Chapter 15
Study of the Bipartite Edge Frustration of Graphs

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Abstract The smallest number of edges that have to be deleted from a graph to obtain a bipartite spanning subgraph is called the bipartite edge frustration of G and denoted by $\varphi(G)$. This topological index is related to the well-known Max − cut problem, and has important applications in computing stability of fullerenes. In this paper we determine the bipartite edge frustration of some classes of composite graphs. Moreover, this quantity for four classes of graphs arising from a given graph under different types of edge subdivisions is investigated.

15.1 Introduction

The problem of finding large bipartite spanning subgraphs of a given non-bipartite graph has a long and rich history. The first results were obtained by Erdös (Erdös 1965) and Edwards (Edwards 1973), who showed that every graph G on $|V(G)|$ vertices and $|E(G)|$ edges contains a bipartite subgraph with at least $|E(G)|/2 + (|V(G)| - 1)/4$ edges. Those bounds were further improved for various classes of graphs; for example, the lower bound of $(4/5)|E(G)|$ was established for cubic triangle-free graphs (Hopkins and Staton 1982) and also for sub-cubic triangle-free graphs (Bondy and Locke 1986). The best currently known (Cui and Wang 2009) lower bound for cubic, planar, and triangle-free graphs is

$$\frac{39}{539}\left|V(G)\right| - \frac{9}{16}.$$

Instead of looking for large bipartite subgraphs of a given graph G, it is sometimes more convenient to look at the equivalent problem of finding a smallest

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set of edges that must be deleted from G in order to make the remaining graph bipartite. Borrowing from the terminology of the antiferromagnetic Ising model, the cardinality of any such set is then called the bipartite edge frustration of a graph. More formally, let G be a graph with the vertex and edge sets $|V(G)|$ and $|E(G)|$, respectively. The bipartite edge frustration of G is then defined as the minimum number of edges that have to be deleted from G to obtain a bipartite spanning subgraph. We denote it by $\varphi(G)$. Clearly, if G is bipartite, then $\varphi(G) = 0$ and $\varphi(G)$ is a topological index. It can be easily shown that $\varphi(G) \le \frac{|E(G)|}{2}$ and that the complete graph on n vertices has the maximum possible bipartite edge frustration among all graphs on n vertices. Hence, the bipartite edge frustration has properties that make it useful as a measure of non-bipartivity of a given graph.

Schmalz et al. (1986) observed that the isolated pentagon fullerenes have the best stability. Because of this success, it is natural to study its vertex version. The bipartite vertex frustration of G, $\varphi(G)$, is defined as the minimum number of vertices that have to be deleted from G to obtain a bipartite subgraph H of G (Yarahmadi and Ashrafi 2011a). Obviously, if G is not bipartite, then $\underline{H}$ is not a spanning subgraph of G and so, H is not in general a large bipartite subgraph of G. The quantity $\varphi(G)$ is, in general, difficult to compute; it is NP-hard for general graphs. Hence, it makes sense to search for classes of graphs that allow its efficient computation. Some results in this direction are reported in (Došlić and Vukičević 2007) for fullerenes and other polyhedral graphs and in (Ghojavand and Ashrafi 2008) for some classes of nanotubes. For mathematical properties of this new topological index, we refer to (Yarahmadi and Ashrafi 2011b, 2013; Yarahmadi et al. 2010; Yarahmadi 2010; Ashrafi et al. 2013).

In this chapter, we will present explicit formulas for the bipartite edge frustration for the Cartesian product, chain, bridge, extended bridge graphs, splice, link, hierarchical product and its generalization. Also, some inequalities of the Nordhaus-Gaddum type will be presented. Moreover, four types of graphs resulting from edge subdivision will be introduced. Two of them, the subdivision graph and the total graph, belong to the folklore, while the other two were introduced in (Cvetković et al 1980) and further investigated in (Yan et al. 2007).

15.2 Definitions and Preliminaries

A *graph* is a pair $G = (V,E)$ of *points* and *lines*. The points and lines of G are also called *vertices* and *edges* of the graph, respectively. If e is an edge of G, connecting the vertices u and v, then we write $e = uv$ and say "u and v are adjacent."

Suppose G is a connected graph and $x, y \in V(G)$. The length of a minimal path connecting x and y is denoted by $d_G(x, y)$. It is easy to see that $(V(G), d_G(x, y))$ is a metric space.

It is well known that the bipartite edge frustration is a measure of stability for the fullerene molecules; see (Došlić 2005a, b; Fajtlowicz and Larson 2003). Here, a fullerene is a planar, 3-regular, and 3-connected molecular graph, 12 of whose faces are pentagons, and any remaining faces are hexagons. Such molecules are entirely constructed from carbon atoms. A fullerene is called an isolated pentagon (IP for short) if its pentagons do not have a common edge.The citation Fajtlowicz (2003) have been changed to Fajtlowicz and Larson (2003) as per the reference list. Please check if okay.It is okay.

In this section we introduce the composite graphs that will be considered here and recall their basic properties relevant for our goal. We start by composite graphs that arise by splicing, i.e., by identifying certain vertices.

Following Imrich and Klavžar (2000), the *Cartesian product* $G \times H$ of two graphs G and H is defined on the Cartesian product $V(G) \times V(H)$ of the vertex sets of the factors. The edge set $E(G \times H)$ is the set of all edges $(u, v)(x, y)$ for which either $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$. Thus, the vertex and edge sets of $G \times H$ are the following sets: $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(\mathrm{u}, \mathrm{v})(\mathrm{x}, \mathrm{y}) \mid \mathrm{u} = \mathrm{x},\ \mathrm{vy} \in \mathrm{E(H)},\ \mathrm{or},\ \mathrm{ux} \in \mathrm{E(G)},\ \mathrm{v} = \mathrm{y}\}$.

For a sequence $G_1, G_2, \ldots, G_n$ of graphs, we denote $G_1 \times \cdots \times G_n$ by $\prod_{i=1}^{n} G_i$. In the case that $G_1 = G = \cdots = G_n = G$, we denote $\prod_{i=1}^{n} G_i$ by G^n.

The *join*, $G + H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$. If $G = H + \cdots + H$, then we denote G by nH. The graph ∇G is obtained from G by adding a new vertex and making it adjacent to all vertices of G. The graph ∇G is called suspension of G. Obviously $\nabla G = G + K_1$. A join of two graphs is bipartite if and only if both graphs are empty, i.e., without edges. Hence, $\varphi(G + H) > 0$ if at least one of components contains an edge.

Both Cartesian product and join are standard graph operations. We refer the reader to monograph of Imrich (2000) for more information on those products.

