Chapter 4
Three-Dimensional Continuum Mechanics

The objective of continuum mechanics is to develop mathematical models to analyze the behavior of idealized three-dimensional bodies. The idealization is related to the hypothesis of a continuum, that is the matter is continuously distributed and fills the entire region of a body, e.g. Haupt (2002). The continuum mechanics is based on balance equations and assumptions regarding the kinematics of deformation and motion. Inelastic behavior is described by means of constitutive equations which relate multi-axial deformation and stress states. Topological details of microstructure are not considered. Processes associated with microstructural changes like hardening, recovery, ageing and damage can be taken into account by means of hidden or internal state variables and corresponding evolution equations. Various models developed within the continuum mechanics of solids can be applied to the structural analysis in the inelastic range.

The classical continuum mechanics of solids takes into account only translational degrees of freedom for motion of material points. The local mechanical interactions between material points are characterized by forces. Moment interactions are not considered. Furthermore, it is assumed that the stress state at a point in the solid depends only on the deformations and state variables of a vanishingly small volume element surrounding the point. To account for the heterogeneous deformation various extensions to the classical continuum mechanics were proposed. Micropolar theories assume that a material point behaves like a rigid body, i.e. it has translation and rotation degrees of freedom. The mechanical interactions are due to forces and moments. Constitutive equations are formulated for force and moment stress tensors. Micropolar theories of plasticity are presented in Forest et al. (1997), Altenbach and Eremeyev (2012), Eremeyev et al. (2012), Altenbach and Eremeyev (2014), among others. Inelastic deformation process is highly heterogeneous at the microscale and several effects cannot be described by the classical continuum mechanics accurately. For example, the dependence of the yield strength on the mean grain size and on the mean size of precipitates, see Sect. 1.3, are not considered within the classical theories since they do not possess intrinsic length scales. To analyze such effects, phase mixture, non-local and gradient-enhanced continuum theories are developed.

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Examples for phase mixture models of inelastic deformation are presented in Naumenko and Gariboldi (2014), Naumenko et al. (2011). Strain gradient and micromorphic theories are discussed in Fleck and Hutchinson (1997), Gao et al. (1999), Forest (2009). Here a gradient or the rotation (curl) of the inelastic strain are considered as additional degrees of freedom. Non-local and phase field theories of damage and fracture were recently advanced to capture initiation and propagation of cracks in solids (Miehe et al. 2010; Schmitt et al. 2013).

This chapter provides basic equations of the classical three-dimensional continuum mechanics. To keep the presentation brief and transparent many details of mathematical derivations are omitted. Several rules of the direct tensor calculus, tensor analysis and special topics related to the theory of tensor functions and invariants are presented in Appendices A–B.5.

With regard to non-linear continuum mechanics there is a number of textbooks, for example Altenbach (2015), Bertram (2012), Eglit and Hodges (1996), Haupt (2002), Lai et al. (1993), Maugin (2013), Smith (1993), Truesdell and Noll (1992).

4.1 Motion, Derivatives and Deformation

4.1.1 Motion and Derivatives

Let $\mathcal{R}$ be the position vector for a point P in a reference state of a solid, Fig. 4.1 and $\boldsymbol{r}$ be the position vector of this point (designated by P') in the actual configuration. The displacement vector $\boldsymbol{u}$ connects the points P and P', Fig. 4.1. The position vector $\mathcal{R}$ can be parameterized with the Cartesian coordinate system including the orthonormal basis $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and the coordinates X, Y, Z, i.e.

$$\mathcal{R}(X, Y, Z) = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$$

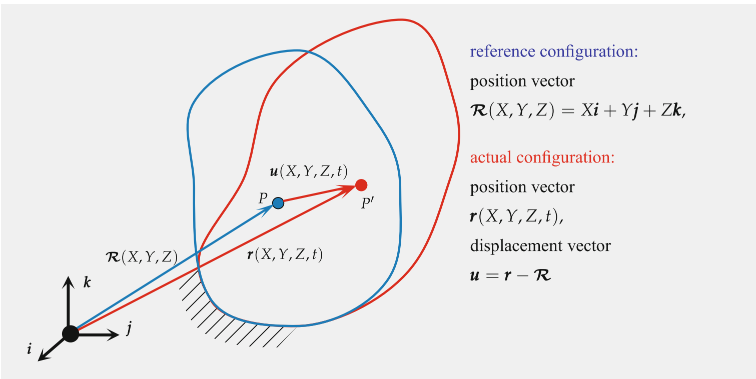

Fig. 4.1 Position vectors and displacement vector

In many cases it is more convenient to use curvilinear coordinates, for example cylindrical, spherical, skew etc. Specifying the curvilinear coordinates by $X^1 = q^1, X^2 = q^2, X^3 = q^3$, see Appendix B.1 the position vector is parameterized as follows

$$\boldsymbol{\mathcal{R}}(X^1, X^2, X^3) = X(X^1, X^2, X^3)\boldsymbol{i} + Y(X^1, X^2, X^3)\boldsymbol{j} + Z(X^1, X^2, X^3)\boldsymbol{k}, \quad (4.1.1)$$

The directed line element in a differential neighborhood of P is

$$\mathrm{d}\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}_i \mathrm{d}X^i, \quad \mathrm{d}X^i = d\boldsymbol{\mathcal{R}} \cdot \boldsymbol{\mathcal{R}}^i, \quad \boldsymbol{\mathcal{R}}_i = \frac{\partial \boldsymbol{\mathcal{R}}}{\partial X^i}, \quad i = 1, 2, 3, \quad (4.1.2)$$

where $\boldsymbol{\mathcal{R}}_i$ is the local basis and $\boldsymbol{\mathcal{R}}^i$ is the dual basis, Appendix B.1. The motion of the continuum is defined by the following mapping

$$\boldsymbol{r} = \boldsymbol{\Phi}(\boldsymbol{\mathcal{R}}, t) \quad (4.1.3)$$

The basic problem of continuum mechanics is to compute the function $\boldsymbol{\Phi}$ for all vectors $\boldsymbol{\mathcal{R}}$ within the body in the reference configuration, for the given time interval $t_0 \leq t \leq t_n$ as well as for given external loads and temperature. It is obvious that $\boldsymbol{\mathcal{R}} = \boldsymbol{\Phi}(\boldsymbol{\mathcal{R}}, t)$. The displacement vector $\boldsymbol{u}$ is defined as follows (Fig. 4.1)

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{\mathcal{R}} \quad (4.1.4)$$

The vector $\boldsymbol{r}$ can be specified with the basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as follows

$$\boldsymbol{r}(X^1, X^2, X^3, t) = x(X^1, X^2, X^3, t)\boldsymbol{i} + y(X^1, X^2, X^3, t)\boldsymbol{j} + z(X^1, X^2, X^3, t)\boldsymbol{k}$$

with the actual Cartesian coordinates x, y, z. The directed line element in a differential neighborhood of P' in the actual configuration is

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{r}_i \mathrm{d}X^i, \quad \mathrm{d}X^i = d\boldsymbol{r} \cdot \boldsymbol{r}^i, \quad \boldsymbol{r}_i = \frac{\partial \boldsymbol{r}}{\partial X^i}, \quad i = 1, 2, 3 \quad (4.1.5)$$

where $\boldsymbol{r}_i$ is the local basis and $\boldsymbol{r}^i$ is the dual basis in the actual configuration.

To analyze the motion it is useful to introduce the rates of change of functions with respect to the coordinates X^i and time t. Consider a tensor-valued function $\boldsymbol{f}(X^i, t)$. The total differential of $\boldsymbol{f}$ for the fixed time variable is

$$\mathrm{d}\boldsymbol{f} = \mathrm{d}X^1 \frac{\partial \boldsymbol{f}}{\partial X^1} + \mathrm{d}X^2 \frac{\partial \boldsymbol{f}}{\partial X^2} + \mathrm{d}X^3 \frac{\partial \boldsymbol{f}}{\partial X^3} = \mathrm{d}X^k \frac{\partial \boldsymbol{f}}{\partial X^k}$$

From Eq. (4.1.2)$_2$ we obtain

$$\mathrm{d}\boldsymbol{f} = \mathrm{d}\boldsymbol{\mathcal{R}} \cdot \boldsymbol{\mathcal{R}}^k \otimes \frac{\partial \boldsymbol{f}}{\partial X^k} = \mathrm{d}\boldsymbol{\mathcal{R}} \cdot \overset{0}{\nabla} \boldsymbol{f} \quad (4.1.6)$$

The operator $\overset{0}{\nabla}$ is the Hamilton (nabla) operator with dual basis vectors of the reference configuration

$$\overset{0}{\nabla} = \mathcal{R}^k \otimes \frac{\partial}{\partial X^k} \tag{4.1.7}$$

With $\boldsymbol{f} = \mathcal{R}$ Eq. (4.1.6) yields

$$\mathrm{d}\mathcal{R} = \mathrm{d}\mathcal{R} \cdot \overset{0}{\nabla} \mathcal{R}, \quad \overset{0}{\nabla} \mathcal{R} = \mathcal{R}^k \otimes \mathcal{R}_k = \boldsymbol{I} \tag{4.1.8}$$

Alternatively, one may use the spatial description by considering $\boldsymbol{f}$ to be the function of $\boldsymbol{r}$ and t. For the fixed time variable we may compute the total differential of $\boldsymbol{f}$ as follows

$$\mathrm{d}\boldsymbol{f} = \mathrm{d}X^k \frac{\partial \boldsymbol{f}}{\partial X^k} = \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{r}^k \otimes \frac{\partial \boldsymbol{f}}{\partial X^k} = \mathrm{d}\boldsymbol{r} \cdot \nabla \boldsymbol{f}, \tag{4.1.9}$$

where

$$\nabla = \boldsymbol{r}^k \otimes \frac{\partial}{\partial X^k} \tag{4.1.10}$$

is the Hamilton (nabla) operator with the dual basis of the actual configuration.

The velocity field $\boldsymbol{v}$ is defined as follows

$$\boldsymbol{v} = \frac{\partial \boldsymbol{\Phi}}{\partial t} = \dot{\boldsymbol{u}} = \dot{\boldsymbol{r}} \tag{4.1.11}$$

The description where $\boldsymbol{f}$ is a function of $\boldsymbol{R}$ and t is sometimes called Lagrangian or material. On the other hand if $\boldsymbol{f}$ is a function of $\boldsymbol{r}$ and t, the description is called Eulerian or spatial. As the mapping $\boldsymbol{\Phi}$ is assumed invertible

$$\mathcal{R} = \boldsymbol{\Phi}^{-1}(\boldsymbol{r}, t), \tag{4.1.12}$$

both the descriptions are equivalent in the sense that if $\boldsymbol{f}$ is known as a function of $\mathcal{R}$ and t, one may use the transformation (4.1.12) to get

$$\boldsymbol{f}(\mathcal{R}, t) = \boldsymbol{g}(\boldsymbol{r}, t)$$

Assuming that both $\boldsymbol{v}$ and $\boldsymbol{f}$ are functions of $\boldsymbol{r}$ and t the material time derivative is defined as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f} = \frac{\partial}{\partial t}\boldsymbol{f} + \boldsymbol{v} \cdot \nabla \boldsymbol{f} \tag{4.1.13}$$

4.1.2 Deformation Gradient and Strain Tensors

Setting $\boldsymbol{f} = \boldsymbol{r}$ into Eq. (4.1.6) we obtain

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}\boldsymbol{\mathcal{R}} \cdot \overset{0}{\nabla} \boldsymbol{r}, \quad \overset{0}{\nabla} \boldsymbol{r} = \boldsymbol{\mathcal{R}}^k \otimes \boldsymbol{r}_k \tag{4.1.14}$$

