Chapter 25
Enhancing Gauge Symmetries Via the Symplectic Embedding Approach

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Abstract One of the best ways to increase the fundamental symmetries of the physical systems with singular Lagrangian is the gauging of those models with the help of symplectic formalism of constrained systems. The main idea of this approach is based on the embedding of the model in an extended phase-space. After the gauging process had done, we can obtain generators of gauge transformations of the model.

25.1 Introduction

Every high energy physicist is aware of the importance of gauge theories. As a matter of fact, gauge invariance is the most significant and practical concept in high energy physics. The standard model of elementary particles is founded on this concept. Gauge invariance occurred due to the presence of the physical variables which are called gauge-invariant variables, and they are independent of local reference frames (Henneaux and Teiltelboim 1992).

It is very important to know that the quantization of gauge theories is as simple as it is thought to be due to the presence of internal symmetries which are called gauge symmetries. These symmetries exist some nonphysical degrees of freedom, that must be wiped out before and after the quantization is applied (Abreu et al. 2012).

On the other hand, in gauge theory, using the equations of motion, the dynamics of the system cannot be determined completely at every moment. Hence, one of the features of gauge theory is the advent of arbitrary time-dependent functions in general solutions of the equations of motion. The emergence of such functions forms some relations between phase-space coordinates, called constraints (Bergmann and Goldberg 1955).

To quantize such systems, Dirac classified the identities between phase-space coordinates into two main groups (Dirac 1967). The first group is identities that

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present in the phase-space, like coordinates or momentum variables, which transform the physical system without any changes in the phase-space. These identities are called first-class constraints. Dirac, as the pioneer of constrained systems, named them as generators of the gauge transformations in the phase-space. The second group is not related to any degrees of freedom and must be eliminated. Presence of such identities indicates the absence of the gauge symmetry in the system. These identities are called second-class constraints. Therefore, to gauge a system which contains second-class constraints, we must convert them to first-class ones (Dirac 1950; Shirzad and Monemzadeh 2004).

To achieve this aim, there have been invented some approaches like BFT method (Batalin and Fradkin 1987; Batalin et al. 1989; Batalin and Tyutin 1991; Shirzad and Monemzadeh 2005; Ebrahimi and Monemzadeh 2014) and F-J approach or its newer version the symplectic formalism (Abreu et al. 2012; Faddeev and Jackiw 1988; Woodhouse 1992; Neto et al. 2001). The main strategy which these methods are based on is embedding a non-invariant system in an extended phase-space (Becchi et al. 1976; Batalin and Vilkovisky 1981; Monemzadeh and Ebrahimi 2012).

25.2 The Symplectic Formalism

The F-J formalism which was formulated first by Faddeev and Jackiw (Faddeev and Jackiw 1988) existed to prevent us from the consistency problems which deviate Poisson brackets algebra from common one, which consequently spoils all quantization techniques in constrained systems (Abreu et al. 2013; Monemzadeh et al. 2014). This method is mathematically founded on the symplectic structure of phase-space. Thus, it is different from approaches with similar usage. Moreover, the interpretation and classification of constraints which are presented by this formalism are different from other congener ones. As we mentioned before, in order to solve the quantization problems of any system, Dirac presented a theory and classified constraints into primary and secondary and first- and second-class constraints (Dirac 1967), whereas in symplectic formalism, all constraints are presumed to be equal and there is no dissimilarity between first- and second-class constraints. However, both formalisms are verified to be equivalent (Govaerts 1990; Montani 1993).

To use the symplectic method, we should start using the first-order Lagrangian, whose corresponding equation of motion does not imbue any acceleration. Thus, one can obtain the Hamiltonian equation of motion from the variational principle (Abreu et al. 2013; Jackiw 1994). So, we must start with first-order Lagrangian, and any other second-order Lagrangian should be transformed into a first-order one by expanding the configuration space, including conjugate momentum and coordinate variables. Also, one can use the Legendre transform to pass from Lagrangian to Hamiltonian (Paschalis and Porfyriadis 1996). We consider a non-invariant mechanical model with the dynamics which are described by the Lagrangian $L\left(q_\mu, \dot{q}_\mu, t\right)$ with $\mu = 1, \ldots, N$ and its corresponding spatial variables q_μ and velocities $\dot{q}_\mu$.

