Chapter 1
New Analytical Solutions of Selected Electromagnetic Problems in Wave Diffraction Theory

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Abstract The chapter presents explicit analytical solutions for some sophisticated electromagnetic problems. The analysis of these solutions made it possible, in particular, to explain the physics of a cycle slipping phenomenon when very long electromagnetic waves propagate in the Earth-ionosphere waveguide, to establish the rigorous criterion of the boundary 'sharpness' for transient radiation and to show that the well-known negative refraction phenomenon in isotropic double-negative media is a direct consequence of the energy conservation law and Maxwell's equations.

1.1 Introduction

Exact analytical solutions of the basic problems of physics—boundary value and initial boundary value—are important not only as a reference for verifying numerical results but also as an effective tool for a deeper understanding of the nature of the model under study. To obtain such solutions for new physical problems, one should invoke, as a rule, new mathematical methods or significantly modify the available ones. Thus, for example, in quantum mechanics, novel approaches have resulted in a sharp increase in the number of exactly solvable problems and raised interest in the subject in the recent years [1]. In theoretical radio physics, this was the case in mid-twentieth century, after publishing of the book by Wiener and Hopf [2]. This work has been of vital importance, which is why the method presented therein takes its name from the authors—the *Wiener-Hopf method*. As applied to diffraction problems, it was first used in [3–5]. In the review [6] the authors attempted to describe the areas of application and discussed the future development of this method.

Mention should be made of the detailed study of the *integral convolution equations* in the book by Gakhov and Cherskiy [7], which although not mentioned

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in [6] can be considered as part of the development of this method. In the first two sections of this chapter, we apply their methodology of solving integral convolution equations to the new problems on wave propagation near a plane surface of varying conductivity, thereby reducing those problems to exactly solvable boundary value ones. Thus, in Sect. 1.2 of this chapter, using the technique suggested in [7] for solving the so-called smooth transition equation, we obtain analytical solutions for two two-dimensional problems, namely, we find analytical expressions for the field generated by a linear current above a plane surface whose impedance varies continuously from Z_1 to Z_2 in a given direction, and for the field generated by the same source in a planar waveguide with a wall of the same impedance distribution. These solutions generalize the known ones in which the surface impedance changes stepwise. In Sect. 1.3 we investigate a model of a ring waveguide of constant cross-section with variable in azimuth impedance of one of the walls. We have found a class of distributions of these impedances, for which the analytical solution of the excitation problem for this waveguide had been obtained. This result is used for simulation of the known *cycle slipping phenomenon* occurring when very long electromagnetic waves propagate in the *Earth-ionosphere waveguide*. A possible cause of this phenomenon is discussed.

The remaining sections of the chapter are not associated with the Wiener-Hopf method. In Sect. 1.4 a novel technique is suggested for the analysis of a transient electromagnetic field generated by a pulsed line current that is located near a planar interface between two dielectric nonabsorbing and nondispersive media. As distinct from the *Cagniard-de Hoop method*, which is widely used for the study of transient fields both in electrodynamics and in the theory of acoustic and seismic waves, our approach is based on the transformation of the domain of integration in the integral expression for the field in the space of two complex variables. As a result, it will suffice to use the standard procedure of finding the roots of the algebraic equation rather than construct auxiliary Carniard's contours. A new representation for the field has been derived in the form of an integral along a finite contour.

In Sect. 1.5 we discuss the transient radiation of a *moving longitudinal magnetic dipole* whose trajectory crosses a soft boundary between two media. The obtained analytical representation for the dipole field ensures a rigorous *criterion of the boundary 'sharpness'* thus significantly improving the now known approximate version.

In Sect. 1.6 the isotropic *Epstein transition layer* was generalized to the case of a *biisotropic plane stratified medium*. An explicit analytical solution to the problem of normal incidence of a linearly polarized electromagnetic plane wave onto the Epstein layer was obtained for this extension. The derived transmission and reflection coefficients are indicative of the presence of the total transmission mode in such media.

In Sect. 1.7 we suggest a model for a smoothly inhomogeneous isotropic flat-layered medium that includes domains with *double-positive* and *double-negative media*. The analytical solution derived for a plane wave propagating through this medium shows that the well-known *negative refraction phenomenon* in the isotropic double-negative medium is a direct consequence of Maxwell's equations and of the *energy conservation law*.

In Sect. 1.8, using as an example a perfectly conducting sphere, we rigorously prove the possibility of drastic *distortion* of its *radar image* by applying a *metamaterial coating* on the sphere surface. We have found such radial distributions of the coating dielectric and magnetic permeabilities that the scattered field everywhere outside the object coincides with the field scattered by a perfectly conducting sphere of any given smaller radius. Requirements on the material parameters of such distorting coating are smaller than they are in the case of a masking coating.

1.2 Wave Propagation Near an Irregular Impedance Structure

One of the problems solved at the early stage of the development of the Wiener-Hopf method was related to the electromagnetic wave propagation above a plane whose impedance changed step-wise from Z_1 to Z_2 in a given direction [8]. A waveguide analog of this problem was studied in [9] for acoustic waves and in [10] for electromagnetic waves. The electromagnetic model presented in [8] was given the name *'the coastal refraction problem'* since it was used for calculation of a radar error arising when the radar crosses a shoreline.

It is well known that in the case of the stratified medium, whose permittivity is given by the hyperbolic tangent or by hyperbolic secant, the solution of the wave propagation problem can be written in explicit form. These two media have been named asymmetric and symmetric Epstein layers, respectively. In this section we will show that the problem of wave propagation near a plane surface, whose impedance is given by the hyperbolic tangent, is also explicitly resolvable. At the same time, attempts to obtain similar results for an impedance analog of the symmetric Epstein layer (the permittivity is given by the hyperbolic secant) were unsuccessful, because in this case we are led to *three-element Carleman's problem* whose solution is unknown.

1.2.1 Wave Propagation Over a Plane Surface of Variable Conductivity

Electrical properties of real underlying surfaces vary smoothly and the assumption as to their step-wise change (for example, when crossing the boundary land/sea) can only be justified for sufficiently large values of the wavelength λ. However, the discontinuity of the function $Z(x)$, which characterizes the surface impedance distribution on the plane $z = 0$ in classical two-dimensional $(\partial/\partial y \equiv 0)$ problems, is incompatible with a mere concept of the surface impedance.

The question arises as to the existence of such continuous and reasonable (from the physical point of view) surface impedance distributions that they allow an exact analytical solution of the problems like those discussed in [8–10].

It has been shown [11] that such a distribution does exist. It is the impedance version of the Epstein transition layer [12]

$$Z(x) = \frac{Z_2 + Z_1 \exp(-\tau x)}{1 + \exp(-\tau x)}; \quad -\infty < x < \infty, \tag{1.1}$$

where $Z_1 = Z(-\infty)$ and $Z_2 = Z(+\infty)$ are the limiting values of impedance. The parameter $0 < \tau < \infty$ determines the width of the transition region in the impedance distribution. The Grinberg-Fock model of the step-wise change in impedance [8] represents the limiting case $\tau \to \infty$.

Let us consider the following two-dimensional problem: a field generated by a filament of linear magnetic current $\vec{J}^{(m)} = I^{(m)}\delta(g - g_0)\exp(-i\omega t)\vec{y}$, which is parallel to the impedance plane $z = 0$, is to be found. Here, $\delta(\ldots)$ is the δ-Dirac function; $g = \{x, z\}$ and $g_0 = \{x_0, z_0\}$ are the points of the space R^2; $\vec{x}$, $\vec{y}$, and $\vec{z}$ are the Cartesian basis vectors. The current self-field can be represented as $\vec{E}^0 = i\omega\mu\mu_0 \mathrm{rot}\, \vec{\Pi}^{(m)}$, where $\Pi_y^{(m)} = -I^{(m)}(4\omega\mu\mu_0)^{-1} H_0^{(1)}(k|g - g_0|)$ and $\Pi_x^{(m)} = \Pi_z^{(m)} = 0$ are the components of the magnetic Hertz potential; $H_0^{(1)}(\ldots)$ is the Hankel function; $k = \omega\sqrt{\varepsilon\varepsilon_0\mu\mu_0}$; ε and μ are the relative dielectric permittivity and magnetic permeability of the medium. The surface impedance is given by relation (1.1).

Basing, as in [8], on the *integral Green formula* and using the impedance boundary condition $\partial E_z(g)/\partial z = -ik\, Z(x) E_z(g)|_{z=0}$ [13, 14], we arrive at the following 1-D integral equation:

$$f(x) = q(x) - \frac{k}{2} Z(x) \int_{-\infty}^{\infty} f(\tilde{x}) H_0^{(1)}(k|x - \tilde{x}|) d\tilde{x}, \tag{1.2}$$

where $f(x) = Z(x)E_z(x, 0)$, $q(x) = 2Z(x)E_z^0(x, 0)$; $E_z^0(x, z)$ and $E_z(x, z)$ are the vertical components of the primary and total electrical fields, respectively.

Equation (1.2) belongs to the class of the so-called *smooth transition equations* introduced by Cherskiy [7]:

$$\begin{aligned} & f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_1(x - \tilde{x}) f(\tilde{x}) d\tilde{x} - q(x) \\ & \quad + e^{-x} \left\{ f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_2(x - \tilde{x}) f(\tilde{x}) d\tilde{x} - q(x) \right\} = 0; \quad -\infty < x < \infty. \end{aligned} \tag{1.3}$$

For this equation to be normally solvable in the space $L_2(-\infty,\infty)$ and have a finite index it is necessary and sufficient to have $1+\tilde{K}_j(\xi)\neq 0,\ j=1,2,$ where $\tilde{K}_j(\xi)$ is the Fourier transform of $K_j(x)$. In our case, we have

$$\tilde{K}_j(\xi)=\kappa Z_j\int\limits_0^\infty H_0^{(1)}(\kappa s)\cos\xi s ds=\kappa Z_j\left(\kappa^2-\xi^2\right)^{-1/2},$$

where $\kappa=k/\tau$, and $\sqrt{\kappa^2-\xi^2}\to i\xi$ with $\xi\to+\infty$.

In [7], the authors prove the solvability in quadratures of (1.3) in the space $L_2(-\infty,\infty)$ with the complementary condition that $q(x)\in L_2(-\infty,\infty)$.

Let us apply the Fourier transform to (1.2), following [7]. Then we are led to *Carleman's two-element boundary value problem* for a strip $0<\mathrm{Im}\xi<1$. Later on, with the use of some conformal mapping $\upsilon=\exp(2\pi\xi)$, we will rearrange this problem to yield the *Riemann problem*, which is as follows: on the real axis of the complex plane of variable $\upsilon=\upsilon'+i\upsilon''$ two functions, $D(\upsilon')$ and $H(\upsilon')$, are given; it is required to find two functions $F^{\pm}(\upsilon)$, which are analytic in the upper complex half-plane $(\upsilon''>0)$ and in the lower complex half-plane $(\upsilon''<0)$, respectively, and which also satisfy the boundary condition $F^+(\upsilon')=D(\upsilon')F^-(\upsilon')+H(\upsilon')$. The value $\chi=(2\pi i)^{-1}[\ln D(\upsilon')]|_{-\infty}^{\infty}$ is known as the index of the Riemann problem. For the two problems considered in this section, we have $\chi=0$. Using the well-known solution of this problem [7], we can write the solution of (1.2) in the following form:

$$f(x)=-\frac{i}{4}I^{(m)}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left[\tau Q(\xi)+e^{\pi\xi}\omega^+\left(e^{2\pi\xi}\right)\right]\frac{\sqrt{\kappa^2-\xi^2}\cdot e^{-i\bar{x}\xi}}{\sqrt{\kappa^2-\xi^2}+\kappa Z_1}d\xi,\tag{1.4}$$

where $\bar{x}=x\tau$, $\bar{z}=z\tau$, and

$$Q(\xi)=\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial\bar{x}_0}\int\limits_{-\infty}^{\infty}\frac{Z_2+Z_1e^{-\eta}}{1+e^{-\eta}}H_0^{(1)}\left(\kappa\sqrt{(\eta-\bar{x}_0)^2+\bar{z}_0^2}\right)e^{i\xi\eta}d\eta,$$

$$\omega^+\left(e^{2\pi\xi}\right)=-i\tau X^+\left(e^{2\pi\xi}\right)\kappa(Z_1-Z_2)\int\limits_{-\infty}^{\infty}\frac{Q(\varsigma)e^{\pi\varsigma}d\varsigma}{\left(\sqrt{\kappa^2-\varsigma^2}+\kappa Z_2\right)X^+\left(e^{2\pi\varsigma}\right)\left(e^{2\pi\varsigma}-e^{2\pi\xi}\right)},$$

$$X^{\pm}\left(e^{2\pi\varsigma}\right)=\exp\left\{\left(1-ie^{2\pi\varsigma}\right)\int\limits_{-\infty}^{\infty}\ln\frac{\sqrt{\kappa^2-\sigma^2}+\kappa Z_1}{\sqrt{\kappa^2-\sigma^2}+\kappa Z_2}\frac{e^{2\pi\sigma}d\sigma}{\left(e^{2\pi\sigma}+i\right)\left(e^{2\pi\sigma}-e^{2\pi\varsigma}\right)}\right\}.$$

The contour of integration passes below the pole for the functions marked by '+' and above the pole for the functions marked by '−'.

These relations represent an explicit expression for the vertical component of the electric field on an impedance plane considered without any restriction on the parameters of the model.

In the case of grazing propagation of a plane wave ($x_0 \to -\infty$) and for $Z_1 = 0$, the integral in the representation of the function $\omega^+[\exp(2\pi\xi)]$ can be calculated. To do this, let us transform the formula for $Q(\xi)$ using the *Parseval equality* for Fourier integrals and then apply the *saddle-point technique*. As a result we get the following asymptotic estimate for $k|x_0| \gg 1$:

$$Q(\xi) = 2\exp(-i\pi/4)\frac{\exp(ik|x_0|)}{\sqrt{k|x_0|}}\frac{1}{\mathrm{sh}[\pi(\kappa+\xi)]}\left(1+O\left(\frac{1}{k|x_0|}\right)\right).$$

Hence, for the vertical component of the total electric field we have

$$E_z(x,0) = 2\mathrm{e}^{ikx} - \frac{i\kappa Z_2}{X^-[\exp(2\pi(-\kappa+i))]}\int_{-\infty}^{\infty}\frac{X^+[\exp(2\pi\xi)]\exp(-i\bar{x}\xi)d\xi}{\sqrt{\kappa^2-\xi^2}\,\mathrm{sh}[\pi(\kappa+\xi)]}, \quad (1.5)$$

where the integration contour passes above the pole $\xi = -\kappa$. The representation in the form of (1.5) is convenient for $x < 0$. The first term represents the plane wave on a perfectly conducting planar surface, while the integral term describes the field scattered by the impedance inhomogeneity.

Taking into account characteristics of the *factorization function* $X^{\pm}[\exp(2\pi\xi)]$, we obtain the representation, which is convenient for the area $x > 0$:

$$E_z(x,0) = -\frac{i\kappa Z_2}{X^-[\exp(2\pi(-\kappa+i))]}\int_{-\infty}^{\infty}\frac{X^-[\exp(2\pi(\xi+i))]\exp(-i\bar{x}\xi)d\xi}{\left(\sqrt{\kappa^2-\xi^2}+\kappa Z_2\right)\mathrm{sh}[\pi(\kappa+\xi)]}, \quad (1.6)$$

where the integration contour passes below the pole $\xi = -\kappa$. Using the following decomposition

$$\frac{1}{\mathrm{sh}[\pi(\kappa+\xi)]} = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\xi+\kappa+in},$$

it is easy to show that (1.6) transforms for $\tau \to \infty$ into the well-known formula [8] for $E_z(x,0)$ on the plane whose impedance equals Z_2 for $x > 0$ and is zero for $x < 0$.

Notice that the solution obtained in [8] represents the dominant term of the long-wave asymptotic of the solution to the problem considered by us. This is the case, where the wavelength of the source is much grater than the width of the transition region on the impedance surface ($2k \ll \tau$).

1.2.2 A Field of Linear Magnetic Current in a Plane Waveguide with Smoothly Varying Impedance of Its Walls

In this section, we construct the exact *Green function* of the *Helmholtz equation* for a band with the non-homogeneous boundary condition of the third kind on one of its boundaries. The coefficient $Z(x)$ in this boundary condition is an impedance analogue for the permittivity of the known Epstein transition layer [12]. We use this Green function below for analyzing the electromagnetic field induced by a linear magnetic current in a gradient junction between two regular impedance waveguides. This solution comprises the stepped impedance distribution as a limiting case [10]. In [15], we considered a related problem of the electromagnetic *TM*-wave propagation in a planar waveguide with the perfectly conducting upper wall and the lower wall with conductivity changing as $\text{th}\tau x$.

In Sect. 1.2.2.1, the boundary value problem is reduced to the integral equation of the second kind. In the next section, we derive the analytical solution by reducing this equation to the Riemann problem of the linear conjugation of two analytical functions on the real axis. For this purpose we invoke the Fourier transform and the conforming mapping. In Sect. 1.2.2.3, the Green function is expressed as the double Fourier integral, which is transformed further, by employing the *Cauchy-Poincaré theorem*, into series in residues. Section 1.2.2.4 is devoted to the analysis of these series as applied to the transformation of the eigenwaves of the regular section of the waveguide junction. We also rigorously estimate the *adiabatic approximation* for the considered waveguides.

1.2.2.1 Reduction of the Problem to an Integral Equation

A Solution to the Following Two-Dimensional Boundary Value Problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \varepsilon\varepsilon_0 \mu\mu_0\right) G^t = -\delta(g - g_0), \tag{1.7a}$$

$$\frac{\partial}{\partial z} G^t = 0 \quad \text{for} \quad z = 0, \tag{1.7b}$$

$$\frac{\partial}{\partial z} G^t + i\omega\varepsilon\varepsilon_0 Z(x) G^t = 0 \quad \text{for} \quad z = d \tag{1.7c}$$

is to be found in the band $\{0 < z < d, -\infty < x < \infty\}$ (see Fig. 1.1). Here $g = \{x, z\}$, $g_0 = \{x_0, z_0\}$, and the function

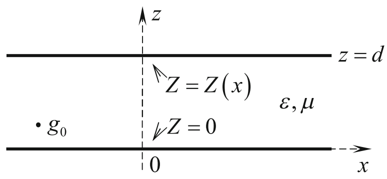


Fig. 1.1 The geometry of the problem

$$Z(x) = \frac{Z_2 + Z_1 \exp(-\tau x)}{Z + \exp(-\tau x)}; \quad \tau > 0, \quad Z = \exp(i\varphi), \quad -\pi < \varphi < \pi \tag{1.8}$$

is the complex-valued function describing the gradient transition from $Z(-\infty) = Z_1 = Z_l$ to $Z(+\infty) = Z_2/Z = Z_r$. Its hodograph represents a circular arc having the angular size of $|2\varphi|$ and joining the points Z_l and Z_r. In the course of solution, the imaginary part of the wave number $k = \omega\sqrt{\varepsilon\varepsilon_0\mu\mu_0}$ is assumed to be positive, whereas in the final formulas we put it equal to zero.

We seek the solution to the problem (1.7a, 1.7b, 1.7c) in the form of a sum

$$G^t(g, g_0) = G^0(g, g_0) + G(g, g_0), \tag{1.9}$$

where

$$G^0(g, g_0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d(\eta, z, z_0)}{R_l(\eta)} \exp[-i(\bar{x} - \bar{x}_0)\eta] d\eta$$

is the solution to (1.7a, 1.7b, 1.7c) with the fixed $Z(x) = Z_l$, and

$$G(g, g_0) = \int_{-\infty}^{\infty} F_0(\eta, \bar{g}_0) \cos(v\bar{z}) \exp[-i(\bar{x} - \bar{x}_0)\eta] d\eta \tag{1.10}$$

is the solution of the homogeneous equation (1.7a) with condition (1.7b). Here, $d(\eta, z, z_0) = \cos(v\bar{z}_<)[\cos(v(\delta - \bar{z}_>)) - i\bar{Z}_l \sin(v(\delta - \bar{z}_>))/v]$, $R_\alpha(\eta) = v \sin v\delta + i\bar{Z}_\alpha \cos v\delta$, $v = v(\eta) = \sqrt{\kappa^2 - \eta^2}$, $\bar{Z}_\alpha = Z_\alpha \omega\varepsilon/\tau$, $\alpha = l$ or r, $\kappa = k/\tau$, $\delta = d\tau$, $\bar{z}_< = \min(\bar{z}, \bar{z}_0)$, $\bar{z}_> = \max(\bar{z}, \bar{z}_0)$, $\bar{x} = x\tau$, $\bar{z} = z\tau$, and d is the waveguide height. With this representation of the function G^t, the requirements (1.7a), (1.7b) are satisfied automatically. The condition (1.7c) leads to the following integral equation

$$\begin{gathered} \int_{-\infty}^{\infty} F(\eta)\left[e^{2iv\delta} - 1 - \bar{Z}(\bar{x})\left(e^{2iv\delta} + 1\right)/v\right] \exp(-i\bar{x}\eta) d\eta = -\sqrt{2\pi}\, Q(\bar{x}); \\ Q(\bar{x}) = i(2\pi)^{-3/2}(\bar{Z}_l - \bar{Z}(\bar{x})) \int_{-\infty}^{\infty} \frac{\cos(v\bar{z}_0)}{R_l(\eta)} \exp[-i(\bar{x} - \bar{x}_0)\eta] d\eta, \quad -\infty < \bar{x} < \infty \end{gathered} \tag{1.11}$$

with respect to the unknown function

$$F(\eta) = \frac{1}{2i} v \exp[-i(v\delta - \eta\bar{x}_0)] F_0(\eta, \bar{g}_0). \tag{1.12}$$

By using the known formula [16]

$$2\exp(iv\delta) = v \int_{-\infty}^{\infty} H_0^{(1)}\left(\kappa\sqrt{\xi^2 + \delta^2}\right)\exp(i\eta\xi)d\xi,$$

we can easily go from (1.11) to the equation of the second kind

$$f(\bar{x}) + \int_{-\infty}^{\infty} \tilde{K}(\bar{x}, \bar{x} - \xi)f(\xi)d\xi = Q(\bar{x}); \quad -\infty < \bar{x} < \infty \tag{1.13}$$

with respect to the Fourier transform of $F(\eta)$

$$f(\bar{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\eta)\exp(-i\eta\bar{x})d\eta. \tag{1.14}$$

The kernel looks like

$$\begin{aligned}\tilde{K}(\bar{x}, \bar{x} - \xi) = \frac{1}{2}\bar{Z}(\bar{x})H_0^{(1)}\left[\kappa\sqrt{(\bar{x} - \xi)^2 + (2\delta)^2}\right] + \frac{1}{2}\bar{Z}(\bar{x})H_0^{(1)}(\kappa|\bar{x} - \xi|) \\ - \frac{1}{4i}\frac{\partial}{\partial\delta}H_0^{(1)}\left[\kappa\sqrt{(\bar{x} - \xi)^2 + (2\delta)^2}\right].\end{aligned}$$

Rewrite finally (1.13) in the form

$$\begin{aligned}Zf(\bar{x}) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_2(\bar{x} - \xi)f(\xi)d\xi - ZQ(\bar{x}) \\ + \exp(-\bar{x})\left\{f(\bar{x}) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_1(\bar{x} - \xi)f(\xi)d\xi - Q(\bar{x})\right\} = 0; \quad -\infty < \bar{x} < \infty,\end{aligned} \tag{1.15}$$

where

$$\frac{1}{\sqrt{2\pi}}K_j(\bar{x}-\xi)=\frac{1}{2}\bar{Z}_j\left[H_0^{(1)}\left[\kappa\sqrt{(\bar{x}-\xi)^2+(2\delta)^2}\right]+H_0^{(1)}(\kappa|\bar{x}-\xi|)\right]-\frac{1}{4i}Z^{j-1}\frac{\partial}{\partial\delta}H_0^{(1)}\left[\kappa\sqrt{(\bar{x}-\xi)^2+(2\delta)^2}\right];\quad j=1,2.$$

1.2.2.2 Solution of the Integral Equation

For $Z=1$, a similar equation was discussed in [17], where a method of obtaining its analytical solution was proposed. Following the basic ideas introduced in this work, let us find the analytical solution of the more general equation (1.15) by reducing it to the Riemann conjugation problem. To this end, we introduce a new unknown function

$$\Phi(\bar{x})=f(\bar{x})+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K_1(\bar{x}-\xi)f(\xi)dt-Q(\bar{x});\quad -\infty<\bar{x}<\infty. \tag{1.16}$$

By subjecting (1.15) and (1.16) to the Fourier transform, we obtain a system of functional equations

$$\begin{cases} ZF(\bar{\xi})+\tilde{K}_2(\bar{\xi})F(\bar{\xi})-Z\tilde{Q}(\bar{\xi})+\tilde{\Phi}(\bar{\xi}+i)=0\\ \tilde{\Phi}(\bar{\xi})=F(\bar{\xi})+\tilde{K}_1(\bar{\xi})F(\bar{\xi})-\tilde{Q}(\bar{\xi}),\end{cases} \tag{1.17}$$

where $\tilde{K}_1(\bar{\xi})$, $\tilde{K}_2(\bar{\xi})$, $\tilde{\Phi}(\bar{\xi})$, and $\tilde{Q}(\bar{\xi})$ are the Fourier transforms of the functions $K_1(\bar{x})$, $K_2(\bar{x})$, $\Phi(\bar{x})$, and $Q(\bar{x})$, respectively. Eliminating $F(\bar{\xi})$, we arrive at the equation

$$\tilde{\Phi}(\bar{\xi})=-D(\bar{\xi})\tilde{\Phi}(\bar{\xi}+i)+H(\bar{\xi});\quad -\infty<\bar{\xi}<\infty, \tag{1.18}$$

where

$$D(\bar{\xi})=R_l(\bar{\xi})/[ZR_r(\bar{\xi})]\quad\text{and}\quad H(\bar{\xi})=i(\bar{Z}_l-\bar{Z}_r)\cos(\nu\delta)\tilde{Q}(\bar{\xi})/R_r(\bar{\xi}).$$

This is the Carleman problem: to find the analytical function $\tilde{\Phi}(\bar{\zeta})$ in the band $0<\mathrm{Im}\bar{\zeta}<1$ of the complex plane $\bar{\zeta}=\bar{\xi}+i\bar{\varsigma}$ from the condition (1.18) on the band boundary. Applying the conformal mapping $\zeta=\exp(2\pi\bar{\zeta})$ to (1.18), we pass to the new unknown function $\omega(\zeta)=\zeta^{-1/2}\Phi(\ln\zeta/2\pi)$. Then this problem is transformed into the Riemann problem of finding two analytical functions $\omega^{\pm}(\zeta)$ (in the upper and lower half-planes of the complex plane $\zeta=\xi+i\varsigma$) from the boundary condition on the real axis ξ

$$\omega^{+}(\xi) = \bar{D}(\xi)\omega^{-}(\xi) + \bar{H}(\xi); \quad -\infty < \xi < \infty \tag{1.19}$$

with the discontinuous coefficient

$$\begin{aligned} &\bar{D}(\xi) = \left\{D\left(\bar{\xi}\right) \text{ for } \xi > 0; 1 \text{ for } \xi < 0\right\} \quad \text{and} \\ &\bar{H}(\xi) = \left\{\mathrm{e}^{-\pi\bar{\xi}} H\left(\bar{\xi}\right) \text{ for } \xi > 0; 0 \text{ for } \xi < 0\right\}; \quad \bar{\xi} = \ln \xi/2\pi. \end{aligned} \tag{1.20}$$

The branches of the functions $\ln \zeta$ and $\sqrt{\zeta}$ are determined by the value $\arg \zeta = 0$ on the upper edge of the cut made along the ray $\xi \geq 0$.

The analytical solution to the homogeneous Riemann problem

$$\omega^{+}(\xi) = \bar{D}(\xi)\omega^{-}(\xi); \quad -\infty < \xi < \infty \tag{1.21}$$

in the case where the function $\bar{D}(\xi)$ is continuous along the whole of the real axis, including the infinitely distant point, is well known [7]. The function in (1.20) is discontinuous at the points $\xi = 0$ and $\xi = \infty$. Represent it as a product

$$\bar{D}(\xi) = \bar{D}_1(\xi)\bar{D}_2(\xi)$$

of the continuous function

$$\bar{D}_1(\xi) = \left\{R_l\left(\bar{\xi}\right)/R_r\left(\bar{\xi}\right) \text{ for } \xi > 0; 1 \text{ for } \xi < 0\right\} \tag{1.22}$$

and the discontinuous function

$$\bar{D}_2(\xi) = \left\{Z^{-1} \text{for } \xi > 0; 1 \text{ for } \xi < 0\right\}.$$

Obviously, if the solutions $\omega_j(\zeta)$ of the problems

$$\omega_j^{+}(\xi) = \bar{D}_j(\xi)\omega_j^{-}(\xi); \quad j = 1, 2, \quad -\infty < \xi < \infty \tag{1.23}$$

are known, then $\omega(\zeta) = \omega_1(\zeta)\omega_2(\zeta)$ is a solution to the problem in (1.21). Let us find $\omega_2(\zeta)$. Since

$$\ln \omega_2^{+}(\xi) = \ln \omega_2^{-}(\xi) - i\varphi\{1 \text{ for } \xi > 0; 0 \text{ for } \xi < 0\}; \quad -\infty < \xi < \infty,$$

then the desired function is analytical in the plane ζ containing a cut along the real positive semiaxis; the discontinuity value on it is $-i\varphi = -\ln Z$. We take for such a function the function

$$\omega_2(\zeta) = \exp\left\{\frac{\varphi}{2\pi} \cdot \ln \zeta\right\}.$$

The solution of the problem in (1.23) for $\omega_1(\zeta)$ can be derived by using the known mathematical technique of factorizing the Riemann problem coefficient [7, 17]

$$\omega_1^+(\zeta) = \exp\{\Gamma^+(\zeta)\},$$

where

$$\Gamma^+(\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \ln \bar{D}_1(\xi)\frac{(\zeta+i)d\xi}{(\xi-\zeta)(\xi+i)}; \quad \text{Im}\zeta > 0.$$

Let us introduce a function

$$X^+(\bar{\zeta}) = \omega_1^+\left[\exp\left(2\pi\bar{\zeta}\right)\right] = \exp\left\{\Gamma^+\left[\exp\left(2\pi\bar{\zeta}\right)\right]\right\},$$

$$\Gamma^+\left[\exp\left(2\pi\bar{\zeta}\right)\right] = \frac{1}{2i}\int_{-\infty}^{\infty} \ln\frac{R_l(\eta)}{R_r(\eta)}\cdot\frac{\text{ch}\left[\pi\left(\bar{\zeta}-i/4\right)\right]d\eta}{\text{ch}[\pi(\eta-i/4)]\text{sh}\left[\pi\left(\eta-\bar{\zeta}\right)\right]}; \quad \text{Im}\,\bar{\zeta} > 0.$$

With the representation

$$\frac{R_l(\eta)}{R_r(\eta)} = \prod_{n=0}^{\infty}\frac{\eta^2-\left(\eta_n^l\right)^2}{\eta^2-\left(\eta_n^r\right)^2},$$

it can be shown that

$$X^+\left(\bar{\zeta}\right) = \prod_{n=0}^{\infty}\frac{\gamma\left(\bar{\zeta},\eta_n^l,\eta_n^r\right)}{\gamma\left(\bar{\zeta},\eta_n^r,\eta_n^l\right)}, \tag{1.24}$$

where $\gamma\left(\eta,\eta_n^1,\eta_n^2\right) = \Gamma\left[1-i\left(\eta_n^1-\eta\right)\right]\cdot\Gamma\left[-i\left(\eta_n^2+\eta\right)\right], \Gamma(\ldots)$ is the gamma-function [16], and $\eta_n^\alpha = \sqrt{\kappa^2-\left(v_n^\alpha\right)^2}$, $\text{Im}\,\eta_n^\alpha \geq 0$, where v_n^α are the roots of the following dispersion equation for a regular waveguide with the impedance $\bar{Z}_\alpha$ of one of the waveguide walls:

$$v_n^\alpha \text{tg}\left(v_n^\alpha\delta\right) + i\bar{Z}_\alpha = 0; \quad \alpha = l \text{ or } r. \tag{1.25}$$

The expression for $X^-\left(\bar{\zeta}\right)$ is evident from (1.23), (1.24).

