee$ of base spaces and for a generic $b \in \boldsymbol{B}$, the fibres $h^{-1}(b)$ and $(h^\vee)^{-1}(\varXi(b))$ are dual Lagrangian tori. If G is simply connected, then $\mathcal{M}_{\mathrm{H}}(C, G)$ is connected but $\mathcal{M}_{\mathrm{H}}(C, G^\vee)$ is not, and a connected component of $\mathcal{M}_{\mathrm{H}}(C, G^\vee)$ corresponds to a flat, torsion gerbe (or B-field) on $\mathcal{M}_{\mathrm{H}}(C, G)$. In general, both $\mathcal{M}_{\mathrm{H}}(C, G)$ and $\mathcal{M}_{\mathrm{H}}(C, G^\vee)$ are disconnected and carry non-trivial flat B-fields.

In [12], quantization of $M := \mathcal{M}(C, G)$ with the symplectic form $k\omega$ $(k \in \mathbb{Z}, k > 0)$ is studied by using the complexification $Y := \mathcal{M}_{\mathrm{H}}(C, G) = \mathcal{M}(C, G^{\mathbb{C}})$ with the complex structure J. The involution σ is anti-holomorphic in J and M is in the fixed-point set (cf. §3.1). Moreover, σ lifts to the line bundle $\ell := \mathcal{L}_I^{\otimes k}$ on Y, which restricts to $\mathcal{L}^{\otimes k}$ on M (cf. §3.2). Let $\mathcal{B}_{\mathrm{cc}} := (Y, \ell) = \mathcal{B}_I^{\otimes k}$ and let $\mathcal{B}'$ be the brane supported on M with a trivial line bundle. Both $\mathcal{B}_{\mathrm{cc}}$ and $\mathcal{B}'$ are branes of type (B, A, A) and hence their mirrors are of type (B, B, B) [21]. The space $\mathcal{H}$ of quantum states is given by $\operatorname{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}')$ on the A-side and by $\operatorname{Ext}(\mathcal{B}_{\mathrm{cc}}^\vee, (\mathcal{B}')^\vee)$ on the B-side. The mirrors $\mathcal{B}_{\mathrm{cc}}^\vee$, $(\mathcal{B}')^\vee$ can be calculated by Fourier–Mukai transform. The brane $(\mathcal{B}')^\vee$ is supported on the most singular point of $\mathcal{M}_{\mathrm{H}}(C, G^\vee)$ represented by the trivial connection and $\varPhi^\vee = 0$, while $\mathcal{B}_{\mathrm{cc}}^\vee$ is space-filling, and its rank is equal to the volume of the Hitchin fibres in $\mathcal{M}_{\mathrm{H}}(C, G^\vee)$ [12].

5.2. Quantization of Hitchin's moduli space (non-orientable case)

Let C be a closed non-orientable surface and let $\pi\colon \tilde{C} \to C$ be its orientation double covering, with a non-trivial deck transformation τ on $\tilde{C}$. Let G be a compact semisimple Lie group. Then Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(C, G)$ is Kähler [19] with complex structure J and symplectic form ω_J (cf. §4.1) and has a prequantum line bundle $\mathcal{L}_J$ (cf. §4.3). We consider the quantization of $\mathcal{M}_{\mathrm{H}}(C, G)$ using branes and mirror symmetry, in the framework of [12, 13].

For the oriented cover $\tilde{C}$, Hitchin's moduli space $\mathcal{M}_{\mathrm{H}}(\tilde{C}, G)$ is hyper-Kähler with complex structures $\tilde{I}, \tilde{J}, \tilde{K}$ and symplectic forms $\tilde{\omega}_I, \tilde{\omega}_J, \tilde{\omega}_K$, and τ acts on $\mathcal{M}_{\mathrm{H}}(\tilde{C}, G)$ reversing $\tilde{I}, \tilde{K}$ but preserving $\tilde{J}$. In particular, the τ-invariant part

$\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$ is Kähler with respect to $\tilde{J}$, $\tilde{\omega}_J$, and moreover, there is a Kähler local diffeomorphism from $\mathcal{M}_{\mathrm{H}}(C,G)$ to $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$, which is a covering map onto $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau_0$ [19] (cf. §4.1). The prequantum line bundle $\tilde{\mathcal{L}}_J$ on $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ restricts to that on $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$, which then pulls back to the prequantum line bundle $\mathcal{L}_J$ on $\mathcal{M}_{\mathrm{H}}(C,G)$ (cf. §4.3).

There is a close relation between quantization of $\mathcal{M}_{\mathrm{H}}(C,G)$ and that of $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau_0$. Recall that for each $e\in(Z(G)_{[2]})^\vee$, there is a flat line bundle ℓ'_e on $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau_0$. Let $\mathcal{H}(C,G)$ denote the space of quantum states by quantizing $(\mathcal{M}_{\mathrm{H}}(C,G),\mathcal{L}_J)$ and $\mathcal{H}_e(C,G)$, that by quantizing $(\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau_0,\tilde{\mathcal{L}}_J\otimes(\ell'_e)^{-1})$. By (14),we have $\mathcal{H}(C,G)=\bigoplus_{e\in(Z(G)_{[2]})^\vee}\mathcal{H}_e(C,G)$.

We regard $Y:=\mathcal{M}_{\mathrm{H}}(\tilde{C},G)$ as a complexification of $M:=\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau_0$. The action of τ on Y is anti-holomorphic in $\tilde{I}$, and there is a $(2,0)$-form $\Omega_I:=\tilde{\omega}_J+\sqrt{-1}\tilde{\omega}_K$ in the complex structure $\tilde{I}$ that satisfies $\tau^*\Omega_I=\bar{\Omega}_I$. The line bundle $\ell:=\mathcal{L}_J$ has a unitary connection whose curvature is $\tilde{\omega}_J/\sqrt{-1}=\operatorname{Re}\Omega_I/\sqrt{-1}$. Further, the action of τ lifts to ℓ preserving the connection. Therefore by choosing a flat line bundle ℓ' on M, we obtain the quantization of M as $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}},\mathcal{B}')$ in the A-model of $(Y,\tilde{\omega}_K)$, where $\mathcal{B}_{\mathrm{cc}}=:(Y,\ell)$ and $\mathcal{B}':=(M,\ell')$ are branes of type (A,B,A). Thus $\mathcal{H}_e(C,G)=\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}},\mathcal{B}'_e)$, where $\mathcal{B}'_e$ is defined by ℓ'_e for each $e\in(Z(G)_{[2]})^\vee$. The algebra $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}},\mathcal{B}_{\mathrm{cc}})$ remains the same if we quantize a manifold fixed by another involution that is anti-holomorphic in $\tilde{I}$ (for example, the involutions in [5]).

On the mirror side, we consider the B-model of $Y^\vee:=\mathcal{M}_{\mathrm{H}}(\tilde{C},G^\vee)$ in the complex structure $\tilde{J}^\vee$. The mirrors of $\mathcal{B}_{\mathrm{cc}}$ and $\mathcal{B}'_e$ are again of type (A,B,A) [21]. Since the restriction of ℓ to each fibre of the Hitchin fibration $h\colon Y\to\boldsymbol{B}$ is a flat line bundle, the mirror of $\mathcal{B}_{\mathrm{cc}}$ is supported on a section of the dual fibration $h^\vee$ of $Y^\vee$ (at least in the generic fibres) determined by these flat line bundles. To describe the mirror of $\mathcal{B}'_e$, we consider a special case when G is simply connected. Then M is connected but has non-trivial flat bundles ℓ'_e, whereas in the mirror manifold $Y^\vee$, $\mathcal{M}_{\mathrm{H}}(\tilde{C},G^\vee)^\tau_0$ is disconnected but each component is simply connected. The connected components are labeled by elements in $\pi_1(G^\vee)/2\,\pi_1(G^\vee)$ and let $\check{\mathcal{B}}'_{m^\vee}$ be the brane supported on the component labeled by $m^\vee$. It turns out that $\pi_1(G^\vee)/2\,\pi_1(G^\vee)\cong(Z(G)_{[2]})^\vee$ and the mirror of $\mathcal{B}'_e$ is $\check{\mathcal{B}}'_e$ [39].[4] Thus we have $\mathcal{H}_e(C,G)\cong\mathrm{Ext}(\mathcal{B}^\vee_{\mathrm{cc}},\check{\mathcal{B}}'_e)$ from the B-side. The general case is quite delicate [39] and involves the coboundary map $\delta\colon H^1(\mathbb{Z}_2,Z(G))\cong Z(G)_{[2]}\to H^2(\mathbb{Z}_2,\pi_1(G))\cong\pi_1(G)/2\,\pi_1(G)$.

Of course, we can choose to quantize $\mathcal{M}_{\mathrm{H}}(C,G)$ with the symplectic form $s\omega_J$ $(s\neq0)$. Then we should use $\mathcal{B}^{\otimes s}_{\mathrm{cc}}=(Y,\ell^{\otimes s})$ (cf. §4.3) while keeping $\mathcal{B}'_e$ the same. The mirror $(\mathcal{B}^{\otimes s}_{\mathrm{cc}})^\vee$ is supported on the same subspace in $Y^\vee$. Let $\mathcal{H}^s(C,G)$ be the

[4]The branes $\mathcal{B}'_e$ and $\check{\mathcal{B}}'_e$ are supported on some connected components of $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$ and $\mathcal{M}_{\mathrm{H}}(\tilde{C},G^\vee)^\tau$, respectively. In [3], treating branes as submanifolds, it was argued that the mirror of $\mathcal{M}_{\mathrm{H}}(\tilde{C},G)^\tau$ is $\mathcal{M}_{\mathrm{H}}(\tilde{C},G^\vee)^\tau$.

space of quantum states. Then $\mathcal{H}^s(C,G)$ is the direct sum of $\mathcal{H}^s_e(C,G)$ over $e \in (Z(G)_{[2]})^\vee$, where $\mathcal{H}^s_e(C,G)$ is $\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}^{\otimes s}, \mathcal{B}'_e)$ on the A-side and $\mathrm{Ext}((\mathcal{B}_{\mathrm{cc}}^{\otimes s})^\vee, \check{\mathcal{B}}'_e)$ from the B-side.

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Siye Wu
Department of Mathematics
National Tsing Hua University
Hsinchu 30013, Taiwan
e-mail: swu@math.nthu.edu.tw

Part VI

Complex Geometry

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Ramadanov Theorem for Weighted Bergman Kernels on Complex Manifolds

Zbigniew Pasternak-Winiarski and Paweł M. Wójcicki

Abstract. We study the limit behavior of weighted Bergman kernels on a sequence of domains in a manifold M and show that under some conditions on domains and weights, weighted Bergman kernel converges uniformly on compact sets.

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Keywords. Weighted Bergman kernel; admissible weight; sequence of domains.

