Some Generalizations of Fixed-Point Theorems on S-Metric Spaces

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Abstract In this paper we prove new fixed-point theorems on complete S-metric spaces. Our results generalize and extend some fixed-point theorems in the literature. We give some examples to show the validity of our fixed-point results.

1 Introduction

Metric spaces are very important in the various areas of mathematics such as analysis, topology, applied mathematics, etc. So it has been studied new generalizations of metric spaces. Recently in 2012, Sedghi et al. have defined the concept of S-metric spaces [13].

Many authors have defined some contractive mappings on complete metric spaces as a generalization of the well-known Banach's contraction principle. In 1974, Ciric studied a generalization of Banach's contraction principle and gave quasi-contractions [3]. In 1979, Fisher proved new fixed-point theorems for quasi-contractions and continuous self-mappings [5]. In 1977, Rhoades investigated some comparisons of various contractive mappings and introduced a new contractive mapping called a Rhoades' mapping [11]. He studied some fixed-point theorems. But he did not have any fixed-point theorem for a Rhoades' mapping. Hence in 1986, Chang introduced the concept of a C-mapping and obtained some fixed-point theorems using this mapping for a Rhoades' mapping [1]. In 1988, Liu et al. defined the notion of L-mapping to give necessary and sufficient conditions for the existence of a fixed point for a Rhoades' mapping [8]. In 1990, Chang and Zhong proved some fixed-point theorems using the notion of periodic point [2].

The fixed-point theory in various metric spaces was also studied. For example, in 2013 Gupta presented the concept of cyclic contraction on S-metric spaces [6]. In 2014, Sedghi and Dung proved some fixed-point theorems and gave some analogues of fixed-point theorems in metric spaces for S-metric spaces [12]. Hieu et al. gave the relation between a metric and an S-metric [7]. In 2014, Dung et al. proved some generalized fixed-point theorems for g-monotone maps on partially ordered S-metric

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spaces [4]. The present authors defined Rhoades' condition on S-metric spaces and proved some fixed-point theorems satisfying Rhoades' condition [9]. Also they introduced some new contractive mappings on S-metric spaces and investigated their relationships with the Rhoades' condition [10].

Similar to the Banach's contraction principle, now we recall the following result on S-metric spaces given in [13]:

Let (X, S) be a complete S-metric space, T be a self-mapping of X, and

$$S(Tx, Tx, Ty) \leq aS(x, x, y), \tag{1}$$

for some $0 \leq a < 1$ and all $x, y \in X$. Then T has a unique fixed point in X and T is continuous at the fixed point.

Notice that there exists a self-mapping T which has a fixed point, but it does not satisfy Banach's contraction principle on S-metric spaces as we have seen in the following example:

Let $\mathbb{R}$ be the S-metric space which is not generated by any metric with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ defined in [10]. Let

$$Tx = 1 - x.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. T has a fixed point $x = \frac{1}{2}$, but T does not satisfy the Banach's contraction principle (1). Hence it is important to study some new fixed-point theorems.

In this paper, we investigate some generalized fixed-point theorems on S-metric spaces. In Sect. 2 we recall some concepts, lemmas, and corollaries which are useful in the sequel. In Sect. 3 we prove new fixed-point theorems on complete S-metric spaces. Our results generalize and extend some fixed-point theorems in the literature. Also we give some examples to show the validity of our fixed-point theorems.

2 Preliminaries

The following definitions, lemmas, and corollaries can be found in the paper referred to.

Definition 1 ([13]). Let X be a nonempty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

(S1) $S(x, y, z) \geq 0$,
(S2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Definition 2 ([13]). Let (X, S) be an S-metric space.

1. A sequence $(x_n) \subset X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \varepsilon$.
2. A sequence $(x_n) \subset X$ is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(x_n, x_n, x_m) < \varepsilon$.
3. The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 1 ([13]). *Let (X, S) be an S-metric space and $x, y \in X$. Then we have*

$$S(x, x, y) = S(y, y, x).$$

Lemma 2 ([13]). *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then we have*

$$S(x_n, x_n, y_n) \to S(x, x, y).$$

Lemma 3 (See Corollary 2.4 in [12]). *Let $(X, S), (Y, S')$ be two S-metric spaces and $f : X \to Y$ be a function. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

The relation between a metric and an S-metric is given in [7] as follows:

Lemma 4 ([7]). *Let (X, d) be a metric space. Then the following properties are satisfied*:

1. *$S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).*
3. *(x_n) is Cauchy in (X, d) if and only if (x_n) is Cauchy in (X, S_d).*
4. *(X, d) is complete if and only if (X, S_d) is complete.*

Now we recall the following fixed-point results.

Corollary 1 (See Corollary 2.12 in [12]). *Let (X, S) be a complete S-metric space, T be a self-mapping of X, and*

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, x) + cS(Ty, Ty, y), \tag{2}$$

for some $a, b, c \geq 0$, $a + b + c < 1$, and all $x, y \in X$. Then T has a unique fixed point in X. Moreover, if $c < \frac{1}{2}$ then T is continuous at the fixed point.

Corollary 2 (See Corollary 2.14 in [12]). *Let (X, S) be a complete S-metric space, T be a self-mapping of X, and*

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, y), S(Ty, Ty, x)\}, \tag{3}$$

for some $h \in [0, \frac{1}{3})$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 3 (See Corollary 2.10 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, x), S(Ty, Ty, y)\}, \tag{4}$$

for some $h \in [0, 1)$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover, if* $h \in [0, \frac{1}{2})$ *then* T *is continuous at the fixed point.*

Corollary 4 (See Corollary 2.17 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, y) + cS(Ty, Ty, x), \tag{5}$$

for some $a, b, c \geq 0$, $a + b + c < 1$, $a + 3c < 1$, *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 5 (See Corollary 2.19 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$\begin{aligned} S(Tx, Tx, Ty) \leq\ & aS(x, x, y) + bS(Tx, Tx, x) + cS(Tx, Tx, y) \\ & + dS(Ty, Ty, x) + eS(Ty, Ty, y), \end{aligned} \tag{6}$$

for some $a, b, c, d, e \geq 0$ *such that* $\max\{a + b + 3d + e, a + c + d, d + 2e\} < 1$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 6 (See Corollary 2.21 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$\begin{aligned} S(Tx, Tx, Ty) \leq h \max\{& S(x, x, y), S(Tx, Tx, x), S(Tx, Tx, y), \\ & S(Ty, Ty, x), S(Ty, Ty, y)\}, \end{aligned} \tag{7}$$

for some $h \in [0, \frac{1}{3})$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 7 (See Corollary 2.15 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, y) + S(Ty, Ty, x)), \tag{8}$$

for some $a \in [0, \frac{1}{3})$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 8 (See Corollary 2.8 in [12]). *Let* (X, S) *be a complete S-metric space,* T *be a self-mapping of* X, *and*

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)), \tag{9}$$

for some $a \in [0, \frac{1}{2})$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

In the next section we give some generalizations of the above results.

