Themistocles M. Rassias ·
Panos M. Pardalos *Editors*

Essays in Mathematics and its Applications

In Honor of Vladimir Arnold

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Preface

Essays in Mathematics and its Applications: In Honor of Vladimir Arnold focuses on various important areas of Mathematical research. The contributed papers have been written by eminent scientists and experts from the international Mathematical community. These papers deepen our understanding of some of the current research problems and theories which have their origin or have been influenced by V. Arnold.

The presentation of concepts and methods featured in this volume makes it an invaluable reference for a wide readership.

We are indebted to all of the scientists who contributed to this volume, and we would also like to acknowledge the superb assistance that the staff of Springer has provided for this publication.

Athens, Greece — Themistocles M. Rassias
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Contents

A New Way to Compute the Rodrigues Coefficients of Functions of the Lie Groups of Matrices

Dorin Andrica and Oana Liliana Chender

Abstract In Theorem 1 we present, in the case when the eigenvalues of the matrix are pairwise distinct, a direct way to determine the general Rodrigues coefficients of a matrix function for the general linear group $\mathbf{GL}(n, \mathbb{R})$ by reducing the Rodrigues problem to the system (7). Then, Theorem 2 gives the explicit formulas in terms of the fundamental symmetric polynomials of the eigenvalues of the matrix. Our formulas permit to consider also the degenerated cases (i.e., the situations when there are multiplicities of the eigenvalues) and to obtain nice determinant formulas. In the cases $n = 2, 3, 4$, the computations are effectively given, and the formulas are presented in closed form. The method is illustrated for the exponential map and the Cayley transform of the special orthogonal group $\mathbf{SO(n)}$, when $n = 2, 3, 4$.

AMS Subject Classification (2010): 22Exx, 22E60, 22E70

1 Introduction

The exponential map $\exp : gl(n, \mathbb{R}) = M_n(\mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$, where $\mathbf{GL}(n, \mathbb{R})$ denotes the Lie group of real invertible $n \times n$ matrices, is defined by (see, for instance, Chevalley [8], Marsden and Raţiu [15], or Warner [22])

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{1}$$

According to the well-known Hamilton–Cayley theorem, it follows that every power X^k, $k \geq n$, is a linear combination of $X^0, X^1, \ldots, X^{n-1}$; hence, we can write

$$\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k, \tag{2}$$

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where the real coefficients $a_0(X), \ldots, a_{n-1}(X)$ are uniquely defined and depend on the matrix X. From this formula, it follows that $\exp(X)$ is a polynomial of X. The problem to find a reasonable formula for $\exp(X)$ is reduced to the problem to determine the coefficients $a_0(X), \ldots, a_{n-1}(X)$. We will call this general question the *Rodrigues problem* and the numbers $a_0(X), \ldots, a_{n-1}(X)$ the *Rodrigues coefficients* of the exponential map with respect to the matrix $X \in M_n(\mathbb{R})$.

The origin of this problem is the classical Rodrigues formula obtained in 1840 for the special orthogonal group $\mathbf{SO(3)}$:

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta}X + \frac{1-\cos\theta}{\theta^2}X^2,$$

where $\sqrt{2}\theta = \|X\|$ and $\|X\|$ denotes the Frobenius norm of the matrix X (for details, see Sect. 3.1). There are numerous arguments pointing out the importance of this formula, and we mention here the following two: the study of the rigid body rotations in $\mathbb{R}^3$ and the parameterization of the rotations in $\mathbb{R}^3$.

The general idea of construction of matrix function generalizing the exponential map is to consider an analytic function $f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m + \cdots$, such that the induced series $\tilde{f}(X) = \alpha_0 I_n + \alpha_1 X + \cdots + \alpha_m X^m + \cdots$ is convergent in an open subset of $M_n(\mathbb{R})$. Then, via the well-known Hamilton–Cayley–Frobenius theorem, we can write a reduced form for the matrix $\tilde{f}(X)$, that is,

$$\tilde{f}(X) = \sum_{k=0}^{n-1} a_k^{(f)}(X)X^k. \tag{3}$$

We call the above relation the *Rodrigues formula* with respect to $\tilde{f}$. The numbers $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are the *Rodrigues coefficients* of the map $\tilde{f}$ with respect to the matrix $X \in M_n(\mathbb{R})$. Clearly, the real coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are uniquely defined, they depend on the matrix X, and $\tilde{f}(X)$ is a polynomial of X.

An important property of the Rodrigues coefficients is the invariance under the matrix conjugacy, i.e., the following result holds:

Proposition 1. *For every invertible matrix U, the following relations hold*

$$a_k^{(f)}(UXU^{-1}) = a_k^{(f)}(X), k = 0, \ldots, n-1. \tag{4}$$

Proof. Assume that we have

$$\tilde{f}(UXU^{-1}) = \sum_{k=0}^{n-1} b_k^{(f)}(UXU^{-1})^k,$$

where $b_k^{(f)} = b_k^{(f)}(UXU^{-1}), k = 0, \ldots, n-1$. Then, we have

$$\tilde{f}(UXU^{-1}) = \sum_{j=0}^{\infty}\alpha_j(UXU^{-1})^k = \sum_{j=0}^{\infty}\alpha_j(UX^kU^{-1})$$

$$= U\left(\sum_{j=0}^{\infty}\alpha_j X^k\right)U^{-1} = U\tilde{f}(X)U^{-1}.$$

Hence, we can write

$$\tilde{f}(UXU^{-1}) = U\tilde{f}(X)U^{-1} = U\left(\sum_{k=0}^{n-1} a_k^{(f)}(X)X^k\right)U^{-1} = \sum_{k=0}^{n-1} a_k^{(f)}(X)(UXU^{-1})^k,$$

and the property immediately follows from the uniqueness of the Rodrigues coefficients.

In Sect. 2 of this paper, we present a new method to determine the general Rodrigues coefficients $a_0^{(f)}(X), \dots, a_{n-1}^{(f)}(X)$ when the eigenvalues of the matrix X are pairwise distinct. Formula (12) gives an explicit formula in terms of the symmetric fundamental polynomials of the eigenvalues. Section 3 illustrates the particular cases $n = 2, 3, 4$, and in Sect. 4, the possible cases of degeneration are considered. Sections 5 and 6 are devoted to the special case of the exponential map and the Cayley transform of the special orthogonal group. We mention that in the paper [5], the same method was used to derive the Rodrigues formula for the Lorentz group $\mathbf{O}(1,3)$.

2 The Rodrigues Formula for $\tilde{f}(X)$

In this section, we will present a new way to determine the general Rodrigues coefficients $a_0^{(f)}(X), \dots, a_{n-1}^{(f)}(X)$ introduced in (3). Following the paper [4], our main idea consists in the reduction of relation (3) to a linear system with the unknowns $a_0^{(f)}(X), \dots, a_{n-1}^{(f)}(X)$.

In this respect, we multiply both sides of (3) by the matrix power X^j, $j = 0, \dots, n-1$, and we obtain the matrix relations

$$X^j\tilde{f}(X) = \sum_{k=0}^{n-1} a_k^{(f)} X^{k+j},\ j = 0, \dots, n-1, \tag{5}$$

where $a_k^{(f)} = a_k^{(f)}(X)$, $k = 0, \dots, n-1$. Now, considering the matrix trace in the both sides of (5), we obtain the linear system

$$\sum_{k=0}^{n-1} \operatorname{tr}(X^{k+j})a_k^{(f)} = \operatorname{tr}(X^j\tilde{f}(X)),\ j = 0, \dots, n-1, \tag{6}$$

where the coefficients are functions of the matrix X. Now, assume that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of matrix X. Then, it is well known that the matrix X^{k+j} has the eigenvalues $\lambda_1^{k+j}, \dots, \lambda_n^{k+j}$ and the matrix $X^j\tilde{f}(X)$ has the eigenvalues $\lambda_1^j f(\lambda_1), \dots, \lambda_n^j f(\lambda_n)$.

Indeed, the function $f_j : \mathbb{C} \to \mathbb{C}, f_j(z) = z^j f(z)$ is analytic; hence, the eigenvalues of the matrix $f_j(X)$ are $f_j(\lambda_1), \ldots, f_j(\lambda_n)$. But, clearly, we have $f_j(\lambda_s) = \lambda_s^j f(\lambda_s), s = 1, \ldots, n$, and the property is proved.

According to the considerations above, the system (6) is equivalent to

$$\sum_{k=0}^{n-1} \left(\sum_{s=1}^{n} \lambda_s^{k+j} \right) a_k^{(f)} = \sum_{s=1}^{n} \lambda_s^j f(\lambda_s), j = 0, \ldots, n-1. \tag{7}$$

From the system (7), we obtain the following result concerning the solution to the general Rodrigues problem with respect to the function f.

Theorem 1.

1) *The Rodrigues coefficients in formula (3) are solutions to the system (7).*
2) *If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct, then the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are perfectly determined by the system (7), and they are given by the formulas*

$$a_k^{(f)}(X) = \frac{V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)}{V_n(\lambda_1, \ldots, \lambda_n)}, k = 0, \ldots, n-1, \tag{8}$$

where $V_n(\lambda_1, \ldots, \lambda_n)$ is the Vandermonde determinant of order n and $V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ is the determinant of order n obtained from $V_n(\lambda_1, \ldots, \lambda_n)$ by replacing the line $k+1$ by $f(\lambda_1), \ldots, f(\lambda_n)$.
3) *If the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct, then the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are linear combinations of $f(\lambda_1), \ldots, f(\lambda_n)$ having the coefficients rational functions of $\lambda_1, \ldots, \lambda_n$, i.e., we have*

$$a_k^{(f)} = b_k^{(1)} f(\lambda_1) + \cdots + b_k^{(n)} f(\lambda_n), k = 0, \ldots, n-1. \tag{9}$$

Proof. The first statement was already proved.

For the second statement, observe that the determinant of the system (7) is

$$D_n = \det \begin{pmatrix} S_0 & S_1 & \ldots & S_{n-1} \\ S_1 & S_2 & \ldots & S_n \\ \ldots & \ldots & \ldots & \ldots \\ S_{n-1} & S_n & \ldots & S_{2n-1}, \end{pmatrix}$$

where $S_l = S_l(\lambda_1, \ldots, \lambda_n) = \lambda_1^l + \cdots + \lambda_n^l, l = 0, \ldots, 2n-1$.

It is clear that

$$D_n = \det \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \ldots & \ldots & \ldots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix} \cdot \det \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{pmatrix}$$

$$= V_n^2 = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)^2 \ne 0,$$

where $V_n = V_n(\lambda_1, \ldots, \lambda_n)$ is the Vandermonde determinant of order n.

Now, it is important to observe that we have

$$\Delta_{a_k^{(f)}} = V_n \cdot W_{n,k}^{(f)}, k = 0, \ldots, n-1, \tag{10}$$

where $W_{n,k}^{(f)} = W_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ is the transpose of the determinant $V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ and the conclusion follows from well-known formulas giving the unique solution to the system by using the property $W_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n) = V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n), k = 0, \ldots, n-1$.

The last property immediately follows from formula (8) by expanding the determinant $V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ with respect to the line $k+1$. □

Expanding the determinant $V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ in Theorem 1 2) with respect to the line $k+1$, it follows

$$a_k^{(f)}(X) = \frac{1}{V_n} \sum_{j=1}^{n} (-1)^{k+j+1} \mathrm{LV}_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) f(\lambda_j), \tag{11}$$

where $\mathrm{LV}_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) f(\lambda_j)$ is the $k+1$ lacunary Vandermonde determinant in the variables $\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n$, i.e., the determinant obtained from $V_n(\lambda_1, \ldots, \lambda_n)$ by cutting the row $k+1$ and the column j. Applying the well-known formula (see the reference [21]),

$$\mathrm{LV}_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) = s_{n-k-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) V_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n),$$

where s_l is the l-th symmetric polynomial in the $n-1$ variables $\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n$, where λ_j is missing, we obtain the following result which completely solves the general problem in the case when the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct.

Theorem 2. *For every $k = 0, \ldots, n-1$, the following formulas hold:*

$$a_k^{(f)} = \sum_{j=1}^{n} (-1)^{k+j+1} \frac{V_{n-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n) s_{n-k-1}(\lambda_1, \ldots, \widehat{\lambda_j}, \ldots, \lambda_n)}{V_n(\lambda_1, \ldots, \lambda_n)} f(\lambda_j), \tag{12}$$

where s_l denotes the l-th symmetric polynomial and $\widehat{\lambda_j}$ means that in the Vandermonde determinant V_{n-1}, the variable λ_j is omitted.

3 Illustrating the Cases $n = 2, 3, 4$

Clearly, when $X = O_n$, we have $\tilde{f}(X) = \alpha_0 I_n$, and in this situation $a_0^{(f)} = \alpha_0, a_1^{(f)} = \cdots = a_{n-1}^{(f)} = 0$. In this section, we assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct.

3.1 The Case $n = 2$

We have $V_2(\lambda_1, \lambda_2) = \lambda_2 - \lambda_1$ and

$$\begin{aligned}
V_1(\widehat{\lambda_1}, \lambda_2) &= V_1(\lambda_1, \widehat{\lambda_2}) = 1, \\
s_1(\widehat{\lambda_1}, \lambda_2) &= \lambda_2, s_1(\lambda_1, \widehat{\lambda_2}) = \lambda_1, \\
s_0(\widehat{\lambda_1}, \lambda_2) &= s_0(\lambda_1, \widehat{\lambda_2}) = 1.
\end{aligned}$$

From (8) and (12), it follows

$$a_0^{(f)} = \frac{V_{2,0}^{(f)}(\lambda_1, \lambda_2)}{V_2(\lambda_1, \lambda_2)} = \frac{\begin{vmatrix} f(\lambda_1) & f(\lambda_2) \\ \lambda_1 & \lambda_2 \end{vmatrix}}{\lambda_2 - \lambda_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1} f(\lambda_1) - \frac{\lambda_1}{\lambda_2 - \lambda_1} f(\lambda_2)$$

and

$$a_1^{(f)} = \frac{V_{2,1}^{(f)}(\lambda_1, \lambda_2)}{V_2(\lambda_1, \lambda_2)} = \frac{\begin{vmatrix} 1 & 1 \\ f(\lambda_1) & f(\lambda_2) \end{vmatrix}}{\lambda_2 - \lambda_1} = -\frac{1}{\lambda_2 - \lambda_1} f(\lambda_1) + \frac{1}{\lambda_2 - \lambda_1} f(\lambda_2).$$

It follows the general Rodrigues formula

$$\tilde{f}(X) = \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} f(\lambda_1) - \frac{\lambda_1}{\lambda_2 - \lambda_1} f(\lambda_2) \right) I_2 + \left(-\frac{1}{\lambda_2 - \lambda_1} f(\lambda_1) + \frac{1}{\lambda_2 - \lambda_1} f(\lambda_2) \right) X. \tag{13}$$

3.2 The Case $n = 3$

We have $V_3(\lambda_1, \lambda_2, \lambda_2) = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ and

$$V_2(\widehat{\lambda_1}, \lambda_2, \lambda_3) = \lambda_3 - \lambda_2, V_2(\lambda_1, \widehat{\lambda_2}, \lambda_3) = \lambda_3 - \lambda_1, V_2(\lambda_1, \lambda_2, \widehat{\lambda_3}) = \lambda_2 - \lambda_1.$$

Moreover,

$$\begin{aligned}
&s_2(\widehat{\lambda_1}, \lambda_2, \lambda_3) = \lambda_2\lambda_3, s_2(\lambda_1, \widehat{\lambda_2}, \lambda_3) = \lambda_1\lambda_3, s_2(\lambda_1, \lambda_2, \widehat{\lambda_3}) = \lambda_1\lambda_2, \\
&s_1(\widehat{\lambda_1}, \lambda_2, \lambda_3) = \lambda_2 + \lambda_3, s_1(\lambda_1, \widehat{\lambda_2}, \lambda_3) = \lambda_1 + \lambda_3, s_1(\lambda_1, \lambda_2, \widehat{\lambda_3}) = \lambda_1 + \lambda_2, \\
&s_0(\widehat{\lambda_1}, \lambda_2, \lambda_3) = s_0(\lambda_1, \widehat{\lambda_2}, \lambda_3) = s_0(\lambda_1, \lambda_2, \widehat{\lambda_3}) = 1.
\end{aligned}$$

Using again formulas (8) and (12), it follows

$$a_0^{(f)} = \frac{V_{3,0}^{(f)}(\lambda_1,\lambda_2,\lambda_3)}{V_3(\lambda_1,\lambda_2,\lambda_3)} = \frac{\begin{vmatrix} f(\lambda_1) & f(\lambda_2) & f(\lambda_3) \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$
$$= \frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} f(\lambda_1) - \frac{\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_2)$$
$$+ \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_3)$$

$$a_1^{(f)} = \frac{V_{3,1}^{(f)}(\lambda_1,\lambda_2,\lambda_3)}{V_3(\lambda_1,\lambda_2,\lambda_3)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ f(\lambda_1) & f(\lambda_2) & f(\lambda_3) \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$
$$= -\frac{\lambda_2+\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} f(\lambda_1) + \frac{\lambda_3+\lambda_1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_2)$$
$$-\frac{\lambda_1+\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_3)$$

$$a_2^{(f)} = \frac{V_{3,2}^{(f)}(\lambda_1,\lambda_2,\lambda_3)}{V_3(\lambda_1,\lambda_2,\lambda_3)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ f(\lambda_1) & f(\lambda_2) & f(\lambda_3) \end{vmatrix}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$
$$= \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} f(\lambda_1) - \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_2)$$
$$+ \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_3)$$

and the corresponding general Rodrigues formula

$$\tilde{f}(X) = \left(\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} f(\lambda_1) - \frac{\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_2) \right.$$
$$\left. + \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_3) \right) I_3 + \left(-\frac{\lambda_2+\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} f(\lambda_1) \right.$$
$$\left. + \frac{\lambda_3+\lambda_1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_2) - \frac{\lambda_1+\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_3) \right) X$$
$$+ \left(\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} f(\lambda_1) - \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_2) \right.$$
$$\left. + \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} f(\lambda_3) \right) X^2.$$

3.3 The Case $n = 4$

We have $V_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)$ and

$$V_3(\widehat{\lambda_1},\lambda_2,\lambda_3,\lambda_4) = (\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3),$$
$$V_3(\lambda_1,\widehat{\lambda_2},\lambda_3,\lambda_4) = (\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_3),$$
$$V_3(\lambda_1,\lambda_2,\widehat{\lambda_3},\lambda_4) = (\lambda_2-\lambda_1)(\lambda_4-\lambda_1)(\lambda_4-\lambda_2),$$
$$V_3(\lambda_1,\lambda_2,\lambda_3,\widehat{\lambda_4}) = (\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2).$$

Here, we have the corresponding symmetric sums

$$s_3(\widehat{\lambda_1},\lambda_2,\lambda_3,\lambda_4) = \lambda_2\lambda_3\lambda_4, s_3(\lambda_1,\widehat{\lambda_2},\lambda_3,\lambda_4) = \lambda_1\lambda_3\lambda_4,$$
$$s_3(\lambda_1,\lambda_2,\widehat{\lambda_3},\lambda_4) = \lambda_1\lambda_2\lambda_4, s_3(\lambda_1,\lambda_2,\lambda_3,\widehat{\lambda_4}) = \lambda_1\lambda_2\lambda_3,$$
$$s_2(\widehat{\lambda_1},\lambda_2,\lambda_3,\lambda_4) = \lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4, s_2(\lambda_1,\widehat{\lambda_2},\lambda_3,\lambda_4) = \lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_3\lambda_4,$$
$$s_2(\lambda_1,\lambda_2,\widehat{\lambda_3},\lambda_4) = \lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4, s_2(\lambda_1,\lambda_2,\lambda_3,\widehat{\lambda_4}) = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3,$$
$$s_1(\widehat{\lambda_1},\lambda_2,\lambda_3,\lambda_4) = \lambda_2+\lambda_3+\lambda_4, s_1(\lambda_1,\widehat{\lambda_2},\lambda_3,\lambda_4) = \lambda_1+\lambda_3+\lambda_4,$$
$$s_1(\lambda_1,\lambda_2,\widehat{\lambda_3},\lambda_4) = \lambda_1+\lambda_2+\lambda_4, s_1(\lambda_1,\lambda_2,\lambda_3,\widehat{\lambda_4}) = \lambda_1+\lambda_2+\lambda_3,$$
$$s_0(\widehat{\lambda_1},\lambda_2,\lambda_3,\lambda_4) = s_0(\lambda_1,\widehat{\lambda_2},\lambda_3,\lambda_4) = s_0(\lambda_1,\lambda_2,\widehat{\lambda_3},\lambda_4) = s_0(\lambda_1,\lambda_2,\lambda_3,\widehat{\lambda_4}) = 1.$$

From formulas (8) and (9), we obtain

$$a_0^{(f)} = \frac{V_{4,0}^{(f)}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{V_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} = \frac{\begin{vmatrix} f(\lambda_1) & f(\lambda_2) & f(\lambda_3) & f(\lambda_4) \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{vmatrix}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}$$
$$= \frac{\lambda_2\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}f(\lambda_1) - \frac{\lambda_1\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}f(\lambda_2)$$
$$+\frac{\lambda_1\lambda_2\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)}f(\lambda_3) - \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}f(\lambda_4),$$

$$a_1^{(f)} = \frac{V_{4,1}^{(f)}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{V_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ f(\lambda_1) & f(\lambda_2) & f(\lambda_3) & f(\lambda_4) \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{vmatrix}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}$$

$$
\begin{aligned}
&= -\frac{\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f(\lambda_1) + \frac{\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} f(\lambda_2) \\
&\quad -\frac{\lambda_1\lambda_2+\lambda_1\lambda_4+\lambda_2\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_3) + \frac{\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_4),
\end{aligned}
$$

$$
\begin{aligned}
a_2^{(f)} &= \frac{V_{4,2}^{(f)}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{V_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ f(\lambda_2) & f(\lambda_2) & f(\lambda_3) & f(\lambda_4) \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{vmatrix}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \\
&= \frac{\lambda_2+\lambda_3+\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f(\lambda_1) - \frac{\lambda_1+\lambda_3+\lambda_4}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} f(\lambda_2) \\
&\quad + \frac{\lambda_1+\lambda_2+\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_3) - \frac{\lambda_1+\lambda_2+\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_4),
\end{aligned}
$$

$$
\begin{aligned}
a_3^{(f)} &= \frac{V_{4,3}^{(f)}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{V_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} = \frac{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ f(\lambda_2) & f(\lambda_2) & f(\lambda_3) & f(\lambda_4) \end{vmatrix}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \\
&= -\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)} f(\lambda_1) + \frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)} f(\lambda_2) \\
&\quad - \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_3) + \frac{1}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} f(\lambda_4)
\end{aligned}
$$

and the corresponding general Rodrigues formula but we don't write it here because of the space reason.

4 Degeneration in Cases $n = 2, 3, 4$

In this section, we show how to obtain the general Rodrigues coefficients when the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the matrix X are not distinct, when $n = 2, 3, 4$.

4.1 The Case $n = 2$

Assume that $\lambda_1 = \lambda_2$. Then the corresponding general Rodrigues coefficients can be obtained from the formulas in Sect. 3.1 for $\lambda_2 \to \lambda_1$. Using the formula of the derivative of a functional determinant, we get

$$a_0^{(f)} = \begin{vmatrix} f(\lambda_1) & f'(\lambda_1) \\ \lambda_1 & 1 \end{vmatrix} = f(\lambda_1) - \lambda_1 f'(\lambda_1)$$

$$a_1^{(f)} = \begin{vmatrix} 1 & 0 \\ f(\lambda_1) & f'(\lambda_1) \end{vmatrix} = f'(\lambda_1).$$

4.2 The Case $n = 3$

In this case, we have to consider the following two possibilities, if we don't take into account the permutations of the eigenvalues λ_1, λ_2, and λ_3.

The Case $\lambda_1 = \lambda_2 \neq \lambda_3$

The corresponding general Rodrigues coefficients can be obtained from the formulas in Sect. 3.2 for $\lambda_2 \to \lambda_1$. Using again the formula of the derivative of a functional determinant, we get

$$a_0^{(f)} = \frac{\begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{vmatrix}}{(\lambda_3 - \lambda_1)^2} = \frac{\lambda_3^2 - 2\lambda_1\lambda_3}{(\lambda_3 - \lambda_1)^2} f(\lambda_1) - \frac{\lambda_1\lambda_3}{\lambda_3 - \lambda_1} f'(\lambda_1) + \frac{\lambda_1^2}{(\lambda_3 - \lambda_1)^2} f(\lambda_3)$$

$$a_1^{(f)} = \frac{\begin{vmatrix} 1 & 0 & 1 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) \\ \lambda_1^2 & 2\lambda_1 & 2\lambda_3^2 \end{vmatrix}}{(\lambda_3 - \lambda_1)^2} = \frac{2\lambda_1}{(\lambda_3 - \lambda_1)^2} f(\lambda_1) + \frac{\lambda_3^2}{(\lambda_3 - \lambda_1)^2} f'(\lambda_1) + \frac{2\lambda_1}{(\lambda_3 - \lambda_1)^2} f(\lambda_3)$$

$$a_2^{(f)} = \frac{\begin{vmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) \end{vmatrix}}{(\lambda_3 - \lambda_1)^2} = -\frac{1}{(\lambda_3 - \lambda_1)^2} f(\lambda_1) - \frac{1}{\lambda_3 - \lambda_1} f'(\lambda_1) + \frac{1}{(\lambda_3 - \lambda_1)^2} f(\lambda_3).$$

The Case $\lambda_1 = \lambda_2 = \lambda_3$

We use the formulas obtained in Section "The Case $\lambda_1 = \lambda_2 \neq \lambda_3$" for $\lambda_3 \to \lambda_1$, and we obtain

$$a_0^{(f)} = \lim_{\lambda_3 \to \lambda_1} \frac{\begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f'(\lambda_3) \\ \lambda_1 & 1 & 1 \\ \lambda_1^2 & 2\lambda_1 & 2\lambda_3 \end{vmatrix}}{2(\lambda_3 - \lambda_1)} = \lim_{\lambda_3 \to \lambda_1} \frac{1}{2} \begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f''(\lambda_3) \\ \lambda_1 & 1 & 0 \\ \lambda_1^2 & 2\lambda_1 & 2 \end{vmatrix}$$

$$= f(\lambda_1) - \lambda_1 f'(\lambda_1) + \frac{1}{2}\lambda_1^2 f''(\lambda_1),$$

$$a_1^{(f)} = \lim_{\lambda_3 \to \lambda_1} \frac{\begin{vmatrix} 1 & 0 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f'(\lambda_3) \\ \lambda_1^2 & 2\lambda_1 & 2\lambda_3 \end{vmatrix}}{2(\lambda_3 - \lambda_1)} = \lim_{\lambda_3 \to \lambda_1} \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_3) \\ \lambda_1^2 & 2\lambda_1 & 2 \end{vmatrix}$$

$$= f'(\lambda_1) - \lambda_1 f''(\lambda_1),$$

$$a_2^{(f)} = \lim_{\lambda_3 \to \lambda_1} \frac{\begin{vmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 1 \\ f(\lambda_1) & f'(\lambda_1) & f'(\lambda_3) \end{vmatrix}}{2(\lambda_3 - \lambda_1)} = \lim_{\lambda_3 \to \lambda_1} \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_3) \end{vmatrix}$$

$$= \frac{1}{2} f''(\lambda_1),$$

and the corresponding Rodrigues formula.

4.3 *The Case* $n = 4$

In this case, we have to consider the following four possibilities, without taking into account the permutations of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and λ_4.

The Case $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$

The general Rodrigues coefficients can be obtained from the formulas in Sect. 3.3 for $\lambda_2 \to \lambda_1$, and using the formula of the derivative of a functional determinant, we get

$$a_0^{(f)} = \frac{1}{(\lambda_3 - \lambda_1)^2(\lambda_4 - \lambda_1)^2(\lambda_4 - \lambda_3)} \begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) & f(\lambda_4) \\ \lambda_1 & 1 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_3^3 & \lambda_4^3 \end{vmatrix}$$

$$= \frac{\lambda_3\lambda_4(3\lambda_1^2 + \lambda_3\lambda_4 - 2\lambda_1(\lambda_3 + \lambda_4))}{(\lambda_3 - \lambda_1)^2(\lambda_4 - \lambda_1)^2} f(\lambda_1) + \frac{-\lambda_1\lambda_3\lambda_4}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} f'(\lambda_1)$$

$$+ \frac{\lambda_1^2\lambda_4}{(\lambda_3 - \lambda_1)^2(\lambda_4 - \lambda_3)} f(\lambda_3) + \frac{-\lambda_1^2\lambda_3}{(\lambda_4 - \lambda_1)^2(\lambda_4 - \lambda_3)} f(\lambda_4),$$

$$a_1^{(f)} = \frac{1}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}\begin{vmatrix} 1 & 0 & 1 & 1 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) & f(\lambda_4) \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_3^3 & \lambda_4^3 \end{vmatrix}$$

$$= \frac{-\lambda_1(3\lambda_1(\lambda_3+\lambda_4)-2(\lambda_3^2+\lambda_3\lambda_4+\lambda_4^2))}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2}f(\lambda_1) + \frac{\lambda_3\lambda_4+\lambda_1(\lambda_3+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}f'(\lambda_1)$$

$$+\frac{-\lambda_1(\lambda_1+2\lambda_4)}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_3)}f(\lambda_3) + \frac{\lambda_1(\lambda_1+2\lambda_3)}{(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}f(\lambda_4),$$

$$a_2^{(f)} = \frac{1}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}\begin{vmatrix} 1 & 0 & 1 & 1 \\ \lambda_1 & 1 & \lambda_3 & \lambda_4 \\ f(\lambda_2) & f'(\lambda_1) & f(\lambda_3) & f(\lambda_4) \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_3^3 & \lambda_4^3 \end{vmatrix}$$

$$= \frac{3\lambda_1^2-\lambda_3^2-\lambda_3\lambda_4-\lambda_4^2}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2}f(\lambda_1) + \frac{-(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}f'(\lambda_1)$$

$$+\frac{2\lambda_1+\lambda_4}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_3)}f(\lambda_3) + \frac{-2\lambda_1-\lambda_3}{(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}f(\lambda_4),$$

$$a_3^{(f)} = \frac{1}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2(\lambda_4-\lambda_3)}\begin{vmatrix} 1 & 0 & 1 & 1 \\ \lambda_1 & 1 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 & \lambda_4^2 \\ f(\lambda_2) & f'(\lambda_1) & f(\lambda_3) & f(\lambda_4) \end{vmatrix}$$

$$= \frac{-2\lambda_1+\lambda_3+\lambda_4}{(\lambda_3-\lambda_1)^2(\lambda_4-\lambda_1)^2}f(\lambda_1) + \frac{1}{(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}f'(\lambda_1)$$

$$+\frac{-1}{(\lambda_4-\lambda_3)(\lambda_3-\lambda_1)^2}f(\lambda_3) + \frac{1}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_1)^2}f(\lambda_4).$$

The Case $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$

We use the formulas in the case "$\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$" for $\lambda_3 \to \lambda_1$ and obtain

$$a_0^{(f)} = \frac{1}{2(\lambda_4-\lambda_1)^3}\begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f(\lambda_4) \\ \lambda_1 & 1 & 0 & \lambda_4 \\ \lambda_1^2 & 2\lambda_1 & 2 & \lambda_4^2 \\ \lambda_1^3 & 3\lambda_1^2 & 6\lambda_1 & \lambda_4^3 \end{vmatrix}$$

$$= \frac{\lambda_1^3+(\lambda_4-\lambda_1)^3}{(\lambda_4-\lambda_1)^3}f(\lambda_1) + \frac{\lambda_1\lambda_4(2\lambda_1-\lambda_4)}{(\lambda_4-\lambda_1)^2}f'(\lambda_1)$$

$$+\frac{\lambda_1^2\lambda_4}{2(\lambda_4-\lambda_1)}f''(\lambda_1) + \frac{-\lambda_1^3}{(\lambda_4-\lambda_1)^3}f(\lambda_4),$$

$$a_1^{(f)} = \frac{1}{2(\lambda_4-\lambda_1)^3}\begin{vmatrix} 1 & 0 & 0 & 1 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f(\lambda_4) \\ \lambda_1^2 & 2\lambda_1 & 2 & \lambda_4^2 \\ \lambda_1^3 & 3\lambda_1^2 & 6\lambda_1 & \lambda_4^3 \end{vmatrix}$$

$$= \frac{-3\lambda_1^2}{(\lambda_4-\lambda_1)^3} f(\lambda_1) + \frac{-2\lambda_1^2 - 2\lambda_1\lambda_4 + \lambda_4^2}{(\lambda_4-\lambda_1)^2} f'(\lambda_1)$$

$$+ \frac{-\lambda_1(\lambda_1+2\lambda_4)}{2(\lambda_4-\lambda_1)} f''(\lambda_1) + \frac{3\lambda_1^2}{(\lambda_4-\lambda_1)^3} f(\lambda_4),$$

$$a_2^{(f)} = \frac{1}{2(\lambda_4-\lambda_1)^3}\begin{vmatrix} 1 & 0 & 0 & 1 \\ \lambda_1 & 1 & 0 & \lambda_4 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f(\lambda_4) \\ \lambda_1^3 & 3\lambda_1^2 & 6\lambda_1 & \lambda_4^3 \end{vmatrix}$$

$$= \frac{3\lambda_1}{(\lambda_4-\lambda_1)^3} f(\lambda_1) + \frac{3\lambda_1}{(\lambda_4-\lambda_1)^2} f'(\lambda_1)$$

$$+ \frac{2\lambda_1+\lambda_4}{2(\lambda_4-\lambda_1)} f''(\lambda_1) + \frac{-3\lambda_1}{(\lambda_4-\lambda_1)^3} f(\lambda_4),$$

$$a_3^{(f)} = \frac{1}{2(\lambda_4-\lambda_1)^3}\begin{vmatrix} 1 & 0 & 0 & 1 \\ \lambda_1 & 1 & 0 & \lambda_4 \\ \lambda_1^2 & 2\lambda_1 & 2 & \lambda_4^2 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f(\lambda_4) \end{vmatrix}$$

$$= \frac{-1}{(\lambda_4-\lambda_1)^3} f(\lambda_1) + \frac{-1}{(\lambda_4-\lambda_1)^2} f'(\lambda_1) + \frac{-1}{2(\lambda_4-\lambda_1)} f''(\lambda_1)$$

$$+ \frac{1}{(\lambda_4-\lambda_1)^3} f(\lambda_4).$$

The Case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$

We use the formulas obtained in the previous case for $\lambda_4 \to \lambda_1$ and get

$$a_0^{(f)} = \frac{1}{6}\begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f'''(\lambda_1) \\ \lambda_1 & 1 & 0 & 0 \\ \lambda_1^2 & 2\lambda_1 & 2 & 0 \\ \lambda_1^3 & 3\lambda_1^2 & 6\lambda_1 & 6 \end{vmatrix}$$

$$= 2f(\lambda_1) - 2\lambda_1 f'(\lambda_1) + \lambda_1^2 f''(\lambda_1) + \frac{-\lambda_1^3}{3} f'''(\lambda_1),$$

$$a_1^{(f)} = \frac{1}{6}\begin{vmatrix} 1 & 0 & 0 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f'''(\lambda_1) \\ \lambda_1^2 & 2\lambda_1 & 2 & 0 \\ \lambda_1^3 & 3\lambda_1^2 & 6\lambda_1 & 6 \end{vmatrix}$$

$$= 2f'(\lambda_1) - 2\lambda_1 f''(\lambda_1) + \lambda_1^2 f'''(\lambda_1),$$

$$a_2^{(f)} = \frac{1}{6}\begin{vmatrix} 1 & 0 & 0 & 0 \\ \lambda_1 & 1 & 0 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f'''(\lambda_1) \\ \lambda_1^3 & 3\lambda_1^2 & 6\lambda_1 & 6 \end{vmatrix} = f''(\lambda_1) - \lambda_1 f'''(\lambda_1),$$

$$a_3^{(f)} = \frac{1}{6}\begin{vmatrix} 1 & 0 & 0 & 0 \\ \lambda_1 & 1 & 0 & 0 \\ \lambda_1^2 & 2\lambda_1 & 2 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f''(\lambda_1) & f'''(\lambda_1) \end{vmatrix} = \frac{1}{3}f'''(\lambda_1).$$

The Case $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$ and $\lambda_2 \neq \lambda_4$

The general Rodrigues coefficients can be obtained from the formulas in Section "The Case $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$" for $\lambda_4 \to \lambda_3$. We obtain

$$a_0^{(f)} = \frac{1}{(\lambda_3 - \lambda_1)^4}\begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) & f'(\lambda_3) \\ \lambda_1 & 1 & \lambda_3 & 1 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 & 2\lambda_3 \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_3^3 & 3\lambda_3^2 \end{vmatrix}$$

$$= \frac{\lambda_3^2(-3\lambda_1 + \lambda_3)}{(\lambda_3 - \lambda_1)^3} f(\lambda_1) + \frac{-\lambda_1\lambda_3^2}{(\lambda_3 - \lambda_1)^2} f'(\lambda_1)$$

$$+ \frac{-\lambda_1^2(\lambda_1 - 3\lambda_3)}{(\lambda_3 - \lambda_1)^3} f(\lambda_3) + \frac{-\lambda_1^2\lambda_3}{(\lambda_3 - \lambda_1)^2} f'(\lambda_3),$$

$$a_1^{(f)} = \frac{1}{(\lambda_3 - \lambda_1)^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) & f'(\lambda_3) \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 & 2\lambda_3 \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_3^3 & 3\lambda_3^2 \end{vmatrix}$$

$$= \frac{6\lambda_1\lambda_3}{(\lambda_3 - \lambda_1)^3} f(\lambda_1) + \frac{\lambda_3(2\lambda_1 + \lambda_3)}{(\lambda_3 - \lambda_1)^2} f'(\lambda_1)$$

$$+ \frac{-6\lambda_1\lambda_3}{(\lambda_3 - \lambda_1)^3} f(\lambda_3) + \frac{\lambda_1(\lambda_1 + 2\lambda_3)}{(\lambda_3 - \lambda_1)^2} f'(\lambda_3),$$

$$a_2^{(f)} = \frac{1}{(\lambda_3-\lambda_1)^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_3 & 1 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) & f'(\lambda_3) \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_3^3 & 3\lambda_3^2 \end{vmatrix}$$

$$= \frac{-3(\lambda_1+\lambda_3)}{(\lambda_3-\lambda_1)^3}f(\lambda_1) + \frac{-\lambda_1-2\lambda_3}{(\lambda_3-\lambda_1)^2}f'(\lambda_1)$$

$$+\frac{3(\lambda_1+\lambda_3)}{(\lambda_3-\lambda_1)^3}f(\lambda_3) + \frac{-2\lambda_1-\lambda_3}{(\lambda_3-\lambda_1)^2}f'(\lambda_3),$$

$$a_3^{(f)} = \frac{1}{(\lambda_3-\lambda_1)^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_3 & 1 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 & 2\lambda_3 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_3) & f'(\lambda_3) \end{vmatrix}$$

$$= \frac{2}{(\lambda_3-\lambda_1)^3}f(\lambda_1) + \frac{1}{(\lambda_3-\lambda_1)^2}f'(\lambda_1) + \frac{-2}{(\lambda_3-\lambda_1)^3}f(\lambda_3) + \frac{1}{(\lambda_3-\lambda_1)^2}f'(\lambda_3).$$

5 The Exponential Map on the Special Orthogonal Group SO(n)

It is easy to check that the set of the real $n \times n$ orthogonal matrices forms a Lie group under multiplication, denoted by $\mathbf{O}(n)$. The subset of $\mathbf{O}(n)$ consisting of those matrices having the determinant equal to $+1$ is a subgroup, denoted by $\mathbf{SO}(n)$ and called the *special orthogonal group* of the Euclidean space $\mathbb{R}^n$. $\mathbf{SO}(n)$ is an important group used in Mechanics (see the famous book of Arnold [6]) and other research directions. Due to geometric reasons, the matrices in $\mathbf{SO}(n)$ are also called *rotation matrices*.

It is well known that the Lie algebra $\mathfrak{so}(n)$ of $\mathbf{SO(n)}$ consists in all skew-symmetric matrices in $M_n(\mathbb{R})$ and the Lie bracket is the standard matrix commutator $[A, B] = AB - BA$. The exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ is defined by the same formula (1) because it is given by the restriction $\exp|_{\mathfrak{so}(n)}$ of the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$. It is known that for every compact connected Lie group, the exponential map is surjective (see Bröcker and tom Dieck [7], Andrica and Casu [1] for the standard proof, or Rohan [20] for applying a new idea of proof given by T. Tao), that is, every compact connected Lie group is exponential (see the monograph of Wüstner [23] for details about the exponential groups). Because the group $\mathbf{SO(n)}$ is compact, it follows that the exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ is surjective. The surjectivity of exp for the group $\mathbf{SO(n)}$ is an important property. Indeed, it implies the existence of a locally inverse function $\log : \mathbf{SO}(n) \to \mathfrak{so}(n)$, and this has interesting applications. In the paper of Gallier and Xu [10] is mentioned that the functions exp and log for the group $\mathbf{SO(n)}$ can be used for motion interpolation (see Kim and Shin [13, 14] and Park and Ravani [16, 17]). Motion interpolation and rational motions have also been investigated by Jüttler [11, 12]. Also, the surjectivity of the exponential map for the group $\mathbf{SO(n)}$ gives the possibility to describe the rotations of the Euclidean space $\mathbb{R}^n$ (see Rohan [20]). The connection with the noncommutative differential geometry is given in the paper of Piscoran [18]. The problem of describing the image of the exponential map in the general setting is discussed in the paper Andrica and Rohan [3]. The exponential map on other groups of matrices is presented in details in Gallier [9].

In what follows, we apply the results obtained in Sects. 2–4 to get the Rodrigues formulas for the exponential map on the special orthogonal group $\mathbf{SO(n)}$. The matrices in the Lie algebra $\mathfrak{so}(n)$ have two essential properties which simplify the computation of the Rodrigues coefficients:

- If n is odd, then they are singular, i.e., they have one eigenvalue equal to 0 (possible with a multiplicity).
- The nonzero eigenvalues are purely imaginary and, of course, conjugated.

5.1 Illustrating the Classical Cases $n = 2, 3$

When $n = 2$, a skew-symmetric matrix $X \neq O_2$ can be written as

$$X = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*,$$

having the eigenvalues $\lambda_1 = ai$, $\lambda_2 = -ai$.

From the formulas derived in Sect. 3.1, we immediately obtain

$$a_0 = \frac{1}{2}\left(e^{ai} + e^{-ai}\right) = \cos a,$$
$$a_1 = \frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} = \frac{e^{ai} - e^{-ai}}{2ai} = \frac{\sin a}{a},$$

and then the corresponding Rodrigues formula is

$$\exp(X) = (\cos a)I_2 + \frac{\sin a}{a}X.$$

When $n = 3$, a real skew-symmetric matrix X is of the form

$$X = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

having the characteristic polynomial

$$p_X(t) = t^3 + (a^2 + b^2 + c^2)t = t^3 + \theta^2 t,$$

where $\theta = \sqrt{a^2 + b^2 + c^2}$. The eigenvalues of X are $\lambda_1 = \theta i$, $\lambda_2 = -\theta i$, $\lambda_3 = 0$. It is clear that $X = O_3$ if and only if $\theta = 0$; hence, it suffices to consider only the situation $\theta \neq 0$. Because $\theta \neq 0$, using the formulas obtained in Sect. 3.2, it follows that

$$a_0 = 1,\ a_1 = \frac{\sin\theta}{\theta},\ a_2 = \frac{1-\cos\theta}{\theta^2},$$

giving the well-known classical formula due to Rodrigues

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta}X + \frac{1-\cos\theta}{\theta^2}X^2.$$

5.2 The Case $n = 4$

The general skew-symmetric matrix $X \in so(4)$ is

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix},$$

and the corresponding characteristic polynomial is given by

$$p_X(t) = t^4 + (a^2 + b^2 + c^2 + d^2 + e^2 + f^2)t^2 + (af - be + cd)^2.$$

Let $\lambda_{1,2} = \pm\alpha i$, $\lambda_{3,4} = \pm\beta i$ be the eigenvalues of the matrix X, where $\alpha, \beta \in \mathbb{R}$. It is clear that the real numbers α and β can be effectively determined in terms of a, b, c, d, e, f by solving the equation $p_X(t) = 0$.

We consider the following three cases:

Case 1. If $|\alpha| \neq |\beta|$, $\alpha, \beta \in \mathbb{R}^*$, then using the formulas in Sect. 3.3, after simple computations, we obtain the Rodrigues coefficients

$$a_0 = \frac{\beta^2\cos\alpha - \alpha^2\cos\beta}{\beta^2 - \alpha^2}, a_1 = \frac{\beta^3\sin\alpha - \alpha^3\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)},$$

$$a_2 = \frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2}, a_3 = \frac{\beta\sin\alpha - \alpha\sin\beta}{\alpha\beta(\beta^2 - \alpha^2)}.$$

In this case, it follows the corresponding Rodrigues formula in the form:

$$\exp(X) = \frac{\beta^2 \cos\alpha - \alpha^2 \cos\beta}{\beta^2 - \alpha^2} I_4 + \frac{\beta^3 \sin\alpha - \alpha^3 \sin\beta}{\alpha\beta(\beta^2 - \alpha^2)} X + \frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2} X^2 + \frac{\beta \sin\alpha - \alpha \sin\beta}{\alpha\beta(\beta^2 - \alpha^2)} X^3. \tag{14}$$

Case 2. If $\alpha \neq 0$ and $\beta = 0$, then we will use the formulas in Section "The Case $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$" when $\lambda_1 \neq \lambda_2 \neq \lambda_3 = \lambda_4$ and obtain

$$a_0 = 1, a_1 = 1, a_2 = \frac{1 - \cos\alpha}{\alpha^2}, a_3 = +\frac{\alpha - \sin\alpha}{\alpha^3}. \tag{15}$$

Therefore, the corresponding Rodrigues formula to this case is

$$\exp(X) = I_4 + X + \frac{1 - \cos\alpha}{\alpha^2} X^2 + \frac{\alpha - \sin\alpha}{\alpha^3} X^3. \tag{16}$$

Case 3. If $\alpha = \beta \neq 0$, then we will use the formulas in Section "The Case $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$ and $\lambda_2 \neq \lambda_4$" for $\lambda_1 = \lambda_3, \lambda_2 = \lambda_4, \lambda_1 \neq \lambda_2$, and after simple computations, we get

$$a_0 = \frac{\alpha \sin\alpha + 2\cos\alpha}{2}, a_1 = \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha}, a_2 = \frac{\sin\alpha}{2\alpha}, a_3 = \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3}. \tag{17}$$

Hence, the Rodrigues formula is

$$\exp(X) = \frac{\alpha \sin\alpha + 2\cos\alpha}{2} I_4 + \frac{3\sin\alpha - \alpha\cos\alpha}{2\alpha} X + \frac{\sin\alpha}{2\alpha} X^2 + \frac{\sin\alpha - \alpha\cos\alpha}{2\alpha^3} X^3. \tag{18}$$

Note that in the paper [4], the formulas (16) and (18) are derived by using the so-called Putzer's method (see [19] for the original reference).

6 The Cayley Transform and the Rodrigues-Type Formulas

As we have already mentioned in the previous section, the matrices of the $\mathbf{SO}(n)$ group describe the rotations as movements in the space $\mathbb{R}^n$. If the matrix A belongs to the Lie algebra $\mathfrak{so}(n)$ of the Lie group $\mathbf{SO}(n)$, then the matrix $I_n - A$ is invertible.

Indeed, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix A are 0 or purely imaginary, so eigenvalues of the matrix $I_n - A$ are $1 - \lambda_1, \ldots, 1 - \lambda_n$. They are clearly different from 0; therefore, we have $\det(I_n - A) = (1 - \lambda_1) \ldots (1 - \lambda_n) \neq 0$, so $I_n - A$ is invertible.

The map Cay : $\mathfrak{so}(n) \to \mathbf{SO}(n)$, defined by

$$\text{Cay(A)} = (I_n + A)(I_n - A)^{-1},$$

is called the *Cayley transform* of the group $\mathbf{SO}(n)$. Let us show that this map is well defined. Let be $\text{Cay}(A) = R$. We have

$$\begin{aligned} R^t R &= (I_n + A)(I_n - A)^{-1t}[(I_n + A)(I_n - A)^{-1}] \\ &= (I_n + A)(I_n - A)^{-1t}[(I_n - A)^{-1}]^t (I_n + A) \\ &= (I_n + A)(I_n - A)^{-1}(I_n -^t A)^{-1}(I_n +^t A) \\ &= (I_n + A)(I_n - A)^{-1}(I_n + A)^{-1}(I_n - A) = I_n, \end{aligned}$$

because matrices and their inverses commute. Therefore, $R \in \mathbf{SO}(n)$. The map Cay is obviously continuous, and we have $\text{Cay}(O_n) = I_n \in \mathbf{SO}(n)$; hence, necessarily we have $R \in \mathbf{SO}(n)$.

Denote by $\sum$ the set of the group $\mathbf{SO}(n)$ containing the matrices with eigenvalue -1. Clearly, we have $R \in \sum$ if and only if the matrix $I_n + R$ is singular.

Theorem 3. *The map* Cay : $\mathfrak{so}(n) \to \mathbf{SO}(n) \setminus \sum$ *is bijective and its inverse is* $\text{Cay}^{-1} : \mathbf{SO}(n) \setminus \sum \to \mathfrak{so}(n)$, *where* $\text{Cay}^{-1}(R) = (R + I_n)^{-1}(R - I_n)$.

Proof. If $R \in \mathbf{SO}(n) \setminus \sum$, then the relation $\text{Cay}(A) = R$ is equivalent to

$$R = (I_n + A)(I_n - A)^{-1} = (2I_n - (I_n - A))(I_n - A)^{-1} = 2(I_n - A)^{-1} - I_n.$$

Because $R \in \mathbf{SO}(n) \setminus \sum$, it follows that the matrix $R + I_n$ is invertible, and from the above relation, we obtain that its inverse is $(R + A)^{-1} = \frac{1}{2}(I_n - A)$. Using this relation, we have

$$(R + I_n)^{-1}(R - I_n) = \frac{1}{2}(I_n - A)(2(I_n - A)^{-1} - 2I_n) = I_n - I_n + A = A,$$

so $\text{Cay}^{-1}(R) = (R + I_n)^{-1}(R - I_n)$.

In addition, a simple computation shows that if the matrix R is orthogonal, then the matrix $A = (R + I_n)^{-1}(R - I_n)$ is antisymmetric. Indeed, we have

$$\begin{aligned} {}^t A &= ({}^t R - I_n)({}^t R + I_n)^{-1} = (R^{-1} - I_n)(R^{-1} + I_n)^{-1} \\ &= (I_n - R)R^{-1}R(I_n + R)^{-1} = -(R + I_n)^{-1}(R - I_n) = -A, \end{aligned}$$

because the matrices $R - I_n$ and $(R + I_n)^{-1}$ commute.

Clearly, the Cayley transform is obtained from the analytic map

$$f(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots, |z| < 1.$$

Therefore, we can apply the results derived in Sects. 2–4. Because the inverse of the matrix $I_n - A$ can be written in the form

$$(I_n - A)^{-1} = I_n + A + A^2 + \cdots$$

for a sufficiently small neighborhood of O_n, from the Hamilton–Cayley theorem, it follows that the Cayley transform of A can be written in the polynomial form

$$\text{Cay}(A) = b_0(A)I_n + b_1(A)A + \cdots + b_{n-1}(A)A^{n-1} \tag{19}$$

where the coefficients $b_0, \ldots, b_{n-1}$ are uniquely determined and depend on the matrix A. We will call these numbers, as in the general setting, the Rodrigues coefficients of A with respect to the application Cay.

6.1 Illustrating the Cases $n = 2, 3$

Following the paper [2] we will continue by the presentation of the particular cases $n = 2$ and $n = 3$. If $A = O_n$, then $\text{Cay}(A) = I_n$, and so $b_0(O_n) = 1, b_1(O_n) = \cdots = b_{n-1}(O_n) = 0$.

In the case $n = 2$, consider the antisymmetric matrix $A \neq O_2$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, a \in \mathbb{R}^*,$$

with eigenvalues $\lambda_1 = ai, \lambda_2 = -ai$. From the formulas derived in Sect. 3.1, we obtain

$$b_0 = \frac{1-a^2}{1+a^2} \text{ and } b_1 = \frac{1}{1+a^2}.$$

Thus, the Rodrigues-type formula for the Cayley transform is

$$\text{Cay}(A) = \frac{1-a^2}{1+a^2}I_2 + \frac{2}{1+a^2}A. \tag{20}$$

For $n = 3$, any real antisymmetric matrix X is of the form

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

with the characteristic polynomial $p_A(t) = t^3 + \theta^2 t$, where $\theta = \sqrt{a^2 + b^2 + c^2}$. The eigenvalues of the matrix A are $\lambda_1 = \theta i, \lambda_2 = -\theta i, \lambda_3 = 0$. We have $A = O_3$ if and only if $\theta = 0$, so it is enough to consider only the situation in which $\theta \neq 0$. Using the formulas obtained in Sect. 3.2, it follows

$$b_0 = 1, b_1 = \frac{2}{1+\theta^2}, b_2 = \frac{2}{1+\theta^2}$$

and the Rodrigues-type formula for the Cayley transform of group $\mathbf{SO}(3)$

$$\mathrm{Cay}(A) = I_3 + \frac{2}{1+\theta^2}A + \frac{2}{1+\theta^2}A^2. \tag{21}$$

Formula (21) offers the possibility to obtain another form for the inverse of Cayley transform. Indeed, let be $R \in \mathbf{SO}(3)$ such that

$$R = I_3 + \frac{2}{1+\theta^2}A + \frac{2}{1+\theta^2}A^2,$$

where A is an antisymmetric matrix. Considering the matrix transpose in both sides of the above relation and taking into account that ${}^tA = -A$, we obtain

$$R -{}^t R = \frac{4}{1+\theta^2}A. \tag{22}$$

On the other hand, we have

$$\mathrm{tr}(R) = 3 - \frac{4\theta^2}{1+\theta^2} = -1 + \frac{4}{1+\theta^2},$$

and by replacing in the relation (22), we get the formula

$$\mathrm{Cay}^{-1}(R) = \frac{1}{1+\mathrm{tr}(R)}(R -{}^t R). \tag{23}$$

Formula (23) makes sense for rotations $R \in \mathbf{SO}(3)$ for which $1 + \mathrm{tr}(R) \neq 0$. If R is a rotation of angle α, then we have $\mathrm{tr}(R) = 1 + 2\cos\alpha$, so application Cay^{-1} is not defined for the rotations of angle $\alpha = \pm\pi$. Because in the domain where it is defined the application Cay is bijective, it follows that the antisymmetric matrices from $\mathfrak{so}(3)$ can be used as coordinates for rotations. Considering the Lie algebra isomorphism "$\widehat{\ }$" between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\cdot,\cdot])$, where "$\times$" denote the vector product, defined by $v \in \mathbb{R}^3 \to \widehat{v} \in \mathfrak{so}(3)$, where

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$\widehat{v} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

by composing the applications

$$\mathbb{R}^3 \widehat{\to} \mathfrak{so}(3) \xrightarrow{\text{Cay}} \mathbf{SO}(3)$$

we get a vectorial parameterization of rotations from $\mathbf{SO}(3)$.

6.2 The Case $n = 4$

As in Sect. 5.2, for a skew-symmetric matrix $A \in so(4)$, let $\lambda_{1,2} = \pm\alpha i$, $\lambda_{3,4} = \pm\beta i$ be the eigenvalues of the matrix A, where $\alpha, \beta \in \mathbb{R}$. We consider the following three situations.

Case 1. If $|\alpha| \neq |\beta|$, $\alpha, \beta \in \mathbb{R}^*$, then using the formulas in Sect. 3.3, we obtain

$$b_0 = \frac{1+\alpha^2+\beta^2-\alpha^2\beta^2}{(1+\alpha^2)(1+\beta^2)}, b_1 = \frac{2(1+\alpha^2+\beta^2)}{(1+\alpha^2)(1+\beta^2)}$$
$$b_2 = \frac{2}{(1+\alpha^2)(1+\beta^2)}, b_3 = \frac{2}{(1+\alpha^2)(1+\beta^2)}$$

and the corresponding Rodrigues formula

$$\begin{aligned}\text{Cay}(A) = {} & \frac{1+\alpha^2+\beta^2-\alpha^2\beta^2}{(1+\alpha^2)(1+\beta^2)} I_4 + \frac{2(1+\alpha^2+\beta^2)}{(1+\alpha^2)(1+\beta^2)} A \\ & + \frac{2}{(1+\alpha^2)(1+\beta^2)} A^2 + \frac{2}{(1+\alpha^2)(1+\beta^2)} A^3.\end{aligned}$$

Case 2. If $\alpha \neq 0$ and $\beta = 0$, then we will use the formulas in Section "The Case $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$" when $\lambda_1 \neq \lambda_2 \neq \lambda_3 = \lambda_4$ and obtain

$$b_0 = \frac{1}{-2i\alpha^5} \begin{vmatrix} 1 & 2 & \frac{1+\alpha i}{1-\alpha i} & \frac{1-\alpha i}{1+\alpha i} \\ 0 & 1 & \alpha i & -\alpha i \\ 0 & 0 & -\alpha^2 & -\alpha^2 \\ 0 & 0 & -i\alpha^3 & i\alpha^3 \end{vmatrix} = 1, b_1 = \frac{1}{-2i\alpha^5} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & \frac{1+\alpha i}{1-\alpha i} & \frac{1-\alpha i}{1+\alpha i} \\ 0 & 0 & -\alpha^2 & -\alpha^2 \\ 0 & 0 & -i\alpha^3 & i\alpha^3 \end{vmatrix} = 2,$$

$$b_2 = \frac{1}{-2i\alpha^5}\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \alpha i & -\alpha i \\ 1 & 2 & \frac{1+\alpha i}{1-\alpha i} & \frac{1-\alpha i}{1+\alpha i} \\ 0 & 0 & -i\alpha^3 & i\alpha^3 \end{vmatrix} = \frac{2}{1+\alpha^2},$$

$$b_3 = \frac{1}{-2i\alpha^5}\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & \alpha i & -\alpha i \\ 0 & 0 & -\alpha^2 & -\alpha^2 \\ 1 & 2 & \frac{1+\alpha i}{1-\alpha i} & \frac{1-\alpha i}{1+\alpha i} \end{vmatrix} = \frac{2}{1+\alpha^2}.$$

The Rodrigues formula in this case is

$$\mathrm{Cay}(A) = I_4 + 2A + \frac{2}{(1+\alpha^2)}A^2 + \frac{2}{(1+\alpha^2)}A^3.$$

Case 3. If $\alpha = \beta \neq 0$, then we will use the formulas in Section "The Case $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$ and $\lambda_2 \neq \lambda_4$" for $\lambda_1 = \lambda_3, \lambda_2 = \lambda_4, \lambda_1 \neq \lambda_2$, and after simple computations, we get

$$b_0 = \frac{1}{16\alpha^4}\begin{vmatrix} \frac{1+\alpha i}{1-\alpha i} & \frac{2}{(1-\alpha i)^2} & \frac{1-\alpha i}{1+\alpha i} & \frac{2}{(1+\alpha i)^2} \\ \alpha i & 1 & -\alpha i & 1 \\ -\alpha^2 & 2\alpha i & -\alpha^2 & -2\alpha i \\ -i\alpha^3 & -3\alpha^2 & i\alpha^3 & -3\alpha^2 \end{vmatrix} = \frac{1+2\alpha^2-\alpha^4}{(1+\alpha^2)^2},$$

$$b_1 = \frac{1}{16\alpha^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ \frac{1+\alpha i}{1-\alpha i} & \frac{2}{(1-\alpha i)^2} & \frac{1-\alpha i}{1+\alpha i} & \frac{2}{(1+\alpha i)^2} \\ -\alpha^2 & 2\alpha i & -\alpha^2 & -2\alpha i \\ -i\alpha^3 & -3\alpha^2 & i\alpha^3 & -3\alpha^2 \end{vmatrix} = \frac{2(2\alpha^2+1)}{(1+\alpha^2)^2},$$

$$b_2 = \frac{1}{16\alpha^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ \alpha i & 1 & -\alpha i & 1 \\ \frac{1+\alpha i}{1-\alpha i} & \frac{2}{(1-\alpha i)^2} & \frac{1-\alpha i}{1+\alpha i} & \frac{2}{(1+\alpha i)^2} \\ -i\alpha^3 & -3\alpha^2 & i\alpha^3 & -3\alpha^2 \end{vmatrix} = \frac{2}{(1+\alpha^2)^2},$$

$$b_3 = \frac{1}{16\alpha^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ \alpha i & 1 & -\alpha i & 1 \\ -\alpha^2 & 2\alpha i & -\alpha^2 & -2\alpha i \\ \frac{1+\alpha i}{1-\alpha i} & \frac{2}{(1-\alpha i)^2} & \frac{1-\alpha i}{1+\alpha i} & \frac{2}{(1+\alpha i)^2} \end{vmatrix} = \frac{2}{(1+\alpha^2)^2}$$

and the corresponding Rodrigues formula

$$\mathrm{Cay}(A) = \frac{1+2\alpha^2-\alpha^4}{(1+\alpha^2)^2}I_4 + \frac{2(2\alpha^2+1)}{(1+\alpha^2)^2}A + \frac{2}{(1+\alpha^2)^2}A^2 + \frac{2}{(1+\alpha^2)^2}A^3.$$

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Quasimodes in Integrable Systems and Semi-Classical Limit

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Abstract Quasimodes are long-living quantum states that are localized along classical orbits. They can be considered as resonances, whose wave functions display semi-classical features. In some integrable systems, they have been constructed mainly by the coherent state method, and their connection with the classical motion has been extensively studied, in particular as a tool to perform the semi-classical limit of a quantum system. In this work, we present a method to construct quasimodes in integrable systems. Although the method is based on elementary procedures, it is quite general. It is shown that the requirement of a long lifetime and strong localization implies that the quasimode must be localized around a closed classical orbit. At a fixed degree of localization, the lifetime of the quasimode can be made arbitrarily longer with respect to the classical period in the asymptotic limit of large quantum numbers. It turns out that the coherent state method is a particular case of this general scheme.

1 Introduction

The semi-classical limit has been one of the basic issues that attracted continuous interest since the foundation of quantum mechanics. The eigenvalues of integrable systems can be obtained from the Bohr–Sommerfeld semi-classical quantization method, which is valid in the large quantum number limit, i.e. large actions with respect to the Planck constant h. More difficult is to obtain the semi-classical limit of the corresponding wave functions. It is usually assumed that the wave functions of the eigenstates cover uniformly the whole available phase space, which is the Liouville torus determined by the values of the quantum numbers. As such, they have no resemblance with any classical behaviour of the system. This is of course

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due to the fact that the eigenstates are stationary states which cannot be connected with any classical trajectory, which requires both localization and time dependence. The alternative method for a semi-classical description is through the introduction of wave packets, following the celebrated theorem by Ehrenfest. In the case of a particle motion, the time-dependent localized wave packet follows the classical trajectories. Its size actually spreads with time, but for macroscopic objects, the spreading time becomes asymptotically so large that the particle behaves in a completely classical fashion. For a particle moving in a potential, the theorem can be applied as a valid method for performing the semi-classical limit if the typical distance over which the potential is changing turns out to be much larger than the wavelengths that characterize the wave packet, that is, as it is well known, in the short wavelength limit.

For non-completely macroscopic objects, this means that the wave packet cannot move inside a container with sharp boundaries, noticeably a billiard; otherwise, the wave packet would spread rapidly as it hits one of the boundaries. However, it has been found that also in this typical case, localization can occur because of the presence of caustics [1], and if there is an exact degeneracy among different sites of the billiard, an exact eigenstate can be obtained by a linear combination of the wave functions localized in each site. These wave functions are generically indicated as "quasimodes". They exist also in a generic system. They have a counterpart in classical sound phenomena, like the whispering modes in an auditorium. However, the term quasimode has been used to indicate generically to states that are "close" to stationary mode, i.e. to eigenstates of the system [2, 12, 20, 22, 26]. We will indicate by quasimodes a more specific class of states. For a generic system, we will consider as quasimodes states that have the wave function sharply localized around a classical periodic orbit and have a long lifetime, in some sense to be specified later. Many authors have proposed different methods to construct quasimodes, either on the basis of particular eigenstate superposition suggested by Ehrenfest theorem [21] or on geometrical optics methods [5, 18]. In the latter work, it has been shown that in a billiard of generic shape, two types of quasimodes can be indeed present, the "whispering gallery" and "bouncing ball" modes.

Several authors [11, 25] have developed general methods to construct long-living quasimodes that asymptotically for $\hbar \rightarrow 0$ "are close" to eigenstates, in a mathematically well-defined sense. All these methods build up the quasimode around a stable periodic orbit, corresponding to an elliptical fixed point. For a Riemann manifold [11], the quasimode is localized around a stable geodesic. The general case for a smooth dynamics has been treated in [25]. An extensive analysis of the mathematical basis of quasimodes can be found in [3].

It has to be stressed that quasimodes have not to be confused with the phenomenon of scars in chaotic systems [15]. In this case, one refers to an exact eigenstate whose wave function shows some enhancement around an unstable classical periodic orbit. This phenomenon is especially present in billiards. It has been studied extensively in the literature [6, 15–17, 23, 27, 28]. Partial explanations of scars have been presented by many authors. In [7], an approach based on the Gutzwiller trace formula [14] for chaotic systems has been used, where the density of states is determined, in the semi-classical limit, solely by the classical (unstable) periodic orbits. If around a classical periodic orbit one takes also the contribution from closed non-periodic orbits, the average wave function can display some enhancement in the vicinity of the periodic orbit.

However, it is possible to relate quasimodes and scars. A method [23, 27] that has been developed to show this link is based on the so-called Gaussian beams, where a Gaussian wave packet is launched along an unstable periodic orbit. The short-time dynamics of the wave packet provides the correct superposition of eigenstates, belonging to a band in the smoothed spectrum, that gives a wave function localized along the unstable periodic orbit. In this way, one gets a short-living quasimode. At the same time, the substantial overlap of the eigenstates, involved in the superposition, with the quasimode gives an explanation of the presence of scars. These results establish a link between quasimode and scars in chaotic systems. Since the periodic orbit is unstable, the quasimode so constructed has a short lifetime, determined at least by the Lyapunov exponents of the orbit. It seems clear that a long-living quasimode, sharply localized around an unstable classical orbit, with a lifetime arbitrarily larger than the classical period, cannot exist, even in the semi-classical limit. In particular, this method can have a limited application to billiards, since, as it is well known, a wave packet that hits a sharp edge spreads out rapidly. However, in [27], it was shown that following the evolution of the wave packet, one can obtain a wave function close to an eigenstate that in some cases displays scarred structure.

In integrable systems, the situation is substantially different. Periodic orbits can be grouped in families that can be obtained by a smooth variation of an orbital parameter. As an example, in the circular billiard, one can smoothly vary the orientation of the orbit. Furthermore, each periodic orbit corresponds to a parabolic fixed point in the Poincaré map, since neighbouring trajectories diverge linearly with time. The systems have symmetries that the eigenfunctions must respect. On the contrary, a given generic periodic orbit can have only discrete symmetries, different from the ones of the system, and therefore, a quasimode, localized around a periodic orbit, cannot approach any eigenstate, even asymptotically for $\hbar \to 0$, while this is possible in the case of an isolated stable periodic orbit (in a generic system, chaotic or not). Of course a suitable linear combination of such quasimodes can approach asymptotically an eigenstate, but this cannot be considered a semi-classical object. This does not prevent the quasimode to have a long lifetime, possibly arbitrarily larger than the classical period. At the same time, localization around the orbit can be in principle achieved by a proper superposition of almost degenerate eigenstates, like in the case of isolated unstable orbits in a chaotic system. To which extent localization and long lifetime can be reached in integrable systems is the subject of the present paper.

For rectangular billiards, a method to construct quasimodes was developed in [9], based on the introduction of coherent states, in analogy with the case of the harmonic oscillator.

In this paper, we present a general procedure to build quasimodes in integrable systems that, although based on elementary methods, allows for a systematic study of their properties. In particular, we will construct quasimodes that, in the proper asymptotic limit, are localized with arbitrary precision around a periodic orbit and at the same time have a lifetime that can be made arbitrarily larger than the period of the orbit. In this way, the quasimodes can be considered as resonances within the point spectrum of the system, as we are going to consider systems completely confined in a restricted region like billiards. Notice however that they are particular resonances, since the wave function is localized around a periodic orbit and they do not respect in general the symmetries of the system and of the eigenfunctions. In any case, because of their properties, they are surely semi-classical objects.

The quasimodes could be the basis for a different way of performing the semi-classical limit, but a systematic study of this possibility has not been fully developed.

Finally, one has to mention that there are exceptional cases where quasimodes are also eigenstates because of the asymptotically large and exact degeneracy present in the spectrum, like for the harmonic oscillator. This feature is clearly connected with the well-known fact that a wave packet does not spread indefinitely, but on the contrary, its size oscillates indefinitely while following the classical trajectory (a closed orbit) [24].

In Sect. 2, we present the method in its general form for two-dimensional systems and for integrable systems, either billiards or Hamiltonian ones. In Sect. 3, we present results for the quasimodes in billiards of different shapes, and we discuss their lifetimes. The connection with coherent states is also discussed. In Sect. 4, we consider Hamiltonian systems. Besides the special case of the harmonic oscillators, we analyse a generic two-dimensional system, and we discuss the quasimodes associated with trajectories that close after several revolutions. Section 5 is devoted to the conclusions and prospects.

2 The General Method

We describe the proposed method by recalling for completeness some elementary results for classical integrable systems. The treatment will be restricted, as in the rest of the paper, to two degrees of freedom. The extension to higher dimension looks possible but not obvious. In the semi-classical limit, we quantize the two-dimensional integrable system by the Bohr–Sommerfeld scheme, where each action integral along a topological distinct path in the invariant torus is imposed to be an integer multiple of the Planck constant h. Therefore, in two-dimensional integrable systems, the energy levels $E(n, l)$ are a function of two integers (quantum numbers) n, l. We are looking for linear combinations of almost degenerate levels that are localized as much as possible. Starting from a particular pair of quantum numbers n_0, l_0 and the corresponding energy $E_0 = E(n_0, l_0)$, we look for the set of quantum numbers that in linear approximation correspond to levels degenerate with E_0. Formally,

$$\begin{aligned} n &= n_0 + \delta n \\ l &= l_0 + \delta l \\ \delta E &= \tfrac{\partial E}{\partial n}\delta n + \tfrac{\partial E}{\partial l}\delta l = 0 \end{aligned} \tag{1}$$

Since n and l are integers, in order to fulfil the condition $\delta E = 0$, it is necessary that the variations δn and δl be in a constant fractional ratio

$$\frac{\delta n}{\delta l} = \frac{p}{q} = -\frac{\partial E}{\partial l} \bigg/ \frac{\partial E}{\partial n} \tag{2}$$

where p and q are two integers that are prime with each other. The partial derivatives of the energy are the frequencies of the classical motion along each degree of freedom

$$\omega_n = \frac{\partial E}{\partial n} \quad ; \quad \omega_l = \frac{\partial E}{\partial l} \tag{3}$$

It follows that the classical orbit associated with (n_0, l_0) closes after $N = pq$ periods of the faster degrees of freedom. Notice however that the trajectory is closed only to order $1/n$, since the frequencies around the tori are discrete upon quantization. This shows the well-known result that sets of quasidegenerate levels, otherwise called "shells" [8], are associated with closed classical orbits. Notice that the condition (2) is also a constraint on the reference quantum numbers (n_0, l_0). Because the semi-classical limit corresponds to asymptotically large quantum numbers, this condition can be fulfilled to any degree of precision.