The *complement* G of graph $\overline{G}$ has $V(G)$ as its vertex set, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Let $\{G_i\}_{i=1}^{n}$ be a set of finite pair wise disjoint graphs with $v_i \in V(G_i)$. The *bridge graph* $B = B(G_1, G_2, \ldots, G_n, v_1, v_2, \ldots, v_n)$ is the graph obtained from the graphs $G_1, G_2, \ldots, G_n$ by connecting the vertices v_i and v_{i+1} by an edge, for all $i = 1, 2, \ldots, n$, as shown in Fig. 15.1. We abbreviate the notation to $B(G_1, G_2, \ldots, G_n)$ when the vertices v_i are clear from context.

The *extended bridge graph* $EB(G, H_1, H_2, \ldots, H_n; v_1, \ldots, v_n)$ of G and $\{H_i\}_{i=1}^{n}$ with respect to $\{v_i\}_{i=1}^{n}$ is constructed by identifying the vertex v_i in G and H_i, for all $i = 1, 2, \cdots, n$. An example is shown in Fig. 15.2.

Let $\{G_i\}_{i=1}^{n}$ be a set of finite pairwise disjoint graphs with $v_i, w_i \in V(G_i)$. The *chain graph* $C = C(G_1, G_2, \ldots, G_n, v_1, w_1, \ldots, v_n, w_n)$ of $\{G_i\}_{i=1}^{n}$ with respect to the vertices $\{v_i, w_i\}_{i=1}^{n}$ is the graph obtained from graphs $G_1, G_2, \ldots, G_n$ by identifying the vertex w_i with v_{i+1}, for all $i = 1, 2, \ldots, n$, as shown in Fig. 15.3.

Again, the dependence on $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ will be often omitted in notation. The above classes of graphs were considered in Mansour and Schork (2009).

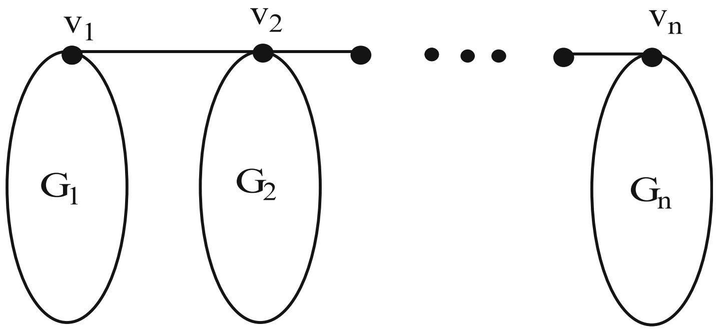

Fig. 15.1 The bridge graph

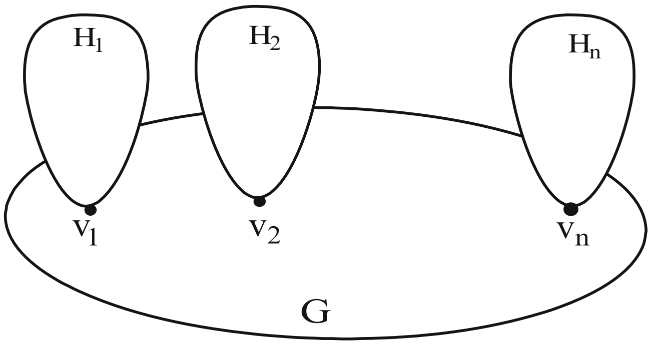

Fig. 15.2 The extended bridge graph

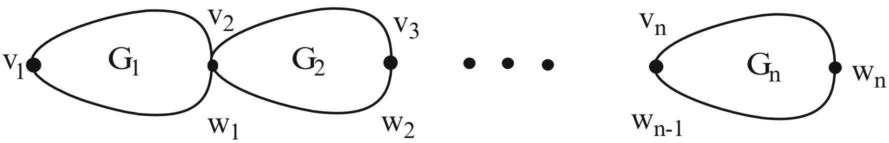

Fig. 15.3 The chain graph

Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a \in V(G)$ and $b \in V(H)$, a *splice* of G and H is defined as the graph, $(G.H)(a,b)$ obtained by identifying the vertices a and b. Similarly, a *link* of G and H is defined as the graph $(G \sim H)(a, b)$ obtained by joining a and b by an edge. The splices and links considered in (Došlić 2005a, b) could be viewed as their special cases.

The following theorem immediately concludes.

Theorem 15.2.1 *Let G and H be two simple and connected graphs with disjoint vertex sets. For each $a \in V(G)$ and $b \in V(H)$, the bipartite edge frustration of splice and link of G and H are obtained as follows:*

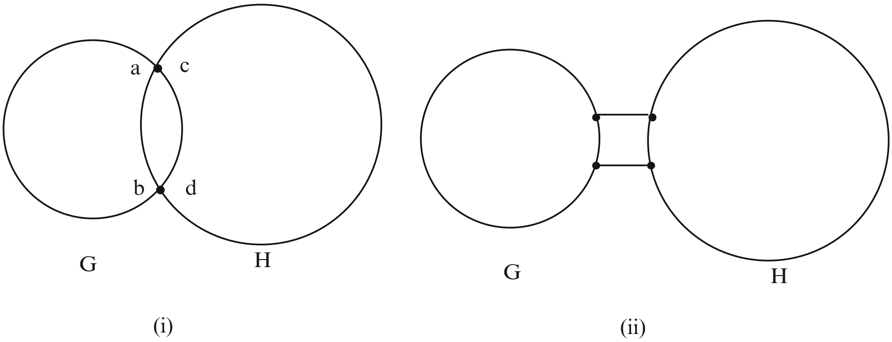

Fig. 15.4 The double splice and double link

1. $\varphi((G.H)(a, b)) = \varphi(G) + \varphi(H)$,
2. $\varphi((\mathrm{G}^{\sim}\mathrm{H})(\mathrm{a, b})) = \varphi(G) + \varphi(H)$.

Now we extend the above operations, for splice of G *and* H *by identifying two vertices and for link* G *and* H *by joining two vertices as the following definition.*

Definition 15.2.2 Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a, b \in V(G)$ and $c, d \in V(H)$, a *double splice* of G and H is defined as the graph $(G : H)(a, b : c, d)$ obtained by identifying the vertices a and c and vertices b and d. Similarly, a *double link* of G and H is defined as the graph $(G \approx H)(a, b : c, d)$ obtained by joining a and c by an edge and b and d by another edge. A double splice and double link of two graphs are shown schematically in Fig. 15.4.

