

# On Strain Rate Tensors and Constitutive Equations of Inelastic Micropolar Materials

Holm Altenbach and Victor A. Eremeyev

**Abstract** Following Altenbach and Eremeyev (Int J Plast 63:3–17, 2014) we introduce a new family of strain rate tensors for micropolar materials. With the help of introduced strain rates we discuss the possible forms of constitutive equations of the nonlinear inelastic micropolar continuum, that is micropolar viscous and viscoelastic fluids and solids, hypo-elastic and viscoelastoplastic materials. Considering the fact that some of strain rates are not true tensors but pseudotensors we obtain some constitutive restrictions following from the material frame indifference principle. Using the theory of tensorial invariants we present the general form of constitutive equations of some types of inelastic isotropic micropolar materials including several new constitutive equations.

**Keywords** Micropolar continua · Strain rate · Constitutive equations · Finite deformations

## 1 Introduction

Nonlinear micropolar continuum model allows to describe complex micro-structured media, for example, polycrystals, foams, cellular solids, lattices, masonries, particle assemblies, magnetic rheological fluids, liquid crystals, etc., for which the rotational degrees of freedom of material particles are important. In the case of inelastic behavior the constitutive equations of the micropolar continuum have more complicated structure, the stress and couple stress tensors as well as other quantities depend

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on the history of strain measures. For the basics of micropolar mechanics we refer to Nowacki (1986), Eringen (1999), Eringen and Kafadar (1976), Eringen (2001), Eremeyev et al. (2013), Łukaszewicz (1999).

Below we discuss the constitutive equations of the nonlinear micropolar continuum considering strain rates. The discussion of strain measures for polar elastic materials presented by Pietraszkiewicz and Eremeyev (2009a, b), where the natural Lagrangian and Eulerian strain measures are introduced. Strain–stress pairs within the framework of micropolar mechanics are discussed in Ramezani and Naghdabadi (2007). Using strain rate tensors incremental equations of the micropolar hypo-elasticity were presented in Ramezani and Naghdabadi (2010), Ramezani et al. (2008). The interrelations between strain rates in discrete and continual models are discussed in Trovalusci and Masiani (1997), Pau and Trovalusci (2012). Here we discuss the Rivlin–Ericksen analogues of strain rate tensors for micropolar mechanics and several types of the constitutive equations of inelastic micropolar solids are summarized.

## 2 Basic Relations of the Micropolar Mechanics

Following Eringen (1999), Pietraszkiewicz and Eremeyev (2009a), Eremeyev et al. (2013) let us recall the basic equations of micropolar mechanics under finite deformations. In what follows we use the standard direct tensor notations (Lebedev et al. 2010; Truesdell 1966; Truesdell and Noll 2004). For example, the gradient and divergence operators in the actual and reference configurations are defined as follows

$$\begin{aligned} \text{grad}(\bullet) &= \frac{\partial(\bullet)}{\partial x_k} \otimes \mathbf{r}^k, & \text{div}(\bullet) &= \frac{\partial(\bullet)}{\partial x_k} \cdot \mathbf{r}^k, & \mathbf{r}_i &= \frac{\partial \mathbf{r}}{\partial x_i}, & \mathbf{r}_i \cdot \mathbf{r}^k &= \delta_i^k, \\ \text{Grad}(\bullet) &= \frac{\partial(\bullet)}{\partial X_k} \otimes \mathbf{R}^k, & \text{Div}(\bullet) &= \frac{\partial(\bullet)}{\partial X_k} \cdot \mathbf{R}^k, & \mathbf{R}_i &= \frac{\partial \mathbf{R}}{\partial x_i}, & \mathbf{R}_i \cdot \mathbf{R}^k &= \delta_i^k, \end{aligned}$$

where  $x_i$  and  $X_i$  are the Eulerian and Lagrangian coordinates, respectively, and  $\delta_i^j$  is the Kronecker symbol.

### 2.1 Kinematics

We describe the micropolar continuum deformation by the following relations:

$$\mathbf{r} = \mathbf{r}(\mathbf{R}, t), \quad \mathbf{H} \equiv \mathbf{d}_k \otimes \mathbf{D}_k = \mathbf{H}(\mathbf{R}, t). \quad (1)$$

Vector  $\mathbf{r}(t)$  describes the position of the particle of the continuum at time  $t$ , whereas  $\mathbf{H}(t)$  defines its rotation. The linear velocity is given by the relation

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad (2)$$

the angular velocity vector  $\boldsymbol{\omega}$  can be presented by

$$\frac{d\mathbf{d}_k}{dt} = \boldsymbol{\omega} \times \mathbf{d}_k, \quad k = 1, 2, 3,$$

where  $\times$  is the vector (cross) product.  $\boldsymbol{\omega}$  can be also expressed using the derivative of  $\mathbf{H}$  as follows

$$\boldsymbol{\omega} = -\frac{1}{2} \left( \frac{d\mathbf{H}}{dt} \cdot \mathbf{H}^T \right)_{\times}, \quad (3)$$

where subindex  $\times$  stands for the vectorial invariant of second-order tensor (Lebedev et al. 2010).