3 Some Fixed-Point Theorems on S-Metric Spaces

In this section we give some definitions and generalizations of fixed-point theorems for self-mappings on complete S-metric spaces.

Definition 3. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN1**) There exist real numbers a, b satisfying $a + 3b < 1$ with $a, b \geq 0$ such that

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + b \max\{S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, y), S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 1. *Let* (X, S) *be a complete S-metric space and* T *be a self-mapping of* X. *If* T *satisfies the condition* (**SN1**), *then* T *has a unique fixed point* x *in* X *and* T *is continuous at* x.

Proof. Let $x_0 \in X$ and define the sequence (x_n) as follows:

$$Tx_0 = x_1, Tx_1 = x_2, \ldots, Tx_n = x_{n+1}, \ldots.$$

Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN1**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq aS(x_{n-1}, x_{n-1}, x_n) \\ &\quad + b \max\{S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_n), S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n-1})\} \\ &= aS(x_{n-1}, x_{n-1}, x_n) + b \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n-1})\}. \end{aligned} \tag{10}$$

By the condition (**S3**) we have

$$S(x_{n+1}, x_{n+1}, x_{n-1}) \leq S(x_{n+1}, x_{n+1}, x_n) + S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)$$

$$= 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n). \quad (11)$$

Then using Lemma 1 and the conditions (10) and (11), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq aS(x_{n-1}, x_{n-1}, x_n) + b \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), \\ &\qquad 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)\} \\ &\leq aS(x_{n-1}, x_{n-1}, x_n) + 2bS(x_{n+1}, x_{n+1}, x_n) + bS(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

and so

$$(1 - 2b)S(x_n, x_n, x_{n+1}) \leq (a + b)S(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{a + b}{1 - 2b} S(x_{n-1}, x_{n-1}, x_n). \quad (12)$$

Let $p = \dfrac{a + b}{1 - 2b}$. Then we have $p < 1$ since $a + 3b < 1$ (notice that $b \neq \dfrac{1}{2}$ since we have $0 \leq b < \dfrac{1}{3}$ by the conditions $a + 3b < 1$ and $a, b \geq 0$).

Repeating this process in the condition (12), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \quad (13)$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the condition (13) and the condition (**S3**), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + 2S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2(p^n + p^{n+1} + \ldots + p^{m-1})S(x_0, x_0, x_1) \\ &\leq 2p^n(1 + p + p^2 + \ldots + p^{m-n-1})S(x_0, x_0, x_1) \\ &\leq 2p^n \frac{1 - p^{m-n}}{1 - p} S(x_0, x_0, x_1) \\ &\leq \frac{2p^n}{1 - p} S(x_0, x_0, x_1). \end{aligned} \quad (14)$$

Hence $\lim\limits_{n,m \to \infty} S(x_n, x_n, x_m) = 0$ since $\lim\limits_{n,m \to \infty} \dfrac{2p^n}{1 - p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) &\leq aS(x_{n-1}, x_{n-1}, x) \\ &\quad + b \max\{S(x_n, x_n, x_{n-1}), S(x_n, x_n, x), S(Tx, Tx, x), S(Tx, Tx, x_{n-1})\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(x, x, Tx) = S(Tx, Tx, x) \leq bS(Tx, Tx, x),$$

which is a contradiction since $0 \leq b < \frac{1}{3}$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition (**SN1**) and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) = S(x, x, y) &\leq aS(x, x, y) \\ &+ b \max\{S(x, x, x), S(x, x, y), S(y, y, y), S(y, y, x)\} \\ &= aS(x, x, y) + bS(x, x, y) = (a + b)S(x, x, y), \end{aligned}$$

which implies $x = y$ since $a + b < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) \\ &+ b \max\{S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x), S(Tx, Tx, x), S(Tx, Tx, x_n)\} \\ &= aS(x_n, x_n, x) + b \max\{S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x), S(x, x, x_n)\}. \end{aligned} \tag{15}$$

Using the condition (**S3**) we have

$$\begin{aligned} S(Tx_n, Tx_n, x_n) &\leq S(Tx_n, Tx_n, x) + S(Tx_n, Tx_n, x) + S(x_n, x_n, x) \\ &= 2S(Tx_n, Tx_n, x) + S(x_n, x_n, x). \end{aligned} \tag{16}$$

Then using the conditions (15), (16) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) \\ &+ b \max\{2S(Tx_n, Tx_n, x) + S(x_n, x_n, x), S(Tx_n, Tx_n, x), S(x, x, x_n)\} \\ &= aS(x_n, x_n, x) + b\{2S(Tx_n, Tx_n, x) + S(x_n, x_n, x)\} \\ &= aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x), \end{aligned}$$

which implies

$$S(Tx_n, Tx_n, Tx) = S(Tx_n, Tx_n, x) \leq \frac{a + b}{1 - 2b} S(x_n, x_n, x). \tag{17}$$

So using the condition (17), for $n \to \infty$ we have

$$\lim_{n \to \infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 1 is a generalization of the Banach's contraction principle (1). Indeed, if we take $b = 0$ in Theorem 1, we obtain the Banach's contraction principle (1).

Now we give an example of a self-mapping satisfying the condition **(SN1)** such that the condition of the Banach's contraction principle (1) is not satisfied.

Example 1. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let us define

$$Tx = \begin{cases} x + 50 & \text{if } |x - 1| = 1 \\ 45 & \text{if } |x - 1| \neq 1 \end{cases}.$$

Then T is a self-mapping on the complete S-metric space $\mathbb{R}$ and satisfies the condition (**SN1**) for $a = 0$ and $b = \frac{1}{4}$. Then T has a unique fixed point $x = 45$. But T does not satisfy the condition of the Banach's contraction principle (1). Indeed, for $x = 0, y = 2$ we obtain

$$S(Tx, Tx, Ty) = 4 \leq aS(x, x, y) = 4a,$$

which is a contradiction since $a < 1$.