An estimate of the level of degeneracy can be obtained by calculating the second derivative of the energy along the direction defined by Eq. (2). Here we are of course treating the quantum numbers as continuous variables, which is justified in the semi-classical limit. Along this direction, one can take, e.g. the quantum number n as a linear function of the other quantum number l. As a consequence, also the energy is a function only of the quantum number l. Let us assume for simplicity that the energy appears explicitly only in the action integral J corresponding to the quantum number n, that is, the corresponding Bohr–Sommerfeld quantization ($\hbar = 1$)

$$J(E, l) = 2\pi n \tag{4}$$

is the equation that determines the semi-classical energy for a given pair (n, l) of quantum numbers (assuming for simplicity no exact quantal degeneracy). This will be the case in all explicit models considered in the rest of the paper. The more general case can be treated along the same lines. In Eq. (4), both E and n are considered functions of l. Taking the first derivative of Eq. (4), one gets

$$\frac{dJ}{dl} = \left(\frac{\partial J}{\partial E}\right)\frac{dE}{dl} + \frac{\partial J}{\partial l} = -2\pi\frac{\omega_l}{\omega_n} \tag{5}$$

which fixes the condition on the quantum numbers (n_0, l_0). At the reference point (n_0, l_0), the derivative of the energy is zero by construction. Taking into account this fact, the second derivation at (n_0, l_0) of the equation reads

$$\frac{d^2J}{dl^2} = \left(\frac{\partial J}{\partial E}\right)\frac{d^2E}{dl^2} + \frac{\partial^2 J}{\partial l^2} = 0 \tag{6}$$

where we have used the vanishing of the second derivative of n with respect to l due to linear dependence of n on l, according to Eq. (2). From this, one gets at (n_0, l_0)

$$\frac{d^2E}{dl^2} = -\frac{\partial^2 J}{\partial l^2} \bigg/ \frac{\partial J}{\partial E} \tag{7}$$

The spread ΔE in energy within a range of values Δl of the quantum number l around the reference value l_0 can be estimated up to second order as

$$\Delta E = \left|\frac{1}{2}\frac{d^2E}{dl^2}\right| \Delta l^2 \tag{8}$$

It follows that a linear combination of eigenstates within this range will correspond to a state with a lifetime τ of order $1/\Delta E$, $(\hbar = 1)$. Notice that the energy derivative of J is associated with the characteristic time T of the corresponding closed classical orbit, in particular to its period, and therefore,

$$\frac{\tau}{T} = 1 \bigg/ \left|\left(\frac{\partial^2 J}{\partial l^2}\Delta l^2\right)\right| \tag{9}$$

Within the same range Δl, one can construct a linear combination of eigenstates $\psi_{n,l}$ to obtain a wave function Ψ localized in coordinate space. To some extent, the type of linear combination is arbitrary. The standard choice is a Gaussian superposition

$$\Psi(\mathbf{r}) = \sum_l \exp[-(l - l_0)^2/\Delta l^2]\psi_l(\mathbf{r}) \tag{10}$$

where the summation is only over l because it is performed along the direction defined by Eq. (2) (i.e. n is a function of l) and $\mathbf{r}$ is the two-dimensional position vector (coordinate space). The localization will be in the coordinate (cyclic) variable ϕ, canonical conjugated to l. The localization $\Delta\phi$ will be then of the order of $1/\Delta l$. The main goal is now to see to what extent this localization put constraints to the lifetime τ of this state. It turns out that the second derivative of J with respect to l is asymptotically of the order of $1/l$, and therefore,

$$\frac{\tau}{T} \approx l/(\Delta l)^2 \tag{11}$$

which means that, at a fixed localization $\Delta\phi$, the ratio between the lifetime and the classical orbital period is asymptotically arbitrarily large. This result is valid for all the particular systems we are going to consider. The extension of these properties to a general system looks likely but not obvious.

It remains to demonstrate that the localization is around a definite classical orbit. This can be shown by introducing the standard semi-classical expression for the wave functions

$$\Psi_l(\mathbf{r}) \approx \exp(iS_l(r)) \exp(il\phi) \tag{12}$$

where $S_l(r)$ is the reduced action, i.e. the wave function is the exponential of the total action. Expanding the action around $l = l_0$ and taking the stationary phase approximation of the superposition in l of Eq. (10) gives (apart from an irrelevant phase)

$$\Psi(\mathbf{r}) \approx \exp\left(-\frac{|\phi - \phi_S(\mathbf{r})|^2}{2\Delta\phi^2}\right) \tag{13}$$

where

$$\phi_S(r) = -\left(\frac{dS_l(r)}{dl}\right)_{l=l_0} \tag{14}$$

and $\phi = \phi_S(r)$ is indeed the equation of the trajectory. It has to be noticed that the superposition (10) is invariant under a shift of ϕ by a multiple of the quantity $\delta\phi$

$$\delta\phi = \frac{2\pi}{q} \tag{15}$$

because the summation over l is performed with a step q. This implies that the quasimode has a discrete symmetry of $\Delta\phi$. This means also that there are actually multiple points of stationary phase, regularly spaced in ϕ by $\delta\phi$. The approximate expression of Eq. (13) must be then summed up over these q stationary points, which gives the discrete symmetry. All that will be more clear in the explicit applications of the method, where the meaning of this shift $\delta\phi$ will be more evident.

The superposition of Eq. (10) can be performed also numerically, as we will do in the specific examples where the eigenfunctions are analytically known. Since in this case no approximation is used for calculating the quasimode wave function (10), the symmetry discussed above is automatically included. It has to be stressed that in this case there is some freedom in the choice of the eigenfunctions Ψ_{nl}, since they can be normalized but they can still be multiplied by a phase which eventually can be dependent on the quantum numbers (n, l). This can modify the superposition (10), since the phases will result in a different interference pattern. This is a quantum feature that cannot be eliminated even in the semi-classical limit. The choice of the phases can modify the region where the quasimode is localized. To get the proper choice, one can look at the approximate expression (12) and check if in the asymptotic limit this expression is indeed recovered. As a particular case, we can associate to each wave function a phase factor $\exp(il\phi_0)$. According to Eq. (13), this would just simply shift the variable ϕ by a fixed amount ϕ_0. Since ϕ_0 is arbitrary, one can see that one can associate to each classical orbit a family of orbits with similar characteristics but with a different geometry.

3 Quasimodes in Integrable Billiards

The billiards are the simplest systems where quasimodes can be constructed. At classical level, they cannot be described easily by means of a Hamiltonian, because of the discontinuity of the trajectory velocity at the point of bounce on the billiard boundary. However, between two bounces, the motion is free, and one can describe the trajectory as piecewise continuous. Moreover, one can still consider the Liouville–Arnold tori in phase space.

3.1 The Rectangular Billiard

Despite billiards not being Hamiltonian systems, they can be treated within the general method we have introduced. Let us start with the simplest billiard, the square billiard (SB). This case has been extensively studied in [9]. It has also been shown [10] that the quasimodes present wave functions characterized by a vortex structure. In that work, the quasimodes were constructed by coherent states, defined as in the case of the harmonic oscillator. We will treat briefly this case, following the scheme of Sect. 2, and show that one can construct quasimodes more generally, being the coherent states a particular class of them.

For the square billiard, the constants of motion are the (quasi)momenta k^x and k^y along the two side directions, and quantization gives

$$\begin{aligned} k^x{}_n &= n\tfrac{\pi}{R} \\ k^y{}_m &= m\tfrac{\pi}{R} \end{aligned} \tag{16}$$

where R is the side length of the square and n, m two integers, positive or negative. The motion is free, and therefore, the energy levels $E(n,m)$ are readily obtained

$$E(n,m) \;=\; n^2 \;+\; m^2 \tag{17}$$

where for simplicity we put $\pi/R \;=\; 1$. The condition (2) of stationary energy around the point (n_0, m_0) in this case is

$$\frac{\delta n}{\delta m} \;=\; -\frac{m_0}{n_0} \;=\; -\frac{p}{q} \tag{18}$$

Eq. (18) gives in a straightforward way the conditions on both the reference quantum numbers (n_0, m_0) and the direction $(\delta n, \delta m)$ along which the quantum numbers must move. The eigenfunctions are just the products of standing waves along x and y, which are sine (cosine) functions for even (odd) quantum numbers (taking the origin at the centre of the square). The two quantum numbers n and m are equivalent, and the superposition of Eq. (10) can be done in anyone of the two. For illustration, let us take $m_0 = n_0$ and a step of one unit for both quantum numbers in the superposition of Eq. (10). One can write

$$\Psi(x, y) = \sum_l \exp(i(m_0 + l)y + l\phi_0)\psi_l[(n_0 - l)x] \tag{19}$$

where for even values of $n_0 - l$, the eigenfunction ψ_l is a sine function, while for odd ones, it is a cosine function. The additional phase ϕ_0 has a given value. Here, the summation on l is extended in an interval between $l = -N_0$ and $l = N_0$, being N_0 large enough but much smaller than n_0 and m_0. Notice that we took a superposition with a constant factor rather than the Gaussian form of Eq. (10). The summation can be calculated exactly, since it involves geometrical series. The result is the superposition of four terms of the type

$$S_{\pm\pm}(x, y) = \sin[(N_0 + 1)(y \pm x \pm \phi_0)] / \sin(y \pm x \pm \phi_0) \tag{20}$$

where the choices of the signs are independent of each other. The wave function is therefore concentrated along the four straight lines $y \pm x \pm \phi_0 = 0$. For the choice $\phi_0 = \pi/2$, the numerical calculation gives the result depicted in Fig. 1. This has been obtained for $(n_0, m_0) = (400, 400)$ and $N_0 = 10$. The wave function is clearly concentrated along a classical trajectory. Notice that the considered superposition with a constant weight produces oscillations outside the region of maximum contribution, as it can be seen from Eq. (20). A similar procedure can be obtained with a Gaussian weight, as in Eq. (10). The approximate result is expected to be the superposition of the same four branches, but with smoother profiles. The numerical evaluation produces a plot practically indistinguishable from the one depicted in Fig. 1. This indicates that the type of weight for the superposition is not crucial, provided of course that the width of the superposition is similar. Other choices of ϕ_0 correspond to different classical trajectories. At classical level, this phase can be interpreted as fixing the time laps of the motions along the x and y directions. For $\phi_0 = \pi/3$, the result is reported in Fig. 2.

As ϕ_0 is varied, a family of classical trajectories is generated.

Other trajectories with a different topology are generated by different steps in the summation for the quantum numbers n an m. For $\delta n/\delta m = 1/2$ and $\phi_0 = \pi/2$ and $\phi_0 = \pi/3$, the result is reported in Fig. 3 and in Fig. 4, respectively.

The connection of the present method with the one in [9], based on the coherent states, can be obtained by considering the asymptotic form of the combinatorial factors that are used in the superposition. In fact, one has

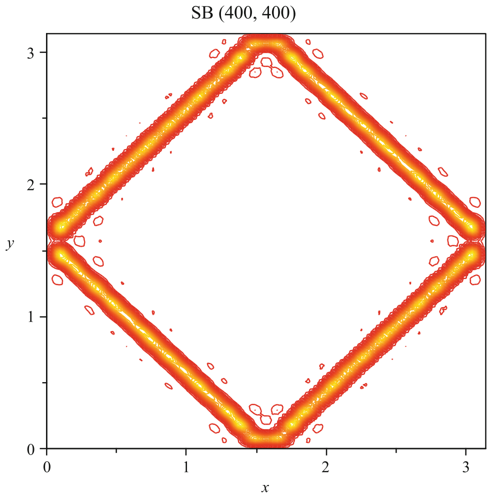


Fig. 1 Quasimode in the square billiard corresponding to the quantum numbers $(n_0, m_0) = (400, 400)$ and $p = q = 1$. The phase is $\phi_0 = \pi$ [see Eq. (19)]

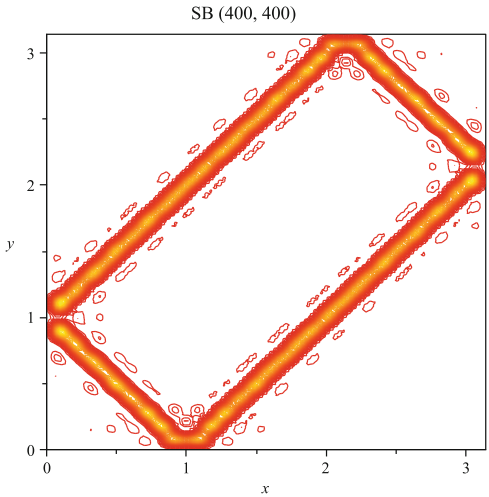


Fig. 2 The same as in Fig. 1, but for $\phi_0 = \pi/4$

$$\begin{pmatrix} N \\ K \end{pmatrix} \approx \exp[-(K - K_0)^2/K_0] \tag{21}$$

where we expanded the Stirling formula for the factorials around the value $K_0 = N/2$, where the combinatorial factor has its maximum. One can see that the coherent state representation corresponds to a Gaussian superposition with a particular choice for the width of the Gaussian.

Finally, for the energy spread of Eq. (8) and lifetime τ of Eq. (9), one gets

$$\Delta E = 1/n_0 \qquad ; \qquad \frac{\tau}{T} = n_0^3/E \approx n_0 \tag{22}$$

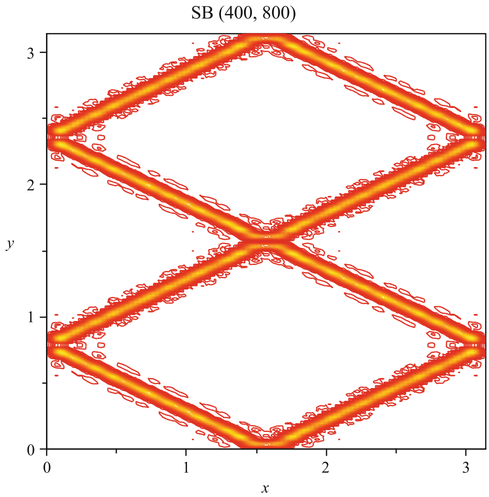


Fig. 3 Quasimode corresponding to $p = 1$ and $q = 2$ with the values of (n_0, m_0) reported in the title. The phase $\phi_0 = \pi$. See the text for details

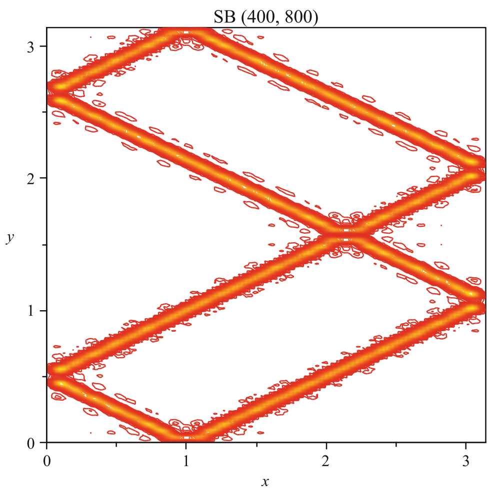


Fig. 4 The same as in Fig. 3, but with $\phi_0 = \pi/2$

which asymptotically for large quantum numbers have the anticipated trend. In the particular cases considered above, one has $\tau/T \approx 4$. With the same degree of localization, one can get an increasing value of τ/T as the quantum numbers increase.

3.2 The Circular Billiard

The next example we are considering is the circular billiard (SB), that was partially analysed in [4]. Here, we report a more extensive analysis, a set of results that illustrate the general method and a study of the localization around a classical trajectory. The semi-classical Bohr–Sommerfeld quantization of the circular billiard leads to a formula of the type of Eq. (4), which explicitly reads [18] ($\hbar = 2M = 1$, being M the mass of the particle)

$$\sqrt{k_{nl}^2 R^2 - l^2} - l\beta_0 = \left(n + \frac{1}{2}\right)\pi + \frac{\pi}{4} \tag{23}$$

where in this case l is the angular momentum, n is the quantum number associated with the radial motion, R is the radius of the billiard and k_{nl} is the momentum, which is then an implicit function of l and n, i.e. it is the eigenvalue. This formula can be obtained by the usual action integral along a closed path on the Liouville torus. Notice that this action integral is actually twice the LHS expression. The additional term $1/2$ at the RHS is introduced as a minimal quantum correction, and the additional term $\pi/4$ is needed because of the presence of reflections at a sharp boundary. These are standard corrections, but in any case, they can be safely neglected since we are working in the large quantum number limit. In this case, the motion is free, and the corresponding energy is just the kinetic energy

$$E_{nl} = k_{nl}^2 \tag{24}$$

In Eq. (23), the angle β_0 is related to the momentum by

$$\cos(\beta_0) = l/k_{nl}R \tag{25}$$

and therefore, β_0 is also an implicit function of the quantum numbers n, l. At the classical level, the angle $2\beta_0$ can be interpreted as the angle spanned by the vector **r** that fixes the position of the particle, between two bouncing on the billiard wall. If this angle is a rational fraction of 2π, i.e. $2\beta_0 = \frac{p}{q}2\pi$, then the particle orbit will close after pq hits on the wall (here we assume that p and q have no common factor). If instead it is an irrational fraction of 2π, the orbit will never close, and in the long time limit, the position of the hits will fill uniformly the circular boundary. The general Eq. (2) in this case reads

$$\frac{\beta_0}{\pi} = \cos^{-1}\left(\frac{l_0}{k_{nl}R}\right) = \frac{\delta n}{\delta l} = -\frac{p}{q} \tag{26}$$

and the corresponding classical orbit around which the quasimode is localized is indeed a closed orbit which closes after q bounces. The localization can be constructed according to the general prescription of Eq. (10). The action integral $S_l(r)$ in this case can be calculated analytically. After some manipulations, it reads

$$\begin{aligned} S_l(r) &= \int^r dr' \sqrt{E - \tfrac{l^2}{r'^2}} \\ &= \sqrt{k^2 r^2 - l^2} - l\beta(r) + C \end{aligned} \tag{27}$$

where C is a constant and

$$\cos(\beta(r)) = \frac{l}{kr} \tag{28}$$

Taking the lower limit of the integral as $r_0 = l/kR$ and the upper limit $r = R$, one gets the LHS of Eq. (23). In fact, this expression, multiplied by 2, is just the action integral over a closed loop along the proper Liouville torus. Following the general procedure, one can consider the superposition of Eq. (10), and one gets for the quasimode wave function (13) in the semi-classical limit

$$\Psi(r,\phi) \approx \exp\left(-\frac{[\phi - \beta(r)]^2}{2\Delta\phi^2}\right) \tag{29}$$

where (r,ϕ) are the cylindrical coordinates and $\Delta\phi = 1/\Delta_l$, being Δ_l the spread in l values considered along the direction specified by Eq. (26). The wave function is therefore concentrated along the curve $\phi = \beta(r)$, which is the straight line between two successive bounces. As discussed in Sect. 2, this expression should be summed up over the series of shifted phases, resulting in the total wave function of the quasimode Ψ_{tot}

$$\Psi_{tot}(r,\phi) = \sum_{j=1}^{q} \Psi(r,\phi + j\Delta\phi) \tag{30}$$

where in this case the discrete symmetry is indeed a rotational symmetry in ordinary space. After q bounces, the trajectory closes, while p has the meaning of "winding number", i.e. the number of times the trajectory performs a complete rotation around the centre of the billiard before closing.

Notice that the spatial width can be estimated as $\Delta s \approx R\Delta\phi \approx R/\Delta_l$. Here, we have implicitly assumed that the superposition is centred around $\phi = 0$. Shifting ϕ by a certain angle ϕ_0 would rotate the wave function by ϕ_0.

The energy spread ΔE can be calculated according to Eq. (8). One finds for ΔE and the corresponding lifetime τ of the quasimode

$$\frac{\Delta E}{E} \approx \left(\frac{\Delta_l}{k_{n_0 l_0} R}\right)^2 \quad ; \quad \frac{\tau}{T} \approx \left(\frac{\Delta s}{R}\right)^2 l_0 \tag{31}$$

One can see that the lifetime is asymptotically large in the semi-classical limit with respect to the period of the corresponding classical orbit at a fixed value of the spatial width.

Also in this system, the eigenfunctions are analytically known, and the eigenvalue equation (23) can be also obtained by considering their asymptotic expression for large quantum numbers. The eigenfunctions are given by the cylindrical Bessel function J_l

$$\Psi_{nl} = \exp(il\phi) J_l(k_{nl} r) \tag{32}$$

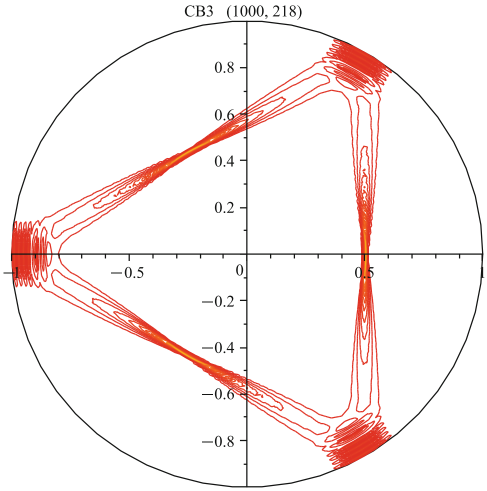


Fig. 5 Quasimode in the circular billiard corresponding to $p = 1$ and $q = 3$ and the central values (l_0, n_0) reported in the title

The exact eigenvalues are given by the zeros of the Bessel function at the boundary

$$J_l(k_{nl}R) \;=\; 0 \tag{33}$$

If one uses the asymptotic expression [13] for the Bessel function, one recovers the Bohr–Sommerfeld quantization of Eq. (23). The superposition of Eq. (10) can be done numerically with the exact eigenfunctions and eigenvalues, while of course the constraint on (n_0, l_0) implicitly implied by Eq. (2) can be only approximately satisfied, in principle with arbitrary precision in the large quantum number limit.

Let us consider some applications. If we take $p = 1$ and $q = 3$, we get the trajectory of triangular shape depicted in Fig. 5 for a specific case. A preliminary study of this case was already considered in [4]. In general, for $p = 1$ one gets, as it can be easily checked, the trajectory along a polygon of q sides, as in the case of Fig. 6, corresponding to $q = 7$. For large q, these quasimodes merge into the "whispering modes", studied in [5] for a generic billiard. For $p > 1$, a trajectory of "star" shape is generated. For $(p, q) = (2, 7)$ and $(p, q) = (3, 7)$, one gets the trajectories of Fig. 7 and of Fig. 8, respectively. For sake of illustration, we report in Table 1 the values of the quantum numbers used in the superposition and the corresponding energies in the case of Fig. 7.

For the star cases, it seems that a large concentration of the wave function is along a polygon on the points of the self-intersections of the classical trajectories. However, this is an artefact of the plotting system. In fact, close to the centre, the Bessel functions display an extremely oscillating behaviour that any plotting system is not able to follow, but instead it samples randomly the sharp and very high peaks of the wave function. This feature is not connected with the self-intersections that occur in the wave function, as it can be checked by looking at the square billiard, analysed in the previous subsection, where self-intersections occur but they do not display any similar behaviour. However, the interference that must occur between

Fig. 6 Quasimode in the circular billiard corresponding to $p = 1$ and $q = 7$, with the central values (l_0, n_0) reported in the title. The quasimode is localized around the classical trajectory that is the polygon of seven sites inscribed in the circle. See the text

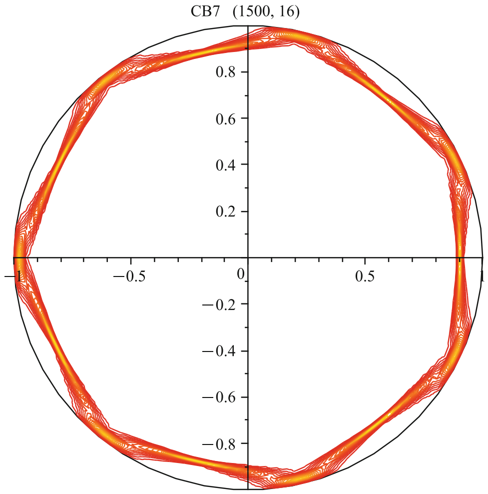
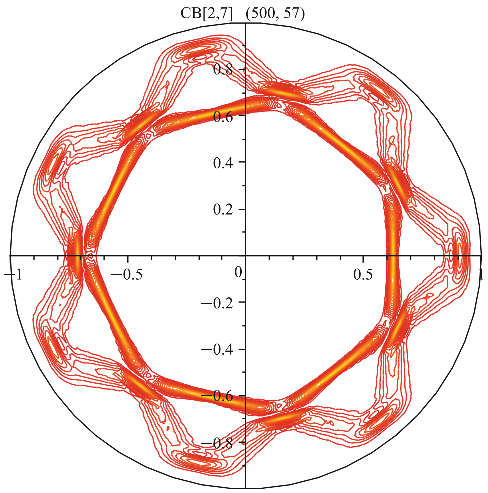


Fig. 7 Quasimode in the circular billiard corresponding to $p = 2$ and $q = 7$ and the central values (l_0, n_0) reported in the title. The quasimode is localized around the classical trajectory of "*star*" shape, in which the particle bounces seven times and performs two turns around the centre before closing



the two branches of the wave function that intersect seems to emphasize this effect. To illustrate the oscillations, we report in Fig. 9 the plot of the Bessel function corresponding to $l = 1000$ and $n = 69$. One can see the suppression of the wave function below the centrifugal barrier and the corresponding sharp rise at the barrier. In any case, it is important to realize that these wild oscillations will persist even in the extreme semi-classical limit (i.e. very large quantum numbers) and they are quantum phenomena that cannot be eliminated. This touches the well-known problem of the de-coherence that should occur in the classical limit, as suggested by several authors [29].

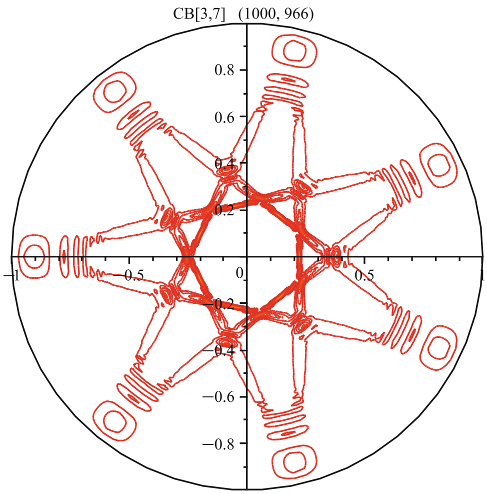


Fig. 8 Quasimode in the circular billiard corresponding to $p = 3$ and $q = 7$ and the central values (l_0, n_0) reported in the title. The quasimode is localized around the classical trajectory of "*star*" shape, in which the particle bounces seven times and performs three turns around the centre before closing

Table 1 Quantum numbers used for the quasimode of Fig. 7 and the corresponding energy (last column)

l	n	$E(n, l)$
430	77	797.5520632
437	75	798.3765400
444	73	799.1238211
451	71	799.7920827
458	69	800.3793629
465	67	800.8835484
472	65	801.3023598
479	63	801.6333358
486	61	801.8738136
493	59	802.0209079
500	**57**	802.0714854
507	55	802.0221372
514	53	801.8691447
521	51	801.6084410
528	49	801.2355660
535	47	800.7456120
542	45	800.1331610
549	43	799.3922075
556	41	798.5160678
563	39	797.4972677
570	37	796.3274064

In bold face are indicated the central quantum numbers (l_0, n_0) of the superposition (see text)

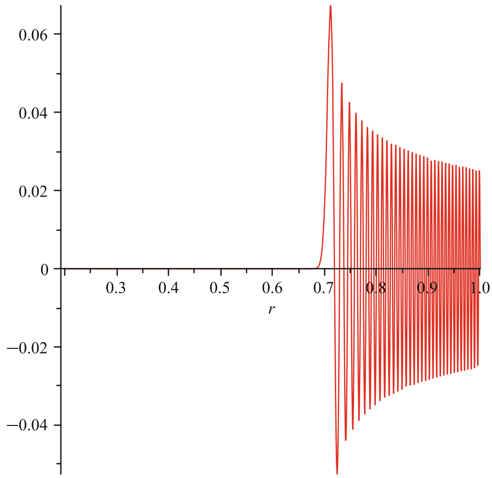


Fig. 9 Plot of the cylindrical Bessel function corresponding to the quantum numbers $(l_0, n_0) = (1000, 69)$. Notice the classically forbidden region and the rapid oscillation at the corresponding boundary

In all these examples, the lifetime τ of the quasimodes is in the range $\tau/T = 10$–15. It has to be noticed that the wave function is concentrated along a classical orbit only if the condition of quasidegeneracy of Eq. (1) is satisfied. In fact, localization can be easily obtained by a suitable superposition of eigenfunctions, but if that condition is not satisfied, the localization is not along a classical trajectory, even if the wave function has the same symmetry. An example is shown in Fig. 10, where the superposition is not taken along the direction fixed by Eq. (1), even if the considered eigenfunctions are chosen to be approximately degenerate, but not with the same degree of accuracy, and the steps are taken as for Fig. 7. In other words, the reference values n_0 and l_0 of the quantum numbers do not satisfy the condition implicit in Eq. (2). One can see that the localization is along some “curved star”, which is not of course a classical trajectory, but with the symmetry fixed by the values of p and q.

4 Hamiltonian Systems

We now consider two-dimensional systems described by an integrable smooth Hamiltonian. We will concentrate on the motion of a particle in a central potential. In this case, the natural choices of the quantum numbers are the angular momentum and the one associated with the radial motion. In general, the classical trajectories are not closed for a generic potential. For particular values of the constant of motion, the trajectory can close after a certain number of turns around the centre of the potential. According to the general scheme, quasimodes are associated with such trajectories, on which they are concentrated. A special case is represented by the circular orbits, which needs a separate treatments since the radial quantum number vanishes. There are few central potentials that admit an exact quantum solution. The

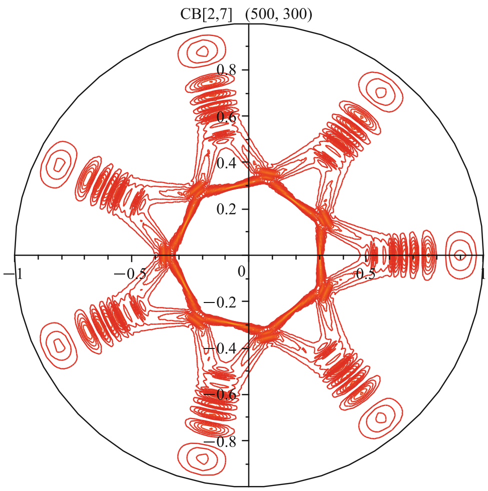


Fig. 10 Wave function localized around a non-classical path, which is not a quasimode. See the text for explanations

harmonic oscillator and the Kepler motion are the most known. However, these are very particular cases, since the trajectories are all closed and the spectrum displays very large exact degeneracy. We will consider briefly these cases that have been already considered in the literature in connection with quasimodes. We will then treat in an approximate way the general case and show the resemblance with the circular billiard.

4.1 The Harmonic Oscillator

It is well known that for the harmonic oscillator, the hypotheses of the Ehrenfest theorem are verified, i.e. a wave packet that moves inside the potential has its centre of mass moving indefinitely along a classical (closed) trajectory and its width does not spread but only oscillates with the frequency of the harmonic oscillator. In [10, 24], quasimodes were constructed using the coherent state representation, and it has been shown they concentrate indeed along a classical trajectory and furthermore the quasimode wave function has the largest strength in the positions where the classical motion is slower, e.g. at the point of maximum radial position ("periastron"), and actually, it is proportional to this time. As already noticed for the square billiard, the coherent state method is a particular case of the general method we have described. As we will show in the next subsection, in this case, the energy width of Eq. (8) vanishes, which is in line with the large degree of degeneracy in the spectrum. This shows the connection between the time-dependent treatment of Ehrenfest and the quasimodes.

We only illustrate in Fig. 11 a quasimode constructed with a uniform superposition of eigenfunctions, with a specific width, belonging to a given degenerate shell. The result is in line with [24].

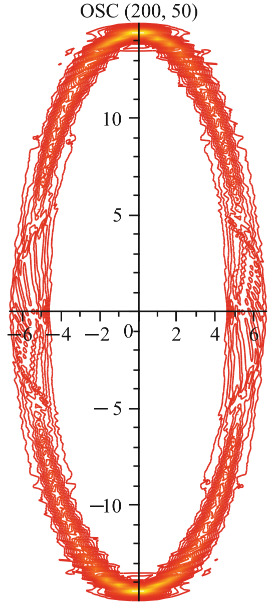


Fig. 11 Quasimode for the harmonic oscillator for the central values (l_0, n_0) reported in the title. The superposition is performed with exactly degenerate eigenfunctions, and therefore, it is actually an eigenstate of the harmonic oscillator

A similar treatment could be followed for the Kepler motion, for which no simple coherent states exist. It can be suggestive to guess a connection between quasimodes and the Rydberg states [19] in hydrogen-like atoms. The study of this case is left to future work.

4.2 *The General Potential Case*

Let us consider the bound motion in a generic central potential $V(r)$. For simplicity, we will assume that the potential is monotonically increasing with r. The action integral J_r for the radial motion can be written as

$$J_r(E,l) = 2\int_{r_m}^{r_M} dr\sqrt{2m(E - V(r)) - l^2/r^2} \tag{34}$$

where r_m and r_M are the smaller and the larger radial coordinates where the square root vanishes. They correspond to the turning points of the trajectory. The derivative of J_r turns out to be the angle $\Delta\theta$ spanned by the radial vector as the particle moves from r_m to r_M and back

$$\Delta\theta = -\frac{\partial}{\partial l}J_r(E,l) = 2\int_{r_m}^{r_M}\frac{dr}{r^2}\frac{l}{\sqrt{2m(E - V(r)) - l^2/r^2}} \tag{35}$$

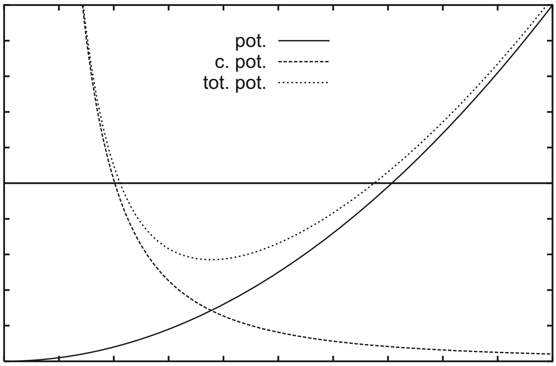


Fig. 12 Schematic representation of a generic central potential (pot) with a possible centrifugal potential (c.pot) and their sum (tot.pot). The *thick horizontal line* indicates a possible value of the total energy

For a generic potential, this angle depends both on the energy E and the angular momentum l. According to Eqs. (2) and (5), this angle must be a rational fraction of π, which is the condition on (n_0, l_0) and implies that the corresponding classical trajectory is closed. For the harmonic oscillator and the Kepler motion, this angle is however exactly π, independent of E and l. As a consequence, all the trajectories are closed (they close after two radial oscillations), and the second derivative of Eq. (7), which determines the energy spread, vanishes, as it was anticipated in the previous subsection.

In Fig. 12 is reported a schematic representation of the three terms appearing in the square root, the energy E, the potential $V(r)$ and the centrifugal potential l^2/r^2. Under our assumptions, the effective potential $U(r)$, the sum of $V(r)$ and the centrifugal potential, has a single minimum at a given radial distance r_0, and the energy must be larger than $U(r_0)$.

As shown in the appendix, after some manipulations of the integral, one finally gets

$$\Delta\theta = G(E, z_l) \qquad ; \qquad z_l = l/\sqrt{2mE} \tag{36}$$

where G is a smooth function, i.e. its derivatives with respect to the arguments are bounded. This term is identically π for the harmonic oscillator. The derivative of G with respect to l at a fixed energy E is therefore of the order at most of $1/\sqrt{E}$, which in the semi-classical limit becomes vanishing small. Furthermore, if the ratio $l/\sqrt{2mE}$ tends to a finite value, the derivative is equally of the order $1/l$. The result is valid for a generic potential, provided some reasonable conditions on the potential are fulfilled (see appendix). This finding is in line with the general statement about the lifetime of the quasimode with respect of the classical period, according to Eq. (9).

The motion in the potential can become close to the motion in a circular billiard if the potential resembles a container with a steep boundary. For the billiard, the ratio $l/\sqrt{2mE}$ is bound by the radius R of the billiard. The limiting values R and zero correspond to the condition on the angle $2\beta_0$ to approach asymptotically a value equal to zero (circular orbit) or to π (radial oscillation passing through the centre).

5 Conclusion

We have presented a general method to construct semi-classical quasimodes in integrable two-dimensional systems. These quantum states are localized around a given classical closed orbit, and at the same time, their lifetime τ can be made asymptotically large with respect to the classical period T of the orbit. At a fixed value of the localization width around the classical closed orbit, the ratio τ/T diverges for asymptotically large quantum numbers. This means that the system is considered first in the asymptotically large time limit, and then eventually in the limit usually indicated as the $\hbar \to 0$ limit. As it is well known, the two limits cannot be inverted in general, and therefore, the quasimodes have to be constructed with a definite procedure.

The method is general, and it can be applied both to billiards and to the Hamiltonian system. It includes as a particular case the one based on the generalized coherent states. Several examples have been illustrated for billiards, and, besides the harmonic oscillator, the treatment of a generic Hamiltonian system has been discussed. The construction of quasimodes suggests a method to perform the semi-classical limit in integrable systems, which however keeps some quantum features since these states are uniformly distributed along a classical periodic orbit. They can be considered as particular resonance states within the spectrum of the system. Their connection with the Ehrenfest semi-classical limit, based on (time-dependent) localized wave packets, has still to be clarified.

Appendix

In this appendix, we evaluate for a generic potential the dependence of the angle $\Delta\theta$ in Eq. (36) on the angular momentum l at a fixed energy E. In the expression of Eq. (35), it is convenient to introduce the new variable $y^2 = l^2/2mEr^2$. In the new variable, one gets

$$\Delta\theta = 2\int_{y_m}^{y_M} \frac{dy}{\sqrt{1 - [y^2 + V(r)/E]}} \tag{37}$$

where

$$r = r(y) = \frac{l}{\sqrt{2mE}\, y} \tag{38}$$

The limits of integration y_m and y_M are the values at which the square root vanishes. Under the assumption of a monotonically increasing potential at increasing r, there are only two values, corresponding to the extremes of the radial oscillations

$$\frac{1}{E}V(r(y_0)) + y_0^2 = 1 \tag{39}$$

where y_0 is either y_m or y_M. If the potential is smooth and the radial motion has a non-zero amplitude (i.e. the trajectory is not exactly circular), y_0 is a smooth function of l, which actually appears only in the combination $l/\sqrt{2mE}$. However, the integrand is singular at the integration limits, although the integral is of course converging. This does not allow to do any derivative with respect to l inside the integral to calculate the derivative of $\Delta\theta$. We have then to analyse the contribution to the integral from an interval close to the limits of integration. If the trajectory is not circular, at y_M, the function $R(y)$ inside the square root vanishes linearly, and the integrand can be written as

$$\frac{1}{\sqrt{1 - R(y)}} = \frac{1}{\sqrt{R'(y_M)(y_M - y)}} + S(y) \tag{40}$$

where R' is the derivative of R and the remainder $S(y)$ is a regular smooth function. The contribution to the integral from an interval $y_1 < y < y_M$, with y_1 some value close to y_M, can then be written as

$$\int_{y_1}^{y_M} \frac{dy}{\sqrt{R(y)}} = \frac{2}{\sqrt{R'(y_M)}}\sqrt{y_M - y_1} + \int_{y_1}^{y_M} dy\, S(y) \tag{41}$$

Explicitly, the derivative R' is

$$R'(y_M) = -\frac{1}{E} V'\left(\frac{l}{\sqrt{2mE}\, y_M}\right) \frac{l}{\sqrt{2mE}\, y_M^2} + 2y_M \tag{42}$$

where $V'(x) = dV/dx$. The same procedure can be followed for the lower limit y_m. At fixed E, the integral is therefore a smooth function of $l/\sqrt{2mE}$.

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Manifolds Which Are Complex and Symplectic But Not Kähler

Giovanni Bazzoni and Vicente Muñoz

Abstract The first example of a compact manifold admitting both complex and symplectic structures but not admitting a Kähler structure is the renowned Kodaira–Thurston manifold. We review its construction and show that this paradigm is very general and is not related to the fundamental group. More specifically, we prove that the simply connected *eight*-dimensional compact manifold of Fernández and Muñoz (Ann Math (2), 167(3):1045–1054, 2008) admits both symplectic and complex structures but does not carry Kähler metrics.

1 Introduction

A complex manifold M is a topological space modeled on open subsets of $\mathbb{C}^n$ and with change of charts being complex differentiable (i.e., biholomorphisms). Here we say that n is the complex dimension of M. Complex manifolds are the objects that appear naturally in algebraic geometry: a projective variety is the zero locus of a collection of polynomials in the complex projective space $\mathbb{C}P^N$. When a projective variety is smooth and of complex dimension n, it is a complex manifold of dimension n.

A complex manifold M of complex dimension n is in particular a smooth differentiable manifold of real dimension $2n$. Multiplication by $\mathbf{i}$ on each complex tangent space T_pM, $p \in M$, gives an endomorphism $J\colon TM \to TM$ such that $J^2 = -\mathrm{Id}$. An endomorphism $J\colon TM \to TM$ with $J^2 = -\mathrm{Id}$ is called an *almost complex* structure. For a complex manifold M, J satisfies that the Nijenhuis tensor

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

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vanishes, $N_J(X, Y) = 0$ for all vector fields X, Y. In this case, we say that the almost complex structure is *integrable*. The celebrated Newlander–Nirenberg theorem [30] says that an almost complex structure with $N_J = 0$ is equivalent to a complex structure. Hence, for a smooth manifold M to admit a complex structure, we need to check if there exist almost complex structures (this is a topological question) and then to find an integrable one (this is an analytic problem).

Projective varieties have further geometric properties. The complex projective space $\mathbb{C}P^N$ has a natural Hermitian metric, the Fubini–Study metric. This is the natural metric when one views $\mathbb{C}P^N$ as the homogeneous space $U(N+1)/U(1)\times U(N)$. Therefore, a projective variety $M \subset \mathbb{C}P^N$ inherits this Hermitian metric. Denote by h the Hermitian metric on M and write $h = g + \mathbf{i}\,\omega$, where $g(X, Y) = \mathrm{Re}(h(X, Y))$ and $\omega(X, Y) = \mathrm{Im}(h(X, Y)) = \mathrm{Re}(-\mathbf{i}h(X, Y)) = \mathrm{Re}(h(JX, Y)) = g(JX, Y)$. Then g is a Riemannian metric for which J is an isometry ($g(JX, JY) = g(X, Y)$) and ω turns out to be skew-symmetric; hence, it is a 2-form with $\omega(JX, JY) = \omega(X, Y)$ and $g(X, Y) = \omega(X, JY)$. We say that ω is the fundamental form of (M, h). This 2-form is positive, in the sense that $\omega^n > 0$ (it gives the natural complex orientation). The Fubini–Study metric h_{FS} has a fundamental form $\omega_{FS} \in \Omega^2(\mathbb{C}P^N)$. It is easy to see, using the $U(N + 1)$-invariance, that $d\omega_{FS} = 0$. Therefore, for $\omega = \omega_{FS}|_M$, it also holds $d\omega = 0$.

We say that (M, h) is a Kähler manifold when M is a complex manifold and the fundamental form ω satisfies $d\omega = 0$. A smooth projective variety is a Kähler manifold. Actually the converse holds when $[\omega] \in H^2(M, \mathbb{R})$ is an integral cohomology class, by Kodaira's theorem [39].

A different weakening of the Kähler condition (forgetting J but keeping ω) is that of a symplectic structure. A symplectic structure on a smooth $2n$-dimensional manifold M is given by a 2-form $\omega \in \Omega^2(M)$ which is closed ($d\omega = 0$) and nondegenerate (ω^n is nowhere zero). Let M be an even-dimensional manifold endowed with a complex structure J and a symplectic structure ω. Then J is said to be *compatible* with ω if, for vector fields X, Y on M, the bilinear form

$$g(X, Y) = \omega(X, JY) \tag{1}$$

is a Riemannian metric. Therefore, a Kähler manifold is a symplectic manifold endowed with a compatible complex structure, and $h = g + \mathbf{i}\,\omega$ is the Kähler metric. The existence of a Kähler metric on a compact manifold constraints the topology. In particular, if (M, J, ω) is a compact Kähler manifold of dimension $2n$, then (see [1, 12, 18, 39]):

1. The fundamental group $\pi_1(M)$ belongs to a very restricted class of groups, called *Kähler groups*.
2. $b_{2i-1}(M)$ is even for $i = 1, \ldots, n$.
3. The Lefschetz map $\mathcal{L}^{n-p}\colon H^p(M; \mathbb{R}) \to H^{2n-p}(M; \mathbb{R})$, $a \mapsto [\omega]^{n-p} \wedge a$, is an isomorphism.
4. M is formal in the sense of Sullivan (see Sect. 2 for details).

So it is natural to ask if the classes of smooth manifolds admitting complex, symplectic, and Kähler structures coincide under some topological constraints.

The lack of examples in symplectic geometry has been haunting this area of mathematics for many years now (pretty much since its *début* as a discipline in its own). Indeed, the main source of examples of symplectic manifolds is algebraic geometry. This led to the belief that symplectic and Kähler conditions coincided in the compact case (see, for instance, [21]). There was a discrete breakthrough, in 1976, when Thurston [38] gave the first example of a compact symplectic manifold with no Kähler structure. Thurston's example had already been discovered, as a complex manifold, by Kodaira during his work on the classification of compact complex surfaces [23]. We call it the *Kodaira–Thurston* manifold KT. Since KT is a compact complex and symplectic manifold without Kähler structure, we obtain:

Theorem 1. *There exist compact manifolds which admit complex and symplectic structures but carry no Kähler metrics.*

This means that the complex and symplectic structures that KT admits cannot be compatible. The manifold KT is in the place shown in Fig. 1.

The next natural question is whether some topological constraints may force the symplectic category to reduce to the Kähler one [32]. Regarding the fundamental group, it is natural to look for simply connected symplectic compact manifolds. In [28], McDuff constructed a compact, simply connected, symplectic manifold with $b_3 = 3$, hence not Kähler. For a detailed study on the relationship between formality and Lefschetz property on symplectic manifolds, we refer to [10]. In [9], Bock constructed nonformal symplectic manifolds with arbitrary Betti numbers.

The construction of simply connected symplectic nonformal (compact) manifolds turned out to be a more difficult problem. In fact, it was conjectured in 1994 (see [26]) that a compact simply connected symplectic (compact) manifold should be formal: this is the so-called Lupton–Oprea conjecture on the formalizing tendency of a symplectic structure. This conjecture was proven false by Babenko and Taĭmanov in 2000 (see [2]). For every $n \geq 5$, they constructed an example of a simply connected, symplectic nonformal compact manifold of real dimension $2n$. On the other hand, by a result of Miller [16, 29], simply connected compact manifolds of dimension ≤ 6 are formal. Hence, a remarkable gap in dimension 8 was left. This gap was filled by M. Fernández and the second author in 2008 (see [17]).

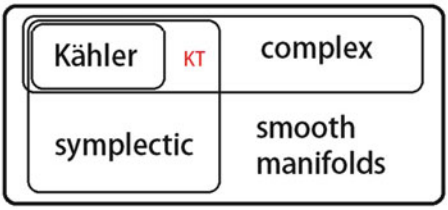

Fig. 1 Diagram of the different classes of manifolds, including KT

Here we shall prove that the manifold constructed in [17] admits a complex structure, thereby giving a new example fitting in the scheme of Theorem 1. The precise result is:

Theorem 2. *There exists an* 8*-dimensional, compact, simply connected, symplectic, and complex manifold which is nonformal and does not satisfy the Lefschetz property. In particular, it does not admit Kähler structures.*

This paper is organized as follows. In Sect. 2, we recall the basics of rational homotopy theory and formality. In Sect. 3, we give a description of KT, construct explicit complex and symplectic structures on it, and show that it carries no Kähler metric. In Sect. 4, we review the construction of the symplectic manifold $(\widetilde{M}, \tilde{\omega})$ of Fernández and Muñoz [17]. This is constructed by resolving symplectically the singularities of a symplectic orbifold $(\widehat{M}, \hat{\omega})$, a quotient of a compact symplectic nilmanifold (M, ω) by a certain $\mathbb{Z}_3$-action. In Sect. 5, we describe a complex structure $\widehat{J}$ on the orbifold $\widehat{M}$ and construct a complex resolution of singularities $(\overline{M}, \overline{J})$. Finally, in Sect. 6, we show that $\widetilde{M}$ and $\overline{M}$ are diffeomorphic.

2 Formality

Formality is a property of the rational homotopy type of a space X. We present here a rough introduction, referring to [14, 15, 19] for more details. By space, we mean a connected CW complex of finite type (we allow a finite number of cells in each dimension) which is nilpotent (its fundamental group is nilpotent and acts nilpotently on higher homotopy groups). A space X is rational if $\pi_i(X)$ is a rational vector space for every $i \geq 1$ (recall that a nilpotent group has a well-defined rationalization). The rationalization of a space X is a rational space $X_{\mathbb{Q}}$ together with a map $f\colon X \to X_{\mathbb{Q}}$ such that $f_i\colon \pi_i(X) \otimes \mathbb{Q} \to \pi_i(X_{\mathbb{Q}})$ is an isomorphism for every $i \geq 1$. We identify two spaces if they have a common rationalization. By rational homotopy type of a space X, we mean the homotopy type of its rationalization. Quillen and Sullivan proposed two different approaches to capture the rational homotopy type of a space in an algebraic model (see [33, 37]). Here we review briefly Sullivan's ideas.

A *commutative differential graded algebra* $(\mathcal{A}, d)$ over a field $\mathbf{k}$ of zero characteristic ($\mathbf{k}$-cdga for short) is a graded algebra $\mathcal{A} = \oplus_{i\geq 0} A^i$ which is graded commutative, together with a $\mathbf{k}$-linear map $d\colon A^i \to A^{i+1}$, the differential, which satisfies $d^2 = 0$ and which is a graded derivation, i.e., for homogeneous elements $a \in A^p$ and $b \in A^q$,

$$d(a \cdot b) = (da) \cdot b + (-1)^p a \cdot (db).$$

The cohomology of a $(\mathcal{A}, d)$, denoted $H^*(\mathcal{A})$, is a $\mathbf{k}$-cdga with trivial differential. A $\mathbf{k}$-cdga is *connected* if $H^0(\mathcal{A}) \cong \mathbf{k}$.

The de Rham algebra $\Omega(M)$ of a smooth manifold M, together with the exterior differential, is an $\mathbb{R}$-cdga. The piecewise linear forms $A_{PL}(X)$ on a PL manifold X, endowed with a suitable differential combining the exterior differential and the boundary of simplices, form a $\mathbb{Q}$-cdga (see [19]). There is a de Rham-type theorem for both cdgas; hence, we have isomorphisms

$$H^*(\Omega(M)) \cong H^*(M;\mathbb{R}) \quad \text{and} \quad H^*(A_{PL}(X)) \cong H^*(X;\mathbb{Q}).$$

Let X be a space. The idea of Sullivan is to replace $A_{PL}(X)$ by another $\mathbb{Q}$-cdga, which has the same cohomological information as $A_{PL}(X)$ but is algebraically more tractable: the *minimal model*. A $\mathbf{k}$-cdga $(\mathcal{A}, d)$ is *minimal* if

- $\mathcal{A}$ is the free graded algebra over a graded vector space $V = \oplus_i V^i$; this means that $\mathcal{A}$ is the tensor product of the exterior algebra on the odd-degree generators and the symmetric algebra on the even-degree generators, $\mathcal{A} = \mathrm{Ext}(V^{\mathrm{odd}}) \otimes \mathrm{Sym}(V^{\mathrm{even}})$. The standard notation is $\mathcal{A} = \Lambda V$.
- There exists a collection $\{x_i\}_{i\in\mathcal{I}}$ of generators of V, indexed by a well-ordered set $\mathcal{I}$, such that $|x_i| \leq |x_j|$ if $i < j$ and the differential of a generator x_j is an element of $\Lambda(V^{<j})$. Here, $|\cdot|$ denotes the degree and $V^{<j}$ consists of the generators x_i with $i < j$. Notice, in particular, that d does not have linear part.

We denote a minimal $\mathbf{k}$-cdga by $(\Lambda V, d)$. A *minimal model* for a $\mathbf{k}$-cdga $(\mathcal{A}, d)$ is a minimal $\mathbf{k}$-cdga $(\Lambda V, d)$ together with a $\mathbf{k}$-cdga morphism $\phi\colon (\Lambda V, d) \to (\mathcal{A}, d)$ which induces an isomorphism in cohomology (such a morphism is called *quasi-isomorphism*).

We have the following fundamental result:

Theorem 3 ([14], Theorem 14.12). *Any connected* $\mathbf{k}$*-cdga has a minimal model, which is unique up to isomorphism.*

By definition, the rational minimal model of a space X, $(\Lambda V_X, d)$, is the minimal model of the $\mathbb{Q}$-cdga $(A_{PL}(X), d)$. One can show that, when M is a smooth manifold, the real minimal model of M can be computed from the de Rham algebra $(\Omega(M), d)$. A central result in rational homotopy theory is the following:

Theorem 4 ([37]). *Two spaces have the same rational homotopy type if and only if their rational minimal models are isomorphic.*

In particular, PL forms (resp. smooth forms) contain all the rational homotopic (resp. real-homotopic) information of a space (smooth manifold). It is often difficult to know the whole de Rham algebra of a manifold; it would be very convenient if the (say, real) minimal model could be constructed directly from the de Rham cohomology. A space for which this happens is called *formal*. More precisely, a space X is formal if there exists a quasi-isomorphism $(\Lambda V_X, d) \to (H^*(X;\mathbb{Q}), 0)$. In particular, the rational homotopy type of a formal space X is a formal consequence of its rational cohomology. Many spaces are known to be formal: compact Lie groups, H-spaces, symmetric spaces, etc. For us, the relevant result is the following:

Theorem 5 ([12]). *A smooth compact manifold M admitting a Kähler structure is formal.*

A very useful criterion for establishing formality is the following:

Theorem 6 ([12], Theorem 4.1). *Let X be a space and let $(\Lambda V_X, d)$ be its minimal model. Then X is formal if and only if we can write $V_X = C \oplus N$ with $d = 0$ on C and d injective on N, in such way that every closed element in the ideal generated by N is exact.*

Let $(\mathcal{A}, d)$ be a **k**-cdga and let $H^*(\mathcal{A})$ be its cohomology. Let $a \in H^{|a|}(\mathcal{A})$, $b \in H^{|b|}(\mathcal{A})$, and $c \in H^{|c|}(\mathcal{A})$ such that $a \cdot b = b \cdot c = 0$. Then $a \cdot b \cdot c$ is zero for two reasons. Consequently, a difference element $\langle a, b, c\rangle \in H^{|a|+|b|+|c|-1}(\mathcal{A})/\mathcal{J}$ can be formed, where $\mathcal{J}$ is the ideal generated by a and c in $H^*(\mathcal{A})$. Take cocycles $\alpha, \beta, \gamma \in \mathcal{A}$ representing a, b, c, respectively. Then $\alpha \cdot \beta = d\xi$ and $\beta \cdot \gamma = d\eta$; hence, $\xi \cdot \gamma + (-1)^{|a|+1}\alpha \cdot \eta$ is a closed $(|a| + |b| + |c| - 1)$-form whose cohomology class is well-defined modulo $\mathcal{J}$. We set $\langle a, b, c\rangle = [\xi \cdot \gamma + (-1)^{|a|+1}\alpha \cdot \eta]$. Then $\langle a, b, c\rangle$ is called the *triple Massey product* of the cohomology classes a, b, c.

The definition of higher Massey products is as follows (see [24, 27]). The Massey product $\langle a_1, a_2, \dots, a_t\rangle$, $a_i \in H^{|a_i|}(\mathcal{A})$, $1 \le i \le t$, $t \ge 3$, is defined if there are $\alpha_{i,j} \in \mathcal{A}$, with $1 \le i \le j \le t$, except for the case $(i,j) = (1,t)$, such that

$$a_i = [\alpha_{i,i}], \qquad d\alpha_{i,j} = \sum_{k=i}^{j-1} (-1)^{|\alpha_{i,k}|}\alpha_{i,k} \wedge \alpha_{k+1,j}. \tag{2}$$

Then the Massey product is

$$\langle a_1, a_2, \dots, a_t\rangle = \left\{ \left[\sum_{k=1}^{t-1} (-1)^{|\alpha_{1,k}|}\alpha_{1,k} \wedge \alpha_{k+1,t} \right] \right\} \subset H^{|a_1|+\dots+|a_t|-(t-2)}(\mathcal{A}) \,, \tag{3}$$

where the $\alpha_{i,j}$ are as in (2). We say that the Massey product is trivial if $0 \in \langle a_1, a_2, \dots, a_t\rangle$. Note that for $\langle a_1, a_2, \dots, a_t\rangle$ to be defined, it is necessary that both $\langle a_1, \dots, a_{t-1}\rangle$ and $\langle a_2, \dots, a_t\rangle$ are defined and trivial.

Proposition 1. *If X is formal, then all (higher) Massey products of $(\Lambda V_X, d)$ are zero.*

Proof. The proof can be found in [3]. We shall give a simple proof for the case of triple and quadruple Massey products, which suffices for this paper.

As X is formal, Theorem 6 guarantees that we can write $V_X = C \oplus N$ with $d = 0$ on C and d injective on N, in such way that every closed element in the ideal $I(N)$ generated by N is exact. Note that there is a decomposition $\Lambda V = \Lambda C \oplus I(N)$. Let $a_i \in H^{|a_i|}(\mathcal{A})$, $1 \le i \le t$. By definition of Massey product, there are $\alpha_{i,i} \in \Lambda V$ with $a_i = [\alpha_{i,i}]$, and for each $i < j$, $(i,j) \ne (1,t)$, there are $\alpha_{i,j}$ with $d\alpha_{i,j} = \sum_{k=i}^{j-1} (-1)^{|\alpha_{i,k}|}\alpha_{i,k} \wedge \alpha_{k+1,j}$.

Write $\alpha_{i,j} = \beta_{i,j} + \eta_{i,j}$ with $\beta_{i,j} \in \Lambda C$, $\eta_{i,j} \in I(N)$. As $d\alpha_{i,j} = d\eta_{i,j}$, we can use in the case of triple Massey products (i.e., $t = 3$) the elements η_{12} and η_{23}. Then the triple Massey product $\langle a_1, a_2, a_3\rangle$ contains $(-1)^{|\eta_{12}|}\eta_{12}\alpha_{33} + (-1)^{|\alpha_{11}|}\alpha_{11}\eta_{23}$ which is in $I(N)$, hence exact.

In the case of quadruple Massey products (i.e., $t = 4$), we use $\eta_{12}, \eta_{23}, \eta_{34}$ instead of $\alpha_{12}, \alpha_{23}, \alpha_{34}$. The equation

$$\begin{aligned} d\alpha_{13} &= (-1)^{|\alpha_{12}|}\alpha_{12}\alpha_{33} + (-1)^{|\alpha_{11}|}\alpha_{11}\alpha_{23} \\ &= (-1)^{|\alpha_{12}|}\eta_{12}\alpha_{33} + (-1)^{|\alpha_{11}|}\alpha_{11}\eta_{23} + (-1)^{|\alpha_{12}|}\beta_{12}\alpha_{33} + (-1)^{|\alpha_{11}|}\alpha_{11}\beta_{23} \end{aligned}$$

implies that $(-1)^{|\alpha_{12}|}\eta_{12}\alpha_{33} + (-1)^{|\alpha_{11}|}\alpha_{11}\eta_{23}$ is closed, hence exact (as it lives in $I(N)$). Write it as $d\psi_{13}$ with $\psi_{13} \in I(N)$. Analogously define $\psi_{24} \in I(N)$. Thus, the quadruple Massey product $\langle a_1, a_2, a_3, a_4\rangle$ contains $(-1)^{|\psi_{13}|}\psi_{13}\alpha_{44} + (-1)^{|\eta_{12}|}\eta_{12}\eta_{34} + (-1)^{|\alpha_{11}|}\alpha_{11}\psi_{24}$ which is in $I(N)$, hence exact.

3 The Kodaira–Thurston Manifold

The Kodaira–Thurston manifold can be described in various ways. For Kodaira, *KT* was a compact quotient of $\mathbb{C}^2$ by a certain group acting co-compactly. Thurston interpreted it as a symplectic T^2-bundle over T^2. In this section, we describe it as a nilmanifold, write down explicit symplectic and complex structures on *KT*, and show that *KT* carries no Kähler metric.

A nilmanifold is a compact quotient of a simply connected, nilpotent Lie group G by a lattice Γ. Since Γ is a subgroup of a nilpotent group, it is also nilpotent. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism; hence, $G \cong \mathbb{R}^n$ for some n. Therefore, if $N = \Gamma\backslash G$ is a compact nilmanifold, $G \to N$ is the universal cover, $\pi_1(N) \cong \Gamma$ and $\pi_i(N) = 0$ for $i \geq 2$. Hence, a nilmanifold is a nilpotent space.

Nilmanifolds are interesting because they are a rich source of answers to many questions in different areas of mathematics. As we already mentioned, *KT* was the first example of a compact symplectic non-Kähler manifold. From the point of view of complex geometry, there exist complex nilmanifolds for which the Frölicher spectral sequence is arbitrarily nondegenerate (see [35]).

Kähler nilmanifolds are very special:

Theorem 7 (Benson–Gordon, Hasegawa [7, 22]). *Let N be a compact symplectic nilmanifold endowed with a Kähler structure. Then N is diffeomorphic to a torus.*

Benson and Gordon proved that a symplectic nilmanifold N for which the Lefschetz map $\mathcal{L}^{n-1} : H^1(N;\mathbb{R}) \to H^{2n-1}(N;\mathbb{R})$ is an isomorphism is diffeomorphic to a torus. Hasegawa showed that a formal nilmanifold N is diffeomorphic to a torus. Notice, however, that there exist many examples of non-toral symplectic and complex nilmanifolds (see [6, 20, 36]).

Let H denote the Heisenberg group, i.e.:

$$H = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

and let $H_{\mathbb{Z}}$ denote the subgroup of matrices with entries in $\mathbb{Z}$. Then H is a nilpotent Lie group, diffeomorphic to $\mathbb{R}^3$, $H_{\mathbb{Z}} \subset H$ is a lattice, and $N = H_{\mathbb{Z}} \backslash H$ is a compact nilmanifold. Let $G = H \times \mathbb{R}$ and $G_{\mathbb{Z}} = H_{\mathbb{Z}} \times \mathbb{Z}$. The Kodaira–Thurston manifold is $\mathrm{KT} = G_{\mathbb{Z}} \backslash G$.

Let $\mathfrak{k}$ be a Lie algebra over a field $\mathbf{k}$ of characteristic zero. The exterior algebra $\Lambda\mathfrak{k}^*$ is endowed with a differential $d \colon \Lambda^p\mathfrak{k}^* \to \Lambda^{p+1}\mathfrak{k}^*$, defined as follows: $d \colon \mathfrak{k}^* \to \Lambda^2\mathfrak{k}^*$ is the dualization of the bracket, i.e., $(d\alpha)(X, Y) = -\alpha([X, Y])$ if $\alpha \in \mathfrak{k}^*$ and $X, Y \in \mathfrak{k}$. d is then extended to $\Lambda\mathfrak{k}^*$ by imposing the graded Leibnitz rule: for $\alpha \in \Lambda^p\mathfrak{k}^*$ and $\beta \in \Lambda^q\mathfrak{k}^*$, $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta)$. The vanishing of d^2 is equivalent to the Jacobi identity in $\mathfrak{k}$. In the language of Sect. 2, $(\Lambda\mathfrak{k}^*, d)$ is a $\mathbf{k}$-cdga, known as Chevalley–Eilenberg complex of $\mathfrak{k}$.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}^*$ be its dual. We identify tensors on $\mathfrak{g}$ and $\mathfrak{g}^*$ with left-invariant objects on G, which therefore descend to KT. It is easy to check that $\mathfrak{g}$ has a basis $\langle X_1, X_2, X_3, X_4 \rangle$ in which the only nonzero bracket is $[X_1, X_2] = -X_3$. Let $\langle x_1, x_2, x_3, x_4 \rangle$ be the dual basis of $\mathfrak{g}^*$. The only nonzero differential on $\mathfrak{g}^*$ is computed to be $dx_3 = x_1 \wedge x_2$.

The element $\omega = x_1 \wedge x_4 + x_2 \wedge x_3 \in \Lambda^2\mathfrak{g}^*$ is closed and nondegenerate. By abuse of notation, we denote by ω the corresponding left-invariant symplectic structure on KT as well. Thus, (KT, ω) is a compact symplectic 4-manifold.

Recall that if $\mathfrak{k}$ is an even-dimensional Lie algebra, $J \colon \mathfrak{k} \to \mathfrak{k}$ is a complex structure if $J^2 = -\mathrm{Id}$ and it satisfies the integrability condition

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0, \text{ for } X, Y \in \mathfrak{k}. \tag{4}$$

In our situation, define $J \colon \mathfrak{g} \to \mathfrak{g}$ by

$$J(X_1) = X_2,\ J(X_2) = -X_1,\ J(X_3) = X_4 \text{ and } J(X_4) = -X_3.$$

A straightforward computation shows that (4) holds; hence, J is a complex structure on $\mathfrak{g}$. Again by abuse of notation, we denote by J the corresponding left-invariant complex structure on KT. Thus, (KT, J) is a compact complex surface.

Let $N = \Gamma \backslash G$ be a compact nilmanifold. Considering $\Lambda\mathfrak{g}^*$ as left-invariant forms on N, we have a natural inclusion $\iota \colon (\Lambda\mathfrak{g}^*, d) \to (\Omega(N), d)$. By a result of Nomizu (see [31]), ι is a quasi-isomorphism; hence, the de Rham cohomology of N is isomorphic to the cohomology of the Chevalley–Eilenberg complex of $\mathfrak{g}$. In our case, three of the four generators of $\mathfrak{g}^*$ are closed; hence, we get $b_1(\mathrm{KT}) = 3$.

Since KT has an odd Betti number which is odd, we see that it does not carry any Kähler metric. We also see explicitly that KT does not satisfy the Lefschetz property. Indeed, take $[x_2] \in H^1(\mathrm{KT}; \mathbb{R})$. Then $\mathcal{L} \colon H^1(\mathrm{KT}; \mathbb{R}) \to H^3(KT; \mathbb{R})$ sends $[x_2]$ to $-[x_1 \wedge x_2 \wedge x_4] = -[d(x_3 \wedge x_4)] = 0$.

The Lie algebra $\mathfrak{g}$ is endowed with a complex structure J and a symplectic structure ω. Define a tensor $g : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ by

$$g(X, Y) = \omega(X, JY), \; X, Y \in \mathfrak{g}.$$

It is easy to see that the matrix of g in the basis $\langle X_1, X_2, X_3, X_4\rangle$ is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

g is not a scalar product on $\mathfrak{g}$; hence, the corresponding left-invariant tensor on KT is not a Riemannian metric.

Let M be a manifold endowed with a complex structure J and a symplectic structure ω. One could in principle relax condition (1) above and ask J to be only *tamed* by ω, which means $\omega(X, JX) > 0$ for $X \in \mathfrak{X}(M)$. A symplectic manifold (M, ω) endowed with a tamed complex structure J is called *Hermitian symplectic*. There are no known examples of compact Hermitian symplectic non-Kähler manifolds.

We see that (KT, J, ω) is not Hermitian symplectic. Indeed, $\omega(X_1, JX_1) = 0$. It is proved in [13] that a compact nilmanifold endowed with a Hermitian-symplectic structure is actually Kähler. Hence, we see that KT does not carry *any* Hermitian-symplectic structure (not just left invariant).

To see explicitly that KT is nonformal, we need to compute the minimal model of a nilmanifold.

Theorem 8 ([22]). *Let $N = \Gamma\backslash G$ be a compact nilmanifold. Then $(\Lambda\mathfrak{g}^*, d)$ is the rational minimal model of N.*

Since a nilmanifold is a nilpotent space, Theorem 4 holds and the rational homotopy of a compact nilmanifold is codified in the corresponding minimal model. Here $(\Lambda\mathfrak{g}^*, d)$ is a minimal algebra generated in degree 1. By a result of Mal'cev (see [34, Theorem 2.12]), a simply connected nilpotent Lie group G admits a lattice if and only if $\mathfrak{g}$ admits a basis such that the structure constants are rational numbers. Hence, if $N = \Gamma\backslash G$ is a compact nilmanifold, $(\Lambda\mathfrak{g}^*, d)$ is automatically a $\mathbb{Q}$-cdga.

Applying Theorem 8, the minimal model of KT is

$$(\Lambda^*\langle x_1, x_2, x_3, x_4\rangle, dx_3 = x_1 \wedge x_2).$$

In the notation of Theorem 6, we have $C = \langle x_1, x_2, x_4\rangle$ and $N = \langle x_3\rangle$. The element $x_1 \wedge x_3$ belongs to the ideal generated by N and is closed, but not exact. A nonzero Massey product is constructed as follows. Take $a = b = [x_1]$ and $c = [x_2]$ in $H^1(\mathrm{KT}; \mathbb{Q})$. The recipe given after Theorem 6 tells us that the triple Massey product $\langle [x_1], [x_1], [x_2]\rangle = [x_1 \wedge x_3]$ is a well-defined element of $H^2(\mathrm{KT}; \mathbb{Q})$, which is nonzero modulo the ideal generated in $H^*(\mathrm{KT}; \mathbb{Q})$ by $[x_1]$ and $[x_2]$.

4 A Simply Connected Symplectic Nonformal 8-Manifold

In this section, we recall the construction of a simply connected, 8-dimensional symplectic nonformal manifold performed in [17]. Although quite involved, this construction also starts with a nilmanifold, showing the importance of such manifolds in the whole theory.

Let $H_{\mathbb{C}}$ be the complex Heisenberg group, defined as

$$H_{\mathbb{C}} = \left\{ A = \begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix} \quad \middle| \quad u_1, u_2, u_3 \in \mathbb{C} \right\}.$$

The map $H_{\mathbb{C}} \to \mathbb{C}^3$, $A \mapsto (u_1, u_2, u_3)$, gives a global system of holomorphic coordinates on $H_{\mathbb{C}}$. Set $G = H_{\mathbb{C}} \times \mathbb{C}$, with global coordinates (u_1, u_2, u_3, u_4). Let $\Gamma \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi \mathbf{i}/3}$. Also, let $G_\Gamma \subset G$ be the discrete subgroup of matrices with entries in Γ. We let G_Γ act on G on the left and set $M = G_\Gamma \backslash G$. Then M is a compact complex parallelizable nilmanifold. Notice that M can be seen as a principal torus bundle

$$T^2 = \Gamma \backslash \mathbb{C} \hookrightarrow M \to T^6 = (\Gamma \backslash \mathbb{C})^3$$

using the projection $(u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_4)$. M is a complex version of the Kodaira–Thurston manifold.

We interpret $\mathbb{Z}_3$ as the group of cubic roots of unity and consider the right $\mathbb{Z}_3$-action $\rho : \mathbb{Z}_3 \times G \to G$ given, in terms of a generator $\zeta = e^{2\pi \mathbf{i}/3}$, by

$$(\zeta, (u_1, u_2, u_3, u_4)) \mapsto (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4). \tag{5}$$

This action preserves the group operation on G and the lattice, hence descends to an action of $\mathbb{Z}_3$ on M. Set $\widehat{M} = M/\mathbb{Z}_3$. Then $\widehat{M}$ is not smooth; it has 81 isolated quotient singularities.

A basis for left-invariant 1-forms on G is given by

$$\mu = du_1, \quad \nu = du_2, \quad \theta = du_3 - u_2 du_1 \quad \textit{and} \quad \eta = du_4$$

(over the complex numbers), with

$$d\mu = d\nu = d\eta = 0, \quad d\theta = \mu \wedge \nu.$$

The action of $\mathbb{Z}_3$ on left-invariant 1-forms is given by

$$\rho^*\mu = \zeta\mu, \quad \rho^*\nu = \zeta\nu, \quad \rho^*\theta = \zeta^2\theta \quad \text{and} \quad \rho^*\eta = \zeta\eta.$$

The 2-form

$$\omega = \mathbf{i}\mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta} + \mathbf{i}\eta \wedge \bar{\eta} \tag{6}$$

on M satisfies $\bar{\omega} = \omega$, so it is real. It is closed and satisfies $\omega^4 \neq 0$. Thus, ω is a symplectic form. Notice also that

$$\rho^*\omega = \zeta^3(\mathbf{i}\mu \wedge \bar{\mu} + \nu \wedge \theta + \mathbf{i}\eta \wedge \bar{\eta}) + \zeta^{-3}\bar{\nu} \wedge \bar{\theta} = \omega.$$

Hence, ω is $\mathbb{Z}_3$−invariant and descends to a symplectic form $\hat{\omega}$ on the quotient $\widehat{M}$. Therefore, $(\widehat{M}, \hat{\omega})$ is a symplectic orbifold. In [17], a desingularization procedure for the symplectic orbifold is given, producing a symplectic manifold.