## 2.2 Motion Equations

The Eulerian equations of motion of micropolar media are

$$\rho \frac{d\mathbf{v}}{dt} = \text{div } \mathbf{T} + \rho \mathbf{f}, \quad j \frac{d\boldsymbol{\omega}}{dt} = \text{div } \mathbf{M} - \mathbf{T}_{\times} + \rho \mathbf{m}, \quad (4)$$

where  $\mathbf{T}$  and  $\mathbf{M}$  are the stress and couple stress tensors of Cauchy type which are non-symmetric, in general,  $\rho$  is the density in the actual configuration,  $j$  is the measure of rotatory inertia of particles of micropolar medium,  $\mathbf{f}$  and  $\mathbf{m}$  are the external forces and couples, respectively.

## 2.3 Constitutive Equations

In the pure mechanical theory of the micropolar continuum with memory the constitutive equations consist of dependence of the stress and the couple stress tensors of the history of deformations. As a result,  $\mathbf{T}$  and  $\mathbf{M}$  take the following form:

$$\mathbf{T}(t) = \mathcal{A}_1[\mathbf{F}^t(s), \mathbf{H}^t(s), \text{Grad } \mathbf{H}^t(s)], \quad \mathbf{M}(t) = \mathcal{A}_2[\mathbf{F}^t(s), \mathbf{H}^t(s), \text{Grad } \mathbf{H}^t(s)], \quad (5)$$

where we introduced the histories of the deformation gradient

$$\mathbf{F}^t(s) = \mathbf{F}(t - s), \quad \mathbf{H}^t(s) = \text{Grad } \mathbf{r}(t), \quad s \geq 0,$$

and of the microrotation tensor

$$\mathbf{H}^t(s) = \mathbf{H}(t - s), \quad s \geq 0.$$

Here  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are operators describing the micropolar material behavior.

The further reduction of (5) is possible using the principle of material frame-indifference. The stress measures  $\mathbf{T}$  and  $\mathbf{M}$  should be indifferent (objective) quantities. In classical mechanics, two motions  $\mathbf{r}$  and  $\mathbf{r}^*$  are called equivalent if they relate as follows

$$\mathbf{r}^* = \mathbf{a}(t) + \mathbf{O}(t) \cdot (\mathbf{r} - \mathbf{r}_0), \quad (6)$$

where  $\mathbf{O}(t)$  is an arbitrary orthogonal tensor,  $\mathbf{a}(t)$  is an arbitrary vector function and the constant vector  $\mathbf{r}_0$  represents a fixed point position (a pole). We assume that in the equivalent motion the directors  $\mathbf{d}_k$  transform similarly to  $\mathbf{r}$ :

$$\mathbf{d}_k^* = \mathbf{O}(t) \cdot \mathbf{d}_k \quad \text{or} \quad \mathbf{H}^* = \mathbf{O}(t) \cdot \mathbf{H}. \quad (7)$$

Denoting by superscript “\*” the stress tensors in the equivalent motions we formulate the property of objectivity for  $\mathbf{T}$  and  $\mathbf{M}$  as follows

$$\mathbf{T}^* = \mathbf{O}(t) \cdot \mathbf{T} \cdot \mathbf{O}(t)^T, \quad \mathbf{M}^* = \det \mathbf{O}(t) \mathbf{O}(t) \cdot \mathbf{M} \cdot \mathbf{O}(t)^T \quad (8)$$

for any orthogonal tensor  $\mathbf{O}(t)$ . Let us note that  $\mathbf{M}$  is an pseudotensor, that is a reason of difference in transformation rules (8).

Thus, operators  $\mathcal{A}_1$  and  $\mathcal{A}_2$  satisfy the relations

$$\begin{aligned} \mathcal{A}_1 [\mathbf{O}^t(s) \cdot \mathbf{F}^t(s), \mathbf{O}^t(s) \cdot \mathbf{H}^t(s), \mathbf{O}^t(s) \cdot \text{Grad } \mathbf{H}^t(s)] \\ = \mathbf{O}(t) \cdot \mathcal{A}_1[\mathbf{F}^t(s), \mathbf{H}^t(s), \text{Grad } \mathbf{H}^t(s)] \cdot \mathbf{O}^T(t), \end{aligned} \quad (9)$$

$$\begin{aligned} \mathcal{A}_2 [\mathbf{O}^t(s) \cdot \mathbf{F}^t(s), \mathbf{O}^t(s) \cdot \mathbf{H}^t(s), \mathbf{O}^t(s) \cdot \text{Grad } \mathbf{H}^t(s)] \\ = \det \mathbf{O}(t) \mathbf{O}(t) \cdot \mathcal{A}_2[\mathbf{F}^t(s), \mathbf{H}^t(s), \text{Grad } \mathbf{H}^t(s)] \cdot \mathbf{O}^T(t) \end{aligned} \quad (10)$$

Finally, we can prove that (5) can be represented as follows

$$\mathbf{T}(t) = \mathbf{H}(t) \cdot \mathcal{B}_1[\mathbf{E}^t(s), \mathbf{K}^t(s)] \cdot \mathbf{H}^T(t), \quad \mathbf{M}(t) = \mathbf{H}(t) \cdot \mathcal{B}_2[\mathbf{E}^t(s), \mathbf{K}^t(s)] \cdot \mathbf{H}^T(t), \quad (11)$$

where  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are operators depending on histories of two Lagrangian strain measures  $\mathbf{E}$  and  $\mathbf{K}$  defined by formulas (Pietraszkiewicz and Eremeyev 2009a)

$$\mathbf{E} = \mathbf{H}^T \cdot \mathbf{F} - \mathbf{I}, \quad \mathbf{K} = -\frac{1}{2} \boldsymbol{\epsilon} : (\mathbf{H}^T \cdot \text{Grad } \mathbf{H}). \quad (12)$$