The singularity nature of the Lagrangian due to its configuration constraint $\phi_i(q_\mu)$ can be imposed by a new dynamical variable (say undetermined Lagrange multiplier) λ_i in such a way that adds the constraints to free Lagrangian:

$$L^{(0)} = \dot{q}_\mu p^\mu - H_c - \lambda_i \phi^i\left(q_\mu\right). \tag{25.1}$$

Mutually we can calculate H_c as

$$H_c = \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}_\mu - L^{(0)}. \tag{25.2}$$

The symplectic variables and symplectic one-form can be read off from the model straightforwardly:

$$\begin{aligned} \xi_\alpha^{(0)} &= \left(q_\mu, p_\mu, \lambda_i\right). \\ \mathcal{A}_\alpha^{(0)} &= \left(p_\nu, 0_\nu, 0_j\right). \end{aligned} \tag{25.3}$$

Then, the symplectic two-form, $f_{\alpha\beta}^{(0)} = \partial_\alpha \mathcal{A}_\beta^{(0)} - \partial_\beta \mathcal{A}_\alpha^{(0)}$, will be obtained in the form of the following matrix:

$$f_{\alpha\beta}^{(0)} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & 0_{\mu j} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu j} \\ 0_{i\nu} & 0_{i\nu} & 0_{ij} \end{pmatrix}. \tag{25.4}$$

This matrix is apparently singular, and so, it has some zero-modes which are defined by $n_\alpha^{i(0)}$. As a matter of fact, because of our knowledge from linear algebra, we know that the linear combination of these null vectors is also a zero-mode.

Using the zero iterative potential,

$$\mathcal{V}^{(0)} = H_c + \lambda_i \phi^i. \tag{25.5}$$

Primary constraints will be obtained from the following relation:

$$\phi^i = n_\alpha^{i(0)} \partial^\alpha \mathcal{V}^{(0)}, \tag{25.6}$$

where ∂^α is the derivation with respect to the symplectic variables.

We can put the constraint into the kinetic part of the Lagrangian by substituting these constraints, obtained from (25.6) into the original Lagrangian. It means that we make primary constraints ϕ_i, as momenta conjugate to the variables λ_i. In other words, we convert strongly nonlinear constraints, ϕ_i, into the momenta (linear constraint) of the phase-space. Hence, the first iterative Lagrangian will be obtained as

$$L^{(1)} = \dot{q}_\mu p^\mu - \dot{\lambda}_i \phi^i - H_c, \tag{25.7}$$

and the first iterative potential we have

$$\mathcal{V}^{(1)} = H_c. \tag{25.8}$$

New symplectic variables and one-form are defined as follows:

$$\begin{aligned} \xi_\alpha^{(1)} &= \left(q_\mu, p_\mu, \lambda_i\right). \\ \mathcal{A}_\alpha^{(1)} &= \left(p_\nu, 0_\nu, \phi_j\right). \end{aligned} \tag{25.9}$$

The corresponding symplectic two-form is

$$f_{\alpha\beta}^{(1)} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & u_{i\nu}^T \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu j} \\ -u_{i\nu} & 0_{i\nu} & 0_{ij} \end{pmatrix}, \tag{25.10}$$

where

$$u_{i\mu} = \frac{\partial \phi_i}{\partial q_\mu}. \tag{25.11}$$

This matrix (25.10) is a singular matrix, and it has some null vectors $n_\alpha^{i(1)}$. Using (25.6), we obtain secondary constraints as

$$\phi_{i'} = \mathcal{A}_\alpha^{(0)} \partial^\alpha \phi_i. \tag{25.12}$$

We can write the second iterative Lagrangian as

$$L^{(2)} = \dot{q}_\mu p^\mu - \dot{\lambda}_i \phi^i - \dot{\lambda}_{i'} \phi^{i'} - H_c. \tag{25.13}$$