The coefficient of problem (1.19) can be written now as

$$\bar{D}(\xi) = \frac{\omega_1^+(\xi)\omega_2^+(\xi)}{\omega_1^-(\xi)\omega_2^-(\xi)},$$

whereas (1.19) takes the form

$$\frac{\omega^+(\xi)}{\omega_1^+(\xi)\omega_2^+(\xi)} = \frac{\omega^-(\xi)}{\omega_1^-(\xi)\omega_2^-(\xi)} + \frac{\bar{H}(\xi)}{\omega_1^+(\xi)\omega_2^+(\xi)}; \quad -\infty < \xi < \infty.$$

The solution of this problem on the discontinuity [7] is the Cauchy integral

$$\Psi^+(\zeta) \equiv \frac{\omega^+(\zeta)}{\omega_1^+(\zeta)\omega_2^+(\zeta)} = \frac{1}{2\pi i}\int_0^\infty \frac{H(\bar{\xi})\exp(-\pi\bar{\xi})}{\omega_1^+(\zeta')\omega_2^+(\zeta')(\zeta'-\zeta)}d\zeta';$$
$$\bar{\xi} = \ln\zeta'/2\pi, \quad \mathrm{Im}\zeta > 0.$$

Hence,

$$\exp(\pi\bar{\zeta})\Psi^+\left[\exp(2\pi\bar{\zeta})\right] = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{H(\bar{\xi}')\exp(-\varphi\bar{\xi}')}{X(\bar{\xi}')\mathrm{sh}\left[\pi(\bar{\xi}'-\bar{\zeta})\right]}d\bar{\xi}'; \quad \mathrm{Im}\bar{\zeta} > 0, \quad (1.26)$$

where $X(\bar{\xi}) = \omega_1(\exp(2\pi\bar{\xi}))$ and

$$H(\bar{\xi}) = \frac{-i(\bar{Z}_l - \bar{Z}_r)^2\exp(\varphi\bar{\xi})\cos\left[v(\bar{\xi})\delta\right]}{4\pi R_r(\bar{\xi})}\int_{-\infty}^{\infty}\frac{\cos[v(\eta)\bar{z}_0]\exp(i\bar{x}_0\eta)d\eta}{R_l(\eta)\exp(\varphi\eta)\mathrm{sh}\left[\pi(\bar{\xi}-\eta)\right]}.$$

The pole at the point $\eta = \bar{\xi}$ lies above the integration contour. Since according to (1.22) we have $\omega_1^+(\xi) = \omega_1^-(\xi)$ for $\xi < 0$, therefore the functions $\omega_1^{\pm}(\zeta)$ represent a unified analytical function $\omega_1(\zeta)$. Hence in what follows, we will not use the superscripts '$\pm$'.

When calculating the function in (1.26), the following integral arises

$$U(\eta,\bar{\xi}) = \int_{-\infty}^{\infty}\frac{\cos\left[v(\bar{\xi}')\delta\right]d\bar{\xi}'}{R_r(\bar{\xi}')X(\bar{\xi}')\mathrm{sh}\left[\pi(\bar{\xi}'-\eta)\right]\mathrm{sh}\left[\pi(\bar{\xi}'-\bar{\xi})\right]},$$

in which the integration contour passes above the pole $\bar{\xi}' = \eta$ and below the pole $\bar{\xi}' = \bar{\xi}$. Let us consider the auxiliary integral $\tilde{U}(\eta,\bar{\xi})$ along the boundary of the band $0 < \mathrm{Im}\bar{\zeta} < 1$. From the above we have

$$\tilde{U}(\eta,\bar{\xi}) = \int \frac{d\bar{\xi}'}{X(\bar{\xi}')\mathrm{sh}\left[\pi(\bar{\xi}'-\eta)\right]\mathrm{sh}\left[\pi(\bar{\xi}'-\bar{\xi})\right]}$$
$$= \int_{-\infty}^{\infty}\left[1 - \frac{X(\bar{\xi}')}{X(\bar{\xi}'+i)}\right]\frac{d\bar{\xi}'}{X(\bar{\xi}')\mathrm{sh}\left[\pi(\bar{\xi}'-\eta)\right]\mathrm{sh}\left[\pi(\bar{\xi}'-\bar{\xi})\right]} = i(\bar{Z}_r - \bar{Z}_l)U(\eta,\bar{\xi}).$$

Here we have used the equality $X(\bar{\xi})R_r(\bar{\xi}) = X(\bar{\xi}+i)R_l(\bar{\xi})$ following from (1.24). At the same time, the integral $\tilde{U}(\eta,\bar{\xi})$ equals to a sum of residues at the points $\bar{\xi}' = \eta + i$ and $\bar{\xi}' = \bar{\xi}$, and hence

$$U(\eta,\bar{\xi}) = \frac{2(\bar{Z}_r - \bar{Z}_l)^{-1}}{\mathrm{sh}[\pi(\bar{\xi}-\eta)]}\left[\frac{1}{X(\bar{\xi})} - \frac{1}{X(\bar{\xi}+i)}\right].$$

If we substitute this formula into (1.26) and take into consideration that the solution of the Carleman's boundary value problem (1.18) is

$$\widetilde{\Phi}(\bar{\zeta}) = \exp(\pi\bar{\zeta})\Psi[\exp(2\pi\bar{\zeta})]X(\bar{\zeta})\exp(\varphi\bar{\zeta}); \quad 0 \le \mathrm{Im}\,\bar{\zeta} \le 1,$$

we derive from (1.14), (1.17) the desired solution of the integral (1.15):

$$f(\bar{x}) = \frac{i(\bar{Z}_l - \bar{Z}_r)}{4(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{\exp(-i\bar{x}\eta')X(\eta')v(\eta')d\eta'}{\exp(-\varphi\eta')\exp[iv(\eta')\delta]R_l(\eta')}$$
$$\times \int_{-\infty}^{\infty} \frac{\cos[v(\eta)\bar{z}_0]\exp(i\bar{x}_0\eta)d\eta}{R_l(\eta)\exp(\varphi\eta)X(\eta)\mathrm{sh}[\pi(\eta-\eta')]}.$$

1.2.2.3 Residue Series Representation

Having regard to the equality $X(\eta)X(-\eta) = R_l(\eta)/R_r(\eta)$ following from (1.24), we obtain from (1.10), (1.12), and (1.14) that

$$G(g,g_0) = \frac{(\bar{Z}_l - \bar{Z}_r)}{4\pi} \int_{-\infty-\alpha_1}^{\infty-\alpha_1} \frac{\cos[v(x_1)\bar{z}]\exp(i\bar{x}x_1)dx_1}{X(x_1)R_r(x_1)}$$
$$\times \int_{-\infty}^{\infty} \frac{X(x_2)\cos[v(x_2)\bar{z}_0]\exp(-i\bar{x}_0x_2)\exp[\varphi(x_2-x_1)]dx_2}{R_l(x_2)\ \mathrm{sh}\pi[(x_2-x_1)]}, \tag{1.27}$$

where α_1 is a small positive value. In view of equalities (1.9), (1.10), we get the expression for the Green function $G^t(g,g_0)$.

Let us transform the integral representation of $G(g,g_0)$ in (1.27) into residue series. To do this, let us deform the integration surface $\mathrm{S} = \{z_1, z_2 : z_j = x_j + iy_j, j = 1,2, x_j \in \mathrm{R}^1, y_1 = -\alpha_1, y_2 = 0\}$ in the space C×C of two complex variables z_1

and z_2 into the Leray coboundary [18] enclosing the analytical set A of the singularities of the integrand. We rewrite (1.27) in the form

$$G(g, g_0) = \frac{1}{4\pi}(\bar{Z}_l - \bar{Z}_r)\int\limits_{S} \omega, \tag{1.28}$$

where the differential form is given by

$$\begin{aligned} &\omega = f(z_1)q(z_2)h(z_2 - z_1)\exp(i\bar{x}z_1 - i\bar{x}_0 z_2)dz_1 \wedge dz_2,\\ f(z_1) &= \frac{\cos[v(z_1)\bar{z}]}{R_r(z_1)X(z_1)}, \quad q(z_2) = \frac{X(z_2)\cos[v(z_2)\bar{z}_0]}{R_l(z_2)},\\ h(z) &= \exp(\varphi z)/\mathrm{sh}(\pi z). \end{aligned}$$

The set A comprises the following families of planes $z_1 = -\eta^l_{nk}$, $z_1 = \eta^r_{nk}$, $z_2 = \eta^l_{nk}$, $z_2 = -\eta^r_{nk}$, $z_2 - z_1 = \pm im$, $n, k, m = 0, 1, 2, \ldots$, where $\eta^\alpha_{nk} = \eta^\alpha_n + ik$ and $\alpha = \{l \text{ or } r\}$. The behavior of the integrand in (1.28) at infinity is governed by the sign of $\mathrm{Re}(i\bar{x}z_1 - i\bar{x}_0 z_2) = -\bar{x}y_1 + \bar{x}_0 y_2$. Consequently, let us introduce the following three-dimensional chains:

$$\begin{aligned} \mathrm{C}^\pm_1 &= \left\{ z_1, z_2 : x_{1,2} \in \mathrm{R}^1, \begin{pmatrix} y_1 > -\alpha_1 \\ y_1 < -\alpha_1 \end{pmatrix}, y_2 = 0 \right\},\\ \mathrm{C}^\pm_2 &= \left\{ z_1, z_2 : x_{1,2} \in \mathrm{R}^1, y_1 = -\alpha_1, \begin{pmatrix} y_2 > 0 \\ y_2 < 0 \end{pmatrix} \right\}, \end{aligned}$$

for which the integration surface S is a common boundary. If one of four inequalities $\bar{x} > 0$, $\bar{x} < 0$, $\bar{x}_0 > 0$ or $\bar{x}_0 < 0$ is satisfied, then we can use the *Cauchy-Poincare theorem* [18] in C^+_1, C^-_1, C^-_2 or C^+_2, respectively, and deform S into the Leray coboundary enclosing the polar straight lines, along which the analytical planes A and the chains $\mathrm{C}^\pm_j$ intersect.

It suffices to restrict ourselves to the case of $\bar{x}_0 < 0$. In C^+_2, the equations for polar straight lines are

$$\begin{aligned} \mathrm{P}_{nk} &= \{z_1 = s, z_2 = \eta^l_{nk}\}, \quad \mathrm{Q}_m = \{z_1 = s, z_2 = s + im\};\\ &-\infty < s < \infty, \quad n, k = 0, 1, 2, \ldots, \quad m = 1, 2, 3, \ldots, \end{aligned}$$

whereas the equations for their coboundaries are as follows:

$$\begin{aligned} &\delta\mathrm{P}_{nk} = \{z_1 = s, z_2 = \Delta\exp(i\theta) + \eta^l_{nk}\} \quad \text{and}\\ &\delta\mathrm{Q}_m = \left\{z_1 = \left(s - \sqrt{2}\Delta\cos\theta\right)\Big/2,\ z_2 = \left(s + \sqrt{2}\Delta\cos\theta\right) + i\Delta\sin\theta + im\right\};\\ &0 < \Delta \ll 1, \quad 0 \le \theta \le 2\pi. \end{aligned}$$

Therefore, the double integral in (1.28) can be represented as a sum of two single integrals

$$G(g,g_0)=\frac{1}{4\pi}(\bar{Z}_l-\bar{Z}_r)\left[\sum_{n,k=0}^{\infty}I_{nk}+\sum_{m=1}^{\infty}I_m\right],$$

where

$$I_{nk}=\lim_{\delta\to 0}\int_{\delta\mathrm{P}_{nk}}\omega=2\pi i(-1)^k\exp\left(ik\varphi-i\bar{x}_0\eta_{nk}^l\right)\cos\left[v\left(\eta_{nk}^l\right)\bar{z}_0\right]\psi_{nk}^{-1}J_n^1(\bar{x}),$$

$$I_m=\lim_{\delta\to 0}\int_{\delta\mathrm{Q}_m}\omega=2i(-1)^m\exp(im\varphi+m\bar{x}_0)J_m^2(\bar{x}-\bar{x}_0)$$

with

$$\psi_{nk}=\left.\frac{d[R_l(\eta)/X(\eta)]}{d\eta}\right|_{\eta=\eta_{nk}^l},\quad J_n^1(\bar{x})=\int_{-\infty}^{\infty}f(\zeta)h\left(\eta_n^1-\zeta\right)\exp(i\bar{x}\zeta)d\zeta,$$

$$J_m^2(\bar{x})=\int_{-\infty}^{\infty}f(\zeta)q(\zeta+im)\exp[i(\bar{x}-\bar{x}_0)\zeta]d\zeta.$$

With allowance made for the asymptotics of $X(\zeta)$ for $|\zeta|\gg 1$ and the fact that $f(\zeta)$ and $q(\zeta)$ are meromorphic functions, the above integrals can be reduced to residue series. As a result, we obtain the following representation for the Green function of problem (1.7a, 1.7b, 1.7c) in the form of the expansion in a two-parameter family of inhomogeneous plane waves:

$$G^t(g,g_0)=\begin{cases}\sum\limits_{n,k=0}^{\infty}g_{nk}^+(\bar{g}_0)\cos\left(v_{nk}^r\bar{z}\right)\exp\left(i\eta_{nk}^r\bar{x}\right); & \bar{x}>0\\ \sum\limits_{n,k=0}^{\infty}g_{nk}^-(\bar{g}_0)\cos\left(v_{nk}^l\bar{z}\right)\exp\left(-i\eta_{nk}^l\bar{x}\right) & \\ \quad+\sum\limits_{n,k=0}^{\infty}q_{nk}^+(\bar{g}_0)\cos\left(v_{n,-k}^l\bar{z}\right)\exp\left(i\eta_{n,-k}^l\bar{x}\right); & \bar{x}_0<\bar{x}<0\\ \sum\limits_{n,k=0}^{\infty}g_{nk}^-(\bar{g}_0)\cos\left(v_{nk}^l\bar{z}\right)\exp\left(-i\eta_{nk}^l\bar{x}\right) & \\ \quad+\sum\limits_{n,k=0}^{\infty}q_{nk}^-(\bar{g}_0)\cos\left(v_{nk}^l\bar{z}\right)\exp\left(-i\eta_{nk}^l\bar{x}\right); & \bar{x}<\bar{x}_0.\end{cases}$$

Here,

$$g_{nk}^{+}(\bar{g}_0) = \pi(\bar{Z}_r - \bar{Z}_l)\varphi_{nk}^{-1}\sum_{p,q=0}^{\infty}\left\{(-1)^{q-k}\psi_{pq}^{-1}\exp[i\varphi(q-k)]\frac{\exp\left[\varphi\left(\eta_p^l - \eta_n^r\right)\right]}{\mathrm{sh}\left[\pi\left(\eta_p^l - \eta_n^r\right)\right]}\right.$$
$$\left.\times\cos\left(v_{pq}^l\bar{z}_0\right)\exp\left(-i\eta_{pq}^l\bar{x}_0\right)\right\}, \tag{1.29a}$$

$$g_{nk}^{-}(\bar{g}_0) = \pi(\bar{Z}_r - \bar{Z}_l)\psi_{nk}^{-1}\sum_{p,q=0}^{\infty}\left\{(-1)^{q-k}\psi_{pq}^{-1}\exp[i\varphi(q+k)]\frac{\exp\left[\varphi\left(\eta_p^l + \eta_n^l\right)\right]}{\mathrm{sh}\left[\pi\left(\eta_p^l + \eta_n^l\right)\right]}\right.$$
$$\left.\times\cos\left(v_{pq}^l\bar{z}_0\right)\exp\left(-i\eta_{pq}^l\bar{x}_0\right)\right\}, \tag{1.29b}$$

$$q_{nk}^{+}(\bar{g}_0) = (\bar{Z}_r - \bar{Z}_l)\left[R_r\left(\eta_{n,-k}^l\right)X\left(\eta_{n,-k}^l\right)\right]^{-1}\sum_{q=0}^{\infty}\left\{(-1)^{q-k}\psi_{nq}^{-1}\exp[i\varphi(q+k)]\right.$$
$$\left.\times\cos\left(v_{nq}^l\bar{z}_0\right)\exp\left(-i\eta_{nq}^l\bar{x}_0\right)\right\}, \tag{1.29c}$$

$$q_{nk}^{-}(\bar{g}_0) = (\bar{Z}_r - \bar{Z}_l)\psi_{nk}^{-1}\sum_{q=0}^{\infty}\left\{(-1)^{q-k}\left[R_r\left(\eta_{n,-q}^l\right)X\left(\eta_{n,-q}^l\right)\right]^{-1}\exp[i\varphi(q+k)]\right.$$
$$\left.\times\cos\left(v_{n,-q}^l\bar{z}_0\right)\exp\left(i\eta_{n,-q}^l\bar{x}_0\right)\right\} \tag{1.29d}$$

with $\varphi_{nk} = d[R_r(\eta)X(\eta)]/d\eta|_{\eta=\eta_{nk}^r}$.

Direct substitution of (1.29a, 1.29b, 1.29c, 1.29d) and (1.27) into (1.7a, 1.7b, 1.7c) assures that we have found the desired solutions.

1.2.2.4 Transformation of Eigenmodes on the Waveguide Junction

The Obtained Green Function Determines the Electromagnetic Field

$$H_y = i\omega\varepsilon\varepsilon_0 I^{(m)}G^t, \quad E_x = I^{(m)}\frac{\partial}{\partial z}H_y, \quad E_z = -I^{(m)}\frac{\partial}{\partial x}H_y,$$

generated by a linear magnetic current of density $\vec{J}^{(m)} = I^{(m)}\delta(g - g_0)\exp(-i\omega t)\vec{y}$ in a plane waveguide whose bottom wall is perfectly conducting, while the surface impedance distribution of the top wall is defined by (1.8).

If the source and the observation point are well off the irregular section of the impedance distribution $Z(x)$, $|x_0| > |x| \gg 1/\tau$, then the functions in (1.29a, 1.29b, 1.29c, 1.29d) become expansions in terms of eigenmodes of the regular waveguides:

$$H_n^\alpha(g) = a_n^\alpha \cos\left(v_n^\alpha \tau z\right) \exp\left(\pm i\eta_n^\alpha \tau x\right); \quad \alpha = l \text{ or } r, \quad n = 0, 1, 2, \ldots \tag{1.30}$$

Here, the normalization $a_n^\alpha = i\left[R'_\alpha\left(\eta_n^\alpha\right)\cos\left(v_n^\alpha \tau d\right)\right]^{1/2}$ has been chosen such that the energy transported by each mode (1.30) does not depend on the indices n and α. Taking into account that the modes are orthogonal in these systems, we deduce that in the irregular segment the m-th mode of the left waveguide transforms into the n-th modes of the right and left regular waveguides with the transmission coefficient

$$T_{mn} = \left[-\pi \frac{R_r\left(\eta_m^l\right)R_l\left(\eta_n^r\right)}{R'_r\left(\eta_n^r\right)R'_l\left(\eta_m^l\right)}\right]^{1/2} \frac{X\left(\eta_m^l\right)\exp\left[\left(\eta_m^l - \eta_n^r\right)\varphi\right]}{X\left(\eta_n^r\right)\mathrm{sh}\left[\pi\left(\eta_m^l - \eta_n^r\right)\right]}; \quad n, m = 0, 1, 2, \ldots \tag{1.31}$$

and the reflection coefficient

$$R_{mn} = \left[\pi \frac{R_r\left(\eta_m^l\right)R_r\left(\eta_n^l\right)}{R'_l\left(\eta_n^l\right)R'_l\left(\eta_m^l\right)}\right]^{1/2} X\left(\eta_n^l\right)X\left(\eta_m^l\right) \frac{\exp\left[\left(\eta_m^l + \eta_n^r\right)\varphi\right]}{\mathrm{sh}\left[\pi\left(\eta_m^l + \eta_n^r\right)\right]}; \quad n, m = 0, 1, 2, \ldots, \tag{1.32}$$

where

$$R_\alpha\left(\eta_m^\beta\right) = i\left(\bar{Z}_\alpha - \bar{Z}_\beta\right)\cos\left(v_m^\beta \tau d\right) \text{ for } \alpha \neq \beta,$$
$$R'_\alpha\left(\eta_n^\alpha\right) = -\eta_n^\alpha \gamma_n^\alpha \left(v_n^\alpha\right)^{-2}\cos\left(v_n^\alpha \tau d\right), \quad \text{and} \quad \gamma_n^\alpha = \tau d\left[\left(v_n^\alpha\right)^2 - \bar{Z}_\alpha^2\right] - i\bar{Z}_\alpha;$$
$$\alpha, \beta = l \text{ or } r.$$

It is not hard to prove the invariance of R_{mn} with respect to a permutation of subscripts and the invariance of T_{mn} with respect to a simultaneous permutation of subscripts and impedances $Z_l \leftrightarrow Z_r$, or, in other words, to prove the reciprocity theorem for the waveguide under study.

Let us estimate the error of *adiabatic approximation* with the use of (1.31). This approximate description of wave processes in slightly irregular waveguides with no regard for the mode interconversion [19] is named by analogy with the Born-Oppenheimer method in solid-state physics. Up to now, the error for this approach has not been estimated. For ease of estimation, let us restrict ourselves to the case of purely imaginary limiting values $Z_l = iQ_l$, and $Z_r = iQ_r$, which is the same to the absence of absorption in the walls of the regular sections of the waveguide. Hodographs for the complex-valued surface impedance functions $Z(x,\varphi)$ ($-\pi < \varphi < 0$ for $Q_l < Q_r$ and $0 \le \varphi < \pi$ for $Q_l > Q_r$) represent a family of circular arcs of radius $Q^-/\sin\varphi$ centered at $iQ^+ - Q^-\text{ctg}\varphi$, $2Q^\pm = Q_l \pm Q_r$ (for $\varphi = 0$ it is a straight line) and connecting the points iQ_l and iQ_r in the right half-plane of physically realizable impedances. In this case, the following equalities for the propagation constants are valid:

$$\text{Im}\,\eta_s^\alpha = 0 \text{ for } 0 \le s \le s^\alpha \quad \text{and} \quad \text{Re}\,\eta_s^\alpha = 0 \text{ for } s^\alpha < s, \tag{1.33}$$

where s^α is the maximum number of the mode (1.30) propagating in the α -regular waveguide without attenuation.

Since, by hypothesis, the waveguide properties vary slowly over the distance of a wavelength, then $|\eta_s^\alpha| = |h_s^\alpha \tau^{-1}| \gg 1$, where h_s^α is the longitudinal wavenumber of the s-mode and τ^{-1} is the characteristic dimension of the irregular section of $Z(x)$. Then, with the asymptotic Stirling formula for gamma functions, we obtain from (1.31)

$$T_{mn} \approx \frac{e_m^r(-\eta_m^l)e_n^l(-\eta_n^r)e_m^l(-\eta_m^l)}{e_m^r(\eta_m^l)e_n^l(\eta_n^r)e_n^r(-\eta_n^r)} \cdot \frac{\exp[\varphi(\eta_m^l - \eta_n^r)]}{2\text{sh}[\pi(\eta_m^l - \eta_n^r)]} \cdot \frac{\Pi_m(\eta_m^l)}{\Pi_n(\eta_n^r)}, \tag{1.34}$$

where $e_m^\alpha(\eta) = \exp\{-i(\eta_m^\alpha - \eta)\ln[-i(\eta_m^\alpha - \eta)]\}$, the principal branch of $\ln z$ with a cut joining the points $z = 0$ and $z = -\infty$ has been chosen, and

$$\Pi_m(\eta) = \prod_{\substack{s=0\\ s\neq m}}^{\infty} \frac{e_s^l(\eta)e_s^r(-\eta)}{e_s^l(-\eta)e_s^r(\eta)}.$$

In view of (1.33), we derive from (1.34) the following expression (with a finite number of multipliers) for absolute values of the transmission coefficients for the undumped mode $H_m^l(g)$, $0 \le m \le s^l$ incoming from the left waveguide and transformed into undumped modes $H_n^r(g)$, $0 \le n \le s^r$ of the right waveguide:

$$|T_{mn}| \approx \left\{ \begin{array}{l} \prod_{s=m}^{n-1} \exp[\pi(\eta_{s+1}^l - \eta_s^r)] \text{ for } m < n, 1 \text{ for } m = n, \text{ and} \\ \exp[2\pi(\eta_m^l - \eta_n^r)] \prod_{s=n}^{m-1} \exp[\pi(\eta_s^r - \eta_{s+1}^l)] \text{ for } m > n \end{array} \right\} \exp[\varphi(\eta_m^l - \eta_n^r)]; \tag{1.35}$$

$Q_l < Q_r$

and

$$|T_{mn}| \approx \left\{ \exp\left[2\pi\left(\eta_n^r - \eta_m^l\right)\right] \prod_{s=m}^{n-1} \exp\left[\pi\left(\eta_s^l - \eta_{s+1}^r\right)\right] \text{ for } m<n, 1 \text{ for } m=n, \text{ and } \prod_{s=n}^{m-1} \exp\left[\pi\left(\eta_{s+1}^r - \eta_s^l\right)\right] \text{ for } m>n \right\} \exp\left[\varphi\left(\eta_m^l - \eta_n^r\right)\right]; \quad Q_l > Q_r. \tag{1.36}$$

In particular, in the case of two-mode operation ($s^l = s^r = 1$), as zero mode $H_0^l(g)$ runs against the inhomogeneity, we can write

$$|T_{01}| \approx \exp\left\{-\left[\pi\left(\eta_0^r - \eta_1^l\right) + |\varphi|\left(\eta_0^l - \eta_1^r\right)\right]\right\} < 1; \quad -\pi < \varphi \le 0, \quad Q_l < Q_r \tag{1.37}$$

and

$$|T_{01}| \approx \exp\left[(\varphi - \pi)\left(\eta_0^l - \eta_1^r\right)\right] < 1; \quad 0 \le \varphi < \pi, \quad Q_l > Q_r. \tag{1.38}$$

An interesting feature is exhibited when comparing the amplitudes of zero (principal) mode $H_0^r(g)$ and the first mode $H_1^r(g)$ travelling into the right waveguide:

$$\left|\frac{T_{01}}{T_{00}}\right| \approx \exp\left\{-\left[\pi\left(\eta_0^r - \eta_1^l\right) + |\varphi|\left(\eta_0^r - \eta_1^r\right)\right]\right\} < 1; \quad -\pi < \varphi \le 0, \quad Q_l < Q_r \tag{1.39}$$

and

$$\left|\frac{T_{01}}{T_{00}}\right| \approx \exp\left\{-\left[\pi\left(\eta_0^l - \eta_1^r\right) - \varphi\left(\eta_0^r - \eta_1^r\right)\right]\right\}; \quad 0 \le \varphi < \pi, \quad Q_l > Q_r. \tag{1.40}$$

In the latter case we have $|T_{01}/T_{00}| < 1$ with small φ, whereas for $\varphi \to \pi$ this value tends to $\exp\left[\pi\left(\eta_0^r - \eta_0^l\right)\right]$ and is greater than unity. That is, for $Q_l > Q_r$, starting with the hodograph $Z(x)$ of sufficiently large radius, the efficiency of transformation (when passing the irregular segment) of the zeroth mode into the first mode ($H_0^l \to H_1^r$) is greater than into the zeroth one ($H_0^l \to H_0^r$).

This effect is caused by the familiar phenomenon of the interconversion of two adjacent modes in the vicinity of the degeneracy regime. Among the wave structures with mode degeneracy is a regular impedance waveguide. It is known [20] that in such a waveguide, for each two adjacent modes H_j^α and H_{j+1}^α, the impedance value $Z_{j,j+1}^{\deg}$ exists such that the solutions v_j^α and v_{j+1}^α of the dispersion equation in (1.25) coincide. The analysis of the behavior of these roots on the trajectories passing around the point $Z^{\deg}_{j,j+1}$ reveals [21] that the complete mode interconversion $H^{\alpha}_{j} \leftrightarrow H^{\alpha}_{j+1}$ occurs as a result of this bypass.

In the above case of the two-mode operation (1.40), as φ increases, the arc of the hodograph $Z(x)$ occupies increasingly more space in the right half-plane of physically realizable impedances, into which the point $Z^{\deg}_{01}$ falls starting with some value φ_0. It is then that the transformation $H^{l}_{0} \rightarrow H^{r}_{1}$ becomes dominant, by virtue of the mode interconversion $H^{\alpha}_{0} \leftrightarrow H^{\alpha}_{1}$. These phenomena are of great interest for clarifying the effects of abnormal propagation of radio waves in the Earth-ionosphere waveguide along the paths intersecting the terminator [22]. It is interesting to note that in the case of $Q_l < Q_r$, the asymptotics in (1.39) do not show the effect at all, as well as in the case of a linear hodograph ($\varphi = 0$).

As obvious from the asymptotics in (1.35), (1.36), the adiabatic approximation error is defined by products of the exponentials $\exp\left(-\pi\left|\eta^{\alpha}_{i} - \eta^{\beta}_{j}\right|\right)$, where $\alpha, \beta = \{l \text{ or } r\}$, $i = 0, 1, \ldots, s^{\alpha}$, $j = 0, 1, \ldots, s^{\beta}$. If the arguments of these exponentials are of the order of unity, the adiabatic approximation is impossible. For example, for large positive Q_l and large negative Q_r, the value of $\eta^{l}_{0} - \eta^{r}_{1}$ is small and $|T_{01}|$ in (1.38) is of the order of unity as $\varphi \approx \pi$.

Finally note that rigorous error estimates are also lacking for the main theoretical approach used in the study of irregular waveguides with slowly varying parameters, namely, for the *cross-section method* [23] suggested by Stevenson [24]. The exact Green function derived in the present section provides such estimates as applied to the irregular impedance waveguides of fixed cross-section. In particular, it is seen from (1.29a, 1.29b, 1.29c, 1.29d) that for these structures the fields should be expanded in terms of two-parameter set of functions, whereas the cross-section method is based on the expansion in one-parameter set, namely, in the eigenfunctions of an auxiliary regular waveguide.

1.3 The Cycle Slipping Phenomenon and the Degeneracy of Waveguide Modes

1.3.1 Introduction

Electromagnetic wave propagation in the Earth-ionosphere waveguide has been studied intensively in the last five decades [25–28]. General formulation of the problems arising in the analysis of such waveguide processes is very complicated since it requires the inclusion of both the inhomogeneity of the Earth and the inhomogeneity and anisotropy of the ionosphere. In this section we restrict our analysis by the case of very low-frequency (VLF) waves, i.e. the electromagnetic oscillations whose frequency varies from 1.0 to 60 kHz. The main advantage of the waves of this range is their high stability against random variation of the ionospheric parameters. In particular, the analysis of peculiarities of the wave processes inherent in this range is of importance in developing global navigation systems.

We will examine the diurnal variations of the VLF field occurring when the 'transmitter-receiver' path crosses the dividing 'day-night' line. The propagation conditions vary significantly along this path during 24 h period. The decrease of the electron density in the lower ionosphere at night increases the effective height of the Earth-ionosphere waveguide and changes the properties of the upper wall of the waveguide, which in the modeling are usually characterized by the surface impedance. As a consequence, there is a marked increase in the field amplitude at night; the phase of the received signal changes as well. The standard view of these relationships, which has become known as *the amplitude and phase of trapezoids*, is shown in [29], Fig. 1.1. It is well explained by the simple single-mode propagation model.

However, a significant distinction from the specified standard form of the amplitude and phase dependencies of VLF signals can be observed on long paths [29, 30]. This difference consists in that the initial and final phases of the signal differ by $\pm 2\pi m$ (as a rule, $m = 1$) in diurnal phase records. This kind of abnormal diurnal field dependency at the point of reception is called a *cycle slipping (CS) phenomenon*.

From Fig. 1.2, which shows typical abnormal diurnal field variations, we notice that the CS phenomenon corresponds to an extremely deep fading of the received signal. This phenomenon can be explained qualitatively by assuming [22] that not

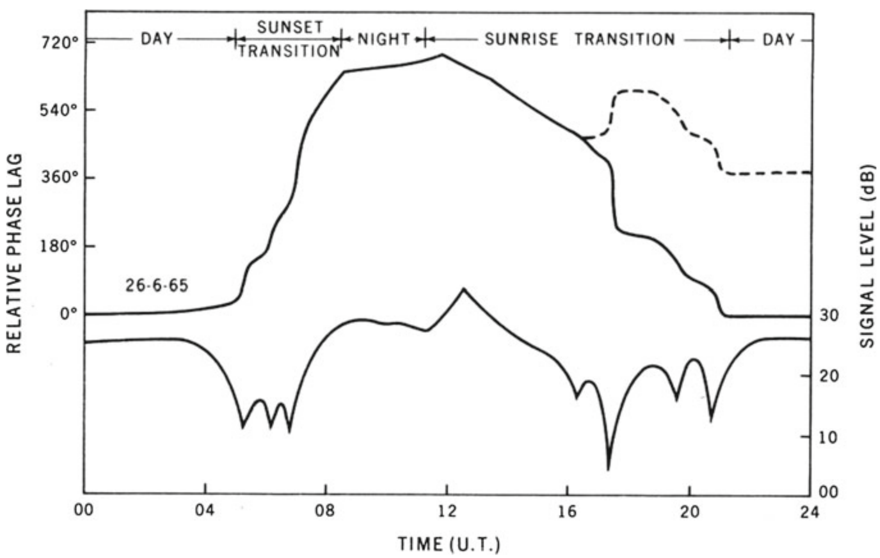


Fig. 1.2 (from paper [29]). Typical diurnal phase and signal level variations in NLK signals received at Smithfield (South Australia). Path length is equal to 13,420 km, f = 18.6 kHz. The broken line shows the phase record when cycle slipping occurs

only the principal (first) mode arrives at the observation point but also do the second mode and the higher-order modes resulting from the transformation of the principal mode on a waveguide discontinuity at the intersection of the path and the terminator (i.e. the sunrise or sunset line).