## 2.4 Elastic Materials

In the case of elastic behaviour Eq. (11) reduce to

$$\mathbf{T}(t) = \mathbf{H}(t) \cdot f_1[\mathbf{E}(t), \mathbf{K}(t)] \cdot \mathbf{H}^T(t), \quad \mathbf{M}(t) = \mathbf{H}(t) \cdot f_2[\mathbf{E}(t), \mathbf{K}(t)] \cdot \mathbf{H}^T(t), \quad (13)$$

where vector functions  $f_1$  and  $f_2$  can be expressed with use of the strain energy function  $W = W(\mathbf{E}, \mathbf{K})$ . For isotropic micropolar elastic solids  $W$  is considered as a function of two strain measures which can be represented as a scalar function depending on 15 joint invariants of  $\mathbf{E}$  and  $\mathbf{K}$ , see Eringen and Kafadar (1976), Eremeyev and Pietraszkiewicz (2012)

$$W = W(I_1, \dots, I_{15}).$$

In particular, assuming  $W$  in the form  $W = W_1(\mathbf{E}) + W_2(\mathbf{K})$  one obtains that functions  $W_1$  and  $W_2$  depend on six invariants of  $\mathbf{E}$  and  $\mathbf{K}$ . An isotropic scalar-valued function of one non-symmetric tensor  $\mathbf{E}$  can be constructed as a function of six invariants  $I_n$ ,  $n = 1, \dots, 6$ , where

$$I_1 = \text{tr } \mathbf{E}, \quad I_2 = \text{tr } \mathbf{E}^2, \quad I_3 = \text{tr } \mathbf{E}^3, \quad I_4 = \text{tr } (\mathbf{E} \cdot \mathbf{E}^T), \quad I_5 = \text{tr } (\mathbf{E}^2 \cdot \mathbf{E}^T), \quad I_6 = \text{tr } (\mathbf{E}^2 \cdot \mathbf{E}^{T2}). \quad (14)$$

An isotropic scalar-valued function of two non-symmetric tensors  $\mathbf{E}$  and  $\mathbf{K}$  depends on the following 15 invariants:

$$\begin{aligned} I_1 &= \text{tr } \mathbf{E}, & I_2 &= \text{tr } \mathbf{E}^2, & I_3 &= \text{tr } \mathbf{E}^3, \\ I_4 &= \text{tr } (\mathbf{E} \cdot \mathbf{E}^T), & I_5 &= \text{tr } (\mathbf{E}^2 \cdot \mathbf{E}^T), & I_6 &= \text{tr } (\mathbf{E}^2 \cdot \mathbf{E}^{T2}), \\ I_7 &= \text{tr } (\mathbf{E} \cdot \mathbf{K}), & I_8 &= \text{tr } (\mathbf{E}^2 \cdot \mathbf{K}), & I_9 &= \text{tr } (\mathbf{E} \cdot \mathbf{K}^2), \\ I_{10} &= \text{tr } \mathbf{K}, & I_{11} &= \text{tr } \mathbf{K}^2, & I_{12} &= \text{tr } \mathbf{K}^3, \\ I_{13} &= \text{tr } (\mathbf{K} \cdot \mathbf{K}^T), & I_{14} &= \text{tr } (\mathbf{K}^2 \cdot \mathbf{K}^T), & I_{15} &= \text{tr } (\mathbf{K}^2 \cdot \mathbf{K}^{T2}) \end{aligned} \quad (15)$$

## 3 Relative Strain Measures

For using the fading memory concept let us introduce relative strain measures. Within the framework of so-called relative description of deformation of continuum we consider the actual configuration  $\chi$  at instant  $t$  as the reference one while the actual configuration  $\chi$  at instant  $\tau$  is considered as actual one. The relative deformation gradient and the relative microrotation tensor are introduced by formulas

$$\mathbf{F}_t(\tau) = \mathbf{F}(\tau) \cdot \mathbf{F}^{-1}(t), \quad \mathbf{H}_t(\tau) = \mathbf{d}_k(\tau) \times \mathbf{d}_k(t) = \mathbf{H}(\tau) \cdot \mathbf{H}^{-1}(t). \quad (16)$$

Obviously,  $\mathbf{F}_t(t) = \mathbf{I}$ ,  $\mathbf{H}_t(t) = \mathbf{I}$ . Using  $\mathbf{F}_t(\tau)$  and  $\mathbf{H}_t(\tau)$  we introduce the relative strain measures  $\mathbf{E}_t(\tau)$ ,  $\mathbf{K}_t(\tau)$  by the relations

$$\mathbf{E}_t(\tau) = \mathbf{H}_t(\tau)^T \cdot \mathbf{F}_t(\tau) - \mathbf{I}, \quad \mathbf{K}_t(\tau) = -\frac{1}{2}\boldsymbol{\epsilon} : [\mathbf{H}_t(\tau)^T \cdot \text{Grad } \mathbf{H}_t(\tau)]. \quad (17)$$