Proposition 2 ([17], Propositions 2.1 and 2.3). *There exists a smooth compact simply connected symplectic manifold $(\widetilde{M}, \tilde{\omega})$ which is isomorphic to $(\widehat{M}, \hat{\omega})$ outside a small neighborhood of the singular points.*

In [17], it is shown that $(\widetilde{M}, \tilde{\omega})$ is nonformal and also that it does not satisfy the Lefschetz property (see Remark 3.3 in [17]). The nonformality is seen in [17] via a modification of the Massey product, named a-Massey product, which is studied extensively in [11]. Here we shall see the nonformality of $\widehat{M}$ by showing that there exists a nonzero quadruple Massey product. Transferring the Massey product from $\widehat{M}$ to the desingularization $\widetilde{M}$ follows the arguments of [17, Theorem 3.2] and it is quite standard.

The complex minimal model of M is $\Lambda V_M = \Lambda(\mu, \nu, \theta, \eta, \bar{\mu}, \bar{\nu}, \bar{\theta}, \bar{\eta})$ with $d\theta = \mu \wedge \nu$ and $d\bar{\theta} = \bar{\mu} \wedge \bar{\nu}$. Our orbifold is $\widehat{M} = M/\mathbb{Z}_3$, where $\mathbb{Z}_3$ acts in the minimal model as $(\mu, \nu, \theta, \eta) \mapsto (\zeta\mu, \zeta\nu, \zeta^2\theta, \zeta\eta)$. A model (i.e., a $\mathbb{C}$-cdga quasi-isomorphic to its minimal model) for $\widehat{M}$ is given by $\mathcal{A} = (\Lambda V_M)^{\mathbb{Z}_3}$. Easily,

$$A^1 = 0,$$

$$A^2 = \big(\langle \mu, \nu, \eta\rangle \wedge \langle \bar{\mu}, \bar{\nu}, \bar{\eta}\rangle\big) \oplus \langle \mu \wedge \theta, \nu \wedge \theta, \eta \wedge \theta, \bar{\mu} \wedge \bar{\theta}, \bar{\nu} \wedge \bar{\theta}, \bar{\eta} \wedge \bar{\theta}, \theta \wedge \bar{\theta}\rangle,$$

$$A^3 = \Lambda^3(\mu, \nu, \eta, \bar{\theta}) \oplus \Lambda^3(\bar{\mu}, \bar{\nu}, \bar{\eta}, \theta).$$

With this, one can check that $H^3(\mathcal{A}) = 0$.

Take now $a_1 = [\nu \wedge \bar{\eta}]$, $a_2 = [\mu \wedge \bar{\mu}]$, $a_3 = [\mu \wedge \bar{\mu}]$, and $a_4 = [\eta \wedge \bar{\nu}]$. We shall compute $\langle a_1, a_2, a_3, a_4\rangle$ and check that it does not contain the zero element. A Massey product $b \in \langle a_1, a_2, a_3, a_4\rangle$ is computed according to formula (3). As $A^1 = 0$, it must be $\alpha_{11} = \nu \wedge \bar{\eta}$, $\alpha_{22} = \alpha_{33} = \mu \wedge \bar{\mu}$, and $\alpha_{44} = \eta \wedge \bar{\nu}$. Then

$$\alpha_{12} = -\theta \wedge \bar{\mu} \wedge \bar{\eta} + z_1,$$

$$\alpha_{13} = \nu \wedge \bar{\eta} \wedge f_2 - f_1 \wedge \mu \wedge \bar{\mu} + w_1,$$

$$\alpha_{23} = z_2,$$

$$\alpha_{24} = \mu \wedge \bar{\mu} \wedge f_3 - f_2 \wedge \eta \wedge \bar{\nu} + w_2,$$

$$\alpha_{34} = -\bar{\theta} \wedge \mu \wedge \eta + z_3,$$

where $z_1, z_2, z_3 \in A^3$ are closed, hence exact; thus, $z_i = df_i$, with $f_i \in A^2$, and $w_1, w_2 \in A^4$ are closed. A computation gives

$$b = [\alpha_{11} \wedge \alpha_{24} - \alpha_{12} \wedge \alpha_{34} + \alpha_{13} \wedge \alpha_{44}]$$

$$= [\theta \wedge \bar{\theta} \wedge \mu \wedge \bar{\mu} \wedge \eta \wedge \bar{\eta} + w_1 \wedge \eta \wedge \bar{\nu} + w_2 \wedge \nu \wedge \bar{\eta}].$$

To check that this is nonzero, we multiply by $[\nu \wedge \bar{\nu}]$. Then the terms with w_1 and w_2 cancel, so $b \wedge [\nu \wedge \bar{\nu}] \neq 0$; hence, $b \neq 0$.

5 The Complex Structure

In this section, we describe the complex structure J on G in two equivalent ways, and we show that it descends to $M = G_\Gamma \backslash G$ and also to the orbifold $\widehat{M} = (G_\Gamma \backslash G)/\mathbb{Z}_3$. Then we construct a complex resolution of singularities, which will give a smooth complex manifold of dimension *four* $(\overline{M}, \overline{J})$.

Let us consider the group $G = H_\mathbb{C} \times \mathbb{C}$ above. G can be realized as a complex Lie subgroup of $\mathrm{GL}(5, \mathbb{C})$ by sending the pair $(A, u_4) \in H_\mathbb{C} \times \mathbb{C}$ to the matrix

$$\begin{pmatrix} 1 & u_2 & u_3 & 0 & 0 \\ 0 & 1 & u_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & u_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathrm{GL}(5, \mathbb{C})$ is an open subset of $\mathbb{C}^{25}$; hence, each tangent space $T_X\mathrm{GL}(5, \mathbb{C}) \cong \mathbb{C}^{25}$, $X \in \mathrm{GL}(5, \mathbb{C})$, inherits the standard complex structure of the ambient space, which is the multiplication by $\mathbf{i} = \sqrt{-1}$. As a complex submanifold of $\mathrm{GL}(5, \mathbb{C})$, G inherits the same complex structure on each tangent space. This means that the complex structure on G is multiplication by $\mathbf{i}$ on each tangent space T_gG, $g \in G$. The left translations $L_g : G \to G$, $h \mapsto gh$ are holomorphic maps, since they are written as polynomials in local coordinates. This shows that G is a complex parallelizable Lie group: the differential of L_g is complex linear, and a parallelization is given by moving T_eG around. Let J denote the complex structure on G induced by the inclusion $G \hookrightarrow \mathrm{GL}(5, \mathbb{C})$. The above considerations show that J is left invariant.

Let us consider the tangent space T_eG, where $e \in G$ is the identity. There is an identification between the Lie algebra $\mathfrak{g}$ of G and the vector space of left-invariant holomorphic vector fields on G, endowed with the natural Lie bracket. The complex structure on $\mathfrak{g}$ is multiplication by $\mathbf{i}$, and $\mathfrak{g}$ is a complex Lie algebra of dimension 4, described as follows:

$$\mathfrak{g} = \{\langle Z_1, Z_2, Z_3, Z_4 \rangle \mid [Z_1, Z_2] = -Z_3\}.$$

By identifying $\mathfrak{g}$ with T_eG, one has $T_gG = d_eL_g(\mathfrak{g}),\ \forall g \in G$. This shows again that the complex structure J_g on T_gG is multiplication by $\mathbf{i}$, for every $g \in G$.

We go through the details of the construction of left-invariant complex structure on G. Let J_e denote the complex structure (i.e., multiplication by $\mathbf{i}$) on $\mathfrak{g}$ and let $g \in G$ be a point. Define the complex structure $J_g : T_gG \to T_gG$ as

$$J_g(X(g)) = d_eL_g(\mathbf{i}x),$$

where X is a left-invariant vector field on G and $x \in \mathfrak{g}$ is such that $d_eL_g(x) = X(g)$. This defines J as a smooth section of the bundle $\mathrm{End}(TG)$. Let us show that $J^2 = -\mathrm{Id}$. Indeed,

$$J_g^2(X(g)) = J_g(J_g(X(g))) = d_eL_g(\mathbf{i}(\mathbf{i}x)) = -d_eL_g(x) = -X(g).$$

Lemma 1. *The (almost) complex structure defined above is left invariant.*

Proof. We must prove that, for every $g \in G$, $(L_g)^*J = J$. So take $X(h) \in T_hG$. Then

$$J_h(X(h)) = d_eL_h(\mathbf{i}x),$$

where $x \in \mathfrak{g}$ is the unique vector satisfying $d_eL_h(x) = X(h)$. On the other hand, we have

$$\begin{aligned}((L_g)^*J)(X(h)) &= d_{gh}L_{g^{-1}} \circ (J_{gh}) \circ (d_hL_g(X(h))) \\ &= d_{gh}L_{g^{-1}} \circ d_eL_{gh}(\mathbf{i}x) = d_eL_h(\mathbf{i}x) \\ &= J_h(X(h)).\end{aligned}$$

Lemma 2. *The (almost) complex structure defined above is integrable.*

Proof. This is trivial. Since J is left invariant, it is enough to work in the Lie algebra $\mathfrak{g}$. But on $\mathfrak{g}$, the complex structure is multiplication by $\mathbf{i}$; hence, the Nijenhuis tensor

$$\begin{aligned}N_J(X,Y) &= [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \\ &= [X,Y] + \mathbf{i}[\mathbf{i}X,Y] + \mathbf{i}[X,\mathbf{i}Y] - [\mathbf{i}X,\mathbf{i}Y] = 0,\end{aligned}$$

for $X, Y \in \mathfrak{g}$.

Lemma 3. *The two complex structures on G coincide.*

Proof. It is enough to notice that the left translations are holomorphic maps; thus, their differential is complex linear. Let $g \in G$ be a point and X a left-invariant vector field on G, such that $X(g) = d_eL_g(x)$, $x \in \mathfrak{g}$. Then

$$\mathbf{i}X(g) = \mathbf{i}d_eL_g(x) = d_eL_g(\mathbf{i}x) = J_g(X(g)).$$

So far we know that the natural complex structure J on the Lie group $G = H_{\mathbb{C}} \times \mathbb{C}$ is left invariant and it is multiplication by $\mathbf{i}$ on each tangent space. As above, let $G_\Gamma \subset G$ be the subgroup of matrices whose elements belong to the lattice $\Gamma = \{a + b\zeta \mid \zeta = e^{2\pi\mathbf{i}/3}, a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Since J is left invariant, it defines a complex structure on the quotient $M = G_\Gamma \backslash G$, which we denote again by J. Hence, (M, J) is a complex nilmanifold.

Next we show that J is compatible with the $\mathbb{Z}_3$-action defined by (5). The complex structure J on M is multiplication by $\mathbf{i}$ at each tangent space T_pM, $p \in M$, since it comes from the complex structure on G. Let $\varphi : M \to M$ denote the $\mathbb{Z}_3$-action, and consider the map

$$d_p\varphi \colon T_pM \to T_{\varphi(p)}M.$$

We claim that the map φ can be lifted to a holomorphic action $\tilde{\varphi}$ of $\mathbb{Z}_3$ on G. By taking global coordinates (u_1, u_2, u_3, u_4) on G, $\tilde{\varphi}$ sends the generator $\zeta \in \mathbb{Z}_3$ to the diagonal matrix $\mathrm{diag}(\zeta, \zeta, \zeta^2, \zeta)$. Since $\tilde{\varphi}$ is linear, it coincides with its differential $d_g\tilde{\varphi} : T_gG \to T_{\tilde{\varphi}(g)}G$. This is clearly a complex linear map, i.e.:

$$d_g\tilde{\varphi} \circ J_g = J_{\tilde{\varphi}(g)} \circ d_g\tilde{\varphi}. \tag{7}$$

This proves the claim. Since the complex structure J on M is multiplication by $\mathbf{i}$ on each tangent space, (7) shows that we can write

$$d_p\varphi \circ J_p = J_{\varphi(p)} \circ d_p\varphi,$$

showing that the complex structure commutes with the $\mathbb{Z}_3$-action, hence descends to the quotient $\widehat{M} = M/\mathbb{Z}_3$. We denote by $\widehat{J}$ the complex structure on $\widehat{M}$. Thus we have proved:

Proposition 3. *Let $M = G_\Gamma \backslash G$ be as above and denote by J the natural complex structure on M. Then $(\widehat{M}, \widehat{J})$ is a complex orbifold.*

Remark 1. The complex nilmanifold M is an example of an 8-dimensional non-simply connected complex, symplectic, and non-Kähler manifold, the symplectic form being given by (6). Indeed, M is nonformal; hence, it cannot be Kähler. One can show that $(\widehat{M}, \widehat{J}, \hat{\omega})$ is simply connected. Therefore, we have an example of an eight-dimensional simply connected complex and symplectic orbifold which is not Kähler. Indeed, one can see that $\widehat{M}$ is not formal [17], while Kähler orbifolds are formal [8].

Proposition 4. *There exists a smooth complex manifold $(\overline{M}, \overline{J})$ which is biholomorphic to $(\widehat{M}, \widehat{J})$ outside a neighborhood of a singular point.*

Proof. Let $p \in M$ be a fixed point of the $\mathbb{Z}_3$-action. Translating with an element $g \in G$, we can suppose that $p = (0, 0, 0, 0)$ in our coordinates. Let $U \subset M$ be a neighborhood of p and let $\phi : U \to B$ be a holomorphic local chart, given by the exponential map (by holomorphic we mean with respect to the complex structure J). Here $B = B_{\mathbb{C}^4}(0, \varepsilon) \subset \mathbb{C}^4$. In these coordinates, the action of $\mathbb{Z}_3$ can be written as

$$(u_1, u_2, u_3, u_4) \mapsto (\zeta u_1, \zeta u_2, \zeta^2 u_3, \zeta u_4).$$

The local model for the singularity is thus $B/\mathbb{Z}_3$. From now on, the desingularization process is formally analogous to that in [17]. We blow up B at p to obtain $\widetilde{B}$. The point p is replaced with a complex projective space $F = \mathbb{P}^3 = \mathbb{P}(T_pB)$ on which $\mathbb{Z}_3$ acts by

$$[u_1 : u_2 : u_3 : u_4] \mapsto [\zeta u_1 : \zeta u_2 : \zeta^2 u_3 : \zeta u_4] = [u_1 : u_2 : \zeta u_3 : u_4].$$

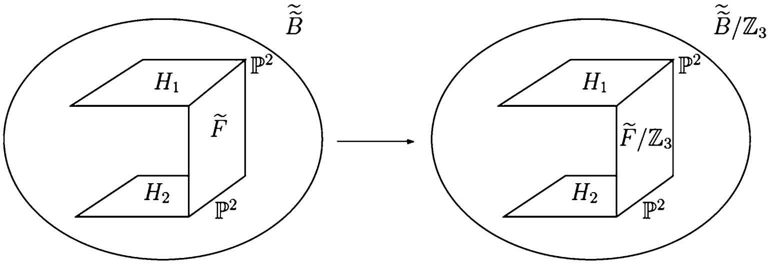

Fig. 2 The second blowup and the $\mathbb{Z}_3$-action

Thus, $\mathbb{Z}_3$ acts on the exceptional divisor F with fixed locus $\{q\} \cup H$ where $q = [0 : 0 : 1 : 0]$ and $H = \{u_3 = 0\}$. Then one blows up $\widetilde{B}$ at q and H to obtain $\widetilde{\widetilde{B}}$. The point q is replaced by a projective space $H_1 \cong \mathbb{P}^3$. The normal bundle to $H \subset F \subset \widetilde{B}$ is the sum of the normal bundle of H in $\mathbb{P}^3$, which is $\mathcal{O}_{\mathbb{P}^2}(1)$, and the restriction to H of the normal bundle of F in $\widetilde{B}$, which is $\mathcal{O}_{\mathbb{P}^2}(-1)$. Hence, the second blowup replaces the projective plane H with a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ defined as $H_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1))$. The strict transform of $F \subset \widetilde{B}$ under the second blowup is the blowup $\widetilde{F}$ of F at q, which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2 = H$, actually $\widetilde{F} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. The resulting situation is depicted in Fig. 2.

The fixed point locus of the $\mathbb{Z}_3$-action on $\widetilde{\widetilde{B}}$ consists of the two disjoint divisors H_1 and H_2. According to [4, p. 82], the quotient $\widetilde{\widetilde{B}}/\mathbb{Z}_3$ is a smooth Kähler manifold. This provides a complex resolution of the singularity $B/\mathbb{Z}_3$. Notice that the blowing up is performed with respect to the natural complex structure inherited from the ambient space. By resolving every singular point, we obtain a smooth complex manifold $(\overline{M}, \overline{J})$.

Proposition 5. *The complex manifold* $(\overline{M}, \overline{J})$ *is simply connected.*

Proof. The proof is analogous to that of [17, Proposition 2.3].

The desingularization process of Proposition 4 is completely similar to the symplectic resolution of [17, Proposition 2.1]. However, the two blowups are performed with respect to *different* complex structures. In the complex resolution, one uses the natural complex structure $\widehat{J}$ of $\widehat{M}$. In the symplectic resolution, one uses a (local) complex structure obtained by using a Kähler model for a neighborhood of a fixed point which is not holomorphically equivalent to a local holomorphic chart for $\widehat{J}$. Indeed, this Kähler model is obtained by performing the following change of coordinates in a small neighborhood of a fixed point of the action:

$$\begin{cases} w_1 = u_1 \\ w_2 = \frac{1}{\sqrt{2}}(u_2 + \mathbf{i}\bar{u}_3) \\ w_3 = \frac{1}{\sqrt{2}}(\mathbf{i}\bar{u}_2 - u_3) \\ w_4 = u_4 \end{cases} \tag{8}$$

Certainly, this is not holomorphic with respect to the natural complex structure $\widehat{J}$ on $\widehat{M}$.

Locally, we have the following situation: on a small neighborhood $\mathcal{U}$ of $0 \in \mathbb{C}^4$ (which is a fixed point of the $\mathbb{Z}_3$-action in suitable coordinates), we have two complex structures, J_1 and J_2. The two complex structures are different, because the change of variables which brings one to the other is not holomorphic. As a consequence, the two blowups are different. In fact, the natural map that one would construct from one resolution to the other would not be even continuous. This becomes particularly clear when the blowup is interpreted as a symplectic cut, following Lerman and McDuff (see, for instance, [25]). The blowup of $\mathbb{C}^n$ at 0 can be thought of as removing a small ball of radius ε centered at the origin and then collapsing the fibers of the Hopf fibration in the boundary of the remaining set. But the fibers of the Hopf fibration (i.e., the intersections of the boundary of the ball, which is a S^{2n-1}, with the "complex" lines through the origin) depend heavily on the complex structure of the ball.

6 Proof of the Main Theorem

In this section, we prove that the smooth manifolds which underlie the two resolutions $\overline{M}$ and $\widetilde{M}$ are diffeomorphic. This completes the proof of Theorem 2.

Proposition 6. *The symplectic and the complex resolution of the orbifold* $(\widehat{M}, \widehat{J}, \hat{\omega})$ *are diffeomorphic.*

Proof. We work locally, in a small neighborhood of each fixed point. We construct a smooth map which is the identity outside this small neighborhood and that does the right job inside the neighborhood. The local model is thus a small ball $B_{\mathbb{C}^4}(0,\delta) \subset \mathbb{C}^4$ endowed with two different complex structures J_1 and J_2. There is a map $\Theta : B_{\mathbb{C}^4}(0,\delta) \to B_{\mathbb{C}^4}(0,\delta)$ which interchanges the two complex structures, namely,

$$\Theta^* J_1 = J_2.$$

Notice that Θ can be composed with biholomorphisms on the right and on the left, thus is not unique. If we take J_1 as the complex structure on the ball induced by the natural complex structure on $\widehat{M}$ and J_2 to be the complex structure associated to the local Kähler model used for the symplectic resolution, then Θ is given by (8). We introduce real coordinates $u_k = x_k + iy_k$ and $w_k = s_k + it_k$, $k = 1, 2, 3, 4$. In such coordinates, (8) is an automorphism of $\mathbb{R}^8$ written as

$$\begin{cases} s_1 = x_1 \\ t_1 = y_1 \\ s_2 = \frac{1}{\sqrt{2}}(x_2 + y_3) \\ t_2 = \frac{1}{\sqrt{2}}(y_2 + x_3) \\ s_3 = \frac{1}{\sqrt{2}}(y_2 - x_3) \\ t_3 = \frac{1}{\sqrt{2}}(x_2 - y_3) \\ s_4 = x_4 \\ t_4 = y_4 \end{cases}$$

The associated matrix is

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix Θ belongs to SO(8). To construct the diffeomorphism, we will find an isotopy $\{\Theta_t\}_{t\in[0,1]}$, such that Θ_0 is the identity $\mathrm{Id} \in \mathrm{SO}(8)$ and $\Theta_1 = \Theta$. In this way we get a path of complex structures $J_{t+1} = \Theta_t^* J_1$ connecting J_1 and J_2. To do this we must produce a smooth path in SO(8) between the identity matrix and Θ, which is furthermore equivariant with respect to the $\mathbb{Z}_3$-action. In fact it is enough to find a smooth $\mathbb{Z}_3$-equivariant path in SO(4) connecting the identity to the matrix

$$\theta = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

In the coordinates (s_2, t_2, s_3, t_3) spanning the $\mathbb{R}^4$ of interest, the $\mathbb{Z}_3$-action can be written as

$$\Upsilon = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

under the natural inclusion $\mathrm{U}(2) \hookrightarrow \mathrm{SO}(4)$. We must ensure that the path $\{\Theta_s\} \subset \mathrm{SO}(4)$ satisfies $\Theta_s \circ \Upsilon = \Upsilon \circ \Theta_s$, for every $s \in [0, 1]$. We do this explicitly. First, notice that $\theta = P\theta'$, where

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \theta' = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The matrix θ' is the image, under the exponential map $\exp : \mathfrak{so}(4) \to \mathrm{SO}(4)$, of the matrix $\frac{\pi}{4}Q$, where

$$Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, a smooth path in SO(4) between the identity and θ' is given by the image of the straight line in $\mathfrak{so}(4)$ joining the zero matrix with Q,

$$\begin{aligned} \gamma : [0, \pi/4] &\to \mathrm{SO}(4) \\ s &\mapsto \exp(sQ) \end{aligned}$$

One sees that, for every $s \in [0, \pi/4]$, $\gamma(s) \circ \Upsilon = \Upsilon \circ \gamma(s)$; hence, $\gamma(s)$ is $\mathbb{Z}_3$-equivariant. Now consider the matrix P. We juxtapose the following three paths in order to join P with the identity matrix:

$$P_1(s) = \begin{pmatrix} 0 & 0 & \sin(\pi s/2) & \cos(\pi s/2) \\ 0 & 0 & \cos(\pi s/2) & -\sin(\pi s/2) \\ \sin(\pi s/2) & \cos(\pi s/2) & 0 & 0 \\ \cos(\pi s/2) & -\sin(\pi s/2) & 0 & 0 \end{pmatrix},$$

$$P_2(s) = \begin{pmatrix} \sin(\pi s/2) & 0 & \cos(\pi s/2) & 0 \\ 0 & \sin(\pi s/2) & 0 & -\cos(\pi s/2) \\ \cos(\pi s/2) & 0 & -\sin(\pi s/2) & 0 \\ 0 & -\cos(\pi s/2) & 0 & -\sin(\pi s/2) \end{pmatrix},$$

$$P_3(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\cos(\pi s) & \sin(\pi s) \\ 0 & 0 & -\sin(\pi s) & -\cos(\pi s) \end{pmatrix}.$$

Again, a computation shows that $P_i(s) \circ \Upsilon = \Upsilon \circ P_i(s)$, $\forall s \in [0, 1]$, $i = 1, 2, 3$. Hence, the path $P(s) = P_1 * P_2 * P_3(s)$ satisfies $P(0) = P$, $P(1) = \mathrm{Id}$ and is $\mathbb{Z}_3$-equivariant. The path $\theta(s) = P(1 - s)\theta'$ satisfies $\theta(0) = \theta'$ and $\theta(1) = \theta$. Finally, the path $\Psi = \gamma * \theta$ satisfies $\Psi(0) = \mathrm{Id}$ and $\Psi(1) = \theta$. However, Ψ is not globally smooth, because at the concatenation points, it is only continuous. To smooth it, we proceed as follows. Let $0 < s_1 < \cdots < s_{n-1} < s_n < 1$ denote the points in which the resulting path has a cusp. Consider a smooth, increasing function $h : [0, 1] \to [0, 1]$ such that there exist intervals $\mathfrak{J}_i = (t_i - \varepsilon, t_i + \varepsilon)$, $0 < t_1 < \cdots < t_{n-1} < t_n < 1$ with $h(t) = s_i$ for $t \in \mathfrak{J}_i$. Define a new path $\Theta_t = \Psi_{h(t)}$. Clearly, Ψ and Θ have the same image. Then Θ_t is a smooth, $\mathbb{Z}_3$-equivariant path in SO(4) connecting θ with the identity matrix. Viewing it as a path in SO(8), we obtain the isotopy Θ_t such that $\Theta_0 = \mathrm{Id}$ and $\Theta_1 = \Theta$. Thus, $\Theta_0^* J_1 = J_1$ and $\Theta_1^* J_1 = J_2$. We also endow the ball with the standard metric. Since $\mathbb{Z}_3 \subset \mathrm{SO}(8)$, $\mathbb{Z}_3$ acts by isometries.

We are ready to define the diffeomorphism between the two resolutions. Notice that the expression of the $\mathbb{Z}_3$-action is the same in the two sets of coordinates $(u_1, \ldots, u_4)$ and $(w_1, \ldots, w_4)$. Thus, when we blow up, we get, in both cases, an exceptional divisor $\mathbb{P}^3$ with one fixed point $q = [0 : 0 : 1 : 0]$ and one fixed hyperplane $H = \{u_3 = 0\} = \{w_3 = 0\}$. The differential of Θ at $0 \in B_{\mathbb{C}^4}(0, \delta)$, which we denote $d_0\Theta$, defines an automorphism of the exceptional divisor (when we projectivize the action), which fixes q and maps H to itself ($d_0\Theta$ is (J_1, J_2) holomorphic, meaning that $d_0\Theta \circ J_1 = J_2 \circ d_0\Theta$). Thus, $d_0\Theta$ also lifts to the second blowup, hence to a map between the two exceptional divisors. Let $\rho : \mathbb{R} \to [0, 1]$ be a cutoff function, i.e., a C^∞ function, which is identically 0 on $(-\infty, 0]$ and identically 1 on $[1, \infty)$. Using the metric on the ball, the diffeomorphism f can then be defined as follows:

$$f(x) = \begin{cases} x & \text{if } |x| > \frac{2\delta}{3} \\ \Theta_t(x) & \text{if } \frac{\delta}{3} < |x| < \frac{2\delta}{3} \\ \Theta(x) & \text{if } |x| < \frac{\delta}{3} \end{cases}$$

where $t = \rho\big(\big(\frac{2\delta}{3} - |x|\big)\frac{3}{\delta}\big)$. By what we have said, $f : \widehat{M} \to \widehat{M}$ lifts to a diffeomorphism $\tilde{f} : \overline{M} \to \widetilde{M}$.

Corollary 1. *The manifold $\widetilde{M}$ is a simply connected, eight-dimensional, nonformal manifold that admits both complex and symplectic structures, but which carries no Kähler metric.*

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Solvability of a Nonclamped Frictional Contact Problem with Adhesion

O. Chau, D. Goeleven, and R. Oujja

Abstract In this paper, we study a class of dynamic thermal problems involving a frictional normal compliance adhesive contact model and a nonclamped condition for viscoelastic materials. The variational formulation of the problem leads to a general system defined by a second-order quasi-variational evolution inequality on the displacement field coupled with two firstorder evolution equations describing the adhesion and temperature fields. Our main result establishes an existence and uniqueness result of these fields.

1 Introduction

Mathematical problems involving contact between deformable bodies play an important role in the engineering sciences. A considerable engineering and mathematical literature has now been devoted to the study of dynamic and quasistatic frictional contact problems. Many results concerning mathematical modeling, mathematical analysis, numerical analysis, and numerical simulations have been published.

An early attempt at the study of contact problems for elastic viscoelastic materials within the mathematical analysis framework was introduced in the pioneering reference works [2, 3, 8, 9, 12]. Infinitesimal frictional models on contact problems with nonlinear viscoelastic or elastoplastic materials were widely studied in [7, 10, 14]. These models are good approximations in the framework of linearized deformations, with some limitations on the impenetrability of mass condition or on the appropriated conservation laws of thermodynamics. The main purpose in these works is to show the cross-fertilization between various frictional new and nonstandard models arising in contact mechanics and numerous types of abstract variational inequalities.

Further extensions to nonconvex contact conditions with non-monotone and possible multivalued constitutive laws led to the active domain of non-smooth

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mechanics within the framework of the so-called hemivariational inequalities, for a mathematical as well as mechanical treatment, we refer to [6] and [13].

This paper is a continuation work of the results obtained in [1]. In this last paper the authors studied a problem for the dynamic contact between a body and an obstacle. The constitution law was assumed viscoelastic of short memory. The contact was considered as clamped on some part of the boundary and assumed to be frictionless. More precisely, it was defined by a normal compliance condition with adhesion. An existence and uniqueness result on displacement and adhesion fields has been established. Moreover, some numerical approximations and simulations have been presented.

Here we investigate a class of dynamic contact problems with adhesive normal compliance condition, friction, and thermal effects, and this for viscoelastic materials of long memory. Moreover, the usual clamped condition has been deleted, which leads to a new nonstandard model resulting in a system defined by a second-order quasi-variational inequality on the displacement field coupled with two differential equations describing the evolution of the adhesion and temperature fields. The main difficulties are that Korn's inequality cannot be applied any more. Here, nonlinearity due to the friction appears strongly. Semi-coercive problems were first studied in [3] for Coulomb's friction models, where the inertial term of the dynamic process has been used in order to balance the loss of coerciveness in the a priori estimates. Then, adopting fixed point methods as used in [7, 14], we prove the existence and uniqueness of displacement, adhesion, and temperature fields.

The paper is organized as follows. In Sect. 2 we describe the mechanical problem and specify the assumptions on the data so as to derive the variational formulation. Then, we state our main existence and uniqueness result. In Sect. 3, we give the proof of the claimed result.

2 The Contact Problem

In this section we study a class of thermal contact problems with nonclamped frictional normal compliance condition, for viscoelastic materials. We describe the mechanical problems, list the assumptions on the data, and derive the corresponding variational formulations. Then, we state an existence and uniqueness result on displacement and temperature fields, which we will prove in the next section.

Let us recall now some classical notations, see, e.g., [3, 11] for further details. We denote by S_d the space of second-order symmetric tensors on $\mathbf{R}^d$, while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d and $\mathbf{R}^d$. Let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel, the indices i and j run from 1 to d, summation over repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable. We also use the following notation:

$$H = \left(L^2(\Omega)\right)^d, \qquad \mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\ 1 \le i,j \le d\},$$

$$H_1 = \{\boldsymbol{u} \in H \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\}, \qquad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \mathrm{Div}\, \boldsymbol{\sigma} \in H\}.$$

Here $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathscr{H}$ and Div $: \mathscr{H}_1 \longrightarrow H$ are the deformation and the divergence operators, respectively, defined by :

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H, $\mathscr{H}$, H_1, and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by:

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx,$$

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathscr{H}}, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathscr{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathscr{H}} + (\text{Div } \boldsymbol{\sigma}, \text{Div } \boldsymbol{\tau})_H.$$

We recall that C denotes the class of continuous functions; C^m, $m \in \mathbf{N}^*$ the set of m times continuously differentiable functions and $W^{m,p}$, $m \in \mathbf{N}$, $1 \leq p \leq +\infty$ for the classical Sobolev spaces.

The physical setting is as follows. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbf{R}^d$ ($d = 1, 2, 3$) with a Lipschitz boundary Γ that is partitioned into two disjoint measurable parts, Γ_F and Γ_c. Let $[0, T]$ be the time interval of interest, where $T > 0$. We assume that a volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0, T)$ and that surface tractions of density $\boldsymbol{f}_F$ apply on $\Gamma_F \times (0, T)$. The body may come in contact with an obstacle, the foundation, over the potential contact surface Γ_c. A gap g exists between the potential contact surface Γ_c and the foundation and is measured along the outward normal vector $\boldsymbol{\nu}$.

We denote by $\boldsymbol{u}(\boldsymbol{x}, t)$ the displacement field, $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ the stress field, and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the small strain tensor. To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$; dots above a quantity represent derivative of the quantity with respect to the time variable, i.e.,

$$\boldsymbol{u}(t) = \boldsymbol{u}(\cdot, t), \quad \dot{\boldsymbol{u}}(t) = \frac{\partial \boldsymbol{u}}{\partial t}(\cdot, t), \quad \ddot{\boldsymbol{u}}(t) = \frac{\partial^2 \boldsymbol{u}}{\partial t^2}(\cdot, t).$$

We assume that the material is viscoelastic and its deformation follows a Kelving–Voigt long-memory thermo-viscoelastic constitutive law of the form

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{B}(t - s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds - \theta(t)\,C_e.$$

Here $\mathscr{A}$ and $\mathscr{G}$ are given nonlinear constitutive functions representing the viscosity operator and the elastic operator, respectively. The term $C_e := (c_{ij})$ represents the thermal expansion tensor, and $\mathscr{B}$ is the so-called tensor of relaxation which defines the long memory of the material, as an important particular case, when $\mathscr{B} \equiv 0$, we find again the usual viscoelasticity of short memory.

Next we describe the condition on the potential contact surface Γ_c. Following Frémond [4], we introduce an internal state variable β, which represents the intensity of *adhesion*, $0 \leq \beta \leq 1$, where $\beta = 1$ means the total adhesion, $\beta = 0$ means the lack of adhesion, and $0 < \beta < 1$ is the case of partial adhesion. Then, we assume that the normal stress satisfies the following general expression of *normal compliance* contact condition with adhesion

$$\sigma_\nu(t) = -p_\nu(u_\nu(t) - g) + H_{st}(\beta(t), u_\nu(t)) \quad \text{on } \Gamma_c,$$

where u_ν represents the normal displacement; g is the gap between the potential contact surface and the foundation, measured along the outward normal vector on the contact surface; p_ν is a normal compliance function such that $p_\nu(r) = 0$ for $r \leq 0$. When $u_\nu - g$ is positive, it represents the penetration of the body into the foundation and $-p_\nu(u_\nu(t) - g)$ represents a compression acted by the foundation to the body.

Different types of prescribed functions p_ν were used in [7] for the study of quasistatic contact problems for viscoelastic materials. As an example, we may consider

$$p_\nu(r) = c_\nu\, r_+,$$

where c_ν is a positive constant and $r_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to +\infty$. We can also consider the normal compliance function

$$p_\nu(r) = \begin{cases} c_\nu\, r_+ & \text{if} \quad r \leq \alpha_0, \\ c_\nu\, \alpha_0 & \text{if} \quad r > \alpha_0, \end{cases}$$

where α_0 is a positive coefficient related to the wear and hardness of the surface. In this case the contact condition means that when the penetration is too large, i.e., it exceeds α_0, the obstacle disintegrates and offers no additional resistance to the penetration.

For example, we may consider

$$H_{st}(\beta, u_\nu) = \gamma_\nu \beta^2 R_\nu(u_\nu),$$

where γ_ν is a coefficient depending on the adhesion facility of the contact surface Γ_c, R_ν a Lipschitz-bounded function such that $\forall r > 0$, $R_\nu(r) = 0$.

Moreover, we assume that during the process, there is friction modeled by a version of Coulomb's dry friction law, that is:

$$\begin{cases} |\boldsymbol{\sigma}_\tau(t)| \leq p_\tau(u_\nu(t) - g), \\ |\boldsymbol{\sigma}_\tau(t)| < p_\tau(u_\nu(t) - g) \implies \dot{\boldsymbol{u}}_\tau(t) = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau(t)| = p_\tau(u_\nu(t) - g) \implies \dot{\boldsymbol{u}}_\tau(t) = -\lambda\, \boldsymbol{\sigma}_\tau(t), \text{ for some } \lambda \geq 0, \end{cases} \quad \text{on } \Gamma_c.$$

Here $\boldsymbol{\sigma}_\tau$ is the tangential stress, $p_\tau(u_\nu(t)-g)$ is the friction bound measuring the maximal frictional resistance, and $\dot{\boldsymbol{u}}_\tau$ is the tangential velocity.

In [7], the friction bound

$$p_\tau(u_\nu) = \mu_\tau\, c_\nu\, (u_\nu)_+ \quad \text{on} \quad \Gamma_c$$

is proportional to the normal stress with some positive coefficient of friction $\mu_\tau\, c_\nu$.

The evolution of the adhesion field is described by the following differential equation

$$\dot{\beta}(t) = H_a(\beta(t), u_\nu(t)).$$

As an example, the model defined by $H_{\mathrm{ad}}(\beta(t), \boldsymbol{u}(t)) = -\gamma_\nu(\beta(t))_+\,[R_\nu(u_\nu(t))]^2$ was used in [1], where the adhesion is always decreasing.

Finally the evolution of the temperature with its associated boundary condition is given by:

$$\begin{cases} \dot{\theta}(t) - \mathrm{div}(K_c\, \nabla\theta(t)) = -c_{ij}\, \frac{\partial \dot{u}_i}{\partial x_j}(t) + q(t) \quad \text{on} \quad \Omega, \\ -k_{ij}\, \frac{\partial \theta}{\partial x_j}(t)\, n_i = k_e\, (\theta(t) - \theta_R) \quad \text{on} \quad \Gamma_c, \qquad\qquad \text{on } \Gamma_c. \\ \theta(t) = 0 \quad \text{on} \quad \Gamma_F, \end{cases}$$

In this system, $K_c := (k_{ij})$ represents the thermal conductivity tensor, $q(t)$ the density of volume heat sources, θ_R is the temperature of the foundation, and k_e is the heat exchange coefficient between the body and the obstacle.

To conclude, we are able now to formulate the mechanical problem as follows.

Problem Q. Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \longrightarrow \mathbf{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \longrightarrow S_d$, an adhesion field $\beta : \Gamma_c \times [0,T] \to \boldsymbol{R}$ and a temperature field $\theta : \Omega \times [0,T] \longrightarrow \mathbf{R}_+$ such that for a.e. $t \in (0,T)$:

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds - \theta(t)\,C_e \quad \text{in} \quad \Omega, \tag{1}$$

$$\ddot{\boldsymbol{u}}(t) = \mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \quad \text{in} \quad \Omega, \tag{2}$$

$$\boldsymbol{\sigma}(t)\nu = \boldsymbol{f}_F(t) \quad \text{on} \quad \Gamma_F, \tag{3}$$

$$\sigma_\nu(t) = -p_\nu(u_\nu(t) - g) + H_{st}(\beta(t), u_\nu(t)) \quad \text{on } \Gamma_c, \tag{4}$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau(t)| \le p_\tau(u_\nu(t) - g), \\ |\boldsymbol{\sigma}_\tau(t)| < p_\tau(u_\nu(t) - g) \implies \dot{\boldsymbol{u}}_\tau(t) = \mathbf{0}, \qquad\qquad \text{on } \Gamma_c, \\ |\boldsymbol{\sigma}_\tau(t)| = p_\tau(u_\nu(t) - g) \implies \dot{\boldsymbol{u}}_\tau(t) = -\lambda\,\boldsymbol{\sigma}_\tau(t), \text{ for some } \lambda \ge 0, \end{cases} \tag{5}$$

$$\dot{\beta}(t) = H_{\text{ad}}(\beta(t), u_\nu(t)), \quad 0 \le \beta(t) \le 1 \quad \text{on } \Gamma_c, \tag{6}$$

$$\dot{\theta}(t) - \text{div}(K_c \nabla\theta(t)) = -c_{ij} \frac{\partial \dot{u}_i}{\partial x_j}(t) + q(t) \quad \text{on} \quad \Omega, \tag{7}$$

$$-k_{ij} \frac{\partial \theta}{\partial x_j}(t)\, n_i = k_e\, (\theta(t) - \theta_R) \quad \text{on} \quad \Gamma_c, \tag{8}$$

$$\theta(t) = 0 \quad \text{on} \quad \Gamma_F, \tag{9}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \theta(0) = \theta_0 \quad \text{in} \quad \Omega, \tag{10}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_c. \tag{11}$$

The equation in (2) is the dynamic model of motion where the mass density $\varrho \equiv 1$. Equation (3) is the traction boundary condition. The data in $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, θ_0, and β_0 in (10)–(11) represent the initial displacement, velocity, temperature, and adhesion, respectively.

In view to derive the variational formulation of the mechanical problems (1)–(9), let us first precise the functional framework. Let

$$V = H_1$$

be the admissible displacement space, endowed with the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{u}, \boldsymbol{v})_H \qquad \forall\, \boldsymbol{u},\ \boldsymbol{v}\ \in V,$$

and let $\|\cdot\|_V$ be the associated norm, i.e.,

$$\|\boldsymbol{v}\|_V^2 = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}^2 + \|\boldsymbol{v}\|_H^2 \qquad \forall\, \boldsymbol{v} \in V.$$

It follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V, and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem, there exists a constant $c_0 > 0$ depending only on Ω, and Γ_c such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_c)} \ \le\ c_0\, \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V.$$

Next, let

$$E = \{\eta \in H^1(\Omega),\ \eta = 0 \quad \text{on} \quad \Gamma_F\}$$

be the admissible temperature space, endowed with the canonical inner product of $H^1(\Omega)$.

We use here two Gelfand evolution triples (see, e.g., [16, pp. 416]) given by

$$V \subset H \equiv H' \subset V', \quad E \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset E',$$

where the inclusions are dense and continuous.

In the study of the mechanical problem (1)–(11), we assume that the viscosity operator $\mathscr{A} : \Omega \times S_d \longrightarrow S_d$ satisfies

$$\begin{cases} \text{(i) there exists } L_{\mathscr{A}} > 0 \text{ such that} \\ \quad \|\mathscr{A}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \leq L_{\mathscr{A}} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|, \quad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S_d, \text{a.e. } \boldsymbol{x} \in \Omega; \\ \text{(ii) there exists } m_{\mathscr{A}} > 0 \text{ such that} \\ \quad (\mathscr{A}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathscr{A}(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_{\mathscr{A}} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2, \\ \qquad \forall\, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega; \\ \text{(iii) the mapping } \mathscr{A}(\cdot, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega, \ \forall \boldsymbol{\xi} \in S_d; \\ \text{(iv)) the mapping } \mathscr{A}(\cdot, \boldsymbol{0}) \in \mathscr{H}. \end{cases} \tag{12}$$

We suppose that the elasticity operator $\mathscr{G} : \Omega \times S_d \longrightarrow S_d$ satisfies the following properties:

$$\begin{cases} \text{(i)} \quad \text{there exists } L_{\mathscr{G}} > 0 \text{ such that} \\ \quad |\mathscr{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathscr{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathscr{G}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|, \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \ a.e.\ \boldsymbol{x} \in \Omega; \\ \text{(ii) the mapping } \mathscr{G}(\cdot, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \quad \forall \boldsymbol{\varepsilon} \in S_d; \\ \text{(iii) the mapping } \mathscr{G}(\cdot, \boldsymbol{0}) \in \mathscr{H}. \end{cases} \tag{13}$$

The relaxation tensor $\mathscr{B} : [0, T] \times \Omega \times S_d \longrightarrow S_d$, $(t, \boldsymbol{x}, \boldsymbol{\tau}) \longmapsto (B_{ijkh}(t, \boldsymbol{x})\, \tau_{kh})$ satisfies

$$\begin{cases} \text{(i) } B_{ijkh} \in L^\infty(0, T; L^\infty(\Omega)); \\ \text{(ii) } \mathscr{B}(t)\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathscr{B}(t)\boldsymbol{\tau} \\ \qquad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d, \text{ a.e. } t \in (0, T), \text{ a.e. in } \Omega \end{cases} \tag{14}$$

The normal compliance functional $p_\nu : \Gamma_c \times \boldsymbol{R} \longrightarrow \boldsymbol{R}^+$ satisfies the following properties:

$$\begin{cases} \text{(i)} \quad \text{there exists } L_\nu > 0 \text{ such that} \\ \quad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_\nu |r_1 - r_2|, \quad \forall r_1, r_2 \in \boldsymbol{R}, \ a.e.\ \boldsymbol{x} \in \Gamma_c; \\ \text{(ii) the mapping } p_\nu(\cdot, r) \text{ is Lebesgue measurable on } \Gamma_c, \quad \forall r \in \boldsymbol{R}; \\ \text{(iii) } p_\nu(\cdot, r) = 0, \text{ on } \Gamma_c, \quad \forall r \leq 0. \end{cases} \tag{15}$$

The tangential compliance functional $p_\tau : \Gamma_c \times \boldsymbol{R} \longrightarrow \boldsymbol{R}^+$ satisfies the following properties:

$$\begin{cases} \text{(i)} \quad \text{there exists } L_\tau > 0 \text{ such that} \\ \quad |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_\tau |r_1 - r_2|, \quad \forall r_1, r_2 \in \boldsymbol{R}, \ a.e.\ \boldsymbol{x} \in \Gamma_c; \\ \text{(ii) the mapping } p_\tau(\cdot, r) \text{ is Lebesgue measurable on } \Gamma_c, \quad \forall r \in \boldsymbol{R}; \\ \text{(iii) } p_\tau(\cdot, r) = 0, \text{ on } \Gamma_c, \quad \forall r \leq 0. \end{cases} \tag{16}$$

The functional $H_{st} : \Gamma_c \times \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ satisfies

$$\begin{cases} \text{(i) the mapping } H_{st}(\cdot, r, s) \in L^\infty(\Gamma_c), \quad \forall\, r,\ s \in \boldsymbol{R}; \\ \text{(ii) } \exists L_{st} > 0,\ \exists b_{st} : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}, \\ \qquad |H_{st}(\boldsymbol{x}, r_1, s_1) - H_{st}(\boldsymbol{x}, r_2, s_2)| \leq L_{st}\, |r_1 - r_2| + b_{st}(r_1, r_2)\, |s_1 - s_2|, \\ \qquad\quad \forall \boldsymbol{x} \in \Gamma_c,\ \forall r_1, s_1, r_2, s_2 \in \boldsymbol{R}. \end{cases} \tag{17}$$

where b_{st} is some function which maps any bounded subset in $\boldsymbol{R}^2$ into a bounded subset in $\boldsymbol{R}$.

The functional $H_{\text{ad}} : \Gamma_c \times \boldsymbol{R} \times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ satisfies

$$\begin{cases} \text{(i) the mapping } H_{\text{ad}}(\cdot, r, s) \in L^\infty(\Gamma_c), \quad \forall\, r,\ s \in \boldsymbol{R}; \\ \text{(ii) } \exists L_{\text{ad}} > 0,\ \exists b_{\text{ad}} : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}, \\ \qquad |H_{\text{ad}}(\boldsymbol{x}, r_1, s_1) - H_{\text{ad}}(\boldsymbol{x}, r_2, s_2)| \leq L_{\text{ad}}\, |r_1 - r_2| + b_{\text{ad}}(r_1, r_2)\, |s_1 - s_2|, \\ \qquad\quad \forall \boldsymbol{x} \in \Gamma_c,\ \forall r_1, s_1, r_2, s_2 \in \boldsymbol{R}; \\ \text{(iii) } \forall \boldsymbol{x} \in \Gamma_c,\ \forall r, s \in \boldsymbol{R} : \\ \qquad r \leq 0 \implies H_{\text{ad}}(\boldsymbol{x}, r, s) \geq 0, \\ \qquad r \geq 1 \implies H_{\text{ad}}(\boldsymbol{x}, r, s) \leq 0. \end{cases} \tag{18}$$

where b_{ad} is some function which maps any bounded subset in $\boldsymbol{R}^2$ into a bounded subset in $\boldsymbol{R}$.

We suppose the body forces and surface tractions satisfy the regularity conditions:

$$\boldsymbol{f}_0 \in L^2(0, T; H), \qquad \boldsymbol{f}_F \in L^2(0, T; L^2(\Gamma_F)^d). \tag{19}$$

The coefficients of the frictional normal compliance verifies.

The gap function verifies

$$g \in L^\infty(\Gamma_c;\ \mathbf{R}^+). \tag{20}$$

Concerning the thermal tensors and the heat sources density, we suppose that:

$$C_e = (c_{ij}), \quad c_{ij} = c_{ji} \in L^\infty(\Omega), \quad q \in L^2(0, T; L^2(\Omega)). \tag{21}$$

We assume that the boundary thermal data satisfy the following regularity properties:

$$k_e \in L^\infty(\Omega;\ \mathbf{R}^+), \quad \theta_R \in W^{1,2}(0, T; L^2(\Gamma_c)). \tag{22}$$

We suppose that the thermal conductivity tensor verifies the usual symmetry end ellipticity : for some $c_k > 0$ and for all $(\xi_i) \in \mathbf{R}^d$,

$$K_c = (k_{ij}), \quad k_{ij} = k_{ji} \in L^\infty(\Omega), \quad k_{ij}\, \xi_i \xi_j \geq c_k\, \xi_i \xi_i. \tag{23}$$

Finally, we assume that the initial data satisfy the conditions

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in V, \quad \theta_0 \in E, \quad \beta_0 \in L^\infty(\Gamma_c), \quad 0 \leq \beta_0 \leq 1. \tag{24}$$

Using Green's formula, we obtain the following weak formulation of the mechanical problem Q, defined by a system of second-order quasi-variational evolution inequality coupled with a first-order evolution equation.

Problem QV**.** Find a displacement field $\boldsymbol{u} : [0,T] \to V$, an adhesion field $\beta : [0,T] \to L^\infty(\Gamma_c)$, and a temperature field $\theta : [0,T] \to E$ satisfying for a.e. $t \in (0,T)$:

$$\begin{cases} \langle \ddot{\boldsymbol{u}}(t) + A\,\dot{\boldsymbol{u}}(t) + B\,\boldsymbol{u}(t) + C\,\theta(t) + \displaystyle\int_0^t \mathscr{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,\ \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V}, \\ +j_\nu(\beta(t), \boldsymbol{u}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}(t)) + j_\tau(\boldsymbol{u}(t), \boldsymbol{w}) - j_\tau(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \\ \qquad \geq \langle \boldsymbol{f}(t),\ \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} \quad \forall \boldsymbol{w} \in V; \\ \dot{\beta}(t) = H_{\mathrm{ad}}(\beta(t), u_\nu(t)), \quad 0 \leq \beta(t) \leq 1 \quad \text{on } \Gamma_c; \\ \dot{\theta}(t) + K\,\theta(t) = R\dot{\boldsymbol{u}}(t) + Q(t) \qquad \text{in} \quad E'; \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \beta(0) = \beta_0, \quad \theta(0) = \theta_0. \end{cases}$$

Here, the operators and functions $A, B : V \longrightarrow V'$, $C : E \longrightarrow V'$, $j_\nu : L^\infty(\Gamma_c) \times V^2 \longrightarrow \mathbf{R}$, $j_\tau : V^2 \longrightarrow \mathbf{R}$, $K : E \longrightarrow E'$, $R : V \longrightarrow E'$, $\boldsymbol{f} : [0,T] \longrightarrow V'$, and $Q : [0,T] \longrightarrow E'$ are defined by $\forall \boldsymbol{v} \in V$, $\forall \boldsymbol{w} \in V$, $\forall \zeta \in E$, $\forall \eta \in E$, $\forall \beta \in L^\infty(\Gamma_c)$:

$$\langle A\,\boldsymbol{v}, \boldsymbol{w}\rangle_{V'\times V} = (\mathscr{A}(\boldsymbol{\varepsilon}\boldsymbol{v}), \boldsymbol{\varepsilon}\boldsymbol{w})_{\mathscr{H}};$$

$$\langle B\,\boldsymbol{v}, \boldsymbol{w}\rangle_{V'\times V} = (\mathscr{G}(\boldsymbol{\varepsilon}\boldsymbol{v}), \boldsymbol{\varepsilon}\boldsymbol{w})_{\mathscr{H}};$$

$$\langle C\zeta, \boldsymbol{w}\rangle_{V'\times V} = -(\zeta\, C_e,\ \boldsymbol{\varepsilon}\boldsymbol{w})_{\mathscr{H}};$$

$$j_\nu(\beta, \boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_c} (p_\nu\,(v_\nu) - H_{st}(\beta, v_\nu)\, w_\nu\, da;$$

$$j_\tau(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Gamma_c} p_\tau(v_\nu)\,|\boldsymbol{w}_\tau|\, da;$$

$$\langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} = (\boldsymbol{f}_0(t), \boldsymbol{w})_H + (\boldsymbol{f}_F(t), \boldsymbol{w})_{(L^2(\Gamma_F))^d};$$

$$\langle K\,\zeta,\eta\rangle_{E'\times E}=\sum_{i,j=1}^{d}\int_{\Omega}k_{ij}\,\frac{\partial\zeta}{\partial x_j}\,\frac{\partial\eta}{\partial x_i}\,dx+\int_{\Gamma_c}k_e\,\zeta\cdot\eta\,da;$$

$$\langle R\,\boldsymbol{v},\eta\rangle_{E'\times E}=-\int_{\Omega}c_{ij}\,\frac{\partial v_i}{\partial x_j}\,\eta\,dx;$$

$$\langle Q(t),\eta\rangle_{E'\times E}=\int_{\Gamma_c}k_e\,\theta_R(t)\,\eta\,dx+\int_{\Omega}q(t)\,\eta\,dx.$$

Our main existence and uniqueness result is the following, which we will prove in the next section.

Theorem 1. *Assume that (12)–(24) hold, and under the condition that L_τ is small enough, then there exists a unique solution $\{\boldsymbol{u},\beta,\theta\}$ to problem QV with the regularity:*

$$\begin{cases}\boldsymbol{u}\in C^1(0,T;H)\cap W^{1,2}(0,T;V)\cap W^{2,2}(0,T;V');\\ \beta\in W^{1,\infty}(0,T;L^\infty(\Gamma_c));\\ \theta\in C(0,T;L^2(\Omega))\cap L^2(0,T;E)\cap W^{1,2}(0,T;E').\end{cases}\tag{25}$$

3 Proof of Theorem 1

The idea is to bring the second order inequality to a first order inequality, using monotone operator, convexity and fixed point arguments, and will be carried out in several steps.

Let us introduce the velocity variable

$$\boldsymbol{v}=\dot{\boldsymbol{u}}.$$

The system in Problem QV is then written for a.e. $t\in(0,T)$:

$$\begin{cases}\boldsymbol{u}(t)=\boldsymbol{u}_0+\displaystyle\int_0^t\boldsymbol{v}(s)\,ds;\\ \langle\dot{\boldsymbol{v}}(t)+A\,\boldsymbol{v}(t)+B\,\boldsymbol{u}(t)+C\,\theta(t)+\displaystyle\int_0^t\mathscr{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,\ \boldsymbol{w}-\boldsymbol{v}(t)\rangle_{V'\times V}\\ +j_\nu(\beta(t),\boldsymbol{u}(t),\boldsymbol{w}-\boldsymbol{v}(t))+j_\tau(\boldsymbol{u}(t),\boldsymbol{w})-j_\tau(\boldsymbol{u}(t),\boldsymbol{v}(t))\\ \qquad\geq\langle\boldsymbol{f}(t),\ \boldsymbol{w}-\boldsymbol{v}(t)\rangle_{V'\times V}\quad\forall\boldsymbol{w}\in V;\\ \dot{\beta}(t)=H_{\rm ad}(\beta(t),u_\nu(t)),\quad 0\leq\beta(t)\leq 1\quad\text{on }\Gamma_c;\\ \dot{\theta}(t)+K\,\theta(t)=R\,\boldsymbol{v}(t)+Q(t)\quad\text{in}\quad E';\\ \boldsymbol{v}(0)=\boldsymbol{v}_0,\quad\beta(0)=\beta_0,\quad\theta(0)=\theta_0,\end{cases}$$

with the regularities

$$\begin{cases} \boldsymbol{v} \in C(0,T;H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V'); \\ \beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_c)); \\ \theta \in C(0,T;L^2(\Omega)) \cap L^2(0,T;E) \cap W^{1,2}(0,T;E'). \end{cases}$$

Let us begin by

Lemma 1. *For all $\eta \in L^2(0,T;V')$, there exists a unique*

$$\boldsymbol{v}_\eta \in C(0,T;H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V')$$

satisfying

$$\begin{cases} \langle \dot{\boldsymbol{v}}_\eta(t) + A\,\boldsymbol{v}_\eta(t),\, \boldsymbol{w} - \boldsymbol{v}_\eta(t)\rangle_{V'\times V} + \langle \eta(t), \boldsymbol{w} - \boldsymbol{v}_\eta(t)\rangle_{V'\times V} \\ \quad + j_\tau(\boldsymbol{u}_\eta(t), \boldsymbol{w}) - j_\tau(\boldsymbol{u}_\eta(t), \boldsymbol{v}_\eta(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{v}_\eta(t)\rangle_{V'\times V}, \\ \qquad\qquad \forall\, \boldsymbol{w} \in V, \quad a.e.\ t \in (0,T); \\ \boldsymbol{v}_\eta(0) = \boldsymbol{v}_0, \end{cases} \tag{26}$$

where

$$\boldsymbol{u}_\eta(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_\eta(s)\,ds.$$

Moreover, if L_τ is small enough, then $\exists c > 0$ such that $\forall \eta_1,\ \eta_2 \in L^2(0,T;V')$, $\forall t \in [0,T]$:

$$\|\boldsymbol{v}_{\eta_2}(t) - \boldsymbol{v}_{\eta_1}(t)\|_H^2 + \int_0^t \|\boldsymbol{v}_{\eta_2} - \boldsymbol{v}_{\eta_1}\|_V^2 \leq c \int_0^t \|\eta_1 - \eta_2\|_{V'}^2. \tag{27}$$

Proof. Given $\eta \in L^2(0,T;V')$ and $x \in C(0,T;V)$, by using a general result on parabolic variational inequality (see, e.g., [5, Chap. 3]), we obtain the existence of a unique $\boldsymbol{v}_{\eta x} \in C(0,T;H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V')$ satisfying

$$\begin{cases} \langle \dot{\boldsymbol{v}}_{\eta x}(t) + A\,\boldsymbol{v}_{\eta x}(t),\, \boldsymbol{w} - \boldsymbol{v}_{\eta x}(t)\rangle_{V'\times V} + \langle \eta(t), \boldsymbol{w} - \boldsymbol{v}_{\eta x}(t)\rangle_{V'\times V} \\ \quad + j_\tau(x(t), \boldsymbol{w}) - j_\tau(x(t), \boldsymbol{v}_{\eta x}(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{v}_{\eta x}(t)\rangle_{V'\times V}, \\ \qquad\qquad \forall\, \boldsymbol{w} \in V, \quad \text{a.e. } t \in (0,T); \\ \boldsymbol{v}_{\eta x}(0) = \boldsymbol{v}_0, \end{cases} \tag{28}$$

Now let us fix $\eta \in L^2(0,T;V')$ and consider $\Lambda_\eta : C(0,T;V) \to C(0,T;V)$ defined by

$$\forall x \in C(0,T;V), \quad \Lambda_\eta x(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_{\eta x}(s)\,ds.$$

We check by algebraic manipulation that for all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{w}_1, \boldsymbol{w}_2 \in V$, we have

$$j_\tau(\boldsymbol{u}_1, \boldsymbol{w}_2) - j_\tau(\boldsymbol{u}_1, \boldsymbol{w}_1) + j_\tau(\boldsymbol{u}_2, \boldsymbol{w}_1) - j_\tau(\boldsymbol{u}_2, \boldsymbol{w}_2) \le c \, \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_V \, \|\boldsymbol{w}_2 - \boldsymbol{w}_1\|_V,$$

where $c > 0$ is some constant proportional to L_τ involving c_0.

Let $x_1, x_2 \in C(0, T; V)$ be given. Putting in (28) the data $x = x_1$ with $\boldsymbol{w} = \boldsymbol{v}_{\eta x_2}$, and $x = x_2$ with $\boldsymbol{w} = \boldsymbol{v}_{\eta x_1}$, adding then the two inequalities and integrating over $(0, T)$, we obtain $\forall t \in [0, T]$:

$$\|\boldsymbol{v}_{\eta x_2}(t) - \boldsymbol{v}_{\eta x_1}(t)\|_H^2 + \int_0^t \|\boldsymbol{v}_{\eta x_2}(s) - \boldsymbol{v}_{\eta x_1}(s)\|_V^2 ds$$
$$\le c \int_0^t \|x_2(s) - x_1(s)\|_V^2 ds + c \int_0^t \|\boldsymbol{v}_{\eta x_2}(s) - \boldsymbol{v}_{\eta x_1}(s)\|_H^2 \, ds.$$

Using Gronwall's inequality, we deduce that $\forall x_1, x_2 \in C(0, T; V), \ \forall t \in [0, T]$,

$$\|\Lambda_\eta(x_2)(t) - \Lambda_\eta(x_1)(t)\|_V^2 \le c \int_0^t \|x_2(s) - x_1(s)\|_V^2 ds.$$

Thus, by Banach's fixed point principle, we know that Λ_η has a unique fixed point denoted by x_η. We then verify that

$$\boldsymbol{v}_\eta = \boldsymbol{v}_{\eta x_\eta}$$

is the unique solution verifying (26).

Now let $\eta_1, \eta_2 \in L^2(0, T; V')$. Putting in (26) the data $\eta = \eta_1$ with $\boldsymbol{w} = \boldsymbol{v}_{\eta_2}$, and $\eta = \eta_2$ with $\boldsymbol{w} = \boldsymbol{v}_{\eta_1}$, adding then the two inequalities and integrating over $(0, T)$, we obtain $\forall t \in [0, T]$:

$$\|\boldsymbol{v}_{\eta_2}(t) - \boldsymbol{v}_{\eta_1}(t)\|_H^2 + \int_0^t \|\boldsymbol{v}_{\eta_2}(s) - \boldsymbol{v}_{\eta_1}(s)\|_V^2 ds$$
$$\le c \int_0^t \|\eta_2(s) - \eta_1(s)\|_{V'}^2 ds + c \int_0^t \|\boldsymbol{u}_{\eta_2}(s) - \boldsymbol{u}_{\eta_1}(s)\|_V^2 \, ds$$
$$+ \, c \int_0^t \|\boldsymbol{v}_{\eta_2}(s) - \boldsymbol{v}_{\eta_1}(s)\|_H^2 \, ds,$$

where $c > 0$ is some constant proportional to L_τ. We deduce (27) from Gronwall's inequality provided that L_τ is small enough.

Here and below, we denote by $c > 0$ a generic constant, which value may change from lines to lines.

Lemma 2. *For all $\eta \in L^2(0,T;V')$, there exists a unique*

$$\beta_\eta \in W^{1,\infty}(0,T;L^\infty(\Gamma_c))$$

satisfying

$$\begin{cases} \text{for a.e. } t \in (0,T), \quad \dot{\beta}_\eta(t) = H_{\text{ad}}(\beta_\eta(t), u_{\eta\nu}(t)), \\ \beta_\eta(0) = \beta_0, \text{ on } \Gamma_c, \\ \forall t \in [0,T], \quad 0 \le \beta_\eta(t) \le 1 \text{ on } \Gamma_c. \end{cases} \tag{29}$$

Moreover, if L_τ is small enough, then $\exists c > 0$ such that $\forall \eta_1,\ \eta_2 \in L^2(0,T;V')$:

$$\|\beta_{\eta_2}(t) - \beta_{\eta_1}(t)\|^2_{L^\infty(\Gamma_c)} \le c \int_0^t \|\eta_1 - \eta_2\|^2_{V'}, \quad \forall t \in [0,T]. \tag{30}$$

Proof. Fix $\eta \in L^2(0,T;V')$.

Let us consider $X = L^\infty(\Gamma_c)$ and $F_\eta : [0,T] \times X \to X$ defined by for all $(t,\beta) \in [0,T] \times X$:

$$F_\eta(t,\beta) = -\gamma_\nu(\beta)_+ [R_\nu(u_{\eta\nu}(t))]^2.$$

We verify that :

(i) For all $t \in [0,T]$, $F_\eta(t,\cdot)$ is Lipschitz continuous;
(ii) For all $\beta \in X$, $F_\eta(\cdot,\beta) \in L^\infty(0,T;X)$.

By using Cauchy-Lipschitz's Theorem (see [15]) we have :
There exists a unique $\beta_\eta \in W^{1,\infty}(0,T;X)$ satisfying for a.e. $t \in (0,T)$,

$$\dot{\beta}(t) = F_\eta(t,\beta(t)), \quad \beta_\eta(0) = \beta_0 \text{ on } \Gamma_c.$$

Let us consider $X = L^\infty(0,T;L^\infty(\Gamma_c))$, and for all $\beta \in X$, define $\Lambda_a(\beta) \in X$ by

$$\forall (\boldsymbol{x},t) \in \Gamma_c \times [0,T], \quad \Lambda_a(\beta)(\boldsymbol{x},t) = \beta_0(\boldsymbol{x}) + \int_0^t H_{\text{ad}}(\boldsymbol{x},\beta(\boldsymbol{x},s),u_{\eta\nu}(\boldsymbol{x},s))\,ds.$$

Using (18) and after some algebraic manipulation, we have $\forall \beta_1,\ \beta_2 \in X$:

$$\|\Lambda_a(\beta_2)(t) - \Lambda_a(\beta_2)(t)\|^2_{L^\infty(\Gamma_c)} \le c \int_0^t \|\beta_1(s) - \beta_2(s)\|^2_{L^\infty(\Gamma_c)}, \quad \forall t \in [0,T].$$

By Banach's contraction principle, Λ_a has a unique fixed point denoted by $\beta_h \in X$. Then, $\beta_\eta \in W^{1,\infty}(0,T;L^\infty(\Gamma_c))$.

Let us show that for all $t \in [0,T]$, $0 \le \beta_\eta(\cdot,t) \le 1$ on Γ_c.

Indeed, we have that for all $t \in [0,T], \boldsymbol{x} \in \Gamma_c$:

$$\beta_\eta(\boldsymbol{x},t) = \beta_0(\boldsymbol{x}) + \int_0^t H_{\mathrm{ad}}(\boldsymbol{x}, \beta(\boldsymbol{x},s), u_{\eta\nu}(\boldsymbol{x},s))\, ds.$$

Now let us fix $t_0 \in [0,T], \boldsymbol{x} \in \Gamma_c$. To simplify denote by

$$F_\eta(\boldsymbol{x},s) = H_{\mathrm{ad}}(\boldsymbol{x}, \beta(\boldsymbol{x},s), u_{\eta\nu}(\boldsymbol{x},s)).$$

Assume that $\beta_\eta(\boldsymbol{x},t_0) < 0$.
Then, $0 < t_0 \le T$ and there exists some $0 \le t_1 < t_0$ such that $\beta_\eta(\boldsymbol{x},t_1) = 0$. Thus, for all $t \in [t_1,T]$:

$$\beta_\eta(\boldsymbol{x},t) = \beta_\eta(\boldsymbol{x},t_1) + \int_{t_1}^t F_\eta(\boldsymbol{x},s)\, ds = \int_{t_1}^t F_\eta(\boldsymbol{x},s)\, ds \le 0.$$

Then, for all $s \in [t_1,T]$, $F_\eta(\boldsymbol{x},s)) = 0$ and for all $t \in [t_1,T]$, $\beta_\eta(\boldsymbol{x},t)) = 0$ which contradicts the fact that $\beta_\eta(\boldsymbol{x},t_0) < 0$. We conclude that for all $t \in [0,T]$, $\beta_\eta(\cdot,t) \ge 0$ on Γ_c.

We prove in the same way that for all $t \in [0,T], \boldsymbol{x} \in \Gamma_c$, $\beta_\eta(\boldsymbol{x},t) \le 1$.

Now for any $\eta_1,\ \eta_2 \in L^2(0,T;V')$, for any $t \in [0,T]$, we have on Γ_c :

$$|\beta_{\eta_2}(t,\cdot) - \beta_{\eta_1}(t,\cdot)| \le c \int_0^t |H_{\mathrm{ad}}(\beta_{\eta_2}(s), u_{\eta_2\nu}(s)) - H_{\mathrm{ad}}(\beta_{\eta_1}(s), u_{\eta_1\nu}(s))|\, ds.$$

Using (18) we obtain

$$|H_{\mathrm{ad}}(\beta_{\eta_2}(s), u_{\eta_2\nu}(s)) - H_{\mathrm{ad}}(\beta_{\eta_1}(s), u_{\eta_1\nu}(s))| \le c\,|\beta_{\eta_2}(s) - \beta_{\eta_1}(s)| + c\,|u_{\eta_2\nu}(s) - u_{\eta_1\nu}(s)|.$$

We deduce that for any $t \in [0,T]$:

$$\|\beta_{\eta_2}(t) - \beta_{\eta_1}(t)\|^2_{L^\infty(\Gamma_c)} \le c \int_0^t \|\beta_{\eta_2}(s) - \beta_{\eta_1}(s)\|^2_{L^\infty(\Gamma_c)}\, ds + c \int_0^t \|\boldsymbol{u}_{\eta_2}(s) - \boldsymbol{u}_{\eta_1}(s)\|^2_V.$$

As

$$\int_0^t \|\boldsymbol{u}_{\eta_2}(s) - \boldsymbol{u}_{\eta_1}(s)\|^2_V \le c \int_0^t \|\boldsymbol{v}_{\eta_2}(s) - \boldsymbol{v}_{\eta_1}(s)\|^2_V,$$

using then Gronwall's inequality and Lemma 1, we deduce the inequality (30) under the condition that L_τ is small enough.

Lemma 3. *For all $\eta \in \eta \in L^2(0,T;V')$, there exists a unique*

$$\theta_\eta \in C(0,T;L^2(\Omega)) \cap L^2(0,T;E) \cap W^{1,2}(0,T;E')$$

satisfying

$$\begin{cases} \dot{\theta}_\eta(t) + K\,\theta_\eta(t) = R\,\boldsymbol{v}_\eta(t) + Q(t), \quad \textit{in} \quad E', \quad \textit{a.e. } t \in (0,T), \\ \theta_\eta(0) = \theta_0. \end{cases} \tag{31}$$

Moreover, if L_τ *is small enough, then* $\exists c > 0$ *such that* $\forall \eta_1,\ \eta_2 \in L^2(0,T;V')$*:*

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|^2_{L^2(\Omega)} \leq c \int_0^t \|\eta_1 - \eta_2\|^2_{V'}, \quad \forall t \in [0,T]. \tag{32}$$

Proof. We verify that the operator $K : E \longrightarrow E'$ is linear continuous and strongly monotone, and from the expression of the operator R, we have

$$\boldsymbol{v}_\eta \in L^2(0,T;V) \Longrightarrow R\,\boldsymbol{v}_\eta \in L^2(0,T;E'),$$

as $Q \in L^2(0,T;E')$ then $R\,\boldsymbol{v}_\eta + Q \in L^2(0,T;E')$. Therefore, the existence and uniqueness result verifying (29) follows from classical result on first-order evolution equation.

Now for $\eta_1,\ \eta_2 \in L^2(0,T;V')$, we have for a.e. $t \in (0;T)$:

$$\begin{aligned}&\langle \dot\theta_{\eta_1}(t) - \dot\theta_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t)\rangle_{E'\times E} + \langle K\,\theta_{\eta_1}(t) - K\,\theta_{\eta_2}(t),\ \theta_{\eta_1}(t) - \theta_{\eta_2}(t)\rangle_{E'\times E}\\ &\quad = \langle R\,\boldsymbol{v}_{\eta_1}(t) - R\,\boldsymbol{v}_{\eta_2}(t),\ \theta_{\eta_1}(t) - \theta_{\eta_2}(t)\rangle_{E'\times E}.\end{aligned}$$

Then integrating the last property over $(0,t)$, using the strong monotonicity of K and the Lipschitz continuity of $R : V \longrightarrow E'$, we deduce

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|^2_{L^2(\Omega)} \leq c \int_0^t \|\boldsymbol{v}_{\eta_1} - \boldsymbol{v}_{\eta_2}\|^2_V, \quad \forall t \in [0,T].$$

The inequality (32) follows then from Lemma 1.