It is not difficult to see [27] that to observe the cycle slipping phenomenon, first of all, the field of the second mode should be greater at some moment of time than the field of the fundamental mode. Indeed, let at the point of reception two oscillations with the complex amplitudes $r_1 \exp(i\varphi_1)$ and $r_2 \exp(i\varphi_2)$ be added up. In order for the diurnal variation in the argument of the amplitude of the total signal $r_1 \exp(i\varphi_1)[1 + (r_2/r_1)\exp(i(\varphi_2 - \varphi_1))]$ be equal to 2π, the variation in the argument of the second factor must be 2π as well. (The phase variation of the first factor is zero, because during 24 h period it makes a symmetric trapezoidal oscillation.) Consequently, it is necessary that the ratio r_2/r_1 is greater than unity, at least, when $\varphi_2 - \varphi_1 = \pi$. It is just the fact that the ratio should be greater than unity, when the first and the second modes are in antiphase, which leads to that the cycle slipping phenomenon is usually accompanied by an abnormally deep minimum of the amplitude (Fig. 1.2). The most important here is the requirement of the large coefficient of conversion of the fundamental mode into the second mode.

A number of different modifications of irregular waveguides have been investigated by employing numerical simulation of the CS phenomenon. For example, the coefficient of conversion from the first into the second mode has been calculated by the method of partial domains for a number of two-dimensional impedance waveguides without considering the reflection from the discontinuity [30, 31]. Even for a stepwise change in the waveguide height, it did not exceed 0.5. In [32], to estimate this coefficient, the authors invoked the method of cross sections [23] developed for waveguide structures with slowly varying parameters over a wavelength distance. A two-dimensional model was used to represent a coaxial waveguide whose cross section and the surface impedance Z of one wall vary in azimuth. The coefficient of conversion reached 1.2, which, as the authors noted, was also too small to explain the CS phenomenon occurring mostly away from the terminator. The approach developed in [31] was extended in a number of papers to the waveguides whose top wall is a flat-layered anisotropic medium [33].

Only in one study [34], in contrast to all the above mentioned papers, the authors provide different *qualitative explanation* for this phenomenon in terms of the crude adiabatic approximation, by linking it with the degeneracy of the fundamental modes.

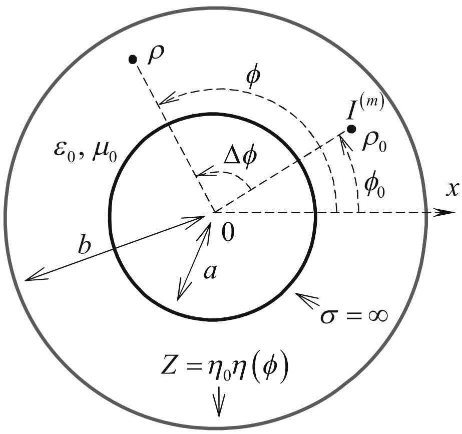
These investigations have cast doubt on the statement that the CS phenomenon can be explained solely by the conversion of the fundamental mode into the higher-order modes in the waveguide of variable cross section. In regular waveguides with walls of finite conductivity, which is constant along the structure, a more efficient mode-interconversion mechanism takes place. It is well known [35] that there exist values of the normalized surface impedance of the walls $\eta_{i,i+1}^{\deg}$ such that the propagation constants v_i and v_{i+1} of two adjacent (i and $i + 1$) waveguide modes coincide. Here $\eta = Z/\eta_0$, where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the wave resistance of vacuum. These modes and the associated impedance values are said to be degenerate. *Mode interconversion* occurs in the neighborhood of the degeneracy regime [21]. For example, by varying the complex-valued impedance $\eta(z)$ of the wall of a regular waveguide such that it draws a closed curve around the degeneracy point $\eta_{i,i+1}^{\deg}$, we get a complete interconversion of the i and $i + 1$ modes. In particular, the degeneracy of two VLF modes in a natural waveguide has been discussed in [36].

Our purpose is to clear up the role of the mode interconversion taking place in the neighborhood of the degeneracy regime in the occurrence of CS [37]. In Sect. 1.3.2, we present a model of the irregular waveguide with a constant cross section and the impedance varying in azimuth, which is a simplified version of the model given in [32]. This model allows us to exclude from consideration the diffraction effect of wave transformation on spatial inhomogeneities of the waveguide walls and to obtain the analytical solution of the associated boundary value problem for some class of surface impedance distributions. In the next section, with the help of the well-known *Watson method*, the solution will be transformed into a rapidly converging series for large wave sizes of the model. In Sect. 1.3.4 we present results of a numerical experiment.

1.3.2 Problem Formulation and Solution

Consider in the cylindrical coordinates ρ, ϕ, z a coaxial waveguide whose inner wall, $\rho = a$, is perfectly conducting and the outer wall, $\rho = b$, has variable surface impedance (Fig. 1.3). A filament of linear magnetic current with the time dependence $\exp(-i\omega t)$ disposed at $g_0 = \{\rho_0, \phi_0\}$ such that it is parallel to the z-axis, generates a field $\vec{E} = i\omega\mu_0\{\partial U/\rho\partial\phi, -\partial U/\partial\rho, 0\}$, $\vec{H} = k^2\{0, 0, U\}$. The Hertz potential U is a solution of the equation

Fig. 1.3 The waveguide cross-section geometry



$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k^2\right]U(g,g_0)=-\frac{iI^{(m)}}{\omega\mu_0}\frac{1}{\rho}\delta(g-g_0);\quad a<\rho,\rho_0<b,\quad -\pi\le\phi,\phi_0\le\pi \tag{1.41}$$

with the boundary conditions

$$\left.\frac{\partial U}{\partial\rho}\right|_{\rho=a}=0,\quad \left.\left[\frac{\partial U}{\partial\rho}-ik\eta(\phi)U\right]\right|_{\rho=b}=0, \tag{1.42}$$

where $k=\omega\sqrt{\varepsilon_0\mu_0}$ is the wavenumber and $I^{(m)}$ is the linear magnetic current density.

Let the normalized surface impedance of the wall $\rho=b$ be given in the form

$$\eta(\phi)=\eta_3\frac{\mathrm{e}^{i\phi}+\eta_1}{\mathrm{e}^{i\phi}+\eta_2} \tag{1.43}$$

with the arbitrary complex parameters η_j, $j=1,2,3$. Then the values of the function $\eta(\phi)$ form in the plane of the complex variable η a circle (the hodograph curve) of radius $r_{\mathrm{imp}}=|\eta_3(\eta_1-\eta_2)|\Big/\left|1-|\eta_2|^2\right|$ centered at the point $\eta_{\mathrm{imp}}=\eta_3(1-\eta_1\eta_2^*)\Big/\left(1-|\eta_2|^2\right)$.

In order to find the function U, we will use the Green formula

$$U(g,g_0)=U_0(g,g_0)+\int\limits_{S}\left[U(g_1,g_0)\frac{\partial}{\partial\vec{n}}G(g_1,g)-\frac{\partial}{\partial\vec{n}}U(g_1,g_0)G(g_1,g)\right]ds_1, \tag{1.44}$$

where $\vec{n}$ is the outer normal to the boundary S of the ring domain $\{a<\rho_1<b,-\pi<\phi_1<\pi\}$. By choosing as the function $G(g_1,g)$ the Green function of the space containing a perfectly conducting cylinder of radius a

$$\begin{aligned}G(g_1,g)&=-\frac{i}{8}\sum_{n=-\infty}^{\infty}\exp[in(\phi_1-\phi)]H_n^{(1,0)}(ka,k\rho_<)\frac{H_n^{(1)}(k\rho_>)}{H_n^{(1)\prime}(ka)}\\&=G(\rho_1,\rho,\phi_1-\phi)\end{aligned} \tag{1.45}$$

and as the function $U_0(g,g_0)$ the Hertz potential of the field generated by a linear magnetic current in the presence of the conducting cylinder $\rho=a$

$$U_0(g,g_0)=-\frac{iI^{(m)}}{k}G(\rho_0,\rho,\phi_0-\phi)=U_0(\rho_0,\rho,\phi_0-\phi), \tag{1.46}$$

we satisfy (1.41) and the first of the boundary conditions (1.42). In (1.45), the following notation is used:

$$H_n^{(j_1,j_2)}(x_1,x_2) = \frac{\partial^{j_1}}{\partial x_1^{j_1}}\frac{\partial^{j_2}}{\partial x_2^{j_2}}\left[H_n^{(1)}(x_1)H_n^{(2)}(x_2) - H_n^{(2)}(x_1)H_n^{(1)}(x_2)\right]; \quad j_1, j_2 = 0, 1,$$

$H_n^{(j)}(\ldots)$ stands for the Hankel functions, $\rho_< = \min(\rho, \rho_1)$, $\rho_> = \max(\rho, \rho_1)$.

In order to satisfy the remained boundary condition from (1.42), one can make in (1.44) the passage $\rho \to b$ and then substitute the value of $\partial U/\partial\rho$ on the boundary $\rho = b$. Then the equality (1.44) turns into an integral equation of the second kind with a strong kernel singularity [38]. To avoid this, let us consider formula (1.44) on the circle $\rho = b - \Delta$, where Δ is a small positive value. Then we have:

$$\begin{aligned} U(b-\Delta, \rho_0, \phi, \phi_0) &= U_0(\rho_0, b-\Delta, \phi_0 - \phi) \\ &+ b\int_{-\pi}^{\pi}\left[\frac{\partial}{\partial\rho}G\left(\rho, b-\Delta, \tilde{\phi}-\phi\right)\Big|_{\rho=b} - ik\eta\left(\tilde{\phi}\right)G\left(b, b-\Delta, \tilde{\phi}-\phi\right)\right]U\left(b, \rho_0, \tilde{\phi}, \phi_0\right)d\tilde{\phi}. \end{aligned} \tag{1.47}$$

Let us denote the direct and inverse Fourier transform operators as

$$W_\phi[a_n] = A(\phi) = \sum_{n=-\infty}^{\infty} a_n \exp(in\phi),$$

$$W_n^{-1}[A(\phi)] = a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\phi)\exp(-in\phi)d\phi.$$

For the inverse Fourier transform the following relationships are valid:

$$W_n^{-1}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} A\left(\tilde{\phi}-\phi\right)B(\phi)d\phi\right] = a_{-n}b_n, \quad W_n^{-1}[\exp(ip\phi)A(\phi)] = a_{n-p}. \tag{1.48}$$

Applying the operator W_n^{-1} to (1.47), we obtain in view of (1.48):

$$\begin{aligned} &W_{n-1}^{-1}\left[\frac{U(b-\Delta, \rho_0, \phi, \phi_0)}{\exp(i\phi)+\eta_2}\right] + \eta_2 W_n^{-1}\left[\frac{U(b-\Delta, \rho_0, \phi, \phi_0)}{\exp(i\phi)+\eta_2}\right] = W_n^{-1}[U_0(\rho_0, b-\Delta, \phi_0-\phi)] \\ &+ 2\pi b W_{-n}^{-1}\left[\frac{\partial G(\rho, b-\Delta, \phi)}{\partial\rho}\Big|_{\rho=b}\right]\left\{W_{n-1}^{-1}\left[\frac{U(b, \rho_0, \phi, \phi_0)}{\exp(i\phi)+\eta_2}\right] + \eta_2 W_n^{-1}\left[\frac{U(b, \rho_0, \phi, \phi_0)}{\exp(i\phi)+\eta_2}\right]\right\} \\ &- 2\pi bik\eta_3 W_{-n}^{-1}[G(b, b-\Delta, \phi)]\left\{W_{n-1}^{-1}\left[\frac{U(b, \rho_0, \phi, \phi_0)}{\exp(i\phi)+\eta_2}\right] + \eta_1 W_n^{-1}\left[\frac{U(b, \rho_0, \phi, \phi_0)}{\exp(i\phi)+\eta_2}\right]\right\}, \end{aligned} \tag{1.49}$$

where

$$W_n^{-1}[U_0(\rho_0, b-\Delta, \phi_0-\phi)] = -\frac{I^{(m)}}{8k}\exp(-in\phi_0)\frac{H_n^{(1,0)}(x, k\rho_0)}{H_n^{(1)'}(x)}H_n^{(1)}[k(b-\Delta)],$$

$$W_{-n}^{-1}[G(b, b-\Delta, \phi)] = -\frac{i}{8}\frac{H_n^{(1,0)}(x, k(b-\Delta))}{H_n^{(1)'}(x)}H_n^{(1)}(y),$$

$$W_{-n}^{-1}\left[\left.\frac{\partial G(\rho, b-\Delta, \phi)}{\partial \rho}\right|_{\rho=b}\right] = -\frac{ik}{8}\frac{H_n^{(1,0)}(x, k(b-\Delta))}{H_n^{(1)'}(x)}H_n^{(1)'}(y); \quad -\infty < n < \infty,$$

and $H_n^{(j)'}(x) = dH_n^{(j)}(x)/dx$. One can pass to the limit $\Delta \to 0$ in these relationships. Considering that $H_{-n}^{(j)}(x) = (-1)^n H_n^{(j)}(x)$, we obtain the following finite-difference equation [7]

$$u_n = -\eta_2^{-1}(1+s_n)u_{n-1} + g_n; \quad -\infty < n < \infty. \tag{1.50}$$

Here

$$(1+s_n) = \frac{H_{n,1}^{(1,1)(1,0)}(x,y)}{H_{n,\delta}^{(1,1)(1,0)}(x,y)}, \quad g_n = -\frac{I^{(m)}\exp(-in\phi_0)}{2\pi iky\eta_2}\frac{H_n^{(1,0)}(x, k\rho_0)}{H_{n,\delta}^{(1,1)(1,0)}(x,y)} \tag{1.51}$$

and $H_{n,\delta}^{(1,1)(1,0)}(x,y) = H_n^{(1,1)}(x,y) - i\eta_3\delta H_n^{(1,0)}(x,y)$, $H_{n,1}^{(1,1)(1,0)}(x,y) = H_{n,\delta}^{(1,1)(1,0)}(x,y)|_{\delta=1}$, $u_n = W_n^{-1}[U(b, \rho_0, \phi, \phi_0)/[\exp(i\phi) + \eta_2]]$, $s_{-n} = s_n$, $x = ka$, $y = kb$, $\delta = \eta_1/\eta_2$.

Let us apply the *factorization method* [7] to solve (1.50). Represent the multiplier in (1.50) in the following form:

$$(1+s_n) = \frac{x_n}{x_{n-1}^{\gamma}}, \tag{1.52}$$

where the exponent $\gamma > 1$ is an auxiliary parameter. Taking the logarithm of (1.52) and then applying the operators W and W^{-1}, we can easily show that

$$\ln x_n = W_n^{-1}[W_\theta[\ln(1+s_n)]/[1-\gamma\exp(i\theta)]] = -\gamma^n \sum_{m=n+1}^{\infty} \ln(1+s_m)\gamma^{-m}. \tag{1.53}$$

Estimate the convergence of this series. Using the known asymptotics

$$J_\nu(z) \approx (2\pi\nu)^{-1/2}\left(\frac{ez}{2\nu}\right)^{\nu}, \quad H_\nu^{(1)}(z) \approx -2i(2\pi\nu)^{-1/2}\left(\frac{ez}{2\nu}\right)^{-\nu}, \tag{1.54}$$

for fixed z and $|v| \gg 1$, $|\arg v| < \pi/2$, one can show that $\ln(1+s_n) = i\eta_3(\delta-1)yn^{-1} + O(n^{-2})$. In other words, the convergence of the series in (1.53) is too weak to pass to the limit $\gamma \to 1$ under the sum sign. The elements of the factorization sequence x_n are defined up to an arbitrary factor without violating the equality (1.52). This allows us to solve the problem of convergence of the series in (1.53). Let us take the logarithm of the right-hand side of (1.52) and rearrange it in the following way:

$$\begin{aligned}
\ln x_n - \gamma \ln x_{n-1} &= -\gamma^n \sum_{m=n+1}^{\infty} \gamma^{-m} \ln(1+s_m) + \gamma^n \sum_{m=n}^{\infty} \gamma^{-m} \ln(1+s_m) \\
&\quad + \gamma^n \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) - \gamma^n \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) \\
&= \gamma^n \sum_{m=0}^{n} \gamma^{-m} \ln(1+s_m) - \gamma^n \sum_{m=0}^{n-1} \gamma^{-m} \ln(1+s_m); \quad n \geq 1, \\
\ln x_0 - \gamma \ln x_{-1} &= -\sum_{m=1}^{\infty} \gamma^{-m} \ln(1+s_m) + \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) \\
&\quad + \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) - \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) = \ln(1+s_0) - 0; \quad n = 0, \\
\ln x_{-1} - \gamma \ln x_{-2} &= -\gamma^{-1} \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) + \gamma^{-1} \sum_{m=-1}^{\infty} \gamma^{-m} \ln(1+s_m) \\
&\quad + \gamma^{-1} \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) - \gamma^{-1} \sum_{m=0}^{\infty} \gamma^{-m} \ln(1+s_m) = 0 - \ln(1+s_{-1})^{-1}; \quad n = -1,
\end{aligned}$$

and

$$\ln x_n - \gamma \ln x_{n-1} = -\gamma^n \sum_{m=-1}^{n+1} \gamma^{-m} \ln(1+s_m) + \gamma^n \sum_{m=-1}^{n} \gamma^{-m} \ln(1+s_m); \quad n \leq -2.$$

So we can pass to the limit $\gamma \to 1$ and get

$$x_n = \left\{ \prod_{m=0}^{n} (1+s_m) \text{ for } n \geq 0;\ 1 \text{ for } n = -1;\ \prod_{m=-1}^{n+1} (1+s_m)^{-1} \text{for } n \leq 2 \right\}. \quad (1.55)$$

It is easy to verify that this sequence satisfies (1.52) with $\gamma = 1$. By substituting (1.52) with $\gamma = 1$ into (1.50), we arrive at the equation

$$\frac{u_n}{x_n} = -\eta_2^{-1} \frac{u_{n-1}}{x_{n-1}} + \frac{g_n}{x_n}; \quad -\infty < n < \infty.$$

The solution of this equation is similar to that of the equation for $\ln x_n$, which can be derived by taking the logarithm of (1.52), and is as follows

$$\frac{u_n}{x_n} = W_n^{-1}\left[\frac{W_\theta(g_n/x_n)}{1+\eta_2^{-1}\exp(i\theta)}\right] = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty}\frac{g_m}{x_m}\frac{\exp[i(m-n)\theta]}{1+\eta_2^{-1}\exp(i\theta)}d\theta. \tag{1.56}$$

The integrand here has no singularities on the path of integration as far as its denominator coincides with the denominator of the function $\eta(\phi)$, while the surface impedance distribution of the waveguide is naturally assumed to be a limited function. Equations (1.44)–(1.46), (1.51), (1.55), and (1.56) allows us to obtain a closed expression for the Hertz potential U. One should distinguish two cases: $|\eta_2| < 1$ and $|\eta_2| > 1$. Let us do the relevant calculations for the first case.

The calculation of the integral in (1.56), by substituting $\exp(i\theta) = z$, is reduced to the calculation of residues at the points $z = 0$ and $z = -\eta_2$. As a result we have

$$\frac{u_n}{x_n} = -\sum_{m=n+1}^{\infty}\frac{g_m}{x_m}(-\eta_2)^{m-n}.$$

Then we find the Hertz potential distribution on the impedance wall $\rho = b$:

$$\begin{aligned} U(b,\rho_0,\phi,\phi_0) &= [\exp(i\phi)+\eta_2]W_\phi[u_n] \\ &= -[\exp(i\phi)+\eta_2]\sum_{n=-\infty}^{\infty}x_n\sum_{m=n+1}^{\infty}\frac{g_m}{x_m}(-\eta_2)^{m-n}\exp(in\phi). \end{aligned}$$

The potential inside the waveguide, as follows from (1.44), is

$$U(\rho,\rho_0,\phi,\phi_0) = U_0(\rho_0,\rho,\phi_0-\phi) + b\int\limits_{-\pi}^{\pi}H(\phi_1,\phi)U(b,\rho_0,\phi_1,\phi_0)d\phi_1, \tag{1.57}$$

where

$$H(\phi_1,\phi) = -\frac{ik}{8}\sum_{l=-\infty}^{\infty}\exp[il(\phi_1-\phi)]H_l^{(1,0)}(x,k\rho)\left[H_l^{(1)\prime}(y) - i\eta(\phi)H_l^{(1)}(y)\right]\Big/H_l^{(1)\prime}(x).$$

The integration in (1.57) results in the following expression for the potential

$$U(g,g_0) = -\frac{I^{(m)}}{8k}[U_0(g,g_0)+U_1(g,g_0)], \tag{1.58}$$

where (see formulas (1.45), (1.46))

$$U_0(g,g_0)=\sum_{n=-\infty}^{\infty}\exp[in(\phi_0-\phi)]H_n^{(1,0)}(x,k\rho_<)H_n^{(1)}(k\rho_>)\Big/H_n^{(1)\prime}(x)$$

and

$$U_1(g,g_0)=-\sum_{n=-\infty}^{\infty}\exp(in\phi)\sum_{m=n+1}^{\infty}\frac{x_n}{x_m}\frac{H_m^{(1,0)}(x,k\rho_0)}{H_{m,\delta}^{(1,1)(1,0)}(x,y)}\exp(-im\phi_0)(-\eta_2)^{m-n-1},$$
$$\times\left[\eta_2\frac{H_n^{(1,0)}(x,k\rho)}{H_n^{(1)\prime}(x)}H_n^{\delta}(y)+\exp(i\phi)\frac{H_{n+1}^{(1,0)}(x,k\rho)}{H_{n+1}^{(1)\prime}(x)}H_{n+1}^{1}(y)\right],$$

$H_n^{\delta}(y)=H_n^{(1)\prime}(y)-i\eta_3\delta H_n^{(1)}(y)$, $H_n^1(y)=H_n^{\delta}(y)\big|_{\delta=1}$, $\rho_<=\min(\rho_0,\rho)$, $\rho_>=\max(\rho_0,\rho)$, and x_n is given by (1.55).

In a similar way, transformations are made for $|\eta_2|>1$. It would be convenient to separate the regular and irregular parts of the potential in (1.58). After lengthy transformations, we arrive at the following expression for the Hertz potential

$$U(g,g_0)=\frac{I^{(m)}}{8k}\left[U_{reg}(g,g_0)+U_{ireg}(g,g_0)\right], \tag{1.59}$$

where its regular part with the simple angular dependence in the form of $\phi-\phi_0$ is

$$U_{\text{reg}}(g,g_0)=\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_0)]\frac{H_n^{(1,0)}(x,k\rho_<)}{H_{n,\alpha}^{(1,1)(1,0)}(x,y)}H_{n,\alpha}^{(1,0)(0,0)}(y,k\rho_>), \tag{1.60}$$

$$\alpha=\{1\ \text{if}|\eta_2|<1,\delta\ \text{if}|\eta_2|>1\},\quad \rho_<=\min(\rho_0,\rho_1),\quad \rho_>=\max(\rho_0,\rho_1),$$

while its irregular part is

$$U_{\text{ireg}}(g,g_0)=-4\frac{\eta_3(1-\delta)}{\pi y}\sum_{n=-\infty}^{\infty}\exp[in(\phi-\phi_0)]\frac{H_n^{(1,0)}(x,k\rho)}{H_{n,1}^{(1,1)(1,0)}(x,y)}U_n(g_0), \tag{1.61}$$

$$U_n(g_0)=\sum_{m=1}^{\infty}\exp(-im\phi_0)(-\eta_2)^m\prod_{j=1}^{m}\frac{H_{n+j,\delta}^{(1,1)(1,0)}(x,y)}{H_{n+j,1}^{(1,1)(1,0)}(x,y)}\frac{H_{n+m}^{(1,0)}(x,k\rho_0)}{H_{n+m,\delta}^{(1,1)(1,0)}(x,y)};\quad |\eta_2|<1, \tag{1.62}$$

$$U_n(g_0)=-\sum_{m=1}^{\infty}\exp(im\phi_0)(-\eta_2)^{-m}\prod_{j=0}^{m-1}\frac{H_{n-j,1}^{(1,1)(1,0)}(x,y)}{H_{n-j,\delta}^{(1,1)(1,0)}(x,y)}\frac{H_{n-m}^{(1,0)}(x,k\rho_0)}{H_{n-m,\delta}^{(1,1)(1,0)}(x,y)};\quad |\eta_2|>1. \tag{1.63}$$

The first term in (1.59) coincides with the solution to the problem where the source excites the regular coaxial waveguide whose reduced surface impedance of the wall $\rho = b$ equals $\eta_3 \alpha$.

It is easy to show the uniform convergence of the series, which determines the second term in (1.59), within the interval $a \le \rho,\ \rho_0 \le b$. Hence in this region the function $U_{\text{ireg}}(g, g_0)$ is analytic and satisfies the homogeneous Helmholtz equation.

Following the methodology in [39], one can make certain that the function $U(g, g_0)$ in (1.59) is really the desired Green function of the Helmholtz (1.41) in the ring region with irregular boundary conditions (1.42).

1.3.3 The Watson Transformation

The series in n in (1.60), (1.61) represent expansions in terms of radially propagating waves. Since the number of the terms contributing significantly to the field are of the order of $O(ka)$ [20, 40], (1.59) is convenient for analysis only for $ka \ll 1$.

For the applications considered in the present section, the range of interest is $ka \gg 1$, where the expansions in terms of azimuthally propagating *'creeping' waves* (alternative to the series in (1.60), (1.61)), obtainable from (1.59) by using the so called *Watson transformation* [20, 39, 41], are rapidly convergent.

The method leading to the Watson transformation was proposed in the early twentieth century in the works of H. Poincare and J.W. Nicholson and was first used in the electromagnetic theory by G.N. Watson [42]. This mathematical apparatus is also used in quantum mechanics, in the theory of potential scattering [42].

As applied to series like in (1.60), (1.61), the initial statement of this method is as follows: if the function of complex variable $B(v)$ is analytic in the neighborhood of the real axis, then the equality is valid

$$\sum_{n=-\infty}^{\infty} \exp(in\phi) B(n) = \frac{i}{2} \int_C \frac{\exp[iv(\phi - \pi)]}{\sin \pi v} B(v) dv, \tag{1.64}$$

, where C is the contour formed by two straight lines $\text{Im}v = \pm\alpha,\ \alpha \ll 1$ and bypassing the real axis in a clockwise direction. Let us first consider the regular part of the field:

$$U_{\text{reg}}(g, g_0) = \sum_{n=-\infty}^{\infty} \exp(in\Delta\phi) B_{\text{reg}}(n), \tag{1.65}$$

$$B_{\text{reg}}(n) = \frac{H_n^{(1,0)}(x, k\rho_<)}{H_{n,\alpha}^{(1,1)(1,0)}(x, y)} H_{n,\alpha}^{(1,0)(0,0)}(y, k\rho_>), \quad \Delta\phi = \phi - \phi_0 > 0. \tag{1.66}$$

If the analytical properties of the function $B_{\text{reg}}(\nu)$ allow the contour of integration C to be deformed to infinity, then the integral in (1.64) can be represented as a series of residues at the poles $B_{\text{reg}}(\nu)$. This series is just the Watson transform of the initial series.

Consider the function $B_{\text{reg}}(\nu)$. Since for the Hankel functions with complex index the following relationships are valid: $H^{(1)}_{-\nu}(z) = \exp(i\pi\nu)H^{(1)}_{\nu}(z)$, $H^{(2)}_{-\nu}(z) = \exp(-i\pi\nu)H^{(2)}_{\nu}(z)$, then we have $B_{\text{reg}}(-\nu) = B_{\text{reg}}(\nu)$; hence it is sufficient to clear up the properties of this function in the half-plane $\text{Re}\nu > 0$. Using the asymptotics (1.54) we find:

$$B_{\text{reg}}(\nu) \approx \frac{2i}{\pi\nu}\left(\frac{\rho_<}{\rho_>}\right)^{\nu} \quad \text{for} \quad |\nu| \gg 1, \quad |\arg \nu| < \pi/2.$$

Hence, the integral in (1.64) is reduced to a sum of the residues at the poles ν_s obtainable from the formula

$$\begin{aligned} H^{(1,1)(1,0)}_{\nu,\alpha}(x,y) \equiv & \left[H^{(1)'}_{\nu}(x)H^{(2)'}_{\nu}(y) - H^{(2)'}_{\nu}(x)H^{(1)'}_{\nu}(y)\right] \\ & - i\eta_3\alpha\left[H^{(1)'}_{\nu}(x)H^{(2)}_{\nu}(y) - H^{(2)'}_{\nu}(x)H^{(1)}_{\nu}(y)\right] = 0; \quad 0 < x < y. \end{aligned} \tag{1.67}$$

Let us determine the location of zeros of this equation in the ν-plane. Following the paper [43], on the assumption that x and y are fixed and $|\nu| \gg 1 + y^2$, we obtain the following approximation:

$$\nu_{\pm 1} \approx \pm\sqrt{i\eta_3\alpha y/\ln(y/x)}, \quad \nu_s \approx \frac{\eta_3\alpha y}{\pi(s-1)} + \frac{i\pi(s-1)}{\ln(y/x)} \quad \text{for} \quad s = 2, 3, \ldots$$

and

$$\nu_s \approx \frac{\eta_3\alpha y}{\pi(s+1)} + \frac{i\pi(s+1)}{\ln(y/x)} \quad \text{for} \quad s = -2, -3, \ldots$$

Thus the roots of (1.67) are located symmetrically in the first ($s = 1, 2, 3, \ldots$) and the third ($s = -1, -2, -3, \ldots$) quadrants of the ν-plane.

By finding the residues at these points, we arrive at the representation

$$U_{\text{reg}}(g, g_0) = -2\pi\sum_{s=1}^{\infty}\frac{\cos[\nu_s(\pi - \Delta\phi)]H^{(1,0)}_{\nu_s}(x, k\rho_<)}{\sin(\pi\nu_s)\tilde{H}^{(1,1)(1,0)}_{\nu_s,\alpha}(x,y)}H^{(1,0)(0,0)}_{\nu_s,\alpha}(y, k\rho_>), \tag{1.68}$$

where the following notation is used: $\tilde{H}^{(1,1)(1,0)}_{\nu_s,\alpha}(x,y) = \partial H^{(1,1)(1,0)}_{\nu,\alpha}(x,y)\Big/\partial\nu\Big|_{\nu=\nu_s}$.

For the irregular part of the field, the manipulations are similar though more cumbersome:

$$U_{\text{ireg}}(g, g_0) = \frac{4\eta_3(1-\delta)}{\pi kb} \sum_{n=-\infty}^{\infty} \exp(in\Delta\phi) B_{\text{ireg}}(n),$$

where

$$B_{\text{ireg}}(v) = -\sum_{l=1}^{\infty} \exp(-il\phi_0)(-\eta_2)^l \frac{H_v^{(1,0)}(x, k\rho)}{H_{v,\delta}^{(1,1)(1,0)}(x,y)} \Pi_l(v) \frac{H_{v+l}^{(1,0)}(x, k\rho_0)}{H_{v+l,\delta}^{(1,1)(1,0)}(x,y)},$$

$$\Pi_l(v) = \prod_{j=0}^{l} \frac{H_{v+j,\delta}^{(1,1)(1,0)}(x,y)}{H_{v+j,1}^{(1,1)(1,0)}(x,y)} \quad \text{for} \quad |\eta_2| < 1$$

and

$$B_{\text{ireg}}(v) = \sum_{l=1}^{\infty} \exp(il\phi_0)(-\eta_2)^l \frac{H_v^{(1,0)}(x, k\rho)}{H_{v,1}^{(1,1)(1,0)}(x,y)} \tilde{\Pi}_l(v) \frac{H_{v-l}^{(1,0)}(x, k\rho_0)}{H_{v-l,1}^{(1,1)(1,0)}(x,y)},$$

$$\tilde{\Pi}_l(v) = \prod_{j=0}^{l} \frac{H_{v-j,1}^{(1,1)(1,0)}(x,y)}{H_{v-j,\delta}^{(1,1)(1,0)}(x,y)} \quad \text{for} \quad |\eta_2| > 1.$$

The poles of the function $B_{\text{ireg}}(v)$ for $|\eta_2| < 1$ are located at the points $v_s - j$, where v_s are the roots of the equation (1.67) with $\alpha = 1$. For $|\eta_2| > 1$, they are located at the points $v_s + j$, where v_s are the roots of the equation (1.67) with $\alpha = \delta$. By finding the residues at these points, we arrive at the following expressions:

$$\begin{aligned} U_{\text{ireg}}(g, g_0) = {} & \frac{4i}{y}\eta_3^2(1-\delta)^2 \sum_{s=1}^{\infty} \frac{H_{v_s}^{(1,0)}(x,y)}{\sin(\pi v_s)\tilde{H}_{v_s,\alpha}^{(1,1)(1,0)}(x,y)} \\ & \times [\exp[iv_s(\Delta\phi - \pi)]U(g, g_0, v_s) + \exp[-iv_s(\Delta\phi - \pi)]U(g, g_0, -v_s)], \end{aligned} \tag{1.69}$$

where

$$\begin{aligned} U(g, g_0, v_s) = {} & \sum_{m=1}^{\infty} \exp(-im\phi_0)(-\eta_2)^m \sum_{l=0}^{m} \exp(-il\Delta\phi)\Pi_m^{(l)}(v_s - l) \\ & \times \frac{H_{v_s-l}^{(1,0)}(x, k\rho) H_{v_s-l+m}^{(1,0)}(x, k\rho_0)}{H_{v_s-l,\delta}^{(1,1)(1,0)}(x,y) H_{v_s-l+m,\delta}^{(1,1)(1,0)}(x,y)} \quad \text{for} \quad |\eta_2| < 1, \end{aligned} \tag{1.70}$$

$$U(g,g_0,v_s)=\sum_{m=1}^{\infty}\exp(im\phi_0)(-\eta_2)^{-m}\sum_{l=0}^{m}\exp(il\Delta\phi)\tilde{\Pi}_m^{(l)}(v_s+l) \\ \times\frac{H_{v_s+l}^{(1,0)}(x,k\rho)H_{v_s+l-m}^{(1,0)}(x,k\rho_0)}{H_{v_s+l,1}^{(1,1)(1,0)}(x,y)H_{v_s+l-m,1}^{(1,1)(1,0)}(x,y)}\quad \text{for}\quad |\eta_2|>1, \tag{1.71}$$

$$\Pi_m^{(l)}(v)=\prod_{\substack{j=0\\ j\neq l}}^{m}\frac{H_{v+j,\delta}^{(1,1)(1,0)}(x,y)}{H_{v+j,1}^{(1,1)(1,0)}(x,y)},\quad \tilde{\Pi}_m^{(l)}(v)=\prod_{\substack{j=0\\ j\neq l}}^{m}\frac{H_{v-j,1}^{(1,1)(1,0)}(x,y)}{H_{v-j,\delta}^{(1,1)(1,0)}(x,y)}, \\ \tilde{H}_{v_s,\alpha}^{(1,1)(1,0)}(x,y)=\tfrac{d}{dv}H_{v,\alpha}^{(1,1)(1,0)}(x,y)\Big|_{v=v_s},\quad \Delta\phi=\phi-\phi_0>0, \tag{1.72}$$

and v_s are the roots of the equation (1.67).