In what follows we denote the histories of relative tensors  $\mathbf{F}_t(\tau)$ ,  $\mathbf{H}_t(\tau)$ , etc., as follows

$$\mathbf{F}_t^s(s) = \mathbf{F}_t(t-s), \quad \mathbf{H}_t^s(s) = \mathbf{H}_t(t-s).$$

From (16) and (17) it follows the following relations:

$$\mathbf{U}^t(s) = \mathbf{H}^T(t) \cdot \mathbf{U}_t^s(s) \cdot \mathbf{H}(t) \cdot \mathbf{U}(t), \quad \mathbf{Y}^t(s) = \mathbf{H}^T(t) \cdot \mathbf{Y}_t^s(s) \cdot \mathbf{H}(t) \cdot \mathbf{U}(t), \quad (18)$$

$$\mathbf{E}_t^s(s) = \mathbf{U}_t^s(s) - \mathbf{I}, \quad \mathbf{K}_t^s(s) = \mathbf{Y}_t^s(s) - \mathbf{B}(t) \quad (19)$$

and

$$\mathbf{U}_t^s(0) = \mathbf{I}, \quad \mathbf{Y}_t^s(0) = \mathbf{B}(t), \quad \mathbf{E}_t^s(0) = \mathbf{0}, \quad \mathbf{K}_t^s(0) = \mathbf{0},$$

where we introduced the histories

$$\mathbf{U}_t^s(s) = \mathbf{U}_t(t-s), \quad \mathbf{Y}_t^s(s) = \mathbf{Y}_t(t-s), \quad \mathbf{E}_t^s(s) = \mathbf{E}_t(t-s), \quad \mathbf{K}_t^s(s) = \mathbf{K}_t(t-s).$$

## 4 Relations of Isotropic Materials with Relative Strain Measures

Substituting Eqs. (18) and (19) into (11) we obtain

$$\begin{aligned} \mathbf{T}(t) &= \mathbf{H}(t) \cdot \mathcal{B}_1 \left[ \mathbf{H}^T(t) \cdot (\mathbf{U}_t^s(s) - \mathbf{I}) \cdot \mathbf{H}(t) \cdot \mathbf{U}(t), \right. \\ &\quad \left. \mathbf{H}^T(t) \cdot (\mathbf{Y}_t^s(s) - \mathbf{B}(t)) \cdot \mathbf{H}(t) \cdot \mathbf{U}(t) \right] \cdot \mathbf{H}^T(t), \\ \mathbf{M}(t) &= \mathbf{H}(t) \cdot \mathcal{B}_2 \left[ \mathbf{H}^T(t) \cdot (\mathbf{U}_t^s(s) - \mathbf{I}) \cdot \mathbf{H}(t) \cdot \mathbf{U}(t), \right. \\ &\quad \left. \mathbf{H}^T(t) \cdot (\mathbf{Y}_t^s(s) - \mathbf{B}(t)) \cdot \mathbf{H}(t) \cdot \mathbf{U}(t) \right] \cdot \mathbf{H}^T(t). \end{aligned}$$

The latter relations transform to

$$\begin{aligned} \mathbf{T}(t) &= \mathbf{H}(t) \cdot \mathcal{C}_1 \left[ \mathbf{U}(t), \mathbf{Y}(t), \mathbf{H}^T(t) \cdot \mathbf{U}_t^s(s) \cdot \mathbf{H}(t), \mathbf{H}^T(t) \cdot \mathbf{Y}_t^s(s) \cdot \mathbf{H}(t) \right] \cdot \mathbf{H}^T(t), \\ \mathbf{M}(t) &= \mathbf{H}(t) \cdot \mathcal{C}_2 \left[ \mathbf{U}(t), \mathbf{Y}(t), \mathbf{H}^T(t) \cdot \mathbf{U}_t^s(s) \cdot \mathbf{H}(t), \mathbf{H}^T(t) \cdot \mathbf{Y}_t^s(s) \cdot \mathbf{H}(t) \right] \cdot \mathbf{H}^T(t) \end{aligned}$$

with new operators  $\mathcal{C}_1$  and  $\mathcal{C}_2$ .

Further reduction of constitutive equation is possible assuming some type of anisotropy as was done by Eremeyev and Pietraszkiewicz (2012). In what follows we restrict ourselves by isotropic behavior. In this case  $\mathcal{C}_1$  and  $\mathcal{C}_2$  should satisfy the restrictions

