Variational Theories of Two-Phase Continuum Poroelastic Mixtures: A Short Survey

Roberto Serpieri, Alessandro Della Corte, Francesco Travascio and Luciano Rosati

Abstract A comprehensive survey is presented on two-phase and multi-phase continuum poroelasticity theories whose governing equations at a macroscopic level are based, to different extents, either on the application of classical variational principles or on variants of Hamilton's least Action principle. As a focal discussion, the 'closure problem' is recalled, since it is widespread opinion in the multiphase poroelasticity community that even the simpler two-phase purely-mechanical problem of poroelasticity has to be regarded as a still-open problem of applied continuum mechanics. This contribution integrates a previous review by Bedford and Drumheller, and covers the period from the early use of variational concepts by Biot, together with the originary employment of porosity-enriched kinematics by Cowin and co-workers, up to variational theories of multiphase poroelasticity proposed in the most recent years.

Keywords Variational poroelasticity · Compressible phases · Generalized continua · Effective stress · VMTPM

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H. Altenbach and S. Forest (eds.), *Generalized Continua as Models for Classical and Advanced Materials*, Advanced Structured Materials 42,
DOI 10.1007/978-3-319-31721-2_17

1 Introduction

Theoretical poroelasticity has a very wide range of applications. Besides the well known applications to soil mechanics (Fillunger 1936; Terzaghi 1936; Biot 1956), poroelastic models have been increasingly applied for the description of complex biological phenomenology such as biological tissue mechanics and remodeling processes (see e.g., Cowin 1999; Giorgio et al. 2014; Andreaus et al. 2014; Madeo et al. 2011; Ateshian and Ricken 2010; Ehlers and Bluhm 2013; Mow et al. 1980). However, most problems of geomechanics and biomechanics require a multiphase continuum description for a proper understanding and prevision of several intertwined mechanical phenomena. As well-known in geomechanics, this is the case of saturated and partially saturated soils (Schrefler 2002; Madeo et al. 2013). Similarly, in biomechanics, cartilaginous tissues have been described as mixtures of a solid phase made up of structural macromolecules plus an interstitial fluid phase consisting of water and solutes (Lai et al. 1991; Gu et al. 1998; Huyghe and Janssen 1997; Travascio et al. 2014).

However, the achievement of a general consistent theory for two-phase continuum poroelasticity, capable of addressing systems with any degree and range of compressibility of the constituent phases, represents a long dated challenge of theoretical and applied continuum mechanics.

The insightful historical retrospective survey by de Boer (1996) provides evidence of the complexity of the construction of a *standard* continuum theory of this type, undertaken over the last century. In particular, by the term *standard* we refer to the formulation of a generally agreed minimal set of mathematically consistent and physically plausible governing equations of two-phase poroelasticity, deducible from the classical principles of physics with assessed predictive capabilities. The review by de Boer covers a large part of the approaches extending from the early Terzaghi–Fillunger dispute (Fillunger 1936; Terzaghi 1936), including the fundamental theoretical contributions by Biot (1956) and the fundamental experimental evidences from geomechanics (Skempton 1954; Nur and Byerlee 1971), up to the more recent group of theories frequently gathered under the term *Theories of Immiscible Mixtures* (TIM).

A comprehensive survey on TIMs proposed until 1983 has been provided by Bedford and Drumheller (1983). Such a review includes mixture theories derived from continuous models more general than classical Cauchy one which are based on continuum mechanics frameworks employing enhanced microstructural descriptions for the solid phase: in particular the theory of linear elasticity with microstructure by Mindlin (1964), Eringen's micromorphic theory (Eringen 1968), as well as Goodman and Cowin's theories which employ an additional equation of motion in either postulated form (Goodman and Cowin 1972) or developed proceeding from a postulated variational principle (Cowin and Goodman 1976).

After the middle of the eighties, driven by the increase of applications in geomechanics, biomechanics, environmental engineering and material engineering, theoretical research efforts have been aimed at developing general and comprehensive multiphase flow theories. Nevertheless, investigators kept searching for a fundamental set of governing equations achieving general consensus. Research in this area has accordingly experienced a proliferation of porous media frameworks which have proceeded quite independently by stressing different arguments in order to achieve the formulation of a standard macroscopic governing set of continuum equations.

To logically organize the research efforts driven by such a *multiplication of languages* from the eighties until current times, classifications of the mainstream approaches can be attempted, without any claim of completeness and of clean-cut separation.

A first classification can be considered by identifying two approaches:

1. *Purely Macroscale Theories* (PMT) which are based on the introduction of kinematic descriptors or constitutive features expressly at the macroscale level;
2. *Upscaling/Averaging Theories* (AT) which proceed from considering a detailed representation of the geometry and flow processes at the microscale.