A new operation on graphs is *hierarchical product*, because of the strong (connectedness) hierarchy of the vertices in the resulting graphs, see Barriére et al. (2009a). In fact, the obtained graphs turn out to be subgraphs of the Cartesian product of the corresponding factors. Some well-known properties of the Cartesian product, such as reduced mean distance and diameter, simple routing algorithms and some optimal communication protocols are inherited by the hierarchical product. Let $G_i = (V_i,, E_i)$ be N graphs with each vertex set Vi, $1 \leq i \leq N$, having a distinguished or root vertex, labeled *0*. The hierarchical product $H = G_N \Pi \ldots \Pi G_2 \Pi G_1$ is the graph with vertices the $N-$tuples $x_N \ldots x_2 x_1$, $x_i \in V_i$, and edges defined by the adjacencies:

$$x_N \ldots x_2 x_1 \approx \begin{cases} x_N \ldots x_3 x_2 y_1 & \text{if } y_1 \approx x_1 \text{ in } G_1 \\ x_N \ldots x_3 y_2 x_1 & \text{if } y_2 \approx x_2 \text{ in } G_2 \text{ and } x_1 = 0, \\ x_N \ldots y_3 x_2 x_1 & \text{if } y_3 \approx x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0, \\ \vdots & \vdots \\ y_N \ldots x_3 x_2 x_1 & \text{if } y_N \approx x_N \text{ in } G_N \text{ and } x_1 = x_2 = \cdots = x_{N-1} = 0 \end{cases}$$

Notice that the structure of the obtained product graph H heavily depends on the root vertices of the factors G_i for $1 \leq i \leq N$. Also, if $|V_i| = n_i$ and $|E_i| = m_i$, the number of vertices of H is $n_N \dots n_2 n_1$ and the number of edges is equal to $m_N + \sum_{i=1}^{N-1} \prod_{j=i+1}^{N} n_j m_i$.

Note also that the hierarchical $H = G_N \Pi \dots \Pi G_2 \Pi G_1$ is simply subgraph of classical Cartesian product $G_N \times \cdots \times G_2 \times G_1$. Although the Cartesian product is both commutative and associative, the hierarchical product has only the second property, provided that the root vertices are conveniently chosen (in the natural way).

A natural generalization of the hierarchical product, proposed in Barriére et al. (2009b), is as follows: Given N graphs $G_i = (V_i, E_i)$ and (nonempty) vertex subsets $U_i \subseteq V_i$, for $1 \le i \le N-1$, the *generalized hierarchical* product $H_g = G_N \Pi \dots \Pi G_2 (U_2) \Pi G_1 (U_1)$ is the graph with vertex set $V_N \times \cdots \times V_2 \times V_1$ and adjacencies:

$$x_N \dots x_2 x_1 \approx \begin{cases} x_N \dots x_3 x_2 y_1 & \text{if } y_1 \approx x_1 \text{ in } G_1, \\ x_N \dots x_3 y_2 x_1 & \text{if } y_2 \approx x_2 \text{ in } G_2 \text{ and } x_1 \in U_1, \\ x_N \dots y_3 x_2 x_1 & \text{if } y_3 \approx x_3 \text{ in } G_3 \text{ and } x_i \in U_i \text{ for } i = 1, 2, \\ \vdots & \vdots \\ y_N \dots x_3 x_2 x_1 & \text{if } y_N \approx x_N \text{ in } G_N \text{ and } x_i \in U_i \text{ for } i = 1, 2, \dots, N. \end{cases}$$

Now, we define four related graphs, for a connected graph G, as follows:

1. $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2.
2. $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G and then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle.
3. $Q(G)$ is obtained from G by inserting a new vertex into each edge of G and then joining with edges those pairs of new vertices on adjacent edges of G.
4. $T(G)$ has as its vertices, the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The graphs $S(G)$ and $T(G)$ are called the *subdivision* and *total* graphs of G, respectively; see Fig. 15.5.

Let F be one of the symbols S, R, Q, or T. The $F-$ sum $G +_F H$ is a graph with the set of vertices $V(G+_F H) = (V(G) \cup E(H)) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G +_F H$ are adjacent if and only if $[u_1 = v_1 \in V(G)$ and $(u_2, v_2) \in E(H)]$ or $[u_2 = v_2$ and $(u_1, v_1) \in E(F(G))]$. In an exact phrase,

$E(G +_F H) = \{((u_1, u_2), (v_1, v_2)) | [u_1 = v_1 \in V(G)$ and $(u_2, v_2) \in E(H)]$ or $[u_2 = v_2$ and $(u_1, v_1) \in E(F(G))]\}$.

Note that $G +_F H$ has $|V(H)|$ copies of the graph $F(G)$, and we may label these copies by vertices of H. The vertices in each copy have two situations: the vertices in $V(H)$ (we refer to these vertices as black vertices) and the vertices in $E(G)$ (we refer to these vertices as white vertices). Now we join only black vertices with the same name in $F(G)$ in which their corresponding labels are adjacent in H. We

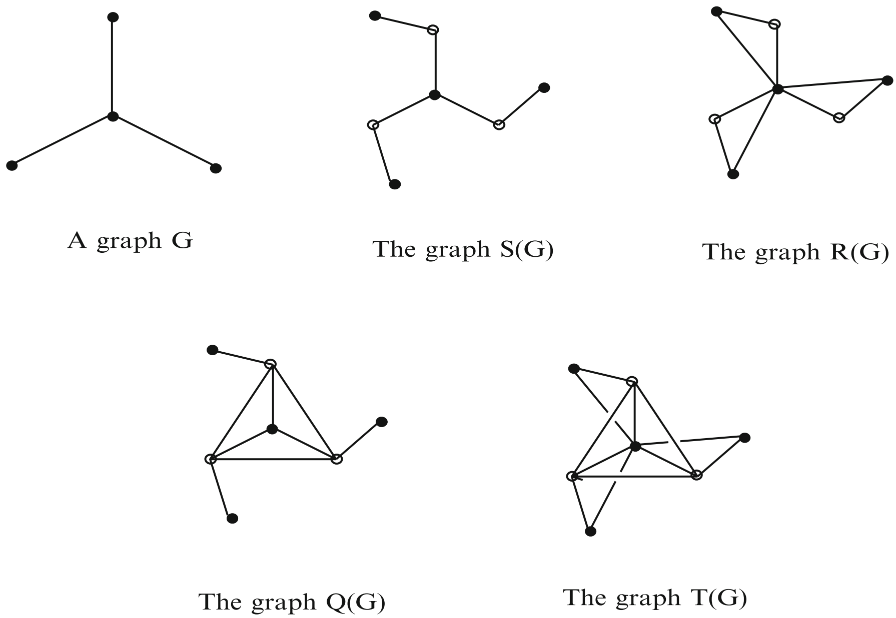

Fig. 15.5 The graph G together with its subdivision $S(G)$, total graph $T(G)$ and the related graphs $R(G)$ and $Q(G)$

illustrate this definition in Fig. 15.6. For more details on these operations we refer the reader to Eliasi and Taeri (2009).