The second rank tensor

$$\boldsymbol{F} = (\overset{0}{\nabla} \boldsymbol{r})^T = \boldsymbol{r}_k \otimes \boldsymbol{\mathcal{R}}^k \tag{4.1.15}$$

is called deformation gradient. With $\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{\mathcal{R}}$ and (4.1.8) the deformation gradient can be expressed trough the displacement gradient as follows

$$\boldsymbol{F} = \boldsymbol{I} + \overset{0}{\nabla} \boldsymbol{u} \tag{4.1.16}$$

Once the deformation gradient is given, one may find the line element $\mathrm{d}\boldsymbol{r}$ in the differential neighborhood of the point P' of the actual configuration for the given line element $\mathrm{d}\boldsymbol{\mathcal{R}}$ of the reference configuration. Consider three line elements $\mathrm{d}\boldsymbol{\mathcal{R}}_a$, $\mathrm{d}\boldsymbol{\mathcal{R}}_b$ and $\mathrm{d}\boldsymbol{\mathcal{R}}_c$ in the neighborhood of P such that

$$(\mathrm{d}\boldsymbol{\mathcal{R}}_a \times \mathrm{d}\boldsymbol{\mathcal{R}}_b) \cdot \mathrm{d}\boldsymbol{\mathcal{R}}_c = \mathrm{d}V_0 > 0,$$

where $\mathrm{d}V_0$ is the elementary volume of the parallelepiped spanned on $\mathrm{d}\boldsymbol{\mathcal{R}}_a$, $\mathrm{d}\boldsymbol{\mathcal{R}}_b$ and $\mathrm{d}\boldsymbol{\mathcal{R}}_c$. With Eqs. (4.1.15) and (A.4.7) we can compute the elementary volume of the actual configuration

$$\begin{aligned}
\mathrm{d}V &= (\mathrm{d}\boldsymbol{r}_a \times \mathrm{d}\boldsymbol{r}_b) \cdot \mathrm{d}\boldsymbol{r}_c = [(\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{\mathcal{R}}_a) \times (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R}_b)] \cdot \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{\mathcal{R}}_c \\
&= \det \boldsymbol{F} (\mathrm{d}\boldsymbol{\mathcal{R}}_a \times \mathrm{d}\boldsymbol{\mathcal{R}}_b) \cdot \mathrm{d}\boldsymbol{R}_c \\
&= \det \boldsymbol{F} \mathrm{d}V_0
\end{aligned}$$

Hence

$$J = \det \boldsymbol{F} = \frac{\mathrm{d}V}{\mathrm{d}V_0} > 0 \tag{4.1.17}$$

The condition $\det \boldsymbol{F} > 0$ guarantees that the inverse $\boldsymbol{F}^{-1}$ exists. It can be computed as follows

$$\boldsymbol{F}^{-1} = \boldsymbol{\mathcal{R}}_k \otimes \boldsymbol{r}^k$$

Indeed

$$\boldsymbol{F}^{-1} \cdot \boldsymbol{F} = \boldsymbol{\mathcal{R}}_k \otimes \boldsymbol{r}^k \cdot \boldsymbol{r}_i \otimes \boldsymbol{\mathcal{R}}^i = \delta_i^k \boldsymbol{\mathcal{R}}_k \otimes \boldsymbol{\mathcal{R}}^i = \boldsymbol{\mathcal{R}}_k \otimes \boldsymbol{\mathcal{R}}^k = \boldsymbol{I}$$

Consider two line elements $\mathrm{d}\boldsymbol{\mathcal{R}}_a$, $\mathrm{d}\boldsymbol{\mathcal{R}}_b$, $\mathrm{d}\boldsymbol{\mathcal{R}}_a \times \mathrm{d}\boldsymbol{\mathcal{R}}_b \neq \mathbf{0}$ in the neighborhood of P. Let

$$\boldsymbol{N}\mathrm{d}A_0 = \mathrm{d}\boldsymbol{\mathcal{R}}_a \times \mathrm{d}\boldsymbol{\mathcal{R}}_b$$

be the infinitesimal oriented area element including the area of the parallelogram $\mathrm{d}A_0$ having $\mathrm{d}\boldsymbol{\mathcal{R}}_a$ and $\mathrm{d}\boldsymbol{\mathcal{R}}_b$ as sides and the unit normal $\boldsymbol{N}$. With the identity (A.4.8)$_1$ one may compute the corresponding area element in the deformed configuration

$$J\boldsymbol{F}^{-T} \cdot (\mathrm{d}\boldsymbol{\mathcal{R}}_a \times \mathrm{d}\boldsymbol{\mathcal{R}}_b) = (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{\mathcal{R}}_a) \times (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{\mathcal{R}}_b)$$

or

$$J\boldsymbol{F}^{-T} \cdot (\boldsymbol{N}\mathrm{d}A_0) = \boldsymbol{n}\mathrm{d}A \qquad (4.1.18)$$

With the deformation gradient the following relations between nabla operators (4.1.7) and (4.1.10) can be derived

$$\overset{0}{\nabla}(\ldots) = \boldsymbol{\mathcal{R}}^k \otimes \frac{\partial(\ldots)}{\partial X^k} = \boldsymbol{\mathcal{R}}^i \otimes \boldsymbol{r}_i \cdot \boldsymbol{r}^k \otimes \frac{\partial(\ldots)}{\partial X^k} = \boldsymbol{F}^T \cdot \nabla(\ldots) \qquad (4.1.19)$$

As a result we obtain

$$\nabla(\ldots) = \boldsymbol{F}^{-T} \cdot \overset{0}{\nabla}(\ldots), \quad \overset{0}{\nabla}(\ldots) = \boldsymbol{F}^T \cdot \nabla(\ldots) \qquad (4.1.20)$$

Once $\boldsymbol{F}$ is given, one may compute the local strains. To this end consider a line element $\mathrm{d}\boldsymbol{\mathcal{R}}_a = \boldsymbol{M}\mathrm{d}l_{a_0}$ in the neighborhood of the point P, where the unit vector $\boldsymbol{M}$ is the direction of the element (direction of the strain measurement) and $\mathrm{d}l_{a_0}$ is the corresponding length. In the actual configuration $\mathrm{d}\boldsymbol{r}_a = \boldsymbol{m}\mathrm{d}l_a$. In the course of deformation both the orientation and the length of the element are changing. The local stretch and the local normal strain can be computed as follows

$$\lambda_{MM} = \frac{\mathrm{d}l_a}{\mathrm{d}l_{a_0}}, \quad \varepsilon_{MM} = \frac{\mathrm{d}l_a - \mathrm{d}l_{a_0}}{\mathrm{d}l_{a_0}} \qquad (4.1.21)$$

With the given deformation gradient and

$$\mathrm{d}\boldsymbol{r}_a = \boldsymbol{\mathcal{R}}_a \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{\mathcal{R}}_a$$

one may compute

$$\mathrm{d}l_a^2 = \mathrm{d}\boldsymbol{r}_a \cdot \mathrm{d}\boldsymbol{r}_a = \mathrm{d}\boldsymbol{\mathcal{R}}_a \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{\mathcal{R}}_a = (\mathrm{d}l_{a_0})^2 \boldsymbol{M} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{M} \qquad (4.1.22)$$

Hence

$$\lambda_{MM}^2 = (1+\varepsilon_{MM})^2 = \boldsymbol{M}\cdot\boldsymbol{C}\cdot\boldsymbol{M}, \quad \boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F}, \tag{4.1.23}$$

where $\boldsymbol{C}$ is the right Cauchy-Green tensor. For three orthogonal directions specified by the unit vectors $\boldsymbol{e}_X$, $\boldsymbol{e}_Y$ and $\boldsymbol{e}_Z$ Eq. (4.1.23) provides the corresponding stretches

$$\lambda_{XX}^2 = \boldsymbol{e}_X\cdot\boldsymbol{C}\cdot\boldsymbol{e}_X, \quad \lambda_{YY}^2 = \boldsymbol{e}_Y\cdot\boldsymbol{C}\cdot\boldsymbol{e}_Y, \quad \lambda_{ZZ}^2 = \boldsymbol{e}_Z\cdot\boldsymbol{C}\cdot\boldsymbol{e}_Z$$

These equations provide three components of the tensor $\boldsymbol{C}$ with respect to the orthonormal basis. To define the remaining components consider two orthogonal line elements given by the vectors $\mathrm{d}\boldsymbol{\mathcal{R}}_a = \boldsymbol{M}\mathrm{d}l_{a_0}$ and $\mathrm{d}\boldsymbol{\mathcal{R}}_b = \boldsymbol{N}\mathrm{d}l_{b_0}$ in the neighborhood of the point P, where $\boldsymbol{N}$ and $\boldsymbol{M}$, $\boldsymbol{N}\cdot\boldsymbol{M} = 0$ are unit vectors. $\mathrm{d}l_{a_0}$ and $\mathrm{d}l_{b_0}$ are reference lengths of the elements. The corresponding line elements in the actual configuration are $\mathrm{d}\boldsymbol{r}_a = \boldsymbol{m}\mathrm{d}l_a$ and $\mathrm{d}\boldsymbol{r}_b = \boldsymbol{n}\mathrm{d}l_b$. Let α_{MN} be the angle between the vectors $\mathrm{d}\boldsymbol{r}_a$ and $\mathrm{d}\boldsymbol{r}_b$. The local shear strain is defined as $\gamma_{MN} = \frac{\pi}{2} - \alpha_{MN}$. The scalar product of the vectors $\mathrm{d}\boldsymbol{r}_a$ and $\mathrm{d}\boldsymbol{r}_b$ yields

$$\begin{aligned}\mathrm{d}\boldsymbol{r}_a\cdot\mathrm{d}\boldsymbol{r}_b &= \mathrm{d}l_a\mathrm{d}l_b\cos\alpha_{MN} = \mathrm{d}l_a\mathrm{d}l_b\sin\gamma_{MN}\\ &= (1+\varepsilon_{MM})(1+\varepsilon_{NN})\sin\gamma_{np}\mathrm{d}l_{a_0}\mathrm{d}l_{b_0}\end{aligned} \tag{4.1.24}$$

For the given deformation gradient $\boldsymbol{F}$

$$\mathrm{d}\boldsymbol{r}_a = \mathrm{d}\boldsymbol{\mathcal{R}}_a\cdot\boldsymbol{F}^T = \mathrm{d}l_{a_0}\boldsymbol{M}\cdot\boldsymbol{F}^T, \quad \mathrm{d}\boldsymbol{r}_b = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{\mathcal{R}}_b = \mathrm{d}l_{b_0}\boldsymbol{F}\cdot\boldsymbol{N} \tag{4.1.25}$$

The scalar product yields

$$\mathrm{d}\boldsymbol{r}_a\cdot\mathrm{d}\boldsymbol{r}_b = \boldsymbol{M}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\boldsymbol{N}\mathrm{d}l_{a_0}\mathrm{d}l_{b_0}$$

With Eq. (4.1.24) we obtain

$$\lambda_{MM}\lambda_{NN}\sin\gamma_{MN} = (1+\varepsilon_{MM})(1+\varepsilon_{NN})\sin\gamma_{MN} = \boldsymbol{M}\cdot\boldsymbol{C}\cdot\boldsymbol{N} \tag{4.1.26}$$

Equation (4.1.26) provides the MN-component of the tensor $\boldsymbol{C}$. Since $\boldsymbol{M}$ and $\boldsymbol{N}$ are two arbitrary orthogonal unit vectors, one, may compute six components of the tensor $\boldsymbol{C}$ by taking the orthogonal unit vectors $\boldsymbol{e}_X$, $\boldsymbol{e}_Y$ and $\boldsymbol{e}_Z$ as directions of the shear strain measurement. Since the tensor $\boldsymbol{C}$ is symmetric only three of them are independent, i.e.

$$\begin{aligned}\lambda_{XX}\lambda_{YY}\sin\gamma_{XY} &= \boldsymbol{e}_X\cdot\boldsymbol{C}\cdot\boldsymbol{e}_Y,\\ \lambda_{XX}\lambda_{ZZ}\sin\gamma_{XZ} &= \boldsymbol{e}_X\cdot\boldsymbol{C}\cdot\boldsymbol{e}_Z,\\ \lambda_{YY}\lambda_{ZZ}\sin\gamma_{YZ} &= \boldsymbol{e}_Y\cdot\boldsymbol{C}\cdot\boldsymbol{e}_Z\end{aligned}$$