New symplectic variables and one-form are as follows:

$$\begin{aligned} \xi_\alpha^{(2)} &= \left(q_\mu, p_\mu, \lambda_i, \lambda_{i'}\right). \\ \mathcal{A}_\alpha^{(2)} &= \left(p_\nu, 0_\nu, \phi_j, \phi_{j'}\right). \end{aligned} \tag{25.14}$$

The corresponding two-form symplectic matrix is non-singular. Thus, it does not have any null vector, and consequently, there is no other constraint:

$$f^{(2)}_{\alpha\beta} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & u^T_{i\nu} & v^T_{i'\nu} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{i\nu} & w^T_{i'\nu} \\ -u_{i\nu} & 0_{\mu j} & 0_{ij} & 0_{ij'} \\ -v_{i'\nu} & -w_{i'\nu} & 0_{i'j} & 0_{i'j'} \end{pmatrix}, \tag{25.15}$$

where v_α and w_α are defined as follows:

$$\begin{aligned} v_{i'\mu} &= \frac{\partial \overline{\phi}_{i'}}{\partial q^\mu}. \\ w_{i'\mu} &= \frac{\partial \overline{\phi}_{i'}}{\partial p^\mu}. \end{aligned} \tag{25.16}$$

25.3 Symplectic Embedding Formalism

The corresponding symplectic two-form is non-singular. Thus, it does not have any null vector and consequently, the iterative process stops and no other constraint will be obtained.

Now, we start the symplectic embedding procedure to convert second-class constraints to first ones. The main idea of this procedure is to adjoin the Wess-Zumino variables to the original phase-space (Wess and Zumino 1971). In order to do that, we expand the original phase-space by introduction of a function G as WZ Lagrangian, depending on the original phase-space variables and the WZ variable θ, as the expansion in terms of the WZ variables, defined by

$$G\big(q_\mu, p_\mu, \lambda_i, \theta\big) = \sum_{n=0}^{\infty} \mathcal{G}^{(n)}. \tag{25.17}$$

This function is gauging potential and satisfies the following boundary condition by vanishing $\mathcal{G}^{(0)}$:

$$G\big(q_\mu, p_\mu, \lambda_i, \theta = 0\big) = 0. \tag{25.18}$$

Introducing the new term G into the Lagrangian (7),

$$\begin{aligned} \widetilde{L}^{(1)} &= L^{(1)} + L_{WZ} \\ &= L^{(1)} + G\big(q_\mu, p_\mu, \lambda_i, \theta\big). \end{aligned} \tag{25.19}$$

With the corresponding symplectic variables and one-form,

$$\begin{aligned}\widetilde{\xi}^{(0)}_{\overline{\alpha}} &= \left(q_\mu, p_\mu, \lambda_i, \theta\right),\\ \widetilde{\mathcal{A}}^{(0)}_{\overline{\alpha}} &= \left(p_\nu, 0_\nu, \phi_j, 0\right),\end{aligned} \tag{25.20}$$

the two-form symplectic matrix will be

$$\widetilde{f}^{(1)}_{\overline{\alpha}\overline{\beta}} = \begin{pmatrix} f^{(1)}_{\alpha\beta} & 0_{\alpha 1} \\ 0_{1\beta} & 0_{1\times 1}\end{pmatrix}, \tag{25.21}$$

which has the following null vectors $\widetilde{n}^{i(1)}_{\overline{\alpha}}$:

$$\begin{aligned}\widetilde{n}^{1(1)}_{\overline{\alpha}} &= \begin{pmatrix}0_\alpha & 1\end{pmatrix}.\\ \widetilde{n}^{2(1)}_{\overline{\alpha}} &= \begin{pmatrix}n^{i(1)}_\alpha & 0\end{pmatrix}.\end{aligned} \tag{25.22}$$