It is obvious from the definition that the bipartite edge frustration of a disconnected graph is equal to the sum of bipartite edge frustration of its components. Hence, it suffices to consider connected graphs. The following observation shows that this type of additive behavior extends also to the graphs with cut-vertices. We will find it useful when dealing with some classes of composite graph introduced above.

Lemma 15.2.3 *Let $v \in V(G)$ be a cut-vertex of a graph G and G_i, $i = 1, \ldots, s$ be the components of $G - \{v\}$. Then $\varphi(G) = \sum_{i=1}^{s} \varphi(G[G_i \cup \{v\}])$. Here $G[G_i \cup \{v\}]$ denotes the graph induced in G by $V(G_i) \cup \{v\}$.*

The notation we used for a graph induced by a certain set of vertices should not be confused with a similar notation used for composition; here in the square brackets is a set of vertices, while in the composition case there is a whole graph. We close the section by formulas for the bipartite edge frustration of cycles and complete graphs.

$$\varphi(C_n) = \frac{1 - (-1)^n}{2} \text{ and } \varphi(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil$$

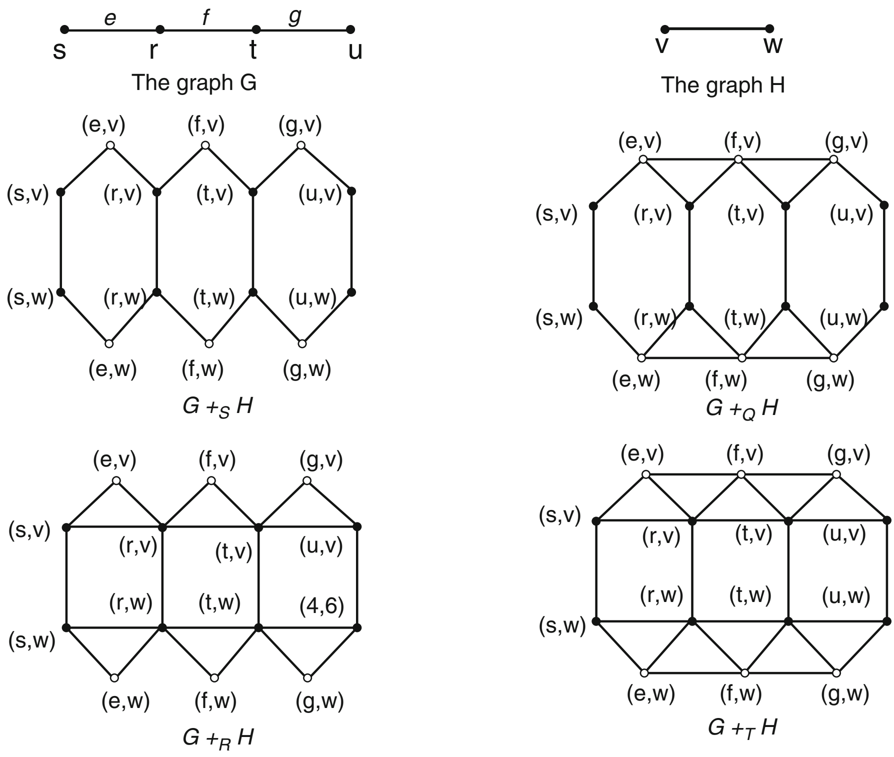

Fig. 15.6 The graphs G, H and $G +_F H$

15.3 The Bipartite Edge Frustration of Some Composite Graphs

The three classes of graphs considered in this section share a certain number of similarities that enable their synoptic treatment. In both chain and bridge graphs their building blocks are so well isolated from each other that their bipartite edge frustrations can be computed separately and then added in order to obtain the bipartite edge frustration of the whole graph. All interaction between components of a bridge graph is via its path backbone, which is itself bipartite. If the backbone is replaced by a non – bipartite scaffold, as in the case of extended bridges, the only additional complication is to compute the bipartite edge frustration of the scaffold graph. This results in (at most) one additional term in the formula for the total bipartite edge frustration.

Theorem 15.3.1

1. Let $G = C(G_1, G_2, \ldots, G_n)$ be a chain graph. Then $\varphi(G) = \sum_{i=1}^{n} \varphi(G_i)$.
2. Let $G = B(G_1, G_2, \ldots, G_n)$ be a bridge graph. Then $\varphi(G) = \sum_{i=1}^{n} \varphi(G_i)$

3. Let $K = EB(G, H_1, H_2, \ldots, H_n)$ be an extended bridge graph. Then $\varphi(K) = \varphi(G) + \sum_{i=1}^{n} \varphi(H_i)$.

The results of this subsection can be specialized in a straightforward way to the cases where all building blocks are identical, yielding the explicit formulas for the bipartite edge frustrations of rooted products of two graphs. Similarly, the results for chain graphs remain valid without any modifications also for splices of two or more graphs and for generalized cactus graphs. The results and proofs follow directly from Lemma 15.2.3, and we leave their formulation and proofs to the reader.

The Cartesian product gives rise to many interesting classes of graphs, such as lattices, tubes, tori, Hamming graphs, and hypercubes, to mention just a few examples.

Theorem 15.3.2. *Let $G_1, G_2, \ldots, G_s$ be connected graphs and $G = \prod_{i=1}^{s} G_i$. Then*

$$\varphi(G) = \prod_{i=1}^{s} |V(G_i)| \sum_{j=1}^{s} \frac{\varphi(G_j)}{|V(G_j)|}.$$

In special case, let B be a bipartite graph on n vertices. Then for any graph G, we have $\varphi(B \times G) = n\varphi(G)$.This case covers the linear polymers $P_n \times G$ induced by an arbitrary graph G. Also, for a non-bipartite graph G on n vertices, $\phi(G^s) = n^{s-1}\phi(G)$.

We present explicit formulas for the bipartite edge frustration of C_4 nanotubes and nanotori in the following example.

Example

(a) $\varphi(P_n \times C_{2m+1}) = n$,
(b) $\varphi(C_{2n} \times C_{2m+1}) = 2n$,
(c) $\varphi(C_{2n+1} \times C_{2m+1}) = 2(m + n + 1)$.