1. Introduction

The Bergman kernel (see for instance [1–4, 6]) has become a very important tool in geometric function theory, both in one and several complex variables. It turned out that not only classical Bergman kernel, but also weighted one can be useful. Let $D \subset \mathbb{C}^N$ be a bounded domain. One of the classic results for unweighted Bergman kernels is Ramadanov's theorem (see [9]):

Theorem 1 (Ramadanov). *Let $D_1 \Subset D_2 \Subset D_3 \cdots$ be an increasing sequence of domains in $\mathbb{C}^N$ and set $D := \bigcup_j D_j$. Then, $K_{D_j} \to K_D$ uniformly on compact subsets of $D \times D$.*

We prove this theorem for weighted Bergman kernels (see [7, 8, 10]) and for sequences of domains in a complex manifold (see also [3] for the definition of the Bergman kernel on a complex manifold). In this last case we consider holomorphic $(n, 0)$-forms instead of holomorphic functions. Because we are considering a sequence of Hilbert spaces $\{L^2_H(D_i, \mu_i)\}_{i=1}^{\infty}$ we give a conditions which allow us to formulate and prove weighted version of the Ramadanov theorem. It turns out that even pointwise convergence of weights gives uniform convergence of kernels. Moreover, the considered weights are of a very general nature, they just provide that the weighted Bergman spaces generated by them are Hilbert, and that weighted Bergman kernels exist. Note that some generalizations of the Ramadanov theorem on complex manifold have been considered recently but only in the special case of weights (see [6]).

2. Definitions and notations

Let M be an n-dimensional complex manifold $\dim_{\mathbb{C}} M = n$ and let D be a domain in M. Any holomorphic coordinate chart $\Theta : U \to V$, where U is an nonempty open subset of M and V is an nonempty open subset of $\mathbb{C}^n$ will be denoted by (U, Θ). Let $W(D)$ be the set of weights (of integration) on D, i.e., $W(D)$ is the set of all Lebesgue measurable, real-valued, positive functions on D ($\mu \in W(D)$ if $\mu : D \to (0, \infty)$ and for any chart (U, Θ) the function $\mu \circ \Theta^{-1}$ is Lebesgue measurable on $V = \Theta(U) \subset \mathbb{C}^n$). We consider two weights $\mu_1, \mu_2 \in W(D)$ as equivalent if they are equal almost everywhere on D (for any chart (U, Θ) the $2n$-dimensional Lebesgue measure $\lambda_{2n}(\{z \in \Theta(U) : \mu_1(\Theta^{-1}(z)) \neq \mu_2(\Theta^{-1}(z))\}) = 0$). If $\mu \in W(D)$, we denote by $L^2(\Lambda^{n,0}T^\star D, \mu)$ the set of all (Lebesgue measurable) $(n, 0)$-forms f such that

$$||f||^2_\mu = (-1)^{n^2/2} \int_D \mu \cdot (f \wedge \overline{f}) < \infty.$$

It is easy to prove that if M has a countable basis of topology then $L^2(\Lambda^{n,0}T^\star D, \mu)$ is a (infinite-dimensional) separable Hilbert space with the norm $|| \cdot ||_\mu$ and the scalar product

$$(f, g)_\mu = (-1)^{n^2/2} \int_D \mu \cdot (f \wedge \overline{g}),\ \ f, g \in L^2(\Lambda^{n,0}T^\star D, \mu)$$

(see [5]). If $M = \mathbb{C}^n$ and $z_1, \ldots, z_n$ are (complex) coordinates in C^n, any element of $L^2(\Lambda^{n,0}T^\star D, \mu)$ can be written in the form

$$f(z) = f_0(z) dz_1 \wedge \cdots dz_n,\ z \in D, \tag{1}$$

where $f_0 : D \to \mathbb{C}$ is a measurable function on D such that

$$||f||^2_\mu = \int_D |f_0(z)|^2 \mu(z) d\lambda_{2n}(z) < \infty.$$

It means that in this case $L^2(\Lambda^{n,0}T^\star D, \mu)$ can be identified with the space $L^2(D, \mu)$ of all Lebesgue measurable functions on D which are square integrable with respect to the weight of integration μ (see [7] or [8]).

From now on we assume that M has a countable basis of topology. Denote by $L^2H(\Lambda^{n,0}T^\star D, \mu)$ the space of all elements of $L^2(\Lambda^{n,0}T^\star D, \mu)$ which are holomorphic $(n, 0)$-forms on D. We have that $f \in L^2H(\Lambda^{n,0}T^\star D, \mu)$ if $f \in L^2(\Lambda^{n,0}T^\star D, \mu)$ and for any holomorphic coordinate chart $(U, \Theta = (z_1, \ldots, z_n))$ the function f_Θ in the representation formula

$$f(p) = f_\Theta(p) dz_{1|p} \wedge \cdots dz_{n|p},\ \ p \in U, \tag{2}$$

is holomorphic on U. Then for any $p \in U$ we can define the functional of evaluation

$$\mathcal{E}_{\Theta,p}(f) := f_\Theta(p),\ \ f \in L^2H(\Lambda^{n,0}T^\star D, \mu).$$

Note that if $(W, \Psi = (w_1, \dots, w_n))$ is an another holomorphic chart on D such that $p \in U \cap W$ then

$$\mathcal{E}_{\Psi,p}(f) = \mathcal{E}_{\Theta,p}(f)\frac{\partial(z_1, \dots, z_n)}{\partial(w_1, \dots, w_n)}(\Psi(p)),\ f \in L^2H(\Lambda^{n,0}T^\star D, \mu) \tag{3}$$

where $\frac{\partial(z_1,\dots,z_n)}{\partial(w_1,\dots,w_n)}(\Psi(p)) \neq 0$ is the holomorphic Jacobian of the map $\Theta \circ \Psi^{-1}$ at the point $\Psi(p) \in \Psi(W) \subset \mathbb{C}^n$.

Definition 2. A weight $\mu \in W(D)$ is called an *admissible weight*, an a-weight for short, if $L^2H(\Lambda^{n,0}T^\star D, \mu)$ is a closed subspace of $L^2(\Lambda^{n,0}T^\star D, \mu)$ and for any $p \in D$ and some holomorphic coordinate chart (U, Θ) on D such that $p \in U$ the evaluation functional $\mathcal{E}_{\Theta,p}$ is continuous on $L^2H(\Lambda^{n,0}T^\star D, \mu)$. The set of all a-weights on D will be denoted by $AW(D)$.

It follows from (3) that if $\mu \in AW(D)$, $(U, \Theta = (z_1, \dots, z_n))$ and $(W, \Psi = (t_1, \dots, t_n))$ are holomorphic charts on D and $p \in U \cap W$ then holomorphic $(n, 0)$-forms $e_{\mu,\Theta,p}$ and $e_{\mu,\Psi,p}$ representing (by Riesz representation theorem) linear functionals $\mathcal{E}_{\Theta,p}$ and $\mathcal{E}_{\Psi,p}$ respectively fulfills the equation

$$e_{\mu,\Theta,p} = \frac{\partial(\overline{z_1}, \dots, \overline{z_n})}{\overline{t_1}, \dots, \overline{t_n}} e_{\mu,\Psi,p}.$$

hence the formula

$$\overline{K_{D,\,\mu}(p, q)} = e_{\mu,\Theta,p}(q) \wedge d\overline{z_1}_{|p} \wedge \dots \wedge d\overline{z_n}_{|p},\ p \in U, q \in D,$$

defines $2n$-form $K_{D,\,\mu}$ on $D \times D$ independently on the choice of the chart $(U, \Theta = (z_1, \dots, z_n))$ such that $p \in U$. Precisely we have

$$K_{D,\,\mu}(p, q) = \overline{e_{\mu,\Theta,p}(q)} dz_1 \wedge \dots \wedge dz_n.$$

If $(W, \Psi = (t_1, \dots, t_n))$ is a holomorphic chart on D such that $q \in W$ and

$$\overline{e_{\mu,\Theta,p}(q)} = k_\mu^{\Theta,\Psi}(p, q) d\overline{t_1}_{|q} \wedge \dots \wedge d\overline{t_n}_{|q}$$

then

$$K_{D,\,\mu}(p, q) = k_\mu^{\Theta,\Psi}(p, q) d\overline{t_1}_{|q} \wedge \dots \wedge d\overline{t_n}_{|q} \wedge dz_{1|p} \cdots \wedge dz_{n|p}. \tag{4}$$

Definition 3. The $2n$-form $K_{D,\,\mu}$ introduced above is called the μ-Bergman kernel $2n$-form of the domain D.

For $\mu \equiv 1$ we obtain the $2n$-form $K_{D,\,\mu} = K_D$ introduced by S. Kobayashi in his famous work [3]. By the very definition of $e_{\mu,\Theta,p}$ we obtain that for any

$f \in L^2H(\Lambda^{n,0}T^\star D, \mu)$

$$\begin{aligned}
&(-1)^{n^2/2} \int_D \mu(\cdot) K_{D,\,\mu}(p,\cdot) \wedge f(\cdot) \\
&\quad = (-1)^{n^2/2} \int_D \mu \cdot \overline{e_{\mu,\Theta,p}} \wedge dz_{1|p} \cdots \wedge dz_{n|p} \wedge f \\
&\quad = [(-1)^{n^2/2} \int_D \mu f \cdot \overline{e_{\mu,\Theta,p}}] \wedge dz_{1|p} \cdots \wedge dz_{n|p} \\
&\quad = (f, e_{\mu,\Theta,p})_\mu dz_{1|p} \cdots \wedge dz_{n|p} = \mathcal{E}_{\Theta,p} dz_{1|p} \cdots \wedge dz_{n|p} \\
&\quad = f_\Theta(p) dz_{1|p} \cdots \wedge dz_{n|p} = f(p).
\end{aligned} \tag{5}$$

The above formula express reproducing property of $K_{D,\,\mu}$. Similarly as in [3] one can prove that if (f_n) is a complete orthonormal system in $L^2H(\Lambda^{n,0}T^\star D, \mu)$ then

$$K_{D,\,\mu}(p,q) = \sum_n f_n(p) \wedge \overline{f_n(q)}, \;\; p,q \in D,$$

where the series on the right-hand side converges locally uniformly on $D \times D$.

For $\mathcal{E}_{\Theta,p} \neq 0$ it follows from the proof of Riesz representation theorem that if $E := \{f \in L^2H(\Lambda^{n,0}T^\star D, \mu) : \mathcal{E}_{\Theta,p}(f) = 1\}$ there exists exactly one $(n,0)$-form $\psi_{\mu,p}$ in E minimizing the norm $||\cdot||_\mu$ ($\{\psi_{\mu,p}\} = E \cap (\operatorname{Ker} \mathcal{E}_{\Theta,p})^\perp$) and

$$\psi_{\mu,p} = \frac{1}{k_\mu^{\Theta,\Theta}(p,p)} e_{\mu,\Theta,p}. \tag{6}$$

Moreover, for any $p \in D$ and any holomorphic chart (U,ψ) such that $p \in U$

$$k_\mu^{\Theta,\Theta}(p,p) = (e_{\mu,\Theta,p}, e_{\mu,\Theta,p})_\mu = ||e_{\mu,\Theta,p}||_\mu^2 \geq 0. \tag{7}$$

If (W,Ψ) is an another holomorphic chart on D and $q \in W$ we have (by Schwarz inequality)

$$|k_\mu^{\Theta,\Psi}(p,q)| = |(e_{\mu,\Psi,q}, e_{\mu,\Theta,p})_\mu| \leq \sqrt{k_\mu^{\Theta,\Theta}(p,p)} \sqrt{k_\mu^{\Psi,\Psi}(q,q)}. \tag{8}$$

3. Ramadanov theorem for weighted Bergman kernels

Theorem 4 (Weighted generalization of the Ramadanov theorem on manifolds). *Let $\{D_i\}_{i=1}^\infty$ be a sequence of domains in a complex n-manifold M and set $D := \bigcup_j D_j$. Let $\mu \in AW(D)$ and $\mu_j \in AW(D_j)$ for $k \in \mathbb{N}$ (extend μ_j by μ on $D \setminus D_j$). Assume moreover that*

a) *For any $j \in \mathbb{N}$ there is $N = N(j)$ s.t. $D_j \subset D_m$ and $\mu_j(z) \leq \mu_m(z) \leq \mu(z)$ for $m \geq N,\ z \in D_j$.*
b) *$\mu_j \xrightarrow[j\to\infty]{} \mu$ pointwise a.e. on D.*

Then for weighted Bergman $2n$-forms $K_{D_i,\,\mu_i}$ and $K_{D,\,\mu}$ we have

$$\lim_{j\to\infty} K_{D_j,\mu_j} = K_{D,\mu}$$

locally uniformly on $D \times D$.