$$\begin{aligned}
& \mathbf{O}^T \cdot \mathcal{C}_1 \left[ \mathbf{U}(t), \mathbf{Y}(t), \mathbf{H}^T(t) \cdot \mathbf{U}_t^t(s) \cdot \mathbf{H}(t), \mathbf{H}^T(t) \cdot \mathbf{Y}_t^t(s) \cdot \mathbf{H}(t) \right] \cdot \mathbf{O} \\
& = \mathcal{C}_1 \left[ \mathbf{O}^T \cdot \mathbf{U}(t) \cdot \mathbf{O}, (\det \mathbf{O}) \mathbf{O}^T \cdot \mathbf{Y}(t) \cdot \mathbf{O}, \right. \\
& \quad \left. \mathbf{O}^T \cdot \mathbf{H}^T(t) \cdot \mathbf{U}_t^t(s) \cdot \mathbf{H}(t) \cdot \mathbf{O}, (\det \mathbf{O}) \mathbf{O}^T \cdot \mathbf{H}^T(t) \cdot \mathbf{Y}_t^t(s) \cdot \mathbf{H}(t) \cdot \mathbf{O} \right], \\
& (\det \mathbf{O}) \mathbf{O}^T \cdot \mathcal{C}_2 \left[ \mathbf{U}(t), \mathbf{Y}(t), \mathbf{H}^T(t) \cdot \mathbf{U}_t^t(s) \cdot \mathbf{H}(t), \mathbf{H}^T(t) \cdot \mathbf{Y}_t^t(s) \cdot \mathbf{H}(t) \right] \cdot \mathbf{O} \\
& = \mathcal{C}_2 \left[ \mathbf{O}^T \cdot \mathbf{U}(t) \cdot \mathbf{O}, (\det \mathbf{O}) \mathbf{O}^T \cdot \mathbf{Y}(t) \cdot \mathbf{O}, \mathbf{O}^T \cdot \mathbf{H}^T(t) \cdot \mathbf{U}_t^t(s) \cdot \mathbf{H}(t) \cdot \mathbf{O}, \right. \\
& \quad \left. (\det \mathbf{O}) \mathbf{O}^T \cdot \mathbf{H}^T(t) \cdot \mathbf{Y}_t^t(s) \cdot \mathbf{H}(t) \cdot \mathbf{O} \right]
\end{aligned} \tag{20}$$

for all orthogonal tensors  $\mathbf{O}$ ,  $\mathbf{O}^{-1} = \mathbf{O}^T$ . As the result we obtain

$$\begin{aligned}
\mathbf{T}(t) &= \mathcal{C}_1 \left[ \mathbf{H} \cdot \mathbf{U}(t) \cdot \mathbf{H}^T(t), \mathbf{H} \cdot \mathbf{Y}(t) \cdot \mathbf{H}^T(t), \mathbf{U}_t^t(s), \mathbf{Y}_t^t(s) \right], \\
\mathbf{M}(t) &= \mathcal{C}_2 \left[ \mathbf{H} \cdot \mathbf{U}(t) \cdot \mathbf{H}^T(t), \mathbf{H} \cdot \mathbf{Y}(t) \cdot \mathbf{H}^T(t), \mathbf{U}_t^t(s), \mathbf{Y}_t^t(s) \right].
\end{aligned}$$

Thus, the constitutive equations of any isotropic micropolar medium with memory take the following form

$$\mathbf{T}(t) = \mathcal{D}_1[\mathbf{e}(t), \mathbf{k}(t), \mathbf{U}_t^t(s), \mathbf{Y}_t^t(s)], \quad \mathbf{M}(t) = \mathcal{D}_2[\mathbf{e}(t), \mathbf{k}(t), \mathbf{U}_t^t(s), \mathbf{Y}_t^t(s)], \tag{21}$$

where  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are isotropic operators and the Eulerian strain measures defined by

$$\mathbf{e} = \mathbf{I} - \mathbf{H} \cdot \mathbf{F}^{-1}, \quad \mathbf{k} = \mathbf{H} \cdot \mathbf{K} \cdot \mathbf{F}^{-1}, \quad \mathbf{u} = \mathbf{H} \cdot \mathbf{F}^{-1}. \tag{22}$$

## 5 Rivlin–Ericksen Tensors

The history of  $\mathbf{U}_t(\tau)$  and  $\mathbf{Y}_t(\tau)$  can be represented as series with respect of two families of tensors  $\mathbf{A}_k$  and  $\mathbf{B}_k$  as follows

$$\mathbf{U}_t^t(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k} s^k \mathbf{A}_k(t), \quad \mathbf{Y}_t^t(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k} s^k \mathbf{B}_k(t). \tag{23}$$

In the micropolar continuum tensors  $\mathbf{A}_k$  and  $\mathbf{B}_k$  play a role of the Rivlin–Ericksen tensors used in the nonlinear viscoelasticity of simple materials. They are given by the recurrent relations

$$\begin{aligned}
\mathbf{A}_{k+1} &= \frac{d}{dt} \mathbf{A}_k + \mathbf{A}_k \cdot \text{grad} \mathbf{v} - \boldsymbol{\omega} \times \mathbf{A}_k, \quad \mathbf{A}_0 = \mathbf{I}, \\
\mathbf{B}_{k+1} &= \frac{d}{dt} \mathbf{B}_k + \mathbf{B}_k \cdot \text{grad} \mathbf{v} - \boldsymbol{\omega} \times \mathbf{B}_k, \quad \mathbf{B}_0 = \mathbf{B}.
\end{aligned}$$

Tensors  $\mathbf{A}_k$  and  $\mathbf{B}_k$  can be also represented using the derivative of  $\mathbf{U}$  and  $\mathbf{Y}$  as follows

$$\mathbf{A}_k = \mathbf{H} \cdot \frac{d^k \mathbf{U}}{dt^k} \cdot \mathbf{F}^{-1}, \quad \mathbf{B}_k = \mathbf{H} \cdot \frac{d^k \mathbf{Y}}{dt^k} \cdot \mathbf{F}^{-1}, \quad (24)$$

or by formulae

$$\mathbf{A}_{k+1} = \mathbf{A}_k^\circ, \quad \mathbf{B}_{k+1} = \mathbf{B}_k^\circ, \quad (25)$$

where the corotational derivative is defined by the relations

$$(\dots)^\circ = \mathbf{H} \cdot \frac{d}{dt} [\mathbf{H}^T \cdot (\dots) \cdot \mathbf{F}] \cdot \mathbf{F}^{-1} \equiv \frac{d}{dt} (\dots) + (\dots) \cdot \text{grad} \mathbf{v} - \boldsymbol{\omega} \times (\dots). \quad (26)$$