In the analysis which follows, we restrict ourselves to the case of $|\eta_2|<1$. The CS phenomenon has been detected for the waves coming to the receiver by the shortest route. Therefore, separating them out in (1.68)–(1.72) and placing the receiver and the source onto the boundary $\rho=a$ at the points with angular coordinates ϕ and ϕ_0, respectively, we arrive at the following expression for the Hertz vector

$$\frac{4k}{I^{(m)}}U(g,g_0)|_{\rho=\rho_0=a}=-\frac{4}{x}\sum_{s=1}^{\infty}\frac{\exp(iv_s\Delta\phi)}{\tilde{H}_{v_s,1}^{(1,1)(1,0)}(x,y)}V_s(\phi,\phi_0), \tag{1.73}$$

$$V_s(\phi,\phi_0)=V_{\text{reg}}(v_s)+V_{\text{ireg}}(\phi,\phi_0,v_s), \tag{1.74}$$

$$V_{\text{reg}}(v_s)=H_{v_s,1}^{(1,0)(0,0)}(y,x), \tag{1.75}$$

$${}_{\text{ireg}}(\phi,\phi_0,v_s)=\frac{-16i\eta_3}{\pi^2xy}(1-\delta)\Big\{u_s^+(\phi_0)+\Big[1+i\eta_3(1-\delta)H_{v_s}^{(1,0)}(x,y)u_s^+(\phi_0)\Big]u_s^-(\phi)\Big\}, \tag{1.76}$$

$$u_s^{\pm}(\phi)=\sum_{m=1}^{\infty}\exp(-im\phi)(-\eta_2)^m\prod_{j=1}^{m}\frac{H_{v_s\pm j,\delta}^{(1,1)(1,0)}(x,y)}{H_{v_s\pm j,1}^{(1,1)(1,0)}(x,y)}\cdot\frac{1}{H_{v_s\pm m,\delta}^{(1,1)(1,0)}(x,y)}. \tag{1.77}$$

To simulate the CS phenomenon let us fix the angular distance $\Delta\phi$ between the receiver and the source. In this case, the function

$$\tilde{U}(\phi)=\Big[4k\Big/I^{(m)}\Big]U(g,g_0)|_{\substack{\rho=\rho_0=a\\ \phi_0=\phi-\Delta\phi}};\quad 0\leq\phi\leq 2\pi$$

may be considered as the 'diurnal dependence' of the received signal. To ensure a nonzero diurnal phase change, the curve $V_s(\phi,\phi_0)$ in the complex plane must enclose the origin of coordinates. Since the regular term V_{reg} in (1.75) does not depend on ϕ, while the irregular term V_{ireg} is proportional to $\exp(-i\phi)$ for $\phi_0=\phi-\Delta\phi$, the inequality

$$\left|V_{\text{reg}}(\nu_s)\right| < \left|V_{\text{ireg}}(\phi, \phi_0, \nu_s)\right| \tag{1.78}$$

is the necessary condition for the CS to occur in the model considered.

1.3.4 A Numerical Experiment

Let us calculate the Hertz potential $\tilde{U}(\phi)$ from (1.73)–(1.77) for the frequency $f = 10\text{kHz}$ and waveguide dimensions $a = 6370\,\text{km}$ and $b - a = 60\,\text{km}$. Since $x = ka = 1335.06 \gg 1$, we will use *Olver's uniform asymptotic representation* [44] to calculate the Hankel functions $H_\nu^{(j)}(x)$ along with their derivatives with respect to the argument and the index. The roots of the transcendental equation in (1.67) for $\alpha = 1$ can be found by the *Newton-Raphson method* [45]. For better understanding of the peculiarities that characterize the waveguide mode interconversion, one should analyze the location of several first roots of the equation (1.67) as a function of the complex parameter η_3.

Figure 1.4 illustrates typical trajectories of the first two roots ν_s, $s = 1, 2$ in the complex ν-plane for several fixed values of arg η_3 as $|\eta_3|$ increases. The real values ν_1^0 and ν_2^0 correspond to zero impedance. The sign '+' indicates the degenerate value ν_{12}^{deg} of these two roots corresponding to the impedance $\eta_{12}^{\text{deg}} \approx 0.1826 - i0.1127$

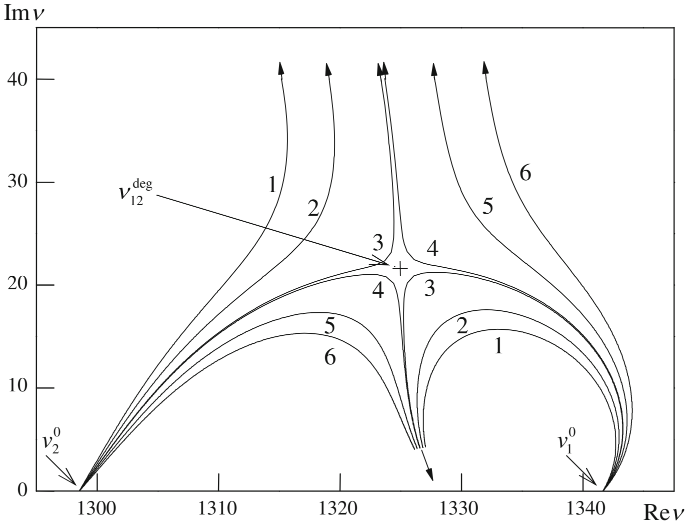


Fig. 1.4 The trajectories of the first two roots, ν_1 and ν_2, of (1.67) in the complex ν-plane for several fixed values of arg η_3 with increasing $|\eta_3|, 0 \le |\eta_3| \le 0.5$: arg $(i\eta_3)$ equals (1) 63.43°, (2) 60.94°, (3) 58.39°, (4) 58.21°, (5) 55.83°, (6) 53.37°

(see [21, 46]). It is easily seen that an abrupt change in the behavior of the eigenvalues of waveguide modes occurs when crossing the ray $\arg \eta_3 = \arg \eta_{12}^{\text{deg}}$.

Let us first consider the case of weakly irregular waveguides ($\delta \approx 1$). Then for $|\eta_2| \ll 1$ we have from (1.76)

$$V_{\text{ireg}}(\phi, \phi_0, v_s) = (\eta_1 - \eta_2) V_0(\phi, \phi_0, v_s), \tag{1.79}$$

$$V_0(\phi, \phi_0, v_s) = -\frac{16}{\pi^2} \frac{i\eta_3}{xy} \exp(-i\phi_0) \left[\frac{1}{H^{(1,1)(1,0)}_{v_s+1,1}(x, y)} + \frac{\exp(-i\Delta\phi)}{H^{(1,1)(1,0)}_{v_s-1,1}(x, y)} \right] + O(1 - \delta). \tag{1.80}$$

In Fig. 1.5, the level curves of the function $|V_0(\phi, \phi_0, v)|$ (for $\phi_0 = \phi - \Delta\phi$) are shown in the complex v-plane for the most interesting domain of variation of the eigenvalues of the first and the second modes for the impedance $i\eta_3 = H_v^{(1,1)}(x, y) \big/ H_v^{(1,0)}(x, y)$ satisfying (1.67).

The angular distance between the receiver and the transmitter is $\Delta\phi = 114.6°$, therefore, as it follows from numerical estimations, the contribution of the third and higher modes can be neglected. Minimal values of $|V_0(\phi, \phi_0, v)|$ are located in the vicinity of the points v_1^0 and v_2^0, while the maximum is close to v_{12}^{deg}. By comparing these results with the level curves of $|V_{\text{reg}}(v)|$ from Fig. 1.6, we can conclude that

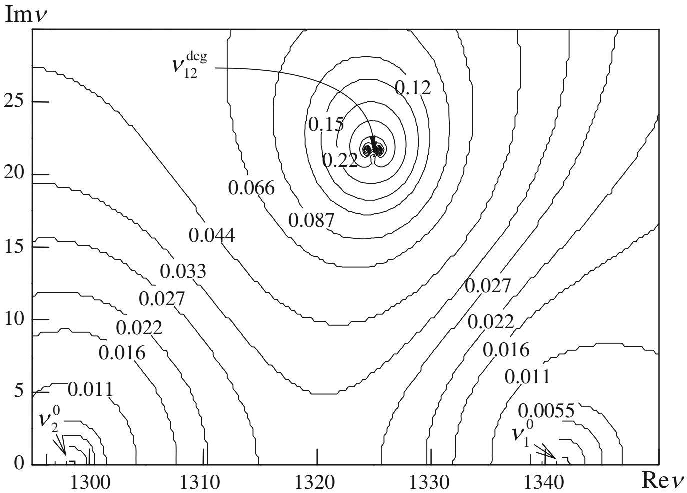


Fig. 1.5 The level curves of the function $|V_0(\phi_0 + \Delta\phi, \phi_0, v)|$ for $\Delta\phi = 114.6°$: $\max_v |V_0(\phi_0 + \Delta\phi, \phi_0, v)| = 3.0517$, $v_{\max} = 1325.5 + i21.75$, $|V_0(\phi_0 + \Delta\phi, \phi_0, v_1^0)| = 1.2593 \cdot 10^{-4}$, $|V_0(\phi_0 + \Delta\phi, \phi_0, v_2^0)| = 4.5397 \cdot 10^{-5}$

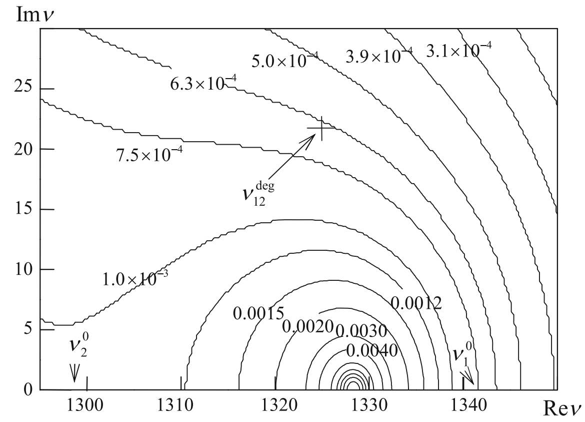


Fig. 1.6 The level curves of the function $|V_{reg}(v)|$, $\max_v|V_{reg}(v)| = 0.1561$, $v_{max} = 1328.25$

for small η_3, for which $r_{imp} \ll 1$ and $v_{1,2} \to v^0_{1,2}$, the inequality $|V_{reg}(v)| > |V_0(\phi, \phi_0, v)|$ holds, and hence, CS is impossible in view of (1.78), (1.79). Let η_3 be increasing and approaching η_{12}^{deg}. At the same time, the center r_{imp} of the impedance circle $\eta_{imp} \to \eta_{12}^{deg}$ increases too, while the eigenvalues of the first and second modes approach the point v_{12}^{deg}, in the vicinity of which the amplitude of the irregular part of $|V_0(\phi, \phi_0, v)|$ is maximal.

Then for not too small values of $|\eta_1 - \eta_2|$, the inequality (1.78) holds. In other words, it follows from the foregoing numerical estimates for the functions $|V_{reg}(v)|$ and $|V_0(\phi, \phi_0, v)|$ for weakly irregular waveguides that there exists a *threshold value* of the hodograph radius r_{imp}^{cs} of the impedance $\eta(\phi)$ (1.43) such that the CS phenomenon is impossible for $r_{imp} < r_{imp}^{cs}$, while for $r_{imp} > r_{imp}^{cs}$ it occurs at least for the hodographs located in the vicinity of η_{12}^{deg}. As the angular distance $\Delta\phi$ increases, the probability that the phenomenon in question will occur is growing too, all factors being equal. A similar situation holds when a degree of the waveguide irregularity grows, i.e. with increasing r_{imp}.

Let us now turn back to the general case of arbitrary index of a waveguide irregularity δ. Figure 1.7 present the simulated diurnal record of the received signal or, in other words, the ϕ-dependencies, $\Delta\phi \le \phi \le 2\pi + \Delta\phi$, of the normalized value

$$W(\phi) = \left[\lg\left(\max_{0\le\phi\le2\pi}|\tilde{U}(\phi)|\right)\tilde{U}(\phi)\right] \Big/ \left[\lg\left(|\tilde{U}(\phi)|\right)|\tilde{U}(\phi)|\right],$$

for the fixed angular distance $\Delta\phi = 114.6°$ between the source and the receiver. On the curves three following values of the received signal are marked: '0' corresponds to the initial moment of the record ($\phi = 0$), 'r' ('t') corresponds to the moment of time when the receiver (the transmitter) is passing through the waveguide cross section $\phi = \phi_{cr}$, where the surface impedance is closest to η_{12}^{deg} (see Fig. 1.8).

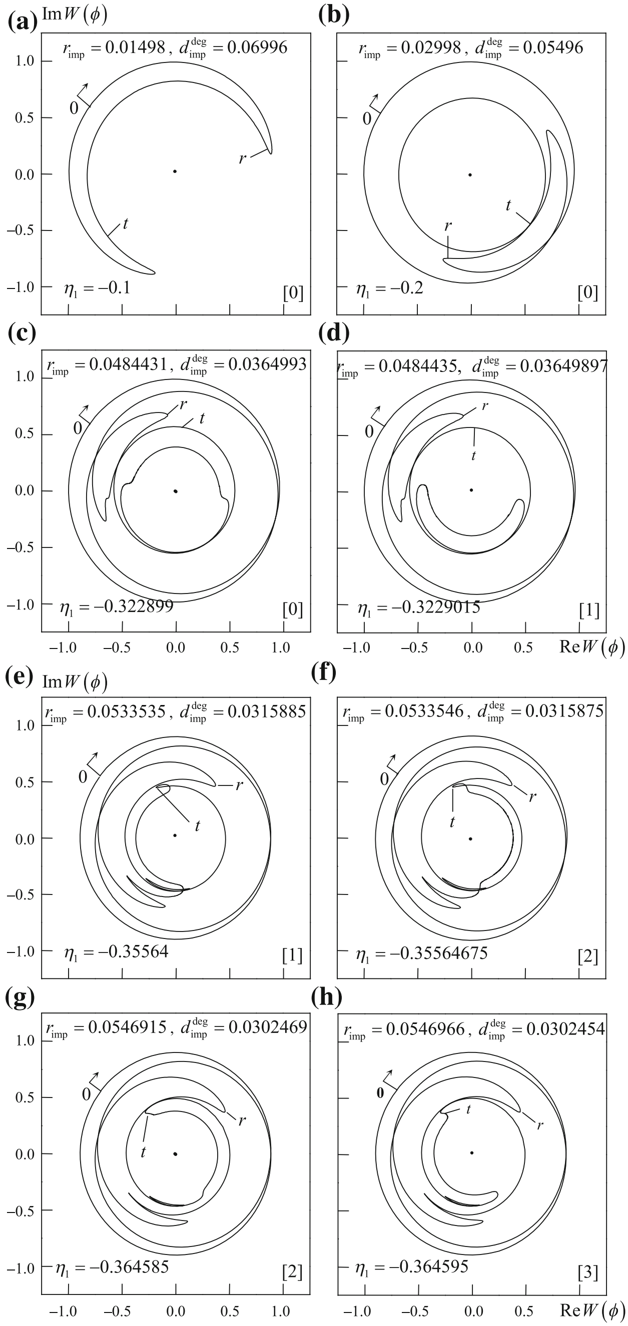


Fig. 1.7 The normalized diurnal records of the received signal$W(\phi)$: $\eta_3 = 0.1455 - i0.03638$, $\eta_2 = 0.0001$, $\Delta\phi = 114.6°$

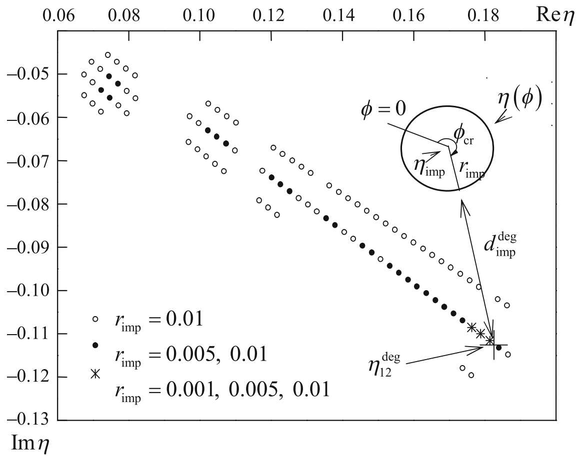


Fig. 1.8 The domains in the complex plane of the impedance η, where the CS occurs with the given radius r_{imp} and the angle $\Delta\phi = 114.6^{\circ}$

The number of lost phase cycles are shown in the figures in square brackets [m]; $d_{\text{imp}}^{\text{deg}}$ is the distance from the impedance circle to the point η_{12}^{deg}. The numerical experiment has shown that the CS phenomenon does not occur for the hodographs of the impedance $\eta(\phi)$ remote from the segment $l_{\text{cs}} = \left\{0 < |\eta| \le \eta_{12}^{\text{deg}},\ \ arg\eta = \arg\eta_{12}^{\text{deg}}\right\}$ (Fig. 1.7a, b, c). As r_{imp} increases, $W(\phi)$ behavior becomes more complex; when the circle $\eta(\phi)$ intersects l_{cs}, CS occurs (Fig. 1.7d) for $\eta_3 = 0.1455 - i0.03638$ and $r_{\text{imp}} \approx 0.0484435$. At the same time, the signal amplitude decreases within a small variation interval of ϕ (of the order of 0.01°).

As r_{imp} grows, at $r_{\text{imp}} \approx 0.0533546$ (Fig. 1.7f), the CS phenomenon for two cycles, at $r_{\text{imp}} \approx 0.0546966$ (Fig. 1.7h) for three cycles, and so forth is observable. A similar situation holds for the circle $\eta(\phi)$, whose center is located in the vicinity of l_{cs} (Fig. 1.8); however, the CS occurs at lesser values of r_{imp}. Each CS phenomenon is accompanied by a sharp decrease in signal amplitude, which is typical for a CS in a natural waveguide [22, 32]. In the context of the given model, the role played by the segment l_{cs} in the initiation of the CS phenomenon can be explained as follows: only for the impedances in the vicinity of this segment, the eigenvalues ν_1 and ν_2 have closely spaced imaginary parts, and consequently, the amplitudes of the first and the second modes are nearly equal. In addition, when η_3 is moving along l_{cs} towards the point η_{12}^{deg}, the real parts of ν_1 and ν_2 come close together (curves 3 or 4 in Fig. 1.4), and consequently, the phase velocities of these modes approach each other.

Of some interest is a localization of the domains in the complex η-plane, for which the CS phenomenon takes place at the given radius r_{imp} and angle $\Delta\phi$. In Fig. 1.8 dots indicate center positions of the hodographs of radiuses 0.001, 0.005 and 0.01, for which CS occurs at $\Delta\phi = 114.6^{\circ}$. It is seen that with increasing r_{imp} the CS phenomenon develops initially in the immediate vicinity of the point η_{12}^{deg}, and then, as r_{imp} grows, this area is extending occupying a constantly increasing part of the segment l_{cs}. For the hodographs with fixed centers, the CS phenomenon having developed at some r_{imp}, persists for larger values of the radius.

In conclusion note the following. We have proposed a model of the ring waveguide of a fixed cross section whose irregularity is caused only by the behavior of the surface impedance of its wall. Hence we have excluded from consideration the diffraction effect of wave transformation on a spatial inhomogeneity of the wall; only the mode degeneracy effect being inherent in the waveguides with finite absorption is analyzed. We have obtained the analytical solution of the corresponding boundary value problem for a class of circular hodographs of surface impedance. It is the first problem of the excitation of a finite irregular waveguide with continuously varying properties, for which the analytical solution is found.

The results of the numerical experiment for widely separated $(1 \leq \Delta\phi \leq \pi)$ transmitter and receiver have shown that the CS phenomenon here is directly related to the degeneracy of the first and the second modes. This phenomenon is threshold-like and it occurs in waveguides with sufficiently high irregularity of the walls whose impedance is distributed in the neighborhood of the degenerate value η_{12}^{deg}. Once the phenomenon is developed, it persists as the radius of the impedance hodograph increases. At the same time, the domain of the complex plane of the impedance, where the CS takes place, is extending occupying a constantly increasing part of the segment joining the origin of coordinates and the point η_{12}^{deg}.

It has been demonstrated with a waveguide of fixed cross section that the CS in irregular lossy waveguides may be caused by the interconversion of two dominant waveguide modes in the neighborhood of their degeneracy rather than by the diffraction effect of rescattering of the principal mode into the higher modes on a spatial inhomogeneity of the waveguide wall, as it is customary to assume.

1.4 Pulsed Radiation from a Line Electric Current Near a Planar Interface

The classical problem of transient electromagnetic fields generated by pulsed currents located near a planar boundary between layered media are the subject of constant theoretical research starting with the B. van der Pol paper [47]. The approaches based on the *Cagniard method* [48, 49] is the most efficient tool in this study. A.T. de Hoop [50] has suggested a modification of Cagniard's method with the help of which exact solutions have been obtained for a number of problems of a dipole or a line source near an interface [51–54]. Various modifications of Cagniard's technique have found wide application in the study of transient acoustic and seismic wave propagation. Following paper [50], the modifications of de Hoop's technique [55, 56] as well as the alternative approaches free from some drawbacks to this method [57, 58] have been suggested.

In this section, following the paper [59], we use the approach alternative to Cagniard's technique to study the transient field generated by line sources located in a flat-layered media. The suggested approach is applied to the already solved problem, namely, the problem of finding the electromagnetic field generated by a pulsed line source located near a planar interface between two nonabsorbing and nondispersive media. The most complete solution to this problem have been obtained and discussed in considerable detail by A.T. de Hoop in [54]. In this paper, were applied the one-sided Laplace transform with respect to time and two-sided Laplace transform with respect to a horizontal spatial variable. The electromagnetic field is represented in the form of a double integral. This integral can be efficiently calculated by the *Cagniard-de Hoop method* (CHM). The essence of the method is as follows. The original path of integration for one of two integrals forming the double integral is deformed into a so-called *modified Cagniard contour*. It is chosen such that upon the corresponding change of the integration variable in the integral along the modified contour, the original double integral turns into a composition of the direct and inverse Laplace transform for the known function. The central problem with this method is to find, generally speaking, numerically, the modified Cagniard contour. It should be noted that the shape of this contour changes as the observation point changes.

The key point of the approach proposed here consists in the following. To calculate the double integral efficiently, we suggest deforming its domain of integration (the real plane) in the C × C-space of two complex variables rather than to deform one contour in the complex C-plane, as has been done in CHM. It is shown that in this case the integral reduces to a sum of residues. The use of powerful apparatus of the residue theory instead of somewhat artificial way used in CHM is a reason to hope that this new approach can be efficient in the situations where the CHM is failed, for example, for anisotropic media. Our method can be extended to multilayered media and arbitrary dipole sources.

1.4.1 Problem Formulation

The field generated by a pulsed line electric current

$$\vec{J}^{(e)} = I^{(e)}\delta(x)\delta(z - z_0)\delta(t)\vec{y}; \quad z_0 > 0, \tag{1.81}$$

which is located near a planar interface (Fig. 1.9), is to be found. The source excites the E-polarized field

$$E_y \neq 0, \quad E_x = E_z = H_y = 0, \quad \frac{\partial H_x}{\partial t} = \frac{1}{\mu\mu_0}\frac{\partial E_y}{\partial z}, \quad \frac{\partial H_z}{\partial t} = -\frac{1}{\mu\mu_0}\frac{\partial E_y}{\partial x}. \tag{1.82}$$

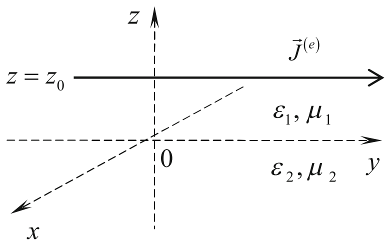


Fig. 1.9 A pulsed line source near the interface between two semi-infinite media

The function E_y is the solution of the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \varepsilon\varepsilon_0\mu\mu_0\frac{\partial^2}{\partial t^2}\right)E_y = \mu_1\mu_0\frac{\partial J_y^{(e)}}{\partial t} \tag{1.83}$$

that satisfies the conditions of continuity of E_y- and H_x-components on the interface $z = 0$ and the causality principle. Here, $\varepsilon = \varepsilon_1$, $\mu = \mu_1$ for $z > 0$ and $\varepsilon = \varepsilon_2$, $\mu = \mu_2$ for $z<0$.

The Fourier transform in time

$$F(x, z, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_y(x, z, t)\mathrm{e}^{i\omega t}dt, \quad E_y(x, z, t) = \int_{-\infty}^{\infty} F(x, z, \omega)\mathrm{e}^{-i\omega t}d\omega \tag{1.84}$$

applied to the boundary value problem in (1.83) results in the following problem

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2\varepsilon_1\varepsilon_0\mu_1\mu_0\right)F^1 = -I_0\delta(x)\delta(z - z_0); \quad z > 0 \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2\varepsilon_2\varepsilon_0\mu_2\mu_0\right)F^2 = 0; \quad z<0 \end{cases} \tag{1.85}$$

with the boundary conditions on $z = 0$

$$F^1 = F^2, \quad \mu_2\frac{\partial F^1}{\partial z} = \mu_1\frac{\partial F^2}{\partial z}, \tag{1.86}$$

where $I_0 = i\omega\mu_1\mu_0 I^{(e)}/2\pi$. The solution of the equations in (1.85) is conveniently represented in the form [41]

$$F^1 = I_0\left(F^0 + F_s^1\right) \text{ for } z > 0 \quad \text{and} \quad F^2 = I_0 F_s^2 \text{ for } z<0, \tag{1.87}$$

where

$$F^0 = \frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{\exp\left[i\xi x + i\sqrt{k_1^2-\xi^2}|z-z_0|\right]}{\sqrt{k_1^2-\xi^2}}d\xi = \frac{1}{4}H_0^{(1)}(k_1R_-), \tag{1.88}$$

$$F_s^1 = \frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{\exp\left[i\xi x + i\sqrt{k_1^2-\xi^2}(z+z_0)\right]}{\sqrt{k_1^2-\xi^2}}\Gamma_1(\xi,\omega)d\xi, \tag{1.89}$$

$$F_s^2 = \frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{\exp\left[i\xi x - i\sqrt{k_2^2-\xi^2}z + i\sqrt{k_1^2-\xi^2}z_0\right]}{\sqrt{k_1^2-\xi^2}}\Gamma_2(\xi,\omega)d\xi, \tag{1.90}$$

$\Gamma_j(\xi,\omega)$ are the unknown functions, $\mathrm{Im}\sqrt{k_j^2-\xi^2}\geq 0$, $k_j^2 = \omega^2 n_j^2$, $n_j^2 = \varepsilon_j\varepsilon_0 \times\mu_j\mu_0$, $j=1,2$, $R_-^2 = x^2+(z-z_0)^2$. From the boundary conditions in (1.86), we have:

$$1+\Gamma_1 = \Gamma_2, \quad -\mu_2\sqrt{k_1^2-\xi^2}+\mu_1\sqrt{k_1^2-\xi^2}\Gamma_1 = -\mu_1\sqrt{k_2^2-\xi^2}\Gamma_2,$$

or

$$\Gamma_1(\xi,\omega) = \frac{\mu_2\sqrt{k_1^2-\xi^2}-\mu_1\sqrt{k_2^2-\xi^2}}{\mu_2\sqrt{k_1^2-\xi^2}+\mu_1\sqrt{k_2^2-\xi^2}}, \tag{1.91}$$

$$\Gamma_2(\xi,\omega) = \frac{2\mu_2\sqrt{k_1^2-\xi^2}}{\mu_2\sqrt{k_1^2-\xi^2}+\mu_1\sqrt{k_2^2-\xi^2}}. \tag{1.92}$$

Thus we obtain the required field in the form of the following double integrals taken over the plane P of real variables ω and ξ:

$$E_y^j(x,z,t) = E_0\frac{\partial}{\partial t}G^j(x,z,t); \quad j=0,1,2, \tag{1.93}$$

$$G^0(x,z,t) = \frac{1}{4\pi i}\iint_{\mathrm{P}}\exp\left[i\xi x + i\sqrt{\omega^2n_1^2-\xi^2}|z-z_0| - i\omega t\right]\frac{d\omega d\xi}{\sqrt{\omega^2n_1^2-\xi^2}}; \quad z>0, \tag{1.94}$$

$$G^1(x,z,t) = \frac{1}{4\pi i} \iint_P \exp\left[i\xi x + i\sqrt{\omega^2 n_1^2 - \xi^2}(z+z_0) - i\omega t\right] \frac{\Gamma_1(\xi,\omega) d\omega d\xi}{\sqrt{\omega^2 n_1^2 - \xi^2}};$$
$$z > 0, \tag{1.95}$$

$$G^2(x,z,t) = \frac{1}{4\pi i} \iint_P \exp\left[i\xi x - i\sqrt{\omega^2 n_2^2 - \xi^2} z + i\sqrt{\omega^2 n_1^2 - \xi^2} z_0 - i\omega t\right]$$
$$\times \frac{\Gamma_2(\xi,\omega) d\omega d\xi}{\sqrt{\omega^2 n_1^2 - \xi^2}}; \quad z<0, \tag{1.96}$$

where $E_0 = I^{(e)}\mu_1\mu_0/2\pi$ and $\mathrm{Im}\sqrt{\omega^2 n_j^2 - \xi^2} \geq 0$, j = 1, 2.

1.4.2 Reduction to Single Integrals

In formulas (1.94)–(1.96), the integrands allow analytic continuation from the real plane $\mathrm{P} = \{\omega, \xi : \ \omega'' = \xi'' = 0\}$ into the C × C-space of two complex variables $\omega = \omega' + i\omega''$ and $\xi = \xi' + i\xi''$. As the previous analysis has shown, there is no need to operate with the whole of real four-dimensional space C × C. To calculate the integrals in (1.94)–(1.96), it is sufficient to restrict our consideration to a three-dimensional space $\mathrm{R}^3 = \{\omega, \xi : \xi'' = 0\} \subset \mathrm{C} \times \mathrm{C}$ containing P. In R^3, one should choose the single-valued branches of two square roots in the integrands.

Consider a function $\kappa(\omega, \xi) = \sqrt{\omega^2 n^2 - \xi^2}$ in R^3 assuming that the refractive index $n = n' + in''$ $(n', n'' > 0)$ is complex-valued.

The surface

$$\mathrm{Re}\kappa^2 = \left(n'^2 - n''^2\right)\omega'^2 - \left(n'^2 - n''^2\right)\omega''^2 - 4n'n''\omega'\omega'' - \xi'^2 = 0 \tag{1.97}$$

has the following invariants [60]: $I = -1$, $J = -|n|^4$, $D = -J$, $A = 0$, $A' = D$. Therefore it represents a two-pole elliptic cone, which is symmetrical with respect to the plane $\xi' = 0$, with its vertex at the origin of coordinates. Let us locate the axis of the cone. The lines of intersection of the cone with the symmetry plane $\xi' = 0$ are two mutually orthogonal straight lines $(n' \mp n'')\omega'' \pm (n' \pm n'')\omega' = 0$ with the bisecting lines $n'\omega'' + n''\omega' = 0$ and $n'\omega' - n''\omega'' = 0$. Consequently, the cone axis is determined by the equations $n'\omega'' + n''\omega' = 0$ and $\xi' = 0$.