A general agreement on the superiority of either PMT and AT has not been reached yet (see e.g. the debates in Gray et al. 2013a, b; Baveye 2013). In any case, PMT are exposed to the criticism of lacking a strong connection with the pore scale physics and of performing implicit approximations, while AT can be criticized for introducing assumptions justifiable only on a heuristic basis, and also for lacking a clear link with the macroscopic measurement processes.

Among PM theories, a further classification can be performed according to the setting employed for the definition of energy potentials of the constituent phases (Gajo 2010). Two families can be identified: the first one includes PMT approaches where a single macroscopic energy potential of the whole saturated mixture gives rise to stresses for both solid and fluid phases (see, e.g., Coussy et al. 1998); the other one includes approaches where the two phases can be treated as superposed continua, each one endowed with a separate energy potential. As observed by several authors, this second and more general approach requires, alongside of linear momentum and mass balances, an additional governing equation (Svendsen and Hutter 1995), usually referred to as *closure equation*.

In this respect, several candidate closure equations have been proposed for the identification of the additional equation (or of the set of additional equations), capable of providing the minimal set of governing balance equations necessary to achieve a general consistent formulation of compressible poroelasticity. Within a wide family of more general formulations (see for instance Bowen 1982; Hassanizadeh and Gray 1990; Schrefler 2002) the closure of the poroelastic problem has been sought by supplementing momentum and mass balance equations with the second law of thermodynamics, in agreement with an early indication by Truesdell (quoted by Bedford and Drumheller) according to which "*the 'missing principle', surely, is a proper generalization of the Clausius–Duhem inequality*".

Similar to Drumheller (1978), Bowen (1982) includes the evolutions equations of volume fractions, as well as momentum of momentum balances, among the governing equations, tracing back to Cosserat's theory (Cosserat and Cosserat 1909;

Eringen and Kafadar 1976). A further closure of the biphasic problem has been proposed by incorporating of the saturation constraint in the entropy inequality, using an incompressibility hypothesis and a Lagrangian multiplier (Svendsen and Hutter 1995; de Boer 1996). Moment of momentum balance is also considered together with a multiplicative decomposition of the strain tensor (Diebels 1999).

Alternatively, porosity was added as an additional independent state field by Albers and Wilmański (2006) and by Wilmański (1998), who investigated several additional balance equations in the form of porosity balance or integrability condition for the deformation of the solid skeleton. A geometric saturation constraint has also been employed, combined with a multiplicative decomposition of the deformation gradient, as a closure equation of the formulation (de Boer 2005).

It is therefore evident that even the simpler two-phase purely-mechanical problem of poroelasticity has to be regarded as a still-open problem of continuum mechanics. This opinion is widely spread in the multiphase poroelasticity community. As stated by de Boer: *"the necessity to attack the problem of developing a consistent general poroelasticity theory is still existent"* (de Boer 2005), as well as by Lopatnikov and Gillespie (2010) *"... in spite of a tremendous number of publications in this field, the discussion continues about physical background of the poroelastic theory. Even the form of basic governing equations are sufficiently different [...] in frame of different approaches that one can find in literature. It seems that there is no final agreement about consistency of proposed different approaches."*

Turning to the objective of the present survey, attention is herein focused on the subclass of two-phase continuum poroelasticity theories which can be classified to be of *variational* type and ascribable to the PMT group. Thus, ruling out the major effort of an accurate updated review of the currently available porous media frameworks, this contribution is aimed at providing an updated survey on the variational subclass of poroelastic multiphase theories, since the authors share the convincement that variational statements are privileged means for the continuum description of physical phenomena ensuring *"a natural and rigorously correct way to think of [...] continuum physics"* (Oden and Reddy 2012).

A further, highly relevant feature of the variational approach is that the principle of minimization is very convenient as a basis for numerical simulations. Indeed, it is well-known that Finite Element (FE) methods are the natural discretization of theories presented in weak formulation. More specifically, numerical investigation of poroelastic continua has greatly benefited from the development of high-regularity FE schemes such as isogeometric analysis (see Hughes et al. 2005, as one of the stem works), a technique that is particularly suitable for generalized continuum theories as recently shown in different contexts (Greco and Cuomo 2014, 2016; Cazzani et al. 2014; Cuomo et al. 2014).

For the purpose of the present contribution, we include under the term *variational theories of poroelasticity* all continuum theories of porous multiphase materials (including single phase theories) which found the derivation of governing equations upon the application of classical variational principles: Hamilton Least Action Principle of mechanics (Landau and Lifshitz 1976; Moiseiwitsch 2013; Berdichevsky 2009), Principle of Virtual Powers (for functionals admitting first differentials), and

Principle of Virtual Works (see also the retrospective by dell'Isola and Placidi 2012 on the application of variational principles to continuum mechanics).

Concerning the focus on PMT, it has to be added that the already mentioned microstructured continua theory can be seen as a general framework in which one can find higher gradient continuum theories as a particular case. Researches on microstructured and higher gradient continua are experiencing a significant intensification, especially in connection with the development of computer-aided manufacturing techniques (as high-precision and multi-material 3D-printing) which allow the designing and the fabrication of micro- and even nanostructures characterized by a high degree of complexity, whose effects at the macroscale cannot be captured by standard continuum theories (dell'Isola et al. 2015).