We define $\widetilde{n}_{\overline{\alpha}}$ as the linear combination of the corresponding null vectors:

$$\widetilde{n}_{\overline{\alpha}} = \sum_i \widetilde{n}^{i(1)}_{\overline{\alpha}} = \begin{pmatrix} n^{i(1)}_\alpha & a\end{pmatrix}. \tag{25.23}$$

Using the following relation,

$$\widetilde{n}_{\overline{\alpha}} \frac{\partial \mathcal{V}^{(1)}}{\partial \widetilde{\xi}^{(0)\overline{\alpha}}} = \frac{\partial \mathcal{G}^{(n)}}{\partial \theta}. \tag{25.24}$$

To start the iterative process, we substitute (25.8) into (25.24), using the zero-mode (25.23) to obtain $\mathcal{G}^{(1)}$ as

$$\mathcal{G}^{(1)} = \theta \phi_{i'}. \tag{25.25}$$

Putting $\mathcal{G}^{(1)}$ into (25.19), the potential will be

$$\widetilde{\mathcal{V}}^{(1)} = H_c - \mathcal{G}^{(1)}. \tag{25.26}$$

Using (25.24) for the second time with respect to the modified first iterative potential (25.26), one can obtain $\mathcal{G}^{(2)}$. Also, with the help of (25.24), one can find the explicit relation which gives $\mathcal{G}^{(n)}$ for $n \geq 2$, as

$$n^{i(1)}_\alpha \left[\frac{\partial \mathcal{G}^{(n-1)}}{\partial \widetilde{\xi}^{(0)\alpha}}\right] + a\frac{\partial \mathcal{G}^{(n)}}{\partial \theta} + b\frac{\partial \mathcal{G}^{(n)}}{\partial p_\theta} = 0. \tag{25.27}$$

Substituting $\mathcal{G}^{(2)}$ into the first iterative Lagrangian, we will obtain the second iterative Lagrangian. Thus,

$$\widetilde{\mathcal{V}}^{(1)} = H_c - \mathcal{G}^{(1)} - \mathcal{G}^{(2)}. \tag{25.28}$$

Again, using (25.24) to obtain, considering the (25.28) as the modified potential, we obtain $\mathcal{G}^{(3)}$. This process should be continued so far forth that $\frac{\partial \mathcal{G}^{(n)}}{\partial \theta}$ become null. Therefore, the zero-mode $\widetilde{n}_{\widetilde{\alpha}}$ does not make a new constraint. Thus, the gauged symplectic potential will be obtained as $\widetilde{\mathcal{V}}$

$$\widetilde{\mathcal{V}}^{(1)}(q_\mu, p_\mu, \lambda_i) = \widetilde{\mathcal{V}}^{(0)}(q_\mu, p_\mu) - \mathcal{G}^{(1)} - \mathcal{G}^{(2)} - \cdots - \mathcal{G}^{(n-1)}, \tag{25.29}$$

and for the canonical Hamiltonian, we have

$$\widetilde{H}_c = H_c + \lambda_i \phi^i - G(q_\mu, p_\mu, \lambda_i). \tag{25.30}$$

The gauged Lagrangian is obtained as

$$\widetilde{L}^{(1)} = L^{(1)} + G(q_\mu, p_\mu, \lambda_i). \tag{25.31}$$

25.4 Gauged Lagrangian

As we mentioned before about the relation (25.19), the gauged Lagrangian of an ungauged system, i.e.,$\widetilde{L}^{(1)}$, will be obtained by adding a Lagrangian-like term to the first-order Lagrangian. This term depends on a new dynamical variable, which is called a WZ variable.

As we have shown in the previous section, an iterative differential equation with the help of zero -modes of the symplectic two-form and the potential of the model in (25.27) (Abreu et al. 2012) has been driven to obtain this added term. As a matter of fact, for most cases, and particularly for the studied model in this chapter, that iteration will not go more than two levels. Thus, a shortcut formula to make the WZ Lagrangian con be introduced here (Abarghouei Nejad et al. 2014).