We have already mentioned that $G_1 + G_2$ is non-bipartite as soon as any of its components contains an edge. It is intuitively clear that joins are "very much" non-bipartite, and our findings confirm this feeling.

Theorem 15.3.3 *Let G_1 and G_2 be two connected bipartite graphs with bipartitions (A_1, B_1) and (A_2, B_2), respectively. Let us denote $a_i = |A_i|$ and $b_i = |B_i|$, $i = 1, 2$ and let $a_i \leq b_i$ for $i = 1, 2$, Fig. 15.7. Then*

$$\varphi(G_1 + G_2) \leq \min\{a_1a_2 + b_1b_2,\ a_1b_2 + b_1a_2,\ a_1|V(G_2)| + |E(G_2)|,\ a_2|V(G_1)| + |E(G_1)|,\ |E(G_1)| + |E(G_2)|\}.$$

Now, the logical thing to do would be to proceed and show that the above upper bound is always achieved. The next example shows that this cannot be done in all

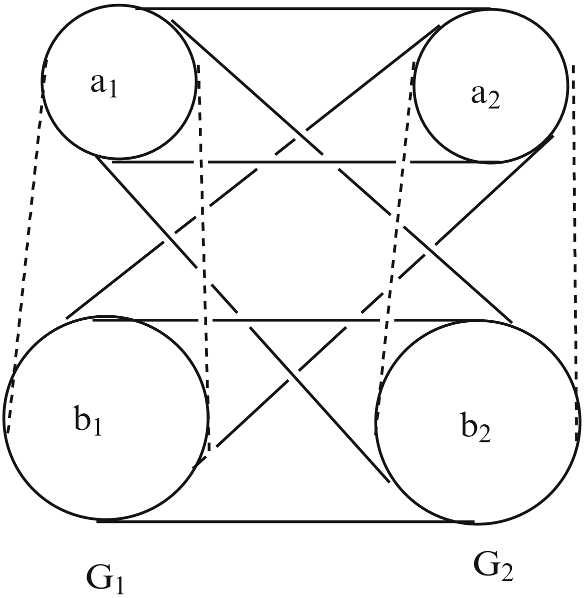

Fig. 15.7 A join of two bipartite graphs.

cases. Take $K_{5,50}$ and attach a path of length 4 to its smaller class by identifying one of its end-vertices with any vertex of the smaller class. Denote the obtained bipartite graph by G_1, and its bipartition by (A_1, B_1). Obviously, $a_1 = 7$, $b_1 = 52$, and $\left|E(G_1)\right| = 254$. Call the vertices from the path exceptional. Take $K_{8,9}$ and call it G_2. Now consider $G = G_1 + G_2$ as shown in Fig. 15.8. By computing all terms of the right-hand side of the inequality of Theorem 15.3.3 it follows that the minimum is achieved

For $a_1\left|V(G_2)\right| + \left|E(G_2)\right| = 191$. Hence G can be made bipartite by deleting 119 edge between A_1 and G_2 and 72 edges of G_2. Let us denote so obtained bipartite graph by G_0. Now take any two vertices of A_2 and connect them to the two exceptional vertices of A_1 by all four possible edges. The new edges are shown by dashed lines in Fig. 15.8. The resulting graph is not bipartite, but it can be made bipartite by removing the three edges connecting the exceptional vertices of A_1 with the exceptional vertices in B_1. The total result is a bipartite spanning subgraph of $G_1 + G_2$ obtained by deleting 190 edges, a strictly smaller number than the minimum of the right-hand side of the inequality of Theorem 15.3.3. With some care the number of vertices in the example could be made smaller, but this is not essential for our conclusion. The inequality of Theorem 15.3.3 can be converted to equality when the minimum of the right-hand side is equal to $|E(G_1)| + |E(G_2)|$.

Theorem 15.3.4 *Let G be a connected bipartite graph on n vertices. Then*

$$\varphi(\nabla S_n) \leq \varphi(\nabla G) \leq \varphi(\nabla P_n).$$

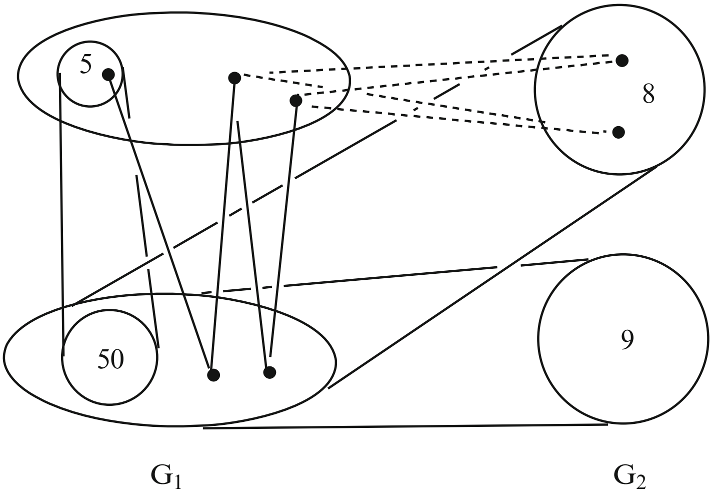

Fig. 15.8 A graph for which the inequality of Theorem 15.3.3 remains strict

The bipartite edge frustration of corona products can be neatly expressed when the non – scaffold graph is bipartite. Again, the result crucially depends on the formula for the bipartite edge frustration of a suspension.

Let us mention to the bipartite edge frustration of double splice and double link of graphs. At first we define a concept that is used for proving the next theorems.

Definition 15.3.5 Let G be a graph. For $a, b \in V(G)$, $\varphi_{a,b}(G)$ is the smallest number of edges that have to be deleted from a graph G to obtain a bipartite spanning subgraph such that a and b are occurred in the same partition. Similarly, we define $\varphi'_{a,b}(G)$, for each a, $b \in V(G)$, as the smallest number of edges that have to be deleted from a graph G to obtain a bipartite spanning subgraph such that a and b are occurred in the different partitions. It is easy to show that $\varphi(G) = \min\{\varphi_{a,b}(G), \varphi'_{a,b}(G)\}$.