2 Variational Theories of the 70's and the 80's

An important remark should be made concerning the notation adopted hereafter. As always happens when a research field gradually produces a considerable amount of literature, there is a general tendency towards a relative uniformity in the notation and conventions which is agreed within the scientific community of specialists in the topic. This fact has obvious advantages, allowing for the immediate understanding of the equations in a paper without requiring a detailed reading of the discursive parts. On the other hand, the literature usually converges only asymptotically towards this status, and there are many cases (especially among pioneering works) in which substantially different notations and conventions are used. Thus, it appeared illogical to the authors not to exploit the advantages of a coherent and uniform notation even though this has meant, in some cases, the modification of the original format of some equations. Therefore, all equations and formulas in the present work have to be intended as conceptually identical (but not philologically accurate) rendering of the original ones in the cited papers.

The first use of some of the ideas of variational approaches in the derivation of a theory of mixtures has been claimed by Truesdell and Toupin (1960) (p. 567) to trace back to Duhem (1893).

It is also important to recall that the seminal and influential poroelastic theory by Biot (1941, 1956, 1962), while originarily obtained on a somewhat intuitive definition of stress measures and elastic relations based on concise mechanical considerations (Bear and Corapcioglu 2012), was subsequently framed by Biot (1972) into a variational theoretical framework in the context of quasi-static and isothermal deformations. Therein, the equilibrium equations are obtained proceeding from the statement of a principle of virtual work, and later extended to account for nonisothermal deformations and to include dynamical forces (Biot 1977).

In particular, variational concepts are applied by Biot (1972), proceeding from the introduction of an 'isothermal free energy density function' which depends on the finite strain of the solid and on a quantity m defined therein as the total mass of fluid added in the pores of the sample during deformation. This particular choice for the descriptors, referred to open mechanical systems where mass can enter or leave, has been criticized by more than one author. For instance, it has been observed that it is not possible to construct a true variational principle as the Biot model contains nonequilibrium variables, see, e.g., Wilmański (2006).

Further works in the seventies, containing some applications of variational concepts to the derivation of multiphase porous media theories, are the papers by Kenyon (1976) where a variational postulate is proposed to justify the linear momentum balance equations introduced by Truesdell (1969) in postulated form. The kinematic descriptors considered are the densities and the deformation gradients of each phase; however, this theory makes no use of volume fractions. Also, an application of variational concepts to formulate a theory of two-phase mixtures is reported by Aizicovici and Aron (1977), although this study proceeds from postulated equations of motion.

2.1 Cowin's Theories Including Porosity

Theories of ideal multiphase mixtures which, in some respects, can be stated to have a variational character are those by Nunziato and Walsh (1980) and Passman (1977). These frameworks consist of extensions of the continuum theory for granular materials (Goodman and Cowin 1972), and rely on the key idea to add the volume fraction of the solid phase (ϕ) as an additional kinematic continuum scalar descriptor. Moreover the frameworks by Nunziato and Walsh (1980) and Passman (1977) use an additional balance scalar equation, proposed by Goodman and Cowin (1972) and denominated therein *equation of balance of equilibrated force*. This additional equation pairs the number of unknown fields, incremented by one as a result of the introduction of ϕ among the kinematic descriptors, with the number of momentum balance scalar PDEs.

Although the theory by Goodman and Cowin (1972) cannot be termed variational, since it is based on thermodynamic arguments and ad-hoc modified forms of the momentum and energy balances, the balance of equilibrated force is motivated by a variational analysis. Later, Cowin and Goodman (1976) have shown that the so called balance of equilibrated forces can be derived proceeding from a postulated variational principle. In particular, such variational theory is derived by addressing the dependence of a *density of stored energy function* upon the solid volume fraction, the true density of the solid porous phase ρ, the solid volume fraction ϕ and its space gradient $\phi\nabla$.

It should be remarked that the variational theory by Cowin and Goodman (1976) is not a standard variational theory in several respects. Actually, Eq. (13) therein presents a postulated condition, directly expressed in the form of first-variations containing two postulated quantities: a quantity H, stated to be a *self-equilibrated stress system*, and a second quantity l, stated to be a *self-equilibrated body force*. There is a potential misunderstanding in this last choice. Indeed, talking about *stress* seems to focus just on classical (Cauchy-type) external actions, while the proposed model entails the presence of more general external actions. Thus, *generalized stress* would probably have been a more appropriate wording in this case. A further uncommon feature of this theory is that the stress tensor of the solid phase is originally defined as a quantity work-associated with the solid true density ρ, instead of being defined as a quantity work-associated with the symmetric part of the displacement gradient in a standard way.