The Cauchy-Green tensor is one example of many strain tensors that can be introduced in the non-linear continuum mechanics. To present several examples let us apply the polar decomposition theorem (see Appendix A.4.18) to the deformation gradient

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R}, \tag{4.1.27}$$

where $\boldsymbol{R}$ is the rotation tensor. $\boldsymbol{U}$ and $\boldsymbol{V}$ are right and left stretch tensors respectively. These positive definite symmetric tensors have the following spectral representations

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i, \quad \boldsymbol{V} = \sum_{i=1}^{3} \lambda_i \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i, \tag{4.1.28}$$

where $\lambda_i > 0$ are principal stretches. The orthonormal unit vectors $\overset{U}{\boldsymbol{N}}_i$ and $\overset{V}{\boldsymbol{n}}_i$ are principal directions of the tensors $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively. From (4.1.27) the following relations can be obtained

$$\overset{V}{\boldsymbol{n}}_i = \boldsymbol{R} \cdot \overset{U}{\boldsymbol{N}}_i, \quad \boldsymbol{R} = \sum_{i=1}^{3} \overset{V}{\boldsymbol{n}}_i \otimes \overset{U}{\boldsymbol{N}}_i \tag{4.1.29}$$

Examples of strain tensors related to $\boldsymbol{U}$ (sometimes called material strain tensors) are the Cauchy-Green strain tensor

$$\boldsymbol{G} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right) = \frac{1}{2}\sum_{i=1}^{3}\left(\lambda_i^2 - 1\right) \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i \tag{4.1.30}$$

the material Biot strain tensor

$$\boldsymbol{E}_{\mathrm{B}} = \boldsymbol{U} - \boldsymbol{I} = \sum_{i=1}^{3} (\lambda_i - 1) \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i \tag{4.1.31}$$

and the material Hencky strain tensor

$$\boldsymbol{H} = \ln \boldsymbol{U} = \sum_{i=1}^{3} \ln \lambda_i \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i \tag{4.1.32}$$

Examples of strain tensors related to $\boldsymbol{V}$ (spatial strain tensors) are the Almansi strain tensor

$$\boldsymbol{E}_{\mathrm{A}} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right) = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{V}^{-2}\right) = \frac{1}{2}\sum_{i=1}^{3}(1 - \lambda_i^{-2}) \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i, \tag{4.1.33}$$

where $\boldsymbol{B} = \boldsymbol{V}^2 = \boldsymbol{F} \cdot \boldsymbol{F}^T$ is the left Cauchy-Green tensor. Further examples are the spacial Biot strain tensor

$$\boldsymbol{E}_{\mathrm{b}} = \boldsymbol{I} - \boldsymbol{V}^{-1} = \sum_{i=1}^{3}(1 - \lambda_i^{-1})\overset{v}{\boldsymbol{n}}_i \otimes \overset{v}{\boldsymbol{n}}_i \tag{4.1.34}$$

and the spacial Hencky strain tensor

$$\boldsymbol{h} = \ln \boldsymbol{V} = \sum_{i=1}^{3} \ln \lambda_i \overset{v}{\boldsymbol{n}}_i \otimes \overset{v}{\boldsymbol{n}}_i \tag{4.1.35}$$

For many structural analysis applications the local strains can be assumed small. With $\varepsilon_{MM} \ll 1$ and $\gamma_{MN} \ll 1$ the left hand side of Eqs. (4.1.23) and (4.1.26) can be linearized as follows

$$\begin{aligned} \lambda_{MM}^2 &= (\varepsilon_{MM} + 1)^2 \approx 2\varepsilon_{MM} + 1, \\ \lambda_{MM}\lambda_{NN} \sin \gamma_{MN} &= (\varepsilon_{MM} + 1)(\varepsilon_{NN} + 1) \sin \gamma_{MN} \approx \gamma_{MN} \end{aligned} \tag{4.1.36}$$

With Eqs. (4.1.23) and (4.1.36) the normal strain in the direction $\boldsymbol{M}$ is

$$\begin{aligned} \varepsilon_{MM} &= \frac{1}{2}\boldsymbol{M} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{M} - \frac{1}{2} \\ &= \frac{1}{2}\boldsymbol{M} \cdot (\boldsymbol{C} - \boldsymbol{I}) \cdot \boldsymbol{M} \end{aligned}$$

The shear strain can be computed as follows

$$\gamma_{MN} = \boldsymbol{M} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{N}$$

With the Green-Lagrange strain tensor

$$\boldsymbol{G} = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}),$$

the strains can be given as follows

$$\varepsilon_{MM} = \boldsymbol{M} \cdot \boldsymbol{G} \cdot \boldsymbol{M}, \quad \gamma_{MN} = 2\boldsymbol{M} \cdot \boldsymbol{G} \cdot \boldsymbol{N}$$

For the given tensor $\boldsymbol{G}$ one may compute the strains with respect to any direction. For three orthogonal directions specified by the unit vectors $\boldsymbol{e}_X$, $\boldsymbol{e}_Y$ and $\boldsymbol{e}_Z$ the six components can be computed as follows

$$\begin{aligned} \varepsilon_{XX} &= \boldsymbol{e}_X \cdot \boldsymbol{G} \cdot \boldsymbol{e}_X, \quad \varepsilon_{YY} = \boldsymbol{e}_Y \cdot \boldsymbol{G} \cdot \boldsymbol{e}_Y, \quad \varepsilon_{ZZ} = \boldsymbol{e}_Z \cdot \boldsymbol{G} \cdot \boldsymbol{e}_Z, \\ \varepsilon_{XY} &= \boldsymbol{e}_X \cdot \boldsymbol{G} \cdot \boldsymbol{e}_Y, \quad \varepsilon_{XZ} = \boldsymbol{e}_X \cdot \boldsymbol{G} \cdot \boldsymbol{e}_Z, \quad \varepsilon_{YZ} = \boldsymbol{e}_Y \cdot \boldsymbol{G} \cdot \boldsymbol{e}_Z \end{aligned}$$

Although the normal and shear strains are assumed small, the difference between the unit vectors like $\boldsymbol{M}$ and $\boldsymbol{m}$ defined in the initial and actual configurations may be essential. To formulate geometrically-linear theory we have additionally to assume infinitesimal rotations.[1] The linearized rotation tensor $\boldsymbol{R}$ can be given as follows

$$\boldsymbol{R} = \boldsymbol{I} + \boldsymbol{\varphi} \times \boldsymbol{I},$$

where $\boldsymbol{\varphi}$ is the vector of infinitesimal rotations. Then with Eqs. (4.1.27) and (4.1.16) the following linearized relations can be established

$$\begin{aligned} &\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{\varepsilon} + \boldsymbol{\varphi} \times \boldsymbol{I}, \quad \overset{0}{\nabla} \boldsymbol{u} = \nabla \boldsymbol{u} = \boldsymbol{\varepsilon} + \boldsymbol{\varphi} \times \boldsymbol{I}, \\ &\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right], \quad \boldsymbol{\varphi} = -\frac{1}{2}\nabla \times \boldsymbol{u} \end{aligned} \tag{4.1.37}$$

The tensor $\boldsymbol{\varepsilon}$ is called tensor of infinitesimal strains.

4.1.3 Velocity Gradient, Deformation Rate, and Spin Tensors

The time derivative of the deformation gradient. can be computed with (4.1.15) as follows

$$\dot{\boldsymbol{F}} = (\overset{0}{\nabla} \dot{\boldsymbol{r}})^{\mathrm{T}} = \dot{\boldsymbol{r}}_k \otimes \boldsymbol{\mathcal{R}}^k \tag{4.1.38}$$

With (4.1.11)

$$\frac{\partial \boldsymbol{v}}{\partial X^k} = \dot{\boldsymbol{r}}_k \tag{4.1.39}$$

Hence

$$\dot{\boldsymbol{F}} = (\overset{0}{\nabla} \boldsymbol{v})^{\mathrm{T}}, \quad \overset{0}{\nabla} \boldsymbol{v} = \boldsymbol{\mathcal{R}}^k \otimes \frac{\partial \boldsymbol{v}}{\partial X^k} \tag{4.1.40}$$

With the relation between nabla operators $(4.1.20)_2$ Eq. (4.1.40) takes the form

$$\dot{\boldsymbol{F}} = (\nabla \boldsymbol{v})^{\mathrm{T}} \cdot \boldsymbol{F} \tag{4.1.41}$$

[1] In many cases strains can be infinitesimal, but rotations finite. One example is a thin plate strip which can be bent into a ring such that the strains remain infinitesimal but cross section rotations are large.

The spatial velocity gradient $\boldsymbol{L} = (\nabla \boldsymbol{v})^{\mathrm{T}}$ can be computed as follows

$$\boldsymbol{L} = (\nabla \boldsymbol{v})^{\mathrm{T}} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{v}}{\partial X^k} \otimes \boldsymbol{r}^k \tag{4.1.42}$$

The tensor $\boldsymbol{L}$ can be additively decomposed into the symmetric and skew symmetric parts (see Appendix A.4.10)

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{\omega} \times \boldsymbol{I}, \tag{4.1.43}$$

where the symmetric part

$$\boldsymbol{D} = \frac{1}{2}\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}}\right]$$

is called the deformation rate tensor[2] while

$$\boldsymbol{\omega} = -\frac{1}{2} \nabla \times \boldsymbol{v}$$

is called vorticity vector.

The time derivative of $J = \det \boldsymbol{F}$ can be computed as follows

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t} \cdot\cdot \left(\frac{\partial J}{\partial \boldsymbol{F}}\right)^{\mathrm{T}}$$

With (B.4.13) we obtain

$$\frac{\partial J}{\partial \boldsymbol{F}} = \det \boldsymbol{F} \boldsymbol{F}^{-\mathrm{T}}$$

Consequently

$$\dot{J} = J \dot{\boldsymbol{F}} \cdot\cdot \boldsymbol{F}^{-1} \tag{4.1.44}$$

Taking the trace of Eq. (4.1.42)

$$\operatorname{tr} \boldsymbol{L} = \nabla \cdot \boldsymbol{v} = \dot{\boldsymbol{F}} \cdot\cdot \boldsymbol{F}^{-1}$$

With Eq. (4.1.44) we obtain

$$\frac{\dot{J}}{J} = \frac{\mathrm{d} \ln J}{\mathrm{d}t} = \nabla \cdot \boldsymbol{v} = \dot{\boldsymbol{F}} \cdot\cdot \boldsymbol{F}^{-1} \tag{4.1.45}$$

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[2]The tensor $\boldsymbol{D}$ is in general not a time derivative of a strain tensor.