Example

1. $\varphi_{a,b}(P_n) = \begin{cases} 0 & 2|d(a,b) \\ 1 & 2|d(a,b) \end{cases}, \quad \varphi'_{a,b}(P_n) = \begin{cases} 1 & 2|d(a,b) \\ 0 & 2|d(a,b) \end{cases},$
2. $\varphi_{a,b}(C_{2n}) = \begin{cases} 0 & 2|d(a,b) \\ 1 & 2|d(a,b) \end{cases}, \quad \varphi'_{a,b}(C_{2n}) = \begin{cases} 1 & 2|d(a,b) \\ 0 & 2|d(a,b) \end{cases},$
3. $\varphi_{a,b}(C_{2n+1}) = \varphi'_{a,b}(C_{2n+1}) = \varphi(C_{2n+1})$, for each a, $b \in V(C_{2n+1})$,
4. $\varphi_{a,b}(K_n) = \varphi'_{a,b}(K_n) = \varphi(K_n)$, for each a, $b \in V(K_n)$.

Remark Let G and H be two graphs. If $ab \in E(G)$ and $cd \in E(H)$, then these edges are identified in double splice graph $(G:H)(a,b:c,d)$. In this case, the number of edges of $(G:H)(a,b:c,d)$ is equal to $|E((G:H)(a,b:c,d))| = |E(G)| + |E(H)| - 1$.

Otherwise, $|E((G:H)(a,b:c,d))|\$\$ = |E(G)|+|E(H)|$. In double splice graph when the vertices a and c are identified, we can certainly assume that $u=a=b$, by similar argument we assume $v=c=d$. Indeed we can assume that, $u,v\in V(G)\cap V(H)$. We abbreviate the notation to $(G:H)$ when the vertices $u,v\in V(G)\cap V(H)$ are clear from context.

In the following theorems formulas for the bipartite edge frustration of double splice and double link of two graphs are computed.

Theorem 15.3.6 *Let G and H be two graphs. For each* $u,v\in V(G)\cap V(H)$ *such that* $|E((G:H)(a,b:c,d))| = |E(G)|+|E(H)|$, *we have*

$$\varphi((G:H)) = \min\left\{\varphi_{u,v}(G)+\varphi_{u,v}(H),\ \varphi'_{u,v}(G)+\varphi'_{u,v}(H)\right\}.$$

Theorem 15.3.7 *Let G and H be two graphs. For each* $u,v\in V(G)\cap V(H)$ *such that* $|E((G:H)(a,b:c,d))| = |E(G)|+|E(H)|-1$, *we have*

$$\varphi((G:H)) = \min\left\{\varphi_{u,v}(G)+\varphi_{u,v}(H)-1,\ \varphi'_{u,v}(G)+\varphi'_{u,v}(H)\right\}.$$

Lemma 15.3.8 *Let G be a connected graph. If* G_0 *be a bipartite subgraph of G by deleting* $\varphi(G)$ *edges, then* G_0 *is connected.*

In the following we obtain formulas for the bipartite edge frustration of double link of graphs.

Theorem 15.3.9 *Let G and H be two graphs. If* $a,b\in V(G)$ *and* $c,d\in V(H)$, *then*

1. If $(\varphi_{a,b}(G)=\varphi(G),\varphi_{c,d}(H)=\varphi(H))$ or $(\varphi'_{a,b}(G)=\varphi(G),\varphi'_{c,d}(H)=\varphi(H))$, then $\varphi((G\approx H)(a,b:c,d)) = \varphi(G)+\varphi(H)$.
2. If $(\varphi_{a,b}(G)=\varphi(G),\varphi'_{c,d}(H)=\varphi(H))$ or $(\varphi'_{a,b}(G)=\varphi(G),\varphi_{c,d}(H)=\varphi(H))$, then $\varphi((G\approx H)(a,b:c,d)) = \varphi(G)+\varphi(H)+1$.

For the sake of completeness, we mention here a theorem of Došlić and Vukičević as follows:

Definition 15.3.10 Let G and H be two connected graphs on disjoint vertex sets, and let $a\in V(G)$ and $b\in V(H)$ An n-link of G and H is a graph obtained by connecting the vertices a and b by a path of length n so that each of these vertices is identified with one of the terminal vertices of P_n. We denote n-link of G and H by $(G\sim_{\mathrm{n}}H)(a,b)$.

Theorem 15.3.11 *Let G and H be two connected graphs with disjoint vertex sets. For each* $a\in V(G)$ *and* $b\in V(H)$, *the bipartite edge frustration of n-link of G and H is obtained as follows:*

$$\varphi\left((G\tilde{}_{\mathrm{n}}H)(a,b)\right) = \varphi(G) + \varphi(H).$$

Definition 15.3.12 Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a, b \in V(G)$ and $c, d \in V(H)$, a $(m, n)-$ link of G and H is defined as the graph $(G \approx_{m,n} H)(a, b : c, d)$ obtained by joining a and c by a path of length m and b and d by another path of length n; see Fig. 15.9.

The following theorem can be proved in much the same way as Theorem 15.3.9.

Theorem 15.3.13 *Let G and H be two graphs and $a, b \in V(G)$ and $c, d \in V(H)$. Then*

1. If $\left(\varphi_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\right)$ or $\left(\varphi'_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\right)$ and $m+n$ be an even number, then $\varphi\left((G \approx_{m,n} H)(a, b : c, d)\right) = \varphi(G) + \varphi(H)$.
2. If $\left(\varphi_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\right)$ or $\left(\varphi'_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\right)$ and m + n be an odd number, then $\varphi\left((G \approx_{m,n} H)(a, b : c, d)\right) = \varphi(G) + \varphi(H) + 1$.
3. If $\left(\varphi_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\right)$ or $\left(\varphi'_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\right)$ and m + n be an even number, then $\varphi\left((G \approx_{m,n} H)(a, b : c, d)\right) = \varphi(G) + \varphi(H) + 1$.
4. If $\left(\varphi_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\right)$ or $\left(\varphi'_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\right)$ and m + n be an odd number, then $\varphi\left((G \approx_{m,n} H)(a, b : c, d)\right) = \varphi(G) + \varphi(H)$.

Finally, we address the study bipartite edge frustration of the hierarchical product of two or more graphs. Some natural generalizations of the hierarchic product are proposed and we consider the bipartite edge frustration of this generalization.