2.2 Mindlin's Variational Single-Phase Theory of Materials with Microstructure

Although not directly applied to multiphase problems, the (single phase) continuum theory of materials with microstructure by Mindlin (1964) has provided a useful (and in some respects 'canonical') background for the subsequent development, on a variational basis, of multiphase poroelastic continuum frameworks which, exploiting ideas similar to those in Cowin and Goodman (1976), Passman (1977), Nunziato and Walsh (1980), employ enhanced kinematics with additional descriptors such as the porosity.

In Mindlin's theory the equations of motion are derived by using Hamilton's principle which can be conveniently written as follows:

$$\delta \int_{t_1}^{t_2} (T - V)\mathrm{d}\, t + \int_{t_1}^{t_2} \delta W \mathrm{d}\, t = 0, \tag{1}$$

where t_1 and t_2 are two arbitrarily assigned time instants, T is the kinetic energy, V is the internal potential energy while the term δW is the virtual work of external body forces, external traction vectors, generalized body forces and generalized surface forces (termed *double forces* by Mindlin).

A quite general framework is considered in which a macroscopic second-order tensor field $\boldsymbol{\psi}$ is added as a further kinematic descriptor, termed *microdeformation*, complementing the displacement field $\mathbf{u}$.

As a consequence of such choice of kinematic descriptors, it is derived from (1) a vector linear momentum balance, expressing the stationarity of (1) with respect to $\mathbf{u}$ plus additional stationarity scalar conditions expressing stationarity with respect to the independent components ψ_{ij}.

The strain measures of this theory are the standard strain tensor $\boldsymbol{\varepsilon} = \mathrm{sym}(\nabla \mathbf{u})$ plus two additional strain measure fields related to $\boldsymbol{\psi}$: the so-called *relative deformation field*, defined as $\boldsymbol{\gamma} = \nabla \mathbf{u} - \boldsymbol{\psi}$, and a *microdeformation gradient* field $\boldsymbol{\kappa} = \nabla \boldsymbol{\psi}$. On this basis, the strain energy is a homogeneous quadratic function of $\boldsymbol{\varepsilon}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$.

2.3 The Variational Theory of Immiscible and Structured Mixtures by Bedford and Drumheller

A fundamental advancement in the derivation of variational theories of multiphase porous media and structured mixtures has been provided by Bedford and Drumheller (1978, 1979, 1983). These authors have extended the ideas laying the basis of single-continuum framework of microstructured continua by Mindlin (1964) and the approaches for the variational treatment of a single continuum in solid and fluid mechanics (Lanczos 1970; Herivel 1955; Eckart 1960; Finlayson 2013; Leech 1977; Oden and Reddy 2012) to derive the balance equations for porous multiphase problems by means of Hamilton's principle. Accordingly, momentum balance equations are derived from a stationarity condition representing a variant of Eq. (1).

A multiphase framework is considered with index of the generic phase ξ hereby indicated by script $(\cdot)^{(\xi)}$. From a constitutive point of view, denoting by $\phi^{(\xi)}$ the volume fraction of the generic ξth phase and by $\hat{\rho}^{(\xi)}$ its 'true density', related to the relevant apparent density $\bar{\rho}^{(\xi)}$ by the usual relation:

$$\hat{\rho}^{(\xi)} = \frac{\bar{\rho}^{(\xi)}}{\phi^{(\xi)}}, \tag{2}$$

it is assumed (Bedford and Drumheller 1979) that each phase ξ has a strain energy density ψ which is only dependent on $\hat{\rho}^{(\xi)}$ while in (Bedford and Drumheller 1978) a dependence of upon $\hat{\rho}^{(\xi)}$ and the (infinitesimal) strain tensor ε is considered.

The primary descriptors of such formulation are fields $\phi^{(\xi)}$ and $\hat{\rho}^{(\xi)}$, together with the placement field $\boldsymbol{\chi}^{(\xi)}$ which operates the association $\mathbf{x}^{(\xi)} = \boldsymbol{\chi}^{(\xi)}\left(\mathbf{X}^{(\xi)}\right)$ between the current position of phase ξ and its material position $\mathbf{X}^{(\xi)}$. This choice of fields amounts to a total of five fields per each phase. In agreement with Leech (1977), the least-action condition is written integrating over a fixed reference volume domain containing a fixed mass of mixture.