Consider the operator $\Lambda : L^2(0,T;V') \to L^2(0,T;V')$ defined by for all $\eta \in L^2(0,T;V')$:

$$\begin{aligned}\langle \Lambda\,\eta\,(t),\ \boldsymbol{w}\rangle_{V'\times V} &= \langle B\,\boldsymbol{u}_\eta(t) + C\,\theta_\eta(t) + \int_0^t \mathscr{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds,\ \boldsymbol{w}\rangle_{V'\times V}\\ &\quad + j_\nu(\beta_\eta(t), \boldsymbol{u}_\eta(t), \boldsymbol{w}), \quad \forall \boldsymbol{w} \in V,\ \text{a.e. } t \in (0,T),\end{aligned}$$

where

$$\boldsymbol{u}_\eta(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_\eta(s)\,ds,\ \forall t \in [0,T].$$

Lemma 4. *Under the condition that* L_τ *is small enough, then* Λ *has a unique fixed point* $\eta^* \in L^2(0,T;V')$.

Proof. We check that the operator $C : E \longrightarrow V'$ is linear and that

$$\exists c > 0,\ \forall \xi \in E,\ \|C\,\xi\|_{V'} \leq c\,\|\xi\|_{L^2(\Omega)}.$$

Let $\eta_1, \eta_2 \in L^2(0,T;V')$ be given. We verify that for a.e. $t \in (0,T)$:

$$\|\Lambda\, \eta_2(t) - \Lambda\, \eta_1(t)\|_{V'} \le c\, \|B\boldsymbol{u}_{\eta_2}(t) - B\boldsymbol{u}_{\eta_1}(t)\|_{V'} + c\, \|\theta_{\eta_2}(t) - \theta_{\eta_1}(t)\|_{L^2(\Omega)}$$
$$+ c\, \|\boldsymbol{u}_{\eta_2}(t) - \boldsymbol{u}_{\eta_1}(t)\|_V + c\, \|\beta_{\eta_2}(t) - \beta_{\eta_1}(t)\|^2_{L^\infty(\Gamma_c)}.$$

Thus, from (13), Lemmas 1, 2, and 3, we deduce that if L_τ is small enough, then $\exists c > 0$ satisfying : for all $\eta_1, \eta_2 \in L^2(0,T;V')$ and for all $t \in [0,T]$,

$$\|\Lambda\, \eta_2(t) - \Lambda\, \eta_1(t)\|^2_{V'} \le c \int_0^t \|\eta_2(s) - \eta_1(s)\|^2_{V'} ds.$$

Then, using again Banach's fixed point principle, we obtain that Λ has a unique fixed point.

Proof of Theorem 1. We have now all the ingredients to prove the Theorem 1.

We verify then that the functions

$$\boldsymbol{u}(t) := \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_{\eta^*}, \ \forall t \in [0,T], \quad \beta := \beta_{\eta^*}, \quad \theta := \theta_{\eta^*}$$

are solutions to problem QV with the regularities (25); the uniqueness follows from the uniqueness in Lemmas 1, 2, and 3.

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The Kolmogorov-Arnold-Moser (KAM) and Nekhoroshev Theorems with Arbitrary Time Dependence

Alessandro Fortunati and Stephen Wiggins

Abstract The Kolmogorov-Arnold-Moser (KAM) theorem and the Nekhoroshev theorem are the two "pillars" of canonical perturbation theory for near-integrable Hamiltonian systems. Over the years there have been many extensions and generalizations of these fundamental results, but it is only very recently that extensions of these theorems near-integrable Hamiltonian systems having explicit, and aperiodic, time dependence have been developed. We will discuss these results, with particular emphasis on the new mathematical issues that arise when treating aperiodic time dependence.

1 Introduction

Vladimir Arnold's contributions to mathematics and mechanics are truly remarkable for both their breadth and depth. In this article we discuss an area where he made contributions that are essential to understand for any student or researcher in the field of Hamiltonian dynamics. In particular, "Arnold" is the middle name on the famous Kolmogorov-Arnold-Moser (KAM) theorem [1–3], which gives sufficient conditions for the existence of quasiperiodic motion in near-integrable Hamiltonian systems (expressed in the action-angle variables of the unperturbed integrable Hamiltonian system). Another theorem in the same field (and with a very similar setup), due to Nekhoroshev [4], describes stability of the action variables over and exponentially long time interval. Together, the KAM and Nekhoroshev theorems are the two "rigorous pillars" that establish canonical perturbation theory of near-integrable Hamiltonian systems. A recent monograph that traces the historical development of this theory in some detail is [5].

Despite the firm establishment of the "KAM theory" and "Nekhoroshev theory" in the mathematics, physical science, and engineering disciplines, there is an important area that has not been addressed. In particular, the development of similar types of perturbation theorems for near-integrable systems having "general" time

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dependence, i.e., when the perturbation of the integrable Hamiltonian (or of a particular motion), is not required to be neither periodic nor quasiperiodic. We shall refer to this class of perturbation as *aperiodic*. The motivation for such results comes from applications, e.g. the study of transport and mixing in fluid mechanics from the dynamical systems point of view (see [6] for a discussion of the issues from the Hamiltonian dynamic point of view that arise in this field).

While our goal here is not to review KAM and Nekhoroshev theory (the monograph of Dumas [5] does an excellent job of this), we do note some of the issues such as results for time-dependent, near-integrable Hamiltonian systems. Essentially all of the literature (with a few notable exceptions that we will mention toward the end of this introduction) concerned with time-dependent Hamiltonian systems deal with periodic or quasiperiodic time dependence. For such time dependence, the problems can often be cast in a form where classical results and approaches can be applied. The monographs [7, 8] discuss some of these topics. The paper [9] develops a KAM type result for quasiperiodically time-dependent systems and the paper [10] develops a Nekhoroshev result for the same class of systems. The first paper to develop a Nekhoroshev type result for Hamiltonian systems with general time dependence was [11]. The form of the system they treated was somewhat different than the classical near-integrable Hamiltonian systems since their goals were somewhat different. The first papers to develop Nekhoroshev type results for systems with general time dependence in the classical setting were [12, 13], and the only paper treating a KAM type result in the classical setting is [14]. The purpose of this paper is to describe the results in these latter papers dealing with aperiodic time dependence, with particular attention on the issues that arise for explicitly time-dependent Hamiltonians and the correspondent regularity hypotheses that the perturbation function is required to satisfy. In Sect. 2 we discuss the Nekhoroshev theorem and in Sect. 3 we discuss the KAM theorem.

2 A Nekhoroshev Theorem with Aperiodic Time Dependence

In this section we describe the setup and strategy for the proof of the theorem. This will provide us with the background and framework for providing a description of the theorem. We follow closely the setup in [11] (but see [15] for a detailed development of the canonical perturbation theory and the Nekhoroshev theory, including historical background).

2.1 The Setup and Assumptions

We consider a near-integrable, slowly varying (to be quantified shortly) time-dependent Hamiltonian system expressed in the action-angle variables of the unperturbed system of the following form:

$$\mathscr{H}(I, \varphi, t) := h(I) + \varepsilon f(I, \varphi, \mu t). \tag{1}$$

We note the following:

- $\varepsilon, \mu > 0$ small parameters.
- $I = (I_1, \ldots, I_n) \in \mathscr{G}$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$ denote action-angle variables, where $\mathscr{G} \subset \mathbb{R}^n$ is an open set.
- The dependence on t is, in general, aperiodic, i.e., it need not be periodic or quasiperiodic.

We will rewrite the time-dependent Hamiltonian (1) as a time-independent Hamiltonian by defining two new conjugate variables in the standard way. If we define $\xi := \mu t$ and η as the new conjugate variable pair, the Hamiltonian (1) takes the autonomous form on $\mathscr{D} := \mathscr{G} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R} \ni (I, \eta, \varphi, \xi)$.

$$H(I, \varphi, \eta, \xi) := h(I) + \mu\eta + \varepsilon f(I, \varphi, \xi). \tag{2}$$

Since we use complex function techniques in the proof of the theorem, we will need to complexify the real domain of the Hamiltonian. Let $\sigma, \rho > 0$ be real numbers. Then we define $\mathscr{D}_{\rho,2\sigma} := \mathscr{G}_\rho \times \mathscr{R}_\rho \times \mathbb{T}^n_{2\sigma} \times \mathscr{S}_\sigma$ to be a *complex neighborhood* of $\mathscr{D}$, where

$$\mathscr{G}_\rho := \bigcup_{I \in \mathscr{G}} \Delta_\rho(I), \qquad \Delta_\rho(I) := \{\hat{I} \in \mathbb{C}^n : |\hat{I} - I| < \rho\},$$
$$\mathscr{R}_\rho := \{\eta \in \mathbb{C} : |\Im\eta| < \rho\}, \quad \mathbb{T}^n_{2\sigma} := \{\varphi \in \mathbb{C}^n : |\Im\varphi| < 2\sigma\},$$
$$\mathscr{S}_\sigma := \{\xi \in \mathbb{C} : |\Im\xi| < \sigma\}.$$

The case $n = 1$ is illustrated in Fig. 1.

Then we assume that $h(I)$ and $f(I, \varphi, \xi)$ are holomorphic on $\mathscr{D}_{\rho,2\sigma}$. Furthermore, we also make a standard assumption on the *unperturbed* Hamiltonian.

Hypothesis 2.1 (Convexity). *There exists two constants $M \geq m > 0$ such that, for all $I \in \mathscr{G}_\rho$*

$$|\partial_I^2 h(I)v| \leq M|v|, \qquad |\langle \partial_I^2 h(I)v, v\rangle| \geq m|v|^2, \tag{3}$$

for all $v \in \mathbb{R}^n$.

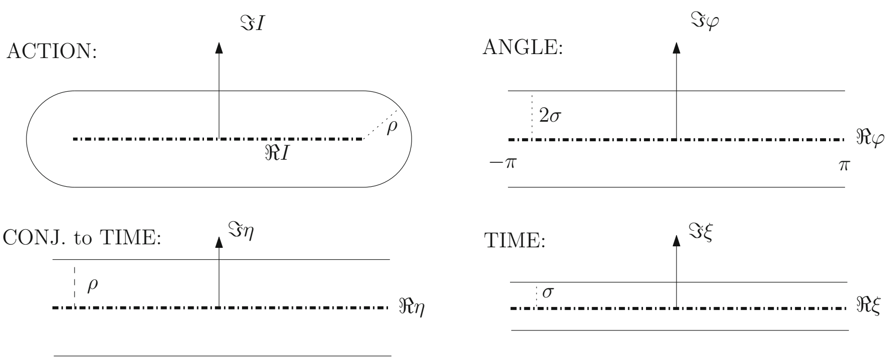

Fig. 1 The sets $\mathscr{D}$ (*dash dotted*) and $\mathscr{D}_{\rho,2\sigma}$ for $n = 1$

2.2 Statement of the Theorem

We can now state a version of the Nekhoroshev theorem for time-dependent Hamiltonian systems having a general time dependence. First, we define

$$\tilde{\mathscr{F}} := \sup_{\mathscr{D}_{\rho,2\sigma}} |f| \left(\frac{1+e^{-\frac{\sigma}{2}}}{1-e^{-\frac{\sigma}{2}}} \right)^n, \qquad \lambda_{\varepsilon,\mu} := \mu + e\tilde{\mathscr{F}}\varepsilon, \tag{4}$$

and we note that the set of parameters $\rho, \sigma, M, m, and \tilde{\mathscr{F}}$ are characteristic of a given Hamiltonian H. Now we state the theorem.

Theorem 2.2 (Aperiodic Nekhoroshev Theorem). *Assume the convexity hypothesis above. Then there exists constants Δ^* and $\mathscr{T}$, depending on $\rho, \sigma, M, m, \tilde{\mathscr{F}}$, and n, such that if ε and μ satisfy*

$$\lambda_{\varepsilon,\mu} < 1/(3^4 \Delta^*), \tag{5}$$

then orbits $(I(t), \varphi(t))$ of the Hamiltonian system (2) starting in $\mathscr{G} \times \mathbb{T}^n$ at t_0, satisfy

$$|I(t) - I(t_0)| < (\Delta^* \lambda_{\mu,\varepsilon})^{\frac{1}{4}} \rho, \qquad \mathit{for} \qquad |t - t_0| < \frac{\mathscr{T}}{\varepsilon} \exp\left[\left(\frac{1}{\Delta^* \lambda_{\mu,\varepsilon}} \right)^{\frac{1}{2n(n+1)}} \right].$$

We note that within the threshold (5), ε and μ are independent. We refer this as *unconditionally slow time dependence*.

2.2.1 Scheme of the Proof

The classical proof of Nekhoroshev is divided into two parts:

Analytic part (normal form lemma): For the analytic part, we construct an ε-close to the identity canonical change $\mathscr{C}_r$ casting H into the *normal form*:

$$H_N := H \circ \mathscr{C}_r = h(I) + \mu\eta + Z^{(r)} + \mathscr{R}^{(r+1)}. \tag{6}$$

We note the following:

- The result is local: it holds on sets called *non-resonance domains*.
- $\mathscr{C}_r$ is the composition of $r < \infty$ canonical transformations.

Geometric part (global result): This is an extremely clever contribution of Nekhoroshev [4] that shows how to cover the entire phase space $\mathscr{D}$ with non-resonance domains where the normal form lemma can be applied. See also [16, 17].

Remark 2.3. It is important to note that the Hamiltonian is not normalized with respect to the variables (ξ, η) (it is a "partial normal form"). Hence the same geometric result of the time-independent classical Nekhoroshev theorem applies.

The construction of the abovementioned normal form is classically achieved in two steps. First a formal perturbation scheme is developed, based on the Lie transform method that yields a normal form on non-resonance domains. Second, we consider the properties of the normal form on the appropriate domains and the "optimal" choice of parameters leading to *exponentially small estimates*.

We give a brief overview of the "formal scheme" for developing the normal form.

Formal Scheme

Step 1: Expand the perturbation as follows:

$$f(I, \varphi, \xi) = \sum_{k \in \mathbb{Z}^n} f_k(I, \xi) e^{ik \cdot \varphi}.$$

Given $K \in \mathbb{N}$ (to be determined afterward in an "optimal" way with respect to all of the parameters in a way that makes the remainder small), we write the expanded Hamiltonian in such a way it is decomposed into suitable "levels" (sets of Fourier harmonics in this case) in order to apply the Lie transform method :

$$H = h(I) + \mu\eta + H_1 + H_2 + \cdots, \qquad H_s := \varepsilon \sum_{(s-1)K \le |k| < sK} f_k(I, \xi) e^{ik \cdot \varphi}.$$

Step 2 (Lie transform method): The aim is to find $\chi^{(r)} := \{\chi_s\}_{s=1,\dots,r}$ such that $T_{\chi^{(r)}} H = H_N$, where

$$T_{\chi^{(r)}} := \sum_{s \ge 0} E_s, \qquad E_s := \begin{cases} \mathrm{id} & s = 0 \\ \dfrac{1}{s} \displaystyle\sum_{j=1}^{s} j \mathscr{L}_{\chi_j} E_{s-j} & s \ge 1 \end{cases}$$

and $\mathscr{L}_f g := \{f, g\} = \partial_\varphi f \partial_I g + \partial_\xi f \partial_\eta g - \partial_\varphi g \partial_I f - \partial_\xi g \partial_\eta f$ is the Lie derivative.

Step 3 Hierarchy of homological equations: Each χ_s is determined as a solution of a *homological equation*:

$$\mathscr{L}_h \chi_s + Z_s = \psi_s, \qquad s = 1, \dots, r,$$

where $Z^{(r)} = Z_1 + \cdots + Z_r$, where Z_s contains the same harmonics as H_s

$$\psi_s := \begin{cases} H_1 & s = 1 \\ H_s + \mu E_{s-1} \eta + \dfrac{1}{s} \displaystyle\sum_{j=1}^{s-1} j [\mathscr{L}_{\chi_j} H_{s-j} + E_{s-j} H_j] & 2 \le s \le r \end{cases}$$

We make the following remarks concerning the solution of the homological equations.

- The solution is found in the Fourier space, by expanding χ_s, Z_s, and ψ_s in a Fourier expansion in the angles.
- The term $\mu E_{s-1}\eta$ is the *extra-term* due to the aperiodic time dependence.

Convergence: Consider on $\mathscr{D}_{\rho,2\sigma}$ the *Fourier norm*

$$\|F\|_{(\rho,\sigma)} := \sum_{k\in\mathbb{Z}^n} \left(\sup_{\mathscr{D}_{\rho,\sigma}} |f_k| \right) e^{|k|\sigma},$$

with f_k Fourier coefficients of F and $|k| := |k_1| + \cdots + |k_n|$. The following lemma of Giorgilli describes the type of estimates that are required in order to establish the convergence of the formal scheme.

Lemma 1 (Giorgilli). *Suppose that there exist $h > 0$ and $\mathscr{F}, b \geq 0$ such that*

$$\|H_s\|_{(\rho,\sigma)} \leq h^{s-1}\mathscr{F}, \qquad \|\psi_s\|_{(1-d)(\rho,\sigma)} \leq \frac{b^{s-1}}{s}\mathscr{F} \tag{7}$$

for all $s \geq 1$ and for all $d \in (0, 1/4)$. Then, if $\mathscr{F}$ and b are sufficiently small, the operator $T_{\chi^{(r)}}$ (and its inverse $T^{-1}_{\chi^{(r)}}$) defines a canonical transformation on the domain $\mathscr{D}_{(1-d)(\rho,\sigma)}$.

After having bounded the above-described extra-term with the tools used in [15], one can see that the constraints imposed by condition (7) lead to more involved estimates with respect to the autonomous case. More precisely, the system of recurrence equations arising from (7) forbids straightforward bounds as in [15] but requires an ad hoc analysis, carried out in this case with the use of the generating function method. See [13] for the details.
The smallness condition of μ required by (5) turns out to be an essential ingredient in order to satisfy condition (7).

3 A KAM Theorem with Aperiodic Time Dependence

In this section we describe the setup and strategy for the proof of the theorem. This will provide us with the background and framework for providing a description of the theorem. Our approach follows closely the Lie transform approach to Kolmogorov's original version of the KAM theorem given in [18].

3.1 The Setup and Assumptions

The setup and assumptions are different than those for the Nekhoroshev theorem. We will comment more in this later on.

We consider a near-integrable, quadratic in P, time-dependent Hamiltonian expressed in the action-angle variables of the unperturbed system of the following form:

$$\mathscr{H}(P, Q, t) = \frac{1}{2}\langle \Gamma P, P\rangle + \varepsilon f(P, Q, t), \tag{8}$$

where:

- Γ is a real non-singular $n \times n$ matrix.
- ε is a small parameter.

We will focus on the preservation of a particular torus (in the spirit of the original Kolmogorov theorem). Therefore, we consider a particular $\hat{P}$, we translate the coordinates $(p, q) := (Q, P - \hat{P})$ so that they are "centered" on the torus of interest, and we transform the time-dependent Hamiltonian to an autonomous Hamiltonian, as above, by introducing a new conjugate pair of coordinates. The Hamiltonian that we obtain in this way has the form:

$$H(p, q, \eta, \xi) = \langle \omega, p\rangle + \frac{1}{2}\langle \Gamma p, p\rangle + \eta + \varepsilon f(p, q, \xi),$$

where:

- $\xi := t$ and $\eta \in \mathbb{R}$ is its conjugate momentum.
- $\omega := \Gamma \hat{P}$.
- $(p, q, \eta, \xi) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^+ =: \mathscr{D}$.

We next define a complex extension to the domain. We let $\sigma, \rho, and\zeta > 0$, and then $\mathscr{D}_{\rho,\sigma,\zeta} := \Delta_\rho \times \mathbb{T}^n_\sigma \times \mathscr{S}_\rho \times \mathscr{R}_\zeta$ is defined to be the *complex neighborhood* of $\mathscr{D}$ where

$$\Delta_\rho := \{p \in \mathbb{C}^n : |p| \leq \rho\}, \qquad \mathbb{T}^n_\sigma := \{q \in \mathbb{C}^n : |\Im q| \leq \sigma\},$$
$$\mathscr{S}_\rho := \{\eta \in \mathbb{C} : |\Im \eta| \leq \rho\}, \qquad \mathscr{R}_\zeta := \{\xi \in \mathbb{C} : \Re \xi \geq -\zeta;\ |\Im \xi| \leq \zeta\}.$$

We endow $\mathscr{D}$ with the Fourier norm defined as

$$\|g\|_{[\rho,\sigma;\zeta]} := \sum_{k\in\mathbb{Z}^n} \sup_{p\in\mathscr{D}_{\rho,\sigma,\zeta}} |g_k(p, \xi)| e^{|k|\sigma}.$$

We make the following assumptions.

Hypothesis 3.1 (I). *There exists* $m \in (0, 1)$ *such that, for all* $v \in \mathbb{C}^n$

$$|\Gamma v| \leq m^{-1}|v|.$$

Hypothesis 3.2 (II, Slow Decay). *The perturbation is an holomorphic function on* $\mathscr{D}$ *satisfying*

$$\|f(q,p,\xi)\|_{[\rho,\sigma;\zeta]} \leq M_f e^{-a|\xi|},$$

for some $M_f > 0$ *and* $a \in (0,1)$.

3.2 Statement of the Theorem

Now we can state the theorem

Theorem 3.3 (Aperiodic Kolmogorov Theorem). *Assume hypotheses I and II and suppose that* $\hat{P}$ *is such that* ω *is a* $\gamma - \tau$ *Diophantine vector. Then, for all* $a \in (0,1)$*, there exists* $\varepsilon_a > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_a]$*, it is possible to find a canonical,* ε*-close to the identity, analytic change of variables* $(q,p,\xi,\eta) = \mathscr{K}(q^{(\infty)}, p^{(\infty)}, \xi, \eta^{(\infty)})$*,* $\mathscr{K} : \mathscr{D}_* \to \mathscr{D}$ *with* $\mathscr{D}_* \subset \mathscr{D}$*, transforming the Hamiltonian (1) into the* Kolmogorov normal form

$$H_\infty(q^{(\infty)}, p^{(\infty)}, \xi, \eta^{(\infty)}) = \langle \omega, p^{(\infty)} \rangle + \eta^{(\infty)} + \mathscr{Q}(q^{(\infty)}, p^{(\infty)}, \xi; \varepsilon),$$

where $\mathscr{Q}$ *is a homogeneous polynomial of degree* 2 *in* p.

Remark 3.4. We note that no restrictions are imposed on a, which implies that the decay of the time dependence can be arbitrarily slow. On the other hand, the threshold is of the form $\varepsilon_a \leq Ca^3$, with C (very small!) constant.

3.2.1 Scheme of the Proof

The proof follows the classical iterative approach following the Lie transform approach of [18] and it is carried out along the lines of [19]. In particular, it is organized as follows:

Step I (Induction basis) We rewrite the Hamiltonian H in the following form:

$$H_j = \langle \omega, p \rangle + \eta + A^{(j)}(q,\xi) + \langle B^{(j)}(q,\xi), p \rangle + \frac{1}{2}\langle C^{(j)}(q,\xi)p, p \rangle, \tag{9}$$

where $j = 0$ denotes zeroth step in the induction process, and for this reason, we set $H_0 := H$.

Step II (Perturbative scheme, formal part) For all j, a generating function χ_j is chosen in such a way the action of $\exp(\mathscr{L}_{\chi_j})$ on H_j removes $A^{(j)}$ and $B^{(j)}$. χ_j is such that $H_{j+1} := \exp(\mathscr{L}_{\chi_j})H_j$ has the same form (9).

Step III (Perturbative scheme, quantitative part) We show that the "unwanted terms" $A^{(j)}$ and $B^{(j)}$ get "smaller and smaller" as j increases. More precisely

$$\max\left\{\left\|A^{(j)}\right\|_{[\sigma_j;\zeta_j]}, \left\|B^{(j)}\right\|_{[\sigma_j;\zeta_j]}\right\} \leq \epsilon_j e^{-a|\xi|},$$

with ϵ_j (quadratically) infinitesimal as $j \to \infty$, while $\sigma_j \geq \sigma_* > 0$. The desired canonical transformation is obtained by setting

$$\mathscr{K} := \lim_{j \to \infty} \exp(\mathscr{L}_{\chi_j}) \circ \exp(\mathscr{L}_{\chi_{j-1}}) \circ \ldots \circ \exp(\mathscr{L}_{\chi_0}).$$

The composition $H \circ \mathscr{K}$ produces the desired *Kolmogorov normal form*.

Time-dependent homological equation: The equation for the determination of χ_j at each stage of the normalization algorithm is of the form

$$\partial_\xi \varphi + \omega \cdot \partial_q \varphi = \psi, \tag{10}$$

with $\psi = \psi(q, \xi)$ given.

Equation (10) is the novelty of our analysis, and it reflects a remarkable conceptual difference with the normalization algorithm used for the Nekhoroshev theorem. Basically, the latter uses the fact that the number of normalization steps is finite: the contribution of the aperiodic term is controlled only over a finite timespan and the constant Δ^* of formula (5) tends to zero as $r \to \infty$. The situation is substantially different in the Kolmogorov scheme, in which the number of normalization steps is infinite, and the only way to control the effect of the time is to annihilate it at each stage of the algorithm with Eq. (10). The properties of its solution are described in the following lemma.

Lemma 2. *Let $\delta \in [0, 1)$ and suppose that ψ satisfies*

$$\|\psi\|_{[(1-\delta)\sigma;\zeta]} \leq K e^{-a|\xi|},$$

*(*exponential decay*). Then for all $d \in (0, 1-\delta)$ and for all ζ such that $2|\omega|\zeta \leq d\sigma$, the solution of (10) exists and satisfies*

$$\|\varphi\|_{[(1-\delta-d)\sigma;\zeta]} \leq \frac{KS}{a(d\sigma)^{2\tau}} e^{-a|\xi|}, \qquad S \geq 0. \tag{11}$$

Remark 3.5. Finally, we note that the exponential rate of the decay is not necessary and is used for simplicity. However a decay hypothesis is essential in order to ensure the existence of the integrals appearing in the bounds which lead to (11).

4 Summary and Outlook

The aim of this paper was to give an overview of the Nekhoroshev and Kolmogorov stability-type results for integrable Hamiltonian systems subject to aperiodic time-dependent perturbations, obtained in the papers [13] and [14]. These are recently added *tesseræ* to the rich mosaic of the Stability Theory of Hamiltonian Systems, one of the several fields in which V.I. Arnold made so many fundamental contributions.

The motivation for generalizing the classical Nekhoroshev and KAM theorems to include explicit, but arbitrary, time dependence arises from many applications. Most notably, applications of the dynamical systems approach to the study of Lagrangian transport in fluid mechanics, as described in [6]. Hopefully, the results in this paper will serve as motivation to analyze other possibilities for the generalization of these fundamental results in Hamiltonian perturbation theory and, thus, extend both the mathematical framework and the range of applications to which these results can be applied.

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Iterative Methods for the Elastography Inverse Problem of Locating Tumors

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Abstract The primary objective of this work is to present a rigorous treatment of various iterative methods for solving the elastography inverse problem of identifying cancerous tumors. From a mathematical standpoint, this inverse problem requires the identification of a variable parameter in a system of partial differential equations. We pose the nonlinear inverse problem as an optimization problem by using an output least-squares (OLS) and a modified output least-squares (MOLS) formulation. The optimality conditions then lead to a variational inequality problem which is solved using various gradient, extragradient, and proximal-point methods. Previously, only a few of these methods have been implemented, and there is currently no understanding of their relative efficiency and effectiveness. We present a thorough numerical comparison of the 15 iterative solvers which emerge from a variational inequality formulation.

1 Introduction

Given the domain Ω as a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ as its boundary, the following system models the response of an isotropic elastic body to the known body forces and boundary traction:

$$-\nabla \cdot \sigma = f \quad \text{in } \Omega, \tag{1a}$$

$$\sigma = 2\mu\epsilon(u) + \lambda \operatorname{div} u I, \tag{1b}$$

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$$u = g \text{ on } \Gamma_1, \tag{1c}$$

$$\sigma n = h \text{ on } \Gamma_2. \tag{1d}$$

In (1), the vector-valued function $u = u(x)$ is the displacement of the elastic body, f is the applied body force, n is the unit outward normal, and $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ is the linearized strain tensor. The resulting stress tensor σ in the stress-strain law (1b) is obtained under the condition that the elastic body is isotropic, and the displacement is sufficiently small so that a linear relationship remains valid. Here μ and λ are the Lamé parameters which quantify the elastic properties of the object.

In this work, our primary objective is to develop a computational framework for the elastography inverse problem of locating soft inclusions in an incompressible object, for example, cancerous tumor in the human body. From a mathematical standpoint, this inverse problem seeks μ from a measurement of the displacement vector u under the assumption that the parameter λ is very large. The key idea behind the elastography inverse problem is that the stiffness of soft tissue can vary significantly based on its molecular makeup and varying macroscopic/microscopic structure, and such changes in stiffness are related to changes in tissue health. In other words, the elastography inverse problem mathematically mimics the practice of palpation by making use of the differing elastic properties of healthy and unhealthy tissue to identify tumors. In most of the existing literature on elastography inverse problem, the human body is modeled as an incompressible elastic object. Although this assumption simplifies the identification process as there is only one parameter μ to identify, it significantly complicates the computational process as the classical finite element methods become quite ineffective due to the so-called locking effect. To describe the difficulties associated with near incompressibility, we first introduce some notation. The dot product of two tensors A_1 and A_2 is denoted by $A_1 \cdot A_2$. Given a sufficiently smooth domain $\Omega \subset \mathbb{R}^2$, the L_2-norm of a tensor-valued function $A = A(x)$ is provided by

$$\|A\|_{L^2}^2 = \|A\|_{L^2(\Omega)}^2 = \int_\Omega A \cdot A = \int_\Omega \left(A_{11}^2 + A_{12}^2 + A_{21}^2 + A_{22}^2\right).$$

On the other hand, for a vector-valued function $u(x) = (u_1(x), u_2(x))^{\mathrm{T}}$, the L_2-norm and the H^1-norm are given by:

$$\|u\|_{L_2}^2 = \|u\|_{L_2(\Omega)}^2 = \int_\Omega \left(u_1^2 + u_2^2\right),$$

$$\|u\|_{H^1}^2 = \|u\|_{H^1(\Omega)}^2 = \|u\|_{L_2}^2 + \|\nabla u\|_{L_2}^2.$$

For the time being, in (1), we set $g = 0$. For this case, the space of test functions, denoted by V, is given by:

$$V = \{\bar{v} \in H^1(\Omega) \times H^1(\Omega) : \ \bar{v} = 0 \text{ on } \Gamma_1\}.$$

By employing the Green's identity and by using (1c) and (1d), we get the following variational form of (1): Find $\bar{u} \in V$ such that

$$\int_{\Omega} 2\mu\epsilon(\bar{u})\cdot\epsilon(\bar{v})+\int_{\Omega}\lambda(\operatorname{div}\bar{u})(\operatorname{div}\bar{v}) = \int_{\Omega} f\bar{v}+\int_{\Gamma_2}\bar{v}h, \quad \text{for every } \bar{v}\in V. \tag{2}$$

For $T : V \times V \to \mathbb{R}$ defined by:

$$T(\bar{u},\bar{v}) = \int_{\Omega} 2\mu\epsilon(\bar{u})\cdot\epsilon(\bar{v}) + \int_{\Omega}\lambda(\operatorname{div}\bar{u})(\operatorname{div}\bar{v}),$$

it can be shown that if μ and $\mu+\lambda$ are bounded away from zero, then there are two positive constants $c_1 > 0$ and $c_2 > 0$ with $c_1 \leq \mu$ and $c_2 \geq \lambda+\mu$ such that

$$c_1\|\bar{v}\|_V^2 \leq T(\bar{v},\bar{v}) \leq c_2\|\bar{v}\|_V^2, \quad \text{for every } \bar{v}\in V.$$

Since $\lambda \gg \mu$, the ratio $c_3 = c_2/c_1$ is large. Given that the constant c_3 determines the error estimates (as defined by Céa's lemma), it follows that the error estimates could easily outweigh the actual approximation error. This situation is well known and has been dubbed the "locking effect."

A wide range of approaches have been given to overcome the locking effect with one of the most popular being the use of mixed finite elements, an approach which we adopt in this work. For this we introduce a "pressure" term $p \in Q = L^2(\Omega)$ by $p = \lambda \operatorname{div}\bar{u}$, which results in the following weak form:

$$\int_{\Omega}(\operatorname{div}\bar{u})q - \int_{\Omega}\frac{1}{\lambda}pq = 0, \quad \text{for every } q \in Q. \tag{3}$$

Using $p = \lambda \operatorname{div}\bar{u}$, the weak form (2) then seeks $\bar{u} \in V$ such that

$$\int_{\Omega} 2\mu\epsilon(\bar{u})\cdot\epsilon(\bar{v}) + \int_{\Omega} p(\operatorname{div}\bar{v}) = \int_{\Omega} f\bar{v} + \int_{\Gamma_2}\bar{v}h, \quad \text{for every } \bar{v}\in V. \tag{4}$$

Thus, we have moved from finding $\bar{u} \in V$ fulfilling (2) to finding $(\bar{u}, p) \in V \times Q$ satisfying the mixed variational problems (3) and (4).

2 An Optimization Framework

Let V and Q be Hilbert spaces and let B be a Banach space. Let A be a nonempty, closed, and convex subset of B. Let $a : B \times V \times V \to \mathbb{R}$ be a trilinear map which is symmetric with respect to the second and the third arguments. Let $b : V \times Q \to \mathbb{R}$ be a bilinear map, let $c : Q \times Q \to \mathbb{R}$ be a symmetric bilinear map, and let $m : V \to \mathbb{R}$ be a linear and continuous map. We assume that there are strictly positive constants

$\kappa_1, \kappa_2, \varsigma_1, \varsigma_2$, and κ_0 such that for every $\mu \in A$, $p, q \in Q$, and $\bar{u}, \bar{v} \in V$, we have

$$a(\mu, \bar{v}, \bar{v}) \geq \kappa_1 \|\bar{v}\|^2, \tag{5a}$$

$$a(\mu, \bar{u}, \bar{v}) \leq \kappa_2 \|\mu\| \|\bar{u}\| \|\bar{v}\|, \tag{5b}$$

$$c(q, q) \geq \varsigma_1 \|q\|^2, \tag{5c}$$

$$c(p, q) \leq \varsigma_2 \|p\| \|q\|, \tag{5d}$$

$$b(\bar{v}, q) \leq \kappa_0 \|\bar{v}\| \|q\|. \tag{5e}$$

We consider the following mixed variational problem: Given $\mu \in A$, find $(\bar{u}, p) \in W := V \times Q$ such that

$$a(\mu, \bar{u}, \bar{v}) + b(\bar{v}, p) = m(\bar{v}), \quad \text{for every } \bar{v} \in V \tag{6a}$$

$$b(\bar{u}, q) - c(p, q) = 0, \quad \text{for every } q \in Q. \tag{6b}$$

Given all the data, the direct problem in the context of (6) is to find $(\bar{u}, p)$. However, our interest is in the inverse problem of finding a parameter $\mu \in A$ that makes (6) true for a measurement $(\bar{z}, \hat{z})$ of $(\bar{u}, p)$.

Clearly, Eqs. (3) and (4) which are connected to the elasticity imaging inverse problem of identifying a variable parameter μ in the system of incompressible linear elasticity can be recovered by setting:

$$a(\mu, \bar{u}, \bar{v}) = \int_\Omega 2\mu \epsilon(\bar{u}) \cdot \epsilon(\bar{v})$$

$$b(\bar{u}, q) = \int_\Omega (\operatorname{div} \bar{u})\, q$$

$$c(p, q) = \int_\Omega \frac{1}{\lambda} pq$$

$$m(\bar{v}) = \int_\Omega f\bar{v} + \int_{\Gamma_2} \bar{v}h.$$

In this work, we will focus on two optimization formulations. The first one is the output least-squares (OLS) functional $J_{\text{OLS}} : A \to \mathbb{R}$ defined by

$$J_{\text{OLS}}(\mu) := \frac{1}{2}\|u(\mu) - z\|_W^2 = \frac{1}{2}\|\bar{u}(\mu) - \bar{z}\|_V^2 + \frac{1}{2}\|p(\mu) - \hat{z}\|_Q^2, \tag{7}$$

where $z = (\bar{z}, \hat{z})$ is the measured data and $u(\mu) = (\bar{u}, p)$ solves (6) for μ.

Due to the known ill posedness of inverse problems, some kind of regularization is necessary for developing a stable computational framework. Therefore, instead of (7), we will use its regularized analogue and consider the following regularized optimization problem: Find $\mu \in A$ by solving

$$\min_{\mu \in A} J_\kappa(\mu) = \frac{1}{2}\|u(\mu) - z\|_W^2 = \frac{1}{2}\|\bar{u}(\mu) - \bar{z}\|_V^2 + \frac{1}{2}\|p(\mu) - \hat{z}\|_Q^2 + \kappa R(\mu), \tag{8}$$

where, given a Hilbert space H, $R : H \to \mathbb{R}$ is a regularizer, $\kappa > 0$ is a regularization parameter, $u(\ell) := (\bar{u}(\ell), p(\ell))$ is the unique solution of (6) that corresponds to the coefficient ℓ, and $z = (\bar{z}, \hat{z})$ is the measured data.

The above optimization problem is a constrained optimization problem where the implicit constraint is the mixed variational problem and the explicit constraint is the set of admissible coefficients A. For nonlinear inverse problems, the output least-squares functional is nonconvex in general and hence can only be used to investigate local minimizers.

Besides (8) we will also consider the following regularized problem

$$\min_{\mu \in A} \widehat{J}_\kappa(\mu) = \widehat{J}(\mu) + \kappa R(\mu), \tag{9}$$

where $\widehat{J} : A \to \mathbb{R}$ is a modified output least squares defined by

$$\widehat{J}(\mu) := \frac{1}{2}a(\mu, \bar{u}(\mu) - \bar{z}, \bar{u}(\mu) - \bar{z}) + b(\bar{u}(\mu) - \bar{z}, p(\mu) - \hat{z}) - \frac{1}{2}c(p(\mu) - \hat{z}, p(\mu) - \hat{z}).$$

Theoretical results dealing with the above and some other optimization formulations for this and simpler inverse problems are given in [10, 16, 17, 19–23, 32].

3 Discrete Formulae for the OLS and the MOLS

In this subsection, we collect some basic information concerning the discretization of the OLS and MOLS functionals defined in (8) and (9). More details can be found in [21]. As usual, we assume that $\mathcal{T}_h$ is a triangulation on Ω, L_h is the space of all piecewise continuous polynomials of degree d_μ relative to $\mathcal{T}_h$, $\bar{U}_h$ is the space of all piecewise continuous polynomials of degree d_u relative to $\mathcal{T}_h$, and Q_h is the space of all piecewise continuous polynomials of degree d_q relative to $\mathcal{T}_h$.

To represent the discrete mixed variational problem in a computable form, we proceed as follows. We represent bases for L_h, $\bar{U}_h$, and Q_h by $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$, $\{\psi_1, \psi_2, \ldots, \psi_n\}$, and $\{\chi_1, \chi_2, \ldots, \chi_k\}$, respectively. The space L_h is then isomorphic to $\mathbb{R}^m$, and for any $\mu \in L_h$, we define $\boldsymbol{\mu} \in \mathbb{R}^m$ by $\boldsymbol{\mu}_i = \mu(x_i)$, $i = 1, 2, \ldots, m$, where the nodal basis $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ corresponds to the nodes $\{x_1, x_2, \ldots, x_m\}$. Conversely, each $\boldsymbol{\mu} \in \mathbb{R}^m$ corresponds to $\mu \in L_h$ defined by $\mu = \sum_{i=1}^m \boldsymbol{\mu}_i \varphi_i$. Analogously, $\bar{u} \in \bar{U}_h$ will correspond to $\bar{U} \in \mathbb{R}^n$, where $\bar{U}_i = u(y_i)$, $i = 1, 2, \ldots, n$, and $\bar{u} = \sum_{i=1}^n \bar{U}_i \psi_i$, where $y_1, y_2, \ldots, y_n$ are the nodes of the mesh defining $\bar{U}_h$. Finally, $q \in Q_h$ will correspond to $Q \in \mathbb{R}^k$, where $Q_i = q(z_i)$, $i = 1, 2, \ldots, k$, and $q = \sum_{i=1}^k Q_i \chi_i$, where $z_1, z_2, \ldots, z_k$ are the nodes of the mesh defining Q_h.

We next define $S : \mathbb{R}^m \to \mathbb{R}^{n+k}$ to be the finite element solution operator that assigns to each coefficient $\mu_h \in A_h$, the unique approximate solution $u_h = (\bar{u}_h, p_h) \in \bar{U}_h \times Q_h$. Then, $S(\boldsymbol{\mu}) = U$, where U is defined by

$$K(\boldsymbol{\mu})U = F, \tag{10}$$

where $K(\boldsymbol{\mu}) \in \mathbb{R}^{(n+k)\times(n+k)}$ is the stiffness matrix and $F \in \mathbb{R}^{n+k}$ is the load vector. We will also use the mass matrices defined by:

$$M^1_{ij} = \langle \psi_i, \psi_j \rangle = \int \psi_i \psi_j$$

$$M^2_{ij} = \langle \chi_i, \chi_j \rangle = \int \chi_i \chi_j.$$

With the above preparation, we obtain the following discrete versions:

$$J(\boldsymbol{\mu}) = \frac{1}{2}(\bar{U}(\boldsymbol{\mu}) - \bar{Z})^T M^1 (\bar{U}(\boldsymbol{\mu}) - \bar{Z}) + \frac{1}{2}(P(\boldsymbol{\mu}) - \widehat{Z})^T M^2 (P(\boldsymbol{\mu}) - \widehat{Z}), \tag{11}$$

$$\widehat{J}(\boldsymbol{\mu}) = \frac{1}{2}(\bar{U}(\boldsymbol{\mu}) - \bar{Z})^{\mathrm{T}} \widehat{K}(\boldsymbol{\mu})(\bar{U}(\boldsymbol{\mu}) - \bar{Z}) + (\bar{U}(\boldsymbol{\mu}) - \bar{Z})^{\mathrm{T}} B^{\mathrm{T}} (P(\boldsymbol{\mu}) - \widehat{Z})$$
$$- \frac{1}{2}(P(\boldsymbol{\mu}) - \widehat{Z})^{\mathrm{T}} C (P(\boldsymbol{\mu}) - \widehat{Z}). \tag{12}$$

4 Gradient, Extragradient, and Proximal-Point Methods

Although we perform numerical tests for some gradient-based methods, the main emphasis of this work is on numerical testing of the so-called extragradient methods and proximal-point methods. We note that extragradient methods, originally proposed to solve minimization and saddle point problems, have received a great deal of attention in recent years, particularly in the context of variational inequalities (see [1, 4, 7–9, 11–13, 26, 28, 30, 35, 37–41, 43–46, 50, 51, 53, 55–57, 59–61]).

Korpelevich [39] introduced the method in the context of smooth optimization and saddle point problems. Earlier developments of these methods were of theoretical nature; however, the recent developments focus not only on the convergence analysis but also on testing their practical usefulness on numerical examples. Therefore, we anticipate that the novel application of these methods to solve inverse problems for partial differential equations will give a plethora of test problems for examining their efficiency and effectiveness.

In this work, we employ variants of projected gradient methods, extragradient methods, and proximal-point methods to solve the inverse problem of parameter identification by first posing it as a variational inequality. We implement numerous algorithms and present a thorough comparison of the projected gradient method, fast projected gradient method [fast iterative shrinkage thresholding (FISTA)], scaled projected gradient method, and several extragradient methods including the Marcotte variants, the He-Goldstein-type method, the projection-contraction methods proposed by Solodov and Tseng, the hyperplane method developed by A. Iusem, and various proximal-point methods. During the last two decades, numerous researchers have focused on these methods, but to the best of our knowledge, this is the first instance where these methods have been thoroughly compared in the context of an applied problem.

Subsequently, we implement and test the numerical performance of the following iterative schemes for solving the elastography inverse problem of tumor identification:

1. Gradient projection using Armijo line search
2. Fast gradient projection using Armijo line search (FISTA)
3. Scaled gradient projection using Barzilai–Borwein rules
4. Khobotov extragradient method using Marcotte rules (three variants)
5. Solodov–Tseng projection-contraction method (two variants)
6. Improved He-Goldstein-type extragradient method
7. Two-step extragradient method
8. Hyperplane extragradient method
9. Hager–Zhang proximal-point method (four variants)

4.1 Basic Gradient-Based Methods

We will solve the elastography inverse problem by formulating the regularized OLS and MOLS functional, henceforth denoted simply by J, as a variational inequality of finding $\boldsymbol{\mu}^* \in K$ such that

$$\langle \nabla J(\boldsymbol{\mu}^*), \boldsymbol{\mu} - \boldsymbol{\mu}^* \rangle \geq 0, \quad \forall\ \boldsymbol{\mu} \in K,$$

where $K \subset \mathbb{R}^m$ is the set of admissible coefficients. In all numerical experiments, we take the set K to be box constrained.

The above variational inequality has a unique solution if ∇J is strongly monotone, that is,

$$\langle \nabla J(\boldsymbol{\mu}_1) - \nabla J(\boldsymbol{\mu}_2), \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 \rangle \geq c\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2, \quad \forall\ \boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in K, \quad c > 0,$$

and Lipschitz continuous

$$\|\nabla J(\boldsymbol{\mu}_1) - \nabla J(\boldsymbol{\mu}_2)\| \leq L\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|, \quad \forall \boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in K, \quad L > 0.$$

As usual, we convert the above variational inequality into a fixed point problem of finding $\boldsymbol{\mu}^* \in K$ such that

$$\boldsymbol{\mu}^* = P_K(\boldsymbol{\mu}^* - \alpha \nabla J(\boldsymbol{\mu}^*)), \quad \alpha > 0,$$

where P_K is the metric projection onto K.

Note that for the variational inequality emerging from the MOLS formulation, the map J is strongly monotone due to the monotonicity of the gradient of the convex MOLS functional and strongly monotone regularizer. On the other hand, for the variational inequality emerging from the OLS formulation, it is necessary to choose a large enough regularization parameter to ensure that the gradient of the regularized OLS is strongly monotone. In fact, it can be shown that the gradient of the OLS functional J_{OLS} satisfies the following inequality:

$$\langle \nabla J_{\text{OLS}}(\boldsymbol{\mu}_1) - \nabla J_{\text{OLS}}(\boldsymbol{\mu}_2), \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 \rangle \geq -c_1 \|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|^2, \quad \forall\, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in K,\ c_1 > 0.$$

Gradient Projection Method

The projected gradient algorithm admits the form: Given $\boldsymbol{\mu}^k \in K$, find $\boldsymbol{\mu}^{k+1} \in K$ by the following scheme

$$\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \alpha \nabla J(\boldsymbol{\mu}^k)).$$

The strong convergence can be established by assuming that

$$\alpha \in \left(0, \frac{2c}{L^2}\right),$$

where c and L are the modulus of strong monotonicity and Lipschitz continuity, respectively.

Note that we do not have information about c and L and hence it is important to use a method to determine the steplength α.

We use Armijo line search to backtrack until the following condition is satisfied:

$$J(\boldsymbol{\mu}^{k+1}) - J(\boldsymbol{\mu}^k) \leq -\alpha\lambda \|\nabla J(\boldsymbol{\mu}^k)\|^2, \quad \text{for } \lambda \in (0, 1).$$

An interesting article devoted to a general projected gradient method for smooth convex optimization problem is given by Dunn [18].

Fast Iterative Shrinkage-Thresholding Algorithm

In this subsection, we describe the "fast iterative shrinkage-thresholding algorithm," which was proposed by Beck and Teboulle for minimizing the sum of two convex, lower-semicontinuous, and proper functions (defined in a Euclidean or Hilbert space), such that one is differentiable with Lipschitz gradient and the proximity operator of the second is easy to compute. This method constructs a sequence of iterates for which the objective is controlled, up to a (nearly optimal) constant, by the inverse of the square of the iteration number. In recent years, this method has received a great deal of attention due to its simplicity and fast convergence properties (see also [6, 15, 58]). We note the convergence of the iterates themselves has only been given recently by Chambolle and Dossal [14].

In the context of linear inverse problems, Beck and Teboulle [3], following the work of Nesterov [47], presented the following fast version of the projected gradient method, which is an optimal first-order method. Here L again is the Lipschitz constant of ∇J:

Algorithm: FISTA

Choose $\bar{\boldsymbol{\mu}}^1 = \boldsymbol{\mu}^0$, $t_1 = 1$, and N, the maximum number of iterations.
For $k = 0, 1, 2, \ldots, N$, perform the following:
  Step 1: $\boldsymbol{\mu}^k = P_K(\bar{\boldsymbol{\mu}}^k - \frac{1}{L}\nabla J(\bar{\boldsymbol{\mu}}^k))$
  Step 2: $t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}$
  Step 3: $\bar{\boldsymbol{\mu}}^{k+1} = \boldsymbol{\mu}^k + \frac{t_k - 1}{t_{k+1}}(\boldsymbol{\mu}^k - \boldsymbol{\mu}^{k-1})$
End

Scaled Gradient Projection

The projected gradient suffers from slow convergence, and many authors have considered a scaled analogue to accelerate the convergence.

The scaled gradient projection (SGP) iteration has the following form

$$\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \alpha_k D_k \nabla J(\boldsymbol{\mu}^k)),$$

where D_k is a scaling matrix. It is a common practice to take the scaling matrix D_k as the main diagonal of the Hessian of $J(\boldsymbol{\mu}^k)$, with all other entries equal to zero (see Harker [27]).

Following Benvenuto et al. [5], we choose α_k using the rules suggested by Barzilai and Borwein [2]. That is, for

$$\begin{aligned} r^{k-1} &= \boldsymbol{\mu}^k - \boldsymbol{\mu}^{k-1} \\ z^{k-1} &= \nabla J(\boldsymbol{\mu}^k) - \nabla J(\boldsymbol{\mu}^{k-1}), \end{aligned}$$

Algorithm: SGP

Choose $\boldsymbol{\mu}^0 \in K$, $\beta, \theta \in (0,1)$, $0 < \alpha_{min} < \alpha_{max}$, $M > 0$
For $k = 0, 1, 2, \ldots,$ perform the following steps:
Step 1: Choose $\alpha_k \in [\alpha_{min}, \alpha_{max}]$ and D_k
Step 2: Projection $Y^k = P_K(\boldsymbol{\mu}^k - \alpha D_k \nabla J(\boldsymbol{\mu}^k))$
If $Y^k = \boldsymbol{\mu}^k$ Stop
Step 3: Descent direction: $d^k = Y^k - \boldsymbol{\mu}^k$
Step 4: Set $\lambda_k = 1$ and $f_{max} = \max_{0 \le j \le \min(k, M-1)} J(\boldsymbol{\mu}^{k-j})$
Step 5: Backtracking loop:
If $J(\boldsymbol{\mu}^k + \lambda_k d^k) \le f_{max} + \beta \lambda_k \nabla J(\boldsymbol{\mu}^k)^T d^k$
Go to Step 6
Else
Set $\lambda_k = \theta \lambda_k$ and go to Step 5
EndIf
Step 6: $\boldsymbol{\mu}^{k+1} = \boldsymbol{\mu}^k + \lambda_k d^k$
End

we compute:

$$\alpha_k^{(1)} = \frac{r^{(k-1)^T} D_k^{-1} D_k^{-1} r^{(k-1)}}{r^{(k-1)^T} D_k^{-1} z(k-1)}$$

$$\alpha_k^{(2)} = \frac{r^{(k-1)^T} D_k z^{(k-1)}}{z^{(k-1)^T} D_k^2 z^{(k-1)}}.$$

Determining α_k: Take a prefixed nonnegative integer M_α and $\tau_1 \in (0,1)$:
If $\alpha_k^{(2)} / \alpha_k^{(1)} \le \tau_k$ then
$\alpha_k = \min \left(\alpha_j^{(2)}, j = \max\left(1, k - M_\alpha\right), \ldots, k \right)$
$\tau_{k+1} = 0.9 \tau_k$
Else
$\alpha_k = \alpha_k^{(1)}$
$\tau_{k+1} = 1.1 \tau_k$
EndIf.

4.2 Extragradient Methods

Korpelevich [39] introduced the extragradient method in the context of saddle point problem studied through a variational inequality formulation. Instead of one projection, her scheme required two projections per iteration. At its purest, the extragradient methods take the following form:

$$\bar{\boldsymbol{\mu}}^k = P_K(\boldsymbol{\mu}^k - \alpha \nabla J(\boldsymbol{\mu}^k))$$

$$\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \alpha \nabla J(\bar{\boldsymbol{\mu}}^k)).$$

Convergence can be proved under the conditions that the solution set is nonempty, ∇J is monotone and Lipshitz (with constant L) and $\alpha \in (0, 1/L)$. In the context of variational inequalities, as opposed to the (single) projection methods, these methods do not require the strong monotonicity of the underlying map. Evidently, the drawback is that when computing the projection on to constraint set is expensive, these methods are, in turn, also quite expensive. In the context of inverse problems, extragradient methods are attractive since the strong monotonicity for the OLS/MOLS can be attained through regularization. These methods then demand relaxed conditions on the regularization parameters. On the other hand, since the constraint sets for the considered inverse problems are typically box constraints, computing the projection is relatively inexpensive and thus not computationally cost prohibitive.

Clearly, when L is unknown, we may have difficulties choosing an appropriate α. Intuitively as with the gradient projection method, if α is too small, then the algorithm will converge slowly and if α is too big, then it may not converge at all.

Khobotov Extragradient Method

In the following, we will consider extragradient methods where α is now an adaptive steplength. We implement the adaptive steplength first introduced in [36] to remove the constraint that ∇J must be Lipshitz continuous. The adaptive algorithm is of the form:

$$\bar{\boldsymbol{\mu}}^k = P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k))$$
$$\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\bar{\boldsymbol{\mu}}^k)).$$

Intuitively, we get better convergence when α gets smaller between iterations, however, it is obvious that we must also control how the sequence of $\{\alpha_k\}$ shrinks.

We use the following reduction rule for α_k given in [36]:

$$\alpha_k > \beta \frac{\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k}{\nabla J(\boldsymbol{\mu}^k) - \nabla J(\bar{\boldsymbol{\mu}}^k)},$$

where $\beta \in (0, 1)$. Results from [53] and [36] show that β is usually 0.8 or 0.9, an observation that is also supported by our results.

The Khobotov extragradient method has the following general form:

Algorithm: Khobotov Extragradient

Choose $\alpha_0 > 0$, $\boldsymbol{\mu}^0$, and $\beta \in (0, 1)$
While $\|\boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k\| > \text{TOL}$
Step 1: Compute $\nabla J(\boldsymbol{\mu}^k)$
Step 2: Compute $\bar{\boldsymbol{\mu}}^k = P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k))$
Step 3: Compute $\nabla J(\bar{\boldsymbol{\mu}}^k)$
If $\nabla J(\bar{\boldsymbol{\mu}}^k) = 0$, Stop
Step 4: If $\alpha_k > \beta \frac{\|\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k\|}{\|\nabla J(\boldsymbol{\mu}^k) - \nabla J(\bar{\boldsymbol{\mu}}^k)\|}$
then reduce α_k and go to Step 5
Step 5: Compute $\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\bar{\boldsymbol{\mu}}^k))$
End.

Marcotte Choices for Steplength

Khobotov's algorithm gives one workable method for reducing α_k but does not rule out other, perhaps more desirable, methods. Marcotte developed a new rule for reducing α_k along with closely related variants [42, 53]. The first Marcotte rule is based on the sequence $\alpha_k = \frac{1}{2}\alpha_{k-1}$ and forces α_k to satisfy Step 4 of Khobotov's algorithm by additionally taking:

$$\alpha_k = \min\left\{\frac{\alpha_{k-1}}{2}, \frac{\|\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k\|}{\sqrt{2}\|\nabla J(\boldsymbol{\mu}^k) - \nabla J(\bar{\boldsymbol{\mu}}^k)\|}\right\}.$$

Both the Khobotov and Marcotte reduction rules can still run the risk of choosing an initial α small enough that α_k is never reduced, resulting in slow convergence. Ideally, α_k should then have the ability to increase if α_{k-1} is smaller than some optimal value. This leads to a modified version of Marcotte's rule where an initial α is selected using the rule

$$\alpha = \alpha_{k-1} + \gamma\left(\beta \frac{\|\boldsymbol{\mu}^{k-1} - \bar{\boldsymbol{\mu}}^{k-1}\|}{\|\nabla J(\boldsymbol{\mu}^{k-1}) - \nabla J(\bar{\boldsymbol{\mu}}^{k-1})\|} - \alpha_{k-1}\right)$$

where $\gamma \in (0, 1)$.

The reduction rule in Step 4 of Khobotov's algorithm is then replaced with

$$\alpha_k = \max\left\{\hat{\alpha}, \min\left\{\xi \cdot \alpha, \beta \frac{\|\boldsymbol{\mu}^{k-1} - \bar{\boldsymbol{\mu}}^{k-1}\|}{\|\nabla J(\boldsymbol{\mu}^{k-1}) - \nabla J(\bar{\boldsymbol{\mu}}^{k-1})\|}\right\}\right\}$$

where $\xi \in (0, 1)$ and $\hat{\alpha}$ is some lower limit for α_k (generally taken as no less than 10^{-4}).

Scaled Extragradient Method

Now we consider a projection-contraction-type extragradient method presented by Solodov and Tseng [52]. It involves a scaling matrix M to accelerate convergence. The main steps read:

$$\begin{aligned} \bar{\boldsymbol{\mu}}^k &= P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k)) \\ \boldsymbol{\mu}^{k+1} &= \boldsymbol{\mu}^k - \gamma M^{-1}(T_\alpha(\boldsymbol{\mu}^k) - T_\alpha(P_K(\bar{\boldsymbol{\mu}}^k)) \end{aligned}$$

where $\gamma \in \mathbb{R}^+$ and $T_\alpha = (I - \alpha \nabla J)$. Here, I is the identity matrix, and α is chosen such that T_α is strongly monotone.

Additional discussion of the scaling matrix is given in [53]; however, in both [53] and [52], test problems take M equal to the identity matrix. In our numerical experiments, we consider the scaling matrix as both the identity matrix and the diagonal of the Hessian of J.

Algorithm: Solodov–Tseng

Choose $\boldsymbol{\mu}^0, \alpha_{-1}, \theta \in (0,2), \rho \in (0,1), \beta \in (0,1), M \in \mathbb{R}^{m\times m}$
Initialize: $\bar{\boldsymbol{\mu}}^0 = 0, k = 0, rx = \text{ones}(m, 1)$
While $\|rx\| > \text{TOL}$
  Step 1: if $\|rx\| < \text{TOL}$ then Stop
    else $\alpha = \alpha_{k-1}$, $flag = 0$
  Step 2: if $\nabla J(\boldsymbol{\mu}^k) = 0$ then Stop
  Step 3: While $\alpha(\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k)^T(\nabla J(\boldsymbol{\mu}^k) - \nabla J(\bar{\boldsymbol{\mu}}^k)) > (1-\rho)\|\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k\|^2$ or $flag = 0$
    If $flag \neq 0$ Then $\alpha = \alpha_{k-1}\beta$ endif
      update $\bar{\boldsymbol{\mu}}^k = P_K(\boldsymbol{\mu}^k - \alpha \nabla J(\boldsymbol{\mu}^k))$, compute $\nabla J(\bar{\boldsymbol{\mu}}^k)$
      $flag = flag + 1$
    endwhile
  Step 4: update $\alpha_k = \alpha$
  Step 5: compute $\gamma = \theta\rho\|\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k\|^2 / \|M^{1/2}(\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k) + \alpha_k \nabla J(\bar{\boldsymbol{\mu}}^k))\|^2$
  Step 6: compute $\boldsymbol{\mu}^{k+1} = \boldsymbol{\mu}^k - \gamma M^{-1}(\boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k) + \alpha_k \nabla J(\bar{\boldsymbol{\mu}}^k))$
  Step 7: $rx = \boldsymbol{\mu}^{k+1} - A^k$, $k = k+1$ go to Step 3
End

The Solodov–Tseng method suggests a more general form for the advanced extragradient methods:

$$\begin{aligned} \bar{\boldsymbol{\mu}}^k &= P_\mu(\boldsymbol{\mu}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k)) \\ \boldsymbol{\mu}^{k+1} &= P_\mu(\boldsymbol{\mu}^k - \eta_k \nabla J(\bar{\boldsymbol{\mu}}^k)), \end{aligned}$$

where α_k and η_k are chosen using different rules.

Goldstein-Type Methods

The classical Goldstein projection method presented in [40] is of the form:

$$\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \beta_k \nabla J(\boldsymbol{\mu}^k)),$$

where $\beta_k > 0$.

The He-Goldstein method, an extragradient method that requires Lipschitz continuity and strong monotonicity of ∇J, is of the form:

$$\bar{\boldsymbol{\mu}}^k = P_K(\nabla J(\boldsymbol{\mu}^k) - \beta_k \boldsymbol{\mu}^k)$$
$$\boldsymbol{\mu}^{k+1} = \boldsymbol{\mu}^k - \frac{1}{\beta_k}\{\nabla J(\boldsymbol{\mu}^k) - \bar{\boldsymbol{\mu}}^k\}.$$

It can also be expressed:

$$r(\boldsymbol{\mu}^k, \beta_k) = \frac{1}{\beta_k}\{\nabla J(\boldsymbol{\mu}^k) - P_K[\nabla J(\boldsymbol{\mu}^k) - \beta_k \boldsymbol{\mu}^k]\}$$
$$\boldsymbol{\mu}^{k+1} = \boldsymbol{\mu}^k - r(\boldsymbol{\mu}^k, \beta_k).$$

We implement the more general version above, presented in [40], as it allows us to control the second projection (i.e., choosing η_k).

Algorithm: Improved He-Goldstein

Initialize: choose $\beta_U > \beta_L > \frac{1}{(4\tau)}, \gamma \in (0, 2), \epsilon > 0, \boldsymbol{\mu}^0, \beta_0 \in [\beta_L, \beta_U], k = 0$
Step 1: Compute:
  $r(\boldsymbol{\mu}^k, \beta_k) = \frac{1}{\beta_k}\{\nabla J(\boldsymbol{\mu}^k) - P_K[\nabla J(\boldsymbol{\mu}^k) - \beta_k \boldsymbol{\mu}^k]\}$
  If $\|r(\boldsymbol{\mu}^k, \beta_k)\| \leq \epsilon$ then Stop
Step 2: $\boldsymbol{\mu}^{k+1} = \boldsymbol{\mu}^k - \gamma \alpha_k r(\boldsymbol{\mu}^k, \beta_k)$ where $\alpha_k := 1 - \frac{1}{4\beta_k \tau}$
Step 3: Update β_k
  $\omega_k = \frac{\|\nabla J(\boldsymbol{\mu}^{k+1} - \nabla J(\boldsymbol{\mu}^k)\|}{\beta_k \|\boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k\|}$
  If $\omega_k < \frac{1}{2}$ Then $\beta_{k+1} = \max\{\beta_L, \frac{1}{2}\beta_k\}$
  Else if $\omega_k > \frac{3}{2}$ Then $\beta_{k+1} = \min\{\beta_U, \frac{6}{5}\beta_k\}$
Step 4: $k = k + 1$, go to Step 1

Two-Step Extragradient Method

Zykina and Melenchuk in [61] considered a three-step projection method which they called a two-step extragradient method. Numerical experiments with mixed variational problem for bilinear functions given in [61] shows that the convergence of this method is faster when compared to the standard extragradient method.

Additional results related to the two-step extragradient method were recently given by Zykina and Melenchuk [62]. The adaptive version of the algorithm is of the form:

$$\bar{\boldsymbol{\mu}}^k = P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k))$$
$$\widetilde{\boldsymbol{\mu}}^k = P_K(\bar{\boldsymbol{\mu}}^k - \eta_k \nabla J(\bar{\boldsymbol{\mu}}^k))$$
$$\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \xi_k \nabla J(\widetilde{\boldsymbol{\mu}}^k)),$$

where α_k, η_k, and ξ_k are suitable step lengths.

Hyperplane Extragradient Method

Within the context of the more general extragradient method, we choose η_k using the following rule from Iusem [29] (see also [31, 53]):

$$\eta_k = \frac{\langle \nabla J(\bar{\boldsymbol{\mu}}^k), \boldsymbol{\mu}^k - \bar{\boldsymbol{\mu}}^k \rangle}{\|\nabla J(\bar{\boldsymbol{\mu}}^k)\|^2}$$

The idea in this method is that the hyperplane of all solutions $\boldsymbol{\mu}$ such that

$$\langle \nabla J(\bar{\boldsymbol{\mu}}^k), \bar{\boldsymbol{\mu}}^k - \boldsymbol{\mu} \rangle = 0,$$

separates all the solutions onto one side of the hyperplane. Using the variational inequality, we know then which side the solutions fall into since:

$$\langle \nabla J(\boldsymbol{\mu}), \bar{\boldsymbol{\mu}}^k - \boldsymbol{\mu} \rangle \geq 0.$$

Consequently, if ∇J is monotone, then we also have

$$\langle \nabla J(\bar{\boldsymbol{\mu}}^k), \bar{\boldsymbol{\mu}}^k - \boldsymbol{\mu} \rangle \geq 0$$

and thus if

$$\langle \nabla J(\bar{\boldsymbol{\mu}}^k), \bar{\boldsymbol{\mu}}^k - \boldsymbol{\mu}^k \rangle < 0,$$

then the solution is on the other side of the hyperplane.

4.3 Proximal-Point Methods

In this section, we now examine several proximal-like optimization algorithms and their application to the optimization frameworks developed in the previous

Algorithm: Hyperplane

Choose: $\boldsymbol{\mu}^0, \epsilon, \hat{\alpha}, \tilde{\alpha}$
Initialize: $k = 0, rx = \text{ones}(m, 1)$
While $\|rx\| > \text{TOL}$
 Step 1: Choose $\tilde{\alpha}_k$ using a finite bracketing procedure
 Step 2: Compute $K^k = P_K(\boldsymbol{\mu}^k - \tilde{\alpha}_k \nabla J(\boldsymbol{\mu}^k))$ and $\nabla J(K^k)$
 Step 3: If $\nabla J(K^k) = 0$ then Stop
 Step 4: If $\|\nabla J(\tilde{\boldsymbol{\mu}}^k) - \nabla J(\boldsymbol{\mu}^k)\| \leq \frac{\|K^k - \boldsymbol{\mu}^k\|^2}{2\tilde{\alpha}_k^2 \|\nabla J(\boldsymbol{\mu}^k)\|}$
  Then $\bar{\boldsymbol{\mu}}^k = K^k$
  Else find $\alpha_k \in (0, \tilde{\alpha}_k)$ such that
  $\epsilon \frac{\|K^k - \boldsymbol{\mu}^k\|^2}{2\tilde{\alpha}_k^2 \|\nabla J(\boldsymbol{\mu}^k)\|} \leq \|\nabla J(P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k))) - \nabla J(\boldsymbol{\mu}^k)\| \leq \frac{\|K^k - \boldsymbol{\mu}^k\|^2}{2\tilde{\alpha}_k^2 \|\nabla J(\boldsymbol{\mu}^k)\|}$
 Step 5: Compute $\bar{\boldsymbol{\mu}}^k = P_K(\boldsymbol{\mu}^k - \alpha_k \nabla J(\boldsymbol{\mu}^k))$
 Step 6: If $\nabla J(\bar{\boldsymbol{\mu}}^k) = 0$ then Stop
 Step 7: Compute η_k
 Step 8: Compute $\boldsymbol{\mu}^{k+1} = P_K(\boldsymbol{\mu}^k - \eta_k \nabla J(\bar{\boldsymbol{\mu}}^k))$
 Step 9: $rx = \boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k, k = k + 1$; go to Step 3;
End

sections. Specifically, we will look at several variants of the adaptive proximal-point algorithms developed by Hager and Zhang [24].

To begin our analysis, we first review the classical proximal-point algorithm. Recall that the solution of the discretized elastography inverse problem is now reposed as a constrained optimization:

$$\min_{\boldsymbol{\mu} \in K} J(\boldsymbol{\mu}) \tag{13}$$

where K is a closed and convex set of feasible parameters and J is either the MOLS or OLS objective functionals.

Now, consider the functional

$$\mathcal{J}_P(\boldsymbol{\mu}) = J(\boldsymbol{\mu}) + \frac{1}{2\lambda^k}\|\boldsymbol{\mu} - \boldsymbol{\mu}^k\|^2, \tag{14}$$

where λ_k is a positive number and $\boldsymbol{\mu}^k \in K$. We note that $\mathcal{J}_P(\boldsymbol{\mu})$ is strictly convex in the MOLS case since J is convex and the term $\frac{1}{2\lambda^k}\|\boldsymbol{\mu} - \boldsymbol{\mu}^k\|_2^2$, known as the *proximal regularization term*, is strictly convex. Thus, we have an optimization subproblem

$$\min_{\boldsymbol{\mu} \in K} \mathcal{J}_P(\boldsymbol{\mu}) \tag{15}$$

with a unique solution and whose optimality conditions give the following variational inequality:

$$\langle \nabla \mathcal{J}_P(\boldsymbol{\mu}^*), \boldsymbol{\mu} - \boldsymbol{\mu}^* \rangle \geq 0 \quad \text{for all } \boldsymbol{\mu} \in K. \tag{16}$$

The classical proximal-point algorithm generates a sequence $\{\boldsymbol{\mu}^k\}$ of solutions to the subproblem (17) with the iteration:

$$\boldsymbol{\mu}^{k+1} = \arg\min_{\mu \in K} \left\{ J(\boldsymbol{\mu}) + \frac{1}{2\lambda^k} \|\boldsymbol{\mu} - \boldsymbol{\mu}^k\|^2 \right\} \tag{17}$$

where $\{\lambda^k\}$ is a sequence of positive real numbers. The iterates $\{\boldsymbol{\mu}^k\}$ can be shown to converge to a solution of (13) under a certain set of assumptions (see [33]).

We note that for convex problems using a Tikhonov-like regularization method (necessary to overcome the general ill posedness of an inverse problem like the one at hand), algorithms like the gradient and extragradient methods are known to converge to a minimal-norm solution. Comparatively for proximal-point methods, no such characterization is possible. However, this also makes their application to inverse problem optimization frameworks appealing in their eliminating the need for the selection of an "ideal" regularization parameter.

In the remaining subsections, we will consider several improvements and variations on the classical proximal method mainly based on the method of Hager and Zhang [24] and applied to the MOLS and OLS frameworks for solving the parameter identification problem. For more details on the development of proximal methods, we refer the reader to [25, 34, 48, 49, 54] and their cited references.

Hager and Zhang's Proximal-Point Method

Hager and Zhang [24] introduce two criteria between subsequent iterates of (17):

$$\mathcal{J}_P(\boldsymbol{\mu}^{k+1}) \leq J(\boldsymbol{\mu}^k) \tag{18}$$

$$\|\nabla \mathcal{J}_P(\boldsymbol{\mu}^{k+1})\| \leq \theta^k \|\nabla J(\boldsymbol{\mu}^k)\|. \tag{19}$$

The proximal regularization parameter is then taken as

$$\theta^k = \tau \|\nabla J(\boldsymbol{\mu}^k)\|^\eta,$$

where $\eta \in [0, 2)$ and $\tau > 0$ are constants. As they show in [24], the iterates converge quadratically to the solution set of (15). This gives rise to the following algorithm:

The minimization of the subproblem in Step 1 is achieved using an unconstrained conjugate-gradient trust-region method.