Let us note that  $\mathbf{A}_1$  and  $\mathbf{B}_1$  coincide with the strain rates used in the theory of micropolar continuum

$$\mathbf{A}_1 = \boldsymbol{\varepsilon} \equiv \text{grad} \mathbf{v} - \mathbf{I} \times \boldsymbol{\omega}, \quad \mathbf{B}_1 = \boldsymbol{\kappa} \equiv \text{grad} \boldsymbol{\omega}. \quad (27)$$

For example, stress power in the micropolar continuum is given by  $w = \mathbf{T} : \boldsymbol{\varepsilon} + \mathbf{M} : \boldsymbol{\kappa}$ .

## 6 Examples of Constitutive Equations

### 6.1 Linear Viscous Micropolar Fluid

The simplest example of an inelastic micropolar material is the micropolar viscous fluid with the constitutive equations (Aero et al. 1965; Eringen 1966)

$$\mathbf{T} = -p(\rho)\mathbf{I} + \alpha_1 \boldsymbol{\varepsilon} + \alpha_2 \boldsymbol{\varepsilon}^T + \alpha_3 \mathbf{I} \text{tr} \boldsymbol{\varepsilon}, \quad \mathbf{M} = \beta_1 \boldsymbol{\kappa} + \beta_2 \boldsymbol{\kappa}^T + \beta_3 \mathbf{I} \text{tr} \boldsymbol{\kappa}, \quad (28)$$

where  $p$  is the pressure,  $\rho$  is the density,  $\alpha_1, \alpha_2, \alpha_3$  and  $\beta_1, \beta_2, \beta_2$  are viscosities.

### 6.2 Non-linear Viscous Micropolar Fluid

The further generalization of (28) is non-linear viscous micropolar fluid with the following constitutive equations:

$$\mathbf{T} = -p(\rho)\mathbf{I} + \mathbf{T}_v(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad \mathbf{M} = \mathbf{M}_v(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad (29)$$

where  $\mathbf{T}_v(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$  and  $\mathbf{M}_v(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$  are non-linear isotropic functions of two non-symmetric 2nd order tensors. Such model may be applied for highly viscous suspensions or



ferrofluids. Assuming the existence of a dissipative potential that is a scalar isotropic function  $\Phi(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \geq 0$  such that

$$\mathbf{T}_v = \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{M}_v = \frac{\partial \Phi}{\partial \boldsymbol{\kappa}},$$

we can apply the theory of invariants for representation of  $\Phi$ . As a result  $\Phi$  depends on 15 invariants of  $I_j(\boldsymbol{\varepsilon}, \boldsymbol{\kappa})$ ,  $j = 1, \dots, 15$  and should satisfy the requirement

$$\begin{aligned} \Phi(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}, I_{12}, I_{13}, I_{14}, I_{15}) \\ = \Phi(I_1, I_2, I_3, I_4, I_5, I_6, -I_7, -I_8, I_9, -I_{10}, I_{11}, -I_{12}, I_{13}, -I_{14}, I_{15}), \end{aligned} \quad (30)$$

since  $I_7, I_8, I_{10}, I_{12}, I_{14}$  are the relative invariants and change sign during the non-proper transformations. For linear viscous fluid  $\Phi$  is a quadratic potential

$$\Phi = \alpha_1 I_1^2 + \alpha_2 I_2 + \alpha_3 I_4 + \beta_1 I_{10}^2 + \beta_2 I_{11} + \beta_3 I_{13}$$

and Eq. (29) reduce to the linear case (28).

### 6.3 Viscoelastic Micropolar Fluids

The model of viscous micropolar fluid can be generalized to the case of viscoelastic behaviour. The viscoelastic micropolar fluid has the following constitutive relations (Yeremeyev and Zubov 1999):

$$\mathbf{T} = \mathcal{H}_1[\rho(t), \mathbf{B}(t), \mathbf{E}'_t(s), \mathbf{K}'_t(s)], \quad \mathbf{M} = \mathcal{H}_2[\rho(t), \mathbf{B}(t), \mathbf{E}'_t(s), \mathbf{K}'_t(s)], \quad (31)$$

where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are isotropic operators. In particular, we define the viscoelastic micropolar fluid of differential type of order  $(m, n)$  as a micropolar fluid with following constitutive dependencies:

$$\mathbf{T} = \mathbf{h}_1(\rho, \mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, \mathbf{B}_1 \dots \mathbf{B}_n), \quad \mathbf{M} = \mathbf{h}_2(\rho, \mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, \mathbf{B}_1 \dots \mathbf{B}_n), \quad (32)$$

where  $\mathbf{h}_1$  and  $\mathbf{h}_2$  are tensor-valued isotropic functions of  $m + n + 1$  tensorial arguments. Since  $\mathbf{M}, \mathbf{B}_k, k = 0, 1, \dots, n$  are pseudo-tensors we prove that  $\mathbf{h}_1$  and  $\mathbf{h}_2$  satisfy the relations

$$\begin{aligned} \mathbf{h}_1(\rho, \mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, \mathbf{B}_1 \dots \mathbf{B}_n) &= \mathbf{h}_1(\rho, -\mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, -\mathbf{B}_1 \dots -\mathbf{B}_n), \\ -\mathbf{h}_2(\rho, \mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, \mathbf{B}_1 \dots \mathbf{B}_n) &= \mathbf{h}_2(\rho, -\mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, -\mathbf{B}_1 \dots -\mathbf{B}_n). \end{aligned}$$