The first step in the proof is to show the monotonicity property of local coefficients of the weighted kernels.

Lemma 5 (Monotonicity property). *For any* $j \in \mathbb{N}$, $t \in D_j$ *and any holomorphic chart* $(U, \Theta = (z_1, \ldots, z_n))$ *on* D_j *such that* $p \in U$ *the inequality* $k_{\mu_j}^{\Theta,\Theta}(p,p) \geq k_{\mu_m}^{\Theta,\Theta}(p,p)$ *holds for* $m \geq N(j)$, *where*

$$K_{D_l,\mu_l}(p,q) = k_{\mu_l}^{\Theta,\Theta}(p,q) \wedge dz_{1|p} \cdots \wedge dz_{n|p} \wedge d\overline{z_1}_{|p} \cdots \wedge d\overline{z_n}_{|p}$$

for any $p, q \in U, l \in \mathbb{N}$.

Proof. Let us fix $j \in \mathbb{N}$, $p \in D_j$ and a holomorphic chart (U, Θ) such that $p \in U$. The inequality in the statement of the lemma is true if $k_{\mu_m}^{\Theta,\Theta}(p,p) = 0$ (i.e., $K_{D_m,\,\mu_m}(p,p) = 0$). Then suppose that $k_{\mu_m}^{\Theta,\Theta}(p,p) > 0$. In the proof we will use the simple remark that by (6)

$$\frac{1}{k_{\mu_j}(p,p)} = (-1)^{n^2/2} \int_{D_j} \mu_j \frac{e_{\mu_j,\Theta,p}}{k_{\mu_j}^{\Theta,\Theta}(p,p)} \wedge \frac{\overline{e_{\mu_j,\Theta,p}}}{k_{\mu_j}^{\Theta,\Theta}(p,p)} dV = \left|\left| \frac{e_{\mu_j,\Theta,p}}{k_{\mu_j}^{\Theta,\Theta}(p,p)} \right|\right|_{\mu_j}^2$$

since $k_{\mu_j}^{\Theta,\Theta}(p,p) > 0$ ($K_{D_j,\,\mu_j}(p,p) \neq 0$). Moreover the term $\dfrac{e_{\mu_j,\Theta,p}}{k_{\mu_j}^{\Theta,\Theta}(p,p)}$ is the only element in the class $\{f \in L^2H(\Lambda^{n,0}T^\star D_j, \mu_j), \mathcal{E}_{\Theta,p}(f) = 1\}$ (see (6)) with the minimal norm. Thus for $m \geq N(j)$ we have

$$\begin{aligned}
\frac{1}{k_{\mu_j}^{\Theta,\Theta}(p,p)} \leq \left|\left| \frac{e_{\mu_m,\Theta,p}}{k_{\mu_m}^{\Theta,\Theta}(p,p)} \right|\right|_{\mu_j}^2 &= (-1)^{n^2/2} \int_{D_j} \mu_j \frac{e_{\mu_j,\Theta,p}}{k_{\mu_j}^{\Theta,\Theta}(p,p)} \wedge \frac{\overline{e_{\mu_j,\Theta,p}}}{k_{\mu_j}^{\Theta,\Theta}(p,p)} dV \\
&\leq (-1)^{n^2/2} \int_{D_m} \mu_m \frac{e_{\mu_m,\Theta,p}}{k_{\mu_m}^{\Theta,\Theta}(p,p)} \wedge \frac{\overline{e_{\mu_m,\Theta,p}}}{k_{\mu_m}^{\Theta,\Theta}(p,p)} dV \\
&= \left|\left| \frac{e_{\mu_m,\Theta,p}}{k_{\mu_m}^{\Theta,\Theta}(p,p)} \right|\right|_{\mu_m}^2 = \frac{1}{k_{\mu_m}^{\Theta,\Theta}(p,p)}.
\end{aligned}$$

□

Remark 6. One can show similarly that $k_{\mu_j}^{\Theta,\Theta}(p,p) \geq k_{\mu}^{\Theta,\Theta}(p,p)$ for $j \in \mathbb{N}$.

Lemma 7 (Uniqueness of the limit). *If* $\lim\limits_{j\to\infty} K_{D_j,\,\mu_j} = K$ *locally uniformly on* $D \times D$, *then* $k = K_{D,\,\mu}$.

Proof. Since the sequence $(K_{D_j,\,\mu_j})$ converges locally uniformly on $D \times D$ and any $2n$-form $K_{D_j,\,\mu_j}$ is continuous we obtain that K is continuous (and then Lebesgue measurable) on $D \times D$. Fix a holomorphic chart $(U, \Theta = (z_1, \ldots, z_n))$ and $p \in U$ such that $U \subset D_j$ for some $j \in \mathbb{N}$. Then the $(n,0)$-forms $e_{\mu_m,\Theta,p}$ for $U \subset D_m$ and the $(n,0)$-form $e_{\mu,\Theta,p}$ are given by the inner products ($m \geq N(j)$)

$$e_{\mu_m,\Theta,p} = \left(\frac{\partial}{\partial \overline{z_1}_{|p}} \wedge \cdots \wedge \frac{\partial}{\partial \overline{z_n}_{|p}} \right) \lrcorner K_{D_m,\mu_m}$$

and

$$e_{\mu,\Theta,p} = \left(\frac{\partial}{\partial \overline{z_1}_{|p}} \wedge \cdots \wedge \frac{\partial}{\partial \overline{z_n}_{|p}}\right) \lrcorner K_{D,\mu}.$$

Denote $e_{\Theta,p} := \left(\frac{\partial}{\partial \overline{z_1}_{|p}} \wedge \cdots \wedge \frac{\partial}{\partial \overline{z_n}_{|p}}\right) \lrcorner K$. Then by Fatou's Lemma

$$(-1)^{n^2/2} \int_{D_j} \mu(z) e_{\Theta,p} \wedge \overline{e_{\Theta,p}} dV \leq \liminf_{m\to\infty} (-1)^{n^2/2} \int_{D_m} \mu_m e_{\mu_m,\Theta,p} \wedge \overline{e_{\mu_m,\Theta,p}} dV$$
$$= \liminf_{m\to\infty} k^{\Theta,\Theta}_{\mu_m}(p,p) = k^{\Theta,\Theta}(p,p).$$

where $K(p,p) = k^{\Theta,\Theta}(p,p) d\overline{z_1}_{|p} \cdots \wedge d\overline{z_n}_{|p} \wedge dz_{1|p} \cdots \wedge dz_{n|p}$. Since there exists an increasing sequence $\{D\}_{i_k}$ such that $D = \bigcup_{k=1}^{\infty} D_{i_k}$ we obtain

$$(-1)^{n^2/2} \int_{D_j} \mu e_{\Theta,p} \wedge \overline{e_{\Theta,p}} = \lim_{k\to\infty} (-1)^{n^2/2} \int_{D_{i_k}} \mu(z) e_{\Theta,p} \wedge \overline{e_{\Theta,p}} \leq k^{\Theta,\Theta}(p,p) < \infty. \tag{9}$$

It means that $e_{\Theta,p} \in L^2(\Lambda^{n,0}T^\star D, \mu)$. On the other hand, by the Weierstrass theorem $e_{\Theta,p}$ is a holomorphic $(n,0)$-form on D. Hence $e_{\Theta,p} \in L^2H(\Lambda^{n,0}T^\star D, \mu)$. By Lemma 5 we get

$$k^{\Theta,\Theta}_{\mu_j}(p,p) \geq k^{\Theta,\Theta}_{\mu}(p,p), \;\; p \in D_j.$$

In the limit $j \to \infty$ we obtain

$$k^{\Theta,\Theta}(p,p) \geq k^{\Theta,\Theta}_{\mu}(p,p), p \in D_j, \;\; p \in D. \tag{10}$$

It suffices to show that for any holomorphic charts $(U, \Theta = (z_1, \ldots, z_n))$ and $(W, \Psi = (t_1, \ldots, t_n))$ if $K_{|U\times W} = k^{\Theta,\Psi} d\overline{t_1}_{|p} \cdots \wedge d\overline{t_n}_{|p} \wedge dz_{1|p} \cdots \wedge dz_{n|p}$ and $K_{D,\mu} = k^{\Theta,\Psi} d\overline{t_1}_{|p} \cdots \wedge d\overline{t_n}_{|p} \wedge dz_{1|p} \ldots \wedge dz_{n|p}$ then for any $p \in U, q \in W$ $k^{\Theta,\Psi}(p,q) = k^{\Theta,\Psi}_{\mu}(p,q)$. We should consider two cases:

1. $K_{D,\mu}(p,p) = 0$, for some $p \in D$.

For a chart (U,Θ) s.t. $p \in U$

$$= k^{\Theta,\Psi}_{\mu}(p,p) = ||e_{\mu,\Theta,p}||^2_{\mu} = (-1)^{n^2/2} \int_D \mu e_{\mu,\Theta,p} \wedge \overline{e_{\mu,\Theta,p}} = 0$$

which implies that $e_{\mu,\Theta,p} = 0$ and by (8) $k^{\Theta,\Psi}_{\mu}(p,q) = 0$ for any $q \in D$ and any chart Ψ defined in a neighborhood of q. Then $K_{D,\mu}(p,\cdot) = 0$ and for any $f \in L^2H(\Lambda^{n,0}T^\star D, \mu)$

$$f(p) = (-1)^{n^2/2} \int_D \mu K_{D,\mu}(p,\cdot) \wedge f = 0$$

(see (5)). In particular $e_{\Theta,p}(p) = 0$ which gives $k^{\Theta,\Theta}(p,p) = 0$. But by (9) $||e_{\Theta,p}||^2_{\mu} = 0$ and therefore

$$e_{\Theta,p}(q) = k^{\Theta,\Psi}(p,q) dt_{1|q} \cdots \wedge dt_{n|q} = 0$$

for any $q \in W$. This implies $k^{\Psi,\Theta}(p,q) = 0 = k^{\Psi,\Theta}_{\mu}(p,q)$.