Theorem 15.3.14 *Let $G_i = (V_i, E_i)$ be N graphs with each vertex set V_i, $i = 1, 2, \ldots, N$, and $H = G_N \Pi \cdots \Pi G_2 \Pi G_1$. Then the bipartite edge frustration of H is obtained as follows:*

$$\varphi(H) = \varphi(G_N) + \sum_{i=1}^{N-1} \left(\prod_{j=i+1}^{N} |V_j| \right) \varphi(G_i).$$

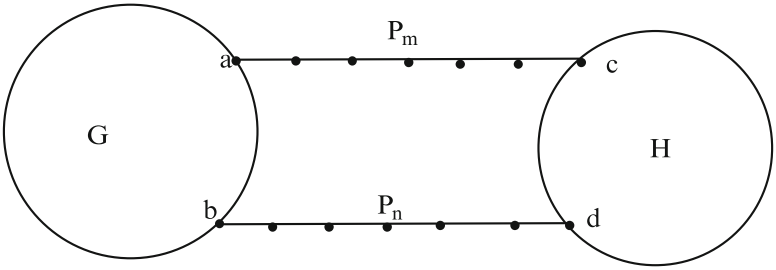

Fig. 15.9 The $(m, n)-$link

Theorem 15.3.15 *Let f↑or i = 1, 2,. . ., N, G_i be a graph and for i = 1, 2, . . ., N − 1, $U_i \subseteq V_i$ and $H_g = G_N \Pi \ldots \Pi G_2(U_2) \Pi G_1(U_1)$. The bipartite edge frustration of H_g is computed as follows:*

$$\varphi\left(H_g\right) = \sum_{i=2}^{N-1} \varphi(G_i) \left(\prod_{k=1}^{i-1} |U_k| \prod_{j=i+1}^{N} |V_j| \right) + \prod_{s=2}^{N} |V_s| \varphi(G_1) + \prod_{s=1}^{N-1} |U_s| \varphi(G_N).$$

Corollary 15.3.16 Let $G_i = (V_i, E_i)$ be N graphs and (nonempty) vertex subsets $U_i \subseteq V_i$, for $1 \leq i \leq N - \underline{1}$. Then the following statements are equivalent:

1. For each $i = 1, 2, \ldots, N$, G_i is bipartite.
2. The graph $H = G_N \Pi \ldots \Pi G_2 \Pi G_1$ is bipartite.
3. The graph $H_g = G_N \Pi \ldots \Pi G_2(U_2) \Pi G_1(U_1)$ is bipartite.

15.4 The Bipartite Edge Frustration of Graphs Under Subdivided Edges and Their Related Sums

In this section, the bipartite edge frustration of two related graphs $S(G)$ and $R(G)$ are computed generally. We also investigate the $\varphi(Q(G))$ in the case that G is a tree or graph with disjoint cycles. A lot of sharp inequalitiesfor $\varphi(Q(G))$ together with a simple inequality for $\varphi(T(G))$ are also proved.

Lemma 15.4.1 *Let G be a graph, then*

1. $\varphi(S(G)) = 0$,
2. $\varphi(R(G)) = |E(G)|$.

Corollary 15.4.2 Let G be a connected graphs on n vertices, then

$$\varphi(R(T_n)) \leq \varphi(R(G)) \leq \varphi(R(K_n)),$$

where T_n is an arbitrary n-vertex tree.

Lemma 15.4.3 *Let G be a graph, then* $\varphi(Q(G)) \geq \sum_{v \in V(G)} \varphi\left(K_{\delta(v)+1}\right)$.

One can show that if G *be a tree or a graph with disjoint cycles, then* $\varphi(Q(G)) = \sum_{v \in V(G)} \varphi\left(K_{\delta(v)+1}\right)$..

A transformation of type ***A*** for a tree is defined as follows. Let T be a tree with n vertices. Choose a maximum path P_{m+1} in T of length, say m. Remove an end vertex of T (which is not in P_{m+1}) and connect a new vertex to one of the end vertices of P_{m+1} to obtain P_{m+2}. This new tree is denoted by T_1. In Fig. 15.10, this

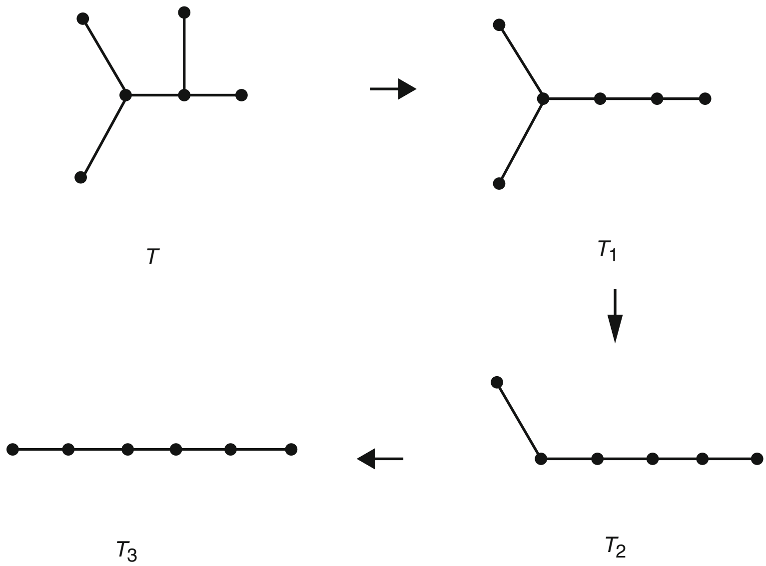

Fig. 15.10 Three transformations of type ***A*** for tree T

process is applied on T (three times) to obtain T_1, T_2 and T_3. Clearly, if T is a path then T_1 is equal to T. Notice that T_i's are not uniquely constructed.

We now define a transformation of type ***B*** for trees. To do this, we assume that T is a tree with n vertices. Suppose v is a vertex of maximum degree and $\delta_T(v) = \Delta(T)$. Omit an end vertex of T which is not adjacent to v and add a new vertex adjacent to this vertex. This new tree is denoted by T^1. It is obvious that $\delta_{T^1}(v) = \Delta(T) + 1$. Clearly, if T is a star then T^1 is isomorphic to T. In Fig. 15.11, this process is applied on T (three times) to obtain T^1, T^2 and T^3. Notice that T^i's are not uniquely constructed, but after finishing the process we will find a star of size $|V(T)|$.

By using these transformations, it is easy to show that, for a tree T,

$$\varphi(Q(T_1)) \leq \varphi(Q(T)) \leq \varphi\left(Q\left(T^1\right)\right),$$

where T_1 is a tree constructed from T transformation of type ***A*** and T^1 is a tree constructed from T transformation of type ***B***. By using this result, one can see that

$$\cdots \leq \varphi(Q(T_i)) \leq \cdots \leq \varphi(Q(T_2)) \leq \varphi(Q(T_1)) \leq \varphi(Q(T))$$

and

$$\varphi(Q(T)) \leq \varphi\left(Q\left(T^1\right)\right) \leq \varphi\left(Q\left(T^2\right)\right) \leq \cdots \leq \varphi\left(Q\left(T^i\right)\right) \leq \cdots$$

By an inductive argument, one can see that there exists a positive integer m such that $T_m = P_n$ and there exists a positive integer k such that $T^k = S_n$. Therefore, this argument proved the following theorem.