The surface

$$\mathrm{Im}\kappa^2 = n'n''\left(\omega'^2 - \omega''^2\right) + \omega'\omega''\left(n'^2 - n''^2\right) = 0 \tag{1.98}$$

has the following invariants: $I = 0$, $D = -|n|^4 \big/ 4$, $A = 0$. Therefore it represents two mutually orthogonal planes intersecting along the ξ'-axis and determined by the equations $n'\omega'' + n''\omega' = 0$ and $n''\omega'' - n'\omega' = 0$. The first plane contains the axis of the cone (1.97) being its another symmetry plane. From (1.97) and (1.98), we derive the following equations for the branch lines of $\kappa(\omega, \xi)$:

$$n'\omega' - n''\omega'' \pm \xi' = 0, \qquad n'\omega'' + n''\omega' = 0.$$

In Fig. 1.10, the distribution of signs for $\mathrm{Re}\kappa^2$ and $\mathrm{Im}\kappa^2$in R^3 is shown. In (1.94)–(1.96), a single-valued branch of the function $\kappa(\omega, \xi)$, for which $\mathrm{Im}\kappa(\omega, \xi) \geq 0$, is determined on the real plane $\mathrm{P} = \{\xi', \omega'\}$. The above mentioned inequality is hold everywhere in R^3, if the following condition is satisfied: $0 \leq \arg \kappa^2 < 2\pi$. In other words, the cut S in R^3 that separates this branch should be determined by the conditions $\mathrm{Re}\kappa^2 \geq 0$, $\mathrm{Im}\kappa^2 = 0$. As is seen from Fig. 1.10, this takes place for a double sector formed by the intersection of the inner part of the cone (1.97) with its symmetry plane $n'\omega'' + n''\omega' = 0$. In R^3, with the cut of this kind (Fig. 1.11), we have $\mathrm{Im}\kappa(\omega, \xi) \geq 0$.

A similar approach to choose a branch of the square root is given in [61] for the case of a single variable. When passing to the lossless medium $\alpha = 0$, the cut surface S is shifted into the plane $\omega'' = 0$ representing the double sector, which contains the ω'-axis and is bounded by the straight branch lines $n'\omega' \pm \xi' = 0$.

Thus we have shown that for a lossless media the cut surface ensuring a choice of the branch, for which we have $\mathrm{Im}\sqrt{\omega^2 n_j^2 - \xi^2} \geq 0$ in R^3, is a double sector S_j (Fig. 1.11), which lies in the plane $\omega'' = 0$, contains the ω'-axis, and is bounded by the branch lines $n_j\omega' \pm \xi' = 0, j = 1, 2$. The root is positive on the upper side of the right-hand sector $\{\omega' > 0,\ \omega'' = 0 + 0\}$ and on the bottom side of the left-hand

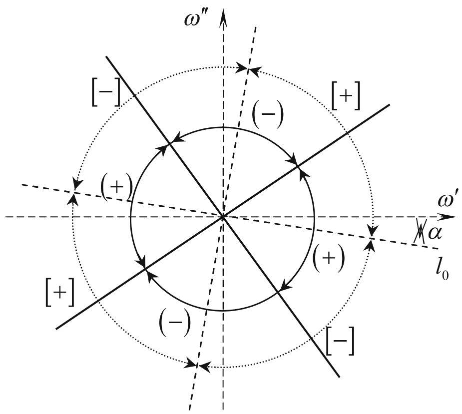


Fig. 1.10 The sign distribution for Re κ^2 and Im κ^2 in the plane $\xi' = 0$. *Straight lines* indicate the lines of intersection with the plane $\xi' = 0$: the *bold lines*—for the cone Re κ^2 = 0, the *dashed lines*—for the planes Im κ^2 = 0. Symbols (±) specify the sign of Re κ^2, while [±] specify the sign of Im κ^2; $\sin\alpha = -n''/|n|$, l_0 is the axis of the cone (1.97)

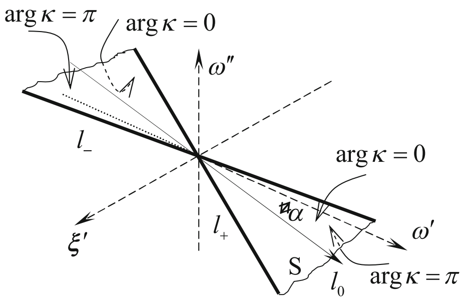


Fig. 1.11 The location of the branch lines $l_{\pm}$ and the cut surface S ensuring the choice of the branch for which $\operatorname{Im} k(\omega, \xi) \geq 0$ in R^3-space; l_0 is the axis of the cone (1.97)

sector $\{\omega' < 0,\ \omega'' = 0 - 0\}$, while it is negative on the other sides. Since the integrands in (1.94)–(1.96) are uniquely defined in the R^3-space with the specified cuts, one can apply the Cauchy-Poincare theorem [18] to deform the surface of integration P in $R^3 \backslash (S_1 \cup S_2)$.

In accordance with the causality principle, the cut surfaces S_1 and S_2 have to adjoin the real plane P from the bottom $(\omega'' = 0 - 0)$. Then, the integrands have no singularities in the half-space $\omega'' > 0$, and we have $E_y(x, z, t) \equiv 0$ for all $t < 0$, according to the mentioned theorem.

For the positive values of t, the P-plane can be deformed to a half-space $\omega'' < 0$. Then we have for E_y^0 an integral over the surface P_{c1}, while for E_y^1, E_y^2 we have integrals over the surface $P_c = P_{c1} \cup P_{c2}$. Here P_{cj} stands for the closed surface enveloping the cut S_j.

Using the function $G^1(x, z, t)$ as an example, let us demonstrate how the integrals describing the secondary field in (1.95), (1.96) can be simplified. Denoting the integrand in (1.95) by $f(\omega, \xi')$, consider the following integral over the surface P_c:

$$\iint_{P_c} f(\omega, \xi')ds = I_1 + I_2, \tag{1.99}$$

where $I_j = \iint_{P_{cj}} f(\omega, \xi')ds$. Let P_{cj}^+ and P_{cj}^- be the right-hand $(\omega' > 0)$ and the left-hand $(\omega' < 0)$ cavities of the surface P_{cj}; $L_{\omega' j}$ is the closed contour generated by the intersection of the surface P_{cj} with the coordinate plane $\omega' = const$. Then we have

$$\begin{aligned} I_1 &= \sum_{\pm} \iint_{P_{c1}^{\pm}} f(\omega, \xi')ds = \int_0^{\infty} d\omega' \int_{L_{\omega' 1}} dl f(\omega, \xi') + \int_{-\infty}^{0} d\omega' \int_{L_{\omega' 1}} dl f(\omega, \xi') \\ &= \int_0^{\infty} d\omega' \left[\int_{L_{\omega' 1}} dl f(\omega, \xi') - \int_{L_{\omega' 1}} dl f(-\omega, -\xi') \right]. \end{aligned} \tag{1.100}$$

In the second integral, we made the following change of variables $\omega \to -\omega$, $\xi' \to -\xi'$. Taking into account the evenness of the chosen branches of the square roots entering the function $f(\omega, \xi')$ with respect to this change of variables and performing another change of variables $\xi' = \omega\eta$, we arrive at the following expression for the integral in (1.95):

$$
\begin{aligned}
I_1 = \int_0^\infty d\omega \int_{\mathrm{L}_1} d\eta \Big(& \exp\Big\{ i\omega \Big[\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) - t \Big] \Big\} \\
& - \exp\Big\{ i\omega \Big[-\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) + t \Big] \Big\} \Big) \tilde{\Gamma}_1(\eta) \Big/ \sqrt{n_1^2 - \eta^2},
\end{aligned}
\tag{1.101}
$$

where

$$
\tilde{\Gamma}_1(\eta) = \frac{\mu_2 \sqrt{n_1^2 - \eta^2} - \mu_1 \sqrt{n_2^2 - \eta^2}}{\mu_2 \sqrt{n_1^2 - \eta^2} + \mu_1 \sqrt{n_2^2 - \eta^2}}. \tag{1.102}
$$

The contour L_1 envelopes the segment $(-n_1, n_1)$ in the complex η-plane. Let us introduce the accessory parameter $\delta > 0$ for the sake of convergence acceleration, then rewrite (1.101) in the form

$$
\begin{aligned}
I_1 &= \lim_{\delta \to 0} \int_0^\infty d\omega \int_{\mathrm{L}_1} d\eta \Big(\exp\Big\{ i\omega \Big[i\delta + \eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) - t \Big] \Big\} \\
&\qquad - \exp\Big\{ i\omega \Big[i\delta - \eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) + t \Big] \Big\} \Big) \tilde{\Gamma}_1(\eta) \Big/ \sqrt{n_1^2 - \eta^2} \\
&= i \lim_{\delta \to 0} \int_{\mathrm{L}_1} \Bigg[\frac{1}{\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) - t + i\delta} \\
&\qquad - \frac{1}{-\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) + t + i\delta} \Bigg] \frac{\tilde{\Gamma}_1(\eta) d\eta}{\sqrt{n_1^2 - \eta^2}}.
\end{aligned}
\tag{1.103}
$$

For the second integral in (1.99), I_2, we obtain a representation similar to (1.103) with L_1 replaced by L_2, where L_2 is the contour enveloping the segment $(-n_2, n_2)$. Thus, for the function given by (1.95), which determines the secondary field in the first medium (see (1.93)), we arrive at the following expression

$$
\begin{aligned}
G^1(x, z, t) = \frac{1}{4\pi} \lim_{\delta \to 0} \int_{\mathrm{L}} \Bigg[& \frac{1}{\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) - t_-} \\
& - \frac{1}{-\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) + t_+} \Bigg] \frac{\tilde{\Gamma}_1(\eta) d\eta}{\sqrt{n_1^2 - \eta^2}},
\end{aligned}
\tag{1.104}
$$

where $t_{\pm} = t \pm i\delta$, L is the contour enveloping the segment $(-n_{\max}, n_{\max})$, $n_{\max} = \max(n_1, n_2)$. The root branches are determined by the inequalities $-\pi < \arg\sqrt{n_j^2 - \eta^2} < \pi$ with zero argument on the bottom side of the cut along the segment $\left(-n_j, n_j\right)$.

Similarly, for the function G^2, describing the field in the second medium, we obtain from (1.96):

$$G^2(x,z,t) = \frac{1}{4\pi}\lim_{\delta\to 0}\int\limits_L \left[\frac{1}{\eta x + \sqrt{n_1^2 - \eta^2}z_0 - \sqrt{n_2^2 - \eta^2}z - t_-} - \frac{1}{-\eta x + \sqrt{n_1^2 - \eta^2}z_0 - \sqrt{n_2^2 - \eta^2}z + t_+}\right]\frac{\tilde{\Gamma}_2(\eta)d\eta}{\sqrt{n_1^2 - \eta^2}}, \tag{1.105}$$

where $\tilde{\Gamma}_2(\eta) = 1 + \tilde{\Gamma}_1(\eta)$. The integrands in (1.104) and (1.105) are analytic in the plane of complex variable η with the specified cut and decreasing at infinity as η^{-2}. Therefore, these integrals can be reduced to the residues determined by zeros of the denominators in the square brackets:

$$\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) - t_- = 0, \quad -\eta x + \sqrt{n_1^2 - \eta^2}(z + z_0) + t_+ = 0 \quad \text{for (1.104)}, \tag{1.106}$$

$$\begin{gathered} \eta x + \sqrt{n_1^2 - \eta^2}z_0 - \sqrt{n_2^2 - \eta^2}z - t_- = 0, \\ -\eta x + \sqrt{n_1^2 - \eta^2}z_0 - \sqrt{n_2^2 - \eta^2}z + t_+ = 0 \quad \text{for} \quad (1.105). \end{gathered} \tag{1.107}$$

1.4.3 The Field in the First Medium

The roots of (1.106) are readily determined and can be written as

$$\begin{gathered} \eta_1^- = \left(xt_- - (z+z_0)\sqrt{n_1^2R_+^2 - t_-^2}\right)R_+^{-2} \quad \text{and} \\ \eta_1^+ = \left(xt_+ + (z+z_0)\sqrt{n_1^2R_+^2 - t_+^2}\right)R_+^{-2}, \end{gathered} \tag{1.108}$$

where $R_+^2 = x^2 + (z+z_0)^2$. For the square root $\sqrt{n_1^2R_+^2 - t^2}$, we determined the same branch in the complex plane of variable t as for $\sqrt{n_j^2 - \eta^2}$ in the η-plane. By calculating the corresponding residues, we get from (1.104):

$$G^1(x,z,t) = \frac{i}{2}\lim_{\delta\to 0}\left\{ \frac{\tilde{\Gamma}_1(\eta)}{x\sqrt{n_1^2-\eta^2}-(z+z_0)\eta}\Bigg|_{\eta=\eta_1^-} + \frac{\tilde{\Gamma}_1(\eta)}{x\sqrt{n_1^2-\eta^2}+(z+z_0)\eta}\Bigg|_{\eta=\eta_1^+} \right\}$$
$$= \frac{i}{2}\lim_{\delta\to 0}\left\{ \frac{\tilde{\Gamma}_1(\eta_1^-)}{\sqrt{n_1^2R_+^2-t_-^2}} + \frac{\tilde{\Gamma}_1(\eta_1^+)}{\sqrt{n_1^2R_+^2-t_+^2}} \right\}. \tag{1.109}$$

Here we used the equality

$$\sqrt{n_1^2-\left(\eta_1^{\mp}\right)^2} = \left[x\sqrt{n_1^2R_+^2-(t_{\mp})^2} \pm (z+z_0)t_{\mp}\right]R_+^{-2}. \tag{1.110}$$

It is easy to verify that the following relationships are hold for the chosen branches of the square roots:

$$\begin{aligned} \sqrt{n_1^2R_+^2-(t^*)^2} &= \exp(i\pi)\left(\sqrt{n_1^2R_+^2-t^2}\right)^*, \\ \sqrt{n_j^2-(\eta^*)^2} &= \exp(i\pi)\left(\sqrt{n_j^2-\eta^2}\right)^* \end{aligned} \tag{1.111}$$

(the asterisk stands for a complex conjugation). Therefore,

$$\begin{aligned} \eta_1^- &= \left[xt_+^* - (z+z_0)\exp(i\pi)\left(\sqrt{n_1^2R_+^2-t_+^2}\right)^*\right]R_+^{-2} \\ &= \left[xt_+ + (z+z_0)\sqrt{n_1^2R_+^2-t_+^2}\right]^*R_+^{-2} = \left(\eta_1^+\right)^*, \\ \sqrt{n_j^2-\left(\eta_1^-\right)^2} &= \exp(i\pi)\left(\sqrt{n_j^2-\left[\left(\eta_1^-\right)^*\right]^2}\right)^* = -\left[\sqrt{n_j^2-\left(\eta_1^+\right)^2}\right]^*, \\ \tilde{\Gamma}_1\left(\eta_1^-\right) &= \left[\tilde{\Gamma}_1\left(\eta_1^+\right)\right]^*. \end{aligned} \tag{1.112}$$

The wave reflected from the interface comes at some point in the first medium at time $t_{\text{ref}} = n_1R_+$. For the time interval $0 < t < t_{\text{ref}}$, in view of (1.112), we obtain

$$G^1(x,z,t) = \frac{i}{2}\lim_{\delta\to 0}\left[\frac{\tilde{\Gamma}_1^*(\eta_1^+)}{\sqrt{n_1^2R_+^2-t_-^2}} + \frac{\tilde{\Gamma}_1(\eta_1^+)}{\sqrt{n_1^2R_+^2-t_+^2}} \right] = \frac{i}{2}\lim_{\delta\to 0}\frac{\left[\tilde{\Gamma}_1^*(\eta_1^+) - \tilde{\Gamma}_1(\eta_1^+)\right]}{\sqrt{n_1^2R_+^2-t^2}}$$
$$= \mathrm{Im}\tilde{\Gamma}_1\left(\eta_1^<\right)\Big/\sqrt{n_1^2R_+^2-t^2}, \tag{1.113}$$

where $\eta_1^< = \eta_1^+\big|_{\delta=0} = \left[xt-(z+z_0)\sqrt{n_1^2R_+^2-t^2}\right]R_+^{-2}$.

For the time interval $t_{\mathrm{ref}} < t$, we have

$$G^1(x,z,t) = \frac{i}{2}\lim_{\delta\to 0}\left[\frac{\tilde{\Gamma}_1^*(\eta_1^+)}{i\sqrt{t^2 - n_1^2R_+^2}} + \frac{\tilde{\Gamma}_1(\eta_1^+)}{i\sqrt{t^2 - n_1^2R_+^2}}\right] = \frac{\mathrm{Re}\tilde{\Gamma}_1(\eta_1^>)}{\sqrt{t^2 - n_1^2R_+^2}}, \quad (1.114)$$

where $\eta_1^> = \eta_1^+|_{\delta=0} = \left[xt + i(z+z_0)\sqrt{t^2 - n_1^2R_+^2}\right]R_+^{-2}$.

The behavior of the secondary field in the first medium for the times $0 < t < t_{\mathrm{ref}}$ essentially depends on the relation between the refractive indices for the first (n_1) and second (n_2) media.

For an arbitrary point in the first medium, both of the roots entering $\tilde{\Gamma}_1(\eta_1^<)$ are real (see (1.110)) if $n_1 < n_2$. Consequently, we have $\mathrm{Im}\tilde{\Gamma}_1(\eta_1^<) = 0$, and the secondary field given by (1.113) is zero ($G^1(x,z,t) \equiv 0$) up to the moment of arrival of the reflected wave.

In the case that $n_1 > n_2$, a more detailed analysis of the function $n_2^2 - (\eta_1^<)^2$ is required. Let us use the following notation: $x/R_+ = \sin\theta$, $(z+z_0)/R_+ = \cos\theta$, $n_2/n_1 = \sin\theta_{\mathrm{tot}}$, where θ_{tot} stands for the angle of total internal reflection [41, 62]. Let us also introduce the parameter $\tau = \arccos(t/t_{\mathrm{ref}})$ such that $\cos\tau = t/t_{\mathrm{ref}}$ and the principal branch $0 < \tau < \pi$ of this function is chosen. Then we arrive at

$$\begin{aligned} n_2^2 - (\eta_1^<)^2 &= n_1^2\left\{\left[\frac{x}{R_+}\sqrt{1 - \left(\frac{t}{n_1R_+}\right)^2} + \frac{(z+z_0)}{R_+}\frac{t}{n_1R_+}\right]^2 - \left(1 - \frac{n_2^2}{n_1^2}\right)\right\} \\ &= n_1^2\left[\cos^2(\tau-\theta) - \cos^2\theta_{\mathrm{tot}}\right] = n_1^2\sin(\theta_{\mathrm{tot}} - \theta + \tau)\sin(\theta_{\mathrm{tot}} + \theta - \tau). \end{aligned} \quad (1.115)$$

Since we have $0 < \theta, \theta_{\mathrm{tot}}, \tau < \pi/2$ for the space-time domain considered, then the arguments of the sine functions in (1.115) find themselves within the interval $(-\pi/2, \pi)$. Therefore, the function given by (1.115) has two roots, $\tau_1 = \theta - \theta_{\mathrm{tot}}$ and $\tau_2 = \theta + \theta_{\mathrm{tot}}$, corresponding to the time points $t_1 = t_{\mathrm{ref}}\cos(\theta - \theta_{\mathrm{tot}})$ and $t_2 = t_{\mathrm{ref}}\cos(\theta + \theta_{\mathrm{tot}})$. There is no difficulty to show (the trajectory $z_0x_1x_2A$ in Fig. 1.12) that

$$t_1 = n_1z_0/\cos\theta_{\mathrm{tot}} + n_2[x - (z+z_0)\mathrm{tg}\theta_{\mathrm{tot}}] + n_1z/\cos\theta_{\mathrm{tot}} = t_{\mathrm{dif}},$$

where t_{dif} is the time of arrival of the so-called side wave [41] (or diffraction wave [62]) at the observation point located in the first medium in the region $\theta > \theta_{\mathrm{tot}}$. For $\theta < \theta_{\mathrm{tot}}$, the variable τ_1 goes to the unphysical sheet of the function $\arccos(t/t_{\mathrm{ref}})$, and the side wave does not occur in this region. By virtue of the causality principle, for the times $t < t_{\mathrm{dif}}$, there is no secondary field and so the other zero (τ_2) is of no importance ($t_2 < t_1$).

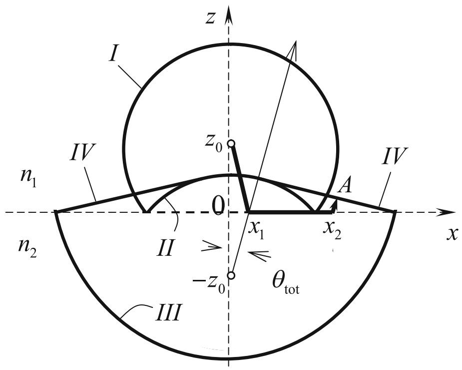


Fig. 1.12 The wave fronts of the field generated by a pulsed line current located near a planar interface for $n_1 > n_2$: the primary (I), reflected (II), transmitted (III), and side (IV) waves; $z_0\, x_1\, x_2\, A$ is the trajectory determining the time of arrival of the side wave at the point A, θ_{tot} is the angle of total internal reflection

Let us find the value of $\text{sign}\left[n_2^2 - \left(\eta_1^<\right)^2\right]$ for $t_{\text{dif}} < t < t_{\text{ref}}$ in the region $\theta > \theta_{\text{tot}}$. Here the following relationships for the arguments of the sine functions in (1.113) are valid:

$$-\pi/2 < \theta_{\text{tot}} - \theta < \theta_{\text{tot}} - \theta + \tau < \theta_{\text{tot}} - \theta + \tau_1 = 0,$$
$$0 < 2\theta_{\text{tot}} = \theta_{\text{tot}} + \theta - \tau_1 < \theta_{\text{tot}} + \theta - \tau < \theta_{tot} + \theta < \pi,$$

which means that $n_2^2 - \left(\eta_1^<\right)^2 < 0$. Considering that $\text{Im}\sqrt{n_1^2 - \left(\eta_1^<\right)^2} = 0$, we have $\text{Im}\tilde{\Gamma}_1\left(\eta_1^<\right) \neq 0$.

Thus for $n_1 > n_2$ and $t_{\text{dif}} < t < t_{\text{ref}}$, in the region $\theta > \theta_{\text{tot}}$, the side wave is generated, which is given by the function in (1.113).

From (1.113), (1.114), through the substitutions $\tilde{\Gamma}_1 \to 1$, $z + z_0 \to z - z_0$, we arrive at the following expression for the function G^0 characterizing the primary field:

$$G^0(x, z, t) = \left[0 \text{ for } 0 < t < t_0;\, 1\Big/\sqrt{t^2 - t_0^2} \;\text{ for }\; t_0 < t\right], \tag{1.116}$$

where $t_0 = n_1 R_-$ is the time of arrival of the primary wave at the observation point in the first medium.

1.4.4 The Field in the Second Medium

Denote the roots of the equation (1.107) by η_2^- and η_2^+. Then the integral in (1.105) takes the form

$$G^2(x,z,t) = \frac{i}{2}\lim_{\delta\to 0}\left[\operatorname*{Res}_{\eta=\eta_2^-} \frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta)}{x\eta + z_0\sqrt{n_1^2-\eta^2} - z\sqrt{n_2^2-\eta^2} - t_-} - \operatorname*{Res}_{\eta=\eta_2^+} \frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta)}{-x\eta + z_0\sqrt{n_1^2-\eta^2} - z\sqrt{n_2^2-\eta^2} + t_+}\right] = \frac{i}{2}\lim_{\delta\to 0}\left[\frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^-)}{x - Z(\eta_2^-)} + \frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^+)}{x + Z(\eta_2^+)}\right], \tag{1.117}$$

where

$$Z(\eta) = \left[\frac{z_0}{\sqrt{n_1^2-\eta^2}} - \frac{z}{\sqrt{n_2^2-\eta^2}}\right]\eta \quad \text{and} \quad \underset{\sim 2}{\tilde{\Gamma}}(\eta) = \frac{\tilde{\Gamma}_2(\eta)}{\sqrt{n_1^2-\eta^2}}. \tag{1.118}$$

The roots $\eta_2^{\pm}$ can be written explicitly, in the form of the solutions of the associated algebraic quartic equations. However, they are too lengthy because of six parameters entering (1.107) and will not be used. In view of the causality principle, we have $G^2(x,z,t) \equiv 0$ for $t < t_{\text{tr}}$, where t_{tr} is the time of arrival of the transmitted wave at the observation point in the second medium. For $t > t_{\text{tr}}$, the roots $\eta_2^{\pm}$ are complex and, as evident from (1.107), in terms of (1.111), we have $\eta_2^- = \left(\eta_2^+\right)^*$. Therefore, taking into account formulas in (1.112), we obtain for $t > t_{\text{tr}}$

$$\begin{aligned} G^2(x,z,t) &= \frac{i}{2}\lim_{\delta\to 0}\left[\frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^+)}{x + Z(\eta_2^+)} - \frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^{+*})}{x - Z(\eta_2^{+*})}\right] = \frac{i}{2}\lim_{\delta\to 0}\left\{\left[\frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^+)}{x + Z(\eta_2^+)}\right] - \left[\frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^+)}{x + Z(\eta_2^+)}\right]^*\right\} \\ &= -\lim_{\delta\to 0}\operatorname{Im}\frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^+)}{x + Z(\eta_2^+)} = -\operatorname{Im}\frac{\underset{\sim 2}{\tilde{\Gamma}}(\eta_2^>)}{x + Z(\eta_2^>)}, \end{aligned} \tag{1.119}$$

where $\eta_2^> = \eta_2^+\big|_{\delta=0}$.

1.4.5 Discussion and Conclusion

Formulas (1.93) and (1.116) for the primary field, formulas (1.113) and (1.114) for the secondary field in the first medium, as well as formula (1.119) for the secondary field in the second medium coincide with the corresponding expressions derived with the help of CHM in [54].

The main result of our study is a new representation for the field generated by a pulsed line current in a two-media configuration in the form of the integrals along finite contours (1.104), (1.105). This method, like the CHM, is applicable to the problems of pulsed electromagnetic radiation from linear sources in the medium formed by an arbitrary finite number N of homogeneous parallel layers with permittivity ε_j and permeability μ_j, $j = 1, 2, \ldots, N$. In this case, for the field in the layers, the integrals along the contour enveloping the interval $(-n_{\max}, n_{\max})$, where $n_{\max} = \max\{n_1, n_2, \ldots, n_N\}$, are similar to representations (1.104), (1.105). Two methods for calculating these integrals are possible.

The first way is to reduce them, by the Cauchy theorem, to a sum of residues at the poles of the integrand. These poles are determined by the roots of algebraic equations that coincide with the equations for the modified Cagniard contours [54]. Therefore this technique, being alternative to the CHM in an analytical sense, is equivalent to it in a calculating sense.

Another way is to estimate numerically the integrals in (1.104), (1.105). It is easy to show that they can be reduced to the integrals over the interval $(0, n_{\max})$. For example, the field in the first medium (1.104) can be represented for $n_2 > n_1$, $t > t_{\text{ref}}$ in the following form:

$$G^1(x,z,t) = -\frac{2}{\pi}t\left[\int_0^{n_1} f(\eta)\,\underset{\sim 1}{\tilde{\Gamma}}(\eta)d\eta + \frac{2}{n_2^2 - n_1^2}\int_{n_1}^{n_2} f(\eta)\sqrt{n_2^2 - \eta^2}d\eta\right],$$

where

$$f(\eta) = \frac{x^2\eta^2 + \left(n_1^2 - \eta^2\right)(z+z_0)^2 - t^2}{\left[x^2\eta^2 - \left(n_1^2 - \eta^2\right)(z+z_0)^2 + t^2\right]^2 - 4x^2t^2\eta^2}, \quad \underset{\sim 1}{\tilde{\Gamma}}(\eta) = \frac{\tilde{\Gamma}_1(\eta)}{\sqrt{n_1^2 - \eta^2}}.$$

We can use a standard integration procedure of any mathematical package to calculate G^1 by this formula. Comparison of the data obtained by this way with the explicit expression given by (1.114) has demonstrated high efficiency and accuracy of the approach.

The key point of the CHM is the solution of the algebraic equation determining the modified Cagniard contour. To do this, the iterative numerical methods are used. The greatest difficulty inherent in these methods is to choose the initial value that is close enough to the required zero of the equation [63]. In the paper [54], such an initial approximation has been proposed for the medium consisting of N isotropic layers. The efficiency of the iterative method has been demonstrated for $N = 2$. For more complex structures containing anisotropic layers, the initial approximation of this kind is unknown. (The CHM allows us to study as yet the simplest situation where the source and the observation point are located on the boundary of an anisotropic medium [64].)

Our approach, being free from such complications, reduces the calculation of the field generated by a line dipole in a multilayered medium to the standard procedure of numerical integration along a finite interval.

1.5 Transition Radiation of a Longitudinal Magnetic Dipole in the Case of Diffuse Interface

In the overwhelming number of studies on *transition radiation* (see reviews [65, 66]) the medium models are used in which spatial properties change abruptly. The transition radiation that occurs when an electric charge moves across the diffuse interface of two media was first discussed in the paper [67]. The authors used the *asymmetric Epstein layer* of relative permittivity $\varepsilon(z) = 1 + \alpha/[1 + \exp(-dz)]$. This model problem is of particular value since its exact solution, if it were obtained, would allow one to determine the conditions under which the transition radiation on the diffuse boundary can be considered approximately the same as in the case of the sharp boundary. This problem in [67] is reduced to the solution of the one-dimensional scalar Helmholtz equation with the coefficient involving $\sqrt{\varepsilon}(1/\sqrt{\varepsilon})''$ instead of $\varepsilon(z)$. Since the analytic solution of this equation is not known, the authors were forced to make an additional assumption about smallness of grad $\varepsilon(z)$. Furthermore, the variation of the function $\varepsilon(z)$ is supposed to be also small since the authors of [67] limited themselves by the case of the radiation from an ultrarelativistic charge at frequencies larger than optical frequencies. These assumptions, weakening the initial rigorous formulation, do not allow one to establish a reliable *criterion of the interface 'sharpness'*, which is free from those restrictions.

In this section, for the medium like an asymmetric Epstein layer, we will show the possibility to solve rigorously the problem of the transition radiation of a longitudinal magnetic dipole [68].