Hager and Zhang's Proximal-Point Method Using φ-Divergence

We can now consider a variant of the Hager–Zhang proximal-point method by replacing the notion of distance between the current point and its proximal "neighbor" with that of φ-divergences (see [33] for a detailed treatment). We consider Φ,

Algorithm: Hager–Zhang Proximal-Point

Initialization Step: Choose an initial guess $\boldsymbol{\mu}^0$, initialize τ and η, and take $k = 0$.
Let $\gamma = 1$.
Step 1: Let $\theta^k = \tau \|\nabla J(\boldsymbol{\mu}^k)\|^{\eta}$.
Find $\boldsymbol{\mu}^{k+1}$ satisfying $\|\nabla \mathcal{J}_P(\boldsymbol{\mu}^{k+1})\| \le \theta^k \gamma \|\nabla J(\boldsymbol{\mu}^k)\|$
Step 2:
If $\boldsymbol{\mu}^{k+1}$ satisfies $\mathcal{J}_P(\boldsymbol{\mu}^{k+1}) \le J(\boldsymbol{\mu}^k)$
Go to Step 3.
Else,
Set $\gamma = 0.1\gamma$ and go to Step 1.
End.
Step 3: Let $\boldsymbol{\mu}^k = \boldsymbol{\mu}^{k+1}$.
Step 4: Set $k = k + 1$ and go to Step 1.

the class of closed, proper, and convex functions $\varphi : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ for which the following properties hold:

1. φ is twice continuously differentiable on $\text{int}(\text{domain}(\varphi)) = (0, +\infty)$.
2. φ is strictly convex on its domain.
3. $\lim\limits_{t \to 0+} \varphi'(t) = -\infty$.
4. $\varphi(1) = \varphi'(1) = 0$ and $\varphi''(1) > 0$.
5. There exists $\nu \in \left(\frac{1}{2}\varphi''(1), \varphi''(1)\right)$ such that

$$\left(1 - \frac{1}{t}\right)\left(\varphi''(1) + \nu(t-1)\right) \le \varphi'(t) \le \varphi''(1)(t-1), \quad \forall t > 0.$$

Using the above definition, for some $\varphi \in \Phi$, the φ-divergence between any two $x, y \in \mathbb{R}^n_+$ is given by:

$$d_{\varphi}(x, y) = \sum_{i=1}^{n} y_i \varphi\left(\frac{x_i}{y_i}\right). \tag{20}$$

As can be easily verified, several φ functions are given by:

$$\varphi_1(t) = t \log t - t + 1$$
$$\varphi_2(t) = -\log t + t - 1$$
$$\varphi_3(t) = \left(\sqrt{t} - 1\right)^2.$$

By way of example, taking φ_1 above then gives the φ-divergence

$$d_{\varphi_1}(x, y) = \sum_{i=1}^{n} x_i \log \frac{x_i}{y_i} + y_i - x_i \tag{21}$$

The φ-divergence can now be used to replace the proximal regularization term in (14) giving

$$\mathcal{J}_{\varphi_1}(\boldsymbol{\mu}) = J(\boldsymbol{\mu}) + \theta^k d_{\varphi_1}(\boldsymbol{\mu}, \boldsymbol{\mu}^k). \tag{22}$$

and subsequently the proximal-like iteration

$$\boldsymbol{\mu}^{k+1} = \arg\min_{\boldsymbol{\mu}\in K} \mathcal{J}_{\varphi_1}(\boldsymbol{\mu}) \tag{23}$$

Substituting $\mathcal{J}_{\varphi_1}(\boldsymbol{\mu})$ for $\mathcal{J}_P(\boldsymbol{\mu})$ into the Hager–Zhang algorithm yields the φ-divergence proximal-like algorithm.

Hager and Zhang's Proximal-Point Method Using Bregman Functions

Similarly, we can also replace the notion of distance in the proximal regularization term using another strictly convex function:

$$D_\psi(x, y) := \psi(x) - \psi(y) - \nabla\psi(y)^T(x - y), \tag{24}$$

where ψ is known as a *Bregman function*.

A Bregman function is defined as follows. Let $S \subset \mathbb{R}^n$ be an open and convex set and a let $\psi : \bar{S} \to \mathbb{R}$ be a given mapping. ψ is a Bregman function if it meets the following criteria:

1. ψ is strictly convex and continuous on $\bar{S}$.
2. ψ is continuously differentiable in S.
3. The partial level set

   $$L_\alpha = \{y \in \bar{S} | D_\psi(x, y) \leq \alpha\}$$

   is bounded for every $x \in \bar{S}$.
4. If $\{y^k\} \subset S$ converges to x, then $\lim_{k\to\infty} D_\psi(x, y^k) = 0$.

It is again easy to verify that the following are all examples of Bregman functions:

$$\psi_1(x) = \frac{1}{2}\|x\|^2 \quad \text{with } S = \mathbb{R}^n,$$

$$\psi_2(x) = \sum_{i=1}^n x_i \log x_i - x_i \quad \text{with } S = \mathbb{R}^n_+,$$

$$\psi_3(x) = -\sum_{i=1}^n \log x_i \quad \text{with } S = \mathbb{R}^n_+.$$

The distance function related to ψ_1 is then defined by:

$$D_{\psi_1}(x, y) := \frac{1}{2}\|x - y\|^2,$$

which, we note, corresponds with the classical proximal regularization approach from (14). Taking ψ_2 we then have

$$D_{\psi_2}(x, y) := \sum_{i=1}^{n} x_i \log \frac{x_i}{y_i} + y_i - x_i$$

which, we likewise note, corresponds to φ-divergence proximal regularization given by (21).

Finally, taking ψ_3 yields

$$D_{\psi_3}(x, y) := \sum_{i=1}^{n} \frac{x_i}{y_i} - \log \frac{x_i}{y_i} - 1$$

with the proximal regularized functional

$$\mathcal{J}_{\psi_3}(\boldsymbol{\mu}) = J(\boldsymbol{\mu}) + \theta^k D_{\psi_3}(\boldsymbol{\mu}, \boldsymbol{\mu}^k) \tag{25}$$

and the corresponding iteration

$$\boldsymbol{\mu}^{k+1} = \arg\min_{\boldsymbol{\mu} \in K} \mathcal{J}_{\psi_3}(\boldsymbol{\mu}). \tag{26}$$

Proximal-Like Methods Using Modified φ-Divergence

All of the proximal methods outlined so far rely on the solution of an optimization subproblem such as one found in Step 1 of the Hager–Zhang algorithm. Second-order methods, like Newton's method, applied to the subproblem provide fast convergence but require calculation of the Hessian of both the objective function and the proximal regularization term.

Problems of conditioning in the Hessian of the proximal regularization term (see [33]) can be overcome with the introduction of a modification to the notion of φ-divergence:

$$\tilde{d}_{\varphi}(x, y) := \sum_{i-1}^{n} y_i^2 \varphi\left(\frac{x_i}{y_i}\right). \tag{27}$$

The Hessian of the proximal term can then be calculated using φ'':

$$\nabla_{xx}^2 \tilde{d}_\varphi(x, y) = \sum_{i=1}^{n} \varphi''\left(\frac{x_i}{y_i}\right) e_i e_i^T \tag{28}$$

where e_i is the ith unit basis vector of $\mathbb{R}^n$.

We can again apply this to the Hager–Zhang algorithm in the context of the elastogrpahy inverse problem by taking

$$\mathcal{J}_{\tilde{\varphi}_1}(\boldsymbol{\mu}) := J(\boldsymbol{\mu}) + \tilde{d}_{\varphi_1}(\boldsymbol{\mu}, \boldsymbol{\mu}^k) \tag{29}$$

where now, however, we can apply full second-order methods to solve the sub problem

$$\boldsymbol{\mu}^{k+1} = \arg\min_{\mu \in K} \mathcal{J}_{\tilde{\varphi}_1}(\boldsymbol{\mu}). \tag{30}$$

5 Performance Analysis

In this section, we present a numerical comparison of the following methods:

1. Gradient projection using Armijo line search
2. Fast gradient projection using Armijo line search
3. Scaled gradient projection using Barzilai–Borwein rules
4. Khobotov extragradient method using Marcotte rules (three versions)
5. Solodov–Tseng (projection-contraction) method
6. Improved He-Goldstein-type extragradient method
7. Hyperplane extragradient method
8. Hager–Zhang proximal-point methods using:
    - Classical proximal regularization
    - φ-divergences
    - Bregman functions
    - Second-order modified φ-divergence

We considered two representative examples of elastography inverse problems for the recovery of a variable μ on a two-dimensional isotropic domain $\Omega = (0, 1) \times (0, 1)$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_L \cup \Gamma_R$. Γ_D, where the Dirichlet boundary conditions hold, was taken as the union of the top and bottom boundary of the square domain. Γ_L and Γ_R, where the Neumann conditions hold, were taken as the left and right boundaries, respectively.

In both examples, the inverse problem was solved on a 40×40 quadrangular mesh with 1681 quadrangles and 5041 total degrees of freedom (3360 degrees of freedom for the solution u and 1681 for the factitious pressure term p).

Due to the near incompressibility of the tumor identification inverse problem, λ was kept as a large constant: $\lambda = 10^6$.

The two numerical examples used in the experiments are defined as follows.

Example 1.

$$\mu(x,y) = 2 + \frac{1}{4}\cos(3\pi xy)\sin(\pi x), \quad g(x,y) = \begin{bmatrix} y^2(1+x) \\ 1+x^2+x^2y \end{bmatrix} \text{ on } \Gamma_D$$

$$h_L(x,y) = \begin{bmatrix} -(\lambda+4)y^2 \\ -4y \end{bmatrix} \text{ on } \Gamma_L, \quad h_R(x,y) = \begin{bmatrix} 4y^2 + \lambda(1+y^2) \\ 4+12y \end{bmatrix} \text{ on } \Gamma_R.$$

Example 2.

$$\mu(x,y) = 1 + x^2 y, \quad g(x,y) = \begin{bmatrix} y^2(1+x) \\ 1+x^2+x^2y \end{bmatrix} \text{ on } \Gamma_D$$

$$h_L(x,y) = \begin{bmatrix} -(\lambda+2)y^2 \\ -2y \end{bmatrix} \text{ on } \Gamma_L, \quad h_R(x,y) = \begin{bmatrix} (2+\lambda)y^2 + 2y^3 + \lambda \\ 6y^2 + 8y + 2 \end{bmatrix} \text{ on } \Gamma_R.$$

5.1 Implementation Remarks

- The gradient and Hessian of the MOLS functional were computed using an adjoint stiffness method (see [32]), while the gradient and Hessian of the OLS functional were computed with an adjoint method and a hybrid adjoint/classical method, respectively (see [10]).
- The Hessian of the OLS functional for both examples was not positive definite in practice (due to the functional's nonconvexity), and this made it unsuitable for direct use as a scaling matrix in the scaled projected gradient or Solodov–Tseng methods due to the appearance of negative values. The absolute value of the diagonal of the Hessian was substituted for scaling.
- A maximum of 50,000 iterations was taken for all algorithms. Iteration counts with this value in Tables 1 or 2 indicate a failure of the algorithm to converge.
- For the proximal-point algorithms, the number of iterations includes both the proximal algorithm iterations and the iterations necessary for solving the optimization subproblems.
- The parameter constraints were taken as either non-active box constraints in the case of the extragradient methods and were ignored in the case of the proximal-point algorithms.

6 Discussion and Concluding Remarks

Here we interpret and discuss the results of the numerical experiments summarized in the figures and in Tables 1 and 2 from the previous section. We would like to emphasize that our remarks on the efficacy of the given methods are based

Table 1 Performance results for MOLS using the gradient, extragradient, and proximal-point methods. The bold values represent the best method of its class

	Example 1			Example 2		
Method	J evals	CPU(s)	Iter.	J evals	CPU (s)	Iter.
FISTA	196,564	40,073.086	50,000	83,137	16,458.189	23,516
Proj. grad.	181,124	36,202.334	50,000	373,449	71,332.026	50,000
Scaled PG	**1081**	**290.895**	**431**	**2274**	**641.224**	**1023**
He-Goldstein	50,000	10,667.586	50,000	50,000	9607.489	50,000
Hyperplane	67,793	13,612.401	21,567	31,420	6013.724	7889
Khobotov	**51,008**	**10,210.173**	**25,502**	**21,032**	**4,180.868**	**10,513**
Korpelevich	100,000	19,416.231	50,000	100,000	19,133.270	50,000
Marcotte 1	100,000	20,294.617	50,000	41,058	8892.693	20,524
Marcotte 2	57,770	11,953.809	21,721	22,370	4202.067	8394
Solodov–Tseng (I)	150,000	63,307.693	50,000	150,000	41,805.362	50,000
Solodov–Tseng (H)	53,762	11,655.049	17,918	29,301	5458.782	9764
Two-step	63,240	12,668.029	20,998	24,630	4751.831	8092
HZ (classical)	185,945	31,973.062	8174	155,188	27,796.726	7110
HZ φ-divergence	161,305	31,256.567	7159	159,162	30,915.996	7019
HZ Bregman	168,501	30,853.027	7152	161,462	31,408.740	7111
HZ Modified φ	**262**	**21,487.738**	**484**	**122**	**10,214.647**	**209**

Table 2 Performance results for OLS using the gradient, extragradient, and proximal-point methods. The bold values represent the best method of its class

	Example 1			Example 2		
Method	J evals	CPU (s)	Iter.	J evals	CPU (s)	Iter.
FISTA	**65,429**	**15,131.008**	**1865**	**34,834**	**8027.285**	**1033**
Proj. grad.	100,000	22,068.355	50,000	100,000	22,824.003	50,000
Scaled PG	100,206	23,073.380	50,000	100,203	21,804.025	50,000
He-Goldstein	100,000	11,552.631	50,000	50,000	11,797.437	50,000
Hyperplane	150,000	34,871.848	50,000	142,185	33,254.525	47,393
Khobotov	40,648	9105.919	20,320	20,954	5647.784	10,474
Korpelevich	100,000	21,876.791	50,000	100,000	22,310.550	50,000
Marcotte 1	100,000	19,163.717	50,000	94,866	22,912.561	47,430
Marcotte 2	**41,640**	**8708.808**	**17,380**	**21,114**	**5,113.853**	**8497**
Solodov–Tseng (I)	150,000	60,630.438	50,000	150,000	63,143.018	50,000
Solodov–Tseng (H)	150,000	25,351.700	50,000	128,164	30,212.637	42,717
Two-step	47,655	10,082.619	15,837	23,145	5,225.002	7660
HZ (classical)	15,897	2915.064	842	8670	1956.415	440
HZ φ-Divergence	17,003	3195.805	917	10,508	2594.147	506
HZ Bregman	18,057	3929.007	886	8630	1735.847	425
HZ Modified φ	451	37,591.002	863	406	39,453.602	777

on (and their scope limited to) our experiments in connection with the parameter identification problem. We begin our discussion with the classical extragradient method and its direct extensions: Korpelevich, Khobotov, and the two related Marcotte variants. Tables 1 and 2 both indicate that the Khobotov and Marcotte variants generally improve over the performance of the Korpelevich method for both the OLS and MOLS approaches. In the MOLS case, the Khobotov algorithm, although the simplest, not only outperforms the more sophisticated Marcotte variants but actually performs best out of all other extragradient methods considered. In particular, we note that even when the Khobotov algorithm results in more overall iterations, it results in fewer objective evaluations and therefore total CPU time. For the OLS case, the second variant of the Marcotte method bests Khobotov's, but with only a marginal improvement. These results suggest that given the second Marcotte variant converges in fewer iterations, the computational overhead of its more sophisticated reduction of α_k can potentially outweigh most of the benefit of its accelerated convergence.

Of the remaining extragradient methods, only the two-step method performs reasonably on par with the more successful methods like Khobotov's. The "extra" extragradient step taken by this method grants faster convergence but also more objective evaluations and subsequent overhead. The hyperplane method performs well for the MOLS functional, but fairs poorly when applied to the OLS approach. The distinct performance difference can perhaps be explained by the existence of instances when the algorithm gets "stuck" near the hyperplane, and this may be exacerbated by the nature of the OLS functional. In particular, this occurs when $\bar{\boldsymbol{\mu}}^k$ is near the hyperplane, making the difference between $\boldsymbol{\mu}^k$ and $\boldsymbol{\mu}^{k+1}$ small. It is possible then that $\alpha's$ or $\eta's$ stops being adaptive and resulting in slow convergence.

The Solodov–Tseng (I) method, scaled using the identity matrix (effectively unscaled), performs poorly in both examples for both the OLS and MOLS approaches. Scaling using the Hessian of the objective function [i.e., Solodov–Tseng (H)] shows markedly different behavior when applied to either OLS or MOLS. In the case of MOLS, the scaling brings performance in line with the Khobotov and Marcotte methods and indicates how proper scaling can significantly enhance the convergence of the algorithm. However, because of the nonpositive definiteness of the OLS functional, scaling matrices derived from the Hessian contained negative values. This made their direct use unfeasible in the context of Solodov–Tseng due to the evaluation of the term $M^{\frac{1}{2}}$. The results presented in Table 2 instead used the absolute value of the Hessian's diagonal to accommodate the algorithm but also indicate that this scaling did not provide any performance benefit over the unscaled algorithm.

For the simpler gradient-based methods, again there were significant discrepancies in performance between the OLS and MOLS approaches. Considering only MOLS, the scaled projected gradient method performed remarkably well, outstripping all other similar methods. Again, compared to the projected gradient method, this shows the dramatic effect of proper scaling on algorithm convergence. Just as in the case of the Solodov–Tseng method, the benefit of scaling disappears when applied to the OLS functional for similar reasons. For OLS, the FISTA algorithm performs best of the gradient methods, but the performance is only commensurate with the Marcotte or Khobotov extragradient method, and no remarkable performance gains are seen as with MOLS.

To gain more insight into the overall performance of these methods beyond what is summarized in Tables 1 and 2, the history of the objective function values (Figs. 1 and 2) was plotted on a logarithmic scale at each iteration for a selection of algorithms. These figures show the smoother or more "direct" convergence of the extragradient methods when compared with the characteristic zigzagging instability

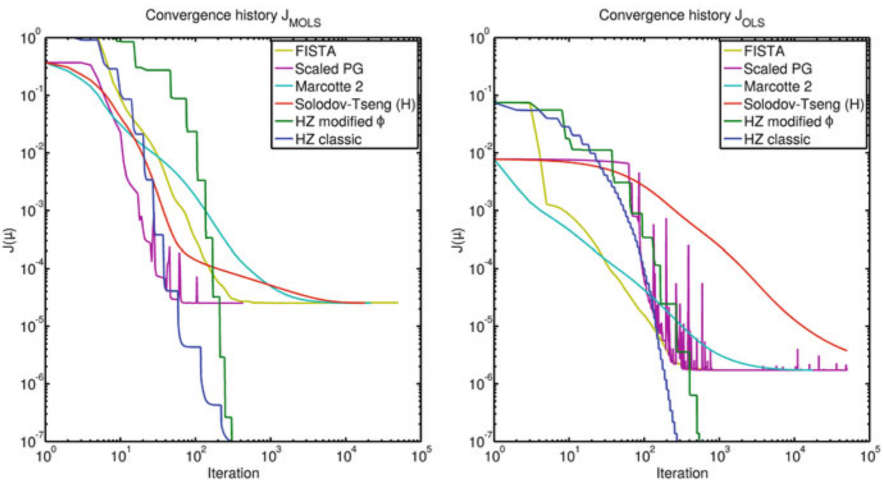

Fig. 1 Example 1: Convergence history comparison for the MOLS and OLS functionals

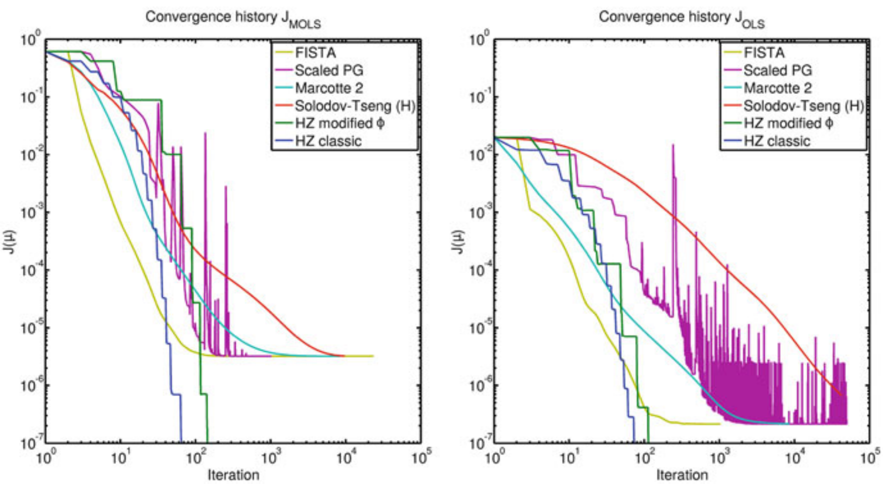

Fig. 2 Example 2: Convergence history comparison for MOLS and OLS functionals

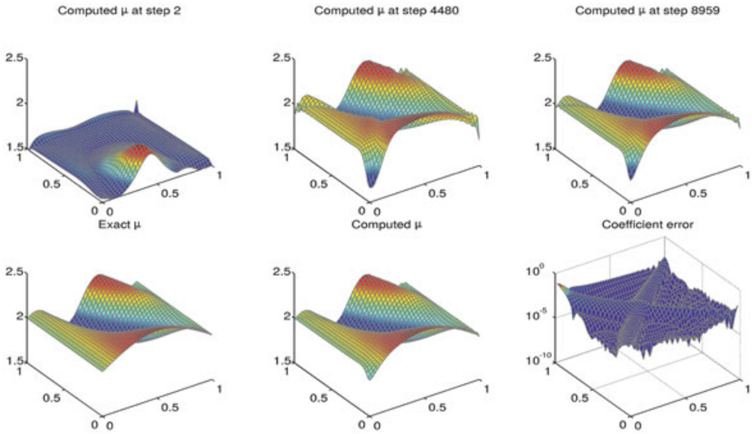

Fig. 3 Example 1: Solodov–Tseng (H) Method (MOLS)

of the projected gradient methods. This behavioral difference can be of significance when considering using the norm of the gradient as a practical algorithmic stopping criteria.

For the proximal-point methods considered, the modified φ-divergence with the MOLS approach performed starkly better than the other proximal methods due to its use of second-order methods for solving the optimization subproblem. However, for the OLS method, this benefit is largely erased due to the computational cost of computing the Hessian of the OLS functional using a hybrid adjoint method [10]. There appears no clear advantage among the proximal methods concerning the OLS approach.

Comparing now all the methods, the proximal-point algorithms roundly outperform all gradient or extragradient methods for either the MOLS or OLS approaches. As can be seen in Figs. 1 and 2, the proximal Hager–Zhang methods not only converge faster (steeper curves) but achieve significantly smaller objective function values in fewer iterations. This performance enhancement can be directly attributed to the nature of proximal regularization and its ability to overcome the error introduced by Tikhonov-type regularization employed by the other methods.

In summary, iterative optimization methods for solving large-scale problems like the elastography inverse problem show significant performance gains when methods such as MOLS are coupled with properly-scaled algorithms like the scaled projected gradient algorithm. In a more general context, when such scaling information is unfeasible or not available, our numerical experiments indicate that careful selection of an initial steplength in a simpler algorithm like Khobotov's can still provide effective "real-world" benefit over more sophisticated but computationally expensive algorithms. Overall, proximal methods coupled with fast solvers for the minimization subproblem show significant performance advantages and overcome inherent drawbacks of other forms of regularization in iterative methods (Figs. 3, 4, 5, 6, 7, and 8).

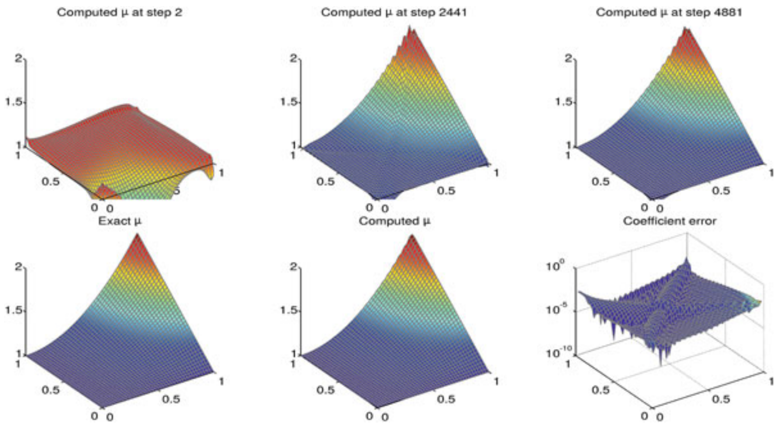

Fig. 4 Example 2: Solodov–Tseng (H) Method (MOLS)

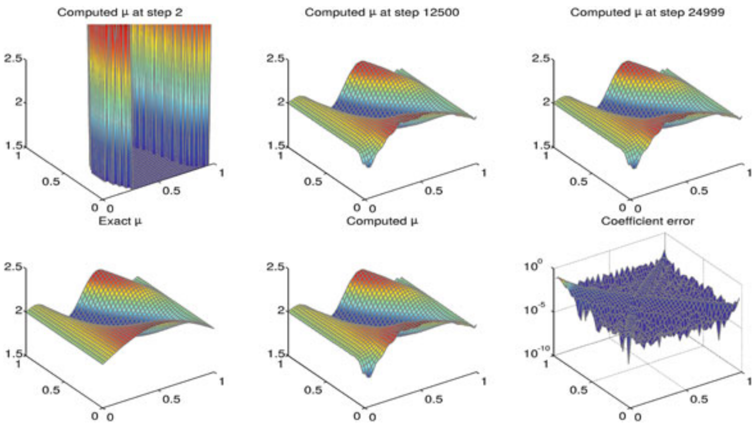

Fig. 5 Example 1: FISTA Method (MOLS)

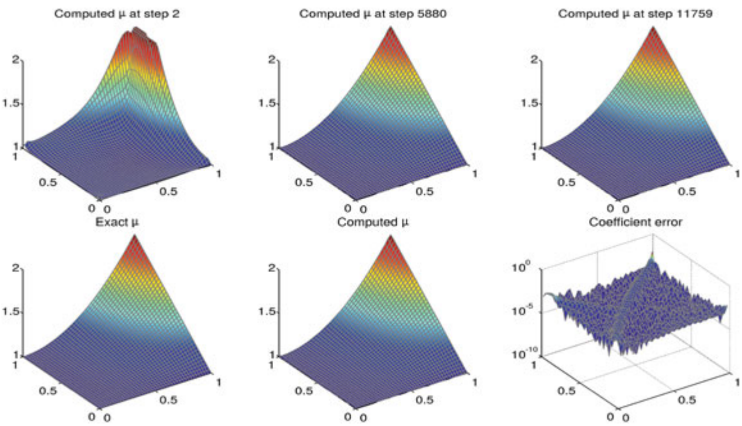

Fig. 6 Example 2: FISTA Method (MOLS)

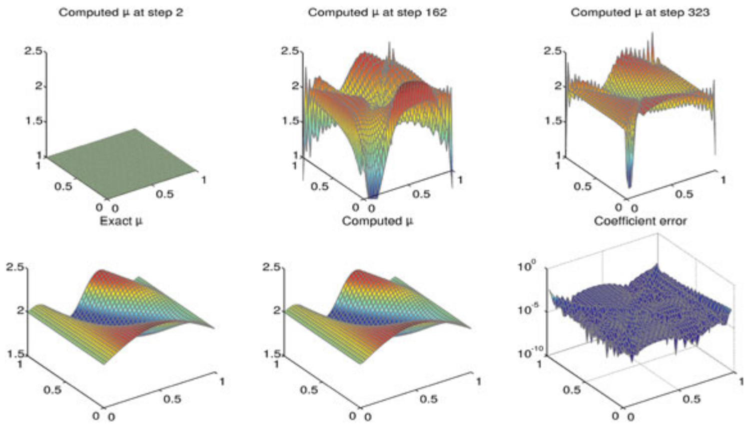

Fig. 7 Example 1: Hager–Zhang Modified φ (MOLS)

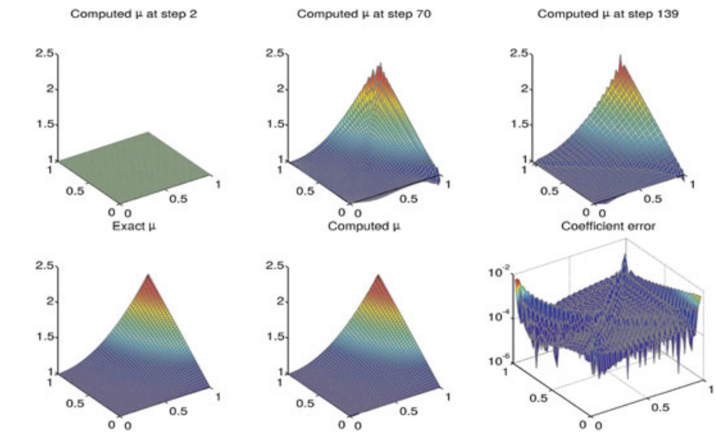

Fig. 8 Example 2: Hager–Zhang Modified φ (MOLS)

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Transversality Theory with Applications to Differential Equations

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Abstract The generic existence of Morse functions in a prescribed family of smooth functionals is investigated. The approach is based on arguments involving the transversality theory. The abstract result is applied to semilinear elliptic boundary value problems. One obtains qualitative information concerning the set of solutions.

1 Introduction

The critical point theory provides powerful tools in studying various nonlinear problems. One of the reasons is that usually the weak solutions in such a problem coincide with the critical points of a suitably constructed smooth functional. The critical points of the associated functional can be located by detecting the change in the homotopy type of the level sets (see, e.g., Marino and Prodi [8], Mawhin [9], Rabinowitz [16], Schwartz [18], Struwe [20], Tanaka [21]). This topological approach is particularly efficient in the case where the functional is a Morse function, that is, it admits only nondegenerate critical points and satisfies the Palais–Smale condition. The reason is that the special properties of a Morse function permit to make use of the powerful tools supplied by the transversality theory.

The aim of this work is to give verifiable criteria for obtaining a Morse function whose critical points be exactly the solutions of the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = p(x,u) + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

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on a bounded domain Ω in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. More precisely, we show that under natural hypotheses upon p, the Euler functional corresponding to (1) (see, e.g., [16, p. 61] and [21]) is generically a Morse function. The stated property simplifies tedious arguments in specific situations. Also it allows to apply different results where the nondegeneracy of solutions is a basic assumption. For a comprehensive discussion in this direction, we refer to the related work of Mawhin [10].

Our approach relies on an abstract result ensuring the generic existence of Morse functions in a given collection of smooth functionals. We recall that a property on a topological space is called generic if it holds on a residual set, i.e., a countable intersection of dense open subsets. The abstract result presented in Theorem 1 below is in fact a version of a general property proved in Motreanu [13] in the setting of differentiable manifolds. The starting idea of our theorem was inspired by the work of Saut and Temam [17] who studied the dependence of solutions with respect to parameters entering the equation. Subsequently, we obtain in Theorem 2 a generic existence result for Morse functions having finitely many critical points.

The abstract result given in Theorem 1 is applied to deduce in Theorem 3 that the Morse functions related to Eq. (1) exist generically with respect to $f \in L^2(\Omega)$ that is regarded as a parameter. In particular, this ensures the density of the set of these generic functions f in $L^2(\Omega)$. Regularity information concerning the dependence of the solutions to (1) with respect to $f \in L^2(\Omega)$ is also available.

A further application of the abstract result given in Theorem 1 treats in Theorem 5 the uniqueness and in generic sense the existence of solutions to (1). This can be seen as an addition to the result due to Saut and Temam [17] establishing the generic finiteness of the set of solutions to a boundary value problem of type (1) with respect to parameters. Note also that, under other hypotheses, the Dirichlet problem (1) may possess infinitely many solutions (see Rabinowitz [16], Struwe [20], Tanaka [21]). A comparison between the different types of assumptions is performed in Sect. 3.

The next objective of the present work is to focus on the natural question of constructing a Morse function whose critical points coincide with the solutions of (1) and be of finite Morse index. In this respect, Theorem 6 below provides a sufficient condition in the case $f = 0$. As an application of Theorem 6, we infer in Corollary 2 the stability with respect to small perturbations of nontrivial solutions to the Dirichlet problem

$$\begin{cases} -\Delta u = p(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

In particular, problem (1) can be seen as a perturbation of (2). Finally, we give in Theorem 7 a positive answer to a question raised in Mawhin [9, p. 160], namely, whether the approach based on Marino and Prodi [8] that was developed by Mawhin [9] for one-dimensional domains Ω and functions $p(x,t) = ar|t|^{r-2}t$ with $a > 0$, $r > 2$, could be extended to problem (1) in multidimensions and in superlinear case, i.e., $p(x,t)t^{-1} \to +\infty$ as $|t| \to +\infty$.

The developments presented in this work use essentially different arguments from the transversality theory. In order to increase the readability of the text, we have included basic elements of transversality theory that are utilized in our proofs.

The rest of the paper is organized as follows. Section 2 sets forth some prerequisites of transversality theory. Section 3 contains the abstract generic result. Section 4 studies the genericity for the existence of Morse functions related to Eq. (1) and the uniqueness in solving (1). Section 5 discusses the Morse functions related to problem (1) with critical points of finite Morse index.

2 Background Material About Transversality

In the sequel we need the basic notion of Fredholm operator. For the sake of clarity, we recall a few essential facts in this direction. A *linear Fredholm operator* means a linear bounded operator $L : X \to Y$ between Banach spaces X and Y such that its kernel $\ker L$ is of finite dimension and its range $R(L)$ is of finite codimension. An important consequence of this definition is that the range $R(L)$ of L is closed in Y (see, e.g., [23, p. 294]). The *index* of a linear Fredholm operator $L : X \to Y$ is defined by

$$\operatorname{ind} L = \dim \ker L - \operatorname{codim} R(L).$$

The linear Fredholm operators of index zero are of special interest, mainly because their class contains the sums of a bijective linear bounded operator and of a linear compact operator.

More generally, a C^1-map $f : X \to Y$ between Banach spaces X and Y is called a *Fredholm operator* if the derivative $f'(x) : X \to Y$ at each $x \in X$ is a linear Fredholm operator. It was shown in [19] that if $f : X \to Y$ is a Fredholm operator, then its index exists being defined by

$$\operatorname{ind} f = \dim \ker f'(x) - \operatorname{codim} R(f'(x))$$

independently of $x \in X$. A fundamental result for Fredholm operators is the theorem of Sard–Smale in [19] (see also [22, pp. 829–830]), which asserts that if a C^k-map $f : X \to Y$ between Banach spaces X and Y is a Fredholm operator with

$$k > \max\{\operatorname{ind} f, 0\},$$

then the set of regular values of f is residual in Y (i.e., a countable intersection of dense open subsets), so dense in Y. The density assertion follows from the Baire theorem. We recall that a *regular value* of a C^1-map $f : X \to Y$ between Banach spaces X and Y means a point $y \in Y$ such that y does not belong to the range of f or for every $x \in X$ with $f(x) = y$ one has that the derivative $f'(x) : X \to Y$ is surjective and its kernel splits X.

Another related fundamental concept is that of transversality. A C^1-map $f : X \to Y$ between Banach spaces X and Y is said to be *transversal* to a C^1-submanifold S of Y if for every $x \in f^{-1}(y)$ with $y \in S$, the range $R(f'(x))$ intersects transversally the tangent space T_yS of S at y, that is,

$$R(f'(x)) + T_yS = Y,$$

and $(f'(x))^{-1}(T_yS)$ splits X. The simplest but highly significant example is that a C^1-map $f : X \to Y$ is transversal to $S = \{y\}$ if and only if $y \in Y$ is a regular value of f.

The key result in this context is that the transversality ensures that the preimage of a submanifold is a submanifold, which goes back to Thom transversality theorem. More precisely, if a C^r-map $f : X \to Y$ ($r \geq 1$) between Banach spaces X and Y is transversal to a C^r-submanifold S of Y, then $f^{-1}(S)$ is a C^r-submanifold of X. Moreover, the tangent spaces are related by the fundamental formula

$$T_x(f^{-1}(S)) = (f'(x))^{-1}(T_{f(x)}S) \text{ for all } x \in X \text{ with } f(x) \in S.$$

In particular, if $y \in Y$ is a regular value of a C^1-map $f : X \to Y$, then $f^{-1}(y)$ is a C^1-submanifold of Y and its tangent space $T_x(f^{-1}(y))$ at any $x \in f^{-1}(y)$ is given by

$$T_x(f^{-1}(S)) = \ker f'(x).$$

For more details regarding the transversality theory, we refer to [7, 22].

Finally, we recall that a C^2-function $f : U \to \mathbb{R}$ defined on an open set U of a Hilbert space X is called a *Morse function* if:

(I) The function f satisfies the *Palais–Smale condition*, i.e., if a sequence $(u_n) \subset U$ is such that $(f(u_n))$ is bounded and $f'(u_n) \to 0$ as $n \to \infty$, then (u_n) contains a convergent subsequence.

(II) f has only *nondegenerate critical points*, that is, if $f'(u) = 0$, then the second derivative $f''(u) : X \to X$ is an isomorphism.

3 Abstract Result

The general problem studied in the present section is to seek Morse functions in a given family $\{F_a : U \to \mathbb{R}\}_{a \in A}$ of smooth functionals defined on an open set U of a Hilbert space. For technical reasons, the family is described by a smooth mapping $I : U \times A \to \mathbb{R}$, with $F_a = I(\cdot, a)$, $a \in A$.

Our main existence result for Morse functions in a prescribed family is formulated in the next theorem. A version of this theorem in the setting of Riemannian manifolds can be found in [13].

Theorem 1. *Let U be an open set in a separable Hilbert space X, let A be a separable Banach space, and let $I : U \times A \to \mathbb{R}$ be a map satisfying the hypotheses:*

(H.1) *The partial derivation $I'_u : U \times A \to X^* = X$ of I with respect to the first argument $u \in U$ exists and is a C^r-mapping, for some $r \geq 1$.*

(H.2) *The second-order partial derivative $I''_{uu}(u, a) : X \to X$ at any $(u, a) \in U \times A$ with*

$$I'_u(u, a) = 0 \tag{3}$$

is a Fredholm operator of index zero.

(H.3) *For every $(u, a) \in U \times A$ satisfying (3), the kernel $\ker(I'_u)'(u, a)$ splits $X \times A$; moreover, the kernels of the linear operators $I''_{uu}(u, a) : X \to X$ and $I''_{ua}(u, a) := (I'_a)'_u(u, a) : X \to A^*$ fulfill the condition*

$$\ker I''_{uu}(u, a) \cap \ker I''_{ua}(u, a) = 0. \tag{4}$$

Then, there hold:

(a) *The set*

$$G := \{a \in A : \ I(\cdot, a) : U \to \mathbb{R} \text{ has only nondegenerate critical points}\} \tag{5}$$

is a residual set in A; hence, it is dense in A.

(b) *If, in addition to* (H.1)–(H.3)*, we assume that*

(H.4) *each function $I(\cdot, a) : U \to \mathbb{R}$ satisfies the Palais–Smale condition,*

then the set G in (5) becomes

$$G = \{a \in A : \ I(\cdot, a) : U \to \mathbb{R} \text{ is a Morse function}\}. \tag{6}$$

So, in particular, the latter set is residual in A.

(c) *In addition to* (H.1)–(H.3)*, we assume that*

(H.4)$'$ *if $(u_n, a_n) \in U \times G$ satisfies (3) for all $n \geq 1$ and if $a_n \to a \in G$, then (u_n) has a convergent subsequence in U.*

Then, for every connected component G_0 of G, the critical points of $I(\cdot, a)$ depend smoothly on $a \in G_0$ in the following sense: there exists an at most countable collection $\{g_i\}_{i \in J}$ of C^r-mappings, each of which maps a neighborhood of G_0 in A into U such that

$$\{u \in U : \ I'_u(u, a) = 0\} = \{g_i(a)\}_{i \in J} \ \text{ for all } a \in G_0. \tag{7}$$

Moreover, $g_i(a) \neq g_j(a)$ for all $a \in G_0$, $i \neq j$. In particular, the number of critical points of $I(\cdot, a)$, with $a \in G$, is constant on each connected component G_0 of G.

We proceed to the proof of Theorem 1 by expressing assumption (H.3) in a form for which an adequate transversality technique can be utilized.

Lemma 1. *Let $U \subset X$ and A be as in Theorem 1. Assume that the function $I : U \times A \to \mathbb{R}$ satisfies* (H.1) *and* (H.2)*. Then, condition* (H.3) *is equivalent to the following statement:*

(H.3)′ $0 \in X$ *is a regular value of the mapping $I'_u : U \times A \to X^* = X$, that is, for every $(u, a) \in U \times A$ which solves (3), $(I'_u)'(u, a)$ is surjective and its kernel splits $X \times A$.*

Proof. Let (u, a) satisfy (3). Then $(I'_u)'(u, a)$ is surjective if and only if for every $v \in X^*$ there is $w \in A$ such that

$$v - I''_{ua}(u, a)(\cdot, w) \in R(I''_{uu}(u, a)). \tag{8}$$

The range of the self-adjoint operator $I''_{uu}(u, a) : X \to X^*$ is characterized by the equality

$$R(I''_{uu}(u, a)) = \overline{R(I''_{uu}(u, a))} = \{z \in X^* : \ z(x) = 0 \ \text{ for all } x \in \ker I''_{uu}(u, a)\}. \tag{9}$$

The first equality above follows from the fact that $I''_{uu}(u, a)$ is a Fredholm operator by (H.2) (see Sect. 2), while, for the second equality, we refer, e.g., to [1, Corollary 2.18]. By (H.2), $\ker I''_{uu}(u, a)$ possesses a finite basis $\{e_i\}_{i=1}^m$. Then, from (8) and (9), it turns out that $(I'_u)'(u, a)$ is surjective if and only if for every $v \in X^*$, there is an element $w \in A$ such that

$$I''_{ua}(u, a)(e_i, w) = v(e_i) \ \text{ for all } i \in \{1, \ldots, m\}.$$

The last equality is equivalent to the surjectivity of the linear operator

$$w \in A \mapsto (I''_{ua}(u, a)(e_i, w))_{1 \le i \le m} \in \mathbb{R}^m.$$

In turn, this is equivalent to the nonexistence of a nonzero vector $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ with

$$\sum_{i=1}^{m} \alpha_i I''_{ua}(u, a)(e_i, w) = 0 \ \text{ for all } w \in A. \tag{10}$$

We have thus shown the equivalence between the surjectivity of $(I'_u)'(u, a)$ and the linear independence of the linear continuous forms on A

$$\{I''_{ua}(u, a)(e_i, \cdot)\}_{i=1}^m. \tag{11}$$

Now, writing (10) as

$$I''_{ua}(u, a)\Big(\sum_{i=1}^{m} \alpha_i e_i, w\Big) = 0 \ \text{ for all } w \in A,$$

the independence of the forms in (11) is seen to be valid if and only if property (4) holds true. Hence, the equivalence between (H.3) and (H.3)′ is proven. □

Proof (Proof of Theorem 1). Lemma 1 ensures that $V = (I'_u)^{-1}(0)$ is a C^r-submanifold of $U \times A$ (see Sect. 2). Denote by $\pi : V \to A$ the restriction to V of the projection $U \times A \to A$ onto the second factor. Applying Lemma A.2 in [17], we obtain that

$$\begin{array}{l} a \in A \text{ is a regular value of} \\ \pi : V \to A \end{array} \iff \begin{array}{l} 0 \in X \text{ is a regular value of} \\ I'_u(\cdot, a) : U \to X \end{array} \tag{12}$$

and

$$\pi : V \to A \text{ is a Fredholm operator of index zero.} \tag{13}$$

In view of (H.2) and the definition of the notion of nondegenerate critical point, one can express (12) as follows:

$$\begin{array}{l} a \in A \text{ is a regular value of} \\ \pi : V \to A \end{array} \iff \begin{array}{l} I(\cdot, a) : U \to \mathbb{R} \text{ admits only} \\ \text{nondegenerate critical points.} \end{array} \tag{14}$$

Property (13) allows us to invoke the Sard–Smale theorem (see Sect. 2) for obtaining that the set

$$G = \{a \in A : a \text{ is a regular value of } \pi : V \to A\}$$

is residual in the Banach space A. The Baire theorem guarantees the density of the set G in A. It is seen from (14) that the above set G coincides with the one introduced in (5). This completes the proof of the first part of Theorem 1.

Part (b) of Theorem 1 is an immediate consequence of part (a) and of the definition of the notion of Morse function.

The proof of part (c) requires some preliminaries.

Claim 1. For every $(u, a) \in U \times G_0$ satisfying (3), there is an open subset $V_{u,a} \subset A$ containing G_0 and a C^r-mapping $g_{u,a} : V_{u,a} \to U$ satisfying:

(i) $g_{u,a}(a) = u$.
(ii) $g_{u,a}(y)$ is a critical point of $I(\cdot, y)$ for all $y \in V_{u,a}$.
(iii) For every $y \in V_{u,a}$, there are neighborhoods $V_y \subset V_{u,a}$ of y and $W_y \subset U$ of $g_{u,a}(y)$ such that $g_{u,a}(y')$ is the unique critical point of $I(\cdot, y')$ in W_y for all $y' \in V_y$.

By (5), we know that $I''_{uu}(u, a) : X \to X$ is a linear isomorphism. Then, according to hypothesis (H.1), the implicit function theorem yields the existence of a maximal open subset $V_{u,a} \subset A$ containing a and a C^r-mapping $g_{u,a} : V_{u,a} \to U$ satisfying (i)–(iii). It remains to verify that $V_{u,a}$ contains G_0. In view of the connectedness of G_0, it suffices to check that $V_{u,a} \cap G_0$ is closed in G_0. So let $a_0 \in \overline{V_{u,a}} \cap G_0$ and let us show that $a_0 \in V_{u,a}$. Let $(a_n) \subset V_{u,a} \cap G_0$ such that $a_n \to a_0$ in A. For every $n \geq 1$, let $u_n = g_{u,a}(a_n)$. By (H.4)′, up to considering a subsequence, we may assume that $u_n \to u_0$ for some $u_0 \in U$. Thus, $I'_u(u_0, a_0) = 0$. Since $a_0 \in G_0$, the implicit function theorem yields neighborhoods $V_0 \subset A$ of a_0, $W_0 \subset U$ of u_0, and a C^r-mapping $g_0 : V_0 \to W_0$ such that $g_0(y)$ is the unique critical point of $I(\cdot, y)$ in W_0 whenever $y \in V_0$. Up to considering V_0 smaller if necessary, we may assume that $V_{u,a} \cap V_0$ is connected. Relying on property (iii) of the mapping $g_{u,a}$, we see that the set $\{y \in V_{u,a} \cap V_0 : g_0(y) = g_{u,a}(y)\}$ is open and closed in $V_{u,a} \cap V_0$, and nonempty (since it contains a_n for n large enough), whence $g_0|_{V_{u,a} \cap V_0} = g_{u,a}|_{V_{u,a} \cap V_0}$. Then, the maximality of $V_{u,a}$ yields $a_0 \in V_{u,a}$. This completes the proof of Claim 1.

Claim 2. If $(u, a), (u', a') \in U \times G_0$ satisfy (3), then we have either $g_{u,a} = g_{u',a'}$ on G_0 or $g_{u,a}(y) \neq g_{u',a'}(y)$ for all $y \in G_0$.

It follows from conditions (i)–(iii) of Claim 1 that the set $\{y \in G_0 : g_{u,a}(y) = g_{u',a'}(y)\}$ is open and closed in G_0. Claim 2 then follows from the connectedness of the set G_0.

We write $\{g_{u,a} : (u, a) \in U \times A \text{ satisfy (3)}\} = \{g_i\}_{i \in J}$ in such a way that $g_i \neq g_j$ whenever $i \neq j$.

Claim 3. For every $a \in G_0$, the map $J \to \{u \in U : I'_u(u, a) = 0\}$, $i \mapsto g_i(a)$ is bijective.

The injectivity follows from Claim 2. Moreover, given $u \in U$ a critical point of $I(\cdot, a)$, we get $g_{u,a}(a) = u$ (by Claim 1). Taking $i \in J$ such that $g_{u,a} = g_i$, we get $u = g_i(a)$, whence the surjectivity. This shows Claim 3.

For every $a \in G_0$, the (nondegenerate) critical points of the function $I(\cdot, a)$ are isolated; hence, the set of critical points of $I(\cdot, a)$ is at most countable. Part (c) of Theorem 1 follows from Claims 1–3 and this observation. □

Remark 1. In (H.2) it suffices to ask that $I''_{uu}(u, a) : X \to X$ be a Fredholm operator because, if this is the case, it is necessarily of null index (cf., Marino–Prodi [8]). Hypothesis (H.2) is always satisfied in the case of a finite dimensional vector space X. The Palais–Smale condition imposed in (H.4) has been extended in Motreanu [11] for studying general constrained minimization problems. Theorem 1, especially the final part, is inspired from Saut and Temam [17], where the dependence of the solutions of equations with respect to parameters is investigated.

We illustrate with a simple situation the possible use of Theorem 1.

Example 1. Let $g : U \to \mathbb{R}$ be a C^2-function on an open subset U of a Hilbert space X with the scalar product $\langle \cdot, \cdot \rangle$. Assume that there is a closed linear subspace A of X such that the following conditions hold:

(a) If $g'(u) \in A$, then $g''(u) : X \to X$ is a Fredholm operator and $\ker g''(u) \subset A$.
(b) If (u_n) is a sequence in U such that $(g(u_n) + \langle u_n, a \rangle)$ is bounded and $g'(u_n) \to -a$ as $n \to \infty$ with $a \in A$, then (u_n) contains a convergent subsequence.

The mapping $I : U \times A \to \mathbb{R}$ given by

$$I(u, a) = g(u) + \langle u, a \rangle \quad \text{for all } (u, a) \in U \times A$$

satisfies the assumptions (H.1)–(H.4) of Theorem 1. Therefore, according to Theorem 1, there exists a residual set G in A such that $I(\cdot, a) = g + \langle \cdot, a \rangle$ is a Morse function on U for all $a \in A$.

In fact, one derives directly from Theorem 1 the density of Morse functions on a finite dimensional manifold (see Motreanu [12]). Moreover, arguing with a suitable modification of the mapping I of Example 1, Theorem 1 leads to the results of Marino and Prodi [8] of approximation by Morse functions on an infinite dimensional Riemannian manifold.

We end this section by pointing out a sufficient condition to have, in addition to the density, the stability of Morse functions under small perturbations. This is expressed by the openness of the set G introduced in (5). At the same time, one obtains the generic finiteness of the set of critical points.

Theorem 2. *Assume that conditions* (H.1)–(H.3) *in Theorem 1 are fulfilled together with*

(H.5) *for each compact subset $C \subset A$, the set $\{u \in U : u$ satisfies (3) for some $a \in C\}$ is compact in U.*

Then the set G defined in (5) is open and dense in A. Furthermore, for every connected component G_0 of G, there exist finitely many C^r-mappings $\{g_i\}_{i=1}^k$ from a neighborhood of G_0 in A into U such that

$$\{u \in U : I'_u(u, a) = 0\} = \{g_i(a)\}_{i=1}^k \ \text{ for all } a \in G_0. \tag{15}$$

Thus, the functional $I(\cdot, a)$ on U, with $a \in G_0$, has a finite number of critical points that is constant on the same connected component G_0 of G.

Proof. We check the openness of the set

$$G = \{a \in A : I(\cdot, a) : U \to \mathbb{R} \text{ possesses only nondegenerate critical points}\}.$$

According to (14), this means the openness of the set of regular values of the mapping $\pi : V \to A$ introduced in the proof of Theorem 1. By a result due to Geba [3], it reduces to show that π is a proper map. If C is a compact subset of A, assumption (H.5) implies that

$$\pi^{-1}(C) = \{(u, a) \in U \times C : I'_u(u, a) = 0\}$$

is a compact subset of V. Therefore, the map π is proper, so the openness claim is established. The density of G is known from Theorem 1 *(a)*.

Hypothesis (H.4)′ of Theorem 1 is implied by (H.5). Therefore, in view of Theorem 1 *(c)*, it remains to show the finiteness of the set of critical points of $I(\cdot, a)$ for each $a \in G$. This set is compact by (H.5). Since it consists only of isolated points, the required finiteness follows. □

4 Morse Functions in Boundary Value Problems

The goal of the present section is to apply the abstract results of Theorems 1 and 2 to the elliptic boundary value problem (1). To this end, we associate to problem (1) the function $I : W_0^{1,2}(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$I(u,f) = \int_\Omega \Big(\frac{1}{2}\|\nabla u\|^2 - P(x,u) - fu\Big)\,dx \text{ for all } (u,f) \in W_0^{1,2}(\Omega) \times L^2(\Omega), \tag{16}$$

where ∇u stands for the gradient of u and P denotes the primitive of p given by

$$P(x,t) = \int_0^t p(x,\tau)\,d\tau \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

It is well known that, under appropriate growth condition on p, the function I in (16) is continuously differentiable and the critical points of $I(\cdot,f) : W_0^{1,2}(\Omega) \to \mathbb{R}$, with $f \in L^2(\Omega)$ fixed, coincide with the weak solutions of the boundary value problem (1) (see Rabinowitz [16] and Tanaka [21]). Therefore, the nondegeneracy of the critical points of $I(\cdot,f)$ becomes a significant qualitative information for the solutions in the study of the Dirichlet problem (1). The following theorem addresses this question.

Theorem 3. (a) *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 3$), whose boundary is a C^2-submanifold of $\mathbb{R}^N$. Assume that the function $p : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(i) $p \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.
(ii) *There exist constants $c_1, c_2 \geq 0$ such that for the partial derivative $p_t'(x,t)$ one has*

$$|p_t'(x,t)| \leq c_1 + c_2|t|^{s-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R}$$

with some $s \in (1, \frac{N+2}{N-2})$.

Then, the set

$$G = \{f \in L^2(\Omega) : I(\cdot,f) : W_0^{1,2}(\Omega) \to \mathbb{R} \text{ has only nondegenerate critical points}\} \tag{17}$$

is residual, so dense, in $L^2(\Omega)$.

(b) *If we further suppose*

(∗) *there are constants* $\mu > 2$*,* $q \geq 0$*, and* $M \in \mathbb{R}$ *such that*

$$p(x,t)t - \mu P(x,t) \geq M \ \textit{for all}\ (x,t) \in \Omega \times \mathbb{R} \ \textit{with}\ |t| \geq q,$$

then the residual set G in (17) coincides with

$$G = \{f \in L^2(\Omega) : I(\cdot,f) : W_0^{1,2}(\Omega) \to \mathbb{R} \textit{ is a Morse function}\}. \tag{18}$$

Proof. (a) Conditions (i) and (ii) imply that the function $I : W_0^{1,2}(\Omega) \times L^2(\Omega) \to \mathbb{R}$ introduced in (16) is differentiable of class C^2 (see Rabinowitz [16, p. 94]). Consequently, hypothesis (H.1) is verified. The first-order partial derivative $I'_u(u,f)$ at any point $(u,f) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ is equal to

$$I'_u(u,f)(v) = \int_\Omega (\langle \nabla u, \nabla v \rangle - p(x,u)v - fv)\, dx \ \text{ for all } v \in W_0^{1,2}(\Omega). \tag{19}$$

Hence, the critical points of $I(\cdot,f)$ are exactly the weak solutions of (1). Differentiating in (19) with respect to u, one obtains

$$I''_{uu}(u,f)(v,w) = \int_\Omega (\langle \nabla v, \nabla w \rangle - p'_u(x,u)vw)\, dx \ \text{ for all } v,w \in W_0^{1,2}(\Omega). \tag{20}$$

Let us check that the continuous linear operator $K : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ given by

$$\langle Kv, w \rangle_{W_0^{1,2}(\Omega)} = \int_\Omega p'_t(x,u)vw\, dx \ \text{ for all } v,w \in W_0^{1,2}(\Omega)$$

is compact for each weak solution u of Eq. (1). By the assumption imposed on the exponent s in hypothesis (ii), we have

$$s < \frac{N+2}{N-2} = 1 + \frac{2N}{N-2}\Big(1 - \frac{N-2}{2N} - \frac{N-2}{2N}\Big).$$

Hence, there are $q, \sigma \in [1, \frac{2N}{N-2}]$ and $\tau \in [1, \frac{2N}{N-2})$ such that $s = 1 + q(1 - \frac{1}{\tau} - \frac{1}{\sigma})$, i.e.,

$$\frac{s-1}{q} + \frac{1}{\tau} + \frac{1}{\sigma} = 1.$$

If (v_n) is a bounded sequence in $W_0^{1,2}(\Omega)$, then it converges in both $L^2(\Omega)$ and $L^\tau(\Omega)$ along a subsequence. Using assumption (ii), the Cauchy–Schwarz inequality, and the generalized Hölder inequality with the exponents $\frac{q}{s-1}$, τ, and σ, we see that for each weak solution u of (1), one has

$$\sup_{\|w\|_{W_0^{1,2}(\Omega)}\leq 1} |\langle Kv_n - Kv_m, w\rangle| = \sup_{\|w\|_{W_0^{1,2}(\Omega)}\leq 1} \left| \int_\Omega p_t'(x,u)(v_n - v_m)w\,dx \right|$$
$$\leq c_1 \sup_{\|w\|_{W_0^{1,2}(\Omega)}\leq 1} \|v_n - v_m\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}$$
$$+ c_2 \sup_{\|w\|_{W_0^{1,2}(\Omega)}\leq 1} \|u\|_{L^q(\Omega)}^{s-1}\|v_n$$
$$- v_m\|_{L^\tau(\Omega)}\|w\|_{L^\sigma(\Omega)}.$$

The Sobolev embedding theorem yields now the following estimate

$$\sup_{\|w\|_{W_0^{1,2}(\Omega)}\leq 1} |\langle Kv_n - Kv_m, w\rangle| \leq C_1\|v_n - v_m\|_{L^2(\Omega)} + C_2\|v_n - v_m\|_{L^\tau(\Omega)}$$

for all $n, m \geq 1$, where $C_1, C_2 \in (0, +\infty)$ are constants independent of the sequence (v_n). It turns out that (Kv_n) contains a convergent subsequence in $W_0^{1,2}(\Omega)$; thus, the claim concerning the compactness of K is valid.

Formula (20) can be written as follows:

$$I''_{uu}(u,f) = \mathrm{id}_{W_0^{1,2}(\Omega)} - K. \tag{21}$$

The compactness of K and equality (21) show that $I''_{uu}(u,f) : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is a Fredholm operator of index zero (see Palais [15, p. 122]). Condition (H.2) is thus satisfied.

Differentiating in (19) with respect to f (and identifying I''_{uf} and I''_{fu}), we find that

$$I''_{uf}(u,f)(v,h) = -\int_\Omega hv\,dx \ \text{ for all } (v,h) \in W_0^{1,2}(\Omega) \times L^2(\Omega).$$

Hence, for the linear operator $I''_{uf}(u,f) : W_0^{1,2}(\Omega) \to L^2(\Omega)$, we get

$$\ker I''_{uf}(u,f) = 0,$$

which shows that (H.3) holds, too. So, Theorem 1 can be applied to the function I of (16). Part (a) of the statement ensues.

(b) We need to show equality (18). To this end, we have to verify hypothesis (H.4) in Theorem 1 for the function $I(\cdot,f) : W_0^{1,2}(\Omega) \to \mathbb{R}$, with an arbitrary $f \in L^2(\Omega)$. The proof relies on assumption $(*)$ and was given by Rabinowitz [16, pp. 10-11] in the case $f = 0$. The present situation can be treated following essentially the same lines, so we omit it. Equality (6) in Theorem 1 implies (18), which completes the proof.

□

Remark 2. (a) Under assumptions (i), (ii), and (∗) of Theorem 3, we deduce that for every $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ and every $f \in G$, with G in (18), problem (1) has finitely many solutions u satisfying $r_1 \leq I(u,f) \leq r_2$. This is the consequence of the Palais–Smale condition and of the property of a Morse function to have only isolated critical points.

(b) Condition (∗) was introduced by Rabinowitz [16]. It is possible to replace hypothesis (∗) in Theorem 3 by other assumptions implying the Palais–Smale condition or a weaker compactness condition of this type, as for example, Cerami condition (see, e.g., the monograph [14]).

The next consequence of Theorem 3 provides an extension of the range of the solution operator for Eq. (1). An abstract result of this type is proven in Mawhin [10].

Corollary 1. *Assume that the hypotheses* (i), (ii), *and* (∗) *of Theorem 3 hold. Let f_0 be in the set G in (17) and let $u_0 \in W_0^{1,2}(\Omega)$ be a solution of (1) with f replaced by f_0. Then there exists a constant $\delta > 0$ such that, if $(u_1,f_1) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ satisfies*

$$\|u_1 - u_0\|_{W_0^{1,2}(\Omega)} < \delta,\ \|f_1 - f_0\|_{L^2(\Omega)} < \delta,$$

and

$$\begin{cases} -\Delta u_1 = p(x,u_1) + f_1(x) & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega \end{cases}$$

then there are a neighborhood V of u_1 in $W_0^{1,2}(\Omega)$ and a constant $\varepsilon > 0$ such that the problem

$$\begin{cases} -\Delta u = p(x,u) + f_1(x) + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in V$ for every h in $(W_0^{1,2}(\Omega))^* = W^{-1,2}(\Omega)$ with $\|h\|_{W^{-1,2}(\Omega)} < \varepsilon$.

Proof. Since f_0 belongs to the set G in (17), it follows that $I''_{uu}(u_0,f_0) : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is a linear topological isomorphism. By the continuity of I''_{uu} and the openness of the set of linear topological isomorphisms, one can find a constant $\delta > 0$ such that, whenever

$$\|u - u_0\|_{W_0^{1,2}(\Omega)} < \delta \quad \text{and} \quad \|f - f_0\|_{L^2(\Omega)} < \delta,$$

the linear operator $I''_{uu}(u,f)$ is an isomorphism of $W_0^{1,2}(\Omega)$. Therefore, for every $(u_1,f_1) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ as in the statement, we get that $I'_u(\cdot,f_1) : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is a local C^1-diffeomorphism from a neighborhood of u_1 in $W_0^{1,2}(\Omega)$ onto a neighborhood of 0 in $W^{-1,2}(\Omega)$. Then, for every $h \in W^{-1,2}(\Omega)$ with $\|h\|_{W^{-1,2}(\Omega)}$ sufficiently small, there is a unique $u \in W_0^{1,2}(\Omega)$ near u_1 satisfying $I'_u(u,f_1) = h$. This yields the conclusion of the corollary. □

It is shown in Theorem 3 that, under the hypotheses therein, generically with respect to $f \in L^2(\Omega)$, problem (1) has a countable number of solutions. It is natural to ask when there is a finite number of solutions, or even a unique solution, for problem (1). In this respect, we notice that there are situations where problem (1) admits an unbounded sequence of weak solutions. For example, this is the case when one supposes $p \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, the growth assumption $|p(x,t)| \le c_1 + c_2|t|^s$ for all $(x,t) \in \Omega \times \mathbb{R}$, with a condition on s more restrictive than the one in (ii), and the additional hypotheses:

(a) The function $p(x,t)$ is odd in the second argument.
(b) There are constants $\mu > 2$ and $q \ge 0$ such that $0 < \mu P(x,t) \le tp(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$ with $|t| \ge q$

(see Rabinowitz [16, p. 61] and also Tanaka [21]). Comparing the conditions $(*)$ and (b), it is seen that in (b) it is required, in addition to $(*)$, the positive sign condition for the primitive $P(x,t)$ of $p(x,t)$ whenever $|t|$ is large enough.

The following result presents a simple situation where problem (1) admits generically a finite number of solutions.

Theorem 4. *Assume that conditions* (i) *and* (ii) *of Theorem 3 hold together with*

$(**)$ *there are constants* $c'_1, c'_2 \in \mathbb{R}$ *and* $r \in [1,2)$ *such that*

$$|p(x,t)| \le c'_1 + c'_2|t|^{r-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

Then:

(a) *The set G in (17) is open and dense in* $L^2(\Omega)$.
(b) *For every connected component* G_0 *of G, the number of weak solutions of (1) is (at most) finite and independent of* $f \in G_0$.
(c) *There exists an (at most) finite family* $\{g_i\}_{i=1}^k$ *of* C^1*-mappings defined on a neighborhood of* G_0 *in* $L^2(\Omega)$ *and taking values in* $W_0^{1,2}(\Omega)$ *such that the weak solutions of (1) for each* $f \in G_0$ *consist of the set* $\{g_i(f)\}_{i=1}^k$.

Proof. As shown in the proof of Theorem 3, conditions *(i)* and *(ii)* imply conditions (H.1)–(H.3) of Theorem 1. Theorem 4 can be deduced from Theorem 2 once we verify condition (H.5). So let $(u_n, f_n) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ such that $f_n \to f$ in $L^2(\Omega)$ and u_n is a solution of (1) with f replaced by f_n, and let us show that (u_n) admits a convergent subsequence. Acting on (1) with u_n as a test function, and taking $(**)$ into account, we find

$$\|u_n\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega (p(x,u_n)u_n + f_n u_n)\,dx \le c''_1 + c''_2\|u_n\|^r_{W_0^{1,2}(\Omega)} + c''_3\|u_n\|_{W_0^{1,2}(\Omega)}$$

for some constants $c_1'', c_2'', c_3'' \geq 0$. Since $r < 2$ (see $(**)$), this implies that (u_n) is bounded in $W_0^{1,2}(\Omega)$. Hence, along a relabeled subsequence, (u_n) converges weakly to some u in $W_0^{1,2}(\Omega)$, and the convergence is strong in $L^2(\Omega)$ and $L^r(\Omega)$. Acting on (1) with the test function $u_n - u$, and passing to the limit, we obtain

$$\lim_{n\to\infty} \langle -\Delta u_n, u_n - u\rangle = 0.$$

Since the negative Laplacian is an operator of $(S)_+$ type (see, e.g., [14, p. 40]), we conclude that $u_n \to u$ in $W_0^{1,2}(\Omega)$. Therefore, condition (H.5) is verified. The proof is complete. □

The below theorem addresses the question asked above to have uniqueness in solving the Dirichlet problem (1). It also gives information for the solutions on their existence, regularity, and stability with respect to small perturbations of $f \in L^2(\Omega)$. We point out that there are no restrictions on s other than the one in (ii).

Theorem 5. *Suppose that conditions* (i) *and* (ii) *of Theorem 3 hold together with the assumptions:*

(iii) *p fulfills the following Lipschitz condition: there is a function $L \in L^\infty(\Omega)$ such that*

$$|p(x,t_1) - p(x,t_2)| \leq L(x)|t_1 - t_2| \text{ for all } x \in \Omega,\ t_1, t_2 \in \mathbb{R}.$$

(iv) *There is a constant $C(\Omega) > 0$ satisfying the Poincaré inequality*

$$\|v\|_{L^2(\Omega)} \leq C(\Omega)\|v\|_{W_0^{1,2}(\Omega)} \text{ for all } v \in W_0^{1,2}(\Omega)$$

such that

$$\|L\|_{L^\infty(\Omega)} < \frac{1}{C(\Omega)^2}.$$

Then, the following assertions are valid:

(a) *The subset G of $L^2(\Omega)$ introduced in (17) is dense and open in $L^2(\Omega)$.*
(b) *Problem (1) has at most one solution whenever $f \in L^2(\Omega)$.*
(c) *If for some $f_0 \in G$ the associated problem (1) has a solution, then for every f belonging to the connected component G_0 of G containing f_0, problem (1) has a unique solution $u = u(f)$. Moreover, the mapping $f \in G_0 \subset L^2(\Omega) \mapsto u(f) \in W_0^{1,2}(\Omega)$ is of class C^2.*

Proof. (a) The result is deduced from Theorem 2 applied to the function $I : W_0^{1,2}(\Omega) \times L^2(\Omega) \to \mathbb{R}$ in (16). Because hypotheses (H.1)–(H.3) were verified in the proof of Theorem 3, it suffices to check assumption (H.5) of Theorem 2. Let $(f_n)_{n\geq 1}$ be a convergent sequence in $L^2(\Omega)$ and, for every $n \geq 1$, let u_n be a solution of (1) with f replaced by f_n, that is,

$$\begin{cases} -\Delta u_n = p(x,u_n) + f_n(x), & x \in \Omega \\ u_n = 0, & x \in \partial\Omega. \end{cases} \tag{22}$$

For all $n, m \geq 1$, we get from (22) that

$$-\Delta(u_n - u_m) = p(x, u_n) - p(x, u_m) + f_n - f_m. \tag{23}$$

Multiplying (23) by $u_n - u_m$ and then integrating over Ω lead to

$$\|u_n - u_m\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega (p(x, u_n) - p(x, u_m))(u_n - u_m)\, dx + \int_\Omega (f_n - f_m)(u_n - u_m)\, dx.$$

Hypothesis (iii) and the Poincaré and Cauchy–Schwarz inequalities imply

$$\left(1 - C(\Omega)^2 \|L\|_{L^\infty(\Omega)}\right) \|u_n - u_m\|_{W_0^{1,2}(\Omega)} \leq C(\Omega)\|f_n - f_m\|_{L^2(\Omega)}.$$

From hypothesis (iv) it turns out that the sequence (u_n) converges in $W_0^{1,2}(\Omega)$. This in conjunction with Theorem 2 yields assertion (a).

The reasoning above with $f_n = f_m = f$ in equality (23) provides the uniqueness of the solution of (1) for every $f \in L^2(\Omega)$. Thus, property (b) is also true.

Finally, since problem (1) is supposed to have a solution for $f = f_0$ and recalling that Theorem 2 guarantees that the number of solutions is constant on the connected component G_0 of G with $f_0 \in G_0$, there exists a solution $u = u(f)$ of (1) for every $f \in G_0$, which is also unique in view of (b). The C^2-differentiability of the mapping $f \mapsto u(f)$ follows once again from Theorem 2. This proves assertion (c). The proof of Theorem 5 is complete. □

Remark 3. (a) If we add to hypotheses (i)–(iv) in Theorem 5 the assumption

$$p(x, 0) = 0 \quad \text{and} \quad p'_t(x, 0) = 0 \ \text{ for all } x \in \Omega,$$

then the Dirichlet problem (1) has a solution whenever f belongs to the connected component in G of the null element $0 \in L^2(\Omega)$. It is so because the trivial solution 0 is a nondegenerate critical point of $I(\cdot, 0)$ as can be seen from (20). Then, one can apply part (c) of Theorem 5. The same remark is also valid for Theorem 4.

(b) Part (b) of Theorem 5 holds true without the need of assumptions (i) and (ii).

Remark 4. Assumption (iii) contrasts with conditions (a), (b) stated before Theorem 5 that have been employed in Rabinowitz [16] and Tanaka [21]. Indeed, the properties of $p(x, t)$ to be odd in t and satisfy the Lipschitz condition (ii) result in

$$|p(x, t)| = \frac{1}{2}|p(x, t) - p(x, -t)| \leq \|L\|_{L^\infty(\Omega)}|t| \ \text{ for all } (x, t) \in \Omega \times \mathbb{R}.$$

Therefore, the primitive $P(x, t)$ is subquadratic in t, while by (b) there exist constants $a_1, a_2 > 0$ such that

$$P(x, t) \geq a_1|t|^\mu - a_2 \ \text{ for all } (x, t) \in \Omega \times \mathbb{R},$$

with $\mu > 2$. The contradiction proves the remark.

Remark 5. The result stated in Theorem 5 has been inspired by the ideas of Saut and Temam in [17], who showed that the finiteness and the smooth dependence of solutions of a second-order quasilinear elliptic boundary value problem

$$\begin{cases} \sum_{i,j} a_{ij}(x)u_{x_i x_j} + g(x, u, \nabla u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{24}$$

is generic with respect to the coefficients a_{ij}, the boundary data φ, and the domain Ω. The dependence of such properties with respect to the nonlinear part g in (24) is not considered in [17]. In this sense, the present paper complements [17]. Here the approach is different involving associated Morse functions.

5 Finite Morse Index and Perturbations

This section deals with the existence of a Morse function as the Euler functional associated to the Dirichlet problem (2) and having all its critical points of finite Morse index. We note that this type of critical points is actually detected by topological tools (see Marino and Prodi [8], Mawhin [9]). We focus here on a different aspect: the presence of critical points of finite Morse index allows to estimate the critical points of the perturbations of the functional. In this respect, for applications it is relevant to consider problem (1) as a perturbation of problem (2). The final part of the section is devoted to a result giving an answer to a question raised in Mawhin [9, p. 160] regarding the superlinear elliptic equations.

The following result provides a sufficient condition ensuring that the Euler functional corresponding to problem (2) is a Morse function possessing only critical points of finite Morse index. In some sense, this is an embodiment in concrete form of Theorem 3.

Theorem 6. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) whose boundary $\partial\Omega$ is a C^2-submanifold of $\mathbb{R}^N$. Assume that the function $p : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies hypotheses* (i) *and* (ii) *in Theorem 3 and in addition*

(v) *if u is a (necessarily classical) solution of the Dirichlet problem (2), then the partial derivative $p'_t(x, t)$ satisfies $p'_t(x, u(x)) \leq 0$ for all $x \in \Omega$.*

Then, the functional $J : W_0^{1,2}(\Omega) \to \mathbb{R}$ given by

$$J(v) = \int_\Omega \left(\frac{1}{2}\|\nabla v\|^2 - P(x, v(x))\right) dx \ \text{ for all } v \in W_0^{1,2}(\Omega), \tag{25}$$

with P as in (16), admits only nondegenerate critical points of finite Morse index. If an additional assumption on the function p to ensure the Palais–Smale condition for J (for example, $(*)$ *in Theorem 3) is supposed, then $J : W_0^{1,2}(\Omega) \to \mathbb{R}$ in (25) is a Morse function whose critical points are of finite Morse index.*

Proof. Hypotheses (i) and (ii) guarantee that the functional J in (25) is twice continuously differentiable (see, e.g., [16, p. 94]). The first- and second-order derivatives of J at any point $u \in W_0^{1,2}(\Omega)$ are expressed as follows:

$$J'(u)(v) = \int_\Omega \big(\langle \nabla u, \nabla v\rangle - p(x,u)v\big)\,dx \text{ for all } v \in W_0^{1,2}(\Omega) \tag{26}$$

and

$$J''(u)(v,w) = \int_\Omega \big(\langle \nabla v, \nabla w\rangle - p_t'(x,u)vw\big)\,dx \text{ for all } v,w \in W_0^{1,2}(\Omega). \tag{27}$$

It is seen from (26) that the critical points of J are exactly the solutions of (2), so assumption (v) can be applied for every critical point $u \in W_0^{1,2}(\Omega)$ of J. As usual, we identify the bilinear form $J''(u) : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ with the corresponding linear operator $J''(u) : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ taking into account that $W_0^{1,2}(\Omega)$ is a Hilbert space.