Applying the polar decomposition (4.1.27) and the relations

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} \quad \Rightarrow \quad \dot{\boldsymbol{F}} = \dot{\boldsymbol{R}} \cdot \boldsymbol{U} + \dot{\boldsymbol{U}} \cdot \boldsymbol{R} \quad \text{and} \quad \boldsymbol{F}^{-1} = \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^{\mathrm{T}}$$

the velocity gradient can be computed as follows

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}} + \boldsymbol{R} \cdot \dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^{\mathrm{T}} \tag{4.1.46}$$

For the rotation tensor $\boldsymbol{R}$ let us introduce the angular velocity vector $\boldsymbol{\Omega}_{\boldsymbol{R}}$ and the spin tensor $\boldsymbol{\Omega}_{\boldsymbol{R}} \times \boldsymbol{I}$ as follows. According to the definition of the orthogonal tensor, see Appendix A.4.17, we obtain

$$\boldsymbol{R} \cdot \boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I} \quad \Rightarrow \quad \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}} + \boldsymbol{R} \cdot \dot{\boldsymbol{R}}^{\mathrm{T}} = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}} = -(\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}})^{\mathrm{T}}$$

The skew-symmetric tensor $\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}}$ is called the left spin tensor or simply spin tensor. With the associated vector $\boldsymbol{\Omega}_{\boldsymbol{R}}$ we obtain

$$\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}} = \boldsymbol{\Omega}_{\boldsymbol{R}} \times \boldsymbol{I}, \quad \boldsymbol{\Omega}_{\boldsymbol{R}} = -\frac{1}{2}(\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}})_{\times}, \tag{4.1.47}$$

where $(\ldots)_{\times}$ denotes the vector invariant or Gibbs cross of the second rank tensor, see Appendix A.4.15. The vector $\boldsymbol{\Omega}_{\boldsymbol{R}}$ is called the left angular velocity vector of rotation or simply angular velocity of rotation. This vector is widely used in the mechanics of rigid bodies, e.g. Altenbach et al. (2007, 2009), Zhilin (1996). Equation (4.1.46) can be given as follows

$$\boldsymbol{D} + \boldsymbol{\omega} \times \boldsymbol{I} = \boldsymbol{\Omega}_{\boldsymbol{R}} \times \boldsymbol{I} + \boldsymbol{R} \cdot \dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^{\mathrm{T}} \tag{4.1.48}$$

Taking the vector invariant of Eq. (4.1.48) the vorticity vector can be computed as follows

$$\boldsymbol{\omega} = \boldsymbol{\Omega}_{\boldsymbol{R}} - \frac{1}{2}(\boldsymbol{R} \cdot \dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^{\mathrm{T}})_{\times} \tag{4.1.49}$$

The symmetric part of Eq. (4.1.48) is

$$\boldsymbol{D} = \frac{1}{2}\boldsymbol{R} \cdot \boldsymbol{U}^{-1} \cdot \dot{\boldsymbol{U}} \cdot \boldsymbol{R}^{\mathrm{T}} = \frac{1}{2}\boldsymbol{R} \cdot (\dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} + \boldsymbol{U}^{-1} \cdot \dot{\boldsymbol{U}}) \cdot \boldsymbol{R}^{\mathrm{T}} \tag{4.1.50}$$

Equation (4.1.50) can be put in the following form

$$\boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{D} \cdot \boldsymbol{R} = \frac{1}{2}(\dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} + \boldsymbol{U}^{-1} \cdot \dot{\boldsymbol{U}})$$

or

$$\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{D} \cdot \boldsymbol{F} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{U}^2) = \frac{1}{2}\dot{\boldsymbol{C}} = \dot{\boldsymbol{G}} \tag{4.1.51}$$

Let us take the time derivatives of stretch tensors applying spectral representations (4.1.28)

$$\begin{aligned}
\dot{\boldsymbol{U}} &= \sum_{i=1}^{3} \left(\dot{\lambda}_i \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i + \lambda_i \frac{\mathrm{d}}{\mathrm{d}t}\overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i + \lambda_i \overset{U}{\boldsymbol{N}}_i \otimes \frac{\mathrm{d}}{\mathrm{d}t}\overset{U}{\boldsymbol{N}}_i \right), \\
\dot{\boldsymbol{V}} &= \sum_{i=1}^{3} \left(\dot{\lambda}_i \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i + \lambda_i \frac{\mathrm{d}}{\mathrm{d}t}\overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i + \lambda_i \overset{V}{\boldsymbol{n}}_i \otimes \frac{\mathrm{d}}{\mathrm{d}t}\overset{V}{\boldsymbol{n}}_i \right)
\end{aligned} \tag{4.1.52}$$

Consider a triple of fixed orthogonal unit vectors $\boldsymbol{e}_i$ and the rotation tensor $\boldsymbol{P}_U$ such that

$$\overset{U}{\boldsymbol{N}}_i = \boldsymbol{P}_U \cdot \boldsymbol{e}_i$$

Hence

$$\overset{V}{\boldsymbol{n}}_i = \boldsymbol{R} \cdot \boldsymbol{P}_U \cdot \boldsymbol{e}_i$$

or

$$\overset{V}{\boldsymbol{n}}_i = \boldsymbol{P}_V \cdot \boldsymbol{e}_i, \quad \boldsymbol{P}_V = \boldsymbol{R} \cdot \boldsymbol{P}_U \tag{4.1.53}$$

For the rotation tensors $\boldsymbol{P}_U$ and $\boldsymbol{P}_V$ the spin tensors and the angular velocity vectors can be introduced as follows

$$\begin{aligned}
\dot{\boldsymbol{P}}_U \cdot \boldsymbol{P}_U^{\mathrm{T}} &= \boldsymbol{\Omega}_U \times \boldsymbol{I}, \quad \dot{\boldsymbol{P}}_U = \boldsymbol{\Omega}_U \times \boldsymbol{P}_U, \\
\dot{\boldsymbol{P}}_V \cdot \boldsymbol{P}_V^{\mathrm{T}} &= \boldsymbol{\Omega}_V \times \boldsymbol{I}, \quad \dot{\boldsymbol{P}}_V = \boldsymbol{\Omega}_V \times \boldsymbol{P}_V
\end{aligned} \tag{4.1.54}$$

The time derivative of Eq. (4.1.53)$_2$ yields

$$\begin{aligned}
\dot{\boldsymbol{P}}_V &= \dot{\boldsymbol{R}} \cdot \boldsymbol{P}_U + \boldsymbol{R} \cdot \dot{\boldsymbol{P}}_U \\
&= \boldsymbol{\Omega}_R \times \boldsymbol{R} \cdot \boldsymbol{P}_U + \boldsymbol{R} \cdot (\boldsymbol{\Omega}_U \times \boldsymbol{I}) \cdot \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{P}_U \\
&= (\boldsymbol{\Omega}_R + \boldsymbol{R} \cdot \boldsymbol{\Omega}_U) \times \boldsymbol{P}_V
\end{aligned}$$

Hence the following relationship between the angular velocity vectors can be established

$$\boldsymbol{\Omega}_V = \boldsymbol{\Omega}_R + \boldsymbol{R} \cdot \boldsymbol{\Omega}_U \tag{4.1.55}$$

With Eqs. (4.1.54) the rates of change of principal directions can be computed as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\overset{U}{\boldsymbol{N}}_i = \boldsymbol{\Omega}_{\boldsymbol{U}} \times \overset{U}{\boldsymbol{N}}_i, \quad \frac{\mathrm{d}}{\mathrm{d}t}\overset{V}{\boldsymbol{n}}_i = \boldsymbol{\Omega}_{\boldsymbol{V}} \times \overset{V}{\boldsymbol{n}}_i$$

Consequently the rates of change of stretch tensors (4.1.52) take the following form

$$\begin{aligned} \dot{\boldsymbol{U}} &= \sum_{i=1}^{3} \dot{\lambda}_i \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i + \boldsymbol{\Omega}_{\boldsymbol{U}} \times \boldsymbol{U} - \boldsymbol{U} \times \boldsymbol{\Omega}_{\boldsymbol{U}}, \\ \dot{\boldsymbol{V}} &= \sum_{i=1}^{3} \dot{\lambda}_i \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i + \boldsymbol{\Omega}_{\boldsymbol{V}} \times \boldsymbol{V} - \boldsymbol{V} \times \boldsymbol{\Omega}_{\boldsymbol{V}} \end{aligned} \tag{4.1.56}$$

With equation (4.1.56) and

$$\boldsymbol{U}^{-1} = \sum_{i=1}^{3} \frac{1}{\lambda_i} \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i,$$

one may compute

$$\boldsymbol{R} \cdot \dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R} \cdot \left[\sum_{i=1}^{3} \dot{\lambda}_i \lambda_i^{-1} \overset{U}{\boldsymbol{N}}_i \otimes \overset{U}{\boldsymbol{N}}_i + \boldsymbol{\Omega}_{\boldsymbol{U}} \times \boldsymbol{I} - (\boldsymbol{U} \times \boldsymbol{\Omega}_{\boldsymbol{U}}) \cdot \boldsymbol{U} \right] \cdot \boldsymbol{R}^{\mathrm{T}}$$

Applying Eqs. (4.1.53), (4.1.55) it can be simplified as follows

$$\begin{aligned} \boldsymbol{R} \cdot \dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^{\mathrm{T}} = &\sum_{i=1}^{3} \dot{\lambda}_i \lambda_i^{-1} \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i \\ &+ (\boldsymbol{\Omega}_{\boldsymbol{V}} - \boldsymbol{\Omega}_{\boldsymbol{R}}) \times \boldsymbol{I} - \boldsymbol{V} \cdot [(\boldsymbol{\Omega}_{\boldsymbol{V}} - \boldsymbol{\Omega}_{\boldsymbol{R}}) \times \boldsymbol{I}] \cdot \boldsymbol{V}^{-1} \end{aligned} \tag{4.1.57}$$

Taking the vector invariant of Eq. (4.1.57) and applying the identities (A.4.15) and (A.4.16) we obtain

$$(\boldsymbol{R} \cdot \dot{\boldsymbol{U}} \cdot \boldsymbol{U}^{-1} \cdot \boldsymbol{R}^{\mathrm{T}})_{\times} = -2(\boldsymbol{\Omega}_{\boldsymbol{V}} - \boldsymbol{\Omega}_{\boldsymbol{R}}) - \boldsymbol{A}_{\boldsymbol{V}} \cdot (\boldsymbol{\Omega}_{\boldsymbol{V}} - \boldsymbol{\Omega}_{\boldsymbol{R}}), \tag{4.1.58}$$

where

$$\boldsymbol{A}_{\boldsymbol{V}} = \sum_{i_1}^{3} \lambda_i \overset{V}{\boldsymbol{n}}_i \times \boldsymbol{V}^{-1} \times \overset{V}{\boldsymbol{n}}_i = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\lambda_i}{\lambda_j} \overset{V}{\boldsymbol{n}}_i \times \overset{V}{\boldsymbol{n}}_j \otimes \overset{V}{\boldsymbol{n}}_j \times \overset{V}{\boldsymbol{n}}_i$$

According to (A.4.16) and the Cayley-Hamilton theorem the tensor $\boldsymbol{A}_{\boldsymbol{V}}$ has the following representations

$$\boldsymbol{A}_{\boldsymbol{V}} = J^{-1} \boldsymbol{V} \cdot [\boldsymbol{V}^2 - (\operatorname{tr} \boldsymbol{V}^2) \boldsymbol{I}] = \boldsymbol{I} + \frac{J_{2V} - J_{1V}^2}{J} \boldsymbol{V} + \frac{J_{1V}}{J} \boldsymbol{V}^2,$$

where J_{1_V}, J_{2_V} and $J = J_{3_V}$ are principal invariants of the tensor $\boldsymbol{V}$ as defined by Eqs. (A.4.11). The spectral form of the tensor $\boldsymbol{A_V}$ is

$$-\boldsymbol{A_V} = \frac{\lambda_2^2 + \lambda_3^2}{\lambda_2\lambda_3}\overset{V}{\boldsymbol{n}}_1 \otimes \overset{V}{\boldsymbol{n}}_1 + \frac{\lambda_3^2 + \lambda_1^2}{\lambda_3\lambda_1}\overset{V}{\boldsymbol{n}}_2 \otimes \overset{V}{\boldsymbol{n}}_2 + \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1\lambda_2}\overset{V}{\boldsymbol{n}}_3 \otimes \overset{V}{\boldsymbol{n}}_3 \tag{4.1.59}$$

With Eqs. (4.1.49) and (4.1.58) the following relationship between the angular velocities can be obtained

$$\boldsymbol{\omega} = \boldsymbol{\Omega_V} + \frac{1}{2}\boldsymbol{A_V} \cdot (\boldsymbol{\Omega_V} - \boldsymbol{\Omega_R}) \tag{4.1.60}$$

The relationship (4.1.60) can also be derived with the following decomposition

$$\boldsymbol{F} = \boldsymbol{V} \cdot \boldsymbol{R} \quad \Rightarrow \quad \boldsymbol{F}^{-1} = \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{V}^{-1}$$

Therefore the velocity gradient is

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = \dot{\boldsymbol{V}} \cdot \boldsymbol{V}^{-1} + \boldsymbol{V} \cdot \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{V}^{-1} \tag{4.1.61}$$

With Eqs. (4.1.28), (4.1.47) and (4.1.56), Eq. (4.1.61) takes the form

$$\boldsymbol{L} = \sum_{i=1}^{3} \dot{\lambda}_i \lambda_i^{-1} \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i + \boldsymbol{\Omega_V} \times \boldsymbol{I} + \boldsymbol{L_\Omega}, \tag{4.1.62}$$

where

$$\boldsymbol{L_\Omega} = \boldsymbol{V} \cdot (\tilde{\boldsymbol{\Omega}} \times \boldsymbol{I}) \cdot \boldsymbol{V}^{-1}, \quad \tilde{\boldsymbol{\Omega}} = \boldsymbol{\Omega_R} - \boldsymbol{\Omega_V}$$

Taking the vector invariant of Eq. (4.1.62) provides the relationship (4.1.60).