## 6.4 Micropolar Hypo-elasticity

Original model of hypoelastic material was introduced in Truesdell (1963). For micropolar solids it was generalized in Tejchman and Bauer (2005), Ramezani et al. (2008), Ramezani and Naghdabadi (2010), Surana and Reddy (2015). Within the framework of the hypo-elastic micropolar solids the constitutive equations for  $\mathbf{T}$  and  $\mathbf{M}$  are formulated as follows

$$\mathbf{T}^\circ = \boldsymbol{\eta}_1(\mathbf{T}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad \mathbf{M}^\circ = \boldsymbol{\eta}_2(\mathbf{M}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad (33)$$

where  $^\circ$  denotes an objective time derivative, and  $\boldsymbol{\eta}_1$  and  $\boldsymbol{\eta}_2$  are isotropic functions of their arguments and linear with respect to strain rates  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\kappa}$ . As a result, Eq. (33) take the form

$$\mathbf{T}^\circ = \mathbf{C}_1(\mathbf{T}) : \boldsymbol{\varepsilon} + \mathbf{C}_2(\mathbf{T}) : \boldsymbol{\kappa}, \quad \mathbf{M}^\circ = \mathbf{C}_3(\mathbf{M}) : \boldsymbol{\varepsilon} + \mathbf{C}_4(\mathbf{M}) : \boldsymbol{\kappa}, \quad (34)$$

where  $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$  are 4th-order tensors, which depend on stress and couple stress tensors, in general. The following restrictions for  $\boldsymbol{\eta}_1$  and  $\boldsymbol{\eta}_2$  and  $\mathbf{C}_k$ :

$$\boldsymbol{\eta}_1(\mathbf{T}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \boldsymbol{\eta}_1(\mathbf{T}, \boldsymbol{\varepsilon}, -\boldsymbol{\kappa}), \quad -\boldsymbol{\eta}_2(\mathbf{M}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \boldsymbol{\eta}_2(\mathbf{M}, \boldsymbol{\varepsilon}, -\boldsymbol{\kappa}), \quad (35)$$

which lead to constraints  $\mathbf{C}_2 = \mathbf{0}$ ,  $\mathbf{C}_3(\mathbf{M}) = -\mathbf{C}_3(-\mathbf{M})$ ,  $\mathbf{C}_4(\mathbf{M}) = \mathbf{C}_4(-\mathbf{M})$ .

Constitutive equations (33) or (34) can be extended as follows

$$\begin{aligned} \mathbf{T}^\circ &= \boldsymbol{\eta}_1(\mathbf{T}, \mathbf{M}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), & \mathbf{M}^\circ &= \boldsymbol{\eta}_2(\mathbf{T}, \mathbf{M}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \\ \mathbf{T}^\circ &= \mathbf{C}_1(\mathbf{T}, \mathbf{M}) : \boldsymbol{\varepsilon} + \mathbf{C}_2(\mathbf{T}, \mathbf{M}) : \boldsymbol{\kappa}, & \mathbf{M}^\circ &= \mathbf{C}_3(\mathbf{T}, \mathbf{M}) : \boldsymbol{\varepsilon} + \mathbf{C}_4(\mathbf{T}, \mathbf{M}) : \boldsymbol{\kappa} \end{aligned} \quad (36)$$

with the following constraints

$$\begin{aligned} \boldsymbol{\eta}_1(\mathbf{T}, \mathbf{M}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) &= \boldsymbol{\eta}_1(\mathbf{T}, -\mathbf{M}, \boldsymbol{\varepsilon}, -\boldsymbol{\kappa}), & -\boldsymbol{\eta}_2(\mathbf{T}, \mathbf{M}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) &= \boldsymbol{\eta}_2(\mathbf{T}, -\mathbf{M}, \boldsymbol{\varepsilon}, -\boldsymbol{\kappa}), \\ \mathbf{C}_1(\mathbf{T}, \mathbf{M}) &= \mathbf{C}_1(\mathbf{T}, -\mathbf{M}), & -\mathbf{C}_2(\mathbf{T}, \mathbf{M}) &= \mathbf{C}_2(\mathbf{T}, -\mathbf{M}), \\ -\mathbf{C}_3(\mathbf{T}, \mathbf{M}) &= \mathbf{C}_3(\mathbf{T}, -\mathbf{M}), & \mathbf{C}_4(\mathbf{T}, \mathbf{M}) &= \mathbf{C}_4(\mathbf{T}, -\mathbf{M}). \end{aligned} \quad (37)$$

## 6.5 Viscoelastic Materials

Considered finite approximation of series (23) we obtain the model of micropolar material of order  $(m, n)$

$$\begin{aligned} \mathbf{T}(t) &= \mathcal{F}_1[\mathbf{e}(t), \mathbf{k}(t), \mathbf{B}(t), \mathbf{A}_1(t) \dots \mathbf{A}_m(t), \mathbf{B}_1(t) \dots \mathbf{B}_n(t)], \\ \mathbf{M}(t) &= \mathcal{F}_2[\mathbf{e}(t), \mathbf{k}(t), \mathbf{B}(t), \mathbf{A}_1(t) \dots \mathbf{A}_m(t), \mathbf{B}_1(t) \dots \mathbf{B}_n(t)], \end{aligned} \quad (38)$$

where  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are isotropic operators.