We claim that this operator is injective for each solution u of problem (2). Indeed, if $v \in \ker J''(u)$, then (27) implies that v must solve the linear Dirichlet problem

$$\begin{cases} -\Delta v = p_t'(x,u)v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{28}$$

Thanks to hypothesis (i), $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The function $p_t'(x,u(x))$ is thus bounded on Ω. Assumption (v) yields then the uniqueness of solution of (28) (see, e.g., Gilbarg and Trudinger [5, p. 180]). It follows that $v = 0$, which proves the injectivity of $J''(u)$.

Note that $J''(u) : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$, with u solving (2), is a Fredholm operator of index zero. This can be seen as in the proof of Theorem 3 because the right-hand sides of relations (27) and (20) coincide and, as shown above, the term $p_t'(x,u(x))$ is bounded. Combining with the injectivity of $J''(u)$, we get from the nullity of the Fredholm index of $J''(u)$ that

$$\dim W_0^{1,2}(\Omega)/\operatorname{Im} J''(u) = \dim\ker J''(u) = 0.$$

Therefore, $J''(u) : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is a linear topological isomorphism for every critical point u of the function J, which means that all the critical points of J are nondegenerate.

It remains to establish the finiteness of the Morse index for every (nondegenerate) critical point u of J. From (27) it is clear that

$$J''(u) = \mathrm{id}_{W_0^{1,2}(\Omega)} - K, \tag{29}$$

with the same linear operator $K : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ appearing in the proof of Theorem 3. Since K is self-adjoint and compact, its spectrum is bounded and contains only real eigenvalues allowing at most 0 as a limit point. The Morse index of J at u is equal to the number of negative eigenvalues of $J''(u)$. From (29) this number coincides with the number of eigenvalues of K greater than 1, which is finite. □

Remark 6. (a) Condition (v) was used by Gidas, Ni, and Nirenberg [4] and by Cheng and Smoller [2] in the study of positive solutions to problem (2).

(b) In the statement of Theorem 6, one can replace $(*)$ by any other hypothesis implying the Palais–Smale condition for the functional J. An example of function $p(x,t)$ satisfying all the hypotheses (i), (ii), (v), and $(*)$ is $p(x,t) = -f(x)(t - \sin t)$, where the function $f : \Omega \to \mathbb{R}$ is nonnegative and $f \in C^1(\Omega) \cap C(\overline{\Omega})$.

(c) The knowledge of Morse indexes of classical solutions of problem (2) permits to obtain L^∞-estimates (see [6]).

Theorem 6 enables us to get an existence and regularity result for a perturbation of problem (2) that is more general than the one considered in (1). This result is useful for obtaining the existence of nontrivial solutions to perturbation problems.

Corollary 2. *Suppose that the function $p : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies hypotheses* (i)*,* (ii)*,* (v)*, and* $(*)$ *of Theorem 6. Let the function $h \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and the positive number $\varepsilon > 0$ fulfill the conditions:*

(A) *There exist constants $b_1, b_2 \geq 0$, and $r \in [0, \frac{N+2}{N-2})$ such that*

$$|h(x,t)| \leq b_1 + b_2|t|^r \ \textit{for all}\ (x,t) \in \Omega \times \mathbb{R}.$$

(B) *Denoting*

$$H(x,t) = \int_0^t h(x,\tau)\, d\tau \ \textit{for all}\ (x,t) \in \Omega \times \mathbb{R},$$

one assumes

$$|H(x,t)| \leq \frac{\varepsilon}{3|\Omega|} \ \textit{for all}\ (x,t) \in \Omega \times \mathbb{R}, \tag{30}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω.

(C) *There are constants $d \geq 0$ and $\eta \in \mathbb{R}$ such that*

$$h(x,t)t - \mu H(x,t) \geq \eta \ \textit{for all}\ (x,t) \in \Omega \times \mathbb{R}\ \textit{such that}\ |t| \geq d$$

with $\mu > 2$ as in $(*)$.

Then, for each (classical) solution u_0 of (2) such that $c := J(u_0)$ is the unique critical value of J in the interval $[c-\varepsilon, c+\varepsilon]$, there exists a weak solution u_1 of the boundary value problem

$$\begin{cases} -\Delta u = p(x,u) + h(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{31}$$

satisfying the estimate

$$J(u_1) - \int_\Omega H(x,u_1)\,dx \in [c-\varepsilon, c+\varepsilon],$$

with J defined in (25).

Proof. Theorem 6 ensures that $J : W_0^{1,2}(\Omega) \to \mathbb{R}$ is a Morse function; hence, its critical values are isolated. So, there exists a constant $\varepsilon > 0$ such that $c = J(u_0)$ is the unique critical value of J in the interval $[c-\varepsilon, c+\varepsilon]$. Hypotheses (i), (ii), and (A) imply that the real-valued function

$$u \in W_0^{1,2}(\Omega) \mapsto J(u) - \int_\Omega H(x,u)\,dx$$

is continuously differentiable and its critical points are the weak solutions of the semilinear elliptic equation (31) (see [16, pp. 90–91]). By assumptions $(*)$ and (C), it verifies the Palais–Smale condition. Since J satisfies the Palais–Smale condition and has only isolated critical points, its set of critical points in the level set $J^{-1}(c)$ is finite. By Theorem 6 it is also known that J possesses only nondegenerate critical points of a finite Morse index. This enables us to apply a stability result due to Marino and Prodi [8] to the function J. Taking condition (B) into account, one achieves the stated conclusion. □

Example 2. A function satisfying the assumptions (i), (ii), (v), and $(*)$ is $p(x,t) = p(t) = -|t|^{p-2}t$ with $2 \le p < \frac{2N}{N-2}$. In order to fulfill $(*)$, we have to choose $\mu > p$. Consequently, Corollary 2 can be applied to the corresponding Dirichlet problem.

We now admit that the function $p : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses (i) and (ii) together with the condition

(vi) there exist constants $\mu > 2$ and $a_1, a_2, M_0, q > 0$ such that

$$p(x,t)t \ge \max\{\mu P(x,t) - M_0, a_1 t^2 - a_2\} \text{ for all } (x,t) \in \Omega \times \mathbb{R} \text{ with } |t| \ge q,$$

where P is the primitive of p in (16).

It is clear that (vi) is stronger than condition $(*)$ in Theorem 3. For example, $p(x,t) = |t|^{r-2}t$ with $2 < r < \frac{2N}{N-2}$ satisfies all the assumptions (i), (ii), and (vi). In order to check (vi), it suffices to choose $2 < \mu \le r$.

Consider the problem (1) with f belonging to the residual subset G of $L^2(\Omega)$ as given in (18) (see Theorem 3). By Theorem 3 *(b)*, the function $I(\cdot, f) : W_0^{1,2}(\Omega) \to \mathbb{R}$ introduced in (16) is a Morse function, so its critical values are isolated. In the following, we shall be concerned with functions $f \in G$ fulfilling the additional hypothesis:

(vii) $I(\cdot,f)$ possesses an unbounded sequence of (positive) critical values $\{c_k = c_k(f)\}_{k\geq 1}$ with

$$c_1 < c_2 < \cdots < c_k < \cdots$$

and

$$\frac{1}{c_{k+1}}(c_{k+1} - c_k)^2 \to +\infty \text{ as } k \to \infty. \tag{32}$$

We present a result exhibiting, under appropriate conditions, the property that (1) has infinitely many solutions under arbitrary perturbations in $L^2(\Omega)$ of the term $f(x)$.

Theorem 7. *Assume that the function $p : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies conditions* (i), (ii), *and* (vi). *Then there exists a residual set G in $L^2(\Omega)$ [namely, the set G in (18)], hence dense in $L^2(\Omega)$, such that there holds: for every $f \in G$ satisfying hypothesis* (vii) *and every $L > 0$, there is a positive integer k_0 such that, for every $k \geq k_0$ and every $h \in L^2(\Omega)$ with $\|h\|_{L^2} \leq L$, the Dirichlet problem*

$$\begin{cases} -\Delta u = p(x,u) + f(x) + h(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{33}$$

has a weak solution u_k with

$$c_k - \varepsilon_k \leq I(u_k, f + h) \leq c_k + \varepsilon_k,$$

where

$$\varepsilon_k = \frac{1}{3}(c_{k+1} - c_k).$$

In particular, problem (33) has infinitely many solutions for every $h \in L^2(\Omega)$.

Proof. We proceed by adapting the reasoning developed in Mawhin [9] in the case of a one-dimensional domain Ω and the function $p(x,t) = ar|t|^{r-2}t$ with constants $a > 0$ and $r > 2$. Actually, we will show that the conclusion of Theorem 7 holds for the residual subset G of $L^2(\Omega)$ given by (18).

Fix $f \in G$ and $h \in L^2(\Omega)$ with $\|h\|_{L^2} \leq L$. Let us choose a nonincreasing C^2-function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfying $\sigma(t) = 1$ for $t \leq 0$ and $\sigma(t) = 0$ for $t \geq 1$. Corresponding to any constant $\rho > 0$, we define the function $I_\rho : W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$I_\rho(u) = I(u,f) - \int_\Omega \sigma\Big(\frac{1}{\rho}(\|u\|^2_{L^2(\Omega)} - \rho)\Big) hu \, dx \text{ for all } u \in W_0^{1,2}(\Omega), \tag{34}$$

with $I(u,f)$ given in (16). Under hypotheses (i) and (ii), the function I is twice continuously differentiable (see [16, pp. 90–91]).

Now we show that $I_\rho : W_0^{1,2}(\Omega) \to \mathbb{R}$ satisfies the Palais–Smale condition. In view of [16, p. 94], it is sufficient to prove that if the sequence $(u_n) \subset W_0^{1,2}(\Omega)$ satisfies

$$|I_\rho(u_n)| \leq M \text{ for all } n \geq 1, \text{ and } (I_\rho)'(u_n) \to 0 \text{ as } n \to \infty$$

(for some constant $M > 0$), then (u_n) is bounded in $W_0^{1,2}(\Omega)$. A direct computation, based on the properties of the function σ, renders that for n sufficiently large one has

$$\begin{aligned} M + \frac{1}{\mu}\|u_n\|_{W_0^{1,2}(\Omega)} &\geq I_\rho(u_n) - \frac{1}{\mu}(I_\rho)'(u_n)(u_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2_{W_0^{1,2}(\Omega)} + \int_\Omega \left(\frac{1}{\mu}p(x,u_n)u_n - P(x,u_n)\right)dx \\ &\quad -\left(1 - \frac{1}{\mu}\right)\|f\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)} - k, \end{aligned}$$

where $k > 0$ is a constant independent of the sequence (u_n). From assumption (vi) it is then clear that the boundedness of (u_n) follows, so I_ρ satisfies the Palais–Smale condition.

Setting

$$\rho_k = \frac{\varepsilon_k^2}{18L^2} \text{ for all } k \geq 1,$$

then (34) yields

$$|I(u,f) - I_{\rho_k}(u)| \leq \frac{\varepsilon_k}{3} \text{ for all } k \geq 1. \tag{35}$$

Since $f \in G$, it follows from (18) that $I(\cdot,f) : W_0^{1,2}(\Omega) \to \mathbb{R}$ is a Morse function. Hence, the set $I(\cdot,f)^{-1}(c_k)$ contains only finitely many critical points, and each critical point of $I(\cdot,f)$ is nondegenerate. Arguing as in the final part of the proof of Theorem 6, one justifies that every (nondegenerate) critical point of the function $I(\cdot,f)$ is of finite Morse index. Therefore, we may apply the stability result of Marino and Prodi [8] to the function $I(\cdot,f)$. By the estimate (35) we then deduce that the function I_{ρ_k} has a critical value d_k in the closed interval $[c_k - \varepsilon_k, c_k + \varepsilon_k]$ for every $k \geq 1$. So, there exists $u_k \in W_0^{1,2}(\Omega)$ such that

$$(I_{\rho_k})'(u_k) = 0 \quad \text{and} \quad I_{\rho_k}(u_k) = d_k \in [c_k - \varepsilon_k, c_k + \varepsilon_k] \text{ for all } k \geq 1. \tag{36}$$

To conclude the proof, it suffices to show the existence of an integer k_0 such that

$$I'_u(u_k, f+h) = 0 \quad \text{and} \quad I(u_k, f+h) = d_k \text{ for all } k \geq k_0. \tag{37}$$

In order to do this, we need the next lemma. For the particular situation mentioned at the beginning of the present proof, it can be found in Mawhin [9, p. 156].

Lemma 2. *Under hypotheses* (i)*,* (ii)*, and* (vi)*, there exist positive constants* α, β *such that if for some* $\rho > 0$ *and* $u \in W_0^{1,2}(\Omega)$ *there hold*

$$(I_\rho)'(u)(u) = 0 \quad \textit{and} \quad I_\rho(u) \leq \alpha\rho - \beta,$$

then the functions $I(\cdot, f + h)$ *and* I_ρ *coincide on a neighborhood of* u *in* $W_0^{1,2}(\Omega)$*.*

Proof (Proof of Lemma 2). The equality $(I_\rho)'(u)(u) = 0$ yields

$$\begin{aligned}I_\rho(u) = I_\rho(u) - \frac{1}{2}(I_\rho)'(u)(u) = &\int_\Omega \Big(\frac{1}{2}p(x,u)u - P(x,u)\Big)\,dx - \frac{1}{2}\int_\Omega fu\,dx\\ &-\frac{1}{2}\sigma\Big(\frac{1}{\rho}(\|u\|^2_{L^2(\Omega)} - \rho)\Big)\int_\Omega hu\,dx\\ &+\frac{1}{\rho}\sigma'\Big(\frac{1}{\rho}(\|u\|^2_{L^2(\Omega)} - \rho)\Big)\|u\|^2_{L^2(\Omega)}\int_\Omega hu\,dx.\end{aligned}$$

On the other hand, by hypothesis (vi), it follows that

$$\begin{aligned}\frac{1}{2}p(x,t)t - P(x,t) &= \frac{1}{\mu}p(x,t)t - P(x,t) + \Big(\frac{1}{2} - \frac{1}{\mu}\Big)p(x,t)t\\ &\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)(a_1t^2 - a_2) - M_0 \ \text{ for all } (x,t) \in \Omega \times \mathbb{R} \text{ with } |t| \geq q.\end{aligned}$$

Then the Cauchy–Schwarz inequality and the property of $\sigma(t)$ to vanish for $t \geq 1$ imply that there exist constants $\alpha, \beta > 0$ independent of u and ρ such that

$$I_\rho(u) > \alpha\|u\|^2_{L^2(\Omega)} - \beta. \tag{38}$$

Let us check that these constants fulfill the required properties. Indeed, assuming the inequality

$$I_\rho(u) \leq \alpha\rho - \beta,$$

if we combine it with (38), we get

$$\|u\|^2_{L^2(\Omega)} < \rho. \tag{39}$$

In view of (39), we consider the following closed ball in $W_0^{1,2}(\Omega)$ centered at u:

$$\Big\{v \in W_0^{1,2}(\Omega) :\ \|v - u\|_{W_0^{1,2}(\Omega)} \leq \frac{1}{C(\Omega)}(\sqrt{\rho} - \|u\|_{L^2(\Omega)})\Big\}, \tag{40}$$

where $C(\Omega)$ denotes a positive constant entering the Poincaré inequality

$$\|w\|_{L^2(\Omega)} \leq C(\Omega)\|w\|_{W_0^{1,2}(\Omega)} \text{ for all } w \in W_0^{1,2}(\Omega).$$

For every v in (40), one sees

$$\|v\|_{L^2(\Omega)} \leq \|v-u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \leq \sqrt{\rho}.$$

Then, the fact that $\sigma(t) = 1$ for $t \leq 0$ readily implies that the mappings I_ρ and $I(\cdot, f+h)$ are equal on the neighborhood of u in $W_0^{1,2}(\Omega)$ defined in (40). The proof is thus complete. □

Proof of Theorem 7 (continued). Let the constants $\alpha, \beta > 0$ be those produced in Lemma 2. We claim that

$$d_k \leq \alpha\rho_k - \beta \text{ for all } k \geq k_0, \tag{41}$$

with a fixed positive integer k_0 to be determined.

In order to prove (41), in view of (36), it is sufficient to find k_0 such that

$$\beta + c_{k+1} \leq \alpha\rho_k \text{ provided } k \geq k_0.$$

This readily follows from hypothesis (vii) on $f \in G$ and the expression of ρ_k. Hence, the claim in (41) is proven.

Then, taking again into account (36), we may apply Lemma 2 for $u = u_k$ and $\rho = \rho_k$, with $k \geq k_0$. In this way we derive from Lemma 2 and (36) that both relations in (37) hold true. Since each critical point u_k of $I(\cdot, f+h) : W_0^{1,2}(\Omega) \to \mathbb{R}$ is a weak solution of the Dirichlet problem (33), the proof of Theorem 7 is complete. □

Remark 7. Theorem 7 answers in the affirmative a question raised in Mawhin [9, p. 160], where it is suggested to study the solutions to perturbations of a superlinear problem (2) (i.e., in the case where $\frac{p(x,t)}{t} \to +\infty$ as $|t| \to +\infty$), in the multidimensional case, via the method of Marino and Prodi [8]. In fact, we showed that, under suitable conditions, the result given in Mawhin [9, p. 158] holds for multidimensional superlinear Dirichlet problems that are generically perturbed in $L^2(\Omega)$.

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Lattice-like Subsets of Euclidean Jordan Algebras

A.B. Németh and S.Z. Németh

Abstract While studying some properties of linear operators in a Euclidean Jordan algebra, Gowda, Sznajder, and Tao have introduced generalized lattice operations based on the projection onto the cone of squares. In two recent papers of the authors of the present paper, it has been shown that these lattice-like operators and their generalizations are important tools in establishing the isotonicity of the metric projection onto some closed convex sets. The results of this kind are motivated by methods for proving the existence of solutions of variational inequalities and methods for finding these solutions in a recursive way. It turns out that the closed convex sets admitting isotone projections are exactly the sets which are invariant with respect to these lattice-like operations, called lattice-like sets. In this paper, it is shown that the Jordan subalgebras are lattice-like sets, but the converse in general is not true. In the case of simple Euclidean Jordan algebras of rank at least 3, the lattice-like property is rather restrictive, e.g., there are no lattice-like proper closed convex sets with interior points.

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1 Introduction

By using and generalizing the extended lattice operations due to Gowda, Sznajder, and Tao [1], in [2] and [3], it has been shown that the projection onto a closed convex set is isotone with respect to the order defined by a cone if and only if the set is invariant with respect to the extended lattice operations defined by the cone.

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We shall call such a set simply invariant with respect to the cone, or, if there is no ambiguity, lattice-like, or shortly l-l. We also showed that the a closed convex set with interior points is l-l if and only if all of its tangent hyperplanes are l-l. These results were motivated by iterative methods for variational inequalities similar to the ones for complementarity problems in [4–7]. More specifically, a variational inequality defined by a closed convex set C and a function f can be equivalently written as the fixed point problem $\mathbf{x} = P_C(\mathbf{x} - f(\mathbf{x}))$, where P_C is the projection onto the closed convex set C. If the Picard iteration $\mathbf{x}_{k+1} = P_C(\mathbf{x}_k - f(\mathbf{x}_k))$ is convergent and f continuous, then the limit of $\mathbf{x}_k$ is a solution of the variational inequality defined by f and C. Therefore, it is important to give conditions under which the Picard iteration is convergent. This idea has been exploited in several papers, such as [8–18]. However, none of these papers used the monotonicity of the sequence $\mathbf{x}_k$. If one can show that $\mathbf{x}_k$ is monotone increasing (decreasing) and bounded from above (below) with respect to an order defined by a regular cone (that is, a cone for which all such sequences are convergent), then it is convergent and its limit is a solution of the variational inequality defined by f and C. In [4–7] the convergence of the sequence $\mathbf{x}_k$ was proved by using its monotonicity. Although they use non-iterative methods, we also mention the paper of Nishimura and Ok [19], where the isotonicity of the projection onto a closed convex set is used for studying the solvability of variational inequalities and related equilibrium problems. To further accentuate the importance of ordered vector structures, let us also mention that recently they are getting more and more ground in studying various fixed point and related equilibrium problems (see the book [20] of S. Carl and S Heikkilä and the references therein). The case of a self-dual cone is of special importance because of the elegant examples for invariant sets with respect to the nonnegative orthant and the Lorentz cone [2]. Moreover, properties of self-dual cones are becoming increasingly important because of conic optimization and applications of the analysis on symmetric cones. Especially important self-dual cones in applications are the nonnegative orthant, the Lorentz cone, and the positive semidefinite cone; however, the class of self-dual cones is much larger [21]. The results of [2] and [3] extend the results of [22] and [19]. G. Isac showed in [22] that the projection onto a closed convex sublattice of the Euclidean space ordered by the nonnegative orthant is isotone. H. Nishimura and E. A. Ok proved an extension of this result and its converse to Hilbert spaces in [19]. The study of invariant sets with respect to the nonnegative orthant goes back to the results of Topkis [23] and Veinott Jr. [24], but it wasn't until quite recently when all such invariant sets have been determined by Queyranne and Tardella [25]. The same results have been obtained in [2] in a more geometric way. Although [2] also determined the invariant sets with respect to the Lorentz cone, it left open the question of finding the invariant sets with respect to the cone $\mathbb{S}^m_+$ of $n \times n$ positive semidefinite matrices, called the positive semidefinite cone.

As a particular case, we show that if $n \geq 3$, then there is no proper closed convex l-l set with nonempty interior in the space $(\mathbb{S}^m, \mathbb{S}^m_+)$ (the space $\mathbb{S}^m$ of $n \times n$ symmetric matrices ordered by the cone $\mathbb{S}^m_+$ of symmetric positive semidefinite matrices).

For this it is enough to show that there are no invariant hyperplanes because the closed convex invariant sets with nonempty interior are the ones which have all tangent hyperplanes invariant.

All these problems can be handled in the unifying context of the Euclidean Jordan algebras. This way we can augment this field to an approach, where the order induced by the cone of squares (the basic notion of the Jordan algebra) becomes emphasized.

To shorten our exposition, we assume the knowledge of basic facts and results on Euclidean Jordan algebras. We strive to be in accordance with the terminology in [26]. A concise introduction of the used basic notions and facts in the field can be found in [1].

2 Preliminaries

Denote by $\mathbb{R}^m$ the m-dimensional Euclidean space endowed with the scalar product $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ and the Euclidean norm $\| \cdot \|$ and topology this scalar product defines.

Throughout this note, we shall use some standard terms and results from convex geometry (see, e.g., [27] and [28]).

Let K be a *convex cone* in $\mathbb{R}^m$, i.e., a nonempty set with (1) $K + K \subset K$ and (2) $tK \subset K, \ \forall \ t \in \mathbb{R}_+ = [0, +\infty)$. The convex cone K is called *pointed*, if $K \cap (-K) = \{\mathbf{0}\}$.

The convex cone K is *generating* if $K - K = \mathbb{R}^m$.

For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, by the equivalence $\mathbf{x} \leq_K \mathbf{y} \Leftrightarrow \mathbf{y} - \mathbf{x} \in K$, the convex cone K induces an *order relation* $\leq_K$ in $\mathbb{R}^m$, that is, a binary relation, which is reflexive and transitive. This order relation is *translation invariant* in the sense that $\mathbf{x} \leq_K \mathbf{y}$ implies $\mathbf{x} + \mathbf{z} \leq_K \mathbf{y} + \mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^m$ and *scale invariant* in the sense that $\mathbf{x} \leq_K \mathbf{y}$ implies $t\mathbf{x} \leq_K t\mathbf{y}$ for any $t \in \mathbb{R}_+$. If $\leq$ is a translation-invariant and scale-invariant order relation on $\mathbb{R}^m$, then $\leq = \leq_K$, where $K = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{0} \leq \mathbf{x}\}$ is a convex cone. If K is pointed, then $\leq_K$ is *antisymmetric* too, that is, $\mathbf{x} \leq_K \mathbf{y}$ and $\mathbf{y} \leq_K \mathbf{x}$ imply that $\mathbf{x} = \mathbf{y}$. The elements $\mathbf{x}$ and $\mathbf{y}$ are called *comparable* if $\mathbf{x} \leq_K \mathbf{y}$ or $\mathbf{y} \leq_K \mathbf{x}$.

We say that $\leq_K$ is a *latticial order* if for each pair of elements $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, there exist the least upper bound $\sup\{\mathbf{x}, \mathbf{y}\}$ and the greatest lower bound $\inf\{\mathbf{x}, \mathbf{y}\}$ of the set $\{\mathbf{x}, \mathbf{y}\}$ with respect to the order relation $\leq_K$. In this case K is said a *latticial or simplicial cone*, and $\mathbb{R}^m$ equipped with a latticial order is called an *Euclidean vector lattice*.

The *dual* of the convex cone K is the set

$$K^* := \{\mathbf{y} \in \mathbb{R}^m : \ \langle \mathbf{x}, \mathbf{y} \rangle \geq 0, \ \forall \ \mathbf{x} \in K\},$$

with $\langle \cdot, \cdot \rangle$ the standard scalar product in $\mathbb{R}^m$.

The convex cone K is called *self-dual*, if $K = K^*$. If K is self-dual, then it is a generating pointed closed convex cone.

In all that follows, we shall suppose that $\mathbb{R}^m$ is endowed with a Cartesian reference system with an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_m$. If $\mathbf{x} \in \mathbb{R}^m$, then we represent it as usual by $\mathbf{x} = (x_1, \dots, x_m)$ with x_i the coordinates of $\mathbf{x}$ with respect to this basis. Then the scalar product of $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$ will be the sum $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^m x_i y_i$.

The set

$$\mathbb{R}^m_+ = \{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : \ x_i \geq 0, \ i = 1, \dots, m\}$$

is called the *nonnegative orthant* of the above introduced Cartesian reference system. A direct verification shows that $\mathbb{R}^m_+$ is a self-dual cone.

Using the above introduced notations, the *coordinate-wise order* $\leq$ in $\mathbb{R}^m$ is defined by

$$\mathbf{x} = (x_1, \dots, x_m) \leq \mathbf{y} = (y_1, \dots, y_m) \ \Leftrightarrow \ x_i \leq y_i, \ i = 1, \dots, m.$$

By using the notion of the order relation induced by a cone, defined in the preceding section, we see that $\leq = \leq_{\mathbb{R}^m_+}$.

With the above representation of $\mathbf{x}$ and $\mathbf{y}$, we define

$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \dots, \min\{x_m, y_m\}), \ \text{ and } \ \mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \dots, \max\{x_m, y_m\}).$$

Then, $\mathbf{x} \wedge \mathbf{y}$ is the greatest lower bound and $\mathbf{x} \vee \mathbf{y}$ is the least upper bound of the set $\{\mathbf{x}, \mathbf{y}\}$ with respect to the coordinate-wise order. Thus, $\leq$ is a lattice order in $\mathbb{R}^m$. The operations $\wedge$ and $\vee$ are called *lattice operations*.

A subset $M \subset \mathbb{R}^m$ is called a *sublattice of the coordinate-wise ordered Euclidean space* $\mathbb{R}^m$, if from $\mathbf{x}, \mathbf{y} \in M$, it follows that $\mathbf{x} \wedge \mathbf{y}, \ \mathbf{x} \vee \mathbf{y} \in M$.

The set

$$\mathscr{L}^{m+1}_+ = \{(\mathbf{x}, x_{m+1}) \in \mathbb{R}^m \oplus \mathbb{R} = \mathbb{R}^{m+1} : \ \|\mathbf{x}\| \leq x_{m+1}\} \tag{1}$$

is a self-dual cone called the $m+1$-*dimensional second-order cone*, or the $m+1$-*dimensional Lorentz cone*, or the $m+1$-*dimensional ice-cream cone* [1].

The nonnegative orthant $\mathbb{R}^m_+$ and the Lorentz cone L defined above are the most important and commonly used self-dual cones in the Euclidean space. But the family of self-dual cones is rather rich [21].

3 Generalized Lattice Operations

Denote by P_D the projection mapping onto a nonempty closed convex set $D \subset \mathbb{R}^m$, that is, the mapping which associates to $\mathbf{x} \in \mathbb{R}^m$ the unique nearest point of x in D [28]:

$$P_D\mathbf{x} \in D \ \text{ and } \ \|\mathbf{x} - P_D\mathbf{x}\| = \inf\{\|\mathbf{x} - \mathbf{y}\| : \ \mathbf{y} \in D\}.$$

The nearest point $P_D\mathbf{x}$ can be characterized by

$$P_D\mathbf{x} \in D \text{ and } \langle P_D\mathbf{x} - \mathbf{x}, P_D\mathbf{x} - \mathbf{y}\rangle \le 0, \ \forall \mathbf{y} \in D. \tag{2}$$

From the definition of the projection and the characterization (2), there follow immediately the relations:

$$P_D(-\mathbf{x}) = -P_{-D}\mathbf{x}, \tag{3}$$

$$P_{\mathbf{x}+D}\mathbf{y} = \mathbf{x} + P_D(\mathbf{y} - \mathbf{x}) \tag{4}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$.

For a closed convex cone K, we define the following operations in $\mathbb{R}^m$:

$$\mathbf{x} \sqcap_K \mathbf{y} = P_{\mathbf{x}-K}\mathbf{y}, \text{ and } \mathbf{x} \sqcup_K \mathbf{y} = P_{\mathbf{x}+K}\mathbf{y}$$

(see [1]). Assume the operations $\sqcup_K$ and $\sqcap_K$ have precedence over the addition of vectors and multiplication of vectors by scalars.

A direct checking yields that if $K = \mathbb{R}^m_+$, then $\sqcap_K = \wedge$, and $\sqcup_K = \vee$. That is $\sqcap_K$ and $\sqcup_K$ are some *generalized lattice operations*. Moreover, $\sqcap_K$ and $\sqcup_K$ *are lattice operations if and only if the self-dual cone used in their definitions is a nonnegative orthant of some Cartesian reference system.* This suggests to call the operations $\sqcap_K$ and $\sqcup_K$ *lattice-like operations*, while a subset $M \subset \mathbb{R}^m$ which is *invariant with respect to* $\sqcap_K$ *and* $\sqcup_K$ (i.e., if for any $\mathbf{x}, \mathbf{y} \in M$, we have $\mathbf{x} \sqcap_K \mathbf{y}, \ \mathbf{x} \sqcup_K \mathbf{y} \in M$) a *lattice-like* or simply an *l-l* subset of $(\mathbb{R}^m, K)$.

The following assertions are direct consequences of the definition of lattice-like operations:

Lemma 1. *The following relations hold for any* $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^m, K)$*:*

$$\mathbf{x} \sqcap_K \mathbf{y} = \mathbf{x} - P_K(\mathbf{x} - \mathbf{y}),$$
$$\mathbf{x} \sqcup_K \mathbf{y} = \mathbf{x} + P_K(\mathbf{y} - \mathbf{x}).$$

A *hyperplane through the origin* is a set of form

$$H(\mathbf{0}, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^m : \ \langle \mathbf{a}, \mathbf{x}\rangle = 0\}, \quad \mathbf{a} \neq \mathbf{0}. \tag{5}$$

For simplicity the hyperplanes through $\mathbf{0}$ will also be denoted by H. The nonzero vector $\mathbf{a}$ in the above formula is called *the normal* of the hyperplane.

A *hyperplane through* $\mathbf{u} \in \mathbb{R}^m$ *with the normal* $\mathbf{a}$ is the set of the form

$$H(\mathbf{u}, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^m : \ \langle \mathbf{a}, \mathbf{x}\rangle = \langle \mathbf{a}, \mathbf{u}\rangle, \ \mathbf{a} \neq \mathbf{0}\}. \tag{6}$$

A hyperplane $H(\mathbf{u}, \mathbf{a})$ determines two *closed halfspaces* $H_-(\mathbf{u}, \mathbf{a})$ and $H_+(\mathbf{u}, \mathbf{a})$ of $\mathbb{R}^m$, defined by

$$H_-(\mathbf{u}, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^m : \ \langle \mathbf{a}, \mathbf{x}\rangle \le \langle \mathbf{a}, \mathbf{u}\rangle\},$$

and

$$H_+(\mathbf{u}, \mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{a}, \mathbf{x} \rangle \geq \langle \mathbf{a}, \mathbf{u} \rangle\}.$$

If K is a nonzero closed convex cone, then the closed convex set $C \subset \mathbb{R}^m$ is called a *K-isotone projection set* or simply *K-isotone* if $\mathbf{x} \leq_K \mathbf{y}$ implies $P_C\mathbf{x} \leq_K P_C\mathbf{y}$. In this case we use equivalently the term *P_C is K-isotone*.

We shall refer next often to the following theorems:

Theorem 1 ([3]). *Let $K \subset \mathbb{R}^m$ be a closed convex cone. Then, C is a lattice-like set, if and only if P_C is K-isotone.*

Theorem 2 ([2]). *The closed convex set C with nonempty interior in $(\mathbb{R}^m, K)$ is lattice-like, if and only if it is of form*

$$C = \bigcap_{i \in \mathbb{N}} H_-(\mathbf{u}_i, \mathbf{a}_i),$$

where each hyperplane $H(\mathbf{u}_i, \mathbf{a}_i)$ through $\mathbf{u}_i$ with the normal $\mathbf{a}_i$ is tangent to C and is lattice-like.

4 Characterization of the Lattice-like Subspaces of $(\mathbb{R}^m, K)$

Denote by K a closed convex cone in $\mathbb{R}^m$ and by $(\mathbb{R}^m, K)$ the resulting ordered vector space.

The notation $G \Subset H$ will mean *H and G are subspaces of $\mathbb{R}^m$ and G is a subspace of H*. Let $H \Subset \mathbb{R}^m$ and $L \subset H$ a closed convex cone. The notation $G \sqsubset_L H$ will mean *G is an l-l subspace of (H, L)*.

We gather some results from Theorem 1 [3] and Lemma 6 [2] and particularize them for subspaces:

Corollary 1. *Let H a subspace in $(\mathbb{R}^m, K)$. the following assertions are equivalent:*

1. *$H \sqsubset_K \mathbb{R}^m$,*
2. *$P_K H \subset H$,*
3. *$P_H K \subset K$.*

Proof. The corollary is in fact a reformulation of Theorem 1 for the case of $D = H$ a subspace. Indeed, condition 2 is nothing else as the l-l property of H since if $\mathbf{x}, \mathbf{y} \in H$, then by Lemma 1, one has

$$\mathbf{x} \sqcap_K \mathbf{y} = \mathbf{x} - P_K(\mathbf{x} - \mathbf{y}) \in H,$$

since $\mathbf{x}$, $\mathbf{x} - \mathbf{y}$, $P_K(\mathbf{x} - \mathbf{y}) \in H$.

Similarly, $\mathbf{x} \sqcup_K \mathbf{y} \in H$.

Condition 3 expresses, by the linearity of P_H, its K-isotonicity.

Corollary 2. *Let $G \Subset H$ and $H \sqsubset_K \mathbb{R}^m$. Then, $G \sqsubset_{K\cap H} H \Leftrightarrow G \sqsubset_K \mathbb{R}^m$.*

Proof. In our proof we shall use without further comments the equivalences in Corollary 1.

Let $G \Subset H$ and $H \sqsubset_K \mathbb{R}^m$.

First suppose that

$$G \sqsubset_K \mathbb{R}^m,$$

which is equivalent to

$$P_G K \subset K.$$

Hence,

$$P_G(H \cap K) \subset P_G K \subset H \cap K,$$

since $P_G(K) \subset G \subset H$. Thus, $G \sqsubset_{K\cap H} H$.

Conversely, assume that $G \sqsubset_{H\cap K} H$. Then

$$P_G(H \cap K) \subset H \cap K \subset K.$$

Whereby, since $P_G = P_G P_H$, one has

$$P_G K = P_G P_H (H \cap K) = P_G(H \cap K) \subset K.$$

Thus, $G \sqsubset_K \mathbb{R}^m$.

Lemma 2. *Suppose that K is a closed convex cone in $\mathbb{R}^m$. Let $H(\mathbf{0}, \mathbf{a}) \subset \mathbb{R}^m$ be a hyperplane through the origin with unit normal vector $\mathbf{a} \in \mathbb{R}^m$. Then, the following assertions are equivalent:*

(i) $P_{H(\mathbf{0},\mathbf{a})}$ *is K-isotone;*
(ii) $P_{H(\mathbf{b},\mathbf{a})}$ *is K-isotone for any* $\mathbf{b} \in \mathbb{R}^m$;
(iii)

$$\langle \mathbf{x}, \mathbf{y} \rangle \geq \langle \mathbf{a}, \mathbf{x} \rangle \langle \mathbf{a}, \mathbf{y} \rangle ,$$

for any $\mathbf{x}, \mathbf{y} \in K$.

Proof. The lemma is a direct consequence of some results in [3]. We give for completeness its proof here.

The equivalence of (i) and (ii) follows from the formulas

$$(\mathbf{x} + \mathbf{z}) \sqcap_K (\mathbf{y} + \mathbf{z}) = \mathbf{x} \sqcap_K \mathbf{y} + \mathbf{z}$$

and

$$(\mathbf{x}+\mathbf{z}) \sqcup_K (\mathbf{y}+\mathbf{z}) = \mathbf{x} \sqcup_K \mathbf{y} + \mathbf{z},$$

which are consequences of Lemma 1.

We shall prove that (i)⇔(iii).

Since P_H is linear, it follows that P_H is isotone if and only if

$$P_H\mathbf{x} = \mathbf{x} - \langle \mathbf{a}, \mathbf{x} \rangle \mathbf{a} \in K, \ \forall \mathbf{x} \in K. \tag{7}$$

By the self-duality of K, it follows that relation (7) is equivalent to

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{a}, \mathbf{x} \rangle \langle \mathbf{a}, \mathbf{y} \rangle + \langle \mathbf{x} - \langle \mathbf{a}, \mathbf{x} \rangle \mathbf{a}, \mathbf{y} \rangle \geq \langle \mathbf{a}, \mathbf{x} \rangle \langle \mathbf{a}, \mathbf{y} \rangle ,$$

for any $\mathbf{y} \in K$.

5 Lattice-like Subspaces of the Euclidean Jordan Algebra

In the particular case of a self-dual cone $K \subset \mathbb{R}^m$, J. Moreau's theorem [29] reduces to the following lemma:

Lemma 3. *Let $K \subset \mathbb{R}^m$ be a self-dual cone. Then, for any $\mathbf{x} \in \mathbb{R}^m$ the following two conditions are equivalent:*

(i) $\mathbf{x} = \mathbf{u} - \mathbf{v}, \ \mathbf{u}, \mathbf{v} \in K, \langle \mathbf{u}, \mathbf{v} \rangle = 0,$
(ii) $\mathbf{u} = P_K\mathbf{x}, \ \mathbf{v} = P_K(-\mathbf{x}).$

In all what follows, we will consider that the ordered Euclidean space is (V, Q), the Euclidean Jordan algebra V of unit $\mathbf{e}$ ordered by the cone Q of squares in V. All the terms concerning V will be equally used for (V, Q).

Since the hyperplanes in Theorem 2 play an important role, and since the l-l property is invariant with respect to translations (Lemma 3, [2]), it is natural to study the l-l subspaces in V which are naturally connected with the algebraic structure of this space.

Theorem 3. *Any Jordan subalgebra of (V, Q) is a lattice-like subspace.*

Proof. Take a Jordan subalgebra L in V and denote by Q_0 its cone of squares. We have

$$Q_0 = \{\mathbf{x}^2 : x \in L\} \subset \{\mathbf{x}^2 : x \in V\} = Q. \tag{8}$$

We shall prove that

$$\mathbf{x} \in L \Rightarrow P_Q\mathbf{x} = P_{Q_0}\mathbf{x} \in L. \tag{9}$$

Indeed, we have, by Lemma 3 applied in the ordered vector space (L, Q_0), that

$$\mathbf{x} = P_{Q_0}\mathbf{x} - P_{Q_0}(-\mathbf{x}), \quad \langle P_{Q_0}\mathbf{x}, P_{Q_0}(-\mathbf{x})\rangle = 0, \tag{10}$$

By (8)

$$P_{Q_0}\mathbf{x}, \quad P_{Q_0}(-\mathbf{x}) \in Q_0 \subset Q,$$

which, by Eqs. (10) and Lemma 3, yield $P_{Q_0}\mathbf{x} = P_Q\mathbf{x}$, or equivalently (9).
Accordingly $P_Q L \subset L$, which by Corollary 1 translates into $L \sqsubset_Q V$.

6 The Pierce Decomposition of the Euclidean Jordan Algebra and Its Lattice-like Subspaces

Let r be the rank of V and $\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$ be an arbitrary Jordan frame in V, that is, $\mathbf{c}_k$ are primitive idempotents such that

$$\mathbf{c}_i\mathbf{c}_j = 0, \quad \text{if } i \neq j, \quad \mathbf{c}_i^2 = \mathbf{c}_i,$$

$$\mathbf{c}_1 + \dots + \mathbf{c}_r = \mathbf{e}.$$

With the notation

$$V_{ii} = V(\mathbf{c}_i, 1) = \mathbb{R}\mathbf{c}_i,$$

$$V_{ij} = V\left(\mathbf{c}_i, \frac{1}{2}\right) \cap V\left(\mathbf{c}_j, \frac{1}{2}\right)$$

(where for $\lambda \in \mathbb{R}$, $V(\mathbf{c}_i, \lambda) = \{\mathbf{x} \in V : \mathbf{c}_i\mathbf{x} = \lambda\mathbf{x}\}$), we have by Theorem IV.2.1. [26] the following orthogonal decomposition (the so-called Pierce decomposition) of V:

$$V = \bigoplus_{i \le j} V_{ij}, \tag{11}$$

where

$$V_{ij}V_{ij} \subset V_{ii} + V_{jj}; \; V_{ij}V_{jk} \subset V_{ik}, \quad \text{if } i \neq k; \; V_{ij}V_{kl} = \{0\}, \quad \text{if } \{i,j\} \cap \{k,l\} = \emptyset. \tag{12}$$

Taking for $1 \le k < r$

$$V^{(k)} = \bigoplus_{i \le j \le k} V_{ij} \tag{13}$$

is a Jordan algebra with the unit

$$\mathbf{e}_k = \mathbf{c}_1 + \cdots + \mathbf{c}_k.$$

Indeed, relations (12) imply the invariance of $V^{(k)}$ with respect to the Jordan product. The same relations and the definitions imply $\mathbf{e}_k\mathbf{x}_{ii} = \mathbf{c}_i\mathbf{x}_{ii} = \mathbf{x}_{ii}$, for any $\mathbf{x}_{ii} \in V_{ii}$ and $i \leq k$; $\mathbf{c}_l V_{ij} = \{0\}$ if $l \notin \{i,j\}$; $\mathbf{e}_k\mathbf{x}_{ij} = (\mathbf{c}_i + \mathbf{c}_j)\mathbf{x}_{ij} = \mathbf{x}_{ij}$, for any $\mathbf{x}_{ij} \in V_{ij}$ and $i,j \leq k$, $i \neq j$. Hence, $\mathbf{e}_k$ is the unity of $V^{(k)}$. These relations also imply that

$$V^{(k)} = V(\mathbf{e}_k, 1) = \{\mathbf{x} \in V : \mathbf{e}_k\mathbf{x} = \mathbf{x}\}. \tag{14}$$

Thus, $V(\mathbf{e}_k, 1)$ is a subalgebra (this follows also by Proposition IV.1.1 in [26] since $\mathbf{e}_k$ is idempotent). Hence, by Theorem 3, $V(\mathbf{e}_k, 1)$ is an l-l subspace in (V, Q).

A Jordan algebra is said *simple* if it contains no nontrivial ideal.

A consequence of the above cited theorem and the content of paragraph IV.2 of [26] is that V is simple if and only if $V_{ij} \neq \{0\}$ for any V_{ij} in (11). By the same conclusion, $V^{(k)}$ given by (13) is simple too, and by Corollary IV.2.6 in [26], the spaces $V_{ij},\ i \neq j$ have the common dimension d; hence, by (13),

$$\dim V^{(k)} = k + \frac{d}{2}k(k-1).$$

The subcone $F \subset Q$ is called a *face of* Q if whenever $0 \leq_Q \mathbf{x} \leq_Q \mathbf{y}$ and $\mathbf{y} \in F$ it follows that $\mathbf{x} \in F$.

It is well known that for an arbitrary face F of Q, one has $P_{\operatorname{span} F} Q \subset Q$ (see, e.g., Proposition II.1.3 in [30]). By Corollary 1 it follows thus the assertion:

Corollary 3. *Each subspace generated by some face of Q is a lattice-like subspace in (V, Q).*

We give an independent proof of this.

Proof. Let $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ be a Jordan frame in V, $k \leq r$. If

$$\mathbf{e}_k = \mathbf{c}_1 + \cdots + \mathbf{c}_k, \quad 0 \leq k \leq r,$$

then by Theorem 3.1 in [31],

$$F = V(\mathbf{e}_k, 1) \cap Q = \{x \in Q : \mathbf{e}_k\mathbf{x} = \mathbf{x}\}$$

is a face of Q, and each face of Q can be represented in this form for some Jordan frame.

The cone $F = V(\mathbf{e}_k, 1) \cap Q$ is the cone of squares in the subalgebra $V(\mathbf{e}_k, 1)$; hence, its relative interior is nonempty, accordingly

$$V(\mathbf{e}_k, 1) = \operatorname{span} F = F - F.$$

Since $V(\mathbf{e}_k, 1)$ is a subalgebra, by Theorem 3, it is an l-l subspace.

7 The Subalgebras and the Lattice-like Subspaces of the Space Spanned by a Jordan Frame

Suppose that the dimension of the Euclidean Jordan algebra V is at least 2. Let $\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$ be a Jordan frame in V. Then,

$$V_r := \operatorname{span}\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$$

is a Jordan subalgebra of V. Obviously, $V_r = V_{11} \oplus \cdots \oplus V_{rr}$. If $\mathbf{x}, \mathbf{y} \in V_r$, then

$$\mathbf{x}\mathbf{y} = (x_1 y_1, \dots, x_r y_r),$$

where x_i and y_i are the coordinates of $\mathbf{x}$ and $\mathbf{y}$, respectively, with respect to the above Jordan frame.

By using the notations of the above section, denote $Q_r = Q \cap V_r$ and let us show that

$$Q_r = \operatorname{cone}\{\mathbf{c}_1, \dots, \mathbf{c}_r\} := \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{c}_i : \lambda_i \geq 0,\ \forall 1 \leq i \leq r \right\}.$$

The inclusion $\operatorname{cone}\{\mathbf{c}_1, \dots, \mathbf{c}_r\} \subset Q_r$ is obvious. Next, we show that $Q_r \subset \operatorname{cone}\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$. Suppose to the contrary that there exists $\mathbf{x} \in Q_r \setminus \operatorname{cone}\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$. It follows that $\langle \mathbf{c}_k, \mathbf{x} \rangle < 0$ for some $k \in \{1, \dots, r\}$. Since Q is self-dual, this implies $\mathbf{x} \notin Q$, which is a contradiction.

The ordered vector space (V_r, Q_r) can be considered an r-dimensional Euclidean vector space ordered with the positive orthant Q_r engendered by the Jordan frame.

Let H_{r-1} be an l-l hyperplane in (V_r, Q_r), with the unit normal $\mathbf{a} \in V_r$. Thus, the results in [2] and [3] applies; hence, if

$$\mathbf{a} = (a_1, \dots, a_r), \tag{15}$$

then we must have

$$a_i a_j \leq 0, \quad \text{if} \quad i \neq j. \tag{16}$$

Then, there are two possibilities:

Case 1. There exists an i such that $a_i = 1$ and $a_j = 0$ for $j \neq i$.
Case 2. There are only two nonzero coordinates, say a_k and a_l with $a_k a_l < 0$.

Ad 1. In **Case 1**,

$$H_{r-1} = \operatorname{span}\{\mathbf{c}_1, \dots, \mathbf{c}_{i-1}, \mathbf{c}_{i+1}, \dots, \mathbf{c}_r\}$$

and H_{r-1} is obviously a Jordan algebra.

Ad 2. In **Case 2**,

$$H_{r-1} = \{\mathbf{x} \in V_r : a_k x_k + a_l x_l = 0\}.$$

We know from the above cited result that H_{r-1} is an l-l subspace in (V_r, Q_r), and since V_r is a subalgebra of V, by Theorem 3, $V_r \sqsubset_Q V$. By using Corollary 2, we have, for the l-l subspace, $H_{r-1} \sqsubset_{Q_r} V_r$, that is,

$$H_{r-1} \sqsubset_Q V.$$

In the case **Ad 1**, the l-l hyperplane H_{r-1} is also a Jordan algebra.

Suppose that **Ad 2** holds. We would like to see under which condition the l-l hyperplane H_{r-1} is a Jordan algebra.

Let us suppose that H_{r-1} is a Jordan algebra and take $\mathbf{x} \in H_{r-1}$, $\mathbf{x} = (x_1, \dots, x_r)$. Then, $\mathbf{x}^2 = (x_1^2, \dots, x_r^2) \in H_{r-1}$. Take $\mathbf{x}$ with $x_l = a_k$ and $x_k = -a_l$. Then, $\mathbf{x} \in H_{r-1}$ and we must have $\mathbf{x}^2 \in H_{r-1}$. Hence

$$a_k a_l^2 + a_l a_k^2 = a_k a_l (a_l + a_k) = 0,$$

and since $a_k a_l \neq 0$, we must have

$$a_k = \frac{\sqrt{2}}{2} \qquad a_l = -\frac{\sqrt{2}}{2},$$

or conversely. In this case

$$H_{r-1} = \{\mathbf{x} : x_k = x_l\} \tag{17}$$

is obviously a subalgebra.

Remark 1. For any hyperplane H_{r-1} in V_r with the unit normal $\mathbf{a}$ having only two nonzero components with opposite signs and different absolute values, H_{r-1} is an l-l subspace, but not a Jordan subalgebra.

If $\mathbf{a} \in V$ is arbitrary, then there exists a Jordan frame $\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$ such that $\mathbf{a}$ can be represented in the form (15) (Theorem III.1.2 in [26]). We will call such a Jordan frame as being *attached to* $\mathbf{a}$.

Corollary 4. *Let H be a lattice-like hyperplane in (V, Q) with the normal* $\mathbf{a}$ *and* $\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$ *be a Jordan frame attached to it. If* $\mathbf{a}$ *is represented by (15), then the coordinates a_i, $i = 1, \dots, r$ of* $\mathbf{a}$ *satisfy the relations (16).*

Proof. If $V_r = \operatorname{span}\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$, then $H_{r-1} = H \cap V_r$ is an l-l hyperplane in (V_r, Q_r) with the normal $\mathbf{a}$ because it is the intersection of two l-l sets: V_r (a subalgebra) and H. Thus, we can apply the characterization of l-l hyperplanes in (V_r, Q_r) described above in this section.

Denote by $\mathscr{F}(Q)$ the family of faces of Q, by $\mathscr{A}$ the family of subalgebras of V and by $\mathscr{L}$ the family of the l-l subspaces in V. Then, by the above reasonings, we conclude:

Corollary 5. *We have the following strict inclusions:*

$$\{\operatorname{span} F : F \in \mathscr{F}(Q)\} \subset \mathscr{A} \subset \mathscr{L}.$$

Proof. The second strict inclusion follows from Remark 1. The first inclusion is strict since, for instance, the subspaces in (17) are subalgebras which are not generated by faces of Q. Indeed, take in V_r the reference system engendered by $\mathbf{c}_1, \ldots, \mathbf{c}_r$ and let

$$H_{r-1} = \{(t, t, x_{r+3}, \ldots, x_r) \in V_r : t, x_j \in \mathbb{R}\}.$$

Take $\mathbf{y} = (1, 1, 0, \ldots, 0)$ and $\mathbf{x} = (1, 0, 0, \ldots, 0)$ in $Q_r = Q \cap V_r$. Since in V_r $\leq_Q = \leq_{Q_r}$ and the latter is coordinate-wise ordering,

$$0 \leq_Q \mathbf{x} \leq_Q \mathbf{y},$$

and we have $\mathbf{y} \in H_{r-1} \cap Q$, but $\mathbf{x} \notin H_{r-1} \cap Q$, which shows that $H_{r-1} \cap Q$, is not a face.

8 The Inexistence of Lattice-like Hyperplanes in Simple Euclidean Jordan Algebras of Rank $r \geq 3$

Theorem 4. *Suppose that V is a simple Euclidean Jordan algebra of rank $r \geq 3$. Then, V does not contain lattice-like hyperplanes.*

Proof. Assume the contrary: H is an l-l hyperplane through 0 in V with the unit normal $\mathbf{a}$.

Consider a Jordan frame $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ attached to $\mathbf{a}$.

The set

$$H_{r-1} = H \cap V_r$$

is obviously a hyperplane through 0 in V_r.

Since by hypothesis $H \sqsubset_Q V$, by Corollary 2, $H_{r-1} \sqsubset_{Q_r} V_r$, where $Q_r = Q \cap V_r$.

If $\mathbf{a} = (a_1, \ldots, a_r)$ is the representation of $\mathbf{a}$ in the reference system engendered by the Jordan frame, then using Corollary 4, the l-l property of H_{r-1} in (V_r, Q_r) implies that one of the following cases must hold:

Case 1. For some i $a_i = 1$ and $a_j = 0$ for $j \neq i$.
Case 2. There are only two nonzero coordinates, say a_i and a_j with $a_i a_j < 0$.

Suppose that $i = 1$, $j = 2$.

Since V is simple, $V_{12} \neq \{0\}$ (by Proposition IV.2.3 [26]); hence, we can take $\mathbf{x} \in V_{12}$ with $\|\mathbf{x}\|^2 = 2$. Then, by Exercise IV. 7 in [26], we have that

$$\mathbf{u} = \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2 + \frac{1}{2}\mathbf{x}, \qquad \text{and} \qquad \mathbf{v} = \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2 - \frac{1}{2}\mathbf{x} \tag{18}$$

are idempotent elements; hence, $\mathbf{u}, \mathbf{v} \in Q$. We further have

$$\mathbf{uv} = \left(\frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2\right)^2 - \frac{1}{4}\mathbf{x}^2,$$

whereby, by using Proposition IV.1.4 in [26], we have

$$\mathbf{x}^2 = \frac{1}{2}\|\mathbf{x}\|^2(\mathbf{c}_1 + \mathbf{c}_2) = \mathbf{c}_1 + \mathbf{c}_2,$$

and after raising to the second power and substitution,

$$\mathbf{uv} = \frac{1}{4}\mathbf{c}_1 + \frac{1}{4}\mathbf{c}_2 - \frac{1}{4}(\mathbf{c}_1 + \mathbf{c}_2) = 0.$$

Hence,

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

Since H_{r-1} is 1-1, we have by Lemma 2,

$$0 = \langle \mathbf{u}, \mathbf{v} \rangle \geq \langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{a}, \mathbf{v} \rangle .$$

If $a_1 = 1$, and $a_j = 0$ for $j \neq 1$, the above relation becomes $0 \geq \frac{1}{4}\|\mathbf{c}_1\|^4$, which is impossible.

Assume $a_1 a_2 < 0$ and $a_j = 0$ for $j > 2$.

Take now

$$\begin{aligned} \mathbf{w} &= \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_3 + \frac{1}{2}\mathbf{y} \\ \mathbf{z} &= \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_3 - \frac{1}{2}\mathbf{y} \end{aligned} \tag{19}$$

with $\mathbf{y} \in V_{13},\ \|\mathbf{y}\|^2 = 2$. Then, $\mathbf{w}, \mathbf{z} \in Q$ (similarly to $\mathbf{u}, \mathbf{v} \in Q$) and, by using the mutual orthogonality of the elements $\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3,\ \mathbf{y}$ and Lemma 2, it follows that

$$\begin{aligned} &0 = \langle \mathbf{w}, \mathbf{z} \rangle \geq \langle \mathbf{a}, \mathbf{w} \rangle \langle \mathbf{a}, \mathbf{z} \rangle \\ &= \left\langle a_1\mathbf{c}_1 + a_2\mathbf{c}_2, \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_3 + \frac{1}{2}\mathbf{y} \right\rangle \left\langle a_1\mathbf{c}_1 + a_2\mathbf{c}_2, \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_3 - \frac{1}{2}\mathbf{y} \right\rangle = \frac{1}{4}a_1^2\|\mathbf{c}_1\|^2, \end{aligned}$$

which is a contradiction.

This theorem collated with Theorem 2 and Lemma 3 in [2] yields

Corollary 6. *In the ordered Euclidean Jordan algebra* (V, Q) *of rank at least* 3, *there is no proper closed convex lattice-like set with nonempty interior. In particular, for* $n \geq 3$ *the ordered space* $(\mathbb{S}^m, \mathbb{S}^m_+)$ *contains no proper, closed, convex lattice-like set with nonempty interior.*

9 The Case of the Simple Euclidean Jordan Algebras of Rank 2

A simple Euclidean Jordan algebra of rank 2 is isomorphic to an algebra associated with a positive definite bilinear form (Corollary IV.1.5 [26]). This is in fact a Jordan algebra associated with the Lorentz cone. Hence, the problem of the existence of l-l hyperplanes in this case is answered positively in [2] and [3]. In this section we use the formalism developed in the preceding sections to this case too.

Lemma 4. *Suppose that* $\mathbf{a}$ *is the unit normal to a lattice-like hyperplane* H *through* 0 *in the simple Euclidean Jordan algebra* V *of rank 2. Let* $\{\mathbf{c}_1, \mathbf{c}_2\}$ *be the Jordan frame attached to* $\mathbf{a}$ *and* $\mathbf{a} = a_1\mathbf{c}_1 + a_2\mathbf{c}_2$. *Then, supposing* $a_1 > 0$, *we obtain*

$$\mathbf{a} = \frac{\sqrt{2}}{2}\mathbf{c}_1 - \frac{\sqrt{2}}{2}\mathbf{c}_2. \tag{20}$$

Proof. Take $\mathbf{u}$ and $\mathbf{v}$ as in the formula (18). Then, $\mathbf{u}, \mathbf{v} \in Q$ and using Lemma 2 we obtain

$$\begin{aligned} 0 &= \langle \mathbf{u}, \mathbf{v} \rangle \geq \langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{a}, \mathbf{v} \rangle \\ &= \left\langle a_1\mathbf{c}_1 + a_2\mathbf{c}_2, \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2 + \frac{1}{2}\mathbf{x} \right\rangle \left\langle a_1\mathbf{c}_1 + a_2\mathbf{c}_2, \frac{1}{2}\mathbf{c}_1 + \frac{1}{2}\mathbf{c}_2 - \frac{1}{2}\mathbf{x} \right\rangle = \frac{1}{4}(a_1 + a_2)^2, \end{aligned}$$

whereby our assumption follows.

Theorem 5. *Let* V *be a simple Euclidean Jordan algebra of rank* 2 *and* H *be a hyperplane through* 0 *with unit normal* $\mathbf{a}$ *in* V. *Then,* H *is lattice-like if and only if* $\mathbf{a} = \sqrt{2}/2\mathbf{c}_1 - \sqrt{2}/2\mathbf{c}_2$ *in its Jordan frame representation. In this case* H *is a subalgebra.*

Proof. Suppose that $H = \ker \mathbf{a}$, $\|\mathbf{a}\| = 1$ is l-l, and that the Jordan frame attached to $\mathbf{a}$ is $\{\mathbf{c}_1, \mathbf{c}_2\}$.

Then, by Lemma 4, it follows that $\mathbf{a}$ is of the form (20).

Suppose that the Jordan frame representation of $\mathbf{a}$ is of the form (20). Then, Eqs. (11) and (12) imply that

$$\ker \mathbf{a} = \{t(\mathbf{c}_1 + \mathbf{c}_2) + \mathbf{x} = t\mathbf{e} + \mathbf{x} : t \in \mathbb{R},\ \mathbf{x} \in V_{12}\}.$$

Then, for two arbitrary elements $\mathbf{u}, \mathbf{v} \in \ker \mathbf{a}$, we have the representations:

$$\mathbf{u} = t_1\mathbf{e} + \mathbf{x};\ \mathbf{v} = t_2\mathbf{e} + \mathbf{y};\ \mathbf{x}, \mathbf{y} \in V_{12};\ t_i \in \mathbb{R},\ i = 1, 2.$$

Then,

$$\mathbf{uv} = t_1t_2\mathbf{e} + t_1\mathbf{y} + t_2\mathbf{x} + \mathbf{xy}.$$

Since $\mathbf{xy} = (1/4)((\mathbf{x} + \mathbf{y})^2 - (\mathbf{x} - \mathbf{y})^2)$, by using Proposition IV.1.4 in [26], we conclude that $\mathbf{xy} = q(\mathbf{c}_1 + \mathbf{c}_2) = q\mathbf{e}$ with $q \in \mathbb{R}$. Hence,

$$\mathbf{uv} = (t_1t_2 + q)\mathbf{e} + t_1\mathbf{y} + t_2\mathbf{x} \in \ker \mathbf{a}.$$

This shows that $H = \ker \mathbf{a}$ is a subalgebra and hence an l-l set.

Remark 2. With the notations in the above proof, we have that $\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2\}$ is a subalgebra of dimension 2 in V.

Similarly to Remark 1, it follows that there exist l-l subspaces of dimension 1 in $\operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2\}$ which are not subalgebras.

Collating Theorem 5 and Theorem 2, it follows the result:

Corollary 7. *The closed convex set with nonempty interior $M \subset V$ is a lattice-like set if and only if it is of the form*

$$M = \bigcap_{i \in \mathbb{N}} H_-(\mathbf{u}_i, \mathbf{a}_i),$$

with the $\mathbf{a}_i$ normal unit vectors represented in their Jordan frame $\mathbf{c}_1^i$, $\mathbf{c}_2^i$ by

$$\mathbf{a}_i = \varepsilon_i \left(\frac{\sqrt{2}}{2}\mathbf{c}_1^i - \frac{\sqrt{2}}{2}\mathbf{c}_2^i \right),\ \varepsilon_i = 1 \text{ or } -1. \tag{21}$$

Example 1. Write the elements of $\mathbb{R}^{m+1}$ in the form $(\mathbf{x}, x_{m+1})$ with $\mathbf{x} \in \mathbb{R}^m$ and $x_{m+1} \in \mathbb{R}$. The Jordan product in $\mathbb{R}^{m+1}$ is defined by

$$(\mathbf{x}, x_{m+1}) \circ (\mathbf{y}, y_{m+1}) = (y_{m+1}\mathbf{x} + x_{m+1}\mathbf{y}, \langle \mathbf{x}, \mathbf{y} \rangle + x_{m+1}y_{m+1}),$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the usual scalar product in $\mathbb{R}^m$. The space $\mathbb{R}^{m+1}$ equipped with the usual scalar product and the operation $\circ$ just defined becomes an Euclidean Jordan algebra of rank 2, denoted by $\mathscr{L}^{m+1}$ [1], with the cone of squares $Q = \mathscr{L}_+^{m+1}$, the Lorentz cone defined by (1).

The unit element in $\mathscr{L}^{m+1}$ is $(\mathbf{0}, 1)$, where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

The Jordan frame attached to $(\mathbf{x}, x_{m+1}) \in \mathscr{L}^{m+1}$ with $\mathbf{x} \neq 0$ is

$$\mathbf{c}_1 = \frac{1}{2}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, 1 \right), \quad \mathbf{c}_2 = \frac{1}{2}\left(-\frac{\mathbf{x}}{\|\mathbf{x}\|}, 1 \right).$$

The unit normal **a** from Lemma 4 will be then parallel with $(\mathbf{b}, 0)$ with some $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{b} \neq \mathbf{0}$. This means that the hyperplanes $H(\mathbf{u}_i, \mathbf{a}_i)$ in Corollary 7 are parallel with the $m + 1$th axis and the closed convex set in the corollary is in fact of the form

$$M = C \times \mathbb{R},$$

with C a closed convex set with nonempty interior in $\mathbb{R}^m$.

This is exactly the result in Example 1 of [3].

10 The General Case

For a general Euclidean Jordan algebra V, gathering the results of Proposition III.4.4, Proposition III.4.5, and Theorem V.3.7, of [26], in Theorem 5 of [1], the following result is stated:

Theorem 6. *Any Euclidean Jordan algebra V is, in unique way, a direct sum*

$$V = \oplus_{i=1}^{k} V_i \tag{22}$$

of simple Euclidean Jordan algebras V_i, $i = 1, \ldots, k$. Moreover, the cone of squares Q in V is, in a unique way, a direct sum

$$Q = \oplus_{i=1}^{k} Q_k \tag{23}$$

of the cones of squares Q_i in V_i, $i = 1, \ldots, k$.

(Here the direct sum (by a difference to that in the Pierce decomposition) means Jordan algebraic and hence also orthogonal direct sum.)

Let $C \subset V$ a closed convex set. From the results in Theorem 6, it follows easily (using the notations in the theorem) that

$$P_C = \sum_{i=1}^{k} P_{C_i}, \tag{24}$$

with $C_i = C \cap V_i$, $i = 1, \ldots, k$.

Collating these results with Corollary 2, we have the following:

Corollary 8. *With the notations in Theorem 6, for the subspace $M \Subset V$, we have the equivalence:*

$$M \sqsubset_Q V \Leftrightarrow M \cap V_i \sqsubset_{Q_i} V_i, \; i = 1, \ldots, k. \tag{25}$$

For the closed convex set C, the projection P_C is Q-isotone if and only if $P_{C \cap V_i}$ is Q_i-isotone in (V_i, Q_i), $i = 1, \ldots k$.

Corollary 9. *If H is a lattice-like hyperplane in V represented as (22) in Theorem 6, then $V_i \Subset H$ for each simple subalgebra in (22) of rank at least 3.*

Proof. Assume the contrary. Then, $H \cap V_i$ is an l-l hyperplane in V_i, which contradicts Theorem 4.

Gathering the results in Theorem 2, Sect. 7, Corollaries 7, and 9, we have:

Theorem 7. *Suppose that V is an Euclidean Jordan algebra of the form (22) with V_i simple subalgebras. Let us write this sum as*

$$V = W_1 \oplus W_2 \oplus W_3 \tag{26}$$

where

$$W_1 = \oplus_{i \in I_1} V_i, \quad W_2 = \oplus_{i \in I_2} V_i, \quad W_3 = \oplus_{i \in I_3} V_i, \tag{27}$$

such that V_i, for $i \in I_1$, are the subalgebras of rank 1; for $i \in I_2$, the subalgebras of rank 2; and, for $i \in I_3$, the subalgebras of rank at least 3. Then, $C \subset V$ is a closed convex lattice-like subset with nonempty interior if and only if the following conditions hold:

$$C = \bigcap_{i \in \mathbb{N}} H_-(\mathbf{u}_i, \mathbf{a}_i), \tag{28}$$

where each hyperplane $H(\mathbf{u}_i, \mathbf{a}_i)$ through $\mathbf{u}_i$ and with the unit normal $\mathbf{a}_i$ is tangent to C and is lattice-like. Let $\{\mathbf{c}_1^i, \dots, \mathbf{c}_r^i\}$ be a Jordan frame attached to $\mathbf{a}_i$. The last conditions hold if and only if

$$\mathbf{a}_i = a_1^i \mathbf{c}_1^i + \dots + a_{r_1}^i \mathbf{c}_{r_1}^i + a_{r_1+1}^i \mathbf{c}_{r_1+1}^i + \dots + a_{r_2}^i \mathbf{c}_{r_2}^i + a_{r_2+1}^i \mathbf{c}_{r_2+1}^i + \dots + a_r^i \mathbf{c}_r^i \tag{29}$$

with $\mathbf{c}_1^i, \dots, \mathbf{c}_{r_1}^i \in W_1$; $\mathbf{c}_{r_1+1}^i, \dots, \mathbf{c}_{r_2}^i \in W_2$, and $\mathbf{c}_{r_2+1}^i, \dots, \mathbf{c}_r^i \in W_3$, and exactly one of the following two cases hold:

(i) There exists a $k \in \{1, \dots, r_1\}$ with $a_k^i \neq 0$, and exactly one of the following two statements is true:

  (i)' The equality $a_j^i = 0$ holds for $j \neq k$.

  (i)" There exists an $l \in \{1, \dots, r_1\}$ such that $a_l^i a_k^i < 0$ and $a_j^i = 0$, for $j \notin \{k, l\}$.

(ii) There exists $k, l \in \{r_1 + 1, \dots, r_2\}$ and $p \in I_2$ such that $\mathbf{c}_k^i, \mathbf{c}_l^i \in V_p$, $a_k^i = \sqrt{2}/2$, $a_l^i = -\sqrt{2}/2$, and $a_j^i = 0$, for $j \notin \{k, l\}$.

Proof. Observe first that using the representation (29) of a_i and the partition (27) of V, we have the following relations:

$$r_1 = |I_1|, \; r_2 - r_1 = 2|I_2|, \; r - r_2 \geq 3|I_3|.$$

The representation (28) follows from Theorem 2. Let us see first that the alternative (i) and (ii), respectively, is sufficient for $H(\mathbf{u}_i, \mathbf{a}_i)$ to be an l-l set.

If (i) holds, then the hyperplane H_{r_1-1} through 0 with the normal $\mathbf{a}^i_{W_1} = a^i_1\mathbf{c}^i_1 + \ldots + a^i_{r_1}\mathbf{c}^i_{r_1}$ is by Sect. 7 an l-l set in $\operatorname{span}\{\mathbf{c}^i_1, \ldots, \mathbf{c}^i_{r_1}\}$ ordered by the orthant engendered by $\mathbf{c}_1, \ldots, \mathbf{c}_{r_1}$. Hence, $H(\mathbf{u}_i, \mathbf{a}_i) = (H_{r_1-1} + \mathbf{u}_i) \bigoplus W_2 \bigoplus W_3$ is l-l in V (by Theorem 1 and Lemma 2).

If (ii) holds, then the hyperplane H' through 0 with the normal $\mathbf{a}^i_0 = a^i_k\mathbf{c}^i_k + a^i_l\mathbf{c}^i_l = (\sqrt{2}/2)(\mathbf{c}^i_k - \mathbf{c}^i_l)$ in V_p is l-l (by Theorem 5, Theorem 1, and Lemma 2); hence, $H(\mathbf{u}_i, \mathbf{a}_i) = (H' + \mathbf{u}_i) \bigoplus (\bigoplus_{j \neq p} V_j)$ is l-l in V.

To complete the proof, we have to show the necessity of the alternatives (i) and (ii). Observe first that if $H(\mathbf{u}_i, \mathbf{a}_i)$ is l-l, then in the representation (29) of $\mathbf{a}_i$, by Corollary 9, we must have $a^i_j = 0$ whenever $j > r_2$. Thus, if $a^i_j \neq 0$, then $j \leq r_2$.

Suppose that $a^i_k \neq 0$ for some $\mathbf{c}^i_k \in W_2$. Then, there exists an $a^i_l \neq 0$ and $\mathbf{c}^i_k$, $\mathbf{c}^i_l \in V_p$, for some V_p in the representation of W_2. Indeed, in the case $a^i_k \neq 0$, it follows that $a^i_k\mathbf{c}^i_k \in V_p \setminus \{0\}$ for some $V_p \subset W_2$; hence, $H(\mathbf{u}_i, \mathbf{a}_i) \cap V_p$ is a hyperplane in V_p, and our assertion follows from Lemma 4 (and in particular one of a^i_k and a^i_l is $\sqrt{2}/2$ and the other is $-\sqrt{2}/2$). From Corollary 4 it follows then that $a^i_j = 0$ for $j \notin \{k, l\}$. Thus, the alternative (ii) must hold.

Suppose now that $a^i_j \neq 0$ for some $j \leq r_1$. Then from the reasoning of the above paragraph and Corollary 4, we must have $a^i_k = 0$ if $k > r_1$. In this case two situations are possible: (i)' $a^i_j = 1$ and $a^i_l = 0$ for $l \neq j$, and (i)" there exists $a^i_l \neq 0$, $(l \leq r_1)$ with $a^i_j a^i_l < 0$ and $a^i_k = 0$ for $k \notin \{j, l\}$. Thus, the alternative (i) must hold.