With the identity $(\mathrm{A.4.8})_2$ the tensor $\boldsymbol{L_\Omega}$ can be represented as follows

$$\begin{aligned} \boldsymbol{L_\Omega} &= \boldsymbol{a} \times \boldsymbol{V}^{-2} = \boldsymbol{V}^2 \times \boldsymbol{b}, \\ \boldsymbol{a} &= J\boldsymbol{V}^{-1} \cdot \tilde{\boldsymbol{\Omega}}, \quad \boldsymbol{b} = J^{-1}\boldsymbol{V} \cdot \tilde{\boldsymbol{\Omega}} \end{aligned} \tag{4.1.63}$$

The right dot product of Eq. (4.1.62) with $\boldsymbol{V}^2$ yields

$$\boldsymbol{L} \cdot \boldsymbol{V}^2 = \sum_{i=1}^{3} \dot{\lambda}_i \lambda_i \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i + \boldsymbol{\Omega_V} \times \boldsymbol{V}^2 + \boldsymbol{a} \times \boldsymbol{I}, \tag{4.1.64}$$

With the decomposition of the velocity gradient (4.1.43), Eq. (4.1.64) takes the following form

$$\boldsymbol{D} \cdot \boldsymbol{V}^2 = \sum_{i=1}^{3} \dot{\lambda}_i \lambda_i \overset{V}{\boldsymbol{n}}_i \otimes \overset{V}{\boldsymbol{n}}_i + (\boldsymbol{\Omega_V} - \boldsymbol{\omega}) \times \boldsymbol{V}^2 + \boldsymbol{a} \times \boldsymbol{I} \tag{4.1.65}$$

Taking the vector invariant of Eq. (4.1.65) yields

$$\frac{1}{2J}\boldsymbol{V}\cdot(\boldsymbol{D}\cdot\boldsymbol{V}^2)_\times = \left(\boldsymbol{I}+\frac{1}{2}\boldsymbol{A}_{\boldsymbol{V}}\right)\cdot\boldsymbol{\Omega}_{\boldsymbol{V}} - \frac{1}{2}\boldsymbol{A}_{\boldsymbol{V}}\cdot\boldsymbol{\omega} - \boldsymbol{\Omega}_{\boldsymbol{R}} \tag{4.1.66}$$

From Eqs. (4.1.60) and (4.1.66) we obtain

$$\frac{1}{2J}\boldsymbol{V}\cdot(\boldsymbol{D}\cdot\boldsymbol{V}^2)_\times = \left(\boldsymbol{I}-\frac{1}{2}\boldsymbol{A}_{\boldsymbol{V}}\right)\cdot(\boldsymbol{\omega}-\boldsymbol{\Omega}_{\boldsymbol{R}})$$

With Eq. (4.1.59) one may verify the tensor $\boldsymbol{I} - 1/2\boldsymbol{A}_{\boldsymbol{V}}$ is non-singular. Hence

$$\boldsymbol{\omega}-\boldsymbol{\Omega}_{\boldsymbol{R}} = \boldsymbol{K}_{\boldsymbol{V}}\cdot(\boldsymbol{D}\cdot\boldsymbol{V}^2)_\times, \quad \boldsymbol{K}_{\boldsymbol{V}} = \frac{1}{2J}\left(\boldsymbol{I}-\frac{1}{2}\boldsymbol{A}_{\boldsymbol{V}}\right)^{-1}\cdot\boldsymbol{V} \tag{4.1.67}$$

Applying Eq. (4.1.59) the following spectral representation of the tensor $\boldsymbol{K}_{\boldsymbol{V}}$ can be established

$$\boldsymbol{K}_{\boldsymbol{V}} = \frac{1}{(\lambda_2+\lambda_3)^2}\overset{V}{\boldsymbol{n}}_1\otimes\overset{V}{\boldsymbol{n}}_1 + \frac{1}{(\lambda_3+\lambda_1)^2}\overset{V}{\boldsymbol{n}}_2\otimes\overset{V}{\boldsymbol{n}}_2 + \frac{1}{(\lambda_1+\lambda_2)^2}\overset{V}{\boldsymbol{n}}_3\otimes\overset{V}{\boldsymbol{n}}_3 \tag{4.1.68}$$

Let us relate the tensor $\boldsymbol{D}$ to the time derivative of the Hencky strain tensor $\boldsymbol{h}$. To this end consider the symmetric part of Eq. (4.1.62)

$$\boldsymbol{D} = \sum_{i=1}^{3}\dot{\lambda}_i\lambda_i^{-1}\overset{V}{\boldsymbol{n}}_i\otimes\overset{V}{\boldsymbol{n}}_i + \frac{1}{2}(\boldsymbol{V}^2\times\boldsymbol{b}-\boldsymbol{b}\times\boldsymbol{V}^2) \tag{4.1.69}$$

The time derivative of the Hencky strain tensor (4.1.35) can be computed as follows

$$\dot{\boldsymbol{h}} = \sum_{i=1}^{3}\dot{\lambda}_i\lambda_i^{-1}\overset{V}{\boldsymbol{n}}_i\otimes\overset{V}{\boldsymbol{n}}_i + \boldsymbol{\Omega}_{\boldsymbol{V}}\times\boldsymbol{h} - \boldsymbol{h}\times\boldsymbol{\Omega}_{\boldsymbol{V}} \tag{4.1.70}$$

Inserting into Eq. (4.1.69) yields

$$\boldsymbol{D} = \dot{\boldsymbol{h}} - \boldsymbol{\Omega}_{\boldsymbol{V}}\times\boldsymbol{h} + \boldsymbol{h}\times\boldsymbol{\Omega}_{\boldsymbol{V}} + \frac{1}{2}(\boldsymbol{V}^2\times\boldsymbol{b}-\boldsymbol{b}\times\boldsymbol{V}^2) \tag{4.1.71}$$

The tensor

$$\boldsymbol{D}_{\boldsymbol{\Omega}} = \frac{1}{2}(\boldsymbol{V}^2\times\boldsymbol{b}-\boldsymbol{b}\times\boldsymbol{V}^2)$$

has the following representation

$$2\boldsymbol{D_\Omega} = \sum_{i=1}^{3}\sum_{j=1}^{3}(\lambda_i^2 - \lambda_j^2)\boldsymbol{b}\cdot(\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j)(\overset{v}{\boldsymbol{n}}_i \otimes \overset{v}{\boldsymbol{n}}_j + \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_i) \tag{4.1.72}$$

Assuming that the tensor $\boldsymbol{V}$ has distinct principal values λ_i let us consider the following identity

$$\begin{aligned}2\boldsymbol{D_\Omega} &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\ln\lambda_i - \ln\lambda_j)\frac{\lambda_i^2 - \lambda_j^2}{(\ln\lambda_i - \ln\lambda_j)}\boldsymbol{b}\cdot(\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j)(\overset{v}{\boldsymbol{n}}_i \otimes \overset{v}{\boldsymbol{n}}_j + \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_i)\\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\ln\lambda_i - \ln\lambda_j)\boldsymbol{c}\cdot(\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j)(\overset{v}{\boldsymbol{n}}_i \otimes \overset{v}{\boldsymbol{n}}_j + \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_i)\\ &= \boldsymbol{h}\times\boldsymbol{c} - \boldsymbol{c}\times\boldsymbol{h}, \quad i \neq j\end{aligned} \tag{4.1.73}$$

where the components of vector $\boldsymbol{c}$ are related to the components of vector $\boldsymbol{b}$ as follows

$$\boldsymbol{c}\cdot(\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j) = \frac{\lambda_i^2 - \lambda_j^2}{(\ln\lambda_i - \ln\lambda_j)}\boldsymbol{b}\cdot(\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j), \quad i \neq j$$

Hence

$$\begin{aligned}&\boldsymbol{c}\cdot\sum_{i=1}^{3}\sum_{j=1}^{3}\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_j \times \overset{v}{\boldsymbol{n}}_i\\ &\qquad = \boldsymbol{b}\cdot\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\lambda_i^2 - \lambda_j^2}{(\ln\lambda_i - \ln\lambda_j)}\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_j \times \overset{v}{\boldsymbol{n}}_i, \quad i \neq j\end{aligned}$$

Applying the identity (A.4.14) we obtain

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_j \times \overset{v}{\boldsymbol{n}}_i = \sum_{i=1}^{3}\overset{v}{\boldsymbol{n}}_i \times \boldsymbol{I} \times \overset{v}{\boldsymbol{n}}_i = \sum_{i=1}^{3}(\overset{v}{\boldsymbol{n}}_i \otimes \overset{v}{\boldsymbol{n}}_i - \overset{v}{\boldsymbol{n}}_i \cdot \overset{v}{\boldsymbol{n}}_i\boldsymbol{I}) = -2\boldsymbol{I}$$

Consequently

$$2\boldsymbol{c} = -\boldsymbol{b}\cdot\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\lambda_i^2 - \lambda_j^2}{(\ln\lambda_i - \ln\lambda_j)}\overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_j \times \overset{v}{\boldsymbol{n}}_i, \quad i \neq j \tag{4.1.74}$$

With Eqs. (4.1.63), (4.1.71), (4.1.73) and (4.1.74) the tensor $\boldsymbol{D}$ is related to the rate of the Hencky strain tensor $\boldsymbol{h}$ as follows

$$\boldsymbol{D} = \dot{\boldsymbol{h}} - \boldsymbol{\Omega_h} \times \boldsymbol{h} + \boldsymbol{h} \times \boldsymbol{\Omega_h}, \quad \boldsymbol{\Omega_h} = \boldsymbol{\Omega_V} + \boldsymbol{A_h} \cdot (\boldsymbol{\Omega_R} - \boldsymbol{\Omega_V}) \qquad (4.1.75)$$

where

$$\boldsymbol{A_h} = -\frac{1}{4J}\boldsymbol{V} \cdot \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\lambda_i^2 - \lambda_j^2}{(\ln\lambda_i - \ln\lambda_j)} \overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_j \times \overset{v}{\boldsymbol{n}}_i, \quad i \neq j$$

The tensor $\boldsymbol{A_h}$ has the following spectral representation

$$2\boldsymbol{A_h} = \frac{\lambda_2^2 - \lambda_3^2}{\lambda_2\lambda_3 \ln\frac{\lambda_2}{\lambda_3}} \overset{v}{\boldsymbol{n}}_1 \otimes \overset{v}{\boldsymbol{n}}_1 + \frac{\lambda_3^2 - \lambda_1^2}{\lambda_3\lambda_1 \ln\frac{\lambda_3}{\lambda_1}} \overset{v}{\boldsymbol{n}}_2 \otimes \overset{v}{\boldsymbol{n}}_2 + \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1\lambda_2 \ln\frac{\lambda_1}{\lambda_2}} \overset{v}{\boldsymbol{n}}_3 \otimes \overset{v}{\boldsymbol{n}}_3$$