2. $K_{D,\mu}(p,p) \neq 0$ (i.e., $k^{\Theta,\Theta}_{\mu}(p,p) > 0$), for some $p \in D$.

By (10) we obtain that $k^{\Theta,\Theta}(p,p) > 0$. We know that the $(n,0)$-form

$$\frac{e_{\mu,\Theta,p}}{k_\mu^{\Theta,\Theta}(p,p)}$$

is the only element in the set

$$E_p = \{f \in L^2H(\Lambda^{n,0}T^\star D,\mu), \mathcal{E}_{\Theta,p}(f) = 1\}$$

which minimizes the norm $||\cdot||_\mu$. Since the $(n,0)$-form $\frac{e_{\Theta,p}}{k^{\Theta,\Theta}(p,p)} \in E_p$ we have by (10) that

$$||e_{\Theta,p}||_\mu \le \sqrt{k^{\Theta,\Theta}(p,p)}.$$

Hence (again by (10))

$$\begin{aligned}\left|\left|\frac{e_{\Theta,p}}{k^{\Theta,\Theta}(p,p)}\right|\right|_\mu &\le \frac{\sqrt{k^{\Theta,\Theta}(p,p)}}{k^{\Theta,\Theta}(p,p)} = \frac{1}{\sqrt{k^{\Theta,\Theta}(p,p)}}\\ &\le \frac{1}{\sqrt{k_\mu^{\Theta,\Theta}(p,p)}} = \left|\left|\frac{e_{\mu,\Theta,p}}{k_\mu^{\Theta,\Theta}(p,p)}\right|\right|_\mu.\end{aligned}$$

So $k^{\Theta,\Theta}(p,p) = k_\mu^{\Theta,\Theta}(p,p)$ and $e_{\Theta,p} = e_{\mu,\Theta,p}$ which implies that for any $q \in D$

$$K(p,q) = K_{D,\,\mu}(p,q).$$ □

Proof of the main theorem. We will show that for $j \in \mathbb{N}$, $\{K_{D_m,\,\mu_m}\}_{m\ge N(j)}$ is locally bounded on $D_j \times D_j$. Choose two holomorphic atlases $\mathcal{A}' = \{(U'_\alpha, \Theta'_\alpha)\}_{\alpha\in A}$ and $\{(U_\alpha, \Theta_\alpha)\}_{\alpha\in A}$ such that for any $\alpha \in A$ $U_\alpha \subset U'_\alpha$ and the closure $cl(U_\alpha)$ in M is a compact subset of U'_α. Then for any $\alpha, \beta \in A$ and any $p \in U_\alpha$ and $q \in U_\beta$ we obtain

$$\begin{aligned}|k_{\mu_m}^{\Theta_\alpha,\Theta_\beta}(p,q)| &\le \sqrt{k_{\mu_m}^{\Theta_\alpha,\Theta_\alpha}(p,p)}\sqrt{k_{\mu_m}^{\Theta_\beta,\Theta_\beta}(q,q)}\\ &\le \sqrt{k_{\mu_j}^{\Theta_\alpha,\Theta_\alpha}(p,p)}\sqrt{k_{\mu_j}^{\Theta_\beta,\Theta_\beta}(q,q)} \le m_\alpha m_\beta,\end{aligned}$$

where $m_\gamma := \max_{s\in cl(U_\gamma)} \sqrt{k_{\mu_j}^{\Theta_\gamma,\Theta_\gamma}(s,s)}$ (see (8) and Lemma 5). By Montel's property any subsequence of $\{K_{D_m,\,\mu_m}\}$ has a subsequence convergent locally uniformly on $D \times D$. By Lemma (7) the limit does not depend on a subsequence and is identically equal to $K_{D,\,\mu}$. Thus

$$\lim_{m\to\infty} K_{D_m,\,\mu_m} = K_{D,\,\mu}$$

locally uniformly on $D \times D$. □

Remark 8. Look that by means of Theorem 4 we may consider the limit behavior of the so-called Skwarczyński distance (which is biholomorphically invariant).

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Zbigniew Pasternak-Winiarski and Paweł M. Wójcicki
Faculty of Mathematics and Information Science
Warsaw University of Technology
Koszykowa 75
PL-00-662 Warsaw, Poland
e-mail: z.pasternak-winiarski@mini.pw.edu.pl
p.wojcicki@mini.pw.edu.pl

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A Characterization of Domains of Holomorphy by Means of Their Weighted Skwarczyński Distance

Zbigniew Pasternak-Winiarski and Paweł M. Wójcicki

Abstract. M. Skwarczyński (†) introduced pseudodistance on domains in $\mathbb{C}^n$ which under some conditions (if the domain is bounded for instance) gives rise to biholomorphically invariant distance, i.e., invariant under biholomorphic transformations. One can find a proof that completeness with respect to Skwarczyński distance implies completeness with respect to Bergman distance, which implies that the considered domain is a domain of holomorphy. In this paper we give a characterization of domains of holomorphy with the help of a weighted version of Skwarczyński pseudodistance. We will work with a special kind of weights, called "admissible weights". Midway, we obtain a new proof (even in the unweighted case) of the theorem that the so-called Kobayashi condition implies Bergman completeness, which may be helpful in answering the (open) question if Bergman completeness and Skwarczyśnki completeness are equivalent or not.

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The Bergman kernel [1, 3, 5, 6, 14, 17] has become a very important tool in geometric function theory, both in one and several complex variables. It turned out that not only classical Bergman kernel, but also weighted one can be useful, particularly from the quantum theory point of view (look at Ref. [7]; see also Introduction of the paper [12] in this volume). Using the theory of Bergman kernels, Maciej Skwarczyński introduced pseudodistance on domains in $\mathbb{C}^n$, which under some conditions (if the domain is bounded for instance) give rise to a distance invariant under biholomorphic transformations which preserve the weight of integration. Explicitly:

$$\varrho_{D,\mu}(p,q) = \left(1 - \frac{|K_{D,\mu}(p,q)|}{\sqrt{K_{D,\mu}(p,p)}\sqrt{K_{D,\mu}(q,q)}}\right)^{1/2},$$

where $K_{D,\mu}$ is a weighted Bergman kernel of a given domain $D \subset \mathbb{C}^n$, and μ is admissible weight on D, $\mu \in AW(D)$ for short (we assume that $K_{D,\mu}(z,z) > 0$ for all $z \in D$, for example, all bounded domains share this property).

We shall study the limit behavior of the weighted Skwarczyński pseudodistance using the results obtained in [11, 12, 18]. We will give a weighted analog of the fact that completeness with respect to Skwarczyński distance implies completeness with respect to Bergman distance, which implies that the considered domain is a domain of holomorphy [3, Th. 12.9.6; 15.1.1]. We shall give conditions implying that the considered domain is complete with respect to weighted Skwarczyński pseudodistance (idea is taken from [4, 15]). Thus we will give a characterization of domains of holomorphy by means of their weighted Skwarczyński distance. We get a new proof of the theorem that Kobayashi condition implies Bergman completeness (look at [4] for the original proof).

1. Definitions and notations

Let $D \subset \mathbb{C}^N$ be a domain, and let $W(D)$ be the set of weights on D, i.e., $W(D)$ is the set of all Lebesgue measurable, real-valued, positive functions on D (we consider two weights as equivalent if they are equal almost everywhere with respect to the Lebesgue measure on D). If $\mu \in W(D)$, we denote by $L^2(D,\mu)$ the space of all Lebesgue measurable, complex-valued, μ-square integrable functions on D, equipped with the norm $||\cdot||_{D,\mu} := ||\cdot||_\mu$ given by the scalar product

$$\langle f|g\rangle_\mu := \int_D f(z)\overline{g(z)}\mu(z)d^{2n}z, \quad f,g \in L^2(D,\mu).$$

The space $L^2H(D,\mu) = H(D) \cap L^2(D,\mu)$ is called the **weighted Bergman space**, where $H(D)$ stands for the space of all holomorphic functions on the domain D. For any $z \in D$ we define the evaluation functional E_z on $L^2H(D,\mu)$ by the formula

$$E_z f := f(z), \quad f \in L^2H(D,\mu).$$

Let us recall the definition [Def. 2.1] of admissible weight given in [10].

Definition 1 (Admissible weight). A weight $\mu \in W(D)$ is called an *admissible weight*, an a-weight for short, if $L^2H(D,\mu)$ is a closed subspace of $L^2(D,\mu)$ and for any $z \in D$ the evaluation functional E_z is continuous on $L^2H(D,\mu)$. The set of all a-weights on D will be denoted by $AW(D)$.

For closed subspace $L^2H(D,\mu)$ an orthogonal projector $P_\mu : L^2(D,\mu) \to L^2H(D,\mu)$ is well defined. Continuity of E_z provides the existence and uniqueness of the weighted Bergman kernel. In [9] the concept of an a-weight was introduced, and in [10] several theorems concerning admissible weights are given. An illustrative one is:

Theorem 2 (10, Cor. 3.1]). *Let $\mu \in W(D)$. If the function μ^{-a} is locally integrable on D for some $a > 0$ then $\mu \in AW(D)$.*

Now, let us fix a point $t \in D$ and minimize the norm $||f||_\mu$ in the class $E_t = \{f \in L^2H(D,\mu); f(t) = 1\}$. It can be proved in a similar way as in the classical case, that if μ is an admissible weight then there exists exactly one function minimizing the norm. Let us denote it by $\phi_\mu(z,t)$.

The weighted Bergman kernel function $K_{D,\mu}$ is defined as follows:

$$K_{D,\mu}(z,t) = \frac{\phi_\mu(z,t)}{||\phi_\mu||_\mu^2}.$$

Let $(H, (\cdot,\cdot))$ be an arbitrary Hilbert space. Let us consider the following relation between non-zero elements: $x \sim y$ if and only if there exists a complex constant $c \neq 0$ such that $x = cy$. The set of equivalence classes forms (generally infinitely dimensional) projective space $P(H)$. This is a complete metric space with respect to the distance

$$d_H([x],[y]) = \mathrm{dist}([x] \cap S_H, [y] \cap S_H),$$

where $S_H \subset H$ is the unit sphere. Explicitly

$$\begin{aligned} d_H^2([x],[y]) &= \inf_{\varphi,\,\psi \in [0;\,2\pi]} \left|\left| \frac{e^{i\varphi}x}{||x||} - \frac{e^{i\psi}y}{||y||} \right|\right|^2 \\ &= \inf_{\varphi,\,\psi \in [0;\,2\pi]} \left[2 - 2\mathrm{Re}\, \frac{e^{i(\varphi-\psi)}(x,y)}{||x||||y||} \right] \\ &= 2 - 2\left[\frac{(x,y)(y,x)}{(x,x)(y,y)} \right]^{1/2}. \end{aligned}$$

Using this fact M. Skwarczyński introduced in [15, p. 20] a pseudodistance on domains in $\mathbb{C}^n$. We will build an analogue of this for weighted Bergman kernels. Let $D \subset \mathbb{C}^n$ be a domain such that $K_{D,\mu}(z,z)$ does not vanish at any point $z \in D$ and $\mu \in AW(D)$. Define the map $\tau : D \to P(L^2H(D,\mu))$ by the formula

$$\tau(z) := [K_{D,\mu}(\cdot, z)].$$

It enables us to introduce the following continuous pseudodistance on $D \times D$:

$$\begin{aligned} \varrho_{D,\mu}(p,q) &:= \frac{1}{\sqrt{2}}\, d_{L^2H(D,\mu)}(\tau(p),\tau(q)) \\ &= \left(1 - \frac{|K_{D,\mu}(p,q)|}{\sqrt{K_{D,\mu}(p,p)}\sqrt{K_{D,\mu}(q,q)}} \right)^{1/2} \end{aligned} \tag{1}$$

$\varrho_{D,1}$ is called the *Skwarczyński pseudodistance.*

Remark 3. Observe that the following conditions are equivalent:

(a) τ is injective;
(b) for each two distinct points $p, q \in D$ the functions $K_{D,\mu}(\cdot,p)$, $K_{D,\mu}(\cdot,q)$ are linearly independent;
(c) $\varrho_{D,\mu}$ is a distance.