It is important to remark that, in the formulation stated by Bedford and Drumheller, the primary descriptors are not unconstrained fields. Actually, fields $\phi^{(\xi)}$, $\hat{\rho}^{(\xi)}$, and $\boldsymbol{\chi}^{(\xi)}$ are constrained by the equations of mass balance:

$$J^{(\xi)}\bar{\rho}^{(\xi)} = \bar{\rho}_0^{(\xi)} \tag{3}$$

and by the volume fraction constraint stating that space is completely saturated by the phases so that the sum of volume fractions equals unity, viz.:

$$\sum_{\xi=1}^{N} \phi^{(\xi)} = 1, \tag{4}$$

where N is the number of phases. In order to respect (3) and (4), the variations $\delta\phi^{(\xi)}$, $\delta\hat{\rho}^{(\xi)}$ and $\delta\mathbf{x}^{(\xi)}$ are also constrained each other. Such constraints are included by Bedford and Drumheller through the addition of (3) and (4) into (1) with the aid of Lagrange multipliers λ and μ_ξ. The resulting equation has the format:

$$\delta\int_{t_1}^{t_2}(T-V)\mathrm{d}t+\int_{t_1}^{t_2}\delta W\mathrm{d}t \\ +\int_{t_1}^{t_2}\left[\sum_{\xi=1}^{N}\int_{\Omega}\mu_\xi\delta\left(J^{(\xi)}-\frac{\phi_0^{(\xi)}\hat{\rho}_0^{(\xi)}}{\phi^{(\xi)}\hat{\rho}^{(\xi)}}\right)\mathrm{d}V_0-\int_{\Omega}\lambda\delta\left(\sum_{\xi=1}^{N}\phi^{(\xi)}\right)\bigg|_{\mathbf{x}}\mathrm{d}V_0\right]\mathrm{d}t=0. \tag{5}$$

The physical interpretation of λ and μ_ξ is also discussed by Bedford and Drumheller (1978). Resorting to the standard notion of Lagrange multipliers as generalized forces ensuring the constraints to be satisfied, and to some considerations on pressure force balances, the authors justify the interpretation of λ as an *interface pressure* between constituents, and infer for μ_ξ the relationship $\mu_\xi=\dfrac{p^{(\xi)}\phi^{(\xi)}}{J^{(\xi)}}$, where $p^{(\xi)}$ indicates the pressure of the ξth constituent.

In this respect, it is important to remark that the mechanical consistency of the choice of incorporating of the effect of constraints in a variational framework has been subjected to debate and objections between researchers. In their valuable review, Bedford and Drumheller (1983) recall a criticism by Truesdell and Toupin (1960) (pp. 594, 595) who have indeed observed that incorporating the effect of constraints in variational principles "... is a somewhat dubious blessing". Bedford and Drumheller have rebutted that the volume fraction constraint does not entail ill-posedness issues and have remarked that the admissibility and usefulness of the volume fraction constraint in multiphase theories can be standardly accepted as a continuum mechanical analogue to the treatment of connections between rigid parts in the variational description of the mechanics of rigid bodies.

3 Most Recent Theories

In more recent years, researches on multiphase theories on a variational macroscopic basis have still continued to appear in the specialized literature. Referring the readers to the original papers for further details, a brief account of these theories and of the key ideas is given in this subsection, proceeding in chronological order.

3.1 Variational Theories by Lopatnikov and Co-workers

The Least Action principle has been employed by Lopatnikov and coworkers to obtain continuum governing equations for binary poroelastic mixtures (Lopatnikov and Cheng 2004; Lopatnikov and Gillespie 2010). Important differences with the framework by Bedford and Drumheller are the following:

- As a peculiar feature of this formulation, distinction is characteristically made between a notion of *internal strain tensor* and a notion of *external strain tensor*. For the definition of these quantities, the reader is referred to the original papers where these concepts are introduced (Lopatnikov and Cheng 2002, 2004; Lopatnikov and Gillespie 2010). Relationships between variations of external and internal parameters of the material are introduced and referred to as *material structural equations*. Such relations have a constitutive nature and, hence, appear to be medium-dependent. Lopatnikov and Gillespie (2010) discuss several options for their definition (see p. 482 therein).
- This theory is essentially formulated in infinitesimal displacements.
- The least Action condition is formulated without making explicit statement of the recourse to Lagrange multipliers, even if the theory contains constraints for the variation fields. Specifically, the system of governing equations contains mass conservation equations. Most importantly, the authors infer from mass conservation relationships between variation of porosity $\delta\phi^{(f)}$ and $\delta\hat{\rho}^{(f)}$ involving also the gradient of porosity $\phi^{(f)}\nabla$.

Lopatnikov and Gillespie remark that the presence of a dependence upon $\phi^{(f)}\nabla$ in their mass conservation relation is an important difference with respect to other previously proposed multiphase variational frameworks such as the one by Bedford and Drumheller. Actually, they show that, in nonhomogenous media, an additional volume force interaction between solid and fluid phases appears in the governing equations. This force, which is proportional to $\phi^{(f)}\nabla$, is traced back by the authors to an interaction force term deduced earlier by Nikolaevskiy (see Nikolaevskiy 2005), based on phenomenological reasoning.

This theory is next deployed to analyze the equilibrium state of a fluid and elastic penetrable material, encapsulated in a rigid volume (Lopatnikov and Gillespie 2011). In Lopatnikov and Gillespie (2012) the derivation of interfacial conditions, compatible with the governing differential equations of the theory, is presented.

