The Kolmogorov-Arnold-Moser (KAM) and Nekhoroshev Theorems with Arbitrary Time Dependence

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Abstract The Kolmogorov-Arnold-Moser (KAM) theorem and the Nekhoroshev theorem are the two "pillars" of canonical perturbation theory for near-integrable Hamiltonian systems. Over the years there have been many extensions and generalizations of these fundamental results, but it is only very recently that extensions of these theorems near-integrable Hamiltonian systems having explicit, and aperiodic, time dependence have been developed. We will discuss these results, with particular emphasis on the new mathematical issues that arise when treating aperiodic time dependence.

1 Introduction

Vladimir Arnold's contributions to mathematics and mechanics are truly remarkable for both their breadth and depth. In this article we discuss an area where he made contributions that are essential to understand for any student or researcher in the field of Hamiltonian dynamics. In particular, "Arnold" is the middle name on the famous Kolmogorov-Arnold-Moser (KAM) theorem [1–3], which gives sufficient conditions for the existence of quasiperiodic motion in near-integrable Hamiltonian systems (expressed in the action-angle variables of the unperturbed integrable Hamiltonian system). Another theorem in the same field (and with a very similar setup), due to Nekhoroshev [4], describes stability of the action variables over and exponentially long time interval. Together, the KAM and Nekhoroshev theorems are the two "rigorous pillars" that establish canonical perturbation theory of near-integrable Hamiltonian systems. A recent monograph that traces the historical development of this theory in some detail is [5].

Despite the firm establishment of the "KAM theory" and "Nekhoroshev theory" in the mathematics, physical science, and engineering disciplines, there is an important area that has not been addressed. In particular, the development of similar types of perturbation theorems for near-integrable systems having "general" time

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dependence, i.e., when the perturbation of the integrable Hamiltonian (or of a particular motion), is not required to be neither periodic nor quasiperiodic. We shall refer to this class of perturbation as *aperiodic*. The motivation for such results comes from applications, e.g. the study of transport and mixing in fluid mechanics from the dynamical systems point of view (see [6] for a discussion of the issues from the Hamiltonian dynamic point of view that arise in this field).

While our goal here is not to review KAM and Nekhoroshev theory (the monograph of Dumas [5] does an excellent job of this), we do note some of the issues such as results for time-dependent, near-integrable Hamiltonian systems. Essentially all of the literature (with a few notable exceptions that we will mention toward the end of this introduction) concerned with time-dependent Hamiltonian systems deal with periodic or quasiperiodic time dependence. For such time dependence, the problems can often be cast in a form where classical results and approaches can be applied. The monographs [7, 8] discuss some of these topics. The paper [9] develops a KAM type result for quasiperiodically time-dependent systems and the paper [10] develops a Nekhoroshev result for the same class of systems. The first paper to develop a Nekhoroshev type result for Hamiltonian systems with general time dependence was [11]. The form of the system they treated was somewhat different than the classical near-integrable Hamiltonian systems since their goals were somewhat different. The first papers to develop Nekhoroshev type results for systems with general time dependence in the classical setting were [12, 13], and the only paper treating a KAM type result in the classical setting is [14]. The purpose of this paper is to describe the results in these latter papers dealing with aperiodic time dependence, with particular attention on the issues that arise for explicitly time-dependent Hamiltonians and the correspondent regularity hypotheses that the perturbation function is required to satisfy. In Sect. 2 we discuss the Nekhoroshev theorem and in Sect. 3 we discuss the KAM theorem.

2 A Nekhoroshev Theorem with Aperiodic Time Dependence

In this section we describe the setup and strategy for the proof of the theorem. This will provide us with the background and framework for providing a description of the theorem. We follow closely the setup in [11] (but see [15] for a detailed development of the canonical perturbation theory and the Nekhoroshev theory, including historical background).

2.1 The Setup and Assumptions

We consider a near-integrable, slowly varying (to be quantified shortly) time-dependent Hamiltonian system expressed in the action-angle variables of the unperturbed system of the following form:

$$\mathscr{H}(I, \varphi, t) := h(I) + \varepsilon f(I, \varphi, \mu t). \tag{1}$$

We note the following:

- $\varepsilon, \mu > 0$ small parameters.
- $I = (I_1, \dots, I_n) \in \mathscr{G}$, $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n$ denote action-angle variables, where $\mathscr{G} \subset \mathbb{R}^n$ is an open set.
- The dependence on t is, in general, aperiodic, i.e., it need not be periodic or quasiperiodic.

We will rewrite the time-dependent Hamiltonian (1) as a time-independent Hamiltonian by defining two new conjugate variables in the standard way. If we define $\xi := \mu t$ and η as the new conjugate variable pair, the Hamiltonian (1) takes the autonomous form on $\mathscr{D} := \mathscr{G} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R} \ni (I, \eta, \varphi, \xi)$.

$$H(I, \varphi, \eta, \xi) := h(I) + \mu\eta + \varepsilon f(I, \varphi, \xi). \tag{2}$$

Since we use complex function techniques in the proof of the theorem, we will need to complexify the real domain of the Hamiltonian. Let $\sigma, \rho > 0$ be real numbers. Then we define $\mathscr{D}_{\rho,2\sigma} := \mathscr{G}_\rho \times \mathscr{R}_\rho \times \mathbb{T}^n_{2\sigma} \times \mathscr{S}_\sigma$ to be a *complex neighborhood* of $\mathscr{D}$, where

$$\begin{aligned} \mathscr{G}_\rho &:= \bigcup_{I \in \mathscr{G}} \Delta_\rho(I), & \Delta_\rho(I) &:= \{\hat{I} \in \mathbb{C}^n : |\hat{I} - I| < \rho\}, \\ \mathscr{R}_\rho &:= \{\eta \in \mathbb{C} : |\Im\eta| < \rho\}, & \mathbb{T}^n_{2\sigma} &:= \{\varphi \in \mathbb{C}^n : |\Im\varphi| < 2\sigma\}, \\ \mathscr{S}_\sigma &:= \{\xi \in \mathbb{C} : |\Im\xi| < \sigma\}. \end{aligned}$$

The case $n = 1$ is illustrated in Fig. 1.

Then we assume that $h(I)$ and $f(I, \varphi, \xi)$ are holomorphic on $\mathscr{D}_{\rho,2\sigma}$. Furthermore, we also make a standard assumption on the *unperturbed* Hamiltonian.

Hypothesis 2.1 (Convexity). *There exists two constants $M \geq m > 0$ such that, for all $I \in \mathscr{G}_\rho$*

$$|\partial_I^2 h(I) v| \leq M|v|, \qquad |\langle \partial_I^2 h(I) v, v\rangle| \geq m|v|^2, \tag{3}$$

for all $v \in \mathbb{R}^n$.

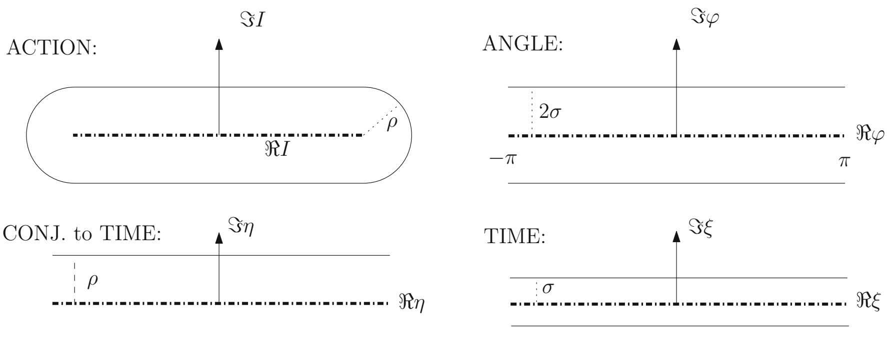

Fig. 1 The sets $\mathscr{D}$ (*dash dotted*) and $\mathscr{D}_{\rho,2\sigma}$ for $n = 1$

2.2 Statement of the Theorem

We can now state a version of the Nekhoroshev theorem for time-dependent Hamiltonian systems having a general time dependence. First, we define

$$\tilde{\mathscr{F}} := \sup_{\mathscr{D}_{\rho,2\sigma}} |f| \left(\frac{1+e^{-\frac{\sigma}{2}}}{1-e^{-\frac{\sigma}{2}}} \right)^n, \qquad \lambda_{\varepsilon,\mu} := \mu + e\tilde{\mathscr{F}}\varepsilon, \tag{4}$$

and we note that the set of parameters $\rho, \sigma, M, m, and \tilde{\mathscr{F}}$ are characteristic of a given Hamiltonian H. Now we state the theorem.

Theorem 2.2 (Aperiodic Nekhoroshev Theorem). *Assume the convexity hypothesis above. Then there exists constants Δ^* and $\mathscr{T}$, depending on $\rho, \sigma, M, m, \tilde{\mathscr{F}}$, and n, such that if ε and μ satisfy*

$$\lambda_{\varepsilon,\mu} < 1/(3^4 \Delta^*), \tag{5}$$

then orbits $(I(t), \varphi(t))$ of the Hamiltonian system (2) starting in $\mathscr{G} \times \mathbb{T}^n$ at t_0, satisfy

$$|I(t) - I(t_0)| < (\Delta^* \lambda_{\mu,\varepsilon})^{\frac{1}{4}} \rho, \qquad for \qquad |t - t_0| < \frac{\mathscr{T}}{\varepsilon} \exp\left[\left(\frac{1}{\Delta^* \lambda_{\mu,\varepsilon}} \right)^{\frac{1}{2n(n+1)}} \right].$$

We note that within the threshold (5), ε and μ are independent. We refer this as *unconditionally slow time dependence*.

2.2.1 Scheme of the Proof

The classical proof of Nekhoroshev is divided into two parts:

Analytic part (normal form lemma): For the analytic part, we construct an ε-close to the identity canonical change $\mathscr{C}_r$ casting H into the *normal form*:

$$H_N := H \circ \mathscr{C}_r = h(I) + \mu\eta + Z^{(r)} + \mathscr{R}^{(r+1)}. \tag{6}$$

We note the following:

- The result is local: it holds on sets called *non-resonance domains*.
- $\mathscr{C}_r$ is the composition of $r < \infty$ canonical transformations.

Geometric part (global result): This is an extremely clever contribution of Nekhoroshev [4] that shows how to cover the entire phase space $\mathscr{D}$ with non-resonance domains where the normal form lemma can be applied. See also [16, 17].

Remark 2.3. It is important to note that the Hamiltonian is not normalized with respect to the variables (ξ, η) (it is a "partial normal form"). Hence the same geometric result of the time-independent classical Nekhoroshev theorem applies.

The construction of the abovementioned normal form is classically achieved in two steps. First a formal perturbation scheme is developed, based on the Lie transform method that yields a normal form on non-resonance domains. Second, we consider the properties of the normal form on the appropriate domains and the "optimal" choice of parameters leading to *exponentially small estimates*.

We give a brief overview of the "formal scheme" for developing the normal form.

Formal Scheme

Step 1: Expand the perturbation as follows:

$$f(I, \varphi, \xi) = \sum_{k \in \mathbb{Z}^n} f_k(I, \xi) e^{ik \cdot \varphi}.$$

Given $K \in \mathbb{N}$ (to be determined afterward in an "optimal" way with respect to all of the parameters in a way that makes the remainder small), we write the expanded Hamiltonian in such a way it is decomposed into suitable "levels" (sets of Fourier harmonics in this case) in order to apply the Lie transform method :

$$H = h(I) + \mu\eta + H_1 + H_2 + \cdots, \qquad H_s := \varepsilon \sum_{(s-1)K \leq |k| < sK} f_k(I, \xi) e^{ik \cdot \varphi}.$$

Step 2 (Lie transform method): The aim is to find $\chi^{(r)} := \{\chi_s\}_{s=1,\ldots,r}$ such that $T_{\chi^{(r)}} H = H_N$, where

$$T_{\chi^{(r)}} := \sum_{s \geq 0} E_s, \qquad E_s := \begin{cases} \mathrm{id} & s = 0 \\ \dfrac{1}{s} \displaystyle\sum_{j=1}^{s} j \mathscr{L}_{\chi_j} E_{s-j} & s \geq 1 \end{cases}$$

and $\mathscr{L}_f g := \{f, g\} = \partial_\varphi f \partial_I g + \partial_\xi f \partial_\eta g - \partial_\varphi g \partial_I f - \partial_\xi g \partial_\eta f$ is the Lie derivative.

Step 3 Hierarchy of homological equations: Each χ_s is determined as a solution of a *homological equation*:

$$\mathscr{L}_h \chi_s + Z_s = \psi_s, \qquad s = 1, \ldots, r,$$

where $Z^{(r)} = Z_1 + \cdots + Z_r$, where Z_s contains the same harmonics as H_s

$$\psi_s := \begin{cases} H_1 & s = 1 \\ H_s + \mu E_{s-1}\eta + \dfrac{1}{s} \displaystyle\sum_{j=1}^{s-1} j[\mathscr{L}_{\chi_j} H_{s-j} + E_{s-j} H_j] & 2 \leq s \leq r \end{cases}$$

We make the following remarks concerning the solution of the homological equations.

- The solution is found in the Fourier space, by expanding χ_s, Z_s, and ψ_s in a Fourier expansion in the angles.
- The term $\mu E_{s-1}\eta$ is the *extra-term* due to the aperiodic time dependence.

Convergence: Consider on $\mathscr{D}_{\rho,2\sigma}$ the *Fourier norm*

$$\|F\|_{(\rho,\sigma)} := \sum_{k\in\mathbb{Z}^n} \left(\sup_{\mathscr{D}_{\rho,\sigma}} |f_k|\right) e^{|k|\sigma},$$

with f_k Fourier coefficients of F and $|k| := |k_1| + \cdots + |k_n|$. The following lemma of Giorgilli describes the type of estimates that are required in order to establish the convergence of the formal scheme.

Lemma 1 (Giorgilli). *Suppose that there exist $h > 0$ and $\mathscr{F}, b \geq 0$ such that*

$$\|H_s\|_{(\rho,\sigma)} \leq h^{s-1}\mathscr{F}, \qquad \|\psi_s\|_{(1-d)(\rho,\sigma)} \leq \frac{b^{s-1}}{s}\mathscr{F} \tag{7}$$

for all $s \geq 1$ and for all $d \in (0, 1/4)$. Then, if $\mathscr{F}$ and b are sufficiently small, the operator $T_{\chi^{(r)}}$ (and its inverse $T^{-1}_{\chi^{(r)}}$) defines a canonical transformation on the domain $\mathscr{D}_{(1-d)(\rho,\sigma)}$.

After having bounded the above-described extra-term with the tools used in [15], one can see that the constraints imposed by condition (7) lead to more involved estimates with respect to the autonomous case. More precisely, the system of recurrence equations arising from (7) forbids straightforward bounds as in [15] but requires an ad hoc analysis, carried out in this case with the use of the generating function method. See [13] for the details.
The smallness condition of μ required by (5) turns out to be an essential ingredient in order to satisfy condition (7).

3 A KAM Theorem with Aperiodic Time Dependence

In this section we describe the setup and strategy for the proof of the theorem. This will provide us with the background and framework for providing a description of the theorem. Our approach follows closely the Lie transform approach to Kolmogorov's original version of the KAM theorem given in [18].

3.1 The Setup and Assumptions

The setup and assumptions are different than those for the Nekhoroshev theorem. We will comment more in this later on.

We consider a near-integrable, quadratic in P, time-dependent Hamiltonian expressed in the action-angle variables of the unperturbed system of the following form:

$$\mathscr{H}(P,Q,t) = \frac{1}{2}\langle \Gamma P, P\rangle + \varepsilon f(P,Q,t), \tag{8}$$

where:

- Γ is a real non-singular $n \times n$ matrix.
- ε is a small parameter.

We will focus on the preservation of a particular torus (in the spirit of the original Kolmogorov theorem). Therefore, we consider a particular $\hat{P}$, we translate the coordinates $(p,q) := (Q, P - \hat{P})$ so that they are "centered" on the torus of interest, and we transform the time-dependent Hamiltonian to an autonomous Hamiltonian, as above, by introducing a new conjugate pair of coordinates. The Hamiltonian that we obtain in this way has the form:

$$H(p,q,\eta,\xi) = \langle \omega, p\rangle + \frac{1}{2}\langle \Gamma p, p\rangle + \eta + \varepsilon f(p,q,\xi),$$

where:

- $\xi := t$ and $\eta \in \mathbb{R}$ is its conjugate momentum.
- $\omega := \Gamma \hat{P}$.
- $(p,q,\eta,\xi) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^+ =: \mathscr{D}$.

We next define a complex extension to the domain. We let $\sigma, \rho, and \zeta > 0$, and then $\mathscr{D}_{\rho,\sigma,\zeta} := \Delta_\rho \times \mathbb{T}^n_\sigma \times \mathscr{S}_\rho \times \mathscr{R}_\zeta$ is defined to be the *complex neighborhood* of $\mathscr{D}$ where

$$\Delta_\rho := \{p \in \mathbb{C}^n : |p| \le \rho\}, \qquad \mathbb{T}^n_\sigma := \{q \in \mathbb{C}^n : |\Im q| \le \sigma\},$$
$$\mathscr{S}_\rho := \{\eta \in \mathbb{C} : |\Im \eta| \le \rho\}, \qquad \mathscr{R}_\zeta := \{\xi \in \mathbb{C} : \Re \xi \ge -\zeta;\ |\Im \xi| \le \zeta\}.$$

We endow $\mathscr{D}$ with the Fourier norm defined as

$$\|g\|_{[\rho,\sigma;\zeta]} := \sum_{k \in \mathbb{Z}^n} \sup_{p \in \mathscr{D}_{\rho,\sigma,\zeta}} |g_k(p,\xi)| e^{|k|\sigma}.$$

We make the following assumptions.

Hypothesis 3.1 (I). *There exists* $m \in (0,1)$ *such that, for all* $v \in \mathbb{C}^n$

$$|\Gamma v| \le m^{-1}|v|.$$

Hypothesis 3.2 (II, Slow Decay). *The perturbation is an holomorphic function on $\mathscr{D}$ satisfying*

$$\|f(q,p,\xi)\|_{[\rho,\sigma;\zeta]} \leq M_f e^{-a|\xi|},$$

for some $M_f > 0$ and $a \in (0,1)$.

3.2 Statement of the Theorem

Now we can state the theorem

Theorem 3.3 (Aperiodic Kolmogorov Theorem). *Assume hypotheses I and II and suppose that $\hat{P}$ is such that ω is a $\gamma - \tau$ Diophantine vector. Then, for all $a \in (0,1)$, there exists $\varepsilon_a > 0$ such that, for all $\varepsilon \in (0, \varepsilon_a]$, it is possible to find a canonical, ε-close to the identity, analytic change of variables $(q,p,\xi,\eta) = \mathscr{K}(q^{(\infty)}, p^{(\infty)}, \xi, \eta^{(\infty)})$, $\mathscr{K} : \mathscr{D}_* \to \mathscr{D}$ with $\mathscr{D}_* \subset \mathscr{D}$, transforming the Hamiltonian (1) into the* Kolmogorov normal form

$$H_\infty(q^{(\infty)}, p^{(\infty)}, \xi, \eta^{(\infty)}) = \langle \omega, p^{(\infty)} \rangle + \eta^{(\infty)} + \mathscr{Q}(q^{(\infty)}, p^{(\infty)}, \xi; \varepsilon),$$

where $\mathscr{Q}$ is a homogeneous polynomial of degree 2 in p.

Remark 3.4. We note that no restrictions are imposed on a, which implies that the decay of the time dependence can be arbitrarily slow. On the other hand, the threshold is of the form $\varepsilon_a \leq Ca^3$, with C (very small!) constant.

3.2.1 Scheme of the Proof

The proof follows the classical iterative approach following the Lie transform approach of [18] and it is carried out along the lines of [19]. In particular, it is organized as follows:

Step I (Induction basis) We rewrite the Hamiltonian H in the following form:

$$H_j = \langle \omega, p \rangle + \eta + A^{(j)}(q,\xi) + \langle B^{(j)}(q,\xi), p \rangle + \frac{1}{2}\langle C^{(j)}(q,\xi)p, p \rangle, \tag{9}$$

where $j = 0$ denotes zeroth step in the induction process, and for this reason, we set $H_0 := H$.

Step II (Perturbative scheme, formal part) For all j, a generating function χ_j is chosen in such a way the action of $\exp(\mathscr{L}_{\chi_j})$ on H_j removes $A^{(j)}$ and $B^{(j)}$. χ_j is such that $H_{j+1} := \exp(\mathscr{L}_{\chi_j})H_j$ has the same form (9).

Step III (Perturbative scheme, quantitative part) We show that the "unwanted terms" $A^{(j)}$ and $B^{(j)}$ get "smaller and smaller" as j increases. More precisely

$$\max\left\{ \left\|A^{(j)}\right\|_{[\sigma_j;\zeta_j]}, \left\|B^{(j)}\right\|_{[\sigma_j;\zeta_j]} \right\} \leq \epsilon_j e^{-a|\xi|},$$

with ϵ_j (quadratically) infinitesimal as $j \to \infty$, while $\sigma_j \geq \sigma_* > 0$. The desired canonical transformation is obtained by setting

$$\mathscr{K} := \lim_{j\to\infty} \exp(\mathscr{L}_{\chi_j}) \circ \exp(\mathscr{L}_{\chi_{j-1}}) \circ \ldots \circ \exp(\mathscr{L}_{\chi_0}).$$

The composition $H \circ \mathscr{K}$ produces the desired *Kolmogorov normal form*.

Time-dependent homological equation: The equation for the determination of χ_j at each stage of the normalization algorithm is of the form

$$\partial_\xi \varphi + \omega \cdot \partial_q \varphi = \psi, \tag{10}$$

with $\psi = \psi(q, \xi)$ given.

Equation (10) is the novelty of our analysis, and it reflects a remarkable conceptual difference with the normalization algorithm used for the Nekhoroshev theorem. Basically, the latter uses the fact that the number of normalization steps is finite: the contribution of the aperiodic term is controlled only over a finite timespan and the constant Δ^* of formula (5) tends to zero as $r \to \infty$. The situation is substantially different in the Kolmogorov scheme, in which the number of normalization steps is infinite, and the only way to control the effect of the time is to annihilate it at each stage of the algorithm with Eq. (10). The properties of its solution are described in the following lemma.

Lemma 2. *Let $\delta \in [0, 1)$ and suppose that ψ satisfies*

$$\|\psi\|_{[(1-\delta)\sigma;\zeta]} \leq K e^{-a|\xi|},$$

*(*exponential decay*). Then for all $d \in (0, 1-\delta)$ and for all ζ such that $2|\omega|\zeta \leq d\sigma$, the solution of (10) exists and satisfies*

$$\|\varphi\|_{[(1-\delta-d)\sigma;\zeta]} \leq \frac{KS}{a(d\sigma)^{2\tau}} e^{-a|\xi|}, \qquad S \geq 0. \tag{11}$$

Remark 3.5. Finally, we note that the exponential rate of the decay is not necessary and is used for simplicity. However a decay hypothesis is essential in order to ensure the existence of the integrals appearing in the bounds which lead to (11).

4 Summary and Outlook

The aim of this paper was to give an overview of the Nekhoroshev and Kolmogorov stability-type results for integrable Hamiltonian systems subject to aperiodic time-dependent perturbations, obtained in the papers [13] and [14]. These are recently added *tesseræ* to the rich mosaic of the Stability Theory of Hamiltonian Systems, one of the several fields in which V.I. Arnold made so many fundamental contributions.

The motivation for generalizing the classical Nekhoroshev and KAM theorems to include explicit, but arbitrary, time dependence arises from many applications. Most notably, applications of the dynamical systems approach to the study of Lagrangian transport in fluid mechanics, as described in [6]. Hopefully, the results in this paper will serve as motivation to analyze other possibilities for the generalization of these fundamental results in Hamiltonian perturbation theory and, thus, extend both the mathematical framework and the range of applications to which these results can be applied.

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References

1. Kolmogorov, A.N.: On conservation of conditionally periodic motions under small perturbations of the Hamiltonian. Dokl. Akad. Nauk. USSR **98**(4), 527–530 (1954)
2. Arnold, V.I.: Proof of A. N. Kolmogorov's theorem on the preservation of quasiperodic motions under small perturbations of the Hamiltonian. Russ. Math. Surv. **18**(5), 9–36 (1963)
3. Moser, J.: On invariant curves of an area preserving mappings of an annulus. Nachr. Akad. Wiss. Göttingen II. Math. Phys. Kl. **25**, 1–20 (1962)
4. Nekhoroshev, N.N.: An exponential estimate on the time of stabilty of nearly-integrable Hamiltonian systems. Russ. Math. Surv, 32:1–65, 1977.
5. Scott Dumas, H.: The KAM Story. World Scientific, Singapore (2014)
6. Wiggins, S., Mancho, A.M.: Barriers to transport in aperiodically time-dependent two-dimensional velocity fields: Nekhoroshev's theorem and "nearly invariant" tori. Nonlinear Process. Geophys. **21**(1), 165–185 (2014)
7. Arnold, V.I., Kozlov, V.V., Neishtadt, A.I.: Mathematical Aspects of Classical and Celestial Mechanics. Dynamical Systems, vol. III. Springer, New York (1988)
8. Broer, H.W., Huitema, G.B., Sevryuk, M.B.: Quasi-Periodic Motions in Families of Dynamical Systems. Lecture Notes in Mathematics, vol. 1645. Springer, New York (1996)
9. Celletti, A., Falcolini, C., Porzio, A.: Rigorous KAM stability statements for non-autonomous one-dimensional Hamiltonian systems. Rend. Sem. Mat. Torino **45**, 43–70 (1987)
10. Jorba, A., Simo, C.: On quasiperiodic perturbations of elliptic equilibrium points. SIAM J. Math. Anal. **27**(6), 1704–1737 (1996)
11. Giorgilli, A., Zehnder, E.: Exponential stability for time dependent potentials. Z. Angew. Math. Phys. **43**, 827–855 (1992)
12. Bounemoura, A.: Effective stability for slow time-dependent near-integrable hamiltonians and application. C. R. Math. Acad. Sci. Paris **351**(17–18), 673–676 (2013)
13. Fortunati, A., Wiggins, S.: Normal form and Nekhoroshev stability for nearly integrable Hamiltonian systems with unconditionally slow aperiodic time dependence. Regul. Chaotic Dyn. **19**(3), 363–373 (2014)
14. Fortunati, A., Wiggins, S.: Persistence of Diophantine flows for quadratic nearly integrable Hamiltonians under slowly decaying aperiodic time dependence. Regul. Chaotic Dyn. **19**(5), 586–600 (2014)
15. Giorgilli, A.: Notes on exponential stability of Hamiltonian systems. In: Dynamical Systems. Part I. Hamiltonian Systems and Celestial Mechanics. Centro di Recerca Matematica Ennio De Giorgi, Scuola Normale Superiore, Pisa (200)

16. Benettin, G., Galgani, L., Giorgilli, A.: A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems. Celest. Mech. **37**, 1–25 (1985)
17. Pöschel, J.: Nekhoroshev estimates for quasi-convex Hamiltonian systems. Math. Z. **213**(2), 187–216 (1993)
18. Benettin, G., Galgani, L., Giorgilli, A., Strelcyn, J.M.: A proof of Kolmogorovs theorem on invariant tori using canonical transformations defined by the Lie method. Nuovo Cimento B (11) **79**(2), 201–223 (1984)
19. Giorgilli, A.: Persistence of invariant tori. http://www.mat.unimi.it/users/antonio/hamsys/hamsys.html