Finally, in order to consider various form of rate-type constitutive equations of micropolar materials we introduce implicit constitutive equations of differential type in the following form

$$\begin{aligned} \mathbf{g}_1[\mathbf{T}^{\circ\{M\}} \dots \mathbf{T}^\circ, \mathbf{M}^{\circ\{N\}} \dots \mathbf{M}^\circ, \mathbf{M}(t), \mathbf{e}(t), \mathbf{k}(t), \mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, \mathbf{B}_1 \dots \mathbf{B}_n] &= \mathbf{0}, \\ \mathbf{g}_2[\mathbf{T}^{\circ\{M\}} \dots \mathbf{T}^\circ, \mathbf{M}^{\circ\{N\}} \dots \mathbf{M}^\circ, \mathbf{M}(t), \mathbf{e}(t), \mathbf{k}(t), \mathbf{B}, \mathbf{A}_1 \dots \mathbf{A}_m, \mathbf{B}_1 \dots \mathbf{B}_n] &= \mathbf{0}, \end{aligned} \quad (39)$$

where  $\mathbf{g}_1$  and  $\mathbf{g}_2$  are isotropic tensor-valued functions of  $M + N + m + n + 3$  tensorial arguments, and  $\circ\{k\}$  stands for  $k$ th objective derivative.

The constitutive equation (21) include various forms of micropolar viscoelastic behaviour under finite deformations. Here we present few examples of constitutive equations of differential type. The constitutive equations of the form

$$\mathbf{T} = \Phi_1(\mathbf{e}, \mathbf{k}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad \mathbf{M} = \Phi_2(\mathbf{e}, \mathbf{k}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad (40)$$

$$\tau_1 \mathbf{T}^\circ + \mathbf{T} = \Psi_1(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad \tau_2 \mathbf{M}^\circ + \mathbf{M} = \Psi_2(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad (41)$$

$$\tau_1 \mathbf{T}^\circ + \mathbf{T} = \Omega_1(\mathbf{e}, \mathbf{k}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \quad \tau_2 \mathbf{M}^\circ + \mathbf{M} = \Omega_2(\mathbf{e}, \mathbf{k}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \quad (42)$$

play a role of Kelvin–Voigt, Maxwell and standard models in micropolar viscoelasticity, respectively. Here  $\circ$  denotes an objective time derivative,  $\tau_1$  and  $\tau_2$  are relaxation time parameters and  $\Phi_1$ ,  $\Phi_2$ ,  $\Psi_1$ ,  $\Psi_2$ ,  $\Omega_1$  and  $\Omega_2$  are constitutive tensor-valued functions. Using higher order objective time derivatives and tensors  $\mathbf{A}_k$ ,  $\mathbf{B}_k$  one can present constitutive equations of differential type of any order.

## 7 Conclusions and Discussion

Following Altenbach and Eremeyev (2014) we presented a family of non-symmetric strain rate tensors for micropolar materials and discussed constitutive equations of inelastic micropolar materials. Using basics principles of continuum mechanics that is the principle of equipresence and the material frame indifference we discussed the constraints for the constitutive equations. In particular, considering the fact that part of strain rates are not true tensors but pseudotensors we obtain some constitutive restrictions following from the material frame indifference principle.

Considering difference between models of classic continua and Cosserat continua let us note that some classic methods of constitutive modelling used in the Cauchy mechanics can be straightforward extended for the micropolar continuum. Among them are the theory of local material symmetry group, invariance properties applied for the micropolar elasticity (Eringen 1999; Ramezani et al. 2009; Pietraszkiewicz and Eremeyev 2009a; Eremeyev and Pietraszkiewicz 2012, 2016), micropolar hypoelasticity (Ramezani and Naghdabadi 2010), and mechanics of viscous micropolar fluids (Aero et al. 1965; Eringen 1966, 2001). Several generalizations of yield criterion for elasto-plastic materials and other models for micropolar elastoplasticity

are given by (Lippmann 1969; de Borst 1993; Steinmann 1994; Grammenoudis and Tsakmakis 2007, 2009). But in some case such straightforward extensions are impossible, let us mention the logarithmic Hencky's strain measure and related logarithmic strain rate (Xiao et al. 1997a, b; Bruhns 2014). Introduction of similar strain tensors based on logarithmic objective derivative in micropolar mechanics is more difficult or impossible, in general.

Similar to introduced above non-symmetric strain measures and strain rates are also used for description of two-level deformations of inelastic materials considering independent spin (Trusov et al. 2015) for derivation of generalized models of elasticity (Lurie et al. 2005).

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