1.5.1 Problem Formulation and Solution

We assume that a longitudinal magnetic dipole with moment $\vec{m} = m_z\vec{z}$ is moving with constant velocity $\vec{V} = V_z\vec{z}$, $V_z > 0$ in an isotropic layered medium with constant relative permeability μ and relative permittivity

$$\varepsilon(z) = \varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{1 + \exp(-\tau z)}; \quad \tau > 0. \tag{1.120}$$

For simplicity, we assume that the condition of *Vavilov-Cherenkov radiation* is not satisfied. The initial equations are [67]:

$$\begin{aligned}
&\operatorname{rot}\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{j}^{(m)}, \quad \operatorname{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{D} = \varepsilon\varepsilon_0\vec{E}, \quad \vec{B} = \mu\mu_0\vec{H},\\
&\vec{j}^{(m)} = \sqrt{1 - V_z^2/c^2}\, m_z[\nabla_t \times \vec{z}]\delta(\vec{r}_\perp)\delta(z - V_z t), c = \left(\sqrt{\varepsilon_0\mu_0}\right)^{-1}, \nabla_t = \frac{\partial}{\partial x}\vec{x} + \frac{\partial}{\partial y}\vec{y},\\
&\vec{r}_\perp = x\vec{x} + y\vec{y}.
\end{aligned}$$

For a plane-layered isotropic medium, these equations with a harmonic time dependence can be reduced [41] to two scalar equations

$$\left(\varepsilon\frac{\partial}{\partial z}\frac{1}{\varepsilon}\frac{\partial}{\partial z}+k^2+\nabla_t^2\right)\left([\nabla_t\times\vec{z}]\cdot\vec{H}_\omega^t\right)=\varepsilon\frac{\partial}{\partial z}\frac{1}{\varepsilon}\left(\nabla_t\cdot\vec{J}_\omega^{(m)}\right),$$
$$\left(\frac{\partial^2}{\partial z^2}+k^2+\nabla_t^2\right)\left([\nabla_t\times\vec{z}]\cdot\vec{E}_\omega^t\right)=-i\omega\mu\mu_0\left([\nabla_t\times\vec{z}]\cdot\vec{J}_\omega^{(m)}\right),$$

where $k^2=k_0^2\varepsilon\mu$, $k_0=\omega\sqrt{\varepsilon_0\mu_0}$, and $\vec{E}_\omega^t$, $\vec{H}_\omega^t$ are the projections of the corresponding vectors on the plane $x0y$. In this case,

$$\left(\nabla_t\cdot\vec{E}_\omega^t\right)=\frac{1}{i\omega\varepsilon\varepsilon_0}\left(\left(\nabla_t\cdot\vec{J}_\omega^{(m)}\right)-\frac{\partial}{\partial z}\left([\nabla_t\times\vec{z}]\cdot H_\omega^t\right)\right),$$
$$\left(\nabla_t\cdot\vec{H}_\omega^t\right)=\frac{1}{i\omega\mu\mu_0}\frac{\partial}{\partial z}\left([\nabla_t\times\vec{z}]\cdot\vec{E}_\omega^t\right),\quad E_{z\omega}=\frac{1}{i\omega\varepsilon}\left([\nabla_t\times\vec{z}]\cdot\vec{H}_\omega^t\right),$$
$$H_{z\omega}=-\frac{1}{i\omega\mu\mu_0}\left([\nabla_t\times\vec{z}]\cdot\vec{E}_\omega^t\right).$$

Since the problem is homogeneous in time and in the direction perpendicular to the velocity of the dipole, we represent all the functions in Maxwell's equations in the form of the Fourier integrals

$$\vec{F}(\vec{r},t)=\int\vec{F}_{\omega,\vec{\kappa}}(z)\exp[i(\vec{\kappa}\cdot\vec{r}_\perp)-i\omega t]\,d\omega\,d\vec{\kappa};\quad\vec{\kappa}=\kappa_x\vec{x}+\kappa_y\vec{y},\quad\vec{r}=\vec{r}_\perp+z\vec{z}$$

with

$$\vec{J}_{\omega,\vec{\kappa}}^{(m)}(z)=\frac{i\sqrt{1-V_z^2/c^2}}{(2\pi)^3V_z}m_z\exp(i\omega z/V_z)[\vec{\kappa}\times\vec{z}].$$

Then, by introducing the scalar function $u(z)=\left([\vec{\kappa}\times\vec{z}]\cdot\vec{E}_{\omega,\vec{\kappa}}^t\right)$, we arrive at the equation

$$\left[\frac{d^2}{dz^2}+k_0^2\varepsilon(z)\mu-\kappa^2\right]u(z)=A\exp(i\omega z/V_z);$$
$$A=\frac{\sqrt{4\pi\varepsilon_0}\omega\mu\mu_0}{(2\pi)^3V_z}\sqrt{1-V_z^2/c^2}m_z\kappa^2. \tag{1.121}$$

The spectral components of the field can be recovered from the solution of this equation by the formulas

$$\vec{E}^t_{\omega,\vec{\kappa}} = -\frac{i}{\kappa^2}[\vec{\kappa}\times\vec{z}]u(z), \quad E_{z,\omega,\vec{\kappa}} = 0, \quad \vec{H}^t_{\omega,\vec{\kappa}} = -\frac{i}{\sqrt{4\pi\varepsilon_0}\omega\mu\mu_0\kappa^2}\vec{\kappa}\frac{d}{dz}u(z),$$
$$H_{z,\omega,\vec{\kappa}} = -\frac{1}{\sqrt{4\pi\varepsilon_0}\omega\mu\mu_0}u(z).$$

The magnetic field vector lies in the radiation plane, which passes through the vectors $\vec{\kappa}$ and $\vec{V}$, that is, the field is an H-polarized wave [69].

We now seek the solution of homogeneous (1.121). By introducing a new independent variable $x = -\exp(-\tau z)$ and a new function $y(x) = (-x)^{-\nu}u(z)$ [12, 67], we pass from (1.121) to the following hypergeometric equation

$$x(1-x)y''(x) + [s-(a+b+1)x]y'(x) - aby(x) = 0 \tag{1.122}$$

with the parameters $a = \nu+\lambda$, $b = \nu-\lambda$, $s = 1+2\nu$, $\nu = (\tau)^{-1}\sqrt{\kappa^2-\omega^2\varepsilon_2\varepsilon_0\mu\mu_0}$, $\lambda = (\tau)^{-1}\sqrt{\kappa^2-\omega^2\varepsilon_1\varepsilon_0\mu\mu_0}$, $\mathrm{Re}\sqrt{\kappa^2-\omega^2\varepsilon_{1,2}\varepsilon_0\mu\mu_0} \geq 0$.

Let us choose two linearly independent solutions of (1.122) that are regular at zero [70]:

$$y_1(x) = F(a,b,s,x), \quad y_5(x) = x^{1-s}F(a+1-s, b+1-s, 2-s, x),$$

where $F(\ldots)$ is the hypergeometric function. The corresponding solutions of homogeneous (1.121) are

$$u_1(z) = \exp(-\nu\tau z)F(\nu+\lambda, \nu-\lambda, 1+2\nu, -\exp(-\tau z)),$$
$$u_5(z) = \exp(\nu\tau z)F(\lambda-\nu, -\lambda-\nu, 1-2\nu, -\exp(-\tau z)).$$

The general solution of inhomogeneous (1.121) is given by

$$\begin{aligned}\frac{W}{A}u(z) = &-u_1(z)\int u_5(z)\exp(i\omega z/V_z)dz + u_5(z)\int u_1(z)\exp(i\omega z/V_z)dz \\ &+ C_1u_1(z) + C_2u_5(z)\end{aligned} \tag{1.123}$$

with the Wronskian $W = \lim\limits_{z\to\infty}\left(u_1u_5' - u_1'u_5\right) = 2\tau\nu$. To calculate these integrals we use the *Barnes representation* [70]

$$F(a,b,s,\xi) = \frac{1}{2\pi i}\frac{\Gamma(s)}{\Gamma(a)\Gamma(b)}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(s+t)}(-\xi)^t dt,$$

where $|\arg(-\xi)| < \pi$, $\gamma > 0$ and all the poles of $\Gamma(-t)$ are located to the right of the contour of integration. Let us introduce the notation $\sigma = i\omega/V_z\tau$, $\zeta = \tau z$ and consider the case where $z < 0$. Then,

$$
\begin{aligned}
I_1 &= \int u_5(z)\exp(i\omega z/V_z)dz \\
&= \frac{1}{\tau}\int F(\lambda - v, -\lambda - v, 1-2v, -\exp(-\varsigma))\exp[(\sigma+v)\varsigma]d\varsigma \\
&= \frac{1}{2\pi i\tau}\frac{\Gamma(1-2v)\exp[(\sigma+v)\varsigma]}{\Gamma(\lambda - v)\Gamma(-\lambda - v)}\int\limits_{\gamma - i\infty}^{\gamma + i\infty}\frac{\Gamma(\lambda - v + t)\Gamma(-\lambda - v + t)\Gamma(-t)}{\Gamma(1-2v+t)(\sigma + v - t)}\exp(-\varsigma t)dt.
\end{aligned}
$$

The integrand allows us to close the integration contour in the left half-plane $\mathrm{Re}\, t < \gamma$. Upon calculating the residues at the poles $t_n = -n - \lambda + v$, $t'_m = -m + \lambda + v$, $n, m = 0, 1, 2, \ldots$ and $t^+ = \sigma + v$, we obtain

$$
\begin{aligned}
I_1 = \frac{1}{\tau}\frac{\exp(\sigma\varsigma)\Gamma(1-2v)}{\Gamma(\lambda - v)\Gamma(-\lambda - v)}\Bigg\{&\exp(\lambda\varsigma)\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(-2\lambda - n)\Gamma(-v+\lambda+n)}{n!\Gamma(1-v-\lambda-n)(\sigma+\lambda+n)}\exp(n\varsigma) \\
&+ \exp(-\lambda\varsigma)\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2\lambda - n)\Gamma(-v-\lambda+n)}{n!\Gamma(1-v+\lambda-n)(\sigma-\lambda+n)}\exp(n\varsigma) \\
&- \frac{\Gamma(\lambda+\sigma)\Gamma(-\lambda+\sigma)\Gamma(-\sigma-v)}{\Gamma(1+\sigma-v)}\exp(-\sigma\varsigma)\Bigg\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \int u_1(z)\exp(i\omega z/V_z)dz = \frac{1}{\tau}\int F(v+\lambda, v-\lambda, 1+2v, -\exp(-\varsigma))\exp[(\sigma - v)\varsigma]d\varsigma \\
&= \frac{1}{2\pi i\tau}\frac{\Gamma(1+2v)\exp[(\sigma - v)\varsigma]}{\Gamma(v+\lambda)\Gamma(v-\lambda)}\int\limits_{\gamma - i\infty}^{\gamma + i\infty}\frac{\Gamma(v+\lambda+t)\Gamma(v-\lambda+t)\Gamma(-t)}{\Gamma(1+2v+t)(\sigma - v - t)}\exp(-\varsigma t)dt.
\end{aligned}
$$

Upon calculating the residues at the poles $t_n = -n - \lambda - v$, $t'_m = -m + \lambda - v$, $n, m = 0, 1, 2, \ldots$, and $t^- = \sigma - v$ in the half-plane $\mathrm{Re} t < \gamma$, we obtain

$$
\begin{aligned}
I_2 = \frac{1}{\tau}\frac{\exp(\sigma\varsigma)\Gamma(1+2v)}{\Gamma(v+\lambda)\Gamma(v-\lambda)}\Bigg\{&\exp(\lambda\varsigma)\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(-2\lambda - n)\Gamma(v+\lambda+n)}{n!\Gamma(1+v-\lambda-n)(\sigma+\lambda+n)}\exp(n\varsigma) \\
&+ \exp(-\lambda\varsigma)\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2\lambda - n)\Gamma(v-\lambda+n)}{n!\Gamma(1+v+\lambda-n)(\sigma-\lambda+n)}\exp(n\varsigma) \\
&- \frac{\Gamma(\lambda+\sigma)\Gamma(\sigma-\lambda)\Gamma(-\sigma+v)}{\Gamma(1+\sigma+v)}\exp(-\sigma\varsigma)\Bigg\}.
\end{aligned}
$$

Applying the formula $\pi(-1)^{n+1} = \sin(\pi\alpha)\Gamma(\alpha + 1 - n)\Gamma(-\alpha + n)$ [70], we find that

$$\frac{W}{A}u(z) = -u_1(z)\exp(\sigma\varsigma)\frac{\Gamma(1-2v)}{\tau\,\Gamma(\lambda-v)\Gamma(-\lambda-v)}\left\{-\exp(\lambda\varsigma)S_1(\varsigma)\frac{\sin[\pi(v+\lambda)]}{\sin(2\pi\lambda)}\right.$$
$$\left.+\exp(-\lambda\varsigma)S_2(\varsigma)\frac{\sin[\pi(v-\lambda)]}{\sin(2\pi\lambda)}-\frac{\Gamma(\lambda+\sigma)\Gamma(-\lambda+\sigma)\Gamma(-\sigma-v)}{\Gamma(1+\sigma-v)}\exp(-\sigma\varsigma)\right\}$$
$$+u_5(z)\exp(\sigma\varsigma)\frac{\Gamma(1+2v)}{\tau\,\Gamma(v+\lambda)\Gamma(v-\lambda)}\left\{-\exp(\lambda\varsigma)S_1(\varsigma)\frac{\sin[\pi(\lambda-v)]}{\sin(2\pi\lambda)}\right.$$
$$\left.-\exp(-\lambda\varsigma)S_2(\varsigma)\frac{\sin[\pi(\lambda+v)]}{\sin(2\pi\lambda)}-\frac{\Gamma(\lambda+\sigma)\Gamma(-\lambda+\sigma)\Gamma(-\sigma+v)}{\Gamma(1+\sigma+v)}\exp(-\sigma\varsigma)\right\}$$
$$+C_1u_1(z)+C_2u_5(z), \tag{1.124}$$

where

$$S_1(\varsigma) = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(-v+\lambda+n)\Gamma(v+\lambda+n)}{n!\Gamma(1+2\lambda+n)(\sigma+\lambda+n)}\exp(n\varsigma) \quad \text{and}$$
$$S_2(\varsigma) = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(v-\lambda+n)\Gamma(-v-\lambda+n)}{n!\Gamma(1-2\lambda+n)(\sigma-\lambda+n)}\exp(n\varsigma).$$

The linearly independent solutions u_1 and u_5 are regular for positive z. We are interested in $z<0$, therefore, let us continue these solutions analytically into this domain [70]: $u_1 = \Gamma_{13}u_3 + \Gamma_{14}u_4, \quad u_5 = \Gamma_{53}u_3 + \Gamma_{54}u_4$. Here

$$\Gamma_{13} = \frac{\Gamma(1+2v)\Gamma(-2\lambda)}{\Gamma(v-\lambda+1)\Gamma(v-\lambda)}, \quad \Gamma_{14} = \frac{\Gamma(1+2v)\Gamma(2\lambda)}{\Gamma(v+\lambda+1)\Gamma(v+\lambda)},$$
$$\Gamma_{53} = \frac{\Gamma(1-2v)\Gamma(-2\lambda)}{\Gamma(-v-\lambda+1)\Gamma(-v-\lambda)}, \quad \Gamma_{54} = \frac{\Gamma(1-2v)\Gamma(2\lambda)}{\Gamma(-v+\lambda+1)\Gamma(-v+\lambda)}.$$

By substituting the above expressions into (1.124), we obtain for $z<0$

$$u(z) = u^m(z) + u^r(z), \tag{1.125}$$

where

$$\frac{W}{A}u^m(z) = \frac{v}{\tau\lambda}\exp(\sigma\varsigma)\left\{-u_4(z)\exp(\lambda\varsigma)S_1(\varsigma)\frac{\Gamma(1+2\lambda)}{\Gamma(\lambda-v)\Gamma(v+\lambda)}\right.$$
$$\left.+u_3(z)\exp(-\lambda\varsigma)S_2(\varsigma)\frac{\Gamma(1-2\lambda)}{\Gamma(-\lambda-v)\Gamma(v-\lambda)}\right\},$$
$$\frac{W}{A}u^r(z) = \frac{1}{\tau}u_3(z)\{\Gamma_{13}[\Gamma(\sigma,\lambda,-v)+\tau C_1]-\Gamma_{53}[\Gamma(\sigma,\lambda,v)-\tau C_2]\}$$
$$+\frac{1}{\tau}u_4(z)\{\Gamma_{14}[\Gamma(\sigma,\lambda,-v)+\tau C_1]-\Gamma_{54}[\Gamma(\sigma,\lambda,v)-\tau C_2]\},$$
$$\Gamma(\sigma,\lambda,v) = \frac{\Gamma(1+2v)\Gamma(\sigma+\lambda)\Gamma(\sigma-\lambda)\Gamma(-\sigma+v)}{\Gamma(v+\lambda)\Gamma(v-\lambda)\Gamma(1+\sigma+v)}.$$

With $z \to -\infty$ we have $u_3(z) \approx \exp(\lambda\varsigma)$, $u_4(z) \approx \exp(-\lambda\varsigma)$, consequently

$$u^m(z) \approx -\frac{\sqrt{4\pi\varepsilon_0}\omega\mu\mu_0 m_z}{(2\pi)^3 V_z}\kappa^2\sqrt{1 - V_z^2/c^2}\frac{\exp(i\omega z/V_z)}{\kappa^2 - \omega^2\varepsilon_1\varepsilon_0\mu\mu_0 + \omega^2/V_z^2},$$

i.e. for $z \to -\infty$ the term $u^m(z)$ changes into the self field of the longitudinal magnetic dipole [69]. The term $u^r(z)$ represents the radiation field. For propagating waves, the inequality $k_0^2\varepsilon_{1,2}\mu > \kappa^2$ holds; choosing the root branch $\arg\sqrt{\kappa^2 - k_0^2\varepsilon_{1,2}\mu} = -\pi/2$, we have

$$v = -\frac{i}{\tau}\sqrt{\omega^2\varepsilon_2\varepsilon_0\mu\mu_0 - \kappa^2}, \quad \lambda = -\frac{i}{\tau}\sqrt{\omega^2\varepsilon_1\varepsilon_0\mu\mu_0 - \kappa^2}.$$

Consequently, for $z \to -\infty$, $u_3(z)$ is the wave outgoing to $-\infty$, while $u_4(z)$ is the wave incoming from $-\infty$. Since the latter should not exist, the coefficient at $u_4(z)$ must be zero:

$$\Gamma_{14}[\Gamma(\sigma, \lambda, -v) + \tau C_1] = \Gamma_{54}[\Gamma(\sigma, \lambda, v) - \tau C_2]. \tag{1.126}$$

Another condition for the constants C_1 and C_2 we obtain from the representation (1.123) for the total field for $z > 0$. In this case, $\exp(-\varsigma) < 1$ in the integral I_1, which allows us to close the contour of integration in the half-plane $\mathrm{Re}\,t > \gamma$, where the poles $t_n = n$, $n = 0, 1, 2, \ldots$ are located. As a result we have for $z > 0$

$$\begin{aligned}\frac{W}{A}u(z) = &-u_1(z)\frac{\exp[(\sigma+v)\varsigma]\Gamma(1-2v)}{\Gamma(\lambda-v)\Gamma(-\lambda-v)\tau}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\lambda-v+n)\Gamma(-\lambda-v+n)}{n!\Gamma(1-2v+n)(\sigma+v-n)}\mathrm{e}^{-n\varsigma}\\ &+ u_5(z)\frac{\exp[(\sigma-v)\varsigma]\Gamma(1+2v)}{\Gamma(v+\lambda)\Gamma(v-\lambda)\tau}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(v+\lambda+n)\Gamma(v-\lambda+n)}{n!\Gamma(1+2v+n)(\sigma-v-n)}\\ &+ C_1u_1(z) + C_2u_5(z).\end{aligned}$$

With $z \to \infty$, $u_1 \approx \exp(-v\varsigma)$ is the wave outgoing to $+\infty$, while $u_5 \approx \exp(v\varsigma)$ is the wave incoming from $+\infty$. That is why the coefficient at $u_5(z)$ must be zero, $C_2 = 0$, and (1.126) turns into

$$\begin{aligned}C_1 &= [\Gamma_{54}\Gamma(\sigma, \lambda, v) - \Gamma_{14}\Gamma(\sigma, \lambda, -v)]\frac{1}{\Gamma_{14}\tau}\\ &= -\frac{\Gamma(1+v+\lambda)\Gamma(\sigma+\lambda)\Gamma(-\sigma+v)\Gamma(-\sigma-v)}{\tau\,\Gamma(2v)\Gamma(\lambda-v)\Gamma(1-\sigma+\lambda)}.\end{aligned}$$

If we introduce, by analogy with [67], the function

$$\upsilon(v, \lambda, \sigma, \exp(-\varsigma)) = F(-v+\lambda, -v-\lambda, 1-2v, -\exp(-\varsigma)) \times \frac{\Gamma(1+2v)}{\Gamma(v-\lambda)\Gamma(v+\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(v-\lambda+n)\Gamma(v+\lambda+n)}{n!\Gamma(1+2v+n)(-\sigma+v+n)} \exp(-n\varsigma),$$

then, for $z > 0$ the field can be written as

$$u(z) = \frac{A}{2\tau^2 v} \exp(\sigma\varsigma)\{-\upsilon[v, -\lambda, \sigma, \exp(-\varsigma)] + \upsilon[-v, -\lambda, \sigma, \exp(-\varsigma)]\} - \frac{A}{\tau^2} \frac{\Gamma(v+\lambda)\Gamma(1+v+\lambda)}{\Gamma(1+2v)\Gamma(1+2\lambda)} \Gamma(-\sigma, -v, \lambda) u_1(z), \tag{1.127}$$

while for $z<0$ it is

$$u(z) = \frac{A}{2\tau^2 \lambda} \exp(\sigma\varsigma)\{-\upsilon[\lambda, v, -\sigma, \exp(\varsigma)] + \upsilon[-\lambda, v, -\sigma, \exp(\varsigma)]\} - \frac{A}{\tau^2} \frac{\Gamma(v+\lambda)\Gamma(1+v+\lambda)}{\Gamma(1+2v)\Gamma(1+2\lambda)} \Gamma(\sigma, -\lambda, v) u_3(z). \tag{1.128}$$

In what follows, we will be interested only in the radiation field away from the boundary ($|z| \gg 1/\tau$). From (1.128) we obtain for this field

$$u^{\mathrm{rad}}(z) = -\frac{A}{\tau^2} \times \begin{cases} \Gamma(\sigma, \lambda, v) \exp(\lambda\varsigma); & z<0 \\ \Gamma(-\sigma, v, \lambda) \exp(-v\varsigma); & z>0, \end{cases} \tag{1.129}$$

where

$$\Gamma(\sigma, \lambda, v) = \frac{\Gamma(1+v+\lambda)\Gamma(\sigma+\lambda)\Gamma(\sigma-\lambda)\Gamma(-\sigma+v)}{\Gamma(1+2\lambda)\Gamma(v-\lambda)\Gamma(1+v+\sigma)}.$$

The energy of the forward radiation into the half-space $z > 0$ is [69]

$$W_2^r = \frac{m_z^2(1 - V_z^2/c^2)\mu\mu_0}{4\pi^2 V_z^2 \tau^4} \int_0^\infty d\omega \int_0^\infty \omega\kappa^2 |\Gamma(-\sigma, v, \lambda)|^2 \sqrt{\omega^2 \varepsilon_2 \varepsilon_0 \mu\mu_0 - \kappa^2}\, d\kappa^2, \tag{1.130}$$

where the integration is performed over the domain $\kappa^2 < \omega^2 \varepsilon_2 \varepsilon_0 \mu\mu_0$, which corresponds to the waves propagating in the right half-space away from the boundary. For such κ, complex conjugation of the parameters v and μ gives $v^* = -v$, $\lambda^* = -\lambda$. Taking into account the properties of the gamma function $\Gamma^*(z) = \Gamma(z^*)$, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we obtain

$$|\Gamma(-\sigma,\nu,\lambda)|^2 = \frac{\pi}{2\nu}\frac{\left(\lambda^2-\nu^2\right)\sin(2\pi\nu)\sin[\pi(\lambda-\nu)]\sin[\pi(\sigma-\lambda)]}{\left(\sigma^2-\nu^2\right)\left(\sigma^2-\lambda^2\right)\sin[\pi(\nu+\lambda)]\sin[\pi(\sigma+\lambda)]} \times \frac{1}{\sin[\pi(\sigma-\nu)]\sin[\pi(\sigma+\nu)]}. \tag{1.131}$$

Formulas (1.127)–(1.131) are valid for arbitrary values of the parameter τ, which characterizes the degree of boundary diffusiveness.

1.5.2 The Criterion of the Interface 'Sharpness'

Let us consider the transition to the sharp interface: $\tau \to \infty$. The expansion of (1.131) in the power series in the small parameter $1/\tau$ requires the smallness of the absolute values of 2ν, $\lambda-\nu$, $\sigma-\lambda$, $\sigma+\lambda$, $\sigma-\nu$, $\sigma+\nu$, $\lambda+\nu$, which can be expressed via four independent values: $\sigma \pm \nu$, $\sigma \pm \lambda$. Denote

$$L_1^{\pm} = \frac{2\pi}{\tau|\sigma \pm \lambda|} = \frac{2\pi}{\left|\frac{\omega}{V_z} \mp \sqrt{k_0^2\varepsilon_1\mu-\kappa^2}\right|}, \quad L_2^{\pm} = \frac{2\pi}{\tau|\sigma \pm \nu|} = \frac{2\pi}{\left|\frac{\omega}{V_z} \mp \sqrt{k_0^2\varepsilon_2\mu-\kappa^2}\right|}$$

and suppose the following inequalities hold:

$$2\pi/\tau \ll L_1^{\pm}, \quad 2\pi/\tau \ll L_2^{\pm}. \tag{1.132}$$

Then we have

$$\frac{\sin(2\pi\nu)\sin[\pi(\lambda-\nu)]\sin[\pi(\sigma-\lambda)]}{\sin[\pi(\nu+\lambda)]\sin[\pi(\sigma+\lambda)]\sin[\pi(\sigma-\nu)]\sin[\pi(\sigma+\nu)]} = \frac{2\nu(\sigma-\lambda)(\lambda-\nu)}{\pi(\nu+\lambda)(\sigma+\lambda)(\sigma^2-\nu^2)}\left[1+\frac{1}{3}\pi^2(\sigma+\nu)(\sigma-\nu+2\lambda)+\ldots\right]$$

and for $\tau \to \infty$

$$|\Gamma(-\sigma,\nu,\lambda)|^2 = \frac{(\lambda-\nu)^2}{(\sigma+\lambda)^2(\sigma^2-\nu^2)^2}. \tag{1.133}$$

Substituting this formula into (1.130), we obtain the expression for the radiation energy, which coincides exactly with the results given in [69] for the case of a sharp boundary.

In evaluating the sharpness of the interface between two media an important role is played by the notion of *the radiation-forming region*. In the case of a diffuse interface, the radiation can be considered approximately the same as in the case of a sharp interface if the characteristic width of the transition boundary layer Δz is much smaller than the length of the radiation-forming region. During the qualitative evaluation [69] based on the determination of the distance at which the field of a moving source and the radiation field, moving away from the boundary, are separated, the following conditions were obtained

$$\Delta z \ll L_1^-, \quad \Delta z \ll L_2^+. \tag{1.134}$$

They are equivalent to the conditions in (1.132) ($\tau = 2\pi/\Delta z$). Here, L_1^- and L_2^+ are the lengths of the radiation-forming regions for the radiation moving away from the interface in the first and second media, respectively.

These two conditions are not enough to transfer from the general solution (1.131) to the solution (1.133) for a sharp boundary. Two additional conditions in (1.132) estimate the distance from the boundary, at which the field of the source and the radiation field incoming on the boundary are separated. Since $L_{1,2}^+ > L_{1,2}^-$, the conditions under which the interface between two media can be considered sharp, are:

$$\Delta z \ll L_1^-, \quad \Delta z \ll L_2^-. \tag{1.135}$$

The error of the condition (1.134), as compared with the exact condition (1.135), shows itself in the situation where the source moves from a less dense into a more dense medium.

In the paper [67], the following two inequalities were chosen as a criterion of the interface sharpness:

$$\Delta z \ll L_1^+, \quad \Delta z \ll L_2^+, \tag{1.136}$$

which were less restrictive than those in (1.135). Within the frequency range $\omega^2 \gg \omega_{\text{pe}}^2 = 4\pi N_{\text{e}} e^2 / m_{\text{e}}$, considered in [67], where N_{e} is the electron density of the material and m_{e} is the electron mass, the conditions in (1.136) are sufficient for passing to the case of a sharp interface in the general relationships for the spectral density of the radiation, produced at small angles by an ultrarelativistic charge in the medium with a *diffuse boundary*.

Thus, we have formulated the problem of transition radiation for a medium with a diffuse boundary. For the first time, we obtained the rigorous analytical solution of this problem, without imposing any restrictions on the parameters of the model. By analyzing the passage to the limiting case of a sharp boundary in this solution, we have found an exact criterion of the interface 'sharpness' in the form of two inequalities (1.135). It substantially improves the well-known criterion (1.134) and, in contrast to another version of this criterion (1.136), does not require any restrictions on the frequency range, the charge velocity and the change in the permittivity $\varepsilon_2 - \varepsilon_1$.

1.6 The Biisotropic Epstein Transition Layer

The isotropic linear media having the properties of chirality and nonreciprocity are referred to as *biisotropic*. Chirality leads to circular dichroism and optical activity—the rotation of the polarization vector as in the *Faraday effect*, but regardless of the direction of propagation. If a medium has the property of nonreciprocity, the electric and magnetic field vectors are not orthogonal, and the phase velocity depends on the nonreciprocity index [71]. These effects may be important for new microwave applications [72, 73], if such a medium is realized.

Analysis of electromagnetic waves in inhomogeneous biisotropic media began from the works [74, 75]. The authors of these papers considered diffraction of a plane electromagnetic wave on a boundary of the half-space filled with a chiral medium. In [76] a similar problem was solved for the general case of an arbitrary *biisotropic medium*. In a number of works, a similar problem for homogeneous biisotropic layers has been studied in detail [77]. The papers [78, 79], which use numerical and analytical methods, are devoted to the investigation of the electromagnetic scattering in biisotropic stratified media with *continuously* varying parameters. Within the class of inhomogeneous biisotropic media, we proposed in [80] a model of the medium, for which one can write the *analytical* solution to the problem of the plane electromagnetic wave that propagates in this medium along the normal to the layers. Such a medium is a generalization to the biisotropic case of the known [12] isotropic Epstein transition layer, which describes a smooth transition in a plane-layered isotropic medium between the regions with different refractive indices n_1 and n_2. In this section, we discuss in detail the methodology for obtaining this solution. The solution can be expressed in terms of the known hypergeometric series, as well as for the isotropic Epstein layer. The analytical expressions for the reflection and transmission coefficients have been derived, from which it follows that in such a medium the total transmission may occur.

1.6.1 Equations for the Electromagnetic Field in a Biisotropic Medium

It is well known that biisotropic media are marked by the magnetoelectric coupling, in which both electric and magnetic excitation leads simultaneously both to the electric and magnetic polarization. To describe the most general form of such a medium, in addition to the relative permittivity ε and the permeability μ, the nonreciprocity parameter χ and the chirality parameter κ are used. The constitutive equations for this medium, on the assumption of harmonic excitation (time dependence is defined by $\exp(-i\omega t)$), are as follows [71]:

$$\vec{D} = \varepsilon\varepsilon_0\vec{E} + \sqrt{\varepsilon_0\mu_0}(\chi + i\kappa)\vec{H}, \quad \vec{B} = \sqrt{\varepsilon_0\mu_0}(\chi - i\kappa)\vec{E} + \mu\mu_0\vec{H}, \tag{1.137}$$

where ε_0, μ_0 are the permittivities of free space. For lossless media, the dimensionless parameters ε, μ, χ, and κ are the real functions of coordinates.

The Maxwell's equations, in view of (1.137), can be written as

$$\mathrm{rot}\vec{E} = ik_0\eta_0\mu\vec{H} + k_0(\kappa + i\chi)\vec{E}, \quad \mathrm{rot}\vec{H} = -ik_0\eta_0^{-1}\varepsilon\vec{E} + k_0(\kappa - i\chi)\vec{H}, \tag{1.138}$$

where $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$, $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$. By eliminating $\vec{H}$ from (1.138), we arrive at the vector Helmholtz equation

$$\frac{1}{k_0}\mathrm{rot}\frac{1}{\mu}\mathrm{rot}\vec{E} - \mathrm{rot}\frac{\kappa + i\chi}{\mu}\vec{E} - \frac{\kappa - i\chi}{\mu}\mathrm{rot}\vec{E} + k_0\left(\frac{\kappa^2 + \chi^2}{\mu} - \varepsilon\right)\vec{E} = 0. \tag{1.139}$$

The parameters ε, χ, κ of the plane-parallel medium depend only on the coordinate z; the magnetic permeability μ is considered constant in the whole space. The electromagnetic wave that propagates in such a medium perpendicularly to the layers, does not depend on the transversal coordinates x, y. Therefore, (1.139) is transformed into the system:

$$\begin{cases} \dfrac{d^2E_x}{dz^2} - 2k_0\kappa\dfrac{dE_y}{dz} - k_0^2\left(\kappa^2 + \chi^2 - \varepsilon\mu\right)E_x - k_0\dfrac{d(\kappa + i\chi)}{dz}E_y = 0 \\ \dfrac{d^2E_y}{dz^2} + 2k_0\kappa\dfrac{dE_x}{dz} + k_0\dfrac{d(\kappa + i\chi)}{dz}E_x - k_0^2\left(\kappa^2 + \chi^2 - \varepsilon\mu\right)E_y = 0 \\ E_z = 0. \end{cases} \tag{1.140}$$

Hence, introducing the auxiliary functions $E_\pm = E_x \pm iE_y$, we obtain two independent equations

$$\frac{d^2E_\pm}{d\,\tilde{z}^2} \pm 2i\kappa\frac{dE_\pm}{d\,\tilde{z}} + \left[\left(n^2 - \kappa^2 - \chi^2\right) \pm i(\kappa' + i\chi')\right]E_\pm = 0, \tag{1.141}$$

where $\tilde{z} = k_0z$, $n = \sqrt{\varepsilon\mu}$, $\kappa' + i\chi' = d(\kappa + i\chi)/d\tilde{z}$. Removing the term with the first derivative by the substitution

$$E_\pm = \tilde{E}_\pm(z)\mathrm{e}^\mp(\tilde{z}), \quad \mathrm{e}^\mp(\tilde{z}) = \exp\left[\mp i\int\kappa(\tilde{z})d\tilde{z}\right], \tag{1.142}$$

we arrive at the following equation for the function $\tilde{E}_\pm(\tilde{z})$:

$$\frac{d^2\tilde{E}_\pm}{d\,\tilde{z}^2} + \left(n^2 - \chi^2 \mp \chi'\right)\tilde{E}_\pm = 0. \tag{1.143}$$

1.6.2 Problem Formulation and Solution

Consider the following version of the plane-layered medium

$$\begin{aligned} & n^2(\tilde{z}) = \varepsilon(\tilde{z}) = 0.5\left(1+\tilde{n}^2\right) + 0.5\left(1-\tilde{n}^2\right)\mathrm{th}(\tilde{\tau}\tilde{z}), \quad \chi(\tilde{z}) = 0.5\tilde{\chi}[1-\mathrm{th}(\tilde{\tau}\tilde{z})], \\ & \kappa(\tilde{z}) = 0.5\tilde{\kappa}[1-\mathrm{th}(\tilde{\tau}\tilde{z})], \quad \mu = 1, \quad |\tilde{\chi}| \le \tilde{n}, \quad \tilde{\tau} = \tau/k_0, \end{aligned} \tag{1.144}$$

which is the generalization of the known isotropic Epstein transition layer on the biisotropic case [12]. For $\tilde{z} = -\infty$, the inhomogeneous biisotropic medium (1.144) smoothly transits into a homogeneous biisotropic medium with the parameters $\tilde{n}^2$, $\tilde{\chi}$, $\tilde{\kappa}$, while for $\tilde{z} = +\infty$ it transits into the isotropic medium with $\varepsilon = \mu = 1$, $\chi = \kappa = 0$. The value $\Delta = 2k_0/\tau$ can be considered an effective width of the transition layer (1.144), which describes the degree of diffusiveness of the boundary between the isotropic and biisotropic media.