To start with, let's imagine that our model has some primary constraints which are introduced by ϕ_i. First, we should find the constraint which is first class in comparison to other primary constraints. We call this primary first-class constraint as $\overline{\phi}_j$:

$$\{\phi_i, \overline{\phi}_j\} = 0. \tag{25.32}$$

Applying the symplectic approach will give us some secondary constraints, denoted by $\phi_i^{'}$. We construct the WZ Lagrangian by adding two generators $G^{(1)}$ and $G^{(2)}$, as

$$L_{WZ} = G^{(1)} + G^{(2)}, \tag{25.33}$$

where

$$G^{(1)} = \theta\phi_i^{'} \quad ,$$
$$G^{(2)} = -\theta^2\left\{\phi_i^{'}, \overline{\phi}_j\right\}. \tag{25.34}$$

Also, θ is the WZ variable, and its conjugate momentum, p_θ, which will not appear in the gauged model is a first-class constraint. According to Dirac's guess, the presence of the first-class constraint guarantees the presence of a gauge symmetry in the model.

25.5 Constraint Structure of the Gauged Lagrangian

Using the symplectic method, we enhance the gauge symmetry of the primary model. In the following, we derive constraints and phase-space structure of the gauged Lagrangian (25.31). In this gauged model, new dynamical variables λ_i and θ appear first orderly in the Lagrangian. So, their momenta are primary constraints in the phase-space. Thus,

$$\frac{\partial\widetilde{L}^{(0)}}{\partial\dot{\lambda}^i} = 0 :\rightarrow \rho_{1_i} = p_{\lambda_i}, \tag{25.35}$$

$$\frac{\partial\widetilde{L}^{(0)}}{\partial\dot{\theta}} = 0 :\rightarrow \rho_2 = p_\theta. \tag{25.36}$$

So, for the total Hamiltonian, corresponding to Lagrangian (25.31) and redefining the constraints $\rho_s = \left(\rho_{1_i}, \rho_2\right)$, we can write

$$\widetilde{H}_T = \widetilde{H}_c + \omega^s\rho_s. \tag{25.37}$$

In the chain-by-chain method (Shirzad and Mojiri 2001), the consistency of each individual constraint starts a chain and gives the next element of that chain. Also, the consistency of second-class constraints determines some of the Lagrange multipliers, ω^s, while the consistency of first-class ones leads to constraints of the next level:

$$0 = \left\{\rho_s, \widetilde{H}_T\right\}.$$
$$0 = \left\{\rho_s, \widetilde{H}_c\right\} + \omega^r \{\rho_s, \rho_r\}. \tag{25.38}$$

We see that primary constraints are Abelian, i.e., $\{\rho_r, \rho_s\} = 0$. So, we arrive to secondary constraints $\psi_s = \left\{\rho_s, \widetilde{H}_c\right\}$.

The consistency of second level of constraints may give us new constraints, like

$$\left\{\psi_s, \widetilde{H}_c\right\} = \Lambda_s, \tag{25.39}$$

or may determine a Lagrange multiplier due to the fact that $\{\psi_2, \rho_2\} \neq 0$, as

$$\left\{\psi_s, \widetilde{H}_c\right\} \neq 0. \tag{25.40}$$

The relation (25.39) is identically true on the constrained surface. We should check the consistency condition (25.38) for Λ_s to see whether there exists any new constraint in the model or the chain is truncated.

Calculating all Poisson brackets, one can find first-class constraints. As we mentioned before, the first-class constraint, Φ_{FC_i}, is one whose Poisson bracket with other constraints vanishes:

$$\left\{\Phi_{FC_i}, \phi_j\right\} = 0. \tag{25.41}$$

Also, the Poisson bracket matrix of all second-class constraints, Δ_{ij}, must be non-singular.