3.2 Variational Higher Gradient Theories by dell'Isola and Co-workers

An investigation of porous media following a consistent variational approach is the one pursued by Madeo et al. (2013), dell'Isola et al. (2009), Sciarra et al. (2007), dell'Isola et al. (2005a, b, 1998).

In dell'Isola et al. (1998), a micro–macro identification is indeed performed for a compaction of grounds with fluid inclusions with the fluid being confined into the pores. The work particularly focuses on the effect of a length-scale l characterizing pore size. The model is a microstructured continuum of the type introduced by Eringen, and the main result is the dependence of evolution equations on the length l.

In Sciarra et al. (2005), the behavior of a sponge under an increase of the outside fluid pressure is studied by using the Principle of Virtual Power, with second gradients of the displacement included as a further deformation measure. In particular, a simple idea is introduced: the boundary pressure is divided between the solid and fluid pressures, $p_f = d_f p^{ext}$, $p_s = d_s p^{ext}$ with $d_f + d_s = 1$; quantity, p_{ext} is the external pressure and d_f and d_s are coefficients which depend on the constituent apparent densities, regarded as state parameters, under the condition that the work performed by these tractions vanishes in every cyclic process over the parameter space. This condition restricts the permissible constitutive relations for the dividing coefficient, which turns out to be characterized by a single material parameter. Moreover, a stability analysis of the solutions is performed by Sciarra et al. (2005).

In dell'Isola et al. (2009), a (classical) solid fluid mixture is studied in the framework of an extended Hamilton–Rayleigh principle. A general set of boundary conditions at fluid-permeable interfaces between dissimilar fluid-filled porous matrices is established, including jump conditions, friction and inertia effects. In particular, solid and fluid domains $B_s \subset \Re^3$ and $B_f \subset \Re^3$ are introduced, as well as the maps

$$\boldsymbol{\chi}_s : \; B_s \times (0, T) \to \Re^3 \qquad \boldsymbol{\chi}_f : \; B_f \times (0, T) \to \Re^3, \tag{6}$$

which represent the (time dependent) placement of the solid and fluid constituent; the motion of the fluid inside the solid matrix is described by the function

$$\boldsymbol{\chi}_{sf} : \; B_s \times (0, T) \to B_f. \tag{7}$$

General motion equations relative to a representative elementary volume are then derived through lengthy computations:

$$\begin{gathered}
-\left(\bar{\rho}^{(s)}\dot{\mathbf{v}}_s + \bar{\rho}^{(f)}\dot{\mathbf{v}}_f^{Ⓢ}\right) + div\left(\mathbf{F}_s^T \cdot \frac{\partial \Psi}{\partial \mathbf{E}}\right) - \frac{\partial \Psi}{\partial \boldsymbol{\chi}_s} = -div\left(J_s(\boldsymbol{\Pi}_f^{Ⓢ})^T \cdot \mathbf{F}_s^{-T}\right) \\
\bar{\rho}^{(f)}\left[\mathbf{F}_s^T \cdot \dot{\mathbf{v}}_f^{Ⓢ} + \nabla\left(\frac{\partial \Psi}{\partial \bar{\rho}^{(f)}}\right)\right] = \mathbf{F}_s^T \cdot \left[J_s \kappa^{Ⓢ} - div\left(J_s(\boldsymbol{\Pi}^{Ⓢ} - \boldsymbol{\Pi}_f^{Ⓢ})^T \cdot \mathbf{F}_s^{-T}\right)\right]
\end{gathered} \tag{8}$$

with the following boundary conditions:

$$\begin{aligned}
&\left[\left|\mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \mathbf{E}} - (\mathbf{v}_f^{Ⓢ} - \mathbf{v}_s) \otimes \mathbf{D} + J_s(\boldsymbol{\Pi}_f^{Ⓢ})^T \cdot \mathbf{F}_s^{-T}\right|\right] \cdot \mathbf{N}_s = \mathbf{o} \\
&\left[\left|\mathbf{G}_s^{-T} \cdot \left(\bar{\rho}^{(f)} \frac{\partial \Psi}{\partial \bar{\rho}^{(f)}} \mathbf{N}_s - \frac{1}{2}\bar{\rho}^{(f)}\left(\mathbf{v}_f^{Ⓢ}\right)^2 \cdot \mathbf{N}_s + \mathbf{F}_s^T \cdot \mathbf{v}_f^{Ⓢ} \otimes \mathbf{D} \cdot \mathbf{N}_s\right) \cdot \boldsymbol{\Gamma}^{Ⓢ}\right|\right] \\
&+\left[\left|\mathbf{G}_s^{-T} \cdot \mathbf{F}_s^T \cdot \left(J_s\left((\boldsymbol{\Pi}^{Ⓢ} - \boldsymbol{\Pi}_f^{Ⓢ})^T\right) \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s - ||J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s|| \sigma^{Ⓢ}\right) \cdot \boldsymbol{\Gamma}^{Ⓢ}\right|\right] = \mathbf{o}.
\end{aligned}$$