Definition 4. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN2**) There exist real numbers a, b satisfying $a + 3b < 1$ with $a, b \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq aS(x, x, y) + b \max\{S(T^m x, T^m x, x), S(T^m x, T^m x, y), S(T^m y, T^m y, y), S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 1.

Corollary 9. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN2**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. From Theorem 1, it can be easily seen that T^m has a unique fixed point x in X, and T^m is continuous at x. Also we have

$$Tx = TT^m x = T^{m+1} x = T^m Tx$$

and so we obtain that Tx is a fixed point for T^m. We get $Tx = x$ since x is a unique fixed point.

Definition 5. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN3**) There exist real numbers a, b, c, d satisfying $\max\{a+b+c+3d, 2b+d\} < 1$ with $a, b, c, d \geq 0$ such that

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, x) + cS(Ty, Ty, y) \\ +d \max\{S(Tx, Tx, y), S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 2. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN3**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN3**) we have

$$S(x_n, x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) \\ +cS(x_{n+1}, x_{n+1}, x_n) + d \max\{S(x_n, x_n, x_n), S(x_{n+1}, x_{n+1}, x_{n-1})\} \\ = aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) + cS(x_{n+1}, x_{n+1}, x_n) + dS(x_{n+1}, x_{n+1}, x_{n-1}). \quad (18)$$

Then using Lemma 1 and the conditions (11) and (18), we obtain

$$S(x_n, x_n, x_{n+1}) \leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_{n-1}, x_{n-1}, x_n) + cS(x_n, x_n, x_{n+1}) \\ +2dS(x_n, x_n, x_{n+1}) + dS(x_{n-1}, x_{n-1}, x_n)$$

and so

$$(1 - c - 2d)S(x_n, x_n, x_{n+1}) \leq (a + b + d)S(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{a + b + d}{1 - c - 2d} S(x_{n-1}, x_{n-1}, x_n). \quad (19)$$

Let $p = \dfrac{a + b + d}{1 - c - 2d}$. Then we have $p < 1$ since $a + b + c + 3d < 1$.

Repeating this process in the condition (19), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \quad (20)$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (20), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1 - p} S(x_0, x_0, x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) convergent to x. Assume that $Tx \neq x$. Then we have

$$S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) \leq aS(x_{n-1}, x_{n-1}, x) + bS(x_n, x_n, x_{n-1})$$
$$+cS(Tx, Tx, x) + d\max\{S(x_n, x_n, x), S(Tx, Tx, x_{n-1})\}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(x, x, Tx) = S(Tx, Tx, x) \leq (c + d)S(Tx, Tx, x),$$

which is a contradiction since $0 \leq c + d < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN3)** and Lemma 1, we have

$$S(Tx, Tx, Ty) = S(x, x, y) \leq aS(x, x, y) + bS(x, x, x) + cS(y, y, y)$$
$$+d\max\{S(x, x, y), S(x, x, y)\} = (a + d)S(x, x, y),$$

which implies $x = y$ since $a + d < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) + cS(Tx, Tx, x) \\ &\quad +d\max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\} \\ &= aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) \\ &\quad +d\max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\}. \end{aligned} \tag{21}$$

Then using the conditions (16), (21) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x) \\ &\quad +d\max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\} \\ &\leq aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x) \\ &\quad +dS(Tx_n, Tx_n, x) + dS(x_n, x_n, x) \end{aligned}$$

and so

$$(1 - 2b - d)S(Tx_n, Tx_n, Tx) \leq (a + b + d)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a+b+d}{1-2b-d} S(x_n, x_n, x). \tag{22}$$

Using the condition (22) for $n \to \infty$, we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 2 is a generalization of Corollaries 1 and 2. Indeed, if we take $d = 0$ and $c < \frac{1}{2}$ in Theorem 2, we obtain Corollary 1 and if we take $a = b = c = 0, d = h$ in Theorem 2, we obtain Corollary 2.

Now we give an example of a self-mapping satisfying the condition **(SN3)** such that the condition (3) is not satisfied.

Example 2. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{5}{6}(1 - x).$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned}
S(Tx, Tx, Ty) &= \frac{5}{3}|x - y|, \\
S(x, x, y) &= 2|x - y|, \\
S(Tx, Tx, y) &= \left|\frac{5}{3}(1 - x) - 2y\right|, \\
S(Ty, Ty, x) &= \left|\frac{5}{3}(1 - y) - 2x\right|, \\
S(Tx, Tx, x) &= \left|\frac{5}{3}(1 - x) - 2x\right|, \\
S(Ty, Ty, y) &= \left|\frac{5}{3}(1 - y) - 2y\right|.
\end{aligned}$$

T satisfies the condition (**SN3**) for $a = \frac{5}{6}, b = c = 0$, and $d = \frac{1}{20}$. Then T has a unique fixed point $x = \frac{5}{11}$. But T does not satisfy the condition (3). Indeed, for $x = 1, y = 0$ we obtain

$$S(Tx, Tx, Ty) = \frac{5}{3} \leq h \max\{S(Tx, Tx, y), S(Ty, Ty, x)\}$$
$$= h \max\left\{0, \frac{1}{3}\right\} = \frac{h}{3},$$

which is a contradiction since $h < \frac{1}{3}$.

Definition 6. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN4**) There exist real numbers a, b, c, d satisfying $\max\{a+b+c+3d, 2b+d\} < 1$ with $a, b, c, d \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq aS(x, x, y) + bS(T^m x, T^m x, x) + cS(T^m y, T^m y, y)$$
$$+d \max\{S(T^m x, T^m x, y), S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 2.

Corollary 10. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN4**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 2 by the same method used in the proof of Corollary 9.

Definition 7. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN5**) There exist real numbers a, b, c, d satisfying $\max\{a + 3c + 2d, a + b + c, b + 2d\} < 1$ with $a, b, c, d \geq 0$ such that

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, y) + cS(Ty, Ty, x)$$
$$+d \max\{S(Tx, Tx, x), S(Ty, Ty, y)\},$$

for all $x, y \in X$.