In Xiao et al. (1997) the tensor $\boldsymbol{\Omega_h} \times \boldsymbol{I}$ is called logarithmic spin. With Eqs. (4.1.55), (4.1.67) and (4.1.75) the vector $\boldsymbol{\Omega_h}$ can be computed as follows

$$\boldsymbol{\Omega_h} = \boldsymbol{\omega} + \boldsymbol{K_h} \cdot (\boldsymbol{D} \cdot \boldsymbol{V}^2)_\times, \qquad (4.1.76)$$

where

$$2\boldsymbol{K_h} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{\lambda_i^2 - \lambda_j^2}\left(\frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2} - \frac{1}{\ln\frac{\lambda_i}{\lambda_j}}\right) \overset{v}{\boldsymbol{n}}_i \times \overset{v}{\boldsymbol{n}}_j \otimes \overset{v}{\boldsymbol{n}}_j \times \overset{v}{\boldsymbol{n}}_i, \quad i \neq j$$

Equation (4.1.76) is firstly derived by Xiao et al. (1997) in a different notation. The tensor $\boldsymbol{K_h}$ has the following spectral representation

$$\begin{aligned}\boldsymbol{K_h} = & \frac{1}{\lambda_2^2 - \lambda_3^2}\left(\frac{1}{\ln\frac{\lambda_2}{\lambda_3}} - \frac{\lambda_2^2 + \lambda_3^2}{\lambda_2^2 - \lambda_3^2}\right) \overset{v}{\boldsymbol{n}}_1 \otimes \overset{v}{\boldsymbol{n}}_1 \\ & + \frac{1}{\lambda_3^2 - \lambda_1^2}\left(\frac{1}{\ln\frac{\lambda_3}{\lambda_1}} - \frac{\lambda_3^2 + \lambda_1^2}{\lambda_3^2 - \lambda_1^2}\right) \overset{v}{\boldsymbol{n}}_2 \otimes \overset{v}{\boldsymbol{n}}_2 \\ & + \frac{1}{\lambda_3^2 - \lambda_1^2}\left(\frac{1}{\ln\frac{\lambda_1}{\lambda_2}} - \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2}\right) \overset{v}{\boldsymbol{n}}_3 \otimes \overset{v}{\boldsymbol{n}}_3\end{aligned}$$

4.2 Conservation of Mass

The mass of an infinitesimal part of the body is

$$\mathrm{d}m = \rho \mathrm{d}V = \rho_0 \mathrm{d}V_0, \tag{4.2.77}$$

where ρ and ρ_0 is the density in the actual and the reference configurations, respectively. With Eq. (4.1.17) the conservation of mass (4.2.77) takes the form

$$\frac{\rho_0}{\rho} = J \tag{4.2.78}$$

4.3 Balance of Momentum

The momentum of an infinitesimal part of the solid is defined as follows

$$\mathrm{d}\boldsymbol{p} = \boldsymbol{v}\mathrm{d}m = \boldsymbol{v}\rho\mathrm{d}V$$

The momentum for a part of the solid with the volume V_p in the in the actual configuration is

$$\boldsymbol{p}_\mathrm{p} = \int\limits_{V_\mathrm{p}} \boldsymbol{v}\rho\mathrm{d}V \tag{4.3.79}$$

The balance of momentum or the first law of dynamics states that the rate of change of momentum of a body is equal to the total force acting on the body.

4.3.1 *Stress Vector*

Figure 4.2 illustrates a body under the given external loads. To visualize the internal forces let us cut the body in the actual configuration by a plane. The orientation of the plane is given by the unit normal vector $\boldsymbol{n}$. In the differential neighborhood of a point P consider an infinitesimal area element $\mathrm{d}A$. To characterize the mechanical action of the part II on the part I of the body let us introduce the force vector $\mathrm{d}\boldsymbol{T}_{\mathrm{II-I}} = \mathrm{d}\boldsymbol{T}_{(\boldsymbol{n})}$ as shown in Fig. 4.2. On the other hand the force vector $\mathrm{d}\boldsymbol{T}_{\mathrm{I-II}} = \mathrm{d}\boldsymbol{T}_{(-\boldsymbol{n})}$ models the mechanical action of the part I on the part II. The intensity of these mechanical actions can be characterized by the stress vectors $\boldsymbol{\sigma}_{(\boldsymbol{n})}$ and $\boldsymbol{\sigma}_{(-\boldsymbol{n})}$. Both the magnitude

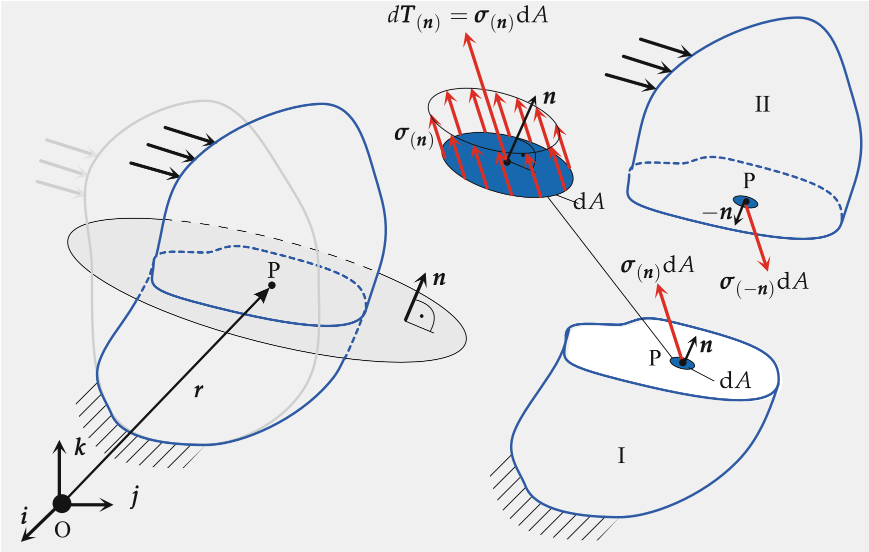

Fig. 4.2 Stress vector for the plane with the normal vector $\boldsymbol{n}$

and the direction of the stress vector depend on the position within the body. Within the infinitesimal area element $\mathrm{d}A$ the stress vector is assumed constant such that

$$\mathrm{d}\boldsymbol{T}_{(\boldsymbol{n})} = \boldsymbol{\sigma}_{(\boldsymbol{n})}\mathrm{d}A, \quad \mathrm{d}\boldsymbol{T}_{(-\boldsymbol{n})} = \boldsymbol{\sigma}_{(-\boldsymbol{n})}\mathrm{d}A$$

One may prove that

$$\mathrm{d}\boldsymbol{T}_{(\boldsymbol{n})} = -\mathrm{d}\boldsymbol{T}_{(-\boldsymbol{n})} \quad \Rightarrow \quad \boldsymbol{\sigma}_{(\boldsymbol{n})} = -\boldsymbol{\sigma}_{(-\boldsymbol{n})} \tag{4.3.80}$$

4.3.2 Integral Form

Let us cut a part with the volume V_p and the surface area A_p from the body, as shown in Fig. 4.3. The mechanical actions on the part of the body can be classified as follows

- body forces, for example force of gravity, electric or magnetic forces acting on a part of the mass $\mathrm{d}m = \rho \mathrm{d}V$. This type of action is described with the force density vector $\boldsymbol{f}$ such that the elementary body force is $\mathrm{d}\boldsymbol{\mathcal{G}} = \boldsymbol{f} dm = \boldsymbol{f} \rho dV$
- surface forces $\mathrm{d}\boldsymbol{T}_{(\boldsymbol{n})} = \boldsymbol{\sigma}_{(\boldsymbol{n})}\mathrm{d}A$ acting on the surface elements $\mathrm{d}A$ of A_p. These forces characterize the mechanical action of the environment (remainder of the body) on the given part V_p.

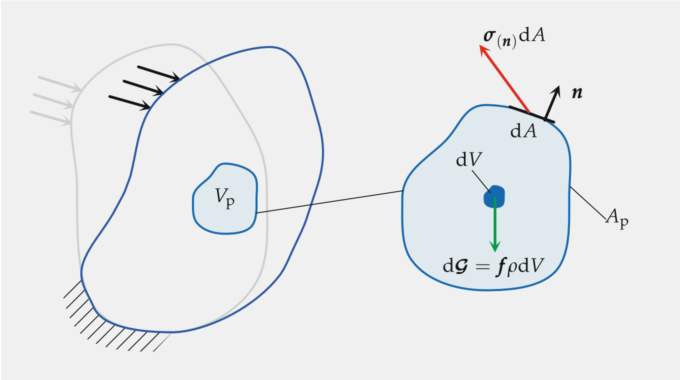

Fig. 4.3 Forces acting on a part of the body with the volume V_p

The resultant force vector is

$$\boldsymbol{\mathcal{F}}_\mathrm{p} = \int\limits_{A_\mathrm{p}} \boldsymbol{\sigma}_{(\boldsymbol{n})}\mathrm{d}A + \int\limits_{V_\mathrm{p}} \boldsymbol{f}\rho\mathrm{d}V$$

The balance of momentum for the part of the solid can be formulated as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_\mathrm{p}} \boldsymbol{v}\rho\mathrm{d}V = \int\limits_{A_\mathrm{p}} \boldsymbol{\sigma}_{(\boldsymbol{n})}\mathrm{d}A + \int\limits_{V_\mathrm{p}} \boldsymbol{f}\rho\mathrm{d}V \tag{4.3.81}$$

4.3.3 Stress Tensor and Cauchy Formula

The balance of momentum (4.3.81) can be applied for any part of the body. Consider an infinitesimal tetrahedron ($A_\mathrm{p} \to 0$, $V_\mathrm{p} \to 0$) as a part of the body, Fig. 4.4. The orthonormal vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are introduced to fix the orientation of the tetrahedron. The mechanical action of the environment on the tetrahedron cut from the body is characterized by forces and corresponding stress vectors. The cut planes, the corresponding areas as well as stress and force vectors are given in the Table 4.1. For the infinitesimal tetrahedron the volume integrals in Eq. (4.3.81) have lower order of magnitude compared to the surface integral such that

$$\int\limits_{A_\mathrm{p}} \boldsymbol{\sigma}_{(\boldsymbol{n})}\mathrm{d}A = \boldsymbol{0} \tag{4.3.82}$$