If D is bounded for instance, then $\varrho_{D,\,\mu}$ is a distance. From now on, we assume that D is bounded. It is easily verified that every linear isometry between two Hilbert spaces

$$L : H \to \widetilde{H}$$

induces an isometry

$$L : P(H) \to P(\widetilde{H})$$

given by the formula $L([h]) = [L(h)]$. Let $\varphi : D \to \widetilde{D}$ be a biholomorphic mapping and $\mu \in AW(D)$, $\widetilde{\mu} \in AW(\widetilde{D})$. Then the map $L : L^2(\widetilde{D},\, \widetilde{\mu}) \to L^2(D,\, \mu)$ defined by the formula

$$Lf = (f \circ \varphi) \det \varphi'$$

is the isometry if only $\widetilde{\mu} \circ \varphi = \mu$. Note that $L(L^2H(\widetilde{D}, \widetilde{\mu})) = L^2H(D,\, \mu)$. The rule of transformation of the weighted Bergman function [2, Lemma 1] can be written in the form

$$K_{D,\,\mu}(\cdot, p) = K_{\widetilde{D},\,\widetilde{\mu}}(\cdot, \varphi(p)) \circ \varphi(\cdot) \det \varphi'(\cdot)\overline{\det \varphi'(p)},\ p \in D. \tag{2}$$

Therefore $K_{D,\,\mu}(\cdot, p) = \mathrm{const} \cdot LK_{\widetilde{D},\,\widetilde{\mu}}(\cdot, \varphi(p))$, so $[K_{D,\,\mu}(\cdot, p)] = L[K_{\widetilde{D},\,\widetilde{\mu}}(\cdot, \varphi(p))]$. We have

$$\begin{aligned} \sqrt{2}\varrho_{D,\,\mu}(p, q) &= d_{L^2H(D,\,\mu)}([K_{D,\,\mu}(\cdot, p)], [K_{D,\,\mu}(\cdot, q)]) \\ &= d_{L^2H(D,\,\mu)}(L[K_{\widetilde{D},\,\widetilde{\mu}}(\cdot, \varphi(p))], L[K_{\widetilde{D},\,\widetilde{\mu}}(\cdot, \varphi(q))]) \\ &= d_{L^2H(\widetilde{D},\,\widetilde{\mu})}([K_{\widetilde{D},\,\widetilde{\mu}}(\cdot, \varphi(p))], [K_{\widetilde{D},\,\widetilde{\mu}}(\cdot, \varphi(q))] \\ &= \sqrt{2}\varrho_{\widetilde{D},\,\widetilde{\mu}}(\varphi(p), \varphi(q)). \end{aligned}$$

Thus $\varrho_{D,\,\mu}$ is invariant under biholomorphic transformations if $\widetilde{\mu} \circ \psi = \mu$ (this can be actually shown directly from formula (1) by the direct use of (2)).

Example. For the unit disc $D \subset \mathbb{C}$

$$\varrho_{D,\,1}(p, q) = \left| \frac{p - q}{1 - p\bar{q}} \right|.$$

is called the pseudohyperbolic metric (see [6]).

2. Skwarczyński distance and domains of holomorphy

From now on, we assume that $\mu \in L^1(D) \cap AW(D)$. The weighted Bergman kernel $K_{D,\,\mu}$ leads to the following positive semidefinite Hermitian form

$$B_{D,\,\mu}(z; X) := \sum_{\lambda,\,\nu=1}^{n} \frac{\partial^2}{\partial z_\lambda \partial \overline{z}_\nu} \log K_{D,\,\mu}(z, z) X_\lambda \overline{X}_\mu, \quad z \in D,\ X \in \mathbb{C}^n.$$

We will assume then (similarly as for D-bounded and $\mu \equiv 1$) $B_{D,\,\mu}$ is positive defined (look in Ref. [2]). The Riemannian metric $\beta_{D,\,\mu}$ induced by $B_{D,\,\mu}$ is called the *weighted Bergman metric* on D. Let $b_{D,\,\mu}$ be the distance defined by $\beta_{D,\,\mu}$ on D. Then $b_{D,\,\mu}$ is called the *weighted Bergman distance on D*.

In [3, Th. 15.1.1] one could find the proof of the fact that any bounded Bergman complete (b-complete for short) domain D is pseudoconvex. Now we will state the same theorem for weighted version of Bergman distance.

Theorem 4. *If D is $b_{D,\mu}$-complete then it is pseudoconvex.*

Proof. Suppose the contrary. Denote $B = B(0,1)$ – the unit Euclidean disc in $\mathbb{C}$. There exist a polydiscs $\Delta = z_0 + \epsilon B^n \subset D$ and $\widetilde{\Delta} = z_0 + RB^n \not\subset D$, $R > \epsilon$, such that for every $f \in H(D)$ the restriction $f\,|_\Delta$ extends holomorphically to $\widetilde{\Delta}$. In particular, by Hartog's theorem, there exists a function $F : \widetilde{\Delta} \times \widetilde{\Delta} \to \mathbb{C}$ satisfying:

(1) $F\,|_{\Delta\times\Delta} = K_{D,\mu}\,|_{\Delta\times\Delta}$,

(2) $F(z,\bar{w})$ is holomorphic on $\widetilde{\Delta} \times \widetilde{\Delta}'$, where $\widetilde{\Delta}' = \{w \in \mathbb{C}^n : \overline{w} \in \widetilde{\Delta}\}$.

By hypothesis there exists $z' \in \widetilde{\Delta} \cap \partial D$ such that the segment $[z_0, z') \subset D \cap \widetilde{\Delta}$. By (1) $\log K_{D,\mu}(z,z) = \log F(z,z)$ near $[z_0, z')$, it follows that any sequence $(z_j)_{j\in\mathbb{N}} \subset [z_0, z')$ such that $\lim_{j\to\infty} z_j = z'$ is a $b_{D,\mu}$-Cauchy sequence, so $z' \in D$; a contradiction. □

Denote by $\mathfrak{D}(D)$ the family of all pseudodistances $\rho : D \times D \to \mathbb{R}_+$ such that

$$\forall a \in D\ \exists M,\, r > 0 : \rho(p,q) \le M\|p-q\|, \quad p,\, q \in B(a,r) \subset D.$$

Now we do define operators ^{i} and $\mathcal{D}$ ([3, p. 140]), by means of which we will establish some inequality between $\varrho_{D,\mu}$ and $b_{D,\mu}$.

Operator $\rho \to \rho^i$. Let $\rho \in \mathfrak{D}(D)$ and let $\alpha : [0,1] \to D$ be a continuous curve. Let

$$L_\rho(\alpha) := \sup\left\{ \sum_{j=1}^{N} \rho(\alpha(t_{j-1}), \alpha(t_j)); \quad N \in \mathbb{N},\, 0 = t_0 < t_1 < \cdots < t_N = 1 \right\}.$$

The number $L_\rho(\alpha) \in [0,\infty]$ is called ρ-length of α. If $L_\rho(\alpha) < \infty$, then we say that α is ρ-rectifiable. Define:

$$\rho^i(p,q) := \inf\{L_\rho(\alpha);\ \alpha \text{ is } \|\cdot\|\text{-rectifiable curve in } D \text{ joining } p \text{ and } q,\ p,q \in D\}.$$

Obviously, $\rho \le \rho^i$.

Remark 5. [3, Prop. 4.3.1] $\|\cdot\|$-rectifiable curve is ρ-rectifiable.

Operator $\rho \to \mathcal{D}\rho$**.** Let $\rho \in \mathfrak{D}(D)$. Define

$$(\mathcal{D}\rho)(a;X) := \limsup_{\lambda\nrightarrow 0, z\to a} \frac{1}{|\lambda|}\rho(z, z+\lambda X), \quad a \in D,\ X \in \mathbb{C}^n.$$

Proposition 6 ([3, Rem. 4.3.8; Prop. 4.3.9]). *If $\rho \in \mathfrak{D}(D)$, is a C^1-pseudodistance, then*

1. $\rho^i(a,b) = \int(\mathcal{D}\rho) = \inf\{\int_C (\mathcal{D}\rho)(c(t), \dot{c}(t))dt;\ c \in C^1([0,1], D),$ $c(0) = a, c(1) = b\}$
2. $(\mathcal{D}\rho)(a;X) = \lim\limits_{p,\,q\to a, p\neq q} \dfrac{\rho(p,\,q)}{\|p-q\|}, \quad a \in D,\ \|X\| = 1$ *and* $\dfrac{p-q}{\|p-q\|} \to X.$

Theorem 7 ([3, Th. 12.9.6]). $\varrho^i_{D,\mu} = (1/\sqrt{2})b_{D,\mu}.$

Proof. According to Proposition 6 we have to only make sure that $\varrho_{D,\,\mu}$ is a C^1-pseudodistance with $D\varrho_{D,\,\mu} = (1/\sqrt{2})\beta_{D,\,\mu}$. So, fix $w \in D$ and take

$$(p_n)_{n\in\mathbb{N}}, (q_m)_{\in\mathbb{N}} \subset D$$

with

$$\lim_{n\to\infty} p_n = \lim_{m\to\infty} q_m = w,\ p_n \neq q_n \quad \text{for all } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} (p_n - q_n)/\|p_n - q_n\| =: X \in \partial B_n,$$

where B_n is the unit Euclidean ball in $\mathbb{C}^n$. We have:

$$\varrho_{D,\,\mu}(p_n, q_n) = \left(1 - \left(\frac{K_{D,\,\mu}(p_n, q_n)K_{D,\,\mu}(q_n, p_n)}{K_{D,\,\mu}(p_n, p_n)K_{D,\,\mu}(q_n, q_n)}\right)^{1/2}\right)^{1/2}$$

$$= \left(\frac{\sqrt{K_{D,\,\mu}(p_n, p_n)}\sqrt{K_{D,\,\mu}(q_n, q_n)} - \sqrt{K_{D,\,\mu}(p_n, q_n)}\sqrt{K_{D,\,\mu}(q_n, p_n)}}{\sqrt{K_{D,\,\mu}(p_n, p_n)}\sqrt{K_{D,\,\mu}(q_n, q_n)}}\right)^{1/2}$$

$$= \left(\frac{\Phi(p_n, q_n)}{\sqrt{K_{D,\,\mu}(p_n, p_n)}\sqrt{K_{D,\,\mu}(q_n, q_n)}}\right)^{1/2}$$

$$\cdot \left(\frac{1}{\sqrt{K_{D,\,\mu}(p_n, p_n)}\sqrt{K_{D,\,\mu}(q_n, q_n)} + |K_{D,\,\mu}(p_n, q_n)|}\right)^{1/2}.$$