In the previous equations, the following definitions are used: $\mathbf{F}_i = \nabla \boldsymbol{\chi}_i$, $\mathbf{G}_s = \nabla \boldsymbol{\chi}_{sf}$, $\mathbf{v}_i = \frac{\partial \boldsymbol{\chi}_i}{\partial t}$, $\mathbf{u}_i = \frac{\partial \boldsymbol{\chi}_{sf}}{\partial t}$. Moreover, in the general case Ψ is the sum of a non homogeneous deformation energy potential $\Psi_i(\mathbf{E}, \bar{\rho}^{(f)}, \mathbf{X}_s)$ and a potential accounting for external body forces $\Psi_g = (\bar{\rho}^{(s)} + \bar{\rho}^{(f)})E_p(\boldsymbol{\chi}_s, \mathbf{X}_s)$. Finally, $J_s = \det \mathbf{F}_s$, $\mathbf{E}$ is the Green-Lagrange strain tensor, $\boldsymbol{\Pi}$ is the Brinkman stress tensor, $\boldsymbol{\Pi}_f$ is the fluid viscous stress tensor and the acceleration fields $\dot{\mathbf{v}}_s$, $\dot{\mathbf{v}}_f$ are the time derivatives of $\mathbf{v}_s$ and $\mathbf{v}_f$ respectively, $\bar{\rho}^{(s)}$ and $\bar{\rho}^{(f)}$ are apparent mass densities for the solid and the fluid, and the symbol Ⓢ denotes the transport of a tensor field from the configuration where it is defined to B_s; for the meaning of the remaining symbols κ, σ, E_p, for the meaning of the $(\cdot)$ operations and for other details we refer the reader to the paper, and especially to the technical Appendices in (dell'Isola et al. 2009).

3.3 The VMTPM Framework and the Extrinsic/Intrinsic Treatment

Among more recent contributions, a two-phase poroelastic formulation, also based on the least-action principle, has been proposed in (Serpieri and Rosati 2011) and in (Serpieri 2011). Chronologically, such formulation is subsequent to the works of Lopatnikov and co-workers, and dell'Isola and co-workers. This theory, hereby abbreviated in VMTPM (Variational Macroscopic Theory of Porous Media), consists of an application to poroelastic problems of a generalized continuum formulation with additional kinematic descriptors, in the wake of the ideas of Mindlin and Beford and Drumheller. However, in contrast to previous applications of generalized continua theories to poroelastic problems, the kinematic of VMTPM is enriched with a so-called scalar field of *intrinsic* volumetric strain $\hat{J}^{(s)}$ in place of a porosity field. More specifically, $\hat{J}^{(s)}$ is an additional macroscopic scalar field, introduced on a purely kinematic rationale, which essentially corresponds to the ratio $\hat{\rho}^{(s)}/\hat{\rho}_0^{(s)}$ between 'true' densities before and after deformation. This field is independent from the primary macroscopic volumetric strain measure $\bar{J}^{(s)} = \det(\nabla \boldsymbol{\chi})$ which remains instead ordinarily defined as the determinant of the macroscopic deformation gradient, and termed *extrinsic* volumetric strain in order to remark its difference with $\hat{J}^{(s)}$. It should be noted that $\hat{J}^{(s)}$ has a direct relation with the porosity field. In a region of a porous medium undergoing a macroscopically homogeneous deformation, the value of $\hat{J}^{(s)}$ can be macroscopically measured by the following relation which links $\hat{J}^{(s)}$ to the porosities before ($\phi_0^{(f)}$) and after deformation ($\phi^{(f)}$):

$$\hat{J}^{(s)} = \bar{J}^{(s)} \left(1 - \phi^{(f)}\right) / \left(1 - \phi_0^{(f)}\right). \tag{9}$$

Hence if the medium is saturated and with completely interconnected pores, the measurement of $\hat{J}^{(s)}$ can be translated into the measurement of the fluid leaving or entering this region as a consequence of a loading-induced deformation.

Based on this choice of extrinsic/intrinsic kinematic descriptors, the associated stress measures consist of an extrinsic stress tensor, work-associated with the extrinsic strain ($\check{\boldsymbol{\sigma}}^{(s)}$), and an scalar intrinsic pressure $\hat{p}^{(s)}$ (Serpieri et al. 2013, 2015; Serpieri and Travascio 2015; Travascio et al. 2015). In such works it is shown that, in undrained conditions (i.e., when no relative solid-fluid motion takes place in a region of a biphasic mixture), VMTPM predicts that the external stress, the fluid pressure, and the stress tensor work-associated with the extrinsic strain of the solid phase are partitioned according to a relation which is formally strictly compliant with Terzaghi's law, irrespective of the microstructural and constitutive features of a given medium.