In order to determine all Dirac brackets of the original and gauged model, we put the inverse of Δ_{ij} in the following formula:

$$\left\{\xi_{\overline{\alpha}}, \xi_{\overline{\beta}}\right\}^* = \left\{\xi_{\overline{\alpha}}, \xi_{\overline{\beta}}\right\} - \{\xi_{\overline{\alpha}}, \Phi_{SC_i}\}\Delta_{ij}^{-1}\left\{\Phi_{SC_j}, \xi_{\overline{\beta}}\right\}, \tag{25.42}$$

where Φ_{SC_i} is the set of all second-class constraints.

Also, by characterizing first-class constraints and Dirac brackets of a classical system, its quantized model, say Hilbert space of the quantum states, is fully available at tree level, according to Dirac prescription:

$$\{A, B\}^* \to \frac{1}{i\text{h}}[A, B], \tag{25.43}$$
$$\hat{\Phi}_{FC_i}\left|phys\right> = 0, \tag{25.44}$$

where $\hat{\Phi}_{FC_i}$ is the quantized version of first-class constraints.

25.6 Gauge Invariance of the Extended Lagrangian

We can obtain all gauge symmetries of the Lagrangian (25.19), using the Poisson brackets of the first-class constraints, Φ_{FC_i}, and symplectic variables (Shirzad and Shabani Moghadam 1999; Henneaux et al. 1990):

$$\delta\widetilde{\xi}_{\overline{\alpha}}^{(0)} = \left\{\widetilde{\xi}_{\overline{\alpha}}^{(0)}, \Phi_{FC_j}\right\}\varepsilon^j \tag{25.45}$$

Also, the generators of infinitesimal gauge transformations can be obtained with the help of zero-modes of the symplectic two-form (25.23), using $\delta\widetilde{\xi}_{\overline{\alpha}}^{(1)} = \varepsilon_i \widetilde{n}_{\overline{\alpha}}$:

$$\begin{aligned} \delta x_i &= 0, \\ \delta p_i &= u_i \varepsilon_1, \\ \delta\lambda &= \varepsilon_2, \\ \delta\theta &= \varepsilon_1, \end{aligned} \tag{25.46}$$

where ε_i is the infinitesimal time-dependent parameter (Abreu et al. 2013; Kim et al. 2004). Thus, the gauge symmetry of the model is determined via these transformations. In other words, the gained model is invariant under these transformations.

Apparently the results obtained from (25.45) are the same as the infinitesimal gauge transformations (25.46). Considering constrained analysis of the Lagrangian (25.31) and detaching its corresponding constraints in the following section, we study the gauge symmetry of the following model easily.

25.7 Particle Model on Hyperplane as a Toy Model

We consider a nonrelativistic particle with unit mass, which is confined on a hyperplane. We try to gauge the model using the symplectic formalism and extract its corresponding generators of infinitesimal gauge transformations.

The Hamiltonian of such a particle is defined as

$$H = \frac{1}{2} p_\mu p^\mu + \lambda_1 \phi^1, \tag{25.47}$$

where ϕ_1 is the constraint which is imposed by the condition of the presence of the particle on the hyperplane. This model has been studied via the Skyrme model (Neto et al. 2001):

$$\phi_1 = q_\mu q^\mu - 1 \tag{25.48}$$

As we mentioned before, we start with the zeroth-iterated first-order Lagrangian as

$$L^{(0)} = \dot{q}_\mu p^\mu - \frac{1}{2} p_\mu p^\mu + \lambda_1 \left(q_\mu q^\mu - 1 \right). \tag{25.49}$$

Symplectic variables and one-form can be read off from the Lagrangian (25.49) as

$$\begin{aligned} \xi_\alpha^{(0)} &= \left(q_\mu, p_\mu, \lambda_1 \right), \\ \mathcal{A}_\alpha^{(0)} &= \left(p_\nu, 0_\nu, 0 \right). \end{aligned} \tag{25.50}$$

Then, the symplectic two-form will be obtained in the form of the following matrix:

$$f_{\alpha\beta}^{(0)} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & 0_{\mu 1} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu 1} \\ 0_{1\nu} & 0_{1\nu} & 0_{1\times 1} \end{pmatrix}, \tag{25.51}$$

which is apparently singular.