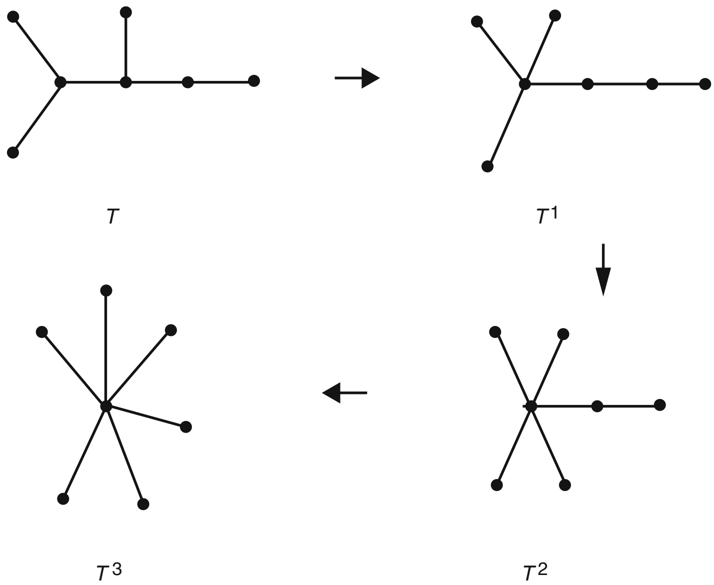

Fig. 15.11 Three transformations of type ***B*** for tree T

Theorem 15.4.4 *Let T be a tree on n vertices, then*

$$\varphi(Q(P_n)) \leq \varphi(Q(T)) \leq \varphi(Q(S_n)).$$

Hence it is clear that $\varphi(Q(P_n)) \leq \varphi(Q(T)) \leq \varphi(Q(K_n))$, and it is straightforward to prove that $\varphi(Q(G)) \leq \varphi(Q(T)) \leq \varphi(Q(G)) + |E(G)|$.

Theorem 15.4.5 *Let G and H be two graphs then*

$$\varphi(G +_F H) = |V(G)|\varphi(H) + |V(H)|\varphi(F(G)).$$

15.5 Results of the Nordhaus-Gaddum Type

A *Nordhaus-Gaddum*-type result is a lower or upper bound on the sum or product of an invariant of a graph and its complement. It is named after a paper (Nordhaus and Gaddum 1956) in which Nordhaus and Gaddum gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results were obtained for many other invariants. A trivial lower bound $\varphi(G) + \varphi(\overline{G}) \geq 1$ is valid for all graphs on more than five vertices. It follows from Ramsey's theorem, since at least one of G and $\overline{G}$ contains a triangle if G has at least six vertices.

We start with the following simple observation: Let G be a graph and e an edge not in $E(G)$. Then $\varphi(G+e) \leq \varphi(G)+1$. In other words, adding an edge to a graph cannot increase its bipartite edge frustration by more than one. By the same reasoning, one can establish that $\varphi(G+e_1+\cdots+e_k) \leq \varphi(G)+k$ for any choice $e_1,\ldots,e_k$ of edges not in $E(G)$.

Now we tackle the reverse problem: what happens to $\varphi(G)$ if we delete edges from G? It is intuitively clear that the answer depends on the density of edges in G; for graphs rich in edges, each edge removal will affect the value of $\varphi(G)$, while for graphs with few edges, the removal is less likely to have an effect.

Theorem 15.5.1 *Let* $G = K_n - \{e_1,\ldots,e_l\}$.

1. If $l \leq \lfloor \frac{n+1}{4} \rfloor$, then $\varphi(G) = \varphi(K_n) - l$.
2. If $l > \lfloor \frac{n+1}{4} \rfloor$, then $\varphi(K_n) - l \leq \varphi(G) \leq \varphi(K_n) - \lfloor \frac{n+1}{4} \rfloor$.

Let us now consider $G = K_n - \{e_1,\ldots,e_l\}$. *Then* $\overline{G} = K_n - \left\{ e_{l+1},\ldots,e_{\binom{n}{2}} \right\}$. *We can relabel the edges of* $\overline{G}$ *by subtracting* l *from their labels, so that* G *can be written as* $\overline{G} = K_n - \{e_1,\ldots,e_s\}$.*, where* $s = \binom{n}{2} - l$. *It is easy to see that at least one of the numbers* l *and* s *must exceed the critical value* $\lfloor \frac{n+1}{4} \rfloor$. *Depending on whether the other one also exceeds it or not, we have three different situations. Two of them are symmetric, hence it suffices to consider only one of them. We look at the case* $l \leq \lfloor \frac{n+1}{4} \rfloor$ *and* $s > \lfloor \frac{n+1}{4} \rfloor$ *first. By Theorem 10 it immediately follows that*

$$\varphi(G) + \varphi(\overline{G}) \leq 2\varphi(K_n) - \left\lfloor \frac{n+1}{4} \right\rfloor - \left|E(\overline{G})\right|.$$

Finally, when both $l, s > \lfloor \frac{n+1}{4} \rfloor$, we obtain

$$\varphi(G) + \varphi(\overline{G}) \leq 2\varphi(K_n) - 2\left\lfloor \frac{n+1}{4} \right\rfloor.$$

In general case, when nothing is known on the value of $E(G)$, we have an upper bound equal to the worst case, i.e., to the maximum of the above three expressions. By plugging in the formula for $\varphi(K_n)$ and rearranging the terms, we obtain an upper bound valid for all graphs.

Theorem 15.5.2

$$\varphi(G) + \varphi(\overline{G}) \leq 2\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil - \left\lfloor \frac{n+1}{4} \right\rfloor - \min\left\{ \left|E(G)\right|, \left|E(\overline{G})\right|, \left\lfloor \frac{n+1}{4} \right\rfloor \right\}.$$

For a bipartite graph $\varphi(G) + \varphi(\overline{G}) = \varphi(\overline{G})$. By combining this fact with the above results, we can determine the bipartite edge frustration of complements of some graphs with (relatively) few edges.

Example

1. Let T_n be a tree $n \geq 7$ vertices. Then $\varphi(\overline{T}_n) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil - (n-1)$.
2. Let C_n be a tree $n \geq 10$ vertices. Then $\varphi(\overline{C}_n) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil - n$.

It is tempting to think of $\varphi(\overline{G})$ as of a measure of bipartivity of a given bipartite graph G: the more frustration a bipartite graph leaves to its complement, the more bipartite it is. Based on this idea, we could say that the trees are the "most bipartite" among all connected graphs on the same number of vertices.

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