Example 2. Let V be a simple Euclidean Jordan algebra with the Pierce decomposition given by (11) and (12) and d the common dimension of V_{ij}, $i \neq j$ (see Corollary IV.2.6 [26]). Denote

$$W_{k,l} = \bigoplus_{k \leq i \leq j \leq l} V_{ij}.$$

Then, $W_{k,l}$ is a subalgebra, hence an l-l subspace. The sum

$$W_{1,k} \bigoplus W_{k+1,r}, \quad k < r$$

is a subalgebra too and hence an l-l subspace. Suppose that $r \geq 4$ and $2 \leq k \leq r-2$. Let H_0 be an l-l hyperplane in $W_{k+1,r}$ which is not a subalgebra. Then,

$$W_{1,k} + H_0$$

is an l-l subspace in V of dimension $k + (d/2)k(k-1) + r - k - 1$ which is not an algebra.

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Simultaneous Diophantine Approximation: Searching for Analogues of Hurwitz's Theorem

Werner Georg Nowak

Abstract Hurwitz's classic theorem of Diophantine approximation tells us that for any irrational number α, there exist infinitely many reduced fractions p/q so that $|\alpha - p/q| < (\sqrt{5}\,q^2)^{-1}$ and that this is no longer true if $\sqrt{5}$ is replaced by some larger constant. Attempting to generalize this to dimensions $s \geq 2$, one is concerned with the problem to determine, resp., to estimate the supremum θ of all reals c so that, for every real but not all rational s-tuple α, there exist infinitely many $\mathbf{p} \in \mathbb{Z}^s$ and positive integers q satisfying $\gcd(\mathbf{p}, q) = 1$ and $|\alpha - q^{-1}\mathbf{p}| < (q\sqrt[s]{cq})^{-1}$, where $|\cdot|$ is any norm in $\mathbb{R}^s$. This survey focuses on the cases of the maximum and the Euclidean norms, giving a survey on the most relevant methods and results on these constants θ.

1 Introduction

For α a real number and $N > 1$ a positive integer, consider the $N + 1$ numbers

$$j\alpha - [j\alpha]\,, \qquad j = 0, 1, \ldots, N\,,$$

where $[\xi]$ denotes throughout the largest integer not exceeding ξ. They are all contained in the half-open unit interval $[0, 1[$; hence, two of them must have a distance less than $\frac{1}{N}$; say, $j\alpha - [j\alpha]$ and $k\alpha - [k\alpha]$, where $j > k$. Writing $j - k =: q$, $[j\alpha] - [k\alpha] =: p$, it follows that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Nq} \leq \frac{1}{q^2}\,, \tag{1}$$

where $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$. This assertion is called *Dirichlet's Theorem*. See, e.g., [11, p. 1]. For α irrational, it readily follows that there exist infinitely many reduced fractions $\frac{p}{q}$, $q > 0$, which satisfy

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}\,. \tag{2}$$

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This result is best possible, apart from a constant factor less than unity on the right-hand side. In fact, the celebrated *Hurwitz's Theorem* tells us that for every irrational number α, there exist infinitely many reduced fractions $\frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\, q^2}\,, \tag{3}$$

and that the constant $\sqrt{5}$ is best possible. See, e.g., [18, pp. 189 and 221].

Applying the above reasoning to a pair of reals (α_1, α_2) and the unit square $[0,1[^2$, we see that, for each positive integer N, there exist integers p_1, p_2, q, with $0 < q \le N^2$, such that

$$\left|\alpha_1 - \frac{p_1}{q}\right| < \frac{1}{Nq} \le \frac{1}{q^{3/2}}\,, \qquad \left|\alpha_2 - \frac{p_2}{q}\right| < \frac{1}{Nq} \le \frac{1}{q^{3/2}}\,.$$

If α_1, α_2 are not both rational, it thus follows that there exist infinitely many pairs of fractions $\left(\frac{p_1}{q}, \frac{p_2}{q}\right)$, with $\gcd(p_1, p_2, q) = 1$ and $q > 0$, such that

$$\left|\alpha_1 - \frac{p_1}{q}\right| < \frac{1}{q(cq)^{1/2}}\,, \qquad \left|\alpha_2 - \frac{p_2}{q}\right| < \frac{1}{q(cq)^{1/2}}\,, \tag{4}$$

as long as we take $c = 1$. The question for the supremum of all numbers c for which this assertion remains true is still to date an open problem!

In fact, on this matter Charles Hermite (1822–1901) wrote:

> La recherche des fractions $p'/p,\ p''/p$ qui approchent le plus de deux nombres donnés n'a cessé depuis plus de 50 ans de me préoccuper et aussi de désespérer.

More generally, for each positive integer s and each constant $1 \le r \le \infty$, one can define $\theta_{r,s}$ as the supremum of all values c with the following property: *For every* $\boldsymbol{\alpha} \in \mathbb{R}^s \setminus \mathbb{Q}^s$, *there exist infinitely many* $(\mathbf{p}, q) \in \mathbb{Z}^s \times \mathbb{Z}_+$ *with* $\gcd(\mathbf{p}, q) = 1$, *such that*

$$\left\|\boldsymbol{\alpha} - \frac{1}{q}\mathbf{p}\right\|_r < \frac{1}{q(cq)^{1/s}}\,. \tag{5}$$

In the literature, research concentrated on the cases $r = \infty$ and $r = 2$ and on lower bounds for the corresponding θ's, since such results guarantee that the italicized assertion above is true for a certain range of values c.

We will give an account on the achievements on this topic obtained so far and on the methods developed and applied with success.

2 The Approach via the Geometry of Numbers

As will be evident from (6), an extremely useful toolkit for simultaneous Diophantine approximation is provided by the *Geometry of Numbers*; see [10] for a standard and very useful textbook reference. Its central concept is that of a *lattice* $\Gamma = A\mathbb{Z}^s$ in $\mathbb{R}^s$, where A is a non-singular real $(s \times s)$ matrix. The *lattice constant* of Γ is defined as

$$d(\Gamma) = |\det A| \ .$$

Another basic notion is that of a *star body* K in $\mathbb{R}^s$, which is a nonempty $\mathbf{o}$-symmetric closed subset of $\mathbb{R}^s$, with the property that for any point $\mathbf{p} \in K$, the closed straight line segment joining $\mathbf{p}$ with $\mathbf{o}$ is contained in the interior of K, with the possible exception of $\mathbf{p}$ itself [10, p. 15].

Further, a lattice Γ in $\mathbb{R}^s$ is called *admissible* for a star body K, if $\mathbf{o}$ is the only lattice point of Γ contained in the interior of K.

Finally, the *critical determinant* $\Delta(K)$ of a star body K is the infimum of all lattice constants $d(\Gamma)$, where Γ ranges over all lattices which are admissible for K.

In 1955, Davenport [6, 8] achieved a major breakthrough, connecting the simultaneous Diophantine approximation problem with the Geometry of Numbers: He was able to show that, for any positive integer s and any $r \in [1, \infty]$,

$$\theta_{r,s} = \Delta(K_{r,s}) \, , \tag{6}$$

where

$$K_{r,s} = \left\{(x_0, \ldots, x_s) \in \mathbb{R}^{s+1} : \ |x_0| \, \|(x_1, \ldots, x_s)\|_r^s \le 1 \right\} .$$

We outline a sketch of a proof of the more important part (in view of the remark at the end of Sect. 1)

$$\theta_{r,s} \ge \Delta(K_{r,s}) \tag{7}$$

of this celebrated result, following [10, p. 481].

For $t > 0$, let

$$K_{r,s}(t) = \{(x_0, \ldots, x_s) \in K_{r,s} : \ \|(x_1, \ldots, x_s)\|_r \le t \} \ .$$

Now, for positive reals t_1, t_2, the linear transformation

$$x_0 \mapsto (t_1/t_2)^s x_0 \, , \quad x_j \mapsto (t_2/t_1) x_j \quad (j = 1, \ldots, s) \, ,$$

has determinant 1 and maps $K_{r,s}(t_1)$ one-one onto $K_{r,s}(t_2)$. Hence $\Delta(K_{r,s}(t))$ is a constant independent of t. From this it follows (although this is actually not trivial!) that, for any $t > 0$,

$$\Delta(K_{r,s}(t)) = \lim_{t \to \infty} \Delta(K_{r,s}(t)) = \Delta(K_{r,s}) \, .$$

Now let $\varepsilon > 0$ arbitrarily small, and $c := \Delta(K_{r,s}) - \varepsilon$. Further, let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s \setminus \mathbb{Q}^s$, and $t > 0$. For $(p_0, p_1, \dots, p_s) \in \mathbb{Z}^{s+1}$, we consider the lattice $\Gamma = \{(\gamma_0, \gamma_1, \dots, \gamma_s)\}$,

$$\gamma_0 = cp_0\,,$$
$$\gamma_j = \alpha_j p_0 - p_j\,, \qquad j = 1, \dots, s\,.$$

Obviously, $d(\Gamma) = c < \Delta(K_{r,s}(t))$. Thus Γ cannot be admissible for $K_{r,s}(t)$; hence, there exists a nonzero lattice point of Γ contained in the interior of $K_{r,s}(t)$. Therefore, there exists a nontrivial $(p_0, p_1, \dots, p_s) \in \mathbb{Z}^{s+1}$, such that

$$|cp_0|\, \|p_0\alpha - (p_1, \dots, p_s)\|_r^s < 1\,,$$
$$\|p_0\alpha - (p_1, \dots, p_s)\|_r \leq t\,.$$

Making $t \to 0$ gives an infinity of $(p_0, p_1, \dots, p_s) \in \mathbb{Z}^{s+1}$ with $p_0 > 0$ for which

$$\left\| \alpha - \frac{1}{p_0}(p_1, \dots, p_s) \right\|_r < \frac{1}{p_0(cp_0)^{1/s}}\,.$$

Since c can be arbitrarily close to $\Delta(K_{r,s})$, the assertion (7) follows.

In what follows, we shall concentrate on the cases $r = \infty$ (maximum norm) and $r = 2$ (Euclidean norm).

3 Bounds for the Simultaneous Diophantine Approximation Constants with Respect to the Maximum Norm

3.1 Simultaneous Approximation of Two Numbers

To start with, we mention a few results on the case $s = 2$ addressed in the above quotation from Hermite. Improving work by Mullender [17], Davenport [7] showed that

$$\theta_{\infty,2} = \Delta(K_{\infty,2}) \geq \sqrt[4]{46} = 2.604\dots \tag{8}$$

This was refined later by Mack [15] and the author [19] who obtained

$$\theta_{\infty,2} = \Delta(K_{\infty,2}) \geq \left(\frac{13}{8}\right)^2 = 2.64062\dots \tag{9}$$

All of these estimates were established by quite intrinsic geometric considerations of very special planar domains.

In the opposite direction, Cassels [4] proved that

$$\theta_{\infty,2} = \Delta(K_{\infty,2}) \leq 3.5\,.$$

3.2 Theorems of Minkowski and of Blichfeldt

For C a bounded convex star body in $\mathbb{R}^s$, Minkowski's Theorem [10, p. 123] in its simplest form tells us that

$$\Delta(C) \geq \mathrm{vol}(C)2^{-s}\,. \tag{10}$$

Following Minkowski, we apply this to derive a crude lower bound for $\theta_{\infty,s} = \Delta(K_{\infty,s})$, $s \geq 2$. By the arithmetic-geometric mean inequality, the body

$$K_{\infty,s}: \quad |x_0|\ (\max(|x_1|,\dots,|x_s|))^s \leq 1$$

contains the convex body

$$C_{s+1}: \quad |x_0| + s\ \max(|x_1|,\dots,|x_s|) \leq s+1\,.$$

By a simple calculus exercise, we see that

$$\mathrm{vol}(C_{s+1}) = \left(\frac{2}{s}\right)^s \int\limits_{|x_0|\leq s+1} (s+1-|x_0|)^s\,dx_0 = 2^{s+1}\left(1+\frac{1}{s}\right)^s\,.$$

Hence, by (10),

$$\theta_{\infty,s} = \Delta(K_{\infty,s}) \geq \Delta(C_{s+1}) \geq \left(1+\frac{1}{s}\right)^s\,.$$

For $s = 2$, this implies that $\theta_{\infty,2} \geq 2.25$, which is much poorer than (8) and (9).

Another classic theorem is more general and, therefore, capable of applications which yield sharper bounds: Due to Blichfeldt [2] (see also [10, p. 123]), for any star body K in $\mathbb{R}^s$ and any measurable set $M \subseteq \mathbb{R}^s$ whose *difference set*

$$\mathscr{D}M = \{\mathbf{m}_1 - \mathbf{m}_2:\ \mathbf{m}_1, \mathbf{m}_2 \in M\}$$

is contained in K, it follows that

$$\Delta(K) \geq \mathrm{vol}(M)\,. \tag{11}$$

Blichfeldt's approach was sharpened by Mullender [16] and made perfect by Spohn [27] who used the calculus of variations to determine, for each star body $K_{\infty,s}$, the set $M_{s+1} \subset \mathbb{R}^{s+1}$ with maximal volume such that $\mathscr{D}M_{s+1} \subseteq K_{\infty,s}$. This turns out to be

$$M_{s+1}: \quad |x_0| \leq 2^s, \quad |x_j| \leq \psi\left(2^{-s}|x_0|\right) \text{ for } j = 1, \ldots, s,$$
$$\psi(z) := \frac{1}{2}\left(1 - z^{1/s}\right)\left(z + (1 - z^{1/s})^s\right)^{-1/s}. \tag{12}$$

Its volume is readily calculated as

$$\operatorname{vol}(M_{s+1}) = s2^{s+1} \int_0^1 \frac{w^{s-1}}{(1+w)^s(1+w^s)} dw. \tag{13}$$

Using this along with (7) and (11), we see that

$$\theta_{\infty,2} \geq 2\pi - 4 = 2.283\ldots, \quad \theta_{\infty,3} \geq 2.449\ldots, \quad \theta_{\infty,4} \geq 2.559\ldots, \quad \theta_{\infty,5} \geq 2.638\ldots.$$

For $s \geq 3$, these are the sharpest explicit lower bounds for the constants $\theta_{\infty,s}$ known to date.

Concerning upper bounds for these constants, the reader is referred to the papers by Cusick cited in the monograph [10].

3.3 The Approach of Siegel and Bombieri

To complete this section, we outline some most elegant ideas due to Siegel [26] and Bombieri [3] which might be the basis of further refinements of the bounds given.

For an integer $s \geq 2$, put $n = s + 1$ for short, and let K be a star body in $\mathbb{R}^n$ with $\Delta(K) < \infty$. Let further $\Gamma = A\mathbb{Z}^n$ be a lattice admissible for K with $d(\Gamma) = \Delta(K)$,[1] and $M \subseteq \mathbb{R}^n$ a bounded measurable set with $\mathscr{D}M \subseteq K$. Denoting by I_M the indicator function of M, we define a function

$$\phi: \mathbb{R}^n \to \mathbb{R}_{\geq 0}, \quad \phi(\mathbf{u}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} I_M\left(A(\mathbf{u} + \mathbf{k})\right)$$

which is periodic in each component of $\mathbf{u}$. It follows from the assumptions that $\phi(\mathbf{u}) = \phi^2(\mathbf{u})$ almost everywhere. Let

$$\phi(\mathbf{u}) \sim \sum_{\mathbf{m} \in \mathbb{Z}^n} \alpha(\mathbf{m}) e(-\mathbf{m}\mathbf{u})$$

[1] Such a lattice is called *critical* for K. For its existence, see [10, p. 187].

the corresponding Fourier series, where $e(z) := e^{2\pi i z}$ as usual and **mu** is the standard inner product. The coefficients are readily calculated as

$$\alpha(\mathbf{m}) = \int_{[0,1]^n} e(\mathbf{mw})\phi(\mathbf{w})d\mathbf{w} = \frac{1}{|\det A|}\int_M e(\mathbf{m}A^{-1}\mathbf{v})d\mathbf{v}\,.$$

In particular,

$$\alpha(\mathbf{o}) = \frac{\mathrm{vol}(M)}{d(\Gamma)}\,.$$

On the other hand, by Parseval's identity,

$$\alpha(\mathbf{o}) = \int_{[0,1]^n} \phi^2(\mathbf{w})d\mathbf{w} = \sum_{\mathbf{m}\in\mathbb{Z}^n} |\alpha(\mathbf{m})|^2\,.$$

Hence,

$$\frac{\mathrm{vol}(M)}{d(\Gamma)} = \left(\frac{\mathrm{vol}(M)}{d(\Gamma)}\right)^2 + \sum_{\mathbf{m}\neq\mathbf{o}} |\alpha(\mathbf{m})|^2\,.$$

Multiplying by $d^2(\Gamma)/\mathrm{vol}(M)$ gives

$$\Delta(K) = d(\Gamma) = \mathrm{vol}(M) + \frac{1}{\mathrm{vol}(M)}\sum_{\mathbf{m}\in\mathbb{Z}^n,\ \mathbf{m}\neq\mathbf{o}} \left|\int_M e(\mathbf{m}A^{-1}\mathbf{v})d\mathbf{v}\right|^2\,. \tag{14}$$

In particular, this implies that $\Delta(K) \geq \mathrm{vol}(M)$, which just proves Blichfeldt's Theorem (11). Furthermore, if we suppose K to be convex and choose $M = \frac{1}{2}K$, Minkowski's Theorem (10) readily follows.

Moreover, with the choice $K = K_{\infty,s}$ and $M = M_{s+1}$, formula (14) applies to our problem of simultaneous Diophantine approximation. In fact, the author [20] was able to show that at least two of the integrals on the right-hand side can be bounded away from zero, by a suitable choice of the lattice Γ. Thus,

$$\theta_{\infty,s} = \Delta(K_{\infty,s}) \geq \mathrm{vol}(M_{s+1}) + \frac{2}{\mathrm{vol}(M_{s+1})}\kappa_s^2\,, \tag{15}$$

with some $\kappa_s \neq 0$. More precisely,

$$\kappa_s = \inf_{\substack{|v_j|\leq c_s\\ j=1,\ldots,s}}\ \sup_{t>0} |F_s(v_1,\ldots,v_s;t)|\,,$$

$$c_s = 2(2\pi)^{1+1/s}\mathrm{vol}(M_{s+1})^{-1/s}\,.$$

The function F_s has an explicit though quite involved representation and satisfies, as $t \to 0+$,

$$F_s(v_1, \dots, v_s; t) \sim -2^{s+1} t^{2s},$$

uniformly in $v_1, \dots, v_s \in [-c_s, c_s]$. This obviously implies that $\kappa_s \neq 0$ and thus in principle improves upon the bound $\Delta(K_{\infty,s}) \geq \mathrm{vol}(M_{s+1})$. But in order to derive an explicit numerical refinement, the technical difficulties are overwhelming.

4 Bounds for the Simultaneous Diophantine Approximation Constants with Respect to the Euclidean Norm

There are at least two good reasons why the Euclidean norm is of particular interest in this context: firstly, because this is the norm we usually measure distances in $\mathbb{R}^s$ with, and secondly, because the very only case where $\theta_{r,s}$ has been determined exactly (apart from Hurwitz's case $s = 1$) is that of $r = s = 2$. In fact, Davenport and Mahler [9] succeeded to prove that

$$\begin{aligned} \theta_{2,2} &= \Delta(K_{2,2}) = \frac{1}{2}\sqrt{23}, \\ K_{2,2}: &\quad |x_0| \left(x_1^2 + x_2^2\right) \leq 1 . \end{aligned} \tag{16}$$

To establish this deep and celebrated result, Mahler's general theory on lattices has been combined with quite intrinsic geometric considerations on special planar domains.

In general, the task is to deduce (lower) bounds for the critical determinants of the $(s + 1)$-dimensional star bodies

$$K_{2,s}: \quad |x_0| \left(x_1^2 + \dots + x_s^2\right)^{s/2} \leq 1, \quad s \geq 3. \tag{17}$$

4.1 Prasad's Method of Inscribing Elliptic Balls

An obvious idea is to observe that, by the mean inequality, the elliptic ball in $\mathbb{R}^{s+1}$,

$$\mathscr{E}_{s+1}: \quad x_0^2 + s\left(x_1^2 + \dots + x_s^2\right) \leq s + 1$$

is contained in $K_{2,s}$. By a suitable linear transformation,

$$\theta_{2,s} = \Delta(K_{2,s}) \geq \Delta(\mathscr{E}_{s+1}) = \frac{(s+1)^{(s+1)/2}}{s^{s/2}} \Delta(\mathscr{S}_{s+1}), \tag{18}$$

where $\mathscr{S}_{s+1}$ is the (spherical) unit ball in $\mathbb{R}^{s+1}$. This was first noticed and applied by Prasad [25]. Fortunately, due to special properties of quadratic forms, the values of $\Delta(\mathscr{S}_n)$ are known up to $n = 8$: See [10, p. 410]. Cf. also Table 1 below.

From this and (18), it follows that

$$\theta_{2,3} \geq 1.5396\ldots, \quad \theta_{2,4} \geq 1.23526\ldots, \quad \theta_{2,5} \geq 0.83656\ldots. \tag{19}$$

4.2 The Method of Mordell and Armitage

Mordell [12–14] had developed a method to relate the critical determinant of a star body with a large group of automorphisms to that of another star body of lower dimension. Using this approach, Armitage [1] proved that

$$\theta_{2,s} = \Delta(K_{2,s}) \geq \left(\Delta(K^{(s,s)})\right)^{(s+1)/(s-1)}, \tag{20}$$

where, for any positive integers p and $s > 2$,

$$K^{(s,p)}: \quad |x_1| \left(x_1^2 + \cdots + x_s^2\right)^{p/2} \leq 1$$

is a star body in $\mathbb{R}^s$. Carrying this argument a bit further, it has been proved that

$$\Delta(K^{(s,p)}) \geq \left(K^{(s-1,p)}\right)^{s/(s-1)} \left(K_*^{(s,p)}\right)^{1/(s-1)}, \tag{21}$$

where

$$K_*^{(s,p)}: \quad \left(x_1^2 + \cdots + x_s^2\right)^{p-s+2} \left(x_2^2 + \cdots + x_s^2\right)^{s-1} \leq 1$$

is another auxiliary star body in $\mathbb{R}^s$, and it is assumed that $p > s - 2$. For $p = s$, this result is due to Armitage [1] and for general p to the author [22]. Armitage combined (20) and (21), in order to estimate $\theta_{2,3}$. He thus deduced that

$$\theta_{2,3} = \Delta(K_{2,3}) \geq \left(\Delta(K^{(3,3)})\right)^2 \geq \left(\Delta(K^{(2,3)})\right)^3 \Delta(K_*^{(3,3)}). \tag{22}$$

Quoting the bound $\Delta(K^{(2,3)}) \geq 1.159$ for the planar domain $K^{(2,3)}$ from his unpublished thesis and estimating $\Delta(K_*^{(3,3)})$ by inscribing an optimal ellipsoid, using the mean inequality, he finally obtained

$$\theta_{2,3} \geq 1.774\ldots. \tag{23}$$

Later on, the author [21] was able to evaluate the critical determinant of the double paraboloid $\mathscr{P}$ in $\mathbb{R}^3$,

$$\mathscr{P}: \quad x_1^2 + x_2^2 + |x_3| \leq 1, \tag{24}$$

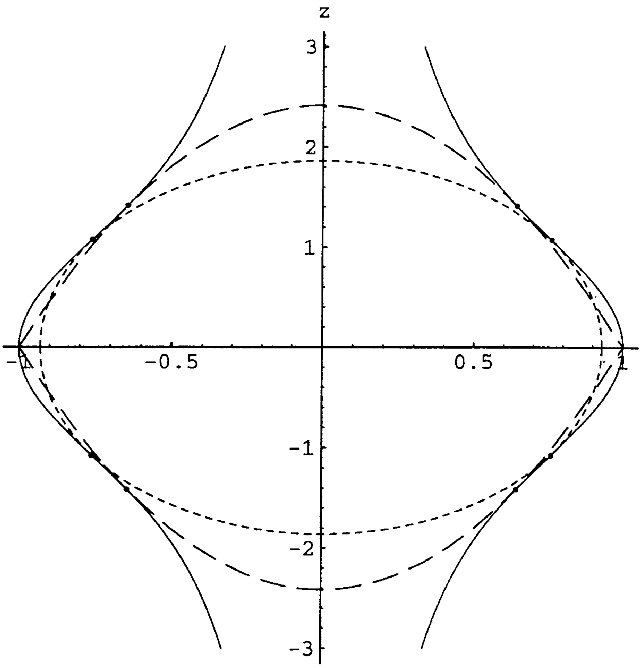

Fig. 1 The body $K_*^{(3,3)}$ with a double paraboloid and an ellipsoid inscribed, in front view

Table 1 Critical determinants of spherical unit balls

n	2	3	4	5	6	7	8
$\Delta(\mathscr{S}_n)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{3}}{8}$	$\frac{1}{8}$	$\frac{1}{16}$

finding that

$$\Delta(\mathscr{P}) = \frac{1}{2}\,. \tag{25}$$

Using (22) and fitting a paraboloid into $K_*^{3,3}$ (Fig. 1), he improved (23) to

$$\theta_{2,3} \geq 1.159^3\,\frac{1+\sqrt{2}}{2} = 1.879\ldots\,. \tag{26}$$

Similarly, the problem of simultaneous approximation in $\mathbb{R}^4$ can be dealt with [21, 22]. Using (20) and (21), one gets

$$\theta_{2,4} = \Delta(K_{2,4}) \geq \Delta(K^{(4,4)})^{5/3} \geq \Delta(K^{(3,4)})^{20/9}\Delta(K_*^{(4,4)})^{5/9}\,. \tag{27}$$

The general strategy now is to fit into three-dimensional bodies a double paraboloid and to use (24), while bodies of higher dimensions are dealt with by inscribing optimal elliptic balls and using the values of Table 1.

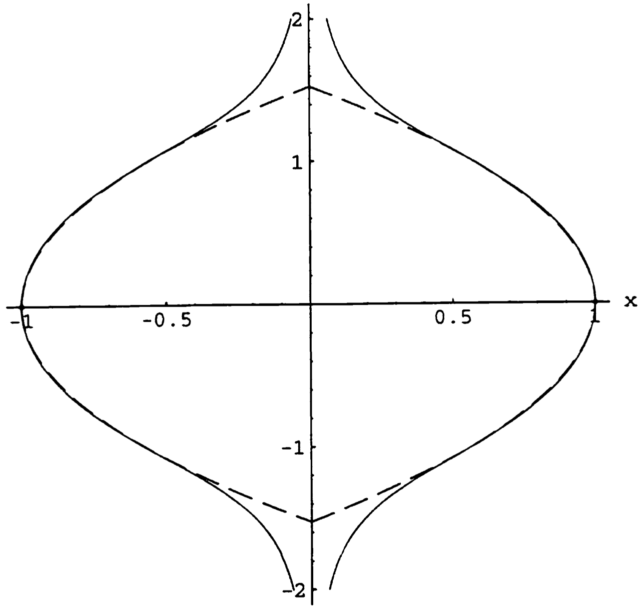

Fig. 2 The bodies $K^{(3,4)}$ and $\mathscr{P}_{(q)}$, $q = 2.3242$, in front view

In this way, one gets [21]

$$\Delta(K^{(3,4)}) \geq 1.1621\,, \quad \Delta(K_*^{(4,4)}) \geq \frac{\sqrt{80}}{6\sqrt{12}}\,.$$

The optimal paraboloid to be inscribed into $K^{(3,4)}$ (Fig. 2) turns out to be

$$\mathscr{P}_{(q)}: \quad |x| + \frac{1}{q}\left(y^2 + z^2\right) \leq 1 \tag{28}$$

with $q = 2.3242$. Thus, by (27),

$$\theta_{2,4} = \Delta(K_{2,4}) \geq 1.3225\ldots\,. \tag{29}$$

Moving on to dimension five, we apply (20) once and (21) twice to conclude that

$$\begin{aligned} \theta_{2,5} = \Delta(K_{2,5}) \geq \Delta(K^{(5,5)})^{3/2} &\geq \Delta(K^{(4,5)})^{15/8} \Delta(K_*^{(5,5)})^{3/8} \\ &\geq \Delta(K^{(3,5)})^{5/2} \Delta(K_*^{(4,5)})^{5/8} \Delta(K_*^{(5,5)})^{3/8}\,. \end{aligned} \tag{30}$$

Using $\mathscr{P}_{(q)}$ with $q = 2.1341$ to estimate $\Delta(K^{(3,5)})$ (Fig. 3), one obtains $\Delta(K^{(3,5)}) \geq 1.067\ldots$. Therefore,

$$\theta_{2,5} = \Delta(K_{2,5}) \geq 0.876 \tag{31}$$

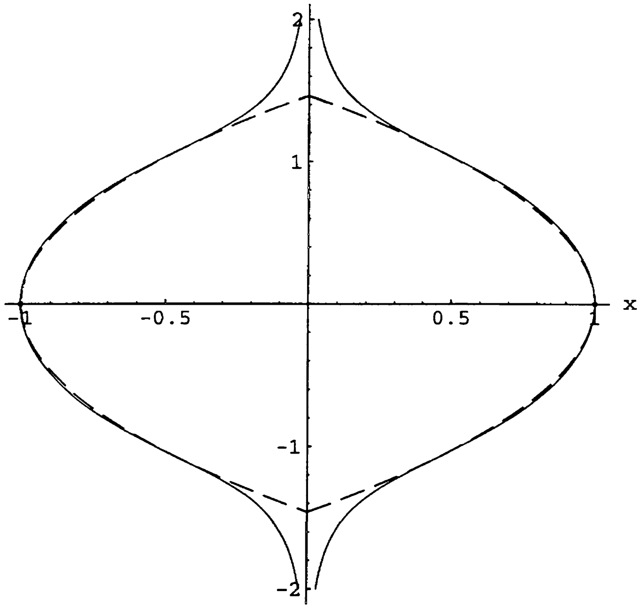

Fig. 3 The bodies $K_*^{(3,5)}$ and $\mathscr{P}_{(q)}$, $q = 2.1341$, in front view

Notice that (26), (29), (31) are significantly sharper than the corresponding bounds in (19).

However, for dimensions $s \geq 6$, it turn out that this approach fails to yield anything sharper than (19).

4.3 A Method of Davenport and Žilinskas

For dimensions $s = 6$ and 7, a different argument can be used to improve upon (18). This is due to Davenport [5, 6] and Žilinskas [28]. It essentially rests on a refinement of *Minkowski's Second Theorem*. The latter says what follows [10, p. 133]:

Let C be any $\mathbf{o}$-symmetric convex body in $\mathbb{R}^n$, $n \geq 2$, given by

$$C: \; G(\mathbf{u}) \leq 1\,,$$

where $G: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is homogeneous of order 1 and is called the *distance function* of C, and let Γ be a lattice in $\mathbb{R}^n$ admissible for C. Then there exist n linearly independent lattice points $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(n)}$ of Γ with $1 \leq G(\mathbf{u}^{(1)}) \leq G(\mathbf{u}^{(2)}) \leq \cdots \leq G(\mathbf{u}^{(n)})$ and

$$G(\mathbf{u}^{(1)})G(\mathbf{u}^{(2)}) \cdots G(\mathbf{u}^{(n)})\,\mathrm{vol}(C)2^{-n} \leq d(\Gamma)\,.$$

Now it is known that if either the dimension is ≤ 3 or C is an **o**-symmetric elliptic ball of arbitrary dimension, then in this inequality, the factor $\mathrm{vol}(C)2^{-n}$ may be replaced by the larger (in view of (10)) $\Delta(C)$; see [10, p. 195]. That is,

$$G(\mathbf{u}^{(1)})G(\mathbf{u}^{(2)})\cdots G(\mathbf{u}^{(n)})\,\Delta(C) \leq d(\Gamma)\,. \tag{32}$$

This can be used to derive sharp lower bounds for $\theta_{2,6}, \theta_{2,7}$ [23]. We will sketch the estimation of $\theta_{2,6} = \Delta(K_{2,6})$, the case $s = 7$ being completely analogous.

By the mean inequality, the elliptic ball in $\mathbb{R}^7$

$$\mathscr{E}: \quad G(\mathbf{u}) := \left(\frac{1}{7}u_0^2 + \frac{6}{7}\sum_{j=1}^{6} u_j^2\right)^{1/2} \leq 1$$

is contained in $K_{2,6}$. Let Γ be a critical lattice of $K_{2,6}$ and $\mathbf{e}$ a point of Γ on the boundary of $K_{2,6}$. Submitting Γ to a suitable automorphism of $K_{2,6}$, if necessary, one can assume that $\mathbf{e} = (1,1,0,0,0,0,0)$. We apply (32), with $C = \mathscr{E}$, and pick $\mathbf{u} = (u_0, u_1, \ldots, u_6) \in \{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$ so that $\mathbf{u} \neq \pm\mathbf{e}$. It follows that

$$\Delta(K_{2,6}) = d(\Gamma) \geq \Delta(\mathscr{E})G(\mathbf{u})^6 \tag{33}$$

and

$$|u_0|\left(\sum_{j=1}^{6} u_j^2\right)^3 \geq 1\,, \quad |u_0 \pm 1|\left((u_1 \pm 1)^2 + \sum_{j=2}^{6} u_j^2\right)^3 \geq 1\,, \tag{34}$$

since $\mathbf{u}$ and $\mathbf{u} \pm \mathbf{e}$ are nonzero lattice points of Γ, hence outside the interior of $K_{2,6}$. Therefore,

$$\frac{\Delta(K_{2,6})}{\Delta(\mathscr{E})} \geq \left(\min_{(34)} G(\mathbf{u})\right)^6 . \tag{35}$$

Now assume that this minimum is attained in some point $\hat{\mathbf{u}} \in \mathbb{R}^7$ and that $G(\hat{\mathbf{u}}) \leq 1$. By the mean inequality and the first part of (34),

$$1 \geq G(\hat{\mathbf{u}}) \geq \left(\hat{u}_0^2\left(\sum_{j=1}^{6} \hat{u}_j^2\right)^6\right)^{1/14} \geq 1\,.$$

Therefore, the mean inequality must hold with equality, which implies that

$$\hat{u}_0^2 = \sum_{j=1}^{6} \hat{u}_j^2\,,$$

hence $\hat{u}_0 = \pm 1$. But this contradicts the right-hand part of (34); thus $G(\hat{\mathbf{u}}) > 1$. Numerical evaluation actually shows that $\min_{(34)} G(\mathbf{u}) > 1.0007\ldots$.

Obviously,

$$\Delta(\mathscr{E}) = \frac{7^{7/2}}{6^3}\Delta(\mathscr{S}_7) = \frac{343}{1728}\sqrt{7}\,, \tag{36}$$

by an appeal to Table 1. Hence [23],

$$\theta_{2,6} = \Delta(K_{2,6}) \geq \frac{343}{1728}\sqrt{7}\left(1 + 9\times 10^{-4}\right) \geq 0.52564\,.$$

Similarly, it can be shown that

$$\theta_{2,7} = \Delta(K_{2,7}) \geq \frac{256}{343}\frac{1}{\sqrt{7}}\left(1 + 3\times 10^{-4}\right) \geq 0.28218\,.$$

For $s > 7$, however, the method fails to be very useful, since $\Delta(\mathscr{S}_{s+1})$ is known for $s \leq 7$ only (Table 1).

4.4 Upper Bounds for the Constants $\theta_{2,s}$

We conclude this section by a short discussion of upper estimates for these Euclidean approximation constants. We follow the last sections of [1] and [22], respectively. Let $\rho = \rho(s) = 2$ if s is odd, and $\rho = \rho(s) = 1$ if s is even. Consider the $(s+1)$-dimensional star body

$$K_{s+1}^{+}: \quad \prod_{j=0}^{\rho-1} x_j^2 \prod_{j=0}^{(s-1-\rho)/2}\left(x_{\rho+2j}^2 + x_{\rho+2j+1}^2\right) \leq 1\,.$$

Let $\mathbb{F}$ be an algebraic number field of degree $s+1$ with exactly ρ real embeddings, and let D denote its discriminant. Following [10, p. 30], consider the lattice

$$\Gamma: \quad \left(\xi^{(0)},\ldots,\xi^{(\rho-1)},\Re\xi^{(\rho)},\Im\xi^{(\rho)},\Re\xi^{(\rho+2)},\Im\xi^{(\rho+2)},\ldots,\Re\xi^{(s-1)},\Im\xi^{(s-1)}\right)\,,$$

where ξ ranges over all algebraic integers of $\mathbb{F}$, and the superscripts denote conjugates. Γ is admissible for K_{s+1}^{+}, and $d(\Gamma) = 2^{-(s+1-\rho)/2}\sqrt{|D|}$; hence,

$$\Delta(K_{s+1}^{+}) \leq 2^{-(s+1-\rho)/2}\sqrt{|D|}\,.$$

By suitable variants of the mean inequality, there exists a linear transformation $\tau: \mathbb{R}^{s+1} \to \mathbb{R}^{s+1}$ of determinant $\det\tau = 2^{(s+1-\rho)/2}s^{-s/2}$, such that

$$K_{2,s} \subseteq \tau\left(K_{s+1}^{+}\right)\,.$$

As a consequence,

$$\theta_{2,s} = \Delta(K_{2,s}) \leq s^{-s/2}\sqrt{|D|}\,. \tag{37}$$

Using, for each degree $s + 1$, a number field which satisfies the conditions stated and has minimal absolute discriminant (see, e.g., the Appendix of the textbook [24]), one thus obtains

$$\begin{aligned}
\theta_{2,1} &= \Delta(K_{2,1}) \leq \sqrt{5}\,,\\
\theta_{2,2} &= \Delta(K_{2,2}) \leq \frac{1}{2}\sqrt{23}\,,\\
\theta_{2,3} &= \Delta(K_{2,3}) \leq 3^{-3/2}\sqrt{275} = 3.1914\ldots\,,\\
\theta_{2,4} &= \Delta(K_{2,4}) \leq 4^{-2}\sqrt{1609} = 2.5070\ldots\,,\\
\theta_{2,5} &= \Delta(K_{2,5}) \leq 5^{-5/2}\sqrt{28037} = 2.9953\ldots\,.
\end{aligned}$$

Here the first two bounds are sharp, according to Hurwitz and Davenport and Mahler [9], respectively. The third estimate is due to Armitage [1].

5 Outlook on Further Research

The observation cannot be denied that the majority of publications in the field (see, e.g., the bibliography in [10]) have been written more than half a century ago. However, it appears that there was no intrinsic reason to stop research on this matter: Of all *simultaneous* Diophantine approximation constants, only that for the *Euclidean norm in dimension two* had been determined exactly (Davenport and Mahler [9], see the beginning of Sect. 4). For all other cases, only more or less crude bounds had been obtained, leaving ample space for further research. It is likely that people just stopped considering these problems because at that time, they seemed to be hopelessly hard, and new lines of attack were not in sight.

However, one must keep in mind that in these old days, personal computers were not available, along with all the modern tools like computer algebra systems and modern graphics devices. In fact, for the heuristic part of work on such problems, the situation is *much* more favorable nowadays. The figures shown in this article (produced throughout with the help of *Mathematica*) may give the reader a feeling of how computer-generated graphics can help us to get the right ideas, in order to make progress on these problems.

This observation—along with the wealth of specific open questions on these approximation constants—should provide enough motivation to reactivate a vigorous research activity on this matter!

There are a lot of problems in the area which still wait to be dealt with. Apart from mere refinements of the work discussed above, it should be pointed out that there are practically no results on the simultaneous Diophantine approximation constants $\theta_{r,s}$ for $2 < r < \infty$. Of course, the approach of Blichfeldt [2] and Spohn [27] discussed in Sect. 3.2 also applies for this case, giving comparatively crude bounds for the constants. But it is to be expected that at least for r close to 2, far better estimates should be true. This might be a promising subject for further research.

Furthermore, there remain a wealth of norms in $\mathbb{R}^s$, $s \geq 2$, which are *not* r-norms. For instance, consider the *cylindric* norm in $\mathbb{R}^3$:

$$\|(x_1, x_2, x_3)\|_{\mathfrak{C}} := \max\left(|x_1|\,, \sqrt{x_2^2 + x_3^2}\right) . \tag{38}$$

Its "unit ball" is the cylinder $|x_1| \leq 1,\ \ x_2^2 + x_3^2 \leq 1$. It gives rise to another approximation constant $\theta_{\mathfrak{C}}$ defined as the supremum of all reals c for which, given any $\alpha \in \left(\mathbb{R}^3 - \mathbb{Q}^3\right)$, there exist infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^3 \times \mathbb{Z}_+$ with $\gcd(\mathbf{p}, q) = 1$, satisfying

$$\left\|\alpha - \frac{1}{q}\mathbf{p}\right\|_{\mathfrak{C}} < \frac{1}{q\,\sqrt[3]{cq}} .$$

As far as the author was able to ascertain, no nontrivial bounds on this constant $\theta_{\mathfrak{C}}$ ever have been established. Of course, this "cylindric norm" readily can be extended to spaces of higher dimensions.

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On the Fixed Points of a Hamiltonian Diffeomorphism in Presence of Fundamental Group

Kaoru Ono and Andrei Pajitnov

Abstract Let M be a weakly monotone symplectic manifold and H be a time-dependent 1-periodic Hamiltonian; we assume that the 1-periodic orbits of the corresponding time-dependent Hamiltonian vector field are non-degenerate. We construct a refined version of the Floer chain complex associated to these data and any regular covering of M and derive from it new lower bounds for the number of 1-periodic orbits. Using these invariants we prove in particular that if $\pi_1(M)$ is finite and solvable or simple, then the number of 1-periodic orbits is not less than the minimal number of generators of $\pi_1(M)$. For a general closed symplectic manifold with infinite fundamental group, we show the existence of 1-periodic orbit of Conley–Zehnder index $1 - n$ for any non-degenerate 1-periodic Hamiltonian system.

1 Introduction

Let M^{2n} be a closed symplectic manifold and denote by ω its symplectic form. Let $H : \mathbb{R}/\mathbb{Z}\times M \to \mathbb{R}$ be a C^∞ function *(the Hamiltonian)*. We will write $H_t(x)$ instead of $H(t,x)$. One associates to H a time-dependent Hamiltonian vector field X_{H_t} on M by the formula:

$$\omega(X_{H_t}, \cdot) = dH_t \text{ for every } \quad t.$$

Assume that every 1-periodic orbit of $\{X_{H_t}\}$ is non-degenerate. Then the set of all 1-periodic orbits is finite. Denote by $\mathscr{P}(H)$ the set of all contractible 1-periodic orbits; and let $p(H)$ be the cardinality of this set. Denote by $\mathscr{M}(M)$ the *Morse number* of M, that is, the minimal possible number of critical points of a Morse function on M.

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The celebrated Arnold conjecture (see [1, Appendix 9], and [2, p. 284]) says that

$$p(H) \geq \mathscr{M}(M). \tag{1}$$

It was proved by V.I. Arnold himself in the case when H is “sufficiently small” function. The Arnold conjecture implies in particular certain homological lower bounds for $p(H)$. Namely, let us denote by $b_i(M)$ the rank of $H_i(M)$ and by $q_i(M)$ the torsion number of $H_i(M)$ (i.e., the minimal possible number of generators of the abelian group $H_i(M)$). Then the conjecture (1) implies the following:

$$p(H) \geq \sum_i \Big(b_i(M) + q_i(M) + q_{i-1}(M)\Big). \tag{2}$$

The inequality (1) implies also the following:

$$p(H) \geq \sum_i b_i^{\mathbb{F}}(M), \tag{3}$$

where $\mathbb{F}$ is any field and we denote by $b_i^{\mathbb{F}}(M)$ the dimension of $H_i(M, \mathbb{F})$ over $\mathbb{F}$.

Floer [7] constructed a chain complex associated with a non-degenerate 1-periodic Hamiltonian $\{H_t\}$; applying this construction he proved the homological version (3) of the Arnold conjecture for any field $\mathbb{F}$ in the case of monotone symplectic manifolds. The obstacle to obtain (2) is the following. Denote by N the minimal Chern number of (M, ω). Then Floer homology is $2N$-periodic in degrees, or Floer homology is $\mathbb{Z}/2N\mathbb{Z}$-graded; torsions in ordinary homology appearing in different degrees but congruent modulo $2N$ with relatively prime orders contribute to the Floer homology in the same degree. This is the reason why (2) does not follow from Floer homology with integer coefficients.

The construction of the Floer chain complex was generalized to wider classes of symplectic manifolds, namely, weakly monotone symplectic manifolds, in [12, 17]. Taking the results on orientation of moduli spaces [7, Sect. 2e] and [8, Sect. 21] into account, the conjecture (3) is verified in the case of weakly monotone symplectic manifolds ([7] for monotone case, [12] for the case when $N = 0$ or $N \geq n$, and [17] for the case of arbitrary weakly monotone manifolds). In the case of a weakly monotone symplectic manifold, the Floer chain complex CF_* is defined over $\mathbb{Z}$ and any field $\mathbb{F}$ and is a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex of free Λ-modules where Λ is a suitable Novikov ring. It is generated in degree k by the periodic orbits of H of Conley–Zehnder index $k \mod 2N$, and there is an isomorphism:

$$H_k(\mathrm{CF}_*) \approx \bigoplus_{i=k+n (\mathrm{mod} 2N)} H_i(M) \otimes \Lambda \tag{4}$$

This implies (3) for any field $\mathbb{F}$.

In the case of spherically Calabi–Yau manifolds (i.e., when the minimal Chern number equals 0), the isomorphism (4) over $\mathbb{Z}$ implies the inequality (2).

The construction over $\mathbb{Q}$ was further generalized to all closed symplectic manifolds in [8, 14]; hence the inequality (3) with $\mathbb{F}$ of characteristic 0, e.g., $\mathbb{F} = \mathbb{Q}$ follows.

These results confirm the homological versions of the Arnold conjecture, i.e., (2) holds when the minimal Chern number of the closed symplectic manifold is zero, (3) with any field $\mathbb{F}$ holds in the case of weakly monotone closed symplectic manifolds, and (3) with a field of characteristic zero holds for general closed symplectic manifolds. As for the initial conjecture (1), it is still unproved in general case. For a simply connected manifold M^{2n} with $n \geq 3$, the statement (1) is equivalent to (2) in view of S. Smale's theorem [22]. However in the non-simply connected case, the number $\mathscr{M}(M)$ can be strictly greater than the right-hand side of (2). A first step to the proof of the geometric Arnold conjecture (1) would be to prove a weaker inequality involving only the invariants of the fundamental group. For a group G, let

$$\mathscr{R} = \{1 \leftarrow G \leftarrow F_1 \leftarrow F_2\} \tag{5}$$

be a presentation of G, where F_1 and F_2 are free groups of ranks $d(\mathscr{R})$ and $r(\mathscr{R})$. Denote by $d(G)$ the minimum of numbers $d(\mathscr{R})$ for all presentations $\mathscr{R}$ and by $D(G)$ the minimum of numbers $d(\mathscr{R}) + r(\mathscr{R})$ for all presentations $\mathscr{R}$. On the occasion of Arnold Fest in Toronto 1997, V.I. Arnold asked the first author whether the development in Floer theory at that time[1] settled the original form of his conjecture, i.e., (1), and, in particular, whether one can show the following weaker assertion, which does not follow from homological version of the conjecture:

$$p(H) \geq D(\pi_1(M)). \tag{6}$$

A weaker form of this conjecture is the following:

$$p(H) \geq d(\pi_1(M)). \tag{7}$$

Since then some progress has been made in this direction, although the conjecture is far from being solved. Damian [5] considered similar questions in the framework of the Hamiltonian isotopies of the cotangent bundle of a compact manifold M (see Sect. 3.6). In a recent preprint [3], J.-F. Barraud suggested a construction of a Floer fundamental group and proved, in particular, that $p_{1-n}(H) \geq 1$ if $\pi_1(M)$ is nontrivial and M is symplectically aspherical, i.e., $c_1(M)$ and $[\omega]$ vanish on $\pi_2(M)$ or monotone. (Here $p_j(H)$ stands for the number of periodic orbits of Conley–Zehnder index j.)

In the present paper, we use the Floer chain complex associated with H and a regular covering $\widetilde{M} \to M$ of the underlying manifold and deduce from it new lower bounds for $p(H)$ in terms of certain invariants $\mu_i(\widetilde{M})$, which depend on the homotopy type of M, the minimal Chern number N of M, and the chosen covering

[1] Fukaya and Ono [8], Liu and Tian [14] appeared as preprints in the previous year.

(see Definition 5.1). These invariants are similar to V.V. Sharko's invariants of chain complexes [21]. The numbers μ_i are indexed by $\mathbb{N}$ in the case of spherical Calabi–Yau manifolds and by $\mathbb{Z}/2N\mathbb{Z}$ in case when the minimal Chern number of M equals N. We have

$$p(H) \geq \sum_i \mu_i(\widetilde{M}).$$

Using these invariants we obtain partial results in the direction of the conjecture (7). For a group G, denote by $\delta(G)$ the minimal number of generators of the augmentation ideal of G as a $\mathbb{Z}[G]$-module. In the case when M is weakly monotone and $\pi_1(M)$ is a finite group, we prove that

$$p(H) \geq \delta(\pi_1(M)).$$

In particular we confirm the conjecture (7) for weakly monotone manifolds whose fundamental groups are finite simple or solvable (Theorem 5.7).

We also show the existence of 1-periodic orbits of Conley–Zehnder index $1-n$ for any non-degenerate 1-periodic Hamiltonian system on any closed symplectic manifold with infinite fundamental group (Theorem 5.9).

2 Floer Complex on the Covering Space

After the preliminary Sect. 2.1 where we gathered the necessary definitions of the Novikov rings, we give the review of Hamiltonian Floer chain complex (Sect. 2.2) and proceed to the definition of the Floer chain complex associated to a regular covering of the underlying manifold.

2.1 Novikov Rings: Definitions

Let T be a finitely generated free abelian group and $\xi : T \to \mathbb{R}$ be a homomorphism. Let R be a ring (commutative or not). Recall that the group ring $R[T]$ is the set of all finite linear combinations:

$$l = \sum_{i=0}^{N} a_i g_i, \qquad \text{with} \quad a_i \in R, \quad g_i \in T$$

with a natural ring structure (determined by the requirement that the elements of R commute with the elements of T).

We denote by $R((T))$ the set of all formal linear combinations (infinite in general):

$$\lambda = \sum_{i=0}^{\infty} a_i g_i, \quad \text{with} \quad a_i \in R, \ \ g_i \in T$$

such that $\xi(g_i) \to -\infty$ with $i \to \infty$. Thus the series λ can be infinite, but for every C, the number of terms of λ with $\xi(g_i) \geq C$ is finite. Usually the homomorphism ξ is clear from the context, so we omit it from the notation. The usual definition of the product of power series endows the abelian group $R((T))$ with the natural ring structure (we require that the elements of R commute with the elements of T). This ring is called *the Novikov completion of the ring* $R[T]$. In this paper we will work with the case $R = \mathbb{Z}[G]$, where G is a group.

The augmentation homomorphism $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ has a natural extension to a ring homomorphism $R((T)) \to \mathbb{Z}((T))$, which will be denoted by the same symbol ε:

$$\varepsilon\left(\sum_i a_i g_i\right) = \sum \varepsilon(a_i) g_i \quad \text{with} \quad a_i \in R, \ \ g_i \in T.$$

Thus the ring $\mathbb{Z}((T))$ acquires a natural structure of $R((T))$-module.

Remark 2.1. If the group G is finite, then the ring $\mathbb{Z}[G]((T))$ coincides with the group ring $\mathbb{Z}((T))[G]$ of the group G with coefficients in $\mathbb{Z}((T))$.

In the case when $\xi : T \to \mathbb{R}$ is a monomorphism, we will use an abbreviated notation. The group ring $\mathbb{Z}[T]$ will be denoted by Λ, and its Novikov completion with respect to a monomorphism ξ will be denoted by $\widehat{\Lambda}$. For a field $\mathbb{F}$, we denote by $\mathscr{F}$ the Novikov completion of the group ring $\mathbb{F}[T]$ with respect to ξ. The ring $\widehat{\Lambda}$ is a principal ideal domain (PID), and $\mathscr{F}$ is a field. We will denote the ring $\mathbb{Z}[G]((T))$ by $\mathscr{L}$ and the ring $\mathbb{F}[G]((T))$ by $\mathscr{L}_{\mathbb{F}}$. These rings will appear frequently in Sects. 3 and 4.

The Novikov rings appear in Hamiltonian dynamics in the following context (see Sect. 2.2). Let M be a closed symplectic manifold. The de Rham cohomology class of the symplectic form determines a homomorphism $[\omega] : \pi_2(M) \to \mathbb{R}$. Consider the group

$$\Gamma = \pi_2(M)/(\operatorname{Ker}[\omega] \cap \operatorname{Ker} c_1(M)),$$

where $c_1(M)$ is the Chern class of the almost complex structure associated to ω. The Novikov completion $\mathbb{Z}((\Gamma))$ will be denoted by $\Lambda^{\mathbb{Z}}_{(M,\omega)}$. The Novikov ring $\mathbb{Z}[G]((\Gamma))$ will be denoted in this context by $\Lambda^{\mathbb{Z}[G]}_{(M,\omega)}$.

The restriction of the homomorphism ω to a smaller group

$$\Gamma_0 = \operatorname{Ker} c_1(M)/(\operatorname{Ker}[\omega] \cap \operatorname{Ker} c_1(M))$$

is a monomorphism, so the corresponding Novikov completion $\mathbb{Z}((\Gamma_0))$ is a PID; it will be denoted by $\Lambda^{(0)\mathbb{Z}}_{(M,\omega)} = \mathbb{Z}((\Gamma_0))$. The Novikov ring $\mathbb{Z}[G]((\Gamma_0))$ will be denoted in this context by $\Lambda^{(0)\mathbb{Z}[G]}_{(M,\omega)}$.

2.2 Review of Hamiltonian Floer Complex

In this subsection, we recall the construction of Hamiltonian Floer complex with integer coefficients[2] following [7, 12, 17]. Here we use homological version. Let (M, ω) be a closed symplectic manifold of dimension $2n$. The minimal Chern number $N = N(M, \omega)$ of (M, ω) is a nonnegative integer such that $\{\langle c_1(M), A\rangle | A \in \pi_2(M)\} = N\mathbb{Z}$. We call (M, ω) weakly monotone (semi-positive) if $\langle [\omega], A\rangle \leq 0$ holds for any $A \in \pi_2(M)$ with $3 - n \leq \langle c_1(M), A\rangle < 0$. This class of symplectic manifolds, in particular, contains the following:

(1) (Monotone case) We call (M, ω) a monotone symplectic manifold, if there exists a positive real number λ such that the following equality holds

$$\langle c_1(M), A\rangle = \lambda\langle [\omega], A\rangle$$

for any $A \in \pi_2(M)$.

(2) (Spherically Calabi–Yau case) We call (M, ω) spherically Calabi–Yau, if the minimal Chern number N is zero.

Let $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ be a smooth function. Set $H_t(p) = H(t, p)$. We denote by X_{H_t} the Hamiltonian vector field of H_t. We call $\ell : \mathbb{R}/\mathbb{Z} \to M$ a 1-periodic solution of $\{X_{H_t}\}$, if ℓ satisfies

$$\frac{d}{dt}\ell(t) = X_{H_t}(\ell(t)).$$

We assume that all contractible 1-periodic solutions of $\{X_{H_t}\}$ are non-degenerate. Denote by $\mathscr{P}(H)$ the set of contractible 1-periodic solutions of $\{X_{H_t}\}$.

Pick a generic t-dependent almost complex structure J compatible with ω. Floer chain complex $(\mathrm{CF}_*(H, J), \delta)$ is constructed for monotone symplectic manifolds in [7] and for weakly monotone case in [12, 17]. For a general closed symplectic manifold, the construction over $\mathbb{Q}$ is due to [8] and [14].

Let $\mathscr{L}(M)$ be the space of contractible loops in M. Consider the set of pairs (ℓ, w), where $\ell : \mathbb{R}/\mathbb{Z} \to M$ is a loop and $w : D^2 \to M$ is a bounding disk of

[2]There is an approach to construct Hamiltonian Floer complex with integer coefficients for non-degenerate periodic Hamiltonian systems on arbitrary closed symplectic manifold [9]. Since the details have not been carried out, we restrict ourselves to the class of weakly monotone symplectic manifolds.

the loop ℓ. We set an equivalence relation $\sim$ by $(\ell, w) \sim (\ell', w')$ if and only if $\ell = \ell'$ and

$$\langle [\omega], w\#(-w') \rangle = 0 \quad \text{and} \quad \langle c_1(M), w\#(-w') \rangle = 0,$$

where $w\#(-w')$ is a spherical two-cycle obtained by gluing w and w' with orientation reversed along the boundaries.

Then the space $\widetilde{\mathscr{L}}(M)$ of equivalence classes $[\ell, w]$ is a covering space of $\mathscr{L}(M)$. Denote by $\Pi : \widetilde{\mathscr{L}}(M) \to \mathscr{L}(M)$ the covering projection, and by Γ the group of the deck transformations of this covering, so that we have

$$\Gamma = \pi_2(M)/(\mathrm{Ker}[\omega] \cap \mathrm{Ker}\, c_1(M)).$$

We have the weight homomorphism:

$$\int \omega : \pi_2(M) \to \mathbb{R},$$

and the corresponding Novikov ring $\Lambda^{\mathbb{Z}}_{(M,\omega)} = \mathbb{Z}((\Gamma))$.

We define the action functional $\mathscr{A}_H : \widetilde{\mathscr{L}}(M) \to \mathbb{R}$ by

$$\mathscr{A}_H([\ell, w]) = \int_{D^2} w^*\omega + \int_0^1 H(t, \ell(t))dt.$$

Then the critical point set $\mathrm{Crit}\mathscr{A}_H$ is equal to $\Pi^{-1}(\mathscr{P}(H))$. For each pair (ℓ, w) of $\ell \in \mathscr{P}(H)$ and its bounding disk w, we have the Conley–Zehnder index $\mu_{\mathrm{CZ}}(\ell, w) \in \mathbb{Z}$:

$$\mu_{\mathrm{CZ}} : \mathrm{Crit}\mathscr{A}_H \to \mathbb{Z}.$$

We define $\mathrm{CF}_k(H, J)$ to be the downward completion of the free module generated by $[\ell, w] \in \mathrm{Crit}\mathscr{A}_H$ with $\mu_{\mathrm{CZ}}([\ell, w]) = k \in \mathbb{Z}$ in the spirit of Novikov complex using the filtration by $\mathscr{A}_H$. Pick and fix a lift $[\ell, w_\ell]$ for each $\ell \in \mathscr{P}(H)$. Then $\mathrm{CF}_*(H, J)$ is a free module generated by $[\ell, w_\ell]$ over the Novikov ring $\Lambda^{\mathbb{Z}}_{(M,\omega)}$.

The boundary operator $\partial : \mathrm{CF}_k(H, J) \to \mathrm{CF}_{k-1}(H, J)$ is defined by counting Floer connecting orbits.

Let $[\ell^\pm, w^\pm] \in \mathrm{Crit}\mathscr{A}_H$. We denote by $\widetilde{\mathscr{M}}([\ell^-, w^-], [\ell^+, w^+])$ the space of the solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ satisfying

$$\frac{\partial u}{\partial \tau} + J(u(\tau, t)) \left(\frac{\partial u}{\partial t} - X_{H_t}(u(\tau, t)) \right) = 0 \tag{8}$$

$$\lim_{\tau \to \pm\infty} u(\tau, t) = \ell^\pm(t) \tag{9}$$

and

$$[\ell^+, w^+] = [\ell^+, w^- \# u]. \tag{10}$$

The group $\mathbb{R}$ acts on the space $\widetilde{\mathscr{M}}([\ell^-, w^-], [\ell^+, w^+])$ of solutions of (8) by shifting the parametrization in τ-coordinate. We denote by $\mathscr{M}([\ell^-, w^-], [\ell^+, w^+])$ the quotient space of $\widetilde{\mathscr{M}}([\ell^-, w^-], [\ell^+, w^+])$ by the $\mathbb{R}$-action. Note that the $\mathbb{R}$-action is free unless $[\ell^+, w^+] = [\ell^-, w^-]$. We have

$$\dim \mathscr{M}([\ell^-, w^-], [\ell^+, w^+]) = \mu_{CZ}([\ell^+, w^+]) - \mu_{CZ}([\ell^-, w^-]) - 1.$$

The moduli spaces $\mathscr{M}([\ell^-, w^-], [\ell^+, w^+])$ are oriented in a compatible way; see [7, Sect. 2e], [8, Sect. 21], and [18, Sect. 5]. For $[\ell^\pm, w^\pm]$ such that $\mu_{CZ}([\ell^+, w^+]) - \mu_{CZ}([\ell^-, w^-]) = 1$, the set $\mathscr{M}([\ell^-, w^-], [\ell^+, w^+])$ is a 0-dimensional compact oriented manifold. We denote by $n([\ell^-, w^-], [\ell^+, w^+])$ the order of $\mathscr{M}([\ell^-, w^-], [\ell^+, w^+])$ counted with signs.

For $[\ell^+, w^+] \in \mathrm{Crit}\mathscr{A}_H$, we define

$$\partial[\ell^+, w^+] = \sum n([\ell^-, w^-], [\ell^+, w^+])[\ell^-, w^-],$$

where the summation is taken over all $[\ell^-, w^-]$ satisfying the condition that

$$\mu_{CZ}([\ell^-, w^-]) = \mu_{CZ}([\ell^+, w^+]) - 1.$$

In [12, 17], the Floer complex is constructed over the Novikov ring $\Lambda^{\mathbb{Z}/2\mathbb{Z}}_{(M,\omega)} \cong \Lambda^{\mathbb{Z}}_{(M,\omega)} \otimes \mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. In order to construct the Floer complex over the Novikov ring with $\mathbb{Z}$-coefficients, we need an appropriate coherent system of orientations on the moduli spaces $\mathscr{M}([\ell^-, w^-], [\ell^+, w^+])$. Taking [7, Sect. 2e] and [8, Sect. 21] into account, the argument in [12, Sect. 5] derives the well definedness of the boundary operator ∂ and the fact that $\partial \circ \partial = 0$.

Namely, if $\mu_{CZ}([\ell_2, w_2]) - \mu_{CZ}([\ell_1, w_1]) = 2$, the moduli space $\mathscr{M}([\ell_1, w_1], [\ell_2, w_2])$ is a compact oriented 1-dimensional manifold with boundary. Its boundary is the union of the sets $\mathscr{M}([\ell_1 w_1], [\ell, w]) \times \mathscr{M}([\ell, w], [\ell_2, w_2])$ where the orbits (l, w) range over the set:

$$S = \{[l, w] \in \ \mathrm{Crit}\ \mathscr{A}_H \mid \mu_{CZ}([\ell, w]) - \mu_{CZ}([\ell_1, w_1]) = 1\}.$$

Therefore the sum of the numbers $n([\ell_1, w_1], [\ell, w]) \cdot n([\ell, w], [\ell_2, w_2])$ over S equals 0, and the coefficient of $[\ell_1, w_1]$ in $\partial \circ \partial([\ell_2, w_2])$ vanishes.

Hence we have

Theorem 2.2. *Let (M, ω) be a closed weakly monotone symplectic manifold. For a non-degenerate 1-periodic Hamiltonian function H and a generic almost complex structure compatible with ω, $(\mathrm{CF}_*(H, J), \partial)$ is a $\mathbb{Z}$-graded chain complex over $\Lambda^{\mathbb{Z}}_{(M,\omega)}$ with integer coefficients.*

We denote by $HF_*(H, J)$ the homology of $(\mathrm{CF}_*(H, J), \partial)$.

Let H_α, H_β be non-degenerate 1-periodic Hamiltonians and J_α, J_β generic almost complex structures compatible with ω. Pick a one-parameter family of smooth functions $\mathscr{H} = \{H^\tau\}$ on $\mathbb{R}/\mathbb{Z} \times M$ and a one-parameter family $\mathscr{J} = \{J^\tau\}$ of almost complex structures compatible with ω such that $H^\tau = H_\alpha$ and $J^\tau = J_\alpha$ for sufficiently negative τ and $H^\tau = H_\beta$ and $J^\tau = J_\beta$ for sufficiently positive τ.

Theorem 2.3. *Let H_α, H_β, and J_α, J_β be as above. Then there exists a chain homotopy equivalence:*

$$\Phi_{\mathscr{H},\mathscr{J}} : \mathrm{CF}_*(H_\alpha, J_\alpha) \to \mathrm{CF}_*(H_\beta, J_\beta).$$

The chain homomorphism $\Phi_{\mathscr{H},\mathscr{J}}$ is constructed by counting *isolated* solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ joining $[\ell^-, w^-] \in \mathrm{Crit}\mathscr{A}_{H_\alpha}$ and $[\ell^+, w^+] \in \mathrm{Crit}\mathscr{A}_{H_\beta}$ of the following equation:

$$\frac{\partial u}{\partial \tau} + J^\tau(u(\tau,t))\Big(\frac{\partial u}{\partial t} - X_{H_t^\tau}(u(\tau,t))\Big) = 0. \tag{11}$$

For two choices $(\mathscr{H}_1, \mathscr{J}_1)$ and $(\mathscr{H}_2, \mathscr{J}_2)$, the chain homomorphisms $\Phi_{\mathscr{H}_1,\mathscr{J}_1}$ and $\Phi_{\mathscr{H}_2,\mathscr{J}_2}$ are chain homotopic. To construct a chain homotopy between them, we pick homotopies $\{\mathscr{H}_s\}$, resp. $\{\mathscr{J}_s\}$, $s \in [0,1]$ between $\mathscr{H}_1$ and $\mathscr{H}_2$, resp. $\mathscr{J}_1$ and $\mathscr{J}_2$, and count *isolated* solutions of Eq. (11) with $\mathscr{J}_s, \mathscr{H}_s$ for some $s \in [0,1]$.

Theorem 2.4 ([19]). *Let (M,ω) be a closed weakly monotone symplectic manifold and f a Morse function f on M. For a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure J compatible with ω, the Floer complex $(\mathrm{CF}_*(H,J), \partial)$ is chain equivalent to the Morse complex $(\mathrm{CM}_{*+n}(f) \otimes \Lambda^{\mathbb{Z}}_{(M,\omega)}, \partial^{\mathrm{Morse}})$.*

For the comparison of orientation of the moduli space of solutions of Eq. (8) and the moduli space of Morse gradient flow lines, see [8, Sect. 21].

Remark 2.5. If (M,ω) is either monotone, spherically Calabi–Yau or the minimal Chern number $N \geq n$, we have a chain homotopy equivalence between $(\mathrm{CF}_*(H,J), \partial)$ and $(\mathrm{CF}_*(f,J), \partial)$ for a sufficiently small Morse function f. The latter is isomorphic to the Morse complex $(\mathrm{CM}_{*+n}(f) \otimes \Lambda^{\mathbb{Z}}_{(M,\omega)}; \partial^{\mathrm{Morse}})$, see [12, Proposition 7.4].

In [17], we introduced modified Floer homology $\widehat{HF}_*(H,J)$, which is computed in the case that (M,ω) is a closed weakly monotone symplectic manifold[3] and showed that

$$\widehat{HF}_*(H,J) \cong H_{*+n}(M; \Lambda^{\mathbb{Z}}_{(M,\omega)}).$$

[3] In [17], $[\omega]$ is a rational cohomology class, i.e., an integral cohomology class after multiplying a suitable integer. Any symplectic form is approximated by rational symplectic forms and the estimate for the number of fixed points of a non-degenerate Hamiltonian diffeomorphism is reduced to the one on a closed symplectic manifolds with rational symplectic class.

(In order to work over integer coefficients, we use the orientation of the moduli space of solutions of Eqs. (8), (11) as in [8, Sect. 21].) In the end of Sect. 6.3 [10], we have an isomorphism between $\widehat{HF}_*(H,J)$ and $HF_*(H,J)$; see also [4, Remark 4], which also yields

$$HF_*(H,J) \cong H_{*+n}\left(M; \Lambda^{\mathbb{Z}}_{(M,\omega)}\right).$$

Let N be the minimal Chern number of (M,ω). We find that the Floer chain complex $(\mathrm{CF}_*(H,J),\delta)$ is $2N$-periodic, i.e.,

$$(\mathrm{CF}_*(H,J),\partial) \cong (\mathrm{CF}_{*+2N}(H,J),\partial).$$

(Pick an element $A \in \pi_2(M)$ such that $\langle c_1(M), A\rangle = N$. Then the action of $[A] \in \pi_2(M)/(\mathrm{Ker}[\omega] \cap \mathrm{Ker}\, c_1(M))$ induces such an isomorphism of chain complexes.) The $\mathbb{Z}/2N\mathbb{Z}$-graded version of Floer complex (CF_*,δ) is a free finitely generated chain complex over the smaller Novikov ring. Namely, put

$$\Gamma_0 = \mathrm{Ker}\, c_1(M)/(\mathrm{Ker}[\omega] \cap \mathrm{Ker}\, c_1(M)).$$

endow it with the homomorphism $\int \omega : \Gamma_0 \to \mathbb{R}$ and consider the corresponding Novikov completion $\Lambda^{(0)\mathbb{Z}}_{(M,\omega)} = \mathbb{Z}((\Gamma_0))$. We may also denote this ring by $\widehat{\Lambda}$ (observe that $\int \omega : \Gamma_0 \to \mathbb{R}$ is a monomorphism).

In general, let R be a ring, and K_* be a free chain complex over the graded ring $\Lambda^R_{(M,\omega)}$. (In our applications $R = \mathbb{Z}$ or $R = \mathbb{Z}G$.) Pick an element A as above. Identifying every module K_i with K_{i+2N} via the isomorphism, induced by A, we obtain a free $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex over the ring $\Lambda^{(0)R}_{(M,\omega)}$. This chain complex will be denoted by K°_*.

Let $f : M \to \mathbb{R}$ be a Morse function. We have the following.

Theorem 2.6. *Let (M,ω) be a closed weakly monotone symplectic manifold with minimal Chern number N and H a non-degenerate 1-periodic Hamiltonian on M:*

(1) Then there exists a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex $(\mathrm{CF}^\circ_*(H,J),\partial)$, which is freely generated by $\{[\ell, w_\ell] | \ell \in \mathscr{P}(H)\}$ over $\Lambda^{(0)\mathbb{Z}}_{(M,\omega)}$.

(2) There is a chain equivalence

$$(\mathrm{CF}^\circ_*(H,J,\partial) \sim \left((\mathrm{CM}_{*+n}(f) \otimes \Lambda^{\mathbb{Z}}_{(M,\omega)})^\circ, \partial^{\mathrm{Morse}}\right)$$

of $\mathbb{Z}/2N\mathbb{Z}$-graded chain complexes over $\Lambda^{(0)\mathbb{Z}}_{(M,\omega)}$.

We can also construct Floer complex with coefficients in a local system on M (see [18, Sect. 6]) and prove corresponding results in the case with coefficients in a local system on M.

Theorem 2.7. *Let (M, ω) be a closed weakly monotone symplectic manifold and $\rho : \pi_1(M) \to GL(r, \mathbb{F})$. For a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure compatible with ω,*

$$HF^{\circ}_{*}(H, J; \rho) \cong \left(H_{*+n}(M; \rho) \otimes \Lambda^{\mathbb{F}}_{(M,\omega)}\right)^{\circ}.$$

For a general symplectic manifold, we have the following

Theorem 2.8. *Let (M, ω) be a closed symplectic manifold with minimal Chern number N and H a non-degenerate 1-periodic Hamiltonian on M:*

(1) Then there exists a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex $(\mathrm{CF}^{\circ}_{*}(H, J), \partial)$, which is freely generated by $\{[\ell, w_\ell] | \ell \in \mathscr{P}(H)\}$ over $\Lambda^{(0)\mathbb{Q}}_{(M,\omega)}$.
(2) There is a chain equivalence

$$(\mathrm{CF}^{\circ}_{*}(H, J, \partial) \sim \left((\mathrm{CM}_{*+n}(f) \otimes \Lambda^{\mathbb{Q}}_{(M,\omega)})^{\circ}, \partial^{\mathrm{Morse}}\right)$$

of $\mathbb{Z}/2N\mathbb{Z}$-graded chain complexes over $\Lambda^{(0)\mathbb{Q}}_{(M,\omega)}$.
(3) For a representation $\rho : \pi_1(M) \to GL(r, \mathbb{Q})$,

$$HF^{\circ}_{*}(H, J; \rho) \cong \left(H_{*+n}(M; \rho) \otimes \Lambda^{\mathbb{Q}}_{(M,\omega)}\right)^{\circ}.$$

2.3 Floer Complex Over a Regular Cover

Let $pr : \widetilde{M} \to M$ be a regular covering of M with the covering transformation group G. Let H be a non-degenerate 1-periodic Hamiltonian M and J a generic t-dependent almost complex structures compatible with ω. Denote by $\widetilde{H}$, resp. $\widetilde{J}$, the pullback of H, resp. J, to $\mathbb{R}/\mathbb{Z} \times \widetilde{M}$. The Floer complex $(\mathrm{CF}_{*}(\widetilde{H}, \widetilde{J}), \partial)$ is constructed in the spirit of [13].