The corresponding some zero-mode is defined as

$$n_\alpha^{(0)} = \left(0_\mu \quad 0_\mu \quad 0 \right). \tag{25.52}$$

Using the relation (25.6), one can again find the primary constraint of the model as

$$\phi_1 = q_\mu q^\mu - 1. \tag{25.53}$$

Now, we redefine the zeroth iterative potential as

$$\mathcal{V}^{(0)} = \frac{1}{2} p_\mu p^\mu - \lambda_1 \left(q_\mu q^\mu - 1 \right). \tag{25.54}$$

The first iterative Lagrangian is

$$L^{(1)} = q_\mu p^\mu + \dot{\lambda}_1 \phi^1 - \frac{1}{2} p_\mu p^\mu. \tag{25.55}$$

Then the symplectic variables and one-form are as follows:

$$\begin{aligned} \xi_\alpha^{(1)} &= \left(q_\mu, p_\mu, \lambda_1 \right). \\ \mathcal{A}_\alpha^{(1)} &= \left(p_\nu, 0_\nu, \phi_1 \right). \end{aligned} \tag{25.56}$$

Therefore, the corresponding symplectic two-form tensor is obtained as

$$f^{(1)}_{\alpha\beta} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & 2q_{\mu 1} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu 1} \\ -2q_{1\nu} & 0_{1\nu} & 0_{1\times 1} \end{pmatrix}. \tag{25.57}$$

Since the above tensor is singular, one can find its associated zero-mode as

$$n^{(1)}_{\alpha} = \begin{pmatrix} 0_{\mu} & q_{\mu} & \frac{1}{2} \end{pmatrix}, \tag{25.58}$$

which is the generator of the following secondary constraint; with the help of (25.6), we have

$$\phi_2 = q_\mu p^\mu. \tag{25.59}$$

Updating our potential, the first iterative potential will be

$$\mathcal{V}^{(1)} = \frac{1}{2} p_\mu p^\mu. \tag{25.60}$$

Now, the second iterative potential can be read off as

$$L^{(1)} = q_\mu p^\mu + \dot{\lambda}_1 \phi^1 + \dot{\lambda}_2 \phi^2 + -\frac{1}{2} p_\mu p^\mu. \tag{25.61}$$

Looking for symplectic variables and one-form, we will have

$$\begin{aligned} \xi^{(2)}_{\alpha} &= \left(q_\mu, p_\mu, \lambda_1, \lambda_2\right). \\ \mathcal{A}^{(2)}_{\alpha} &= \left(p_\nu, 0_\nu, \phi_1, \phi_2\right). \end{aligned} \tag{25.62}$$

The second-iterated symplectic two-form can be obtained as

$$f^{(2)}_{\alpha\beta} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & 2q_{\mu 1} & p_{\mu 1} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{i\nu} & q_{\mu 1} \\ -2q_{1\nu} u_{i\nu} & 0_{\mu j} & 0_{ij} & 0_{ij'} \\ -p_{1\nu} & -q_{1\nu} & 0_{i'j} & 0_{i'j'} \end{pmatrix}. \tag{25.63}$$

This tensor is non-singular. So, it does not have any null vector to generate new constraint. Therefore, the constraint-making process truncates. As we mentioned before, the inverse of (25.63) gives the usual Dirac brackets, using the relation (25.42).

Now, this is the time to start the embedding process. First, we should find the Wess-Zumino Lagrangian as the relation (25.17) with the corresponding boundary condition (25.18).