Here $\Phi(p, q) := K_{D,\,\mu}(p, p)K_{D,\,\mu}(q, q) - K_{D,\,\mu}(p, q)K_{D,\,\mu}(q, p)$, $p, q \in D$, is a C^∞-function with $\Phi \geq 0$ and $\Phi(p, p) = 0, p \in D$. Applying the Taylor formula for $\Phi(\cdot, q_n)$ up to second order and the holomorphicity properties of the weighted Bergman kernel one gets

$$\lim_{n\to\infty} \frac{\varrho_{D,\,\mu}(p_n, q_n)}{\|p_n - q_n\|} = \frac{1}{2K^2_{D,\,\mu}(w, w)}\left(\sum_{\lambda,\,\nu=1}^{n}\left(\frac{\partial^2 K_{D,\,\mu}}{\partial z_\lambda \partial \overline{z}_\nu}(w, w)K_{D,\,\mu}(w, w) - \frac{\partial K_{D,\,\mu}}{\partial z_\lambda}(w, w)\frac{\partial K_{D,\,\mu}}{\partial \overline{z}_\nu}(w, w)\right)X_\lambda \overline{X}_\nu\right)^{1/2}$$

$$= (1/2B_{D,\,\mu}(w; X))^{1/2}.$$

□

Thus we have $\sqrt{2}\varrho_{D,\,\mu} \leq \sqrt{2}\varrho^i_{D,\,\mu} = b_{D,\,\mu}$.

3. Completeness with respect to the weighted Bergman distance

Following the ideas of [4] and particularly [15] we can study the completeness with respect to the weighted invariant distance.

Theorem 8. *A sequence $p_m \in D$, $m = 1, 2, \ldots$, is Cauchy with respect to the distance $\varrho_{D,\,\mu}$ if and only if the sequence $\tau(p_m)$ is Cauchy in $P(L^2H(D, \mu))$.*

Proof. It is immediate consequence of the definition. □

For simplicity we restrict our remarks to bounded domains.

Theorem 9. *A sequence* $p_m \in D,\ m = 1, 2, \ldots,$ *is Cauchy with respect to* $\rho_{D,\,\mu}$ *if and only if there exists an* $f \in L^2H(D,\,\mu)$ *such that* $||f||_\mu = 1$ *and*

$$\lim_{m\to\infty} \frac{|f(p_m)|^2}{K_{D,\,\mu}(p_m, p_m)} = 1. \tag{3}$$

Proof. By the previous theorem, a sequence p_m is Cauchy in D if and only if $\tau(p_m)$ is Cauchy in $P(L^2H(D,\,\mu))$. By completeness of $P(L^2H(D,\,\mu))$, $\{\tau(p_m)\}$ converges to some $[f]$. We may assume that $||f||_\mu = 1$. Now, by the definition we can easily observe that (3) holds iff

$$\lim_{m\to\infty} d_{L^2H(\widetilde{D},\,\widetilde{\mu})}(\tau(p_m), [f]) = 0. \qquad \square$$

Theorem 10. *The Euclidean distance and the weighted distance* $\varrho_{D,\,\mu}$ *induce the same topology in* $D \subset \mathbb{C}^n$*.*

Proof. Assume that $p_j \in D$ converges to $p \in D$ in the Euclidean norm. Then $\lim_{j\to\infty} \varrho_{D,\mu}(p_j,\, p) = 0$ since the weighted Bergman function is continuous. Conversely, $\lim_{j\to\infty} \varrho_{D,\mu}(p_j,\, p) = 0$ implies that

$$\lim_{j\to\infty} \left|\left| \frac{e^{i\theta_j} K_{D,\,\mu}(\cdot, p_j)}{\sqrt{K_{D,\,\mu}(p_j, p_j)}} - \frac{K_{D,\,\mu}(\cdot, p)}{\sqrt{K_{D,\,\mu}(p, p)}} \right|\right|_\mu = 0.$$

Thus, there exist constants $c_j \neq 0,\ j = 1, 2, \ldots,$ such that
$c_j K_{D,\,\mu}(\cdot, p_j) \xrightarrow{\mu} K_{D,\,\mu}(\cdot, p)$. Since $1 \in L^2H(D,\,\mu)$,

$$\lim_{j\to\infty} \overline{c}_j = \lim_{j\to\infty} (1, c_j K_{D,\,\mu}(\cdot, p_j))_\mu = (1, K_{D,\,\mu}(\cdot, p))_\mu = 1.$$

Let π_k denote the kth coordinate function. We have

$$\begin{aligned}\lim_{j\to\infty} \pi_k(p_j) &= \lim_{j\to\infty} (\pi_k(\cdot), K_{D,\,\mu}(\cdot, p_j))_\mu = \lim_{j\to\infty} \frac{1}{\overline{c}_j} (\pi_k(\cdot), c_j K_{D,\,\mu}(\cdot, p_j))_\mu \\ &= (\pi_k(\cdot), K_{D,\,\mu}(\cdot, p))_\mu = \pi_k(p), \quad \text{i.e.,} \quad \lim_{j\to\infty} p_j = p\,. \qquad \square\end{aligned}$$

Hence, the two topologies coincide. In other words, $\varrho_{D,\,\mu}$-completeness implies $b_{D,\,\mu}$-completeness (additionally, we will prove now that the so-called Kobayashi condition implies $\varrho_{D,\,\mu}$-completeness). So from Theorems 4 and 7 follows directly

Theorem 11. *Under assumptions from Theorem* 4*, any* $\varrho_{D,\,\mu}$*-complete domain is a domain of holomorphy.*

Theorem 12 (Weighted version of Kobayashi Th.). *Assume that for every sequence* $p_m \in D$ *without accumulation point in* D *and for every* $f \in L^2H(D,\,\mu)$*,*

$$\lim_{m\to\infty} \frac{|f(p_m)|^2}{K_{D,\,\mu}(p_m, p_m)} = 0. \tag{4}$$

Then D *is* $\varrho_{D,\,\mu}$*-complete, thus is a domain of holomorphy.*

Proof. Suppose that $p_m \in D$ is a Cauchy sequence without limit in D. Thus p_m has no accumulation point in D, and (4) holds. But (4) contradicts (3). Thus there is a limit point of p_m in D. □

Theorem 13. *The assumptions of Theorem* 12 *are equivalent to the following: for every point* $p \in D$ *and every sequence* $p_m,\ m = 1, 2, \dots,$ *without an accumulation point in* D,

$$\lim_{m\to\infty} \varrho_{D,\mu}(p, p_m) = 1. \tag{5}$$

Proof. Assume that for every sequence $\{p_m\} \in D$ without an accumulation point in D and for every $f \in L^2H(D, \mu)$ (4) holds. Take $f = K_{D,\mu}(\cdot, p)$ to get (5).

Conversely, if (5) holds then

$$\lim_{m\to\infty} \frac{|K_{D,\mu}(p_m, p)|^2}{K_{D,\mu}(p_m, p_m)} = 0.$$

Denote by $F_\mu \subset L^2H(D, \mu)$ the set of all f such that (4) holds. We will show that F_μ is a closed linear subspace of $L^2H(D, \mu)$. One can easily observe that F_μ is a linear subspace of $L^2H(D, \mu)$. We show that F_μ is closed. Let $\{f_j\}_{j=1}^{\infty}$, $f_j \in F_\mu$, be a sequence convergent to $f \in L^2H(D, \mu)$. So for any $\epsilon > 0$ there is a j_0 such that $\|f_{j_0} - f\| < \epsilon$, thus

$$\begin{aligned} |(f_{j_0} - f)(p_m)| &= \left| \int_D (f_{j_0} - f)(z)\overline{K_{D,\mu}(z, p_m)}\mu(z)dz \right| \\ &\le \|f_{j_0} - f\|_\mu \sqrt{\int_D |K_{D,\mu}(p_m, z)|^2 \mu(z)dz} \le \epsilon\sqrt{K_{D,\mu}(p_m, p_m)} \end{aligned}$$

for all $m \in \mathbb{N}$. Thus $|(f_{j_0} - f)(p_m)|^2 \le \epsilon^2 K_{D,\mu}(p_m, p_m)$. By (4) there is M such that

$$\frac{|f_{j_0}(p_m)|^2}{K_{D,\mu}(p_m, p_m)} \le \epsilon^2$$

for $m > M$. Thus

$$\frac{|f(p_m)|^2}{K_{D,\mu}(p_m, p_m)} \le \frac{2|(f_{j_0} - f)(p_m)|^2 + 2|f_{j_0}(p_m)|^2}{K_{D,\mu}(p_m, p_m)} \le 4\epsilon^2$$

for $m > M$. Thus $f \in F_\mu$. Since $\{K_{D,\mu}(\cdot, p)\}_{p\in D}$ is the linearly dense set in $L^2H(D, \mu)$, and $K_{D,\mu}(\cdot, p) \in F_\mu$ for all $p \in D$, $F_\mu = L^2H(D, \mu)$. □

Theorem 14. *Suppose that for each boundary point* $p \in \partial D$ *there exists a function* $h \in H(D)$ *s.t.*

(A1) $|h(z)| < 1$ *for* $z \in D$

(A2) $\lim_{z\to p} |h(z)| = 1$.

Then D *is* $\varrho_{D,\mu}$*-complete, thus is a domain of holomorphy.*

Proof. Let p_m be an arbitrary sequence without an accumulation point in D. We want to show that for every $f \in L^2H(D, \mu)$ (4) holds. We may assume that p_m converges to $p \in \partial D$. We may use Lebesgue Dominated Convergence Theorem to show that $\lim_{k\to\infty} ||h^k f||^2_\mu = 0$, so for each $\epsilon > 0$ there is $k \in \mathbb{N}$ s.t. $||h^k f||^2_\mu < \epsilon$. Assumption $(A2)$ implies that $(1-\epsilon) < |h^k(p_m)|^2$ for m large enough, and thus

$$(1-\epsilon)|f(p_m)|^2 \le |h^k(p_m)f(p_m)|^2 \le K_{D,\,\mu}(p_m, p_m)||h^k f||^2_\mu.$$

It follows that for sufficiently large m

$$\frac{|f(p_m)|^2}{K_{D,\,\mu}(p_m, p_m)} \le \frac{\epsilon}{1-\epsilon}.$$
□

Theorem 15. *Suppose that*

(S1) *for each boundary point $p \in \partial D$ $\lim_{z\to p} K_{D,\,\mu}(z, z) = \infty$.*

(S2) *The set of all functions bounded in a neighborhood of ∂D is linearly dense in $L^2H(D, \mu)$.*

Then D is $\varrho_{D,\,\mu}$-complete, thus is a domain of holomorphy.