Theorem 3. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN5**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN5**) we have

$$\begin{aligned}S(x_n,x_n,x_{n+1}) &= S(Tx_{n-1},Tx_{n-1},Tx_n) \leq aS(x_{n-1},x_{n-1},x_n) + bS(x_n,x_n,x_n)\\ &\quad + cS(x_{n+1},x_{n+1},x_{n-1}) + d\max\{S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n)\}\\ &= aS(x_{n-1},x_{n-1},x_n) + cS(x_{n+1},x_{n+1},x_{n-1})\\ &\quad + d\max\{S(x_n,x_n,x_{n-1}),S(x_{n+1},x_{n+1},x_n)\}. \qquad (23)\end{aligned}$$

Then using Lemma 1 and the conditions (11) and (23), we obtain

$$\begin{aligned}S(x_n,x_n,x_{n+1}) &\leq aS(x_{n-1},x_{n-1},x_n) + 2cS(x_{n+1},x_{n+1},x_n) + cS(x_{n-1},x_{n-1},x_n)\\ &\quad + dS(x_n,x_n,x_{n-1}) + dS(x_{n+1},x_{n+1},x_n)\end{aligned}$$

and

$$(1-2c-d)S(x_n,x_n,x_{n+1}) \leq (a+c+d)S(x_{n-1},x_{n-1},x_n),$$

which implies

$$S(x_n,x_n,x_{n+1}) \leq \frac{a+c+d}{1-2c-d}S(x_{n-1},x_{n-1},x_n). \qquad (24)$$

Let $p = \dfrac{a+c+d}{1-2c-d}$. Then we have $p < 1$ since $a + 3c + 2d < 1$.

Repeating this process in the condition (24), we obtain

$$S(x_n,x_n,x_{n+1}) \leq p^n S(x_0,x_0,x_1). \qquad (25)$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (25), we have

$$S(x_n,x_n,x_m) \leq \frac{2p^n}{1-p}S(x_0,x_0,x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n,x_n,x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1-p}S(x_0,x_0,x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned}S(x_n,x_n,Tx) &= S(Tx_{n-1},Tx_{n-1},Tx) \leq aS(x_{n-1},x_{n-1},x) + bS(x_n,x_n,x)\\ &\quad + cS(Tx,Tx,x_{n-1}) + d\max\{S(x_n,x_n,x_{n-1}),S(Tx,Tx,x)\}\end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(x,x,Tx) = S(Tx,Tx,x) \leq (c+d)S(Tx,Tx,x),$$

which is a contradiction since $0 \leq c + d < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition (**SN5**) and Lemma 1, we have

$$S(Tx, Tx, Ty) = S(x, x, y) \leq aS(x, x, y) + bS(x, x, y) + cS(y, y, x) \\ +d \max\{S(x, x, x), S(y, y, y)\} = (a + b + c)S(x, x, y),$$

which implies $x = y$ since $a + b + c < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n) \\ &\quad +d \max\{S(Tx_n, Tx_n, x_n), S(Tx, Tx, x)\} \\ &= aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n) \\ &\quad +dS(Tx_n, Tx_n, x_n). \end{aligned} \tag{26}$$

Then using the conditions (16), (26) and Lemma 1, we obtain

$$S(Tx_n, Tx_n, Tx) \leq aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n) \\ +2dS(Tx_n, Tx_n, x) + dS(x_n, x_n, x)$$

and

$$(1 - b - 2d)S(Tx_n, Tx_n, Tx) \leq (a + c + d)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a + c + d}{1 - b - 2d} S(x_n, x_n, x). \tag{27}$$

So using the condition (27), for $n \to \infty$ we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 3 is a generalization of Corollaries 3 and 4. Indeed, if we take $d = 0$ in Theorem 3, we obtain Corollary 4 and if we take $a = b = c = 0, d = h$ in Theorem 3, we obtain Corollary 3.

Notice that the condition (**SN1**) is the special case of the conditions (**SN3**) and (**SN5**) for $b = c = 0$ and $b = d = 0$, respectively. So we have obtained three generalizations of the Banach's contraction principle (1).

Now we give an example of a self-mapping satisfying the condition (**SN5**) such that the condition (4) is not satisfied.

Example 3. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{x}{2}.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned} S(Tx, Tx, Ty) &= |x - y|, \\ S(x, x, y) &= 2|x - y|, \\ S(Tx, Tx, y) &= 2\left|\frac{x}{2} - y\right|, \\ S(Ty, Ty, x) &= 2\left|\frac{y}{2} - x\right|, \\ S(Tx, Tx, x) &= |x|, \\ S(Ty, Ty, y) &= |y|. \end{aligned}$$

T satisfies the condition (**SN5**) for $a = \frac{1}{2}, b = c = 0$, and $d = \frac{1}{8}$. Then T has a unique fixed point $x = 0$. But T does not satisfy the condition (4). Indeed, for $x = 0, y \in [0, 1]$ we obtain

$$\begin{aligned} S(Tx, Tx, Ty) &= |y| \leq h \max\{S(Tx, Tx, x), S(Ty, Ty, y)\} \\ &= h \max\{|x|, |y|\} = h|y|, \end{aligned}$$

which is a contradiction since $h < 1$.

Definition 8. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN6**) There exist real numbers a, b, c, d satisfying $\max\{a + 3c + 2d, a + b + c, b + 2d\} < 1$ with $a, b, c, d \geq 0$ such that

$$\begin{aligned} S(T^m x, T^m x, T^m y) \leq\ & aS(x, x, y) + bS(T^m x, T^m x, y) + cS(T^m y, T^m y, x) \\ & + d \max\{S(T^m x, T^m x, x), S(T^m y, T^m y, y)\}, \end{aligned}$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 3.

Corollary 11. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* **(SN6)**, *then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 3 by the same method used in the proof of Corollary 9.

Definition 9. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN7**) There exist real numbers a, b, c, d, e, f satisfying $\max\{a + b + 3d + e + 3f, a + c + d + f, 2b + c + 2f\} < 1$ with $a, b, c, d, e, f \geq 0$ such that

$$\begin{aligned} S(Tx, Tx, Ty) \leq\ & aS(x, x, y) + bS(Tx, Tx, x) + cS(Tx, Tx, y) \\ & + dS(Ty, Ty, x) + eS(Ty, Ty, y) + f \max\{S(x, x, y), \\ & S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, x), S(Ty, Ty, y)\}, \end{aligned}$$

for all $x, y \in X$.