In view of (1.144), (1.143) takes the form

$$\frac{d^2\tilde{E}_\pm}{d\,\tilde{z}^2} + n_\pm^2(\tilde{z})\tilde{E}_\pm = 0, \tag{1.145}$$

where

$$\begin{aligned} & n_\pm^2(\tilde{z}) = 1 - N\frac{\exp(-2\tilde{\tau}\tilde{z})}{1+\exp(-2\tilde{\tau}\tilde{z})} - 4M_\pm\frac{\exp(-2\tilde{\tau}\tilde{z})}{[1+\exp(-2\tilde{\tau}\tilde{z})]^2}; \\ & N = 1 - \tilde{n}^2 + \tilde{\chi}^2, \quad 4M_\pm = -\tilde{\chi}(\tilde{\chi} \pm 2\tilde{\tau}). \end{aligned}$$

Let a plane linearly polarized electromagnetic wave of unit amplitude be incident from $\tilde{z} = +\infty$ on the biisotropic medium (1.144). The resulting electromagnetic field is to be found.

It is known [12] that solutions of the equations like the one in (1.145) can be represented as

$$\tilde{E}_\pm(\tilde{z}) = (\zeta')^{-1/2}\zeta^{\gamma/2}(1-\zeta)^{(\alpha+\beta-\gamma+1)/2}u(\zeta), \tag{1.146}$$

where $\zeta = -\exp(-2\tilde{\tau}\tilde{z})$, while the function $u(\zeta)$ is the general solution to the *hypergeometric Gauss equation*

$$\zeta(1-\zeta)\frac{d^2u}{d\zeta^2} - [(\alpha+\beta+1)\zeta - \gamma]\frac{du}{d\zeta} - \alpha\beta u = 0. \tag{1.147}$$

The parameters α, β, γ are representable through the parameters $\tilde{\tau}$, N, $M_\pm$ of the model:

$$\alpha_\pm = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4M_\pm \tilde{\tau}^{-2}} + \frac{i}{2\tilde{\tau}}\left(1 - \sqrt{1-N}\right),$$
$$\beta_\pm = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4M_\pm \tilde{\tau}^{-2}} + \frac{i}{2\tilde{\tau}}\left(1 + \sqrt{1-N}\right).$$

Substituting N and $M_\pm$, we obtain

$$\alpha_\pm = 1 + \frac{1}{2\tilde{\tau}}\left[\pm\tilde{\chi} + i\left(1 - \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)\right], \quad \beta_\pm = 1 + \frac{1}{2\tilde{\tau}}\left[\pm\tilde{\chi} + i\left(1 + \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)\right],$$
$$\gamma = 1 + \frac{i}{\tilde{\tau}}, \tag{1.148}$$

where the imaginary parts of the roots is nonnegative.

Equation (1.147) has three proper critical points $\zeta = 0, 1, \infty$, in the vicinity of which the two linearly independent solutions of this equation can be represented in the form of the converging hypergeometric series u_i, $i = 1, 2, 3, 4, 5, 6$ [70].

The transmitted wave should be outgoing as $\tilde{z} \to -\infty$, i.e. for $\zeta \to -\infty$. In the vicinity of an infinitely distant point of the complex plane ζ, the linearly independent solutions of (1.147) are

$$u_3 = (-\zeta)^{-\alpha} F\left(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \zeta^{-1}\right),$$
$$u_4 = (-\zeta)^{-\beta} F\left(\beta, \beta - \gamma + 1, \beta - \alpha + 1, \zeta^{-1}\right), \tag{1.149}$$

where $-\zeta = \zeta \exp(i\pi)$ and $F(\alpha, \beta, \gamma, \zeta) \equiv {}_2F_1(\alpha, \beta, \gamma, \zeta)$ is a hypergeometric series [70].

The asymptotics as $\tilde{z} \to -\infty$ of the functions in (1.146)

$$\tilde{E}_\pm(z) = \exp[(1-\gamma)\tilde{\tau}\tilde{z}][1 + \exp(-2\tilde{\tau}\tilde{z})]^{(\alpha_\pm + \beta_\pm - \gamma + 1)/2} u(\zeta)$$

that correspond to the solutions (1.149) are

$$\tilde{E}^{(3)}_\pm(\tilde{z}) \approx \exp[(\alpha_\pm - \beta_\pm)\tilde{\tau}\tilde{z}], \quad \tilde{E}^{(4)}_\pm(\tilde{z}) \approx \exp[(\beta_\pm - \alpha_\pm)\tilde{\tau}\tilde{z}].$$

Since $\alpha_\pm - \beta_\pm = -i\sqrt{\tilde{n}^2 - \tilde{\chi}^2}\Big/\tilde{\tau}$ and $\sqrt{\tilde{n}^2 - \tilde{\chi}^2} > 0$, then $\tilde{E}^{(3)}_\pm$ corresponds to the waves outgoing to $\tilde{z} = -\infty$, while $\tilde{E}^{(4)}_\pm$ corresponds to the ones incoming from $\tilde{z} = -\infty$. Consequently, one should take as solutions the functions

$$\tilde{E}^{(3)}_\pm(z) = \exp[(1-\gamma)\tilde{\tau}\tilde{z}][1 + \exp(-2\tilde{\tau}\tilde{z})]^{(\alpha_\pm + \beta_\pm - \gamma + 1)/2} u_3(\zeta),$$

whose behavior at $\tilde{z} \to +\infty$ $(\zeta \to 0)$ is determined by the following *Kummer's formula* [70]:

$$u_3 = \frac{\Gamma(1-\gamma)\Gamma(\alpha_\pm + 1 - \beta_\pm)}{\Gamma(1-\beta_\pm)\Gamma(\alpha_\pm + 1 - \gamma)} u_1 - \frac{\Gamma(\gamma)\Gamma(1-\gamma)\Gamma(\alpha_\pm + 1 - \beta_\pm)}{\Gamma(2-\gamma)\Gamma(\gamma - \beta_\pm)\Gamma(\alpha_\pm)} u_5, \quad (1.150)$$

where

$$u_1 = F(\alpha, \beta, \gamma, \zeta), u_5 = \zeta^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \zeta),$$

and $\Gamma(\ldots)$ is the gamma function. The asymptotics of the solutions of (1.145) that correspond to the functions u_1, u_5 are

$$\begin{aligned} \tilde{E}_\pm^{(1)}(\tilde{z}) &\approx \exp[(1-\gamma)\tilde{\tau}\tilde{z}] = \exp(-i\tilde{z}), \quad \tilde{E}_\pm^{(5)}(\tilde{z}) \approx \exp[(1-\gamma)\tilde{\tau}\tilde{z}] \\ &\times [-\exp(-2\tilde{\tau}\tilde{z})]^{1-\gamma} = (-1)^{1-\gamma}\exp[-(1-\gamma)\tilde{\tau}\tilde{z}] = (-1)^{1-\gamma}\exp(i\tilde{z}). \end{aligned} \quad (1.151)$$

i.e. $\tilde{E}_\pm^{(1)}$ corresponds to the wave incoming from $\tilde{z} = +\infty$, while $\tilde{E}_\pm^{(5)}$ corresponds to the wave outgoing to $\tilde{z} = +\infty$. It follows from (1.150) that

$$\begin{aligned} &\frac{\Gamma(1-\beta_\pm)\Gamma(\alpha_\pm + 1 - \gamma)}{\Gamma(1-\gamma)\Gamma(\alpha_\pm + 1 - \beta_\pm)} \tilde{E}_\pm^{(3)}(\tilde{z}) \\ &\quad = \tilde{E}_\pm^{(1)}(\tilde{z}) - \frac{\Gamma(\gamma)\Gamma(1-\beta_\pm)\Gamma(\alpha_\pm + 1 - \gamma)}{\Gamma(2-\gamma)\Gamma(\gamma-\beta_\pm)\Gamma(\alpha_\pm)} (-1)^{\gamma-1} \tilde{E}_\pm^{(5)}(\tilde{z}). \end{aligned} \quad (1.152)$$

Since for $\tilde{z} \to +\infty$ the behavior of the functions $\tilde{E}_\pm^{(1)}(\tilde{z})$ and $(-1)^{\gamma-1}\tilde{E}_\pm^{(5)}(\tilde{z})$ are determined by the asymptotics (1.151) and for $\tilde{z} \to -\infty$ we have $\tilde{E}_\pm^{(3)}(\tilde{z}) \approx \exp\left(-i\sqrt{\tilde{n}^2 - \tilde{\chi}^2}\tilde{z}\right)$, then these functions define the primary, reflected, and transmitted waves, respectively, while the factors in (1.152) are the transmission coefficients

$$T_\pm = \frac{\Gamma(1-\beta_\pm)\Gamma(\alpha_\pm + 1 - \gamma)}{\Gamma(1-\gamma)\Gamma(\alpha_\pm + 1 - \beta_\pm)} \quad (1.153)$$

and the reflection coefficients

$$R_\pm = \frac{\Gamma(\gamma - 1)\Gamma(1-\beta_\pm)\Gamma(\alpha_\pm + 1 - \gamma)}{\Gamma(1-\gamma)\Gamma(\gamma-\beta_\pm)\Gamma(\alpha_\pm)}. \quad (1.154)$$

The left and right parts of the equation (1.152) are two representations of the solutions of (1.145) $\tilde{E}_\pm(\tilde{z})$ for $\tilde{z} < 0$ and for $\tilde{z} > 0$, respectively.

1.6.3 Analysis of the Reflected and Transmitted Fields

As evident from (1.142) and (1.144), the electromagnetic field components can be written in the form

$$E_x = \frac{1}{2}\left(\tilde{E}_+ \mathrm{e}^- + \tilde{E}_- \mathrm{e}^+\right), \quad E_y = \frac{1}{2i}\left(\tilde{E}_+ \mathrm{e}^- - \tilde{E}_- \mathrm{e}^+\right),$$

where

$$\mathrm{e}^{\mp}(\tilde{z}) = \exp\left\{\mp \frac{i}{2}\tilde{\kappa}\left[\tilde{z} - \tilde{\tau}^{-1}\ln(\exp(\tilde{\tau}\tilde{z}) + \exp(-\tilde{\tau}\tilde{z}))\right]\right\}.$$

Let us consider the field structure away from the region, whose dimensions are determined by the effective width of the layer, $|\tilde{z}| \gg \Delta$.

In the region $\tilde{z} \gg \Delta$, where the medium differs little from the isotropic one, we have

$$\mathrm{e}^{\mp}(\tilde{z}) = 1 + O[\exp(-2\tilde{\tau}\tilde{z})], \quad \tilde{E}_{\pm} \approx \exp(-i\tilde{z}) + R_{\pm}\exp(i\tilde{z}),$$
$$E_x \approx \exp(-i\tilde{z}) + \frac{1}{2}(R_+ + R_-)\exp(i\tilde{z}), \quad E_y \approx \frac{1}{2i}(R_+ - R_-)\exp(i\tilde{z}).$$

In other words, the primary wave is linearly polarized along the x-axis, while the reflected wave is

$$\vec{E}_{\mathrm{ref}} = \left[\frac{1}{2}(R_+ + R_-)\vec{x} - \frac{i}{2}(R_+ - R_-)\vec{y}\right]\exp(i\tilde{z}) = \vec{E}^r_{\mathrm{ref}} + \vec{E}^l_{\mathrm{ref}},$$

where $\vec{E}^{r,l}_{\mathrm{ref}} = 0.5R^{r,l}(\vec{x} \pm i\vec{y})\exp(i\tilde{z})$ (the upper sign is associated with the superscript r, the bottom sign is associated with l), $\vec{E}^r_{\mathrm{ref}}$ is the right-hand circularly polarized wave, $\vec{E}^l_{\mathrm{ref}}$ is the left-hand circularly polarized wave. The reflection coefficients of these two waves can be represented in the form

$$R^{r,l} = R_{\mp} = \frac{\tilde{\chi} \pm i\left(1 + \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)}{\tilde{\chi} \mp i\left(1 - \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)}\frac{\Gamma(\gamma - 1)}{\Gamma(1 - \gamma)}R, \tag{1.155}$$

where

$$R = \frac{\Gamma\left(\Delta\left[\tilde{\chi} - i\left(1 + \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)\right]\right)\Gamma\left(\Delta\left[-\tilde{\chi} - i\left(1 + \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)\right]\right)}{\Gamma\left(\Delta\left[\tilde{\chi} + i\left(1 - \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)\right]\right)\Gamma\left(\Delta\left[-\tilde{\chi} + i\left(1 - \sqrt{\tilde{n}^2 - \tilde{\chi}^2}\right)\right]\right)}. \tag{1.156}$$

The reflected wave takes the form

$$\vec{E}_{\text{ref}} = \frac{R}{1 - 2\sqrt{\tilde{n}^2 - \tilde{\chi}^2} + \tilde{n}^2} \frac{\Gamma(2i\Delta)}{\Gamma(-2i\Delta)} \left[(\tilde{n}^2 - 1)\vec{x} - 2\tilde{\chi}\vec{y} \right] \exp(i\tilde{z}). \tag{1.157}$$

When passing to the sharp boundary ($\Delta = 0$), we get

$$\vec{E}_{\text{ref}} = \left[(1 - \tilde{n}^2)\vec{x} + 2\tilde{\chi}\vec{y} \right] \left(1 + 2\sqrt{\tilde{n}^2 - \tilde{\chi}^2} + \tilde{n}^2 \right)^{-1} \exp(i\tilde{z}). \tag{1.158}$$

As seen from (1.157) and regardless of the width of the layer, when a plane linearly polarized wave is reflected from a biisotropic half-space with $n^2 = \varepsilon = \tilde{n}^2 > 1$, $\chi = \tilde{\chi} > 0$ ($\chi = \tilde{\chi} < 0$), the plane of polarization rotates anticlockwise (clockwise) by the angle of

$$\varphi_{\text{ref}} = \operatorname{arctg} \frac{2\tilde{\chi}}{1 - \tilde{n}^2}, \tag{1.159}$$

if viewed in the direction of the reflected wave.

Expressions (1.158) and (1.159) differ from those obtained in [76] by the sign of $\tilde{\chi}$. It is interesting to note that the reflection from the biisotropic transition layer (1.144), in contrast to the isotropic layer, may disappear completely. Indeed, if the following conditions on the non-reciprocity parameter $\tilde{\chi} = \tilde{\chi}_0$, the refraction index $\tilde{n} = \tilde{n}_0$, and the layer width $\Delta = \Delta_0$ are satisfied:

$$\tilde{\chi}_0 = \sqrt{\tilde{n}_0^2 - 1}, \quad \tilde{\chi}_0 \Delta_0 = m; \quad m = 1, 2, \ldots, \tag{1.160}$$

then the coefficients $R^{r,l}$ (1.155) vanish due to the second gamma function in the denominator (1.156).

With the increase in the width of the transition layer, the coefficients $R^{r,l}$ decay exponentially to zero. Using the known formulas for gamma functions, we obtain from (1.155)

$$\left| R^{r,l} \right| \approx \left\{ \exp(-2\pi\Delta) \text{for} |\tilde{\chi}| < \tilde{\chi}_0, \exp\left(-2\pi\Delta\sqrt{\tilde{n}^2 - \tilde{\chi}^2} \right) \text{for} |\tilde{\chi}| > \tilde{\chi}_0 \right\}; \quad \Delta \gg 1, \tag{1.161}$$

where $\tilde{\chi}_0$ is defined by (1.160).

Consider the field transmitted into the biisotropic media, away from the transition layer. For $\tilde{z} \ll -\Delta$ we have

$$\mathrm{e}^{\mp}(\tilde{z}) = \exp(\mp i\tilde{\kappa}\tilde{z})[1 + O(\exp(2\tilde{\tau}\tilde{z}))], \quad \tilde{E}_{\pm} = T_{\pm}\tilde{E}_{\pm}^{(3)} \approx T_{\pm}\exp\left(-i\tilde{z}\sqrt{\tilde{n}^2-\tilde{\chi}^2}\right),$$
$$E_x \approx \frac{1}{2}\left[T_+\exp\left(-i\tilde{z}\sqrt{\tilde{n}^2-\tilde{\chi}^2} - i\tilde{\kappa}\tilde{z}\right) + T_-\exp\left(-i\tilde{z}\sqrt{\tilde{n}^2-\tilde{\chi}^2} + i\tilde{\kappa}\tilde{z}\right)\right],$$
$$E_y \approx \frac{1}{2i}\left[T_+\exp\left(-i\tilde{z}\sqrt{\tilde{n}^2-\tilde{\chi}^2} - i\tilde{\kappa}\tilde{z}\right) - T_-\exp\left(-i\tilde{z}\sqrt{\tilde{n}^2-\tilde{\chi}^2} + i\tilde{\kappa}\tilde{z}\right)\right].$$

Hence, the wave transmitted through the transition layer is

$$\vec{E}_{\mathrm{tr}} = \vec{E}_{\mathrm{tr}}^r + \vec{E}_{\mathrm{tr}}^l, \tag{1.162}$$

where $\vec{E}_{\mathrm{tr}}^r = 0.5T^r(\vec{x} - i\vec{y})\exp(-ik^+\tilde{z})$ is the right-hand circularly polarized wave, while $\vec{E}_{\mathrm{tr}}^l = 0.5T^l(\vec{x} + i\vec{y})\exp(-ik^-\tilde{z})$ is the left-hand circularly polarized wave with the propagation constants $k^{\pm} = \sqrt{\tilde{n}^2-\tilde{\chi}^2} \pm \tilde{\kappa}$, $T^r = T_+$, $T^l = T_-$. It is easy to verify that

$$\frac{T^r}{T^l} = \frac{1 + \tilde{n}\left(\sqrt{1-\underset{\sim}{\tilde{\chi}}^2} + i\,\underset{\sim}{\tilde{\chi}}\right)}{1 + \tilde{n}\left(\sqrt{1-\underset{\sim}{\tilde{\chi}}^2} - i\,\underset{\sim}{\tilde{\chi}}\right)} = \frac{1 + \tilde{n}\exp(i\theta)}{1 + \tilde{n}\exp(-i\theta)} = \exp(i\psi_{\mathrm{tr}}), \tag{1.163}$$

where $\underset{\sim}{\tilde{\chi}} = \tilde{\chi}/\tilde{n} = \sin\theta$, $|\theta| \le \pi/2$, and

$$\psi_{\mathrm{tr}} = \mathrm{arctg}\,\frac{2\tilde{n}\sin\theta + \tilde{n}^2\sin 2\theta}{1 + 2\tilde{n}\cos\theta + \tilde{n}^2\cos 2\theta}, \quad |\psi_{\mathrm{tr}}| \le \pi - \mathrm{arctg}\,\frac{2\tilde{n}}{\tilde{n}^2 - 1}. \tag{1.164}$$

Thus, regardless of the width of the transition layer Δ, the two waves, $\vec{E}_{\mathrm{tr}}^r$ and $\vec{E}_{\mathrm{tr}}^l$, into which the primary wave is split (when transmitting into the biisotropic medium), have the amplitudes equal in absolute values and shifted in phase by ψ_{tr}. By representing T^l in the form

$$T^l = \frac{\Gamma\left(\Delta\left[\tilde{\chi} - i\left(1 + \sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)\right]\right)}{\Gamma\left(\Delta\left[\tilde{\chi} + i\left(1 + \sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)\right]\right)} \frac{\Gamma\left(2i\Delta\sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)}{\Gamma(-2i\Delta)} \frac{\sin\left(2\pi i\Delta\sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)}{\sin\left\{\pi\Delta\left[\tilde{\chi} + i\left(1 + \sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)\right]\right\}}, \tag{1.165}$$

we find that

$$|T^r| = \left|T^l\right| = \left|\frac{\Gamma\left(2i\Delta\sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)}{\Gamma(-2i\Delta)}\right| \cdot \left|\frac{\sin\left(2\pi i\Delta\sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)}{\sin\left\{\pi\Delta\left[\tilde{\chi} + i\left(1 + \sqrt{\tilde{n}^2-\tilde{\chi}^2}\right)\right]\right\}}\right|, \tag{1.166}$$

and with no reflection (see (1.160)) we get $|T^r| = |T^l| = 1$. From notions of $\vec{E}^r_{\rm tr}$ and $\vec{E}^l_{\rm tr}$, we can see that in this case all the energy of the primary wave is distributed between these two waves.

Thus, the isotropic Epstein transition layer has been generalized to the case of biisotropic medium. We have also found the explicit analytical solution to the problem of a linearly polarized wave normally incident onto the Epstein layer. The main results are as follows.

- The reflected wave does not depend on the chirality of the medium and has, regardless of the width Δ, the polarization shifted by an angle of $\varphi_{\rm ref}$ as compared to the case of the isotropic half-space with the same refraction index $n = \tilde{n}$.
- In a biisotropic half-space, the transmitted wave is split into the right-hand and the left-hand circularly polarized waves that are equal in amplitude and shifted in phase by an angle of $\psi_{\rm tr}$.
- Regimes with zero reflection coefficients, which occur only in the case of the nonreciprocal medium ($\chi \neq 0$) with a diffuse boundary ($\Delta \neq 0$), have been revealed. They are determined by the following relationships between the nonreciprocity index $\tilde{\chi} = \tilde{\chi}_0$, the refraction index $\tilde{n} = \tilde{n}_0$, and the effective width of the transition layer $\Delta = \Delta_0 : \tilde{\chi}_0 = \sqrt{\tilde{n}_0^2 - 1}$, $\tilde{\chi}_0 \Delta_0 = m$, $m = 1, 2, \ldots$

1.7 Negative Refraction in Isotropic Double-Negative Media

1.7.1 Negative Refraction Phenomenon in Homogeneous Double-Negative Media

In recent years, a growing number of publications have analyzed the unusual effects in the propagation of electromagnetic waves in the isotropic media with negative relative permittivity ε and permeability μ—the so-called double negative (DNG) or *left-handed media*. One such effect is the so-called *negative refraction* (NR), in which the beam refracted in a DNG medium lies in the plane of incidence on the same side of the normal to the interface, as the incident beam. At the same time, the wave vector of the transmitted wave is directed towards the interface. Since there is no isotropic media with $\varepsilon < 0$, $\mu < 0$ in the natural environment, then, in the experiments, the artificial composite materials in the form of three-dimensional periodic structures [81] are used as DNG media. As is well known [82], when an electromagnetic wave, whose wavelength is comparable to the period, is propagating through a periodic medium, the NR effect may also occur. In this case, it is impossible to introduce the effective permeabilities of the medium. Since in the experiments [83, 84] revealed the NR, the values of ε and μ, as well as the wavelength inside the material, were not determined directly and were assessed implicitly, the authors of some works expressed doubt [85] about that this effect is inherent in a continuous isotropic medium with negative permittivity rather that it is caused by the periodicity of the material.

In this section, following the approach outlined in [86], we explore the possibility that the NR effect occurs in isotropic media. A model is suggested of an inhomogeneous isotropic flat-layered lossless medium comprising spatial regions with conventional and DNG media and smooth, monotonic transition between them. The analytical description of the plane electromagnetic wave propagating through such a medium is found, which demonstrates the NR effect in the region occupied by a DNG isotropic medium. For the first time, this is shown without any additional assumptions, as a direct consequence of Maxwell's equations and the energy conservation law. In addition, letting the size of the transition region to zero, we verify the conditions on a sharp interface between the conventional and the DNG homogeneous media. The proposed model has allowed us to obtain for the first time the accurate description for the electromagnetic field distribution in the vicinity of the point at which the medium permeabilities ε and μ are zero.

When an electromagnetic wave is passing from a conventional medium to a DNG medium, the NR effect can be seen from the standard *Fresnel formulas*, if we *assume* that they remain valid in the case where one of the media is DNG. If the $E-$polarized wave $\vec{E}^i = \exp\left(i\vec{k}_2\vec{r} - i\omega t\right)\vec{y}$, $\vec{k}_2 = \{k_2 \sin\theta_0, 0, -k_2\cos\theta_0\}$, $\vec{r} = x\vec{x} + y\vec{y} + z\vec{z}$ is incident from the half-space $z > 0$ with the relative permeabilities $\varepsilon_2 > 0$ and $\mu_2 > 0$ at an angle θ_0, the wave transmitted into the half-space $z<0$ with the relative permeabilities $\varepsilon_1 > \varepsilon_2$, $\mu_1 > \mu_2$ has the form [87]

$$\vec{E}^t = \frac{2\mu_1 k_2 \cos\theta_0}{\mu_1 k_2 \cos\theta_0 + \mu_2\sqrt{k_1^2 - k_2^2\sin^2\theta_0}} \exp\left(i\vec{k}_1\vec{r} - i\omega t\right)\vec{y}. \quad (1.167)$$

Its wave vector is

$$\vec{k}_1 = \left\{k_2\sin\theta_0, 0, -\sqrt{k_1^2 - k_2^2\sin^2\theta_0}\right\}; \quad k_j^2 = \omega^2\varepsilon_0\mu_0\varepsilon_j\mu_j, \quad j = 1, 2, \quad (1.168)$$

while the average energy flux is

$$\vec{\Pi}_1 = \left|\vec{E}^t\right|^2 \vec{k}_1 \Big/ 2\mu_1\mu_0\omega. \quad (1.169)$$

In the denominator of (1.167) we have the sum of two positive values.

Let us pass to the case where $\varepsilon_1 < 0$ and $\mu_1 < 0$. Then the first term in the denominator in (1.167) will be negative; and for the denominator not to be zero, we should choose the second branch of the square root, that is replace $\sqrt{k_1^2 - k_2^2\sin^2\theta_0}$ by $-\sqrt{k_1^2 - k_2^2\sin^2\theta_0}$. At the same time, as it seen from (1.168), (1.169), the signs of the longitudinal component of $\vec{k}_1$ and the transversal component of $\vec{\Pi}_1$ change.

Thus, assuming that the Fresnel formulas for DNG media are valid, we arrive at the NR effect. However, this assumption is not obvious, since the boundary conditions for a pair of conventional media and the radiation condition suggesting that the wave vector of the transmitted wave is directed away from the interface are used in the derivation of these formulas from Maxwell's equations.

To avoid any suggestion, one should consider a medium without sharp boundaries, with the permeabilities being smooth (analytic) functions of the spatial variable and changing from positive to negative values.

1.7.2 A Model of Smoothly Inhomogeneous Flat-Layered Double Negative Medium. Solution of the Problem of Transmission of a Plane Wave

The propagation of electromagnetic waves in an inhomogeneous isotropic stratified lossless medium with the permeabilities $\varepsilon(z)$ and $\mu(z)$ is described for E-polarization by the equations:

$$\begin{gathered}\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial z^2}+\left(\mu\frac{d}{dz}\frac{1}{\mu}\right)\frac{\partial E_y}{\partial z}+\omega^2\varepsilon\varepsilon_0\mu\mu_0 E_y=0,\\ \frac{\partial E_y}{\partial z}=-i\omega\mu\mu_0 H_x,\quad \frac{\partial E_y}{\partial x}=i\omega\mu\mu_0 H_z,\quad E_x=E_z=H_y=0.\end{gathered} \tag{1.170}$$

The substitutions $\vec{E}\to\vec{H}$, $\vec{H}\to-\vec{E}$, $\varepsilon\varepsilon_0\to\mu\mu_0$, $\mu\mu_0\to\varepsilon\varepsilon_0$ in (1.170) yield the corresponding equations for H-polarization.

A plane E-polarized wave in such a medium can be represented as

$$E_y=Z(z)\exp(i\kappa_\infty x-i\omega t). \tag{1.171}$$

It follows from (1.170) that the unknown amplitude function $Z(z)$ is the solution of the following equation

$$\frac{d^2 Z}{dz^2}-\frac{1}{\mu}\frac{d\mu}{dz}\frac{dZ}{dz}+\left(\omega^2\varepsilon\varepsilon_0\mu\mu_0-\kappa_\infty^2\right)Z=0;\quad -\infty<z<\infty, \tag{1.172}$$

with the evident condition $z\to+\infty$. Here $\kappa_\infty=k_\infty\sin\theta$, $k_\infty=\omega\sqrt{\varepsilon_\infty\varepsilon_0\mu_\infty\mu_0}$, θ is the angle of incidence of the primary wave, $\varepsilon_\infty=\varepsilon(\infty)>0$, $\mu_\infty=\mu(\infty)>0$.

In order to describe a smooth transition from the conventional medium ($\varepsilon>0$, $\mu>0$ for $z>0$) to the DNG medium ($\varepsilon<0$, $\mu<0$ for $z<0$), consider the following distribution of the permeabilities:

$$\varepsilon=\varepsilon_\infty\alpha(z),\quad \mu=\mu_\infty\alpha(z),\quad \alpha(z)=\mathrm{th}(z/\Delta), \tag{1.173}$$

where the parameter $\Delta>0$ defines the width of the transition region in the vicinity of the point $z=0$. With the help of the substitution

$$Z(z) = \xi^q u(\xi), \quad \xi = \mathrm{ch}^{-2}(z/\Delta), \quad q = \frac{1}{2} ik_\infty \Delta \cos\theta, \tag{1.174}$$

(1.172) is rearranged to the hypergeometric equation

$$\xi(1-\xi)\frac{d^2u}{d\xi^2} + [c - (a+b+1)\xi]\frac{du}{d\xi} - abu = 0 \tag{1.175}$$

with the parameters

$$a = \frac{1}{2} ik_\infty \Delta(1 + \cos\theta), \quad b = -\frac{1}{2} ik_\infty \Delta(1 - \cos\theta), \quad c = 1 + ik_\infty \Delta \cos\theta. \tag{1.176}$$

The function $\xi(z)$ maps the strip $|\mathrm{Im}z| < \pi\Delta/2$ of the complex plane z onto a double-sheeted Riemann surface of the complex variable ξ with the branch points $\xi = 0$ and $\xi = 1$. At the same time, the real semiaxis $(-\infty < z < 0)$ is mapped onto a segment $(0 < \xi < 1, \arg(1-\xi) = 2\pi)$ of the first sheet, while the semiaxis $0 < z < \infty$ is mapped onto a segment $(1 > \xi > 0, \ \arg(1-\xi) = 0)$ of the second sheet.

Following the general theory of hypergeometric equations [70], we find the desired solution of (1.175). It is known that the points $\xi = 0, 1, \infty$ are the singular points of this equation, in the vicinity of which the standard pairs of its linearly independent solutions are determined: u_1 and u_5, u_2 and u_6, u_3 and u_4, respectively.

Let us choose as a solution of (1.175) in the vicinity of the point $\xi = 0$ of the first sheet of the Riemann surface (this point corresponds to the value $z = -\Delta\ln\left[\left(1+\sqrt{1-\xi}\right)/\sqrt{\xi}\right]\big|_{\xi=0} = -\infty$) the function [70]

$$u_1 = F(a,b,c,\xi) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!}\xi^n; \quad |\xi| \le 1. \tag{1.177}$$

The alternative, with a choice of the function u_5 as a solution of (1.175) in the vicinity of the point $\xi = 0$ will be discussed below.

To obtain the solution of (1.175), and, therefore, in view of (1.174), of (1.172) as well, on the entire axis $-\infty < z < \infty$, one should perform the following steps: (i) to continue analytically, on the first sheet of the Riemann surface, the function $u_1(\xi)$ from the neighborhood of the point $\xi = 0$ to the neighborhood of the point $\xi = 1$; (ii) to go onto the second sheet in this neighborhood; (iii) to perform the analytic continuation on this sheet into a vicinity of the point $\xi = 0$.