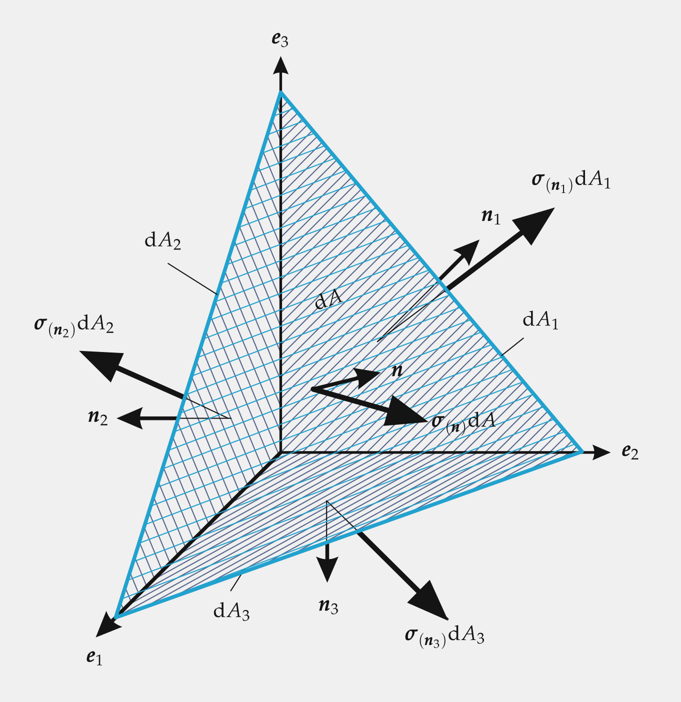

Fig. 4.4 Infinitesimal tetrahedron cut from the body

Table 4.1 Summary of formulae for the infinitesimal tetrahedron

Plane	Area	Stress vector	Force vector
$\boldsymbol{n}_1 = -\boldsymbol{e}_1$	dA_1	$\boldsymbol{\sigma}_{(\boldsymbol{n}_1)} = -\boldsymbol{\sigma}_{(\boldsymbol{e}_1)}$	$\boldsymbol{T}_{(\boldsymbol{n}_1)} = \boldsymbol{\sigma}_{(\boldsymbol{n}_1)}dA_1$
$\boldsymbol{n}_2 = -\boldsymbol{e}_2$	dA_2	$\boldsymbol{\sigma}_{(\boldsymbol{n}_2)} = -\boldsymbol{\sigma}_{(\boldsymbol{e}_2)}$	$\boldsymbol{T}_{(\boldsymbol{n}_2)} = \boldsymbol{\sigma}_{(\boldsymbol{n}_2)}dA_2$
$\boldsymbol{n}_3 = -\boldsymbol{e}_3$	dA_3	$\boldsymbol{\sigma}_{(\boldsymbol{n}_3)} = -\boldsymbol{\sigma}_{(\boldsymbol{e}_3)}$	$\boldsymbol{T}_{(\boldsymbol{n}_3)} = \boldsymbol{\sigma}_{(\boldsymbol{n}_3)}dA_3$
$\boldsymbol{n}$	dA	$\boldsymbol{\sigma}_{(\boldsymbol{n})}$	$\boldsymbol{T}_{(\boldsymbol{n})} = \boldsymbol{\sigma}_{(\boldsymbol{n})}dA$

Hence

$$\boldsymbol{\sigma}_{(\boldsymbol{n}_1)}dA_1 + \boldsymbol{\sigma}_{(\boldsymbol{n}_2)}dA_2 + \boldsymbol{\sigma}_{(\boldsymbol{n}_3)}dA_3 + \boldsymbol{\sigma}_{(\boldsymbol{n})}dA = \boldsymbol{0}$$

Taking into account (4.3.80)

$$-\boldsymbol{\sigma}_{(\boldsymbol{e}_1)}dA_1 - \boldsymbol{\sigma}_{(\boldsymbol{e}_2)}dA_2 - \boldsymbol{\sigma}_{(\boldsymbol{e}_3)}dA_3 + \boldsymbol{\sigma}_{(\boldsymbol{n})}dA = \boldsymbol{0} \tag{4.3.83}$$

or

$$\boldsymbol{\sigma}_{(\boldsymbol{n})}dA = \boldsymbol{\sigma}_{(\boldsymbol{e}_1)}dA_1 + \boldsymbol{\sigma}_{(\boldsymbol{e}_2)}dA_2 + \boldsymbol{\sigma}_{(\boldsymbol{e}_3)}dA_3 \tag{4.3.84}$$

In addition the following equation is valid for any part of the volume[3]

$$\int\limits_{A_{\mathrm{p}}} \boldsymbol{n}\mathrm{d}A = \boldsymbol{0} \tag{4.3.85}$$

Applying (4.3.85) to the tetrahedron yields

$$\boldsymbol{n}_1\mathrm{d}A_1 + \boldsymbol{n}_2\mathrm{d}A_2 + \boldsymbol{n}_3\mathrm{d}A_3 + \boldsymbol{n}\mathrm{d}A = \boldsymbol{0} \tag{4.3.86}$$

Therefore

$$\begin{aligned} &\boldsymbol{n}\mathrm{d}A = \boldsymbol{e}_1\mathrm{d}A_1 + \boldsymbol{e}_2\mathrm{d}A_2 + \boldsymbol{e}_3\mathrm{d}A_3, \\ &\boldsymbol{n}\cdot\boldsymbol{e}_1\mathrm{d}A = \mathrm{d}A_1, \quad \boldsymbol{n}\cdot\boldsymbol{e}_2\mathrm{d}A = \mathrm{d}A_2, \quad \boldsymbol{n}\cdot\boldsymbol{e}_3\mathrm{d}A = \mathrm{d}A_3 \end{aligned} \tag{4.3.87}$$

Inserting $\mathrm{d}A_i$ $(i = 1, 2, 3)$ into Eq. (4.3.84) we obtain

$$\begin{aligned} \boldsymbol{\sigma}_{(\boldsymbol{n})}\mathrm{d}A &= \mathrm{d}A_1\boldsymbol{\sigma}_{(\boldsymbol{e}_1)} + \mathrm{d}A_2\boldsymbol{\sigma}_{(\boldsymbol{e}_2)} + \mathrm{d}A_3\boldsymbol{\sigma}_{(\boldsymbol{e}_3)} \\ &= \boldsymbol{n}\cdot\boldsymbol{e}_1\mathrm{d}A\boldsymbol{\sigma}_{(\boldsymbol{e}_1)} + \boldsymbol{n}\cdot\boldsymbol{e}_2\mathrm{d}A\boldsymbol{\sigma}_{(\boldsymbol{e}_2)} + \boldsymbol{n}\cdot\boldsymbol{e}_3\mathrm{d}A\boldsymbol{\sigma}_{(\boldsymbol{e}_3)} \end{aligned}$$

This can be simplified as follows

$$\begin{aligned} \boldsymbol{\sigma}_{(\boldsymbol{n})} &= \boldsymbol{n}\cdot\boldsymbol{e}_1\boldsymbol{\sigma}_{(\boldsymbol{e}_1)} + \boldsymbol{n}\cdot\boldsymbol{e}_2\boldsymbol{\sigma}_{(\boldsymbol{e}_2)} + \boldsymbol{n}\cdot\boldsymbol{e}_3\boldsymbol{\sigma}_{(\boldsymbol{e}_3)} \\ &= \boldsymbol{n}\cdot[\boldsymbol{e}_1\otimes\boldsymbol{\sigma}_{(\boldsymbol{e}_1)} + \boldsymbol{e}_2\otimes\boldsymbol{\sigma}_{(\boldsymbol{e}_2)} + \boldsymbol{e}_3\otimes\boldsymbol{\sigma}_{(\boldsymbol{e}_3)}] \end{aligned} \tag{4.3.88}$$

With the tensor

$$\boldsymbol{\sigma} = \boldsymbol{e}_1\otimes\boldsymbol{\sigma}_{(\boldsymbol{e}_1)} + \boldsymbol{e}_2\otimes\boldsymbol{\sigma}_{(\boldsymbol{e}_2)} + \boldsymbol{e}_3\otimes\boldsymbol{\sigma}_{(\boldsymbol{e}_3)} \tag{4.3.89}$$

Eq. (4.3.88) takes the following form

$$\boldsymbol{\sigma}_{(\boldsymbol{n})} = \boldsymbol{n}\cdot\boldsymbol{\sigma} \tag{4.3.90}$$

Equation (4.3.90) is the Cauchy formula[4] that allows one to compute the stress vector for any plane with the unit normal $\boldsymbol{n}$ if the Cauchy stress tensor $\boldsymbol{\sigma}$ is given.

---

[3]This can be verified applying the integral theorem $(\mathrm{B.3.4})_1$ with $\varphi = 1$.

[4]In some books of continuum mechanics and applied mathematics the stress tensor is defined as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{(\boldsymbol{e}_1)}\otimes\boldsymbol{e}_1 + \boldsymbol{\sigma}_{(\boldsymbol{e}_2)}\otimes\boldsymbol{e}_2 + \boldsymbol{\sigma}_{(\boldsymbol{e}_3)}\otimes\boldsymbol{e}_3$ such that the Cauchy formula is $\boldsymbol{\sigma}_{(\boldsymbol{n})} = \boldsymbol{\sigma}\cdot\boldsymbol{n}$. Formally this definition differs from (4.3.89) by transpose. It might be more convenient, as it is closer to the matrix algebra. For engineers dealing with internal forces it is more natural to use (4.3.89). Indeed, to analyze a stress state we need to cut the body first and to specify the normal to the cut plane. Only after that we can introduce the internal force. The sequence of these operations is clearly seen in (4.3.89).

4.3.4 Local Forms

With the Cauchy formula (4.3.90) and the integral theorem (B.3.5)$_2$ the surface integral in (4.3.81) is transformed as follows

$$\int\limits_{A_\mathrm{p}} \boldsymbol{\sigma}_{(\boldsymbol{n})} \mathrm{d}A = \int\limits_{A_\mathrm{p}} \boldsymbol{n} \cdot \boldsymbol{\sigma} \mathrm{d}A = \int\limits_{V_\mathrm{p}} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \mathrm{d}V \tag{4.3.91}$$

Now the balance of momentum takes the form

$$\int\limits_{V_\mathrm{p}} (\rho \dot{\boldsymbol{v}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho \boldsymbol{f}) \mathrm{d}V = \boldsymbol{0} \tag{4.3.92}$$

Since Eq. (4.3.92) is valid for any part of the solid, the following local form of the balance of momentum can be established

$$\rho \dot{\boldsymbol{v}} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f} \tag{4.3.93}$$

With the identity (4.1.18) the surface integral (4.3.91) can be transformed as follows

$$\int\limits_{A_\mathrm{p}} \boldsymbol{\sigma}_{(\boldsymbol{n})} \mathrm{d}A = \int\limits_{A_\mathrm{p}} \boldsymbol{n} \cdot \boldsymbol{\sigma} \mathrm{d}A = \int\limits_{A_{\mathrm{p}0}} \boldsymbol{N} \cdot \boldsymbol{P} \mathrm{d}A_0 \int\limits_{V_{\mathrm{p}0}} \overset{0}{\boldsymbol{\nabla}} \cdot \boldsymbol{P} \mathrm{d}V_0, \tag{4.3.94}$$

where

$$\boldsymbol{P} = J \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \tag{4.3.95}$$

is the Piola-Kirchhoff stress tensor. With Eqs. (4.3.94) and (4.2.78) the balance of momentum can be formulated as follows

$$\int\limits_{V_{\mathrm{p}0}} (\rho_0 \dot{\boldsymbol{v}} - \overset{0}{\boldsymbol{\nabla}} \cdot \boldsymbol{P} - \rho \boldsymbol{f}) \mathrm{d}V_0 = \boldsymbol{0} \tag{4.3.96}$$

The corresponding local form is

$$\rho_0 \dot{\boldsymbol{v}} = \overset{0}{\boldsymbol{\nabla}} \cdot \boldsymbol{P} + \rho_0 \boldsymbol{f} \tag{4.3.97}$$

4.4 Balance of Angular Momentum

With respect to the point O the angular momentum and the resultant moment vectors for a part of the body are defined as follows[5]

$$\boldsymbol{q}_{\mathrm{pO}} = \int_{V_\mathrm{p}} \boldsymbol{r} \times \boldsymbol{v}\rho \mathrm{d}V, \quad \boldsymbol{\mathcal{M}}_{\mathrm{pO}} = \int_{A_\mathrm{p}} \boldsymbol{r} \times \boldsymbol{\sigma}_{(\boldsymbol{n})}\mathrm{d}A + \int_{V_\mathrm{p}} \boldsymbol{r} \times \boldsymbol{f}\rho \mathrm{d}V \tag{4.4.98}$$

The balance of angular momentum or the second law of dynamics states that the rate of change of angular momentum of a body is equal to the resultant moment acting on the body. The surface integral in Eq. (4.4.98) can be transformed applying (B.3.5)$_2$ as follows

$$\int_{A_\mathrm{p}} \boldsymbol{r} \times (\boldsymbol{n} \cdot \boldsymbol{\sigma})\mathrm{d}A = -\int_{A_\mathrm{p}} \boldsymbol{n} \cdot \boldsymbol{\sigma} \times \boldsymbol{r}\mathrm{d}A = -\int_{V_\mathrm{p}} \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \times \boldsymbol{r})\mathrm{d}V \tag{4.4.99}$$

Applying the identity (B.2.3) we obtain

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \times \boldsymbol{r}) = (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \times \boldsymbol{r} - \boldsymbol{\sigma}_\times$$

The balance of angular momentum can be formulated as follows

$$\dot{\boldsymbol{q}}_{\mathrm{pO}} = \int_{V_\mathrm{p}} \boldsymbol{r} \times \dot{\boldsymbol{v}}\rho \mathrm{d}V = \int_{V_\mathrm{p}} [\boldsymbol{r} \times (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{\rho} f) + \boldsymbol{\sigma}_\times]\mathrm{d}V$$

or

$$\int_{V_\mathrm{p}} \boldsymbol{r} \times (\dot{\boldsymbol{v}}\rho - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{\rho} f)\mathrm{d}V = \int_{V_\mathrm{p}} \boldsymbol{\sigma}_\times \mathrm{d}V$$

Taking into account the balance of momentum (4.3.93) this results in

$$\boldsymbol{\sigma}_\times = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{4.4.100}$$

[5] With regard to structural analysis applications discussed in this book it is enough to identify the angular momentum as the moment of momentum and the resultant moment as the moment of forces. In contrast, within the micropolar theories material points are equipped by tensor of inertia. The resultant moment includes surface and body moments which are not related to moment of forces, e.g. Altenbach et al. (2003), Eringen (1999), Nowacki (1986).