Theorem 4. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* **(SN7)**, *then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN7**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) \\ &\quad + cS(x_n, x_n, x_n) + dS(x_{n+1}, x_{n+1}, x_{n-1}) + eS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_n), \\ &\qquad S(x_{n+1}, x_{n+1}, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) + dS(x_{n+1}, x_{n+1}, x_{n-1}) \\ &\quad + eS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n-1}), \\ &\qquad S(x_{n+1}, x_{n+1}, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\}. \end{aligned} \tag{28}$$

Then using Lemma 1 and the conditions (11) and (28), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) + 2dS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + dS(x_{n-1}, x_{n-1}, x_n) + eS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n-1}), \\ &\qquad 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= (a + b + d)S(x_{n-1}, x_{n-1}, x_n) + (2d + e)S(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f\{2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)\} \\ &= (a + b + d + f)S(x_{n-1}, x_{n-1}, x_n) + (2d + e + 2f)S(x_{n+1}, x_{n+1}, x_n) \end{aligned}$$

and

$$(1-2d-e-2f)S(x_{n+1},x_{n+1},x_n) \leq (a+b+d+f)S(x_{n-1},x_{n-1},x_n),$$

which implies

$$S(x_n,x_n,x_{n+1}) \leq \frac{a+b+d+f}{1-2d-e-2f}S(x_{n-1},x_{n-1},x_n). \tag{29}$$

Let $p = \dfrac{a+b+d+f}{1-2d-e-2f}$. Then we have $p < 1$ since $a+b+3d+e+3f < 1$.

Repeating this process in the condition (29), we obtain

$$S(x_n,x_n,x_{n+1}) \leq p^n S(x_0,x_0,x_1). \tag{30}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (30), we have

$$S(x_n,x_n,x_m) \leq \frac{2p^n}{1-p}S(x_0,x_0,x_1).$$

Hence $\lim\limits_{n,m\to\infty} S(x_n,x_n,x_m) = 0$ since $\lim\limits_{n,m\to\infty} \dfrac{2p^n}{1-p}S(x_0,x_0,x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n,x_n,Tx) = S(Tx_{n-1},Tx_{n-1},Tx) &\leq aS(x_{n-1},x_{n-1},x) + bS(x_n,x_n,x_{n-1}) \\ &+cS(x_n,x_n,x) + dS(Tx,Tx,x_{n-1}) + eS(Tx,Tx,x) \\ &+f\max\{S(x_{n-1},x_{n-1},x), S(x_n,x_n,x_{n-1}), S(x_n,x_n,x), \\ &\quad S(Tx,Tx,x_{n-1}), S(Tx,Tx,x)\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$\begin{aligned} S(x,x,Tx) = S(Tx,Tx,x) &\leq dS(Tx,Tx,x) + eS(Tx,Tx,x) \\ &+f\max\{S(Tx,Tx,x), S(Tx,Tx,x)\} = (d+e+f)S(Tx,Tx,x), \end{aligned}$$

which is a contradiction since $0 \leq d+e+f < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN7)** and Lemma 1, we have

$$\begin{aligned} S(Tx,Tx,Ty) = S(x,x,y) &\leq aS(x,x,y) + bS(x,x,x) + cS(x,x,y) \\ &+dS(y,y,x) + eS(y,y,y) + f\max\{S(x,x,y), S(x,x,x), S(x,x,y), \\ &\quad S(y,y,x), S(y,y,y)\} = (a+c+d+f)S(x,x,y), \end{aligned}$$

which implies $x = y$ since $a+c+d+f < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) \leq & \; aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) + cS(Tx_n, Tx_n, x) \\ & + dS(Tx, Tx, x_n) + eS(Tx, Tx, x) + f \max\{S(x_n, x_n, x), S(Tx_n, Tx_n, x_n), \\ & \quad S(Tx_n, Tx_n, x), S(Tx, Tx, x_n), S(Tx, Tx, x)\} \\ = & \; aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) + cS(Tx_n, Tx_n, x) + dS(Tx, Tx, x_n) \\ & + f \max\{S(x_n, x_n, x), S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x)\}. \end{aligned} \tag{31}$$

Then using the conditions (16), (31) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) = & \; S(Tx_n, Tx_n, x) \leq aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) \\ & + bS(x_n, x_n, x) + cS(Tx_n, Tx_n, x) + dS(Tx, Tx, x_n) + f \max\{S(x_n, x_n, x) \\ & + 2S(Tx_n, Tx_n, x) + S(x_n, x_n, x), S(Tx_n, Tx_n, x)\} \\ = & \; aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x) + cS(Tx_n, Tx_n, x) \\ & + dS(Tx, Tx, x_n) + 2fS(Tx_n, Tx_n, x) + fS(x_n, x_n, x) \\ = & \; (a + b + d + f)S(x_n, x_n, x) + (2b + c + 2f)S(Tx, Tx, x_n) \end{aligned}$$

and

$$(1 - 2b - c - 2f)S(Tx_n, Tx_n, Tx) \leq (a + b + d + f)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a + b + d + f}{1 - 2b - c - 2f} S(x_n, x_n, x). \tag{32}$$

So using the condition (32) for $n \to \infty$ we have

$$\lim_{n \to \infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 4 is a generalization of Corollaries 5 and 6. Indeed, if we take $f = 0$ in Theorem 4, we obtain Corollary 5 and if we take $a = b = c = d = e = 0, f = h$ in Theorem 4, we obtain Corollary 6. Also the condition $d + 2e < 1$ which is used in Corollary 5 is not necessary condition in Theorem 4.

Now we give an example of a self-mapping satisfying the condition **(SN7)** such that the condition (7) is not satisfied.

Example 4. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{x}{2} + \frac{1}{3}.$$

Then T is a self-mapping on the complete S-metric space [0, 1]. We have

$$\begin{aligned} S(Tx, Tx, Ty) &= |x - y|, \\ S(x, x, y) &= 2|x - y|, \\ S(Tx, Tx, y) &= 2\left|\frac{x}{2} + \frac{1}{3} - y\right|, \\ S(Ty, Ty, x) &= 2\left|\frac{y}{2} + \frac{1}{3} - x\right|, \\ S(Tx, Tx, x) &= 2\left|\frac{-x}{2} + \frac{1}{3}\right|, \\ S(Ty, Ty, y) &= 2\left|\frac{-y}{2} + \frac{1}{3}\right|. \end{aligned}$$

T satisfies the condition (**SN7**) for $a = \frac{1}{2}, b = c = d = e = 0$ and $f = \frac{1}{7}$. Then T has a unique fixed point $x = \frac{2}{3}$. But T does not satisfy the condition (7). Indeed, for $x = 1, y = 0$ we obtain

$$S(Tx, Tx, Ty) = \frac{1}{2} \leq h \max\left\{\frac{5}{6}, 1, \frac{2}{3}, \frac{1}{6}, \frac{1}{3}\right\} = h,$$

which is a contradiction since $h < \frac{1}{3}$.