Since for $z \to -\infty$ we have $\xi \approx 4\exp(2z/\Delta) \approx 0$, $u_1 \approx 1$, then the function

$$Z \approx \xi^q \approx 4^q \exp(ik_\infty z \cos\theta) \tag{1.178}$$

in view of (1.171), will describe the field that is a plane wave, whose phase velocity is directed towards positive z as $z \to -\infty$. For the analysis of the field at small $|z|$, as seen from (1.174), the function $u_1(\xi)$ must be analytically continued into a neighborhood of the point $\xi = 1$. Given that the parameters (1.176) are related by the equation $c - a - b = 1$, to do this, one should use the equality [70]

$$F(a, b, a+b+1, \xi) = \frac{\Gamma(c)}{\Gamma(a+1)\Gamma(b+1)} - \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-\xi)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(n+1)_n n!} \times \left[h_n'' - \ln(1-\xi)\right](1-\xi)^n; \quad |1-\xi| < 1, \tag{1.179}$$

where $h_n'' = \psi(n+1) + \psi(n+2) - \psi(a+n+1) - \psi(b+n+1)$, $\Gamma(\ldots)$ is the gamma function and $\psi(x)$ is the logarithmic derivative of the gamma function. Hence, it follows that when the point $\xi = 1$ is passed around once in the negative direction $(1-\xi) \to (1-\xi)\exp(-2\pi i)$, the following transformation occurs:

$$u_1(\xi) \to \tilde{u}_1(\xi) = u_1(\xi) - 2\pi i a b u_0 u_6(\xi); \quad |1-\xi| < 1, \tag{1.180}$$

where $\tilde{u}_1(\xi)$ stands for the values of the solution $u_1(\xi)$ on the second sheet,

$$u_0 = \frac{\Gamma(c)}{\Gamma(a+1)\Gamma(b+1)}, \quad \text{and} \quad u_6(\xi) = (1-\xi)F(a+1, b+1, 2, 1-\xi).$$

Now continue the function $\tilde{u}_1(\xi)$ from the vicinity of the point $\xi = 1$ on the second sheet into the vicinity of the point $\xi = 0$, using the following Kummer's relationship [70]

$$u_6 = \Gamma_{61}u_1 + \Gamma_{65}u_5, \tag{1.181}$$

where

$$\Gamma_{61} = \frac{\Gamma(c+1-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}, \quad \Gamma_{65} = \frac{\Gamma(c+1-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)}.$$

Upon the substitution of (1.181) into (1.180) we obtain for $|\xi| < 1$ on the second sheet of the *Riemann surface*:

$$\tilde{u}_1 = B_{11}u_1 + B_{15}u_5, \tag{1.182}$$

where

$$u_5 = \xi^{1-c} F(-a, -b, 2-c, \xi), \quad B_{11} = \frac{\operatorname{ch}(\pi k_\infty \Delta) - \exp(-\pi k_\infty \Delta \cos\theta)}{\operatorname{sh}(\pi k_\infty \Delta \cos\theta)},$$

$$B_{15} = 2\pi i \frac{(ik_\infty \Delta \cos\theta)\Gamma^2(ik_\infty \Delta \cos\theta)}{(0.5ik_\infty \Delta \sin\theta)^2 \Gamma^2[0.5ik_\infty \Delta(\cos\theta + 1)]\Gamma^2[0.5ik_\infty \Delta(\cos\theta - 1)]}.$$

Taking into account that $\xi \sim 4\exp(-2z/\Delta) \sim 0$ as $z \to +\infty$, from equalities (1.174) and (1.182) we get

$$Z = \xi^q (B_{11} u_1 + B_{15} u_5) \approx 4^q B_{11} \exp(-ik_\infty z \cos\theta) + 4^{1-c+q} B_{15} \exp(ik_\infty z \cos\theta). \tag{1.183}$$

Since for $z \to +\infty$ the medium (1.173) goes into a conventional medium with constant permeabilities ε_∞ and μ_∞, then the first term in (1.183) describes the wave incoming on the transition region while the second term describes the reflected wave. Expressions (1.178) and (1.183) are the principal terms in the expansions for large $|z|$ of the function $Z(z)$ for $z<0$ and $z>0$, respectively. Normalizing this function by the factor at the first exponent in (1.183), we obtain the coefficients of reflection and transmission for the plane wave (1.171) propagating through the transition layer (1.173):

$$R = 4^{1-c} B_{15} B_{11}^{-1}, \quad T = B_{11}^{-1}. \tag{1.184}$$

In view of the known formula $|\Gamma(iy)|^2 = \pi/y\sin(\pi y)$, we find their absolute values:

$$|R| = \frac{\operatorname{ch}(\pi k_\infty \Delta) - \operatorname{ch}(\pi k_\infty \Delta \cos\theta)}{\operatorname{ch}(\pi k_\infty \Delta) - \exp(-\pi k_\infty \Delta \cos\theta)}, \quad |T| = \frac{\operatorname{sh}(\pi k_\infty \Delta \cos\theta)}{\operatorname{ch}(\pi k_\infty \Delta) - \exp(-\pi k_\infty \Delta \cos\theta)}. \tag{1.185}$$

1.7.3 Analysis of the Expressions for Fields

In going to a sharp boundary ($\varDelta \to 0$), the coefficients behave, as they must [81], like $|R| \to 0$, $|T| \to 1$. As seen from (1.178), (1.183) and (1.184), the field components and the *Poynting vector* away from the transition region ($|z| \gg \Delta$) are as follows:

$$E_y = \exp[ik_\infty(-z\cos\theta + x\sin\theta) - i\omega t], \quad H_x = \eta_\infty \cos\theta E_y, \quad H_z = \eta_\infty \sin\theta E_y,$$
$$\vec{\Pi} = \frac{1}{2}\eta_\infty \{\sin\theta, 0, -\cos\theta\}, \quad \eta_\infty = \sqrt{\varepsilon_\infty/\mu_\infty} \tag{1.186}$$

for the incident wave,

$$E_y = R\exp[ik_\infty(z\cos\theta + x\sin\theta) - i\omega t], \quad H_x = -\eta_\infty \cos\theta E_y, \quad H_z = \eta_\infty \sin\theta E_y,$$
$$\vec{\Pi} = \frac{1}{2}\eta_\infty |R|^2 \{\sin\theta, 0, \cos\theta\} \tag{1.187}$$

for the reflected wave, and

$$E_y = T\exp[ik_\infty(z\cos\theta + x\sin\theta) - i\omega t], \quad H_x = \eta_\infty \cos\theta E_y, \quad H_z = -\eta_\infty \sin\theta E_y,$$
$$\vec{\Pi} = \frac{1}{2}\eta_\infty |T|^2 \{-\sin\theta, 0, -\cos\theta\} \tag{1.188}$$

for the transmitted wave.

It is easy to verify that the NR occurs for the wave transmitted into the region with negative ε and μ. The above relations are also valid for all z in the limiting case $\Delta \to 0$ of a sharp interface between the conventional and DNG homogeneous media. It follows from (1.186), (1.187) and (1.188) that the well-known continuity conditions are fulfilled on this boundary for the tangential components of $\vec{E}$ and $\vec{H}$ and for the normal components of $\vec{D}$ and $\vec{B}$.

Now find, using (1.179), the field in the transition region between two media for small $|z|$. Since for $|z| \ll \Delta$ we have:

$$\xi = 1 - (z/\Delta)^2 + O\left[(z/\Delta)^4\right],$$
$$u_1(\xi) = u_0\left\{1 - ab\left[h_0'' - \ln(1-\xi)\right](1-\xi) + O\left[(1-\xi)^2 \ln(1-\xi)\right]\right\},$$
$$\ln(1-\xi) = \ln(z/\Delta)^2 + 2\pi i\{0 \text{ for } z > 0; 1 \text{ for } z < 0\} + O\left[(z/\Delta)^2\right].$$

Then we arrive at the following representations for the field components:

$$\begin{aligned}
E_y &= u_0\left\{1 - \left[abh^\pm + q - ab\ln z^2\right]\left(\frac{z}{\Delta}\right)^2 + O\left[\left(\frac{z}{\Delta}\right)^4 \ln\left(\frac{z}{\Delta}\right)\right]\right\} \\
&\quad \times \exp(i\kappa_\infty x - i\omega t), \\
H_x &= \frac{2u_0}{i\omega\mu_\infty\Delta}\left\{\left[abh^\pm + q - ab\ln z^2\right] + O\left[\left(\frac{z}{\Delta}\right)^2 \ln\left(\frac{z}{\Delta}\right)\right]\right\} \\
&\quad \times \exp(i\kappa_\infty x - i\omega t), \\
H_z &= \eta_\infty\Delta\frac{u_0\sin\theta}{z}\left\{1 - \left[abh^\pm + q - \frac{1}{3} - ab\ln z^2\right]\left(\frac{z}{\Delta}\right)^2 + O\left[\left(\frac{z}{\Delta}\right)^4 \ln\left(\frac{z}{\Delta}\right)\right]\right\} \\
&\quad \times \exp(i\kappa_\infty x - i\omega t),
\end{aligned} \tag{1.189}$$

where

$$h^{\pm} = h_0'' + 2\ln\Delta - 2\pi i \left\{ \begin{array}{c} 0 \text{ for } z > 0 \\ 1 \text{ for } z < 0 \end{array} \right\}.$$

As seen from these expansions, the components of the magnetic field intensity have singularities at $z = 0 : H_x \approx \ln z$, $H_z \approx 1/z$. It is interesting that in the case of oblique incidence of the H-polarized wave onto the conventional flat-layered medium $(\varepsilon(z), \mu = const > 0)$ in a neighborhood of zero of its dielectric permittivity, the respective components of the electric field have the same singularities, as has been shown in [88–90]. These singularities disappear at normal incidence, as well as for all angles θ when passing to a sharp interface. For arbitrary values of the model parameters, this is the case when taking into account the absorption in the medium.

Returning to formula (1.177), we would like to note that if the function u_5, instead of u_1, is chosen as a solution in the vicinity of the point $\xi = 0$, the we get the usual refraction law for a plane wave transmitted into a medium with negative ε and μ. As this takes place, we have the following expressions for absolute values of the reflection and transmission coefficients:

$$|R| = \frac{|\mathrm{ch}(\pi k_\infty \Delta) - \exp(\pi k_\infty \Delta \cos\theta)|}{\mathrm{ch}(\pi k_\infty \Delta) - \mathrm{ch}(\pi k_\infty \Delta \cos\theta)}, \quad |T| = \frac{\mathrm{sh}(\pi k_\infty \Delta \cos\theta)}{\mathrm{ch}(\pi k_\infty \Delta) - \mathrm{ch}(\pi k_\infty \Delta \cos\theta)}.$$

The nonphysical nature of these formulas is evident: at normal incidence, the coefficients become infinite. That is, the selection of the function u_5 results in the usual refraction law, but violates the energy conservation law.

Thus, we have shown the following. There exist two formal solutions of Maxwell's equations that describe the transmission of a plane wave from a conventional to a DNG media. One of them, which corresponds to the conventional refraction of a plane wave, is inconsistent with the energy conservation law and should be disregarded. The other, correct, solution obeying this law corresponds to the NR in the considered medium.

1.8 Distorting Coatings as an Alternative to Masking Coatings

1.8.1 Transformation Optics, Masking Coatings, Distorting Coatings

One of urgent problems in the applied radio physics is the radar camouflage with the help of special electromagnetic materials. In recent years, there was a conceptual and methodological breakthrough in this field [91]. A novel approach to this problem based on the idea of '*wave flow*' was presented in 2006, in the works [92, 93]. Its physical meaning is that a masking coating has to bend the propagation direction of the electromagnetic radiation incident on it and cause the wave to pass round the masked region, after which the initial direction of propagation is restored maintaining the desired phase. Thus, the electromagnetic waves cannot penetrate into the area bounded by this coating; and any object being placed inside it becomes invisible. To find the parameters of such a coating, the method of coordinate transformations is used, which is based on the fact that Maxwell's equations are invariant with respect to arbitrary coordinate transformations, if the permittivity and the permeability are properly redefined. This approach received the name *transformation optics* (TO) [94]. With the help of the TO, a wide range of masking coatings has been studied [91]. The overwhelming majority of the works listed in the review [91] are based on numerical experiments; for three-dimensional models only the case of spherical surfaces has been studied analytically [95–98]. It has been shown that all these coatings are anisotropic gradient materials, whose permittivity and permeability tensor elements are less than unity. The problem of practical realization of such materials is extremely complex and far from being solved [99]. In addition to the inhomogeneity and anisotropy, for a number of important types of coatings, including spherical, the vanishing of the permittivity and permeability components on its inner surface is also required. That is, the surface consists entirely of critical points, which greatly complicates both the analysis of the corresponding electrodynamic problem and the practical implementation of such coatings.

In this section, we investigate the alternative way of the object masking—the distortion of its image, instead of using masking coatings [100].

1.8.2 Radical Distortion of Radar Image by Applying a Special Coating on the Metamaterial Surface

The geometry of the problem is shown in Fig. 1.13. In the spherical coordinate system r, ϑ, ϕ, a horizontal electric dipole is located at the point $\{r, \vartheta, \phi\} = \{b, 0, 0\}$, $b > R_3$; the time dependence is given by $\exp(-i\omega t)$.

Suppose one should construct the coating in the form of a spherical layer $R_2 < r < R_3$ on a perfectly conducting sphere of radius $r = R_2$. Following the TO methodology, let us consider the coordinate transformation

$$\tilde{r} = \frac{R_3 - R_1}{R_3 - R_2}(r - R_2) + R_1 = f(r), \quad \tilde{\vartheta} = \vartheta, \quad \tilde{\phi} = \phi, \tag{1.190}$$

which maps the spherical layer $0 < R_1 < R_2 \le r \le R_3$ onto the spherical layer $R_1 \le \tilde{r} \le R_3$. Under this transformation, Maxwell's equations for a homogeneous isotropic medium with permittivity ε_0 and permeability μ_0 pass into Maxwell's equations for the inhomogeneous anisotropic medium, whose relative permittivity and permeability are defined by the following diagonal tensors [101]:

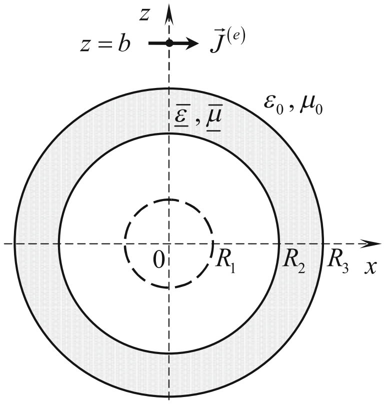


Fig. 1.13 Geometry of the problem

$$\underline{\varepsilon} = \underline{\mu} = diag\left\{Q_{\tilde{\vartheta}}Q_{\tilde{\phi}}\Big/Q_{\tilde{r}}, Q_{\tilde{r}}Q_{\tilde{\phi}}\Big/Q_{\tilde{\vartheta}}, Q_{\tilde{r}}Q_{\tilde{\vartheta}}\Big/Q_{\tilde{\phi}}\right\}, \tag{1.191}$$

where

$$Q_{\tilde{r}} = \frac{h_{\tilde{r}}}{h_r}\frac{\partial\tilde{r}}{\partial r}, \quad Q_{\tilde{\vartheta}} = \frac{h_{\tilde{\vartheta}}}{h_\vartheta}\frac{\partial\tilde{\vartheta}}{\partial\vartheta}, \quad Q_{\tilde{\phi}} = \frac{h_{\tilde{\phi}}}{h_\phi}\frac{\partial\tilde{\phi}}{\partial\phi},$$

and $h_r = 1$, $h_\vartheta = r$, $h_\phi = r\sin\vartheta$, $h_{\tilde{r}} = 1$, $h_{\tilde{\vartheta}} = \tilde{r}$, $h_{\tilde{\phi}} = \tilde{r}\sin\tilde{\vartheta}$.

Thus we get for the permittivity and the permeability:

$$\begin{aligned} &\underline{\varepsilon} = \underline{\mu} = diag\{\alpha_{rr}, \alpha_{\vartheta\vartheta}, \alpha_{\phi\phi}\}, \\ &\alpha_{rr} = \frac{R_3 - R_1}{R_3 - R_2}\left(1 - \frac{R_2 - R_1}{R_3 - R_1}\frac{R_3}{r}\right)^2, \quad \alpha_{\vartheta\vartheta} = \alpha_{\phi\phi} = \frac{R_3 - R_1}{R_3 - R_2}. \end{aligned} \tag{1.192}$$

It is easily seen that $\underline{\varepsilon}$ and $\underline{\mu}$ do not vanish in the layer $R_2 \le r \le R_3$.

Now we use the expansion in vector spherical harmonics $\vec{Y}_{lm}^{(j)}(\vartheta, \phi)$, where $j = -1, 0, 1$, to find the fields [102]. In the region $R_3 < r$, the total electromagnetic field is equal to the sum of the field of a horizontal electric dipole

$$\begin{aligned} \vec{E}^i = \frac{J^{(e)}}{4\sqrt{\pi}\,br}\sqrt{\frac{\mu_0}{\varepsilon_0}}\sum_{l=1}^{\infty}\sum_{m=\pm1}\sqrt{2l+1}\Big\{m\zeta_l^{(1)\prime}(k_0b)\Big[\psi_l'(k_0r)\vec{Y}_{lm}^{(1)}(\vartheta, \phi) \\ + \frac{\sqrt{l(l+1)}}{k_0r}\psi_l(k_0r)\vec{Y}_{lm}^{(-1)}(\vartheta, \phi)\Big] - \zeta_l^{(1)}(k_0b)\psi_l(k_0r)\vec{Y}_{\rm lm}^{(0)}(\vartheta, \phi)\Big\}, \end{aligned} \tag{1.193}$$

$$\vec{H}^i = -\frac{J^{(e)}}{4\sqrt{\pi}\, br}\sum_{l=1}^{\infty}\sum_{m=\pm 1}\sqrt{2l+1}\Big\{m\zeta_l^{(1)\prime}(k_0b)\psi_l(k_0r)\vec{Y}_{lm}^{(0)}(\vartheta,\phi) + \zeta_l^{(1)}(k_0b)\left[\psi_l'(k_0r)\vec{Y}_{lm}^{(1)}(\vartheta,\phi)+\frac{\sqrt{l(l+1)}}{k_0r}\psi_l(k_0r)\vec{Y}_{lm}^{(-1)}(\vartheta,\phi)\right]\Big\} \tag{1.194}$$

and the scattered field

$$\vec{E}^s = \frac{1}{r}\sum_{l=1}^{\infty}\sum_{m=\pm 1}\Big\{\tilde{E}_{lm}\zeta_l^{(1)}(k_0r)\vec{Y}_{lm}^{(0)}(\vartheta,\phi) - \tilde{H}_{lm}\left[\sqrt{\frac{\mu_0}{\varepsilon_0}}\zeta_l^{(1)\prime}(k_0r)\vec{Y}_{lm}^{(1)}(\vartheta,\phi)+\frac{\sqrt{l(l+1)}}{\omega\varepsilon_0 r}\zeta_l^{(1)}(k_0r)\vec{Y}_{lm}^{(-1)}(\vartheta,\phi)\right]\Big\}, \tag{1.195}$$

$$\vec{H}^s = \frac{1}{r}\sum_{l=1}^{\infty}\sum_{m=\pm 1}\Big\{\tilde{H}_{lm}\zeta_l^{(1)}(k_0r)\vec{Y}_{lm}^{(0)}(\vartheta,\phi) + \tilde{E}_{lm}\left[\sqrt{\frac{\varepsilon_0}{\mu_0}}\zeta_l^{(1)\prime}(k_0r)\vec{Y}_{lm}^{(1)}(\vartheta,\phi)+\frac{\sqrt{l(l+1)}}{\omega\mu_0 r}\zeta_l^{(1)}(k_0r)\vec{Y}_{lm}^{(-1)}(\vartheta,\phi)\right]\Big\}. \tag{1.196}$$

Here, $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$, $J^{(e)}$ is the amplitude of the elementary electric current, $\zeta_l^{(1)\prime}(k_0b) = d\zeta_l^{(1)}(x)\big/dx\big|_{x=k_0b}$, $\psi_l'(k_0r) = d\psi_l(x)/dx|_{x=k_0r}$. The *Riccati-Bessel functions* [103] can be expressed in terms of the cylindrical functions as

$$\psi_l(x) = \sqrt{\pi x/2}\, J_{l+1/2}(x), \quad \zeta_l^{(1)}(x) = \sqrt{\pi x/2}\, H_{l+1/2}^{(1)}(x).$$

Formulas (1.193) and (1.194) are given for $r<b$. For $b<r$, one should substitute $\zeta_l^{(1)} \leftrightarrow \psi_l$.

In the anisotropic layer $R_2<r<R_3$, the total field can be represented as the expansion [104]:

$$\vec{E} = \frac{1}{r}\sum_{l=1}^{\infty}\sum_{m=\pm 1}\Big\{\left[\widehat{E}_{lm}^{(1)}f_l^{(1)}(r)+\widehat{E}_{lm}^{(2)}f_l^{(2)}(r)\right]\vec{Y}_{lm}^{(0)}(\vartheta,\phi) - \frac{1}{\omega\varepsilon_t}\left[\widehat{H}_{lm}^{(1)}g_l^{(1)\prime}(r)+\widehat{H}_{lm}^{(2)}g_l^{(2)\prime}(r)\right]\vec{Y}_{lm}^{(1)}(\vartheta,\phi) - \frac{\sqrt{l(l+1)}}{\omega\varepsilon_r r}\left[\widehat{H}_{lm}^{(1)}g_l^{(1)}(r)+\widehat{H}_{lm}^{(2)}g_l^{(2)}(r)\right]\vec{Y}_{lm}^{(-1)}(\vartheta,\phi)\Big\}, \tag{1.197}$$

$$\vec{H} = \frac{1}{r}\sum_{l=1}^{\infty}\sum_{m=\pm 1}\left\{\left[\widehat{H}_{lm}^{(1)} g_l^{(1)}(r) + \widehat{H}_{lm}^{(2)} g_l^{(2)}(r)\right]\vec{Y}_{lm}^{(0)}(\vartheta,\phi)\right.$$
$$+\frac{1}{\omega\mu_t}\left[\widehat{E}_{lm}^{(1)} f_l^{(1)\prime}(r) + \widehat{E}_{lm}^{(2)} f_l^{(2)\prime}(r)\right]\vec{Y}_{lm}^{(1)}(\vartheta,\phi) \tag{1.198}$$
$$\left.+\frac{\sqrt{l(l+1)}}{\omega\mu_r r}\left[\widehat{E}_{lm}^{(1)} f_l^{(1)}(r) + \widehat{E}_{lm}^{(2)} f_l^{(2)}(r)\right]\vec{Y}_{lm}^{(-1)}(\vartheta,\phi)\right\},$$

where $f_l^{(j)}(r)$ are the independent solutions of the equation

$$\mu_r \frac{d}{dr}\frac{1}{\mu_t}\frac{df_l}{dr} + \left[\omega^2\varepsilon_t\mu_r - \frac{l(l+1)}{r^2}\right] f_l = 0, \tag{1.199}$$

while $g_l^{(j)}(r)$ are the independent solutions of the equation

$$\varepsilon_r \frac{d}{dr}\frac{1}{\varepsilon_t}\frac{dg_l}{dr} + \left[\omega^2\varepsilon_r\mu_t - \frac{l(l+1)}{r^2}\right] g_l = 0; \tag{1.200}$$
$$\varepsilon_r = \varepsilon_0\alpha_{rr}, \quad \varepsilon_t = \varepsilon_0\alpha_{\vartheta\vartheta} = \varepsilon_0\alpha_{\phi\phi}, \quad \mu_r = \mu_0\alpha_{rr}, \quad \mu_t = \mu_0\alpha_{\vartheta\vartheta} = \mu_0\alpha_{\phi\phi},$$

and $g_l^{(j)\prime}(r) = dg_l^{(j)}(r)\big/dr, f_l^{(j)\prime}(r) = df_l^{(j)}(r)\big/dr$. Taking into account (1.192), one can easily obtain the independent solutions of (1.199) and (1.200):

$$f_l^{(1)} = g_l^{(1)} = \zeta_l^{(2)}\left(k_0\widehat{r}\right), \quad f_l^{(2)} = g_l^{(2)} = \zeta_l^{(1)}\left(k_0\widehat{r}\right);$$
$$\widehat{r} = \frac{R_3 - R_1}{R_3 - R_2} r - R_3\frac{R_2 - R_1}{R_3 - R_2}. \tag{1.201}$$

The continuity conditions for the tangential components of the total field on the boundary $r = R_3$ yield:

$$\widehat{H}_{lm}^{(1)} g_l^{(1)\prime}(R_3) + \widehat{H}_{lm}^{(2)} g_l^{(2)\prime}(R_3) = \left[\tilde{H}_{lm}\zeta_l^{(1)\prime}(k_0R_3)\right.$$
$$\left.-\frac{J^{(e)}m}{4\sqrt{\pi}b}\sqrt{2l+1}\zeta_l^{(1)\prime}(k_0b)\psi_l'(k_0R_3)\right]k_0\frac{\varepsilon_t}{\varepsilon_0},$$
$$\widehat{E}_{lm}^{(1)} f_l^{(1)}(R_3) + \widehat{E}_{lm}^{(2)} f_l^{(2)}(R_3) = \tilde{E}_{lm}\zeta_l^{(1)}(k_0R_3) - \frac{J^{(e)}}{4\sqrt{\pi}b}\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{2l+1}\zeta_l^{(1)}(k_0b)\psi_l(k_0R_3),$$
$$\widehat{H}_{lm}^{(1)} g_l^{(1)}(R_3) + \widehat{H}_{lm}^{(2)} g_l^{(2)}(R_3) = \tilde{H}_{lm}\zeta_l^{(1)}(k_0R_3) - \frac{J^{(e)}m}{4\sqrt{\pi}b}\sqrt{2l+1}\zeta_l^{(1)\prime}(k_0b)\psi_l(k_0R_3),$$
$$\widehat{E}_{lm}^{(1)} f_l^{(1)\prime}(R_3) + \widehat{E}_{lm}^{(2)} f_l^{(2)\prime}(R_3) = \left[\tilde{E}_{lm}\zeta_l^{(1)\prime}(k_0R_3)\right.$$
$$\left.-\frac{J^{(e)}}{4\sqrt{\pi}b}\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{2l+1}\zeta_l^{(1)}(k_0b)\psi_l'(k_0R_3)\right]k_0\frac{\mu_t}{\mu_0}.$$

While the conditions on the perfectly conducting sphere $r = R_2$ yield:

$$\widehat{E}^{(1)}_{lm} f^{(1)}_l(R_2) + \widehat{E}^{(2)}_{lm} f^{(2)}_l(R_2) = 0, \quad \widehat{H}^{(1)}_{lm} g^{(1)'}_l(R_2) + \widehat{H}^{(2)}_{lm} g^{(2)'}_l(R_2) = 0.$$

Solving the system of all these equations with respect to the unknown values $\widehat{E}^{(1)}_{lm}$, $\widehat{E}^{(2)}_{lm}$, $\widehat{H}^{(1)}_{lm}$, $\widehat{H}^{(2)}_{lm}$, and $\tilde{E}_{lm}$, $\tilde{H}_{lm}$ we obtain, in particular,

$$\tilde{E}_{lm} = \frac{J^{(e)}}{4\sqrt{\pi}b}\sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{2l+1}\,\zeta^{(1)}_l(k_0 b)\frac{\psi_l(k_0R_1)}{\zeta^{(1)}_l(k_0R_1)}, \tag{1.202}$$

$$\tilde{H}_{lm} = \frac{J^{(e)}m}{4\sqrt{\pi}b}\sqrt{2l+1}\,\zeta^{(1)\prime}_l(k_0 b)\frac{\psi'_l(k_0R_1)}{\zeta^{(1)\prime}_l(k_0R_1)}. \tag{1.203}$$

Substituting these values into (1.195), (1.196) gives the expression for the scattered field that results from the interaction of a horizontal electric dipole with a perfectly conducting sphere of radius R_2, coated with a layer of a magneto-dielectric material of thickness $R_3 - R_2$ and with the permittivity and permeability given by (1.192). It is easy to see that outside the layer (that is for $r > R_3$), this field is exactly the same as the field generated by the same source and scattered by the perfectly conducting sphere of radius $R_1 < R_2$ [105].

Thus, by using a perfectly conducting sphere as an example, we have rigorously proved that the application of some special coating onto its surface allows one to obtain the scattered electromagnetic field that will be exactly the same as the field scattered by the perfectly conducting sphere of any smaller radius. At the same time, it is much easier to make such a *distorting coating*, since it does not require the vanishing of certain components of its permittivity and permeability, as in the case of a *masking coating*.

In [106], the authors demonstrate the possibility of a complete *replacement of the image* of the real object with the image of any other object without using the wave flow method (the so-called *illusion optics*). However, this complex procedure, based on the double application of the TO method, can be simulated only numerically.

1.9 Conclusion

In this chapter, analytical solutions have been obtained for the following electromagnetic problems associated with wave propagation.

- *Wave Propagation in a Homogeneous Medium Bounded by a Surface with Variable Impedance*. We proposed a more realistic compared to the known [8] model of electromagnetic wave propagation over a plane surface with impedance that varies smoothly in the given direction; we found the analytic representation for the field generated by a line current located above this plane; the case of rapidly varying impedance function $Z(x)$ (see (1.1)) has been considered

($\tau \gg 2k$); it is shown that the principal term in the asymptotic approximation for the obtained electromagnetic field coincides with the known expression derived in [8] for the case, where the surface impedance changes step-wise.

We constructed the exact Green's function of the Helmholtz equation for a plane waveguide with smoothly varying impedance of its wall. As in the previous problem, the coefficient $Z(x)$ in the boundary condition is an impedance analogue of the permittivity of the Epstein 'transition' layer. The obtained solution was used for the analysis of the field induced by a linear magnetic current in the gradient junction between two regular impedance waveguides. In the limiting case, this solution goes to a well-known expression for the field in the waveguide with the stepped impedance distribution. The error of adiabatic approximation for smoothly irregular waveguides has been estimated. It has been also revealed that there exists a regime with the abnormally efficient transformation of zero fundamental mode into the first mode. The asymptotics, for large dimensions of the transition region, makes it possible to estimate the error of the well-known heuristic approach to the study of the waveguides with slowly varying parameters (the cross-section method).

A model of the irregular circular waveguide of constant cross-section, with variable in azimuth impedance of its wall, has been proposed; it has been found the class of the impedance functions, for which the analytical solution of the excitation problem for such a waveguide is obtained; this solution allowed us to find the cause of the well-known *cycle slipping phenomenon*, which occurs when VLF electromagnetic waves propagate in the Earth-ionosphere waveguide; it is the first exact analytical solution of the excitation problem for the finite irregular waveguide, whose properties vary continuously.

- *Wave Propagation in Inhomogeneous Media*. The problem of the transition radiation in a medium with a diffuse boundary has been formulated; for this problem, a rigorous analytical solution has been obtained for the first time without imposing any restrictions on the model parameters. The limiting transition to the sharp boundary in this solution allowed us to find the precise criterion of boundary sharpness in the form of two inequalities, which essentially clarify the known criterion.

  The known isotropic Epstein transition layer, describing a smooth transition between the regions with different refractive indices n_1 and n_2 in a flat-layered isotropic medium, is extended to the case of biisotropic media. An analytical solution to the problem of a plane electromagnetic wave propagating in such a medium in the normal-to-layer direction has been obtained. The analytical expressions for the reflection and transmission coefficients, which suggest the existence of the total transition mode, are derived.

  A model of a smoothly inhomogeneous isotropic flat-layered medium that involves domains of conventional and double-negative media is proposed. The analytical solution derived for a plane wave travelling through this medium shows that the well-known negative refraction phenomenon in isotropic double-negative medium is a direct consequence of Maxwell's equations and the energy conservation law.

- *Pulsed Radiation from a Line Electric Current near a Planar Interface: a Novel Technique*. A novel technique has been proposed for the analysis of a transient electromagnetic field generated by a pulsed line current that is located near a planar interface between two dielectric nonabsorbing and nondispersive media. As distinct from the Cagniard-de Hoop method, which is widely used for the study of transient fields both in electrodynamics and in the theory of acoustic and seismic waves, our approach is based on the transformation of the domain of integration in the integral expression for the field in the space of two complex variables. As a result, it will suffice to use the standard procedure of finding the roots of the algebraic equation rather than construct auxiliary Carniard's contours. We have represented the field in the form of an integral along a finite contour. The algorithm based on such representation may work as the most effective tool for calculating fields in multilayered media. The suggested method allows extension to the case of arbitrary dipole sources.
- *Radical Distortion of Radar Image Caused by a Special Coating Applied on the Surface of Metamaterial*. We have rigorously proved, by the example of a perfectly conducting sphere, that by applying a special coating on it one can ensure that the scattered electromagnetic field will be exactly the same as the field scattered by a perfectly conducting sphere of any given smaller radius. At the same time, it is much easier to make such a *distorting coating*, since it does not require vanishing of certain components of its permittivity and permeability, as in the case of a *masking coating*.

  Another two papers need to be mentioned. In [107], for a quasi-homogeneous random medium with the dispersion varying in some direction as hyperbolic tangent, the average Green's function is obtained as an exact solution of Dyson's equation in the bilocal approximation. The coherent part of the plane wave, which is incident on a bounded, randomly fluctuating medium with a diffuse boundary, is studied in detail. It is shown that in the case of small-scale fluctuations such a medium is a random analogue of the transient Epstein layer. Paper [108] is devoted to the study of the radiation from a uniformly moving charge in the nonstationary medium, whose time dependence of the permittivity is given by the formula similar to that for the symmetric Epstein layer: $\varepsilon(t) = \varepsilon_0\left[1 + \alpha/\left[\mathrm{ch}^2(t/\Delta t) - \alpha\right]\right]$.

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