Introducing this Lagrangian into the first iterative Lagrangian (25.55), we will obtain

$$\widetilde{L}^{(1)} = \dot{q}_\mu p^\mu + \dot{\lambda}_1 \phi^1 - \frac{1}{2} p_\mu p^\mu + G\left(q_\mu, p_\mu, \theta\right). \tag{25.64}$$

We can read off extended symplectic variable and one-form as follows:

$$\begin{aligned} \widetilde{\xi}_\alpha^{(1)} &= \left(q_\mu, p_\mu, \lambda_1, \theta\right). \\ \widetilde{\mathcal{A}}_\alpha^{(1)} &= \left(p_\nu, 0_\nu, \phi_1, 0\right). \end{aligned} \tag{25.65}$$

Computing the symplectic tensor, $\widetilde{f}_{\alpha\beta}^{(1)}$, as

$$\widetilde{f}_{\alpha\beta}^{(1)} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & 2q_{\mu 1} & 0_{\mu 1} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu 1} & 0_{\mu 1} \\ -2q_{1\nu} & 0_{1\nu} & 0_{1\times 1} & 0_{1\times 1} \\ 0_{1\nu} & 0_{1\nu} & 0_{1\times 1} & 0_{1\times 1} \end{pmatrix}. \tag{25.66}$$

which is exactly in the form of (25.21). This tensor is apparently singular and has the following zero-modes:

$$\begin{aligned} \widetilde{n}_{\bar{\alpha}}^{1(1)} &= \begin{pmatrix} 0_\alpha & 1 \end{pmatrix}. \\ \widetilde{n}_{\bar{\alpha}}^{2(1)} &= \begin{pmatrix} n_\alpha^{(1)} & 0 \end{pmatrix}. \end{aligned} \tag{25.67}$$

Similar to (25.23), we use the linear combination of these zero-modes to start generating constraints.

Using (25.24) and (25.25), one can obtain the first iterative term, depending on θ as

$$\mathcal{G}^{(1)}\left(q_\mu, p_\mu, \theta\right) = \left(q_\mu p^\mu\right)\theta. \tag{25.68}$$

Putting this term in the Lagrangian (25.64), we have

$$\widetilde{L}^{(1)} = \dot{q}_\mu p^\mu + \dot{\lambda}_1 \phi^1 - \frac{1}{2} p_\mu p^\mu + \left(q_\mu p^\mu\right)\theta. \tag{25.69}$$

While the zero-mode (25.67) generated a new constraint, the Lagrangian is not still a gauge-invariant one:

$$\widetilde{n}_\alpha^{(1)} \frac{\partial \mathcal{V}^{(1)}}{\partial \widetilde{\xi}_\alpha^{(1)}} = q_\mu q^\mu \theta. \tag{25.70}$$

Thus,

$$\mathcal{G}^{(2)} = -\frac{1}{2} q_\mu q^\mu \theta^2. \tag{25.71}$$

So, the first iterative Lagrangian will be

$$\widetilde{L}^{(1)} = \dot{q}_\mu p^\mu + \dot{\lambda}_1 \phi^1 - \frac{1}{2} p_\mu p^\mu + \left(q_\mu p^\mu\right)\theta - \frac{1}{2} q_\mu q^\mu \theta^2. \tag{25.72}$$

At this stage, the null vector (25.67) does not produce any new constraint. Thus, correction terms $\mathcal{G}^n$ with $n \geq 3$ vanish. Hence, (25.72) is an invariant Lagrangian. Also, the corresponding canonical Hamiltonian can be obtained as

$$\widetilde{H}_c^{(1)} = \frac{1}{2} p_\mu p^\mu - \lambda_1 \phi^1 - \left(q_\mu p^\mu\right)\theta + \frac{1}{2} q_\mu q^\mu \theta^2. \tag{25.73}$$

Now, according to gauge transformation-generating functions (25.46), we can obtain the infinitesimal variations, under which both Hamiltonian (25.73) and (25.72) are invariant:

$$\begin{aligned} \delta q_\mu &= 0. \\ \delta p_\mu &= \varepsilon_1 q_\mu. \\ \delta \lambda &= \frac{1}{2}\varepsilon_2. \\ \delta \theta &= \varepsilon_1. \end{aligned} \tag{25.74}$$

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