Definition 10. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN8**) There exist real numbers a, b, c, d, e, f satisfying $\max\{a + b + 3d + e + 3f, a + c + d + f, 2b + c + 2f\} < 1$ with $a, b, c, d, e, f \geq 0$ such that

$$\begin{aligned} &S(T^m x, T^m x, T^m y) \leq aS(x, x, y) + bS(T^m x, T^m x, x) + cS(T^m x, T^m x, y) \\ &+dS(T^m y, T^m y, x) + eS(T^m y, T^m y, y) + f \max\{S(x, x, y), \\ &\quad S(T^m x, T^m x, x), S(T^m x, T^m x, y), S(T^m y, T^m y, x), S(T^m y, T^m y, y)\}, \end{aligned}$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 4.

Corollary 12. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN8**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 4 by the same method used in the proof of Corollary 9.

Definition 11. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN9**) There exist real numbers a, b, c satisfying $3a + b + 2c < 1$ with $a, b, c \geq 0$ such that

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, y) + S(Ty, Ty, x)) + bS(x, x, y)$$
$$+c \max\{S(Tx, Tx, x), S(Ty, Ty, y)\},$$

for all $x, y \in X$.

Theorem 5. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN9**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN9**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq a(S(x_n, x_n, x_n) + S(x_{n+1}, x_{n+1}, x_{n-1})) \\ &\quad +bS(x_{n-1}, x_{n-1}, x_n) + c \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= aS(x_{n+1}, x_{n+1}, x_{n-1}) + bS(x_{n-1}, x_{n-1}, x_n) \\ &\quad +c \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\}. \end{aligned} \tag{33}$$

Then using Lemma 1 and the conditions (11) and (33), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq 2aS(x_{n+1}, x_{n+1}, x_n) + aS(x_{n-1}, x_{n-1}, x_n) + bS(x_{n-1}, x_{n-1}, x_n) \\ &\quad +c(S(x_n, x_n, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_n)) \\ &= 2aS(x_{n+1}, x_{n+1}, x_n) + (a + b)S(x_{n-1}, x_{n-1}, x_n) \\ &\quad +cS(x_n, x_n, x_{n-1}) + cS(x_{n+1}, x_{n+1}, x_n) \\ &= (2a + c)S(x_{n+1}, x_{n+1}, x_n) + (a + b + c)S(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

and

$$(1 - 2a - c)S(x_n, x_n, x_{n+1}) \leq (a + b + c)S(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{a + b + c}{1 - 2a - c} S(x_{n-1}, x_{n-1}, x_n). \tag{34}$$

Let $p = \dfrac{a + b + c}{1 - 2a - c}$. Then we have $p < 1$ since $3a + b + 2c < 1$.

Repeating this process in the condition (34), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{35}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (35), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1-p} S(x_0, x_0, x_1).$$

Hence $\lim\limits_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim\limits_{n,m\to\infty} \frac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) &\leq a(S(x_n, x_n, x) + S(Tx, Tx, x_{n-1})) \\ &+ bS(x_{n-1}, x_{n-1}, x) + c \max\{S(x_n, x_n, x_{n-1}), S(Tx, Tx, x)\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \leq (a + c)S(Tx, Tx, x),$$

which is a contradiction since $0 \leq a + c < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition (**SN9**) and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) &= S(x, x, y) \leq a(S(x, x, y) + S(y, y, x)) \\ &\quad + bS(x, x, y) + c \max\{S(x, x, x), S(y, y, y)\} \\ &= (2a + b)S(x, x, y), \end{aligned}$$

which implies $x = y$ since $2a + b < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq a(S(Tx_n, Tx_n, x) + S(Tx, Tx, x_n)) + bS(x_n, x_n, x) \\ &\quad + c \max\{S(Tx_n, Tx_n, x_n), S(Tx, Tx, x)\} \\ &= a(S(Tx_n, Tx_n, x) + S(Tx, Tx, x_n)) + bS(x_n, x_n, x) + cS(Tx_n, Tx_n, x_n). \end{aligned} \tag{36}$$

Then using the conditions (16), (36) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(Tx_n, Tx_n, x) + aS(Tx, Tx, x_n) + bS(x_n, x_n, x) \\ &\quad + 2cS(Tx_n, Tx_n, x) + cS(x_n, x_n, x) \end{aligned}$$

and

$$(1-a-2c)S(Tx_n, Tx_n, Tx) \le (a+b+c)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \le \frac{a+b+c}{1-a-2c} S(x_n, x_n, x). \tag{37}$$

So using the condition (37), for $n \to \infty$ we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 5 is a generalization of Corollary 7. Indeed, if we take $b = c = 0$ in Theorem 5, we obtain Corollary 7.

Now we give an example of a self-mapping satisfying the condition **(SN9)** such that the condition (8) is not satisfied.

Example 5. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{2x}{3} + \frac{1}{4}.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned}
S(Tx, Tx, Ty) &= \frac{4}{3}|x-y|, \\
S(x, x, y) &= 2|x-y|, \\
S(Tx, Tx, y) &= 2\left|\frac{2x}{3} + \frac{1}{4} - y\right|, \\
S(Ty, Ty, x) &= 2\left|\frac{2y}{3} + \frac{1}{4} - x\right|, \\
S(Tx, Tx, x) &= 2\left|\frac{-x}{3} + \frac{1}{4}\right|, \\
S(Ty, Ty, y) &= 2\left|\frac{-y}{3} + \frac{1}{4}\right|.
\end{aligned}$$

T satisfies the condition (**SN9**) for $a = 0, b = \frac{2}{3}$ and $c = \frac{1}{7}$. Then T has a unique fixed point $x = \frac{3}{4}$. But T does not satisfy the condition (8). Indeed, for $x = 1, y = 0$ we obtain

$$S(Tx, Tx, Ty) = \frac{2}{3} \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)) = \frac{5a}{3},$$

which is a contradiction since $a < \frac{1}{3}$.

Definition 12. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN10**) There exist real numbers a, b, c satisfying $3a+b+2c < 1$ with $a, b, c \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq a(S(T^m x, T^m x, y) + S(T^m y, T^m y, x)) + bS(x, x, y) + c \max\{S(T^m x, T^m x, x), S(T^m y, T^m y, y)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 5.

Corollary 13. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN10**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 5 by the same method used in the proof of Corollary 9.