We will now define $\overline{\mathscr{L}}(\widetilde{M})$ similarly to $\overline{\mathscr{L}}(M)$. Consider the set of all pairs (γ, w) where γ is a loop in $\widetilde{M}$ and w is a bounding disk for $pr \circ \gamma$. Introduce in this set the following equivalence relation: (γ, w) and (γ', w') are equivalent if $\gamma = \gamma'$ and the values of both cohomology classes $[\omega]$ and $c_1(M)$ on $w\#(-w')$ are the same. The set $\overline{\mathscr{L}}(\widetilde{M})$ of the equivalence classes is a covering space of $\mathscr{L}(\widetilde{M})$, and the deck transformation group of the covering is isomorphic to

$$\Gamma = \pi_2(M)/(\mathrm{Ker}[\omega] \cap \mathrm{Ker}\, c_1(M)).$$

The action functional

$$\mathscr{A}_{\widetilde{H}} : \overline{\mathscr{L}}(\widetilde{M}) \to \mathbf{R}$$

is defined by the same formula as before, namely,

$$\mathscr{A}_{\widetilde{H}}(\gamma, w) = \mathscr{A}_H(pr \circ \gamma, w).$$

We define $\mathrm{CF}_*(\widetilde{H}, \widetilde{J})$ to be the downward completion of free abelian group generated by $\mathrm{Crit}\mathscr{A}_{\widetilde{H}}$ with respect to the action functional $\mathscr{A}_{\widetilde{H}}$. Pick and fix a lift $\widetilde{\ell}$ of $\ell \in \mathscr{P}(H)$ to a 1-periodic solution of $X_{\widetilde{H}_t}$ on $\widetilde{M}$. Note that $pr^{-1}(\mathscr{P}(H)) = \{g \cdot \widetilde{\ell} | \ell \in \mathscr{P}(H),\ g \in G\}$. We have

$$\mathrm{Crit}\mathscr{A}_{\widetilde{H}} = \{[g \cdot \widetilde{\ell}, w] | [\ell, w] \in \mathrm{Crit}\mathscr{A}_H,\ g \in G\}.$$

We also pick and fix a bounding disk w_ℓ for each $\ell \in \mathscr{P}(H)$. Then we find that $\mathrm{CF}_*(\widetilde{H}, \widetilde{J})$ is isomorphic to a free module generated by $\{[\widetilde{\ell}, w_\ell] | \ell \in \mathscr{P}(H)\}$ over $\Lambda^{\mathbb{Z}[G]}_{(M,\omega)}$.

The boundary operator is defined by counting certain isolated solutions of Eq. (8) as follows. Let $[\gamma^{\pm}, w^{\pm}] \in \mathrm{Crit}_{\widetilde{H}}$. We consider the moduli space $\mathscr{M}([\gamma^-, w^-], [\gamma^+, w^+])$ of solutions of Eq. (8) satisfying Condition (9), with $\ell^{\pm} = pr \circ \gamma^{\pm}$, and Condition (10) and that $u(\tau, 0) : \mathbb{R} \to \widetilde{M}$ lifts to a path joining $\gamma^-(0)$ and $\gamma^+(0)$. We set $n([\gamma^-, w^-], [\gamma^+, w^+])$ the signed count of isolated solutions in $\mathscr{M}([\gamma^-, w^-], [\gamma^+, w^+])$. We have obviously

$$\mathscr{M}\Big([pr \circ \gamma^-, w^-], [pr \circ \gamma^+, w^+]\Big) \cong \bigsqcup_{g \in G} \mathscr{M}([g \cdot \gamma^-, w^-], [\gamma^+, w^+]).$$

For $\mu([\gamma^+, w^+]) - \mu([\gamma^-, w^-]) = 1$ the left-hand side is a finite set; therefore there is only finite number of the nonempty sets in the right-hand side, and each of them is finite.

In other words, for fixed $[\gamma^-, w^-]$, $[\gamma^+, w^+]$, there are at most finitely many $g \in G$ such that $n([g \cdot \gamma^-, w^-], [\gamma^+, w^+]) \neq 0$. The boundary operator ∂ on $\mathrm{CF}_*(\widetilde{H}, \widetilde{J})$ is given by

$$\partial[\gamma^+, w^+] = \sum n([\gamma, w], [\gamma^+, w^+])[\gamma, w],$$

where the sum is over $[\gamma, w] \in \mathrm{Crit}\mathscr{A}_{\widetilde{H}}$ such that $\mu_{\mathrm{CZ}}([\gamma^+, w^+]) - \mu_{\mathrm{CZ}}([\gamma, w]) = 1$. It is clear that ∂ is linear over $\Lambda^{\mathbb{Z}[G]}_{(M,\omega)}$.

Keeping attention on the homotopy classes of paths $u(\tau, 0) : \mathbb{R} \to M$ of solutions of Eqs. (8) and (11), the proofs of Theorems 2.2–2.4 work for the case of $\mathrm{CF}_*(\widetilde{H}, \widetilde{J})$. For example, we show the fact that $\partial \circ \partial = 0$ in the following way. In Sect. 2.2, we recalled that $(\mathrm{CF}_*(H, J), \partial)$ is a chain complex using the moduli space $\mathscr{M}([\ell_1, w_1], [\ell_2, w_2])$ with $\mu_{\mathrm{CZ}}([\ell_2, w_2]) - \mu_{\mathrm{CZ}}([\ell_1, w_1]) = 2$. The projection $pr : \widetilde{M} \to M$ gives an identification of $\mathscr{M}([\gamma_1, w_1], [\gamma_2, w_2])$ and the subspace of $\mathscr{M}([pr \circ \gamma^-, w^-], [pr \circ \gamma^+, w^+])$ consisting of connecting orbits u such that $u(\tau, 0)$ lifts to a path joining $\gamma_1(0)$ and $\gamma_2(0)$. The boundary of this subspace is the union of the direct product $\mathscr{M}([\gamma_1, w_1], [\gamma, w]) \times \mathscr{M}([\gamma, w], [\gamma_2, w_2])$ such that $\mu_{CZ}([\gamma, w]) = \mu_{CZ}([\gamma^+, w^+]) - 1$, which is identified with the union of the space of pairs (u_1, u_2) of $\mathscr{M}([pr \circ \gamma_1, w_1], [pr \circ \gamma, w]) \times \mathscr{M}([pr \circ \gamma, w], [pr \circ \gamma_2, w_2])$ such that the concatenation of the paths $u_1(\tau, 0)$ and $u_2(\tau, 0)$ lifts to a path joining $\gamma_1(0)$ and $\gamma_2(0)$. Hence, by looking at the components of $\mathscr{M}([pr \circ \gamma_1, w_1], [pr \circ \gamma_2, w_2])$ with $u(\tau, 0)$ in the prescribed homotopy class of paths joining $pr \circ \gamma_1(0)$ and $pr \circ \gamma_2(0)$, we find that $\mathrm{CF}_*(\widetilde{H}, \widetilde{J})$ is a chain complex. This Floer complex is periodic with respect to the degree shift by $2N$. Hence we can also obtain a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex, which we denote by $\mathrm{CF}^\circ_*(\widetilde{H}, \widetilde{J})$.

Let $f : M \to \mathbb{R}$ be a Morse function. The Morse complex of the function $f \circ pr : \widetilde{M} \to \mathbb{R}$ on the covering $\widetilde{M}$ is a free chain complex over $\mathbb{Z}[G]$, and the following theorem compares it with the Floer chain complex.

Theorem 2.9. *Let $pr : \widetilde{M} \to M$ be a regular covering of a closed weakly monotone symplectic manifold (M, ω) with minimal Chern number N. Let G be the structure group of the covering:*

(1) For a non-degenerate 1-periodic Hamiltonian H on M, there exists a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex $(\mathrm{CF}^\circ_*(\widetilde{H}, \widetilde{J}), \partial)$ such that $\mathrm{CF}^\circ_*(\widetilde{H}, \widetilde{J})$ is a free module generated by $\{[\widetilde{\ell}, w_\ell] | \ell \in \mathscr{P}(H)\}$ over $\Lambda^{(0)\mathbb{Z}[G]}_{(M,\omega)}$.

(2) Let f be a Morse function on M. Then for a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure J compatible with ω, there is a chain equivalence

$$(\mathrm{CF}^\circ_*(\widetilde{H}, \widetilde{J}), \partial) \sim \left((\mathrm{CM}_{*+n}(f \circ pr) \otimes_{\mathbb{Z}[G]} \Lambda^{(0)\mathbb{Z}[G]}_{(M,\omega)})^\circ, \partial^{\mathrm{Morse}} \right)$$

of $\mathbb{Z}/2N\mathbb{Z}$-graded chain complexes over $\Lambda^{(0)\mathbb{Z}[G]}_{(M,\omega)}$.

In the case of arbitrary closed symplectic manifolds, we have an analog of this result over the field $\mathbb{Q}$.

Theorem 2.10. *Let $pr : \widetilde{M} \to M$ be a regular covering of a closed symplectic manifold (M, ω) with minimal Chern number N. Let G be the structure group of the covering:*

(1) For a non-degenerate 1-periodic Hamiltonian H on M, there exists a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex $(\mathrm{CF}^\circ_*(\widetilde{H}, \widetilde{J}), \partial)$ such that $\mathrm{CF}^\circ_*(\widetilde{H}, \widetilde{J})$ is a free module generated by $\{[\widetilde{\ell}, w_\ell] | \ell \in \mathscr{P}(H)\}$ over $\Lambda^{(0)\mathbb{Q}[G]}_{(M,\omega)}$.

(2) Let f be a Morse function on M. Then for a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure J compatible with ω, there is a chain equivalence

$$(\mathrm{CF}^\circ_*(\widetilde{H}, \widetilde{J}), \partial) \sim \left((\mathrm{CM}_{*+n}(f \circ pr) \otimes_{\mathbb{Q}[G]} \Lambda^{(0)\mathbb{Q}[G]}_{(M,\omega)})^\circ, \partial^{\mathrm{Morse}} \right)$$

of $\mathbb{Z}/2N\mathbb{Z}$-graded chain complexes over $\Lambda^{(0)\mathbb{Q}[G]}_{(M,\omega)}$.

3 Invariants of Chain Complexes: $\mathbb{Z}$-Graded Case

The Sects. 3 and 4 are purely algebraic. We introduce some invariants of chain complexes, which will be applied in Sect. 5 to obtain lower bounds for $p(H)$.

3.1 Definition of Invariants μ_i

Recall that a ring R is called *an IBN-ring*, if the cardinality of a base of a free R-module does not depend on the choice of the base. Any principal ideal domain (PID) is an IBN-ring. The group ring of any group with coefficients in a PID is an IBN-ring. All the rings which we consider in this paper will be IBN-rings.

Definition 3.1. For a free based finitely generated module A over an IBN-ring R, we denote by $m(A)$ the cardinality of any base of A. It will be called *the rank of* A.

Let $C_* = \{C_n\}_{n\in\mathbb{Z}}$ be a free finitely generated chain complex over a ring R. Denote by $m_i(C_*)$ the number $m(C_i)$. The minimum of the numbers $m_i(D_*)$, where D_* ranges over the set of all free based finitely generated chain complexes chain equivalent to C_*, will be denoted by $\mu_i(C_*)$.

Observe that the chain complexes which we consider are not supposed to vanish in negative degrees.

Our aim in this section is to develop efficient tools for computing the invariants $\mu_i(C_*)$ for the case of chain complexes arising in the applications to the Arnold conjecture.

We will use here the terminology from Sect. 2.1. Namely, G is a group, T is a free abelian finitely generated group, and $\xi : T \to \mathbb{R}$ is a monomorphism. We consider the rings

$$\mathscr{L} = \mathbb{Z}[G]((T)), \quad \widehat{\Lambda} = \mathbb{Z}((T)), \quad \mathscr{F} = \mathbb{F}((T)).$$

The ring $\mathscr{L}$ is an IBN-ring since it has an epimorphism onto $\widehat{\Lambda}$. Similarly, $\mathscr{L}_{\mathbb{F}}$ is an IBN-ring.

We will use the following notation throughout the rest of the paper:

X is a connected finite CW-complex, $\widetilde{X} \to X$ is a regular covering with a structure group G, so that we have an epimorphism $\pi_1(X) \to G$,

$$\mathscr{C}_*(X) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z}[G]((T)). \tag{12}$$

Then $\mathscr{C}_*(X)$ is a free finitely generated chain complex over $\mathscr{L} = \mathbb{Z}[G]((T))$. Observe the following isomorphism of $\mathscr{L}$-modules:

$$H_0(\mathscr{C}_*(X)) \approx \widehat{\Lambda}. \tag{13}$$

3.2 Lower Bounds Provided by the Cohomology with Local Coefficients

Let $\rho : G \to \mathrm{GL}(r, \mathbb{F})$ be a representation. We denote by $b_i(X, \rho)$ the Betti numbers of X with respect to the local coefficient system induced by ρ. Put

$$\beta_i(X, \rho) = \frac{1}{r} b_i(X, \rho). \tag{14}$$

We will need the following basic lemma.

Lemma 3.2. *Let D_* be a free finitely generated chain complex over $\mathscr{L}$, chain equivalent to $\mathscr{C}_*(X)$. Let $\rho : G \to \mathrm{GL}(r, \mathbb{F})$ be a representation. Then there is a chain complex E_* over $\mathscr{F}$ such that:*

(1) $\dim_{\mathscr{F}} E_k = r \cdot m_k(D_*)$,
(2) $\dim_{\mathscr{F}} H_k(E_*) = b_k(X, \rho)$.

Proof. The vector space $\mathbb{F}^r$ has the structure of $\mathbb{Z}[G]$-module via the representation ρ. This structure induces a natural structure $\hat{\rho}$ of $\mathscr{L}$-module on $\mathscr{F}^r$ by the following formula:

$$\left(\sum_i a_i g_i\right)\left(\sum_j v_j h_j\right) = \sum_{i,j} a_i(v_j) g_i h_j \quad \text{with} \quad a_i \in \mathbb{Z}[G],\;\; v_j \in \mathbb{F}^r,\; g_i,\; h_j \in T.$$

We have also the representation $\rho_0 : G \to \mathrm{GL}(r, \mathscr{F})$ obtained as the composition of ρ with the embedding $\mathbb{F} \hookrightarrow \mathscr{F}$. Put $E_* = D_* \otimes_{\hat{\rho}} \mathscr{F}^r$. Then $\dim_{\mathscr{F}} E_k = r \cdot m_k(D_*)$, and

$$\mathscr{C}_*(X) \otimes_{\hat{\rho}} \mathscr{F}^r = \left(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[G]} \mathscr{L}\right) \otimes_{\hat{\rho}} \mathscr{F}^r \approx C_*(\widetilde{X}) \otimes_{\rho_0} \mathscr{F}^r \approx \left(C_*(\widetilde{X}) \otimes_{\rho} \mathbb{F}^r\right) \otimes_{\mathbb{F}} \mathscr{F}$$

so that

$$\dim_{\mathscr{F}} H_k(E_*) = \dim_{\mathscr{F}} H_k\left(\mathscr{C}_*(X) \otimes_{\hat{\rho}} \mathscr{F}^r\right) = b_k(X, \rho).$$

□

The next proposition follows.

Proposition 3.3. *We have*

$$\mu_i(\mathscr{C}_*(X)) \geq \beta_i(X, \rho).$$

Proof. Let D_* be any chain complex over $\mathscr{L}$ which is chain equivalent to $\mathscr{C}_*(X)$. Pick a chain complex E_* constructed in the previous Lemma. We have

$$r \cdot m_k(D_*) = \dim_{\mathscr{F}} E_k \geq \dim_{\mathscr{F}} H_k(E_*) = b_k(X, \rho) = r \cdot \beta_i(X, \rho).$$

□

For $i = 1$, we have a slightly stronger version of this estimate. In order to prove it, we need a lemma.

Lemma 3.4. *Let A_* be a free based finitely generated chain complex over a ring R, such that $A_* = 0$ for $* \leq l-2$ and $H_{l-1}(A_*) = 0$. Then there exists a free based finitely generated chain complex B_* such that:*

(1) $B_j \approx A_j$ for $j \geq l+2$, and $j = l$.
(2) $B_{l+1} = A_{l+1} \oplus A_{l-1}$.
(3) $B_j = 0$ for $j \leq l-1$.

Proof. Let T_* be the chain complex

$$\{0 \leftarrow A_{l-1} \overset{Id}{\leftarrow} A_{l-1} \leftarrow 0\}$$

concentrated in degrees l and $l+1$. By the Thickening Lemma [21, Lemma 3.6, p. 56], the chain complex $C_* = A_* \oplus T_*$ is isomorphic to the chain complex:

$$C'_* = \{0 \leftarrow A_{l-1} \overset{\partial_l}{\leftarrow} A_{l-1} \oplus A_l \overset{\partial_{l+1}}{\leftarrow} A_{l-1} \oplus A_{l+1} \leftarrow A_{l+2} \leftarrow \ldots\}$$

with $\partial_l(x, y) = x$. Splitting off the chain complex

$$\{0 \leftarrow A_{l-1} \overset{Id}{\leftarrow} A_{l-1} \leftarrow 0\}$$

concentrated in degrees $l-1$ and l, we obtain the required chain complex B_*. □

Proposition 3.5. *We have*

$$\mu_1(\mathscr{C}_*(X)) \geq \beta_1(X, \rho) + 1$$

for any representation ρ such that $H_0(X, \rho) = 0$.

Proof. Let D_* be any free based finitely generated chain complex over $\mathscr{L}$, chain equivalent to $\mathscr{C}_*(X)$. An easy induction argument using Lemma 3.4 shows that D_* is chain equivalent to a free based finitely generated chain complex D'_* such that $D'_i = 0$ for $i \leq -2$ and $D'_i = D_i$ for $i \geq 1$. Put

$$\alpha = m(D'_{-1}), \quad \beta = m(D'_0), \quad \gamma = m(D'_1).$$

The homology of D'_* and of $D_* \otimes_{\hat{\rho}} \widehat{\Lambda}^r_{\mathbb{F}}$ vanishes in degree -1. Applying Lemma 3.2 to the trivial one-dimensional representation ρ_0, we find a chain complex

$$E_* = \{0 \leftarrow \mathscr{F}^\alpha \overset{\partial_0}{\leftarrow} \mathscr{F}^\beta \overset{\partial_1}{\leftarrow} \mathscr{F}^\gamma \leftarrow \ldots\}$$

such that $H_{-1}(E_*) = 0$ and $H_0(E_*) \approx \mathscr{F}$; this implies $\beta \geq \alpha + 1$. Applying Lemma 3.2 to the representation ρ, we obtain a chain complex

$$E'_* = \{0 \leftarrow \mathscr{F}^{\alpha r} \overset{\partial'_0}{\leftarrow} \mathscr{F}^{\beta r} \overset{\partial'_1}{\leftarrow} \mathscr{F}^{\gamma r} \leftarrow \dots\}$$

with $\dim_{\mathscr{F}} \operatorname{Ker} \partial'_1 \geq b_1(X, \rho)$. Since $H_0(X, \rho)$ and $H_0(E'_*)$ vanish, we obtain

$$r\gamma \geq b_1(X, \rho) + r(\beta - \alpha) \geq b_1(X, \rho) + r. \qquad \square$$

Remark 3.6. Observe that the condition $H_0(X, \rho) = 0$ is true for every nontrivial irreducible representation ρ.

When $\pi_1(X)$ is a perfect group, the above methods allow to obtain a lower bound for $\mu_2(X)$.

Proposition 3.7. *Assume that the covering $\widetilde{X} \to X$ is the universal covering of X, so that in particular $\pi_1(X) \approx G$. Assume that G is a perfect finite group. Then $\mu_2(X) \geq b_2(X) + 2$.*

Proof. Let D_* be any free based finitely generated chain complex over $\mathscr{L}$, chain equivalent to $\mathscr{C}_*(X)$. Consider the chain complex D'_* constructed in the proof of the Lemma 3.5. Applying Lemma 3.2 to the trivial one-dimensional representation ρ_0, we find a chain complex

$$E_* = \{0 \leftarrow \mathscr{F}^{\alpha} \overset{\partial_0}{\leftarrow} \mathscr{F}^{\beta} \overset{\partial_1}{\leftarrow} \mathscr{F}^{\gamma} \overset{\partial_2}{\leftarrow} \mathscr{F}^{\delta} \leftarrow \dots\}$$

such that $H_{-1}(E_*) = 0$ and $H_0(E_*) \approx \mathscr{F}$; we have $\dim \operatorname{Ker} \partial_1 = \gamma - (\beta - \alpha) + 1$. Since G is perfect, we have $\dim H_1(E_*) = b_1(X) = 0$; therefore $\dim \operatorname{Im} \partial_2 = \gamma - (\beta - \alpha) + 1$, so that

$$\delta \geq b_2(X) + \gamma - (\beta - \alpha) + 1.$$

Choose an irreducible representation ρ, such that $b_1(X, \rho) \geq 1$ (this is possible by Theorem 3.8 and the fact that the invariant $\delta(G) > 1$ for noncyclic finite groups as we will mention after Theorem 3.8). Applying Lemma 3.2 to the representation ρ, we deduce $\gamma - (\beta - \alpha) \geq \beta_1(X, \rho) > 0$, so that $\delta \geq b_2(X) + 2$. $\square$

3.3 The Invariant μ_1: The Case of Finite Groups

The results of the previous subsection allow to obtain a complete result for the invariant μ_1 in the case of finite groups. For a group G, denote by $d(G)$ the minimal possible number of generators of G and by $\delta(G)$ the minimal possible number of generators of the augmentation ideal of $\mathbb{Z}[G]$ as a $\mathbb{Z}[G]$-module. The next theorem is a reformulation of a well-known result in the cohomological theory of finite groups (see, e.g., [20, Corollary 5.8, p. 191]).

Theorem 3.8. *Let G be a finite group. Then $\delta(G)$ equals the maximum of two numbers $A(G)$ and $B(G)$, defined below:*

$$A(G) = \max_{p,\rho}\Big(\Big\lceil \frac{1}{r} b_1(G,\rho)\Big\rceil + 1\Big),$$

where the maximum is taken over all prime divisors p of $|G|$ and all the irreducible nontrivial representations $\rho : G \to \mathrm{GL}(r, \mathbb{F}_p)$: [4]

$$B(G) = \max_p b_1(G, \mathbb{F}_p),$$

where the maximum is taken over all prime divisors p of $|G|$.

Remark 3.9. If $\phi : G \to K$ is a group epimorphism, V a K-module, and we endow V with a structure of G-module via ϕ, then the induced homomorphism $H_1(G, V) \to H_1(K, V)$ is surjective.

Recall from (12) that X denotes a connected finite CW-complex, and $\widetilde{X} \to X$ is a regular covering with a structure group G, so that we have an epimorphism $\pi_1(X) \to G$. We denote by $\mathscr{C}_*(X)$ the chain complex $C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} \mathscr{L}$.

Theorem 3.10. *Let G be a finite group with a group epimorphism $\pi_1(X) \to G$. Then $\mu_1(\mathscr{C}_*(X)) \geq \delta(G)$.*

Proof. Let D_* be a free based finitely generated chain complex chain equivalent to $\mathscr{C}_*(X)$. The inequality $\mu_1(\mathscr{C}_*(X)) \geq \delta(G)$ follows immediately from Propositions 3.3 and 3.5 together with Theorem 3.8 and Remarks 3.6 and 3.9. ☐

The invariant $\delta(G)$ of a finite group has the following properties:

(1) $\delta(G) = d(G)$ if G is solvable (K. Grünberg's theorem [11]; see also [20, Theorem 5.9]).
(2) $\delta(G) = 1$ if and only if G is cyclic (see [20, Lemma 5.5]).

The second point implies also that $\delta(G)$ equals 2 for any simple non-abelian group G (since $d(G) = 2$ for such a group).

Corollary 3.11. *Let G be a finite group with a group epimorphism $\pi_1(X) \to G$. We have:*

(1) $\mu_1(\mathscr{C}_*(X)) \geq d(G)$ if G is solvable or simple.
(2) $\mu_1(\mathscr{C}_*(X)) \geq 1$ for every nontrivial group G.
(3) $\mu_1(\mathscr{C}_*(X)) \geq 2$ if G is not cyclic.

---

[4] The symbol $\lceil z \rceil$ denotes the minimal integer $k \geq z$.

3.4 The Invariant μ_1: The Case of Infinite Groups

If the group G is infinite, it is more difficult to give computable lower bounds for $\mu_1(\mathscr{C}_*(X))$. In this section we prove that $\mu_1(\mathscr{C}_*(X)) > 0$. Recall the ring $\mathscr{L} = \mathbb{Z}[G]((T))$ and its subring $\widehat{\Lambda} = \mathbb{Z}((T))$. Observe that the $\widehat{\Lambda}$-module $\mathscr{L}$ has no torsion. The ring $\widehat{\Lambda}$ is also a module over $\mathscr{L}$ via the augmentation homomorphism ε (see Sect. 3.1).

Lemma 3.12. *A free $\mathscr{L}$-module contains no submodule isomorphic to $\widehat{\Lambda}$.*

Proof. Assume that there is an embedding $i : \widehat{\Lambda} \to \mathscr{L}^n$. There is a projection $p : \mathscr{L}^n \to \mathscr{L}$ such that $p \circ i$ is nontrivial. Since $\mathscr{L}$ has no $\widehat{\Lambda}$-torsion, the homomorphism $p \circ i$ is an embedding. Put $a = (p \circ i)(1) \in \mathscr{L}$. Multiplying a by a suitable element of T if necessary, we can consider that $a = \alpha \cdot \mathbf{1} + \alpha'$ where $\alpha \in \mathbb{Z}[G]$ and α' is a power series in monomials $g_i \in T$ with $\xi(g_i) < 0$. For every $g \in G$, we have then $g \cdot a = \lambda \cdot a$ with with some $\lambda \in \widehat{\Lambda}$, which implies $g\alpha = l\alpha$ for some $l \in \mathbb{Z}$. The last property is impossible, since G is infinite, and α is a finite linear combination of elements of G. □

Remark 3.13. The lemma and its proof hold also for the case of the ring $\mathbb{Q}[G]((T))$.

Corollary 3.14. *Let D_* be a free based finitely generated chain complex over $\mathscr{L}$. Assume that $H_0(D_*) \approx \widehat{\Lambda}$. Then $D_1 \neq 0$.*

Proof. If $D_1 = 0$, then $H_0(D_*)$ is isomorphic to the kernel of the boundary operator $\partial_0 : D_0 \to D_{-1}$; therefore $H_0(D_*)$ is a submodule of free $\mathscr{L}$-module, which contradicts to Lemma 3.12. □

The next proposition follows.

Proposition 3.15. *If G is infinite, then $\mu_1(\mathscr{C}_*(X)) \geq 1$.*

Remark 3.16. This proposition is valid also for nontrivial finite groups; see Theorem 3.10.

3.5 The Invariant μ_1: The General Case

Definition 3.17. Let R be a ring, and N be a module over R. The minimal number s such that there exists an epimorphism $R^{s+r} \to N \oplus R^r$ will be called *the stable number of generators of N* and denoted by $\sigma(N)$. The stable number of generators of the augmentation ideal $\operatorname{Ker} \varepsilon : \mathscr{L} \to \widehat{\Lambda}$ of the ring $\mathscr{L}$ will be denoted $\sigma(G)$.

Remark 3.18. Using Theorem 3.8, it is easy to show that $\sigma(G) = \delta(G)$ for any finite group. It does not seem that this equality holds in general, although we do not have a counterexample at present.

Proposition 3.19. $\mu_1(\mathscr{C}_*(X)) \geq \sigma(G)$.

Proof. Let D_* be any free based finitely generated chain complex over $\mathscr{L}$, chain equivalent to $\mathscr{C}_*(X)$. Similarly to Proposition 3.5, we construct a free based finitely generated chain complex D'_* chain equivalent to D_* such that $D'_i = 0$ for $i < -2$ and $D'_i = D_i$ for $i \geq 1$. Denote D'_{-2} by A and D'_{-1} by B. Similarly to the proof of 3.5, we deduce $m(D_0) \geq m(B) - m(A) + 1$. We have $H_i(D_*) = 0$ for $i < 0$. Applying Lemma 3.4, we obtain a chain complex

$$D'_* = \{\ldots 0 \leftarrow D'_{-2} \stackrel{\partial_{-1}}{\leftarrow} D'_{-1} \leftarrow D_0 \leftarrow D_1 \leftarrow \ldots\ \}$$

Applying Lemma 3.4 two more times, we obtain a chain complex

$$D''_* = \{\ldots 0 \leftarrow D_0 \oplus A \leftarrow D_1 \oplus B \leftarrow D_2 \leftarrow \ldots\ \}$$

chain equivalent to D'_*. Since $H_0(D'_*) \approx \widehat{\Lambda}$, we have an exact sequence

$$0 \leftarrow \widehat{\Lambda} \stackrel{\phi}{\leftarrow} D_0 \oplus A \stackrel{\psi}{\leftarrow} D_1 \oplus B.$$

Add to it the exact sequence $\{0 \leftarrow 0 \leftarrow \mathscr{L} \stackrel{Id}{\leftarrow} \mathscr{L} \leftarrow 0\}$. By the Thickening Lemma, the result is isomorphic to the following exact sequence:

$$0 \leftarrow \widehat{\Lambda} \stackrel{\chi}{\leftarrow} \mathscr{L} \oplus D_0 \oplus A \stackrel{\phi}{\leftarrow} \mathscr{L} \oplus D_1 \oplus B$$

where $\chi(f, d, a) = \varepsilon(f)$. Let $J(G) = \mathrm{Ker}(\varepsilon : \mathscr{L} \to \widehat{\Lambda})$. We have $\mathrm{Ker}\,\chi = J(G) \oplus D_0 \oplus A$, so that ϕ is an epimorphism of a free $\mathscr{L}$-module of rank $m(B) + m(D_1) + 1$ onto the sum of $J(G)$ and a free $\mathscr{L}$-module of rank $m(A) + m(D_0)$. Thus

$$\sigma(G) \leq m(B) - m(A) + 1 - m(D_0) + m(D_1) \leq m(D_1). \qquad \square$$

3.6 On the Case of Positively Graded Chain Complexes

If we restrict ourselves to the category of positively graded chain complexes, the lower bound of the Proposition 3.19 can be improved.

Definition 3.20. Let $C_* = \{C_n\}_{n \in \mathbb{N}}$ be a free finitely generated chain complex over a ring R. Denote by $m_i(C_*)$ the number $m(C_i)$. The minimum of the numbers $m_i(D_*)$, where D_* ranges over the set of all free based finitely generated chain complex es concentrated in degrees ≥ 0 and chain equivalent to C_*, will be denoted by $M_i(C_*)$.

The next proposition is proved in M. Damian's work [5].

Proposition 3.21. *Let X be a finite connected CW-complex with $\pi_1(X) \approx G$. Let $C_*(\widetilde{X})$ denote the cellular chain complex of the universal covering $\widetilde{X}$. Then*

$$M_1(C_*(\widetilde{X})) \geq \delta(G).$$

Let Φ_t be a Hamiltonian isotopy of the cotangent bundle $T^*(M)$ of a closed manifold M. The zero section of the cotangent bundle will be denoted by M by an abuse of notation.

Using the theory of generating functions developed by F. Laudenbach and J.-Cl. Sikorav, one easily deduces from the above proposition a lower bound for the number of intersections of $\Phi_t(M)$ and M:

$$\text{card}\,(\Phi_t(M) \cap M) \geq \delta(G).$$

3.7 The Invariant μ_2

The results about this invariant are less complete than for μ_1: we have two different lower bounds for $\mu_2(\mathscr{C}_*(X))$ (Proposition 3.22 and Corollary 3.31); none of them is optimal in general. Denote by $B_1(X)$ the maximum of numbers $\beta_1(X,\rho) - \beta_0(X,\rho)$ where ρ ranges over all representations of G.

Proposition 3.22. *For every representation $\rho : G \to \mathrm{GL}(r, \mathbb{F})$, we have*

$$\mu_2(\mathscr{C}_*(X)) \geq B_1(X) + \beta_2(X,\rho) - \beta_1(X,\rho) + \beta_0(X,\rho). \tag{15}$$

Proof. Let D_* be any free finitely generated chain complex over $\mathscr{L}$, chain equivalent to $\mathscr{C}_*(X)$. Similarly to Proposition 3.5, we can assume that $D_i = 0$ for $i \leq -2$. Put

$$\alpha = m(D_{-1}),\ \beta = m(D_0),\ \gamma = m(D_1),\ \delta = m(D_2).$$

Apply Lemma 3.2 and let E_* be the corresponding chain complex. Denote by Z_0 the space of cycles of degree 0 of this complex; then $\dim_{\mathscr{F}} Z_0 = r(\beta - \alpha)$. Consider the chain complex

$$0 \leftarrow Z_0 \leftarrow E_1 \leftarrow E_2 \leftarrow \dots$$

of vector spaces over $\mathscr{F}$. Its Betti numbers are equal to the Betti numbers of X with coefficients in ρ, and applying the strong Morse inequalities, we obtain the following:

$$\beta - \alpha \geq \beta_0(X,\rho); \tag{16}$$

$$\gamma - (\beta - \alpha) \geq \beta_1(X,\rho) - \beta_0(X,\rho); \tag{17}$$

$$\delta - \gamma + \beta - \alpha \geq \beta_2(X,\rho) - \beta_1(X,\rho) + \beta_0(X,\rho). \tag{18}$$

The inequality (17) implies that $\gamma - \beta + \alpha \geq B_1(X)$. Now the proposition follows from (18). □

Corollary 3.23. *Assume that G is finite and the epimorphism $\pi_1(X) \to G$ is an isomorphism. Then*

$$\mu_2(\mathscr{C}_*(X)) \geq \delta(G) - b_1(X, \mathbb{F}) + b_2(X, \mathbb{F}).$$

Proof. It follows from Theorem 3.8 that $B_1(X) + b_0(X, \mathbb{F}) \geq \delta(G)$. □

Remark 3.24. If G is a finite perfect group, then $b_1(X, \mathbb{F}) = 0$, and $\delta(G) \geq 2$; thus we recover the Proposition 3.7.

Now we will give a lower bound for $\mu_2(X)$ in terms of a numerical invariant depending only on G and related to the invariant $D(G)$ (see Introduction). Up to the end of this section, we assume that G is finite and the epimorphism $\pi_1(X) \to G$ is an isomorphism. In this case the natural inclusion $\widehat{\Lambda}[G] \hookrightarrow \mathbb{Z}[G]((T))$ is an isomorphism. We will make no difference between these two rings; observe also that

$$\mathscr{C}_*(X) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}} \widehat{\Lambda}.$$

Definition 3.25. Let R be a commutative ring and

$$\mathscr{F}_* = \{0 \leftarrow R \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots\}$$

be a free $R[G]$-resolution of the trivial $R[G]$-module R; put $m_i(\mathscr{F}_*) = m(F_i)$. The minimum of $m_i(\mathscr{F}_*)$ over all free resolutions of R will be denoted by $\mu_i(G, R)$.[5] If $R = \mathbb{Z}$, we abbreviate $\mu_i(G, R)$ to $\mu_i(G)$.

The following properties are easy to prove:

(1) For any ring R, we have $\mu_i(G, R) \leq \mu_i(G)$.
(2) $\mu_1(G) = \delta(G)$.
(3) $D(G) \geq \mu_1(G) + \mu_2(G)$.

We will now introduce a similar notion appearing in the context of $\mathbb{Z}$-graded complexes.

Definition 3.26. Let R be a commutative ring. A $\mathbb{Z}$-graded chain complex of free finitely generated $R[G]$-modules

$$\mathscr{E}_* = \{\ldots \leftarrow \mathscr{E}_{-n} \leftarrow \ldots \leftarrow \mathscr{E}_0 \leftarrow \ldots \mathscr{E}_n \leftarrow \ldots\}$$

is called *a $\mathbb{Z}$-graded resolution of the trivial $R[G]$-module R* if

[5] Our terminology here differs from that of Swan's paper [23].

(1) $H_*(\mathscr{E}_*) = 0$ for every $i \neq 0$ and $H_0(\mathscr{E}_*) \approx R$.
(2) $\mathscr{E}_{-n} = 0$ for sufficiently large $n \geq 0$.

The minimum of $m_i(\mathscr{E}_*)$ over all $\mathbb{Z}$-graded resolutions of R will be denoted by $\bar{\mu}_i(G,R)$. If $R = \mathbb{Z}$, we abbreviate $\bar{\mu}_i(G,R)$ to $\bar{\mu}_i(G)$.

We have obviously $\bar{\mu}_i(G,R) \leq \mu_i(G,R)$. The next proposition follows from the fundamental result of Swan [23].

Proposition 3.27. *We have $\mu_i(G) = \bar{\mu}_i(G)$ for $i = 1, 2$.*

Proof. Let $\mathscr{E}_*$ be a $\mathbb{Z}$-graded resolution. Similarly to Proposition 3.5, we can assume that $\mathscr{E}_i = 0$ for $i \leq -2$; put

$$f_0 = m(\mathscr{E}_0) - m(\mathscr{E}_{-1}), \;\; f_i = m(\mathscr{E}_i) \;\text{ for }\; i \geq 1.$$

Let $\rho : G \to \mathrm{GL}(r, \mathbb{F})$ be any irreducible representation. Put $\mathscr{E}_*^\rho = \mathscr{E}_* \otimes_\rho \mathbb{F}^r$, and let Z_0^ρ be the vector space of cycles of degree 0. Then $\dim Z_0^\rho = rf_0$. By the strong Morse inequalities applied to the chain complex,

$$0 \leftarrow Z_0^\rho \leftarrow \mathscr{E}_1^\rho \leftarrow \mathscr{E}_2^\rho \leftarrow \dots$$

we have

$$f_0 \geq \beta_0(\mathscr{E}_*, \rho); \tag{19}$$

$$f_1 - f_0 \geq \beta_1(\mathscr{E}_*, \rho) - \beta_0(\mathscr{E}_*, \rho); \tag{20}$$

$$f_2 - f_1 + f_0 \geq \beta_2(\mathscr{E}_*, \rho) - \beta_1(\mathscr{E}_*, \rho) + \beta_0(\mathscr{E}_*, \rho). \tag{21}$$

By Swan's theory [23, Theorem 5.1, Corollary 6.1, and Lemma 5.2], there exists a free resolution $\mathscr{F}_*$ of $\mathbb{Z}$ over $\mathbb{Z}[G]$ such that $m_i(\mathscr{F}_*) = f_i$ for $i = 0, 1, 2$. Therefore $\mu_i(G) \leq f_i$ for $i = 1, 2$. The proposition follows. □

Remark 3.28. The proposition is valid for all $i \geq 1$, with some mild restrictions on G (see Theorem 5.1 of [23].)

A similar method proves the next proposition.

Proposition 3.29. $\bar{\mu}_i(G, \widehat{\Lambda}) = \mu_i(G)$.

Now we can obtain the estimate for $\mu_2(\mathscr{C}_*(X))$.

Proposition 3.30. *We have $\mu_2(\mathscr{C}_*(X)) \geq \bar{\mu}_2(G, \widehat{\Lambda})$.*

Proof. Let D_* be a free finitely generated chain complex chain equivalent to $\mathscr{C}_*(X)$. Then

$$H_i(D_*) = 0 \;\text{ for }\; i < 0, \;\text{ and }\; H_0(D_*) \approx \widehat{\Lambda}, \;\text{ and }\; H_1(D_*) = 0.$$

Using the standard procedure of killing the homology groups of a chain complex, we embed D_* into a free chain complex $D'_* = D_* \oplus E_*$ such that D'_* is finitely generated in each dimension and

$$E_i = 0 \text{ for } i \leq 2, \text{ and } H_0(D'_*) = \widehat{\Lambda} \text{ and } H_i(D'_*) = 0 \text{ for } i \neq 0.$$

Then $m_2(D_*) = m_2(D'_*) \geq \mu_2(G)$. The proposition follows. □

The next corollary is immediate.

Corollary 3.31. $\mu_2(\mathscr{C}_*(X)) \geq \mu_2(G)$.

4 Invariants of Chain Complexes: $\mathbb{Z}/k\mathbb{Z}$-Graded Case

Definition 4.1. Let R be a ring, and $k \in \mathbb{N}, k \geq 2$. A *$\mathbb{Z}/k\mathbb{Z}$-graded chain complex* is a family of free based finitely generated R-modules A_i indexed by $i \in \mathbb{Z}/k\mathbb{Z}$ together with homomorphisms $\partial_i : A_i \to A_{i-1}$, satisfying $\partial_i \circ \partial_{i+1} = 0$.

Given $k \in \mathbb{N}$, $k \geq 2$, and a free based finitely generated $\mathbb{Z}$-graded chain complex C_*, one constructs a $\mathbb{Z}/k\mathbb{Z}$-graded chain complex $\breve{C}_*$ as follows:

$$\breve{C}_i = \bigoplus_{s \equiv i(k)} C_s.$$

In this section we will be working with the $\mathbb{Z}/k\mathbb{Z}$-graded chain complex induced by $\mathscr{C}_*(X)$ [see Definition (12)]. It will be denoted by $\breve{\mathscr{C}}_*(X)$, where

$$\breve{\mathscr{C}}_i(X) = \bigoplus_{s \equiv i(k)} \mathscr{C}_s(X). \tag{22}$$

Definition 4.2. Let C_* be a $\mathbb{Z}/k\mathbb{Z}$-graded complex and $i \in \mathbb{Z}/k\mathbb{Z}$. The minimal number $m(D_i)$ where D_* is a $\mathbb{Z}/k\mathbb{Z}$-graded complex, chain equivalent to C_*, is denoted by $\mu_i(C_*)$.

4.1 Lower Bounds from Local Coefficient Homology

Similarly to Sect. 3.2, we have the following estimates for the invariants μ_i of $\mathbb{Z}/k\mathbb{Z}$-graded complexes. The proof of the next proposition is similar to 3.3.

Proposition 4.3. *Let $\rho : G \to \mathrm{GL}(r, \mathbb{F})$ a representation. Suppose that there is a group epimorphism $\pi_1(X) \to G$, then*

$$\mu_i(\breve{\mathscr{C}}_*(X)) \geq \sum_{s \equiv i(k)} \beta_s(X, \rho).$$

4.2 Invariant μ_1 in the k-Graded Case

The previous theorem implies the following lower bounds for $\mu_i(\breve{\mathscr{C}}_*(X))$ in terms of the invariants $d(G)$ *and* $\delta(G)$.

Theorem 4.4. *If $\pi_1(X) \to G$ is an epimorphism onto a finite nontrivial group G, then:*

(1) $\mu_1(\breve{\mathscr{C}}_*(X)) \geq \max(\delta(G) - 1, 1)$.
(2) If G is simple or solvable, we have $\mu_0(\breve{\mathscr{C}}_*(X)) + \mu_1(\breve{\mathscr{C}}_*(X)) \geq d(G)$.

Proof. We need only to prove that $\mu_1(\breve{\mathscr{C}}_*(X)) \geq 1$. To this end, observe that if $\mu_1(\breve{\mathscr{C}}_*(X)) = 0$, then the homology of X in degree 1 with all local coefficients vanishes, which implies $\delta(G) = 1$; then G is cyclic, $b_1(G) = 1$, which leads to a contradiction. □

For the case of infinite groups, we have the following result.

Theorem 4.5. *If G is an infinite group, then $\mu_1(\breve{\mathscr{C}}_*(X)) \geq 1$.*

Proof. Let D_* be a $\mathbb{Z}/k\mathbb{Z}$-graded chain complex equivalent to $\breve{\mathscr{C}}_*(X)$. The module $\widehat{\Lambda} = H_0(\mathscr{C}_*(X))$ is a submodule of $H_0(\breve{\mathscr{C}}_*(X))$. The condition $D_1 = 0$ would imply that $H_0(\mathscr{C}_*^\circ(X))$ and $\widehat{\Lambda}$ are submodules of a free $\mathscr{L}$-module $\mathscr{C}_0^\circ(X)$, and this is impossible when G is infinite by Lemma 3.12. □

This estimate can be improved in the case when $k - 2 \geq \dim X$. Note that in this case, the sum in the right-hand side of (22) contains only one term for every i.

Theorem 4.6. *Assume that* $\dim X \leq k - 2$*, and there exists a group epimorphism from $\pi_1(X)$ to a finite group G. Then*

$$\mu_1(\breve{\mathscr{C}}_*(X)) \geq \delta(G).$$

Proof. Similarly to Sect. 3.3, it suffices to prove that

$$\mu_1(\breve{\mathscr{C}}_*(X)) \geq \beta_1(X, \rho) + 1$$

for any representation ρ such that $H_0(X, \rho) = 0$. Let

$$D_* = \{\ldots \leftarrow D_{-2} \overset{\partial_{-1}}{\leftarrow} D_{-1} \overset{\partial_0}{\leftarrow} D_0 \leftarrow \ldots\}$$

be a $\mathbb{Z}/k\mathbb{Z}$-graded complex, chain equivalent to $K_* = \breve{\mathscr{C}}_*(X)$. The chain complex D_* does not necessarily vanish in any degree, and the argument which we used in the proof of the Proposition 3.5 cannot be applied immediately.

Let $D_* \overset{\phi}{\to} K_* \overset{\psi}{\to} D_*$ be the mutually inverse chain equivalences. Since $k \geq \dim X + 2$, the chain complex K_* vanishes in degree -1, hence the map $\psi \circ \phi : D_{-1} \to D_{-1}$ is equal to 0. The existence of chain homotopy from $\psi \circ \phi$ to Id implies that the submodule $\operatorname{Ker} \partial_{-1} = \operatorname{Im} \partial_0$ is a direct summand of D_{-1}, hence a projective $\widehat{\Lambda}[G]$-module. Let us denote it by L. The $\mathbb{Z}/k\mathbb{Z}$-graded chain complex D_* contains a ($\mathbb{Z}$-graded) subcomplex:

$$D'_* = \{0 \leftarrow L \overset{\partial_0}{\leftarrow} D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow 0\},$$

with $H_0(D'_*) \approx \widehat{\Lambda},\ H_{-1}(D'_*) = 0, H_1(D'_*) \approx H_1(D_*)$. Denote by $S \subset \mathbb{N}$ the multiplicative subset of all numbers t, such that $gcd(t, |G|) = 1$. The module $S^{-1}L$ is free by a fundamental result of R. Swan (see [20, Sect. 5]). Thus the chain complex $D''_* = S^{-1}D'_*$ is free over $S^{-1}\widehat{\Lambda}[G]$; put

$$\alpha = m(D''_{-1}),\ \beta = m(D''_0),\ \gamma = m(D''_1).$$

Let p be a prime divisor of $|G|$ and $\rho : G \to \mathrm{GL}(r, \mathbb{F}_p)$ be a representation. The homology of the complex $K_* \otimes_\rho \mathbb{F}_p^r$ is isomorphic to that of $D''_* \otimes_\rho \mathbb{F}_p^r$ in degrees $-1, 0, 1$. Therefore the argument proving Proposition 3.5 applies here as well, and the proof of the theorem is complete. □

5 Estimates for the Number of Closed Orbits

We proceed to the estimates of the number of periodic orbits of a Hamiltonian isotopy induced by a non-degenerate 1-periodic Hamiltonian H on a closed connected symplectic manifold M. We denote by $\widetilde{M} \to M$ a regular covering with a structure group G. Put

$$\mathscr{C}_*(M) = C_*(\widetilde{M}) \otimes_{\mathbb{Z}[G]} \Lambda^{\mathbb{Z}[G]}_{(M,\omega)}. \tag{23}$$

Denote by N the minimal Chern number of M. For $N > 0$, we obtain a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex:

$$\mathscr{C}^\circ_*(M) = \left(C_*(\widetilde{M}) \otimes_{\mathbb{Z}[G]} \Lambda^{\mathbb{Z}[G]}_{(M,\omega)}\right)^\circ.$$

Definition 5.1. If $N = 0$, put

$$\mu_i(\widetilde{M}) = \mu_i(\mathscr{C}_*(M)). \tag{24}$$

(see Definition 3.1; here $i \in \mathbb{N}$).

If $N > 0$ put

$$\mu_i(\widetilde{M}) = \mu_i(\mathscr{C}^\circ_*(M)). \tag{25}$$

(see Definition 4.2; here $i \in \mathbb{Z}/2N\mathbb{Z}$).

The numbers $\mu_i(\widetilde{M})$ are obviously homotopy invariants of M and the chosen covering $\widetilde{M} \to M$.

5.1 The Spherical Calabi–Yau Case

We consider here symplectic manifolds M with $c_1(M)(A) = 0$ for every $A \in \pi_2(M)$. In this case every contractible periodic orbit γ has a well-defined index $i(\gamma) \in \mathbb{Z}$. The Floer chain complex $\mathrm{CF}_*(\widetilde{H}, \widetilde{J})$ is a $\mathbb{Z}$-graded free based finitely generated chain complex over the ring $\Lambda^{\mathbb{Z}[G]}_{(M,\omega)}$, generated in degree k by contractible periodic orbits of the Hamiltonian vector field of index k, and we have

$$\mathrm{CF}_*(\widetilde{H}, \widetilde{J}) \sim \mathscr{C}_{*+n}(M), \quad \text{where} \quad \dim M = 2n.$$

Denote by p_k the number of contractible periodic orbits of period k. The results of the previous sections imply the following lower bound:

$$p_{i-n} \geq \mu_i(\widetilde{M}).$$

Applying the results of Sect. 3, we obtain the following lower bounds for p_i.

Proposition 5.2. *For any field $\mathbb{F}$ and any representation $\rho : G \to \mathrm{GL}(r, \mathbb{F})$, we have*

$$p_{i-n} \geq \frac{1}{r} b_i(M, \rho).$$

Theorem 3.10 and Corollary 3.11 imply some stronger lower bounds for $i = 1$.

Theorem 5.3. *(1) If $\pi_1(M)$ is nontrivial, then $p_{1-n} \geq 1$.*
(2) If $\pi_1(M)$ has an epimorphism onto a finite group G, then:

a. $p_{1-n} \geq \delta(G)$.
b. $p_{1-n} \geq d(G)$ if G is solvable or simple.
c. $p_{1-n} \geq 2$ if G is not cyclic.

Let us proceed to the lower bounds for p_{2-n}. Applying Corollaries 3.23 and 3.31, we obtain the following result.

Theorem 5.4. *Assume that $\pi_1(M)$ is finite and the homomorphism $\pi_1(M) \to G$ is an isomorphism. Then:*

(1) $p_{2-n} \geq \delta(\pi_1(M)) - b_1(M, \mathbb{F}) + b_2(M, \mathbb{F})$ for any field $\mathbb{F}$.
(2) If G is perfect, then $p_{2-n} \geq b_2(M, \mathbb{F}) + 2$.
(3) $p_{2-n} \geq \mu_2(\pi_1(M))$.

Remark 5.5. A recent result of Fine and Panov [6] asserts that for every finitely presented group G, there exists a symplectic manifold M of dimension 6 with the fundamental group G and $c_1(M) = 0$.

5.2 The Weakly Monotone Case

Let us denote by p the total number of the periodic orbits of the Hamiltonian vector field. We have a $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex $\mathrm{CF}^\circ_*(\widetilde{H},\widetilde{J})$, generated by periodic orbits, such that

$$\mathrm{CF}^\circ_*(\widetilde{H},\widetilde{J}) \sim \mathscr{C}^\circ_{*+n}(M), \quad \text{where} \quad 2n = \dim M,$$

(see Theorem 2.9). Therefore

$$p_{i-n} \geq \mu_i(\widetilde{M}) \ \text{ for } \ i \in \mathbb{Z}/2N\mathbb{Z}.$$

Applying the results of the Sect. 4, we obtain the following.

Theorem 5.6. *For every representation* $\rho : G \to \mathrm{GL}(r, \mathbb{F})$*, we have*

$$p_{i-n} \geq \frac{1}{r}\Big(\sum_{s\equiv i(2N)} b_s(M,\rho)\Big) \ \textit{for} \ \ i \in \mathbb{Z}/2N\mathbb{Z}.$$

As for the number p_{1-n}, we have the following.

Theorem 5.7. *(1) If* $\pi_1(M)$ *is nontrivial, then* $p_{1-n} \geq 1$.
(2) If $\pi_1(M)$ *has an epimorphism onto a finite group G, then:*

a. $p_{1-n} \geq \max(1, \delta(G) - 1)$*, and* $p \geq \delta(G)$.
b. if G is simple or solvable, then $p \geq d(G)$.
c. if G is not cyclic, then $p \geq 2$.

For the manifolds where the minimal Chern number N is strictly greater than $n = \dim M/2$, we have the following improvement of Theorem 5.7 (the proof follows from Theorem 4.6).

Theorem 5.8. *Let* $N \geq n + 1$. *Assume that* $\pi_1(M)$ *has an epimorphism onto a finite group G. Then:*

(1) $p_{1-n} \geq \delta(G)$.
(2) If G is simple or solvable, then $p_{1-n} \geq d(G)$.
(3) If G is not cyclic, then $p_{1-n} \geq 2$.

5.3 The General Case

Let M^{2n} be an arbitrary closed connected symplectic manifold. Applying Theorem 2.10 instead of Theorem 2.9, working over $\mathbb{Q}$ and using Remark 3.13, we obtain the following result.

Theorem 5.9. *Assume that $\pi_1(M)$ is infinite. Then $p_{1-n} \geq 1$.*

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Some Generalizations of Fixed-Point Theorems on *S*-Metric Spaces

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Abstract In this paper we prove new fixed-point theorems on complete *S*-metric spaces. Our results generalize and extend some fixed-point theorems in the literature. We give some examples to show the validity of our fixed-point results.

1 Introduction

Metric spaces are very important in the various areas of mathematics such as analysis, topology, applied mathematics, etc. So it has been studied new generalizations of metric spaces. Recently in 2012, Sedghi et al. have defined the concept of *S*-metric spaces [13].

Many authors have defined some contractive mappings on complete metric spaces as a generalization of the well-known Banach's contraction principle. In 1974, Ciric studied a generalization of Banach's contraction principle and gave quasi-contractions [3]. In 1979, Fisher proved new fixed-point theorems for quasi-contractions and continuous self-mappings [5]. In 1977, Rhoades investigated some comparisons of various contractive mappings and introduced a new contractive mapping called a Rhoades' mapping [11]. He studied some fixed-point theorems. But he did not have any fixed-point theorem for a Rhoades' mapping. Hence in 1986, Chang introduced the concept of a *C*-mapping and obtained some fixed-point theorems using this mapping for a Rhoades' mapping [1]. In 1988, Liu et al. defined the notion of *L*-mapping to give necessary and sufficient conditions for the existence of a fixed point for a Rhoades' mapping [8]. In 1990, Chang and Zhong proved some fixed-point theorems using the notion of periodic point [2].

The fixed-point theory in various metric spaces was also studied. For example, in 2013 Gupta presented the concept of cyclic contraction on *S*-metric spaces [6]. In 2014, Sedghi and Dung proved some fixed-point theorems and gave some analogues of fixed-point theorems in metric spaces for *S*-metric spaces [12]. Hieu et al. gave the relation between a metric and an *S*-metric [7]. In 2014, Dung et al. proved some generalized fixed-point theorems for *g*-monotone maps on partially ordered *S*-metric

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spaces [4]. The present authors defined Rhoades' condition on S-metric spaces and proved some fixed-point theorems satisfying Rhoades' condition [9]. Also they introduced some new contractive mappings on S-metric spaces and investigated their relationships with the Rhoades' condition [10].

Similar to the Banach's contraction principle, now we recall the following result on S-metric spaces given in [13]:

Let (X, S) be a complete S-metric space, T be a self-mapping of X, and

$$S(Tx, Tx, Ty) \leq aS(x, x, y), \tag{1}$$

for some $0 \leq a < 1$ and all $x, y \in X$. Then T has a unique fixed point in X and T is continuous at the fixed point.

Notice that there exists a self-mapping T which has a fixed point, but it does not satisfy Banach's contraction principle on S-metric spaces as we have seen in the following example:

Let $\mathbb{R}$ be the S-metric space which is not generated by any metric with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ defined in [10]. Let

$$Tx = 1 - x.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. T has a fixed point $x = \frac{1}{2}$, but T does not satisfy the Banach's contraction principle (1). Hence it is important to study some new fixed-point theorems.

In this paper, we investigate some generalized fixed-point theorems on S-metric spaces. In Sect. 2 we recall some concepts, lemmas, and corollaries which are useful in the sequel. In Sect. 3 we prove new fixed-point theorems on complete S-metric spaces. Our results generalize and extend some fixed-point theorems in the literature. Also we give some examples to show the validity of our fixed-point theorems.

2 Preliminaries

The following definitions, lemmas, and corollaries can be found in the paper referred to.

Definition 1 ([13]). Let X be a nonempty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

(S1) $S(x, y, z) \geq 0$,
(S2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Definition 2 ([13]). Let (X, S) be an S-metric space.

1. A sequence $(x_n) \subset X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \varepsilon$.
2. A sequence $(x_n) \subset X$ is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(x_n, x_n, x_m) < \varepsilon$.
3. The S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 1 ([13]). *Let (X, S) be an S-metric space and $x, y \in X$. Then we have*

$$S(x, x, y) = S(y, y, x).$$

Lemma 2 ([13]). *Let (X, S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ then we have*

$$S(x_n, x_n, y_n) \to S(x, x, y).$$

Lemma 3 (See Corollary 2.4 in [12]). *Let $(X, S), (Y, S')$ be two S-metric spaces and $f : X \to Y$ be a function. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

The relation between a metric and an S-metric is given in [7] as follows:

Lemma 4 ([7]). *Let (X, d) be a metric space. Then the following properties are satisfied*:

1. *$S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
2. *$x_n \to x$ in (X, d) if and only if $x_n \to x$ in (X, S_d).*
3. *(x_n) is Cauchy in (X, d) if and only if (x_n) is Cauchy in (X, S_d).*
4. *(X, d) is complete if and only if (X, S_d) is complete.*

Now we recall the following fixed-point results.

Corollary 1 (See Corollary 2.12 in [12]). *Let (X, S) be a complete S-metric space, T be a self-mapping of X, and*

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, x) + cS(Ty, Ty, y), \tag{2}$$

for some $a, b, c \geq 0$, $a + b + c < 1$, and all $x, y \in X$. Then T has a unique fixed point in X. Moreover, if $c < \frac{1}{2}$ then T is continuous at the fixed point.

Corollary 2 (See Corollary 2.14 in [12]). *Let (X, S) be a complete S-metric space, T be a self-mapping of X, and*

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, y), S(Ty, Ty, x)\}, \tag{3}$$

for some $h \in [0, \frac{1}{3})$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 3 (See Corollary 2.10 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, x), S(Ty, Ty, y)\}, \tag{4}$$

for some $h \in [0, 1)$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover, if* $h \in [0, \frac{1}{2})$ *then* T *is continuous at the fixed point.*

Corollary 4 (See Corollary 2.17 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, y) + cS(Ty, Ty, x), \tag{5}$$

for some $a, b, c \geq 0$, $a + b + c < 1$, $a + 3c < 1$, *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 5 (See Corollary 2.19 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$\begin{aligned} S(Tx, Tx, Ty) \leq\ & aS(x, x, y) + bS(Tx, Tx, x) + cS(Tx, Tx, y) \\ & + dS(Ty, Ty, x) + eS(Ty, Ty, y), \end{aligned} \tag{6}$$

for some $a, b, c, d, e \geq 0$ *such that* $\max\{a + b + 3d + e, a + c + d, d + 2e\} < 1$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 6 (See Corollary 2.21 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$\begin{aligned} S(Tx, Tx, Ty) \leq h \max\{& S(x, x, y), S(Tx, Tx, x), S(Tx, Tx, y), \\ & S(Ty, Ty, x), S(Ty, Ty, y)\}, \end{aligned} \tag{7}$$

for some $h \in [0, \frac{1}{3})$ *and all* $x, y \in X$. *Then* T *has a unique fixed point in* X. *Moreover,* T *is continuous at the fixed point.*

Corollary 7 (See Corollary 2.15 in [12]). *Let* (X, S) *be a complete* S*-metric space,* T *be a self-mapping of* X*, and*

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, y) + S(Ty, Ty, x)), \tag{8}$$

for some $a \in [0, \frac{1}{3})$ *and all* $x, y \in X$*. Then* T *has a unique fixed point in* X*. Moreover,* T *is continuous at the fixed point.*

Corollary 8 (See Corollary 2.8 in [12]). *Let* (X, S) *be a complete S-metric space,* T *be a self-mapping of* X*, and*

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)), \tag{9}$$

for some $a \in [0, \frac{1}{2})$ *and all* $x, y \in X$*. Then* T *has a unique fixed point in* X*. Moreover,* T *is continuous at the fixed point.*

In the next section we give some generalizations of the above results.

3 Some Fixed-Point Theorems on *S*-Metric Spaces

In this section we give some definitions and generalizations of fixed-point theorems for self-mappings on complete S-metric spaces.

Definition 3. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN1**) There exist real numbers a, b satisfying $a + 3b < 1$ with $a, b \geq 0$ such that

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + b \max\{S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, y), S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 1. *Let* (X, S) *be a complete S-metric space and* T *be a self-mapping of* X*. If* T *satisfies the condition* (**SN1**)*, then* T *has a unique fixed point* x *in* X *and* T *is continuous at* x*.*

Proof. Let $x_0 \in X$ and define the sequence (x_n) as follows:

$$Tx_0 = x_1, Tx_1 = x_2, \ldots, Tx_n = x_{n+1}, \ldots.$$

Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN1**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq aS(x_{n-1}, x_{n-1}, x_n) \\ &\quad + b \max\{S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_n), S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n-1})\} \\ &= aS(x_{n-1}, x_{n-1}, x_n) + b \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n-1})\}. \end{aligned} \tag{10}$$

By the condition (**S3**) we have

$$S(x_{n+1}, x_{n+1}, x_{n-1}) \leq S(x_{n+1}, x_{n+1}, x_n) + S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)$$

$$= 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n). \tag{11}$$

Then using Lemma 1 and the conditions (10) and (11), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) \leq aS(x_{n-1}, x_{n-1}, x_n) + b \max\{&S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n), \\ &2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)\} \\ \leq aS(x_{n-1}, x_{n-1}, x_n) &+ 2bS(x_{n+1}, x_{n+1}, x_n) + bS(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

and so

$$(1 - 2b)S(x_n, x_n, x_{n+1}) \leq (a + b)S(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{a + b}{1 - 2b} S(x_{n-1}, x_{n-1}, x_n). \tag{12}$$

Let $p = \frac{a + b}{1 - 2b}$. Then we have $p < 1$ since $a + 3b < 1$ (notice that $b \neq \frac{1}{2}$ since we have $0 \leq b < \frac{1}{3}$ by the conditions $a + 3b < 1$ and $a, b \geq 0$).

Repeating this process in the condition (12), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{13}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the condition (13) and the condition **(S3)**, we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + 2S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2(p^n + p^{n+1} + \ldots + p^{m-1})S(x_0, x_0, x_1) \\ &\leq 2p^n(1 + p + p^2 + \ldots + p^{m-n-1})S(x_0, x_0, x_1) \\ &\leq 2p^n \frac{1 - p^{m-n}}{1 - p} S(x_0, x_0, x_1) \\ &\leq \frac{2p^n}{1 - p} S(x_0, x_0, x_1). \end{aligned} \tag{14}$$

Hence $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1 - p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) &\leq aS(x_{n-1}, x_{n-1}, x) \\ &+b \max\{S(x_n, x_n, x_{n-1}), S(x_n, x_n, x), S(Tx, Tx, x), S(Tx, Tx, x_{n-1})\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(x, x, Tx) = S(Tx, Tx, x) \leq bS(Tx, Tx, x),$$

which is a contradiction since $0 \leq b < \frac{1}{3}$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN1)** and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) = S(x, x, y) &\leq aS(x, x, y) \\ &\quad + b \max\{S(x, x, x), S(x, x, y), S(y, y, y), S(y, y, x)\} \\ &= aS(x, x, y) + bS(x, x, y) = (a + b)S(x, x, y), \end{aligned}$$

which implies $x = y$ since $a + b < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) \\ &\quad + b \max\{S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x), S(Tx, Tx, x), S(Tx, Tx, x_n)\} \\ &= aS(x_n, x_n, x) + b \max\{S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x), S(x, x, x_n)\}. \end{aligned} \tag{15}$$

Using the condition **(S3)** we have

$$\begin{aligned} S(Tx_n, Tx_n, x_n) &\leq S(Tx_n, Tx_n, x) + S(Tx_n, Tx_n, x) + S(x_n, x_n, x) \\ &= 2S(Tx_n, Tx_n, x) + S(x_n, x_n, x). \end{aligned} \tag{16}$$

Then using the conditions (15), (16) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) \\ &\quad + b \max\{2S(Tx_n, Tx_n, x) + S(x_n, x_n, x), S(Tx_n, Tx_n, x), S(x, x, x_n)\} \\ &= aS(x_n, x_n, x) + b\{2S(Tx_n, Tx_n, x) + S(x_n, x_n, x)\} \\ &= aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x), \end{aligned}$$

which implies

$$S(Tx_n, Tx_n, Tx) = S(Tx_n, Tx_n, x) \leq \frac{a + b}{1 - 2b} S(x_n, x_n, x). \tag{17}$$

So using the condition (17), for $n \to \infty$ we have

$$\lim_{n \to \infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 1 is a generalization of the Banach's contraction principle (1). Indeed, if we take $b = 0$ in Theorem 1, we obtain the Banach's contraction principle (1).

Now we give an example of a self-mapping satisfying the condition **(SN1)** such that the condition of the Banach's contraction principle (1) is not satisfied.

Example 1. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let us define

$$Tx = \begin{cases} x + 50 & \text{if } |x - 1| = 1 \\ 45 & \text{if } |x - 1| \neq 1 \end{cases}.$$

Then T is a self-mapping on the complete S-metric space $\mathbb{R}$ and satisfies the condition **(SN1)** for $a = 0$ and $b = \frac{1}{4}$. Then T has a unique fixed point $x = 45$. But T does not satisfy the condition of the Banach's contraction principle (1). Indeed, for $x = 0, y = 2$ we obtain

$$S(Tx, Tx, Ty) = 4 \leq aS(x, x, y) = 4a,$$

which is a contradiction since $a < 1$.

Definition 4. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN2**) There exist real numbers a, b satisfying $a + 3b < 1$ with $a, b \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq aS(x, x, y) + b \max\{S(T^m x, T^m x, x), S(T^m x, T^m x, y), S(T^m y, T^m y, y), S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 1.

Corollary 9. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN2**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. From Theorem 1, it can be easily seen that T^m has a unique fixed point x in X, and T^m is continuous at x. Also we have

$$Tx = TT^m x = T^{m+1} x = T^m Tx$$

and so we obtain that Tx is a fixed point for T^m. We get $Tx = x$ since x is a unique fixed point.

Definition 5. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN3**) There exist real numbers a, b, c, d satisfying $\max\{a+b+c+3d, 2b+d\} < 1$ with $a, b, c, d \geq 0$ such that

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, x) + cS(Ty, Ty, y) + d\max\{S(Tx, Tx, y), S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 2. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN3**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN3**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) \\ &\quad + cS(x_{n+1}, x_{n+1}, x_n) + d\max\{S(x_n, x_n, x_n), S(x_{n+1}, x_{n+1}, x_{n-1})\} \\ &= aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) + cS(x_{n+1}, x_{n+1}, x_n) + dS(x_{n+1}, x_{n+1}, x_{n-1}). \end{aligned} \tag{18}$$

Then using Lemma 1 and the conditions (11) and (18), we obtain

$$S(x_n, x_n, x_{n+1}) \leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_{n-1}, x_{n-1}, x_n) + cS(x_n, x_n, x_{n+1}) + 2dS(x_n, x_n, x_{n+1}) + dS(x_{n-1}, x_{n-1}, x_n)$$

and so

$$(1 - c - 2d)S(x_n, x_n, x_{n+1}) \leq (a + b + d)S(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{a + b + d}{1 - c - 2d} S(x_{n-1}, x_{n-1}, x_n). \tag{19}$$

Let $p = \dfrac{a + b + d}{1 - c - 2d}$. Then we have $p < 1$ since $a + b + c + 3d < 1$.

Repeating this process in the condition (19), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{20}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (20), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1 - p} S(x_0, x_0, x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) &= S(Tx_{n-1}, Tx_{n-1}, Tx) \leq aS(x_{n-1}, x_{n-1}, x) + bS(x_n, x_n, x_{n-1}) \\ &\quad + cS(Tx, Tx, x) + d \max\{S(x_n, x_n, x), S(Tx, Tx, x_{n-1})\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(x, x, Tx) = S(Tx, Tx, x) \leq (c + d)S(Tx, Tx, x),$$

which is a contradiction since $0 \leq c + d < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN3)** and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) &= S(x, x, y) \leq aS(x, x, y) + bS(x, x, x) + cS(y, y, y) \\ &\quad + d \max\{S(x, x, y), S(x, x, y)\} = (a + d)S(x, x, y), \end{aligned}$$

which implies $x = y$ since $a + d < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) + cS(Tx, Tx, x) \\ &\quad + d \max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\} \\ &= aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) \\ &\quad + d \max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\}. \end{aligned} \tag{21}$$

Then using the conditions (16), (21) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x) \\ &\quad + d \max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\} \\ &\leq aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x) \\ &\quad + dS(Tx_n, Tx_n, x) + dS(x_n, x_n, x) \end{aligned}$$

and so

$$(1 - 2b - d)S(Tx_n, Tx_n, Tx) \leq (a + b + d)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a+b+d}{1-2b-d} S(x_n, x_n, x). \tag{22}$$

Using the condition (22) for $n \to \infty$, we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 2 is a generalization of Corollaries 1 and 2. Indeed, if we take $d = 0$ and $c < \frac{1}{2}$ in Theorem 2, we obtain Corollary 1 and if we take $a = b = c = 0, d = h$ in Theorem 2, we obtain Corollary 2.

Now we give an example of a self-mapping satisfying the condition **(SN3)** such that the condition (3) is not satisfied.

Example 2. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{5}{6}(1 - x).$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned}
S(Tx, Tx, Ty) &= \frac{5}{3}|x - y|, \\
S(x, x, y) &= 2|x - y|, \\
S(Tx, Tx, y) &= \left|\frac{5}{3}(1 - x) - 2y\right|, \\
S(Ty, Ty, x) &= \left|\frac{5}{3}(1 - y) - 2x\right|, \\
S(Tx, Tx, x) &= \left|\frac{5}{3}(1 - x) - 2x\right|, \\
S(Ty, Ty, y) &= \left|\frac{5}{3}(1 - y) - 2y\right|.
\end{aligned}$$

T satisfies the condition (**SN3**) for $a = \frac{5}{6}, b = c = 0$, and $d = \frac{1}{20}$. Then T has a unique fixed point $x = \frac{5}{11}$. But T does not satisfy the condition (3). Indeed, for $x = 1, y = 0$ we obtain

$$S(Tx, Tx, Ty) = \frac{5}{3} \leq h \max\{S(Tx, Tx, y), S(Ty, Ty, x)\}$$
$$= h \max\left\{0, \frac{1}{3}\right\} = \frac{h}{3},$$

which is a contradiction since $h < \frac{1}{3}$.

Definition 6. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN4**) There exist real numbers a, b, c, d satisfying $\max\{a+b+c+3d, 2b+d\} < 1$ with $a, b, c, d \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq aS(x, x, y) + bS(T^m x, T^m x, x) + cS(T^m y, T^m y, y)$$
$$+d \max\{S(T^m x, T^m x, y), S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 2.

Corollary 10. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN4**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 2 by the same method used in the proof of Corollary 9.

Definition 7. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN5**) There exist real numbers a, b, c, d satisfying $\max\{a + 3c + 2d, a + b + c, b + 2d\} < 1$ with $a, b, c, d \geq 0$ such that

$$S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, y) + cS(Ty, Ty, x)$$
$$+d \max\{S(Tx, Tx, x), S(Ty, Ty, y)\},$$