4.5 Balance of Energy

The total energy E_p of the part of the body, is defined as a sum of the kinetic energy $\mathrm{K_p}$ and the internal energy $\mathrm{U_p}$ as follows

$$
\begin{aligned}
&\mathrm{E_p} = \mathrm{K_p} + \mathrm{U_p},\\
&\mathrm{K_p} = \int\limits_{V_\mathrm{p}} \rho \mathcal{K} \mathrm{d}V, \quad \mathrm{U_p} = \int\limits_{V_\mathrm{p}} \rho \mathcal{U} \mathrm{d}V, \quad \mathcal{K} = \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v},
\end{aligned}
\tag{4.5.101}
$$

where $\mathcal{K}$ and $\mathcal{U}$ are densities of the kinetic and the internal energy, respectively. The energy balance equation or the first law of thermodynamics states that the rate of change of the energy of a body is equal to the mechanical power plus the rate of change of non-mechanical energy, for example heat, supplied into the body. The energy balance equation is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E_p} = \mathrm{L_p} + \mathrm{Q_p}, \tag{4.5.102}
$$

where $\mathrm{L_p}$ is the mechanical power and $\mathrm{Q_p}$ is the rate of change of non-mechanical energy supply. The mechanical power of forces introduced in Sect. 4.3.2 is defined as follows

$$
\mathrm{L_p} = \int\limits_{A_\mathrm{p}} \boldsymbol{\sigma}_{(\boldsymbol{n})} \cdot \boldsymbol{v} \mathrm{d}A + \int\limits_{V_\mathrm{p}} \boldsymbol{f} \cdot \boldsymbol{v} \rho \mathrm{d}V \tag{4.5.103}
$$

With Eqs. (4.3.90) and $(\mathrm{B.3.5})_2$ the surface integral in (4.5.103) is transformed to

$$
\int\limits_{A_\mathrm{p}} \boldsymbol{\sigma}_{(\boldsymbol{n})} \cdot \boldsymbol{v} \mathrm{d}A \int\limits_{A_\mathrm{p}} \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{v} \mathrm{d}A = \int\limits_{V_\mathrm{p}} \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{v})\mathrm{d}V \tag{4.5.104}
$$

With the identity (B.2.2) we obtain

$$
\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{v}) = (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot\boldsymbol{v} + \boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla}\otimes\boldsymbol{v})^\mathrm{T}
$$

The mechanical power can be now given as follows

$$
\mathrm{L_p} = \int\limits_{V_\mathrm{p}} [(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho\boldsymbol{f})\cdot\boldsymbol{v} + \boldsymbol{\sigma}\cdot\cdot(\boldsymbol{\nabla}\otimes\boldsymbol{v})^\mathrm{T}]\mathrm{d}V \tag{4.5.105}
$$

The energy balance equation (4.5.102) takes the form

$$\int_{V_\mathrm{p}} \rho(\dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \dot{\mathcal{U}})\mathrm{d}V = \int_{V_\mathrm{p}} [(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}) \cdot \boldsymbol{v} + \boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})^\mathrm{T}]\mathrm{d}V + \mathrm{Q}_\mathrm{p} \tag{4.5.106}$$

With the balance of momentum (4.3.93), Eq. (4.5.106) simplifies to

$$\int_{V_\mathrm{p}} \rho\dot{\mathcal{U}}\mathrm{d}V = \int_{V_\mathrm{p}} \boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})^\mathrm{T}\mathrm{d}V + \mathrm{Q}_\mathrm{p} \tag{4.5.107}$$

The rate of change of the energy supply includes the contributions through the outer surface and within the volume of the part p

$$\mathrm{Q}_\mathrm{p} = \int_{A_\mathrm{p}} q_{(\boldsymbol{n})}\mathrm{d}A + \int_{V_\mathrm{p}} r\rho\mathrm{d}V \tag{4.5.108}$$

Equation (4.5.107) takes the following form

$$\int_{V_\mathrm{p}} [\rho\dot{\mathcal{U}} - \boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})^\mathrm{T} - \rho r]\mathrm{d}V = \int_{A_\mathrm{p}} q_{(\boldsymbol{n})}\mathrm{d}A \tag{4.5.109}$$

Equation (4.5.109) is valid for any part of the body. Considering an infinitesimal tetrahedron the energy balance reduces to

$$\int_{A_\mathrm{p}} q_{(\boldsymbol{n})}\mathrm{d}A = 0$$

Applying the procedures discussed in Sect. 4.3.3 one may derive the following equation

$$q_{(\boldsymbol{n})} = -\boldsymbol{n} \cdot \boldsymbol{q}, \tag{4.5.110}$$

where $\boldsymbol{q}$ is the heat flow vector. With $(\mathrm{B}.3.5)_1$ and (4.5.110) the surface integral can be transformed into the volume one as follows

$$\int_{A_\mathrm{p}} q_{(\boldsymbol{n})}\mathrm{d}A = -\int_{V_\mathrm{p}} \boldsymbol{\nabla} \cdot \boldsymbol{q}\mathrm{d}V \tag{4.5.111}$$

Equation (4.5.108) takes the form

$$\int_{V_\mathrm{p}} [\rho\dot{\mathcal{U}} - \boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})^\mathrm{T} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \rho r]\mathrm{d}V = 0 \tag{4.5.112}$$

Equation (4.5.112) is valid for any part of the deformed body. Hence

$$\rho\dot{\mathcal{U}} = \boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})^{\mathrm{T}} - \boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho r \tag{4.5.113}$$

With the identity (4.1.18) the surface integral (4.5.111) can be transformed as follows

$$\int\limits_{A_{\mathrm{p}}} q_{(\boldsymbol{n})} \mathrm{d}A = -\int\limits_{A_{\mathrm{p}}} \boldsymbol{n} \cdot \boldsymbol{q} \mathrm{d}A = -\int\limits_{A_{\mathrm{P}0}} \boldsymbol{N} \cdot \hat{\boldsymbol{q}} \mathrm{d}A_0 - \int\limits_{V_{\mathrm{P}0}} \overset{0}{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{q}} \mathrm{d}V_0, \tag{4.5.114}$$

where by analogy to the Piola-Kirchhoff stress tensor the following heat flow vector can be introduced

$$\hat{\boldsymbol{q}} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{q} \tag{4.5.115}$$

Now it is not difficult to derive the local form of the energy balance per unit volume of the body in the reference configuration

$$\rho_0\dot{\mathcal{U}} = \boldsymbol{P} \cdot\cdot (\overset{0}{\boldsymbol{\nabla}} \otimes \boldsymbol{v})^{\mathrm{T}} - \overset{0}{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{q}} + \rho_0 r \tag{4.5.116}$$

4.6 Entropy and Dissipation Inequalities

The second law of thermodynamics states that the entropy production of a body is non-negative. This statement is given as the Clausius-Planck inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{S}_{\mathrm{p}} - \left(\frac{\mathrm{Q}}{T}\right)_{\mathrm{p}} \geq 0, \tag{4.6.117}$$

where S is the entropy and T is the absolute temperature. The entropy of the part of the body is defined as follows

$$S_{\mathrm{p}} = \int\limits_{V_{\mathrm{p}}} \rho\mathcal{S}\mathrm{d}V, \tag{4.6.118}$$

where $\mathcal{S}$ is the entropy density. For the part of the body we define

$$\left(\frac{\mathrm{Q}}{T}\right)_{\mathrm{p}} = \int\limits_{A_{\mathrm{p}}} \frac{q_{(\boldsymbol{n})}}{T} \mathrm{d}A + \int\limits_{V_{\mathrm{p}}} \frac{r}{T} \rho \mathrm{d}V \tag{4.6.119}$$

Applying Eqs. (4.5.110) and $(\mathrm{B}.3.5)_1$ we obtain

$$\left(\frac{\mathrm{Q}}{T}\right)_{\mathrm{p}} = \int\limits_{V_{\mathrm{p}}} \left[-\nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) + \frac{r}{T}\rho\right] \mathrm{d}V \tag{4.6.120}$$

With Eqs. (4.6.118) and (4.6.120) the entropy inequality (4.6.117) can be formulated as follows

$$\int\limits_{V_{\mathrm{p}}} \left[\rho\dot{\mathcal{S}} + \nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) - \frac{r}{T}\rho\right] \mathrm{d}V \geq 0 \tag{4.6.121}$$

Since (4.6.121) is valid for any part of the body the local form of the entropy inequality is

$$\rho\dot{\mathcal{S}} \geq -\nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) + \frac{\rho r}{T} \tag{4.6.122}$$

With the identity (B.2.1)

$$\nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) = \frac{\boldsymbol{\nabla} \cdot \boldsymbol{q}}{T} - \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla} T}{T^2}$$

Multiplying both sides of (4.6.122) by T yields the Clausius-Duhem inequality

$$\rho\dot{\mathcal{S}}T \geq -\boldsymbol{\nabla} \cdot \boldsymbol{q} + \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla} T}{T} + r\rho \tag{4.6.123}$$

The energy balance equation (4.5.113) can be formulated as follows

$$\rho r - \boldsymbol{\nabla} \cdot \boldsymbol{q} = \rho\dot{\mathcal{U}} - \boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})^{\mathrm{T}} \tag{4.6.124}$$

Inserting into the entropy inequality (4.6.123) yields the dissipation inequality

$$\boldsymbol{\sigma} \cdot\cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})^{\mathrm{T}} - \rho\dot{\mathcal{U}} + \rho\dot{\mathcal{S}}T - \frac{\boldsymbol{q} \cdot \boldsymbol{\nabla} T}{T} \geq 0 \tag{4.6.125}$$

Introducing the Helmholtz free energy density $\Phi = \mathcal{U} - \mathcal{S}T$ the dissipation inequality (4.6.125) can be put into the following form

$$\boldsymbol{\sigma} \cdot\cdot (\nabla \otimes \boldsymbol{v})^{\mathrm{T}} - \rho\dot{\Phi} - \rho\mathcal{S}\dot{T} - \frac{\boldsymbol{q} \cdot \nabla T}{T} \geq 0 \qquad (4.6.126)$$

With Eqs. (4.2.78), (4.3.95) and (4.5.115) as well as the relationships between the gradients (4.1.20) the dissipation inequality (4.6.126) can be given with respect to the reference configuration as follows

$$\boldsymbol{P} \cdot\cdot (\overset{0}{\nabla} \otimes \boldsymbol{v})^{\mathrm{T}} - \rho_0\dot{\Phi} - \rho_0\mathcal{S}\dot{T} - \frac{\hat{\boldsymbol{q}} \cdot \overset{0}{\nabla} T}{T} \geq 0 \qquad (4.6.127)$$

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