Proof. As before, we shall show that condition (4) is satisfied, and we may assume that p_m converges to $p \in \partial D$. It is evident that the linearly dense set from $(S2)$ is a subset of F_μ of all f satisfying (4). Since F_μ is closed linear subspace, it follows that $F_\mu = L^2H(D, \mu)$. □

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Zbigniew Pasternak-Winiarski and Paweł M. Wójcicki
Faculty of Mathematics and Information Science
Warsaw University of Technology
Koszykowa 75
PL-00-662 Warsaw, Poland
e-mail: z.pasternak-winiarski@mini.pw.edu.pl
p.wojcicki@mini.pw.edu.pl

Part VII

Special Talk by Bogdan Mielnik

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Science and its Constraints (an unfinished story)

Bogdan Mielnik

Abstract. It is noticed that in the present day societies the progress of science is too dependent on the mass sociology. Some steps to moderate the phenomenon are briefly discussed.

The science in our world depended always on its economic, political or even religious trends. Enough to mention the Pythagoras theorem, which became the state secret in ancient mini totalitarian regimes. Then the polemics between the geocentric (Ptolemean) and heliocentric doctrines [1]. Also the cosmological ideas about the origin of our universe. One would like to think that the modern societies, at least in the democratic countries, created finally the conditions in which the science can develop without artificial barriers, but it is enough to contemplate our XX and XXI centuries to see that this opinion might be too optimistic.

As it seems, the relatively natural development of science occurred in the first part of XX c. thanks to spontaneous exchange of opinions in the historical congresses (Solvay and others). Their polemics, not limited by an undue ‘correctness’, were crucial for the advance of some (right or wrong) ideas.

So today, one can enjoy Einstein discovery of special relativity. Then his hypothesis about the light quanta, together with the persistent Millican attempts to defeat the quantum model, an effort which turned quite fertile – though not to reject but rather to understand the idea better [2].

We can also follow the early arguments of Pauli about the nonsense of the electron spin. Yet, after changing his opinion he proposed to describe it by 2×2 matrices, today called the *Pauli spin*.

In the next decades, the most fruitful relations (though not quite innocent!) between the scientists and politicians were typically due to the personal contacts between the scientists and government representatives, creating the support for military and technical projects.

For some time the publications of papers seemed to obey certain natural rules of *submissions, referee opinions and public discussions* described, e.g., in [3] Yet, the authors paint an almost ideal equilibrium between the individual effort and the public response. In reality, this equilibrium, since several decades, is in danger.

The Profession Expands
Already in 60s the number of scientists was relatively large and soon, the personal contacts became difficult, if not impossible. A class of intermediaries thus appeared and turned out to be increasingly powerful. The administrators, bureaucrats, etc. had to classify the scientists and their work without reading papers. The same about scientific projects. Part of the information could be deduced from some accessible numerical data, such as the number of publications, citations, graduations etc. The rest was left to various levels of referees, chosen by experience or just intuition of the Editors (and/or the Institute directors). Their opinions were quite important but not always correct. Still, the science was progressing reasonably well.

"Publish or Perish"
I did not succeed to check precisely at which moment appeared the famous slogan: "publish or perish". If you search in Google, you can see that this simplified rule is still quite alive. Some recent observations are made by David Colquhoun, neuroscientist from London, who discusses the problem in his (2014) article: "Publish-or-perish: Peer review and the corruption of science" [4]. The scientists, afraid of their reputation (or survival!), started to hurry up and the number of papers was increasing fast, at cost of their quality. The devastating stress to keep publishing affected many areas of science producing almost an ecological disaster.

The Referees Missing!
As an inevitable consequence, there was soon not enough specialists to evaluate the "productive avalanche". The authors of the medium quality papers had therefore to evaluate the other medium quality papers. Even this could not help. A lot of contributions which could be of interest if elaborated patiently, were not indeed completed, aborted rather than published. The masses of "fast papers" could not be consumed by "fast referees".

An exception among the "publication champions" was Peter Higgs, the Edinburgh professor (Nobel 2014). According to Dekka Aitkenhead report in The Guardian, (Dec. 6, 2013), Higgs confesses that before getting his Nobel Prize, he was an embarrassment to his department in all moments of the research reports. When asked about the list of his recent publications, he could only say: "None". He noticed: "Today, I wouldn't get an academic job. It's as simple as that". He also doubts that his breakthrough could be achieved in today's scientific culture, because of obligations to collaborate and keep churring out papers. "It is difficult to imagine how I would ever have enough of peace and quiet (...) to do what I did in 1964". Skeptical and unbeliever, he regrets that the particle he identified in 1964 is known as "God particle". In 1999 he turned down his knighthood, considering that too many honors are used for political purposes...

Trends and Mainstreams
Of course, the case of Higgs is an exception. What the organism of science must

assure is to create the majority of 'regular workers', i.e., the scientists whose destiny is not necessarily the Nobel prize, but who work in equilibrium between the formal obligations and free moments for some independent ideas. Unfortunately, this equilibrium is also in danger.

The reaction of the science organism were "trends", and "mainstreams". Associated with high reputation journals they create the international power groups. By a natural mechanism of sociology, the authors 'navigating' in the mainstreams discuss mostly with themselves, cite mostly works of their groups. The referees privilege the papers of their "friends in trends", sometimes converting themselves into the subconscious trend guardians, rejecting the outsiders or critical articles. This did not moderate, but focused only the publication efforts, an effect clearly observed in theoretical or mathematical physics, existing also in natural sciences.

As remembered by one colleague, in certain epoch it was rather difficult to publish a work in elementary particle physics without mentioning the Regge poles.

Later on, the similar "magic subject" became *strings*. In several scientific centers it became almost an obligation to cultivate the subject (even with a damage for the rest) "because it is the only game in town". After his 20 years of work on strings the known physicist Lee Smolin published the book "Trouble with physics" [5]. But the trouble persists. To mention that some of your results confirm (or seem to confirm) some string predictions immediately assures the positive interest of a large number of referees.

The price of trends is high; it consists in producing the large numbers of secondary papers which not even the most dedicated archaeologist of XXII century will have patience to read! To dissent or to criticize the mainstream could be rather difficult.

The situation reminds inevitably an aphorism of Jean de La Fontaine from XVII c: "All minds of the world are impotent against any stupidity which became fashionable" [6]. Indeed. Even if not stupidity, the minds are still helpless...

The trends and mainstreams, have some true achievements by focusing the efforts and accelerating the progress in some particular areas. This may be justified if contributes to the training (and graduations) of the young students, but even so, some mass phenomena, like the huge numbers of authors of one publication, can make quite difficult to appreciate the individual merits.

The Editorial Empires

Unable to protect the market from the avalanche of publications the Editors of prestigious journals, quite frequently, take the problem into their hands: they don't even worry to send a paper to a referee, but just decide themselves. I happen to know about a paper submitted to the Physical Review Letters to which the Editor answered: "Sorry, but this is not interesting to our community." Given the situation, this is not an offence, though on margin of this humorous incident the question arises, what is exactly "our community"?

The illustrations are many. On one occasion one of our colleagues submitted to the Physical Review A an article about the operational techniques of controlling the charged particle by the $\delta(t)$ potential pulses in an ion trap. The referee decided that even if the mathematical result were true, the $\delta(t)$ pulses cannot be experimentally achieved, so the subject is outside the areas of the Physical Review A, which traditionally takes care to be close to the experiment. Yet, at this moment, the Physical Review Letters receives gladly the articles on "*bam-bam* control" of quantum evolution, the *bam's* meaning the δ-pulses, though now they are called "separators" or "interruptors", published, by the authors who already had a large number of papers in the Physical Review Letters, Physical Review A, etc., typically from the highly prestigious centers like, e.g., Harvard, MIT, etc., though, till now they do not seem to lead to any efficient experimental methods (even in "our community"?). Of course, these remarks may be premature, since the practical applications of theoretical results are unpredictable and almost never immediate. So, maybe, in some 10 years we shall see a technological bam-bam revolution? On other occasion, I could detect two essential critical articles against a leading trend, one of them waiting about one and a half years and the other about two years before they could be published. This was no longer the symptom of a light disease...

An excessive power of some elite groups certainly is not limited to the *Physical Review Letters* et al. The same phenomenon can be noticed in *Nature, Cells, Science*, etc., criticized recently by the known scientist. Thus, in his article "How journals like *Nature*, *Cell* and *Science* are damaging science" published in The Guardian, Monday, Dec. 9 2013, 19:30 GMT, the known specialist Randy Schekman (one day before receiving his Nobel prize!) wrote: "... These luxury journals are supposed to be the epitomes of quality, publishing only the best research. Because founding and appointment panels often use the place of publication as a proxy for quality of science, appearing in these titles often leads to grants and professorships. But the big journals reputations are only partly warranted... " And later on: "It builds the bubbles of fashionable fields where researchers can make the bold claims these journals want, while discouraging other important work... " (remember Jean de La Fontaine?!),

The criticism of Schekman is just a detail compared with much wider discussions in biological sciences, concerning the evolution of life. For the groups of colleagues sympathizing with the leftist ideas, the Charles Darwin theory of the natural selection [8] is almost sacred. Yet, Darwin never claimed that his theory explains *everything.* In fact, attempts of modifying Darwin appeared in the Soviet regime in form of Michurin and Lysenko doctrines that the living organisms "can learn" to modify themselves (however, forbidding the genetics of Mendel as an antisocialist intrigue). The evolution steps difficult to explain by Darwin's theory indeed exist and were recently explored by the opposite current of the religion sympathizers who launched the idea of the "intelligent design". As a result, in some modern educative centers the theory of Darwin is unwelcome, as offending religious dogmas. It thus seems, that the role of religions in science is not over. In fact, the religious problems of our world are visibly increasing. Curiously, in many modern cultures an obligatory principle seems to be "the respect for religious feelings". How nice! But then, what about any other feelings?

All that are, perhaps, just anecdotes hardly sufficient to reflect the magnitude of the desperate scientists' battle to legitimize themselves by streams of "creativity". Since the numbers of the scientists and their works are still growing (according to the statistics performed by D. Colquhoun [4] about 1.3 millions of papers published in 23.750 journals in 2006), it was inevitable that the social organism had to find some new channels to expand.

The Open Access, Paid-Journals

The answer were the new type journals, accessible *on line* to anybody, in which the authors must however pay for their publications. It is assumed, that the papers submitted to those journals are carefully revised, but it can be easily understood that the main reason to accept an article is money paid by the author rather than the article quality. While in physics few of these journals already gained some positive reputation, the most of them are just business enterprises, caring basically to earn money. The socio-economical study of this phenomenon was recently made by the workers of the (criticized) *Science* by submitting some senseless articles to a huge number of the on-line paid journals. This is what one of them reports [7]:

> On 4 July, good news arrived in the inbox of Ocorrafoo Cobange, a biologist at the Wassee Institute of Medicine in Asmara. It was the official letter of acceptance of the paper he submitted 2 months earlier to the *Journal of Natural Pharmaceuticals*, describing the anticancer properties of a chemical that Cobange extracted from a lichen. In fact, it should have been promptly rejected (. . .) Its experiments are so hopelessly flawed that the results are meaningless. I know because I wrote the paper. Ocorrafoo Cobange does not exist, nor does the Wassee Institute of Medicine. Over the past 10 months, I have submitted 304 versions of the wonder drug paper to open access journals. More than half of the journals accepted the paper, failing to notice its fatal flaws. Beyond this headline result, the data from this sting operation reveal the contours of an emerging Wild West in academic publishing.