Definition 13. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN11**) There exist real numbers a, b, c satisfying $2a+b+3c < 1$ with $a, b, c \geq 0$ such that

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)) + bS(x, x, y) + c \max\{S(Tx, Tx, y), S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 6. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN11**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN11**) we have

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &= S(Tx_{n-1},Tx_{n-1},Tx_n) \le a(S(x_n,x_n,x_{n-1}) + S(x_{n+1},x_{n+1},x_n)) \\ &\quad + bS(x_{n-1},x_{n-1},x_n) + c\max\{S(x_n,x_n,x_n), S(x_{n+1},x_{n+1},x_{n-1})\} \\ &= aS(x_n,x_n,x_{n-1}) + aS(x_{n+1},x_{n+1},x_n) \\ &\quad + bS(x_{n-1},x_{n-1},x_n) + cS(x_{n+1},x_{n+1},x_{n-1}). \end{aligned} \tag{38}$$

Then using Lemma 1 and the conditions (11) and (38), we obtain

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &\le aS(x_n,x_n,x_{n-1}) + aS(x_{n+1},x_{n+1},x_n) + bS(x_{n-1},x_{n-1},x_n) \\ &\quad +2cS(x_{n+1},x_{n+1},x_n) + cS(x_{n-1},x_{n-1},x_n) \\ &= (a+2c)S(x_{n+1},x_{n+1},x_n) + (a+b+c)S(x_{n-1},x_{n-1},x_n) \end{aligned}$$

and

$$(1-a-2c)S(x_n,x_n,x_{n+1}) \le (a+b+c)S(x_{n-1},x_{n-1},x_n),$$

which implies

$$S(x_n,x_n,x_{n+1}) \le \frac{a+b+c}{1-a-2c}S(x_{n-1},x_{n-1},x_n). \tag{39}$$

Let $p = \frac{a+b+c}{1-a-2c}$. Then we have $p < 1$ since $2a+b+3c < 1$.

Repeating this process in the condition (39), we obtain

$$S(x_n,x_n,x_{n+1}) \le p^n S(x_0,x_0,x_1). \tag{40}$$

Then for all $n,m \in \mathbb{N}$, $n < m$, using the conditions (14) and (40), we have

$$S(x_n,x_n,x_m) \le \frac{2p^n}{1-p}S(x_0,x_0,x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n,x_n,x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1-p}S(x_0,x_0,x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \ne x$. Then we have

$$\begin{aligned} S(x_n,x_n,Tx) &= S(Tx_{n-1},Tx_{n-1},Tx) \le a(S(x_n,x_n,x_{n-1}) + S(Tx,Tx,x)) \\ &\quad +bS(x_{n-1},x_{n-1},x) + c\max\{S(x_n,x_n,x), S(Tx,Tx,x_{n-1})\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \leq (a + c)S(Tx, Tx, x),$$

which is a contradiction since $0 \leq a + c < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition (**SN11**) and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) &= S(x, x, y) \leq a(S(x, x, x) + S(y, y, y)) \\ &\quad + bS(x, x, y) + c\max\{S(x, x, y), S(y, y, x)\} \\ &= (b + c)S(x, x, y), \end{aligned}$$

which implies $x = y$ since $b + c < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq a(S(Tx_n, Tx_n, x_n) + S(Tx, Tx, x)) + bS(x_n, x_n, x) \\ &\quad + c\max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\} \\ &= aS(Tx_n, Tx_n, x_n) + bS(x_n, x_n, x) \\ &\quad + c\max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\}. \end{aligned} \tag{41}$$

Then using the conditions (16), (41) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq 2aS(Tx_n, Tx_n, x) + aS(x_n, x_n, x) + bS(x_n, x_n, x) \\ &\quad + cS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n) \\ &= (2a + c)S(Tx_n, Tx_n, x) + (a + b + c)S(Tx, Tx, x_n) \end{aligned}$$

and

$$(1 - 2a - c)S(Tx_n, Tx_n, Tx) \leq (a + b + c)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a + b + c}{1 - 2a - c}S(x_n, x_n, x). \tag{42}$$

So using the condition (42), for $n \to \infty$ we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 6 is a generalization of Corollary 8. Indeed, if we take $b = c = 0$ in Theorem 6, we obtain Corollary 8.

Now we give an example of a self-mapping satisfying the condition (**SN11**) such that the condition (9) is not satisfied.

Example 6. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{3x}{4} + \frac{1}{5}.$$

Then T is a self-mapping on the complete S-metric space [0, 1]. We have

$$\begin{aligned} S(Tx, Tx, Ty) &= \frac{3}{2}|x - y|, \\ S(x, x, y) &= 2|x - y|, \\ S(Tx, Tx, y) &= 2\left|\frac{3x}{4} + \frac{1}{5} - y\right|, \\ S(Ty, Ty, x) &= 2\left|\frac{3y}{4} + \frac{1}{5} - x\right|, \\ S(Tx, Tx, x) &= 2\left|\frac{1}{5} - \frac{x}{4}\right|, \\ S(Ty, Ty, y) &= 2\left|\frac{1}{5} - \frac{y}{4}\right|. \end{aligned}$$

T satisfies the condition (**SN11**) for $a = 0, b = \frac{3}{4}$, and $c = \frac{1}{13}$. Then T has a unique fixed point $x = \frac{4}{5}$. But T does not satisfy the condition (9). Indeed, for $x = 1, y = 0$, we obtain

$$S(Tx, Tx, Ty) = \frac{3}{2} \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)) = \frac{a}{2},$$

which is a contradiction since $a < \frac{1}{2}$.

Definition 14. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN12**) There exist real numbers a, b, c satisfying $2a+b+3c < 1$ with $a, b, c \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq a(S(T^m x, T^m x, x) + S(T^m y, T^m y, y)) + bS(x, x, y) + c \max\{S(T^m x, T^m x, y), S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 6.

Corollary 14. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN12**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 6 by the same method used in the proof of Corollary 9.

Definition 15. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN13**) There exist a real number h satisfying $0 \leq h < \frac{1}{4}$ such that

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, y)+S(Ty, Ty, y), S(Ty, Ty, x)+S(Tx, Tx, x)\},$$

for all $x, y \in X$.