In (Serpieri and Travascio 2015) and (Travascio et al. 2015) the constitutive response for isotropic media is found to be strongly determined by an additional dimensionless parameter $\bar{k}_r$ with bounds $-1 \leq \bar{k}_r \leq 0$, in a way similar to the role played by the Poisson's coefficient in characterizing the isotropic response of a single continuum medium. In particular, $\bar{k}_r$ appears in the expression for linear isotropic media in undrained conditions (i.e., fast loading) of Skempton's coefficient B, defined as the ratio between the induced pressure p of the interstitial fluid and the applied stress, $t_x^{(ext)}$ (Skempton 1954). Specifically:

$$B = \frac{p}{t_x^{(ext)}} = -\frac{\left(1+\bar{k}_r\right)\hat{k}_{sf}}{\left[2\bar{\mu}+\bar{\lambda}+\left(1+\bar{k}_r\right)^2\hat{k}_{sf}\right]}, \tag{10}$$

where $\bar{\mu}$ and $\bar{\lambda}$ are Lamé moduli and $\hat{k}_{sf}$ is a coupling modulus of intrinsic stiffness in series. The reader is again referred to Serpieri and Travascio (2015) for an exhaustive definition of these parameters on a variational basis. Serpieri and Travascio (2015) have found a peculiar mechanical behavior predicted by VMTMP which is discriminated by $\bar{k}_r$: the extrinsic pressure can actually become *tensile* (negative) even in presence of compressive external stresses under specific values of $\bar{k}_r$. Although such behavior may appear counterintuitive at first sight (by drawing a straightforward parallel with the traditional Cauchy stress tensor in single continuum mechanics), it is shown in the referenced work that such condition entails no violation of positive definiteness of strain energy, so that compressive external tractions always induce compressive strains and the interstitial fluid pressures is always positive and compressive.

Also, Travascio et al. (2015) have reported another unique feature characterizing VMTPM mixtures and, once more, modulated by $\bar{k}_r$: during displacement-controlled static compression, the mixture can express either a *stress-relaxing* or a *stress-tensing* behavior. The stress relaxation is a well-known phenomenon in poroelasticity, whose description has been documented in several studies (Mow et al. 1980; Ehlers and Bluhm 2013): the solid stress increases as the compression is applied; subsequently, fluid redistribution within the mixture occurs, and the stress relaxes to an equilibrium value which is held indefinitely, as long as the system is compressed. The behavior of stress tensing mixtures is substantially different: during compression, the solid stress progressively tenses upon reaching, in its drained state, an equilibrium value once again depending on applied deformation and stiffness of the solid phase. As a further confirmation of the important role played by $\bar{k}_r$, Travascio et al. (2013) showed in a numerical study simulating uniaxial stress relaxation tests on bovine articular cartilage that the consolidation time of the tissue reduces three-fold when $\bar{k}_r$ varies from 0 to -0.25.

4 Conclusions

The present survey on variational macroscopic continuum approaches to multiphase poroelasticity highlighted the existence of fundamental features shared by the theories reviewed in this paper. Also, several aspects have been pointed out, where agreement between the surveyed theories is not found. As such, these aspects deserve further investigation by the generalized continua community.

A fundamental feature which almost all the theories herein presented have in common is the resort to kinematics with additional descriptors (i.e., porosity, intrinsic strain, etc.) for a proper formulation of the problem. In this respect, generalized continua models appear to be the natural setting to properly address the multiphase problem, even in absence of a specific focus on microstructural or multi-scale effects.

On the other hand, some important still-open issues can be identified, where further investigation is needed either to assess the higher degree of mechanical consistency and of predictive capabilities of any of the existing frameworks over the others, or to formulate more comprehensive theories. In particular, the following issues are considered to be relevant:

- The role of constraints in relation to the variational treatment, with special reference to mass balance; in particular the well-posedness of the variational statement of the problem in presence of mass balance constraints for the primary fields appears to be a relevant research issue.
- In variational theories making use of Lagrange multipliers, an assessment of the physical meaning of stress quantities in relation to boundary data and to the macroscopic measurement process could be a relevant research endeavor.
- Even if the set of Euler–Lagrange equations for multiphase problems appears to be very broad (as very broad are the possibilities of conceiving enriched kinematics in generalized continua frameworks) an important objective for the generalized continua community should be the agreement on a set of minimal medium-independent equilibrium equations. Moreover, any new theory should be downward compatible with such equations.
- The theory should be based on the minimum possible number of parameters, which should have a clear physical-mechanical meaning. In addition, their experimental characterization should be possible.
- The identification of a generally agreed set of governing balance equations necessary to achieve a consistent formulation of compressible poroelasticity in a vari-

ational multiphase framework could benefit from contributions coming from all disciplines (e.g., theoretical mechanics, geomechanics, biomechanics, etc.) with the aim of identifying appropriate benchmark programs for validation of continuum poroelasticity theories.

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