The business of the open access journals is certainly worth of careful attention. According to *Science* report, the journal described above is one of 270 in one of the largest open access publishers, with 2 millions of its articles downloaded by the researchers every month. In 2011 it was bought by Wolters Kluver Netherlands (the company with the annual revenues of nearly \$ 5 billion). Published in Science [7].

The Support for Scientific Projects?

On this complicated background, specially in the crisis time, the increasingly difficult task is to choose, which scientific projects of the Universities and other science institutes should obtain the financial support. Despite the fact that the funds for the science go recently down, the state organs controlling them don't shrink but rather systematically grow: they need more regulations and personnel to spend their money truly well! (remember Parkinson's law?)

About three years ago I heard a curious story about a group of our colleagues in Prague working hard to formulate an ambitious research project with participation of well-known specialists from several UE countries. In order not to make errors in presenting the 'formats', they asked help of a colleague who was recently working in the European Evaluation Commission. The formats were filled carefully under the expert control and presented to the Commission. As required, they contained the original together with a high quality copy. Yet, one week later, they received the answer that the project was rejected, with no possibility of reclamation. The reason was that the signature on the original was in the blue color and on the xerox copy it was black. No revisions accepted!

Something from my personal observations (as the referee on several occasions). It is of course natural that the project's authors should inform the commissions about their expected work topics, with some approximate timing, also about their planned experiments and measuring devices, some hopes about the results, etc. The phenomenon which I have seen, however, were the data of the kind: after the first year of the project, we shall publish 4 papers, after the second year, again 4 papers and after the third year, the 5 papers... Quite shocking! The authors, after one or more months of working hard on the project, were so afraid that their effort may be in vain, that they were blind to the idiocy of reporting the exact numbers of their future publications (and, as somebody told: blind also to their own blindness!)

Several years ago I had an occasion to ask Bill Phillips (Nobel in physics 1997): "What do you think about the short, two or three years long projects, whose authors are asked to present from the beginning the 'calendar of activities' including even the future publications?". "Yes, we have them too" he answered, "but the power of US is, that we have also a large variety of institutions which can support the science without this kind of nonsense!". (I just wonder whether it is still true?). This is, however, not the end of the story.

Science AND *Technology?*

One of the reasons of so high interest in science are the technical applications which changed completely our lives (for good and for bad!). However, the relations between the scientific and technical progress are seldom free of conflicts.

A known historical example was the collaboration of the famous US inventor, Thomas Edison with his colleague, the equally famous Nicola Tesla [10]. The son of the Serbian family, Tesla, already as a youngster showed exceptional talents. He was specially fascinated by electricity. The electricity generators at this time were producing only the constant voltage (direct current), which could be sent by cables at small distances. In his moment of inspiration Tesla predicted the generators of alternating currents, but in the conservative Austro-Hungary nobody was interested. Tesla therefore decided to go to the U.S.A. He was accepted in the laboratory of T. Edison, who promised him the reward of $ 50,000 if he succeeded to construct a better electric generator. As it seems, Edison did not expect any great success. He already invested millions into an electric plant in the Pearl Str.

which could only send with difficulty the direct current at the 1.5 km distance to Manhattan. Meanwhile, Tesla worked intensely and soon he created the first generator of the alternating current. Edison, however, was completely absorbed in his competition for the better generation of the direct current (the challenge was again of $ millions) Some journals report that he just fired Tesla without paying him the promised reward of $ 50,000. According to a more exact study [10] this was not so. Anyhow, Edison blocked further work on alternating currents. Furious Tesla abandoned the Edison's laboratory and patented his discovery, which was bought by an American millionaire for $ 60,000. Tesla did not even suspect that he lost an enormous fortune for a miserable price.

In this old story Tesla represented the scientific idea and Edison the technology. It can be noticed that some elements of their antagonism were repeated in the future. In fact, after the golden time of historical collaboration, some products of the science were inconvenient for industry: by simplifying the production they could cause the looses if not ruin of the already developed production patterns. A notable phenomenon was that the industrial enterprises were buying the interesting patents, not to apply, but inversely, to avoid their applications! The phenomenon does not seem limited to the past.

Another trouble is the different dynamics of the science and its applications.

It seems essential that the application of the scientific discoveries can be unpredictable; on some occasions it happens by pure accident, not by any carefully elaborated project. The historical examples are multiple. Just to mention one, the discovery of penicillin was not the success but rather an obstacle of a microbiological research program. If Alexander Fleming continued his planned experiments without worrying about his spoiled microbe cultures, he would probably ended up with some routine results without even suspecting what he had lost!

It must be also remembered that the most important technical applications must take their time. The discovery of the electrodynamics of Maxwell and Faraday was seen with extreme skepticism by lords of the British Parliament. "What use there can be out of some partial differential equations?", they asked. According to the existing memories, Faraday answered: "And what use of a child?". In fact, the child was growing. Not immediately, but after about 50 years, the radio and then the radar were invented. Later on the use of optical fibers. The internet seemed a little practical trick, not even patented, but within about 30 years it turned on the most powerful revolution, causing the rebellions and governments fall. Our today civilization is based on fibers. The question arises, could all this be accelerated? Apparently not by bureaucratic pressures.

In physics, and other exact sciences, while the mountains of publications were growing fast, the increase of valuable applications was much slower. Some state administrations, helped by the industry leaders invented the concept of "Science AND Technology". The hopes were that it would convince the successful industries to invest into the science programs. At the beginning it was almost true (just remember the NASA investment into the International Conferences). Soon, however, the dependence was inverted.

In 1989 after the change of regimes in the Eastern Europe, the hope was that the scientific institutes finally will obtain enough grants to develop ambitious research projects. Yet it was not exactly so. In the first years of the “new deal” in Poland, the funds for the universities and research centers did not satisfy the initial hopes. A group of scientist from various institutes, decided to examine the situation. The result was shaking: several million dollars missing! The delegation of scientists visited the Science Ministry demanding the information. “No trouble, nothing disappeared”, they were told. “In the framework of “Science and technology”, several large enterprises also presented their scientific projects which the Ministry considered of interest. Then, they reported quite satisfactory results…

Today, the neologism “Science AND Technology”, in spite of the entire optimism, works as a linguistic tumor which sucks funds out of the scientific work, for enterprises which have never enough. The cases are abundant. About three years ago, our colleagues checked that an enormous part of the “science support funds” all over the world are consumed by industrial establishments which (specially in the crisis time) care only about the short term profits. A shaking example is the sequence of industrial projects with costs notably higher than the customary university projects, approved by one of the world science ministries, including quite costly scientific project of Volkswagen factory. Thanks to this kind of mechanisms, many state bureaucracies can report, e.g.: “We spend around 0.4% of national income to develop the science”. Yes… but of this quantity, how much were the souvenirs to the great (transnational) enterprises, which had nothing against simply consuming the funds and then, no difficulty in presenting the satisfactory reports? The Journal “El Pais” in Spain some weeks ago asked: “And where is the promised 1%?” An extremely naive question. It would be very fortunate if some part of 1% was really invested into the basic science in any world country.

Instead – the scientific communities all over the world are now incessantly bombarded by marketing announcement, how to make, how to organize our own enterprise, in spite of the world crisis… (Should I cite hectares of promising announcements?) In UE e-mail boxes, some of the business proposals are quite difficult to remove: by trying to cancel, the announcement responds by sending you to some new links, which neither want to disappear. To cancel the entire sequence and return to your e-mail, you need some additional computer tricks. All this is no longer an innocent marketing, but a heavy parasitism!… One would like to think that this is the last unpleasant problem, but it isn’t!

The Far East Catastrophe

As if it was not enough, a new challenge is now developing in China. The article “China’s Publication Bazaar” (*Science* [9]) reports the existence of the new lucrative commerce which permits the young desperate scientists to buy the authorship of papers already accepted for publication. The report is so shaking, that I permit myself to quote some fragments.

> The *Science* investigation has uncovered smorgasbord of questionable practices including paying for author's slots on papers written by other scientists and buying papers from online brokers (...).
>
> "There are some authors who don't have much use for their papers after they're published and they can be transferred to you" a sales agent told to a Science reporter posing as a scientist. (...) The company would sell the title of first co-author on the cancer paper for 90,000 yuan ($ 14,800) Adding two names, first co-author and corresponding co-author, would run $ 26,300, with a deposit due upon acceptance and the rest on publication.
>
> (...) On 6 July, a few weeks after our conversation (...) the paper appeared online in the International Journal of Biochemistry & Cell Biology. The printed version followed in September, roughly when the agent said it would. The title and the abstract had undergone minor revisions from the e-mail solicitation. But the list of authors was transformed. On the published paper, two first authors shared the honor. (Our reporter did not pay for authorship). (...) Following an inquiry from *Science*, an investigation by the International Journal of Biochemistry & Cell Biology found that a total of four authors had been added and two dropped (...)
>
> *Science* documented authorship fees ranging from $ 1600 to $ 26,300. At the high end fees exceed the annual salary of some Chinese assistant professors. But SCI papers – particularly those published in journals with a high impact factor – are so critical for getting promotions that researchers shell out.

The section "Paper-pushers" quotes the Chinese dealer: ... 'Several agencies claim they collaborate with specific journals indexed in SCI to guarantee publication, The representative for one company (...) was blunt about the collaboration: "We rely on our guanxi" – a Chinese concept evoking relationships often deepened by exchanging gifts. "To put it simply, we give them money". At least three companies offer to assist the scientists who have written a paper and want to ensure the publication. Other firms claim to purchase a number of pages in journals. Several agencies specified both the journal and issue in which a paper would appear – even though the paper had yet to be written.'

The article quotes an opinion of one of ex-editors that the phenomenon is not too abundant, but "it completely destroys the academic environment". Let us add: already damaged by the "publish or perish". Moreover, the Chinese "Publication bazaar" can be so prosperous only because its brokers have accomplices in the Editorial Boards of some world SCI journals *outside of China*! The corruptive process illustrated here has an almost cancerous mechanism!

Certain uneasy **Conclusions** must follow.

1. The bureaucratic pressure of "publish or perish" must disappear. The scientific results cannot be estimated by numbers of publications.

2. Enough of the scientific projects which promise the number of publications. The scientific research is needed only if the results cannot be predicted. If authors can predict the number of their papers, it means that their work is unnecessary.
3. Be careful with numerical rankings of the scientific institutions. Their significance can be very misleading.
4. Careful with the linguistic tumor of "Science AND Technology". Shouldn't these two concepts be at least partly separated to grant some modest contribution to the basic science?
5. An investigation of the corrupt activities on the Chinese Publication Bazaar as well as their partners in all world journals is urgently needed.
6. Enough of ideologies, religions and 'political correctness'. The scientist should not offend his colleagues, but has no obligation to care that his results won't antagonize anybody.
7. Enough of the trend, mainstreams and obligatory worship of the "excellence groups". Yet, in some near future, we might offer our friendly patience to the "luxury journals" and their leaders (of course, not without some friendly critiques!).

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Bogdan Mielnik
Physics Department, Cinvestav
P.O. Box 14-740, 07000 México D.F., Mexico
e-mail: bogdan@fis.cinvestav.mx