Theorem 7. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN13**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN13**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq h \max\{S(x_n, x_n, x_n) + S(x_{n+1}, x_{n+1}, x_n), \\ &\quad S(x_{n+1}, x_{n+1}, x_{n-1}) + S(x_n, x_n, x_{n-1})\} \\ &= h \max\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n-1}) + S(x_n, x_n, x_{n-1})\}. \end{aligned} \tag{43}$$

Then using Lemma 1 and the conditions (11) and (43), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq h \max\{S(x_{n+1}, x_{n+1}, x_n), 2S(x_{n+1}, x_{n+1}, x_n) + 2S(x_{n-1}, x_{n-1}, x_n)\} \\ &= 2hS(x_{n+1}, x_{n+1}, x_n) + 2hS(x_n, x_n, x_{n-1}) \end{aligned}$$

and

$$(1 - 2h)S(x_n, x_n, x_{n+1}) \leq 2hS(x_n, x_n, x_{n-1}),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{2h}{1-2h} S(x_{n-1}, x_{n-1}, x_n). \tag{44}$$

Let $p = \frac{2h}{1-2h}$. Then we have $p < 1$ since $a < \frac{1}{4}$.

Repeating this process in the condition (44), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{45}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (45), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1-p} S(x_0, x_0, x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) \leq h \max\{S(x_n, x_n, x) + S(Tx, Tx, x), \\ S(Tx, Tx, x_{n-1}) + S(x_n, x_n, x_{n-1})\}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \leq hS(Tx, Tx, x),$$

which is a contradiction since $0 \leq h < \frac{1}{4}$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN13)** and Lemma 1, we have

$$S(Tx, Tx, Ty) = S(x, x, y) \leq h \max\{S(x, x, y) + S(y, y, y), \\ S(y, y, x) + S(x, x, x)\} = hS(x, x, y),$$

which implies $x = y$ since $h < \frac{1}{4}$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$S(Tx_n, Tx_n, Tx) \leq h \max\{S(Tx_n, Tx_n, x) + S(Tx, Tx, x), S(Tx, Tx, x_n) + S(Tx_n, Tx_n, x_n)\}. \tag{46}$$

Then using the conditions (16), (46) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq h \max\{S(Tx_n, Tx_n, x), 2S(x_n, x_n, x) + 2S(Tx_n, Tx_n, x)\} \\ &= 2hS(Tx_n, Tx_n, x) + 2hS(x_n, x_n, x) \end{aligned}$$

and

$$(1 - 2h)S(Tx_n, Tx_n, Tx) \leq 2hS(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{2h}{1 - 2h} S(x_n, x_n, x). \tag{47}$$

So using the condition (47), for $n \to \infty$ we have

$$\lim_{n \to \infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

Definition 16. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN14**) There exist a real number h satisfying $0 \leq h < \frac{1}{4}$ such that

$$\begin{aligned} S(T^m x, T^m x, T^m y) \leq h \max\{&S(T^m x, T^m x, y) + S(T^m y, T^m y, y), \\ &S(T^m y, T^m y, x) + S(T^m x, T^m x, x)\}, \end{aligned}$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 7.

Corollary 15. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN14**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 7 by the same method used in the proof of Corollary 9.

Definition 17. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN15**) There exist a real number h satisfying $0 \leq h < \frac{1}{3}$ such that

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, x) + S(Ty, Ty, y), S(Tx, Tx, y) + S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 8. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN15**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN15**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq h \max\{S(x_n, x_n, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_n), S(x_n, x_n, x_n) \\ &\quad + S(x_{n+1}, x_{n+1}, x_{n-1})\}. \end{aligned} \tag{48}$$

Then using Lemma 1 and the conditions (11) and (48), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq h \max\{S(x_n, x_n, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_n), \\ &\qquad 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)\} \\ &= 2hS(x_{n+1}, x_{n+1}, x_n) + hS(x_n, x_n, x_{n-1}) \end{aligned}$$

and

$$(1 - 2h)S(x_n, x_n, x_{n+1}) \leq hS(x_n, x_n, x_{n-1}),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{h}{1-2h} S(x_{n-1}, x_{n-1}, x_n). \tag{49}$$

Let $p = \frac{h}{1-2h}$. Then we have $p < 1$ since $a < \frac{1}{3}$.

Repeating this process in the condition (49), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{50}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (50), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1-p} S(x_0, x_0, x_1).$$

Hence $\lim_{n,m \to \infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m \to \infty} \frac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) &\leq h \max\{S(x_n, x_n, x_{n-1}) + S(Tx, Tx, x), \\ &\qquad S(x_n, x_n, x) + S(Tx, Tx, x_{n-1})\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \leq hS(Tx, Tx, x),$$

which is a contradiction since $0 \leq h < \frac{1}{3}$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN15)** and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) = S(x, x, y) &\leq h \max\{S(x, x, x) + S(y, y, y), \\ &S(x, x, y) + S(y, y, x)\} = 2hS(x, x, y), \end{aligned}$$

which implies $x = y$ since $h < \frac{1}{3}$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$S(Tx_n, Tx_n, Tx) \leq h \max\{S(Tx_n, Tx_n, x_n)+S(Tx, Tx, x), S(Tx_n, Tx_n, x)+S(Tx, Tx, x_n)\}. \tag{51}$$

Then using the conditions (16), (51) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq h \max\{2S(Tx_n, Tx_n, x) + S(x_n, x_n, x), \\ &\quad S(Tx_n, Tx_n, x) + S(x_n, x_n, x)\} \\ &= 2hS(Tx_n, Tx_n, x) + hS(x_n, x_n, x) \end{aligned}$$

and

$$(1 - 2h)S(Tx_n, Tx_n, Tx) \leq hS(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{h}{1 - 2h} S(x_n, x_n, x). \tag{52}$$

So using the condition (52), for $n \to \infty$ we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

Definition 18. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN16**) There exist a real number h satisfying $0 \leq h < \frac{1}{3}$ such that

$$S(T^m x, T^m x, T^m y) \leq h \max\{S(T^m x, T^m x, x) + S(T^m y, T^m y, y), \\ S(T^m x, T^m x, y) + S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 8.

Corollary 16. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN16**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 8 by the same method used in the proof of Corollary 9.

Notice that the condition (**SN15**) is the special case of the condition (**SN1**) for $a = 0, b = h$.

Example 7. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let us consider the following constant function:

$$Tx = k, k \in [0, 1].$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned} S(Tx, Tx, Ty) &= 0, \\ S(Tx, Tx, y) &= 2|k - y|, \\ S(Ty, Ty, x) &= 2|k - x|, \\ S(Tx, Tx, x) &= 2|k - x|, \\ S(Ty, Ty, y) &= 2|k - y|. \end{aligned}$$

T satisfies the conditions (**SN13**) and (**SN15**) for all $h \in [0, \frac{1}{3})$, respectively. Then T has a unique fixed point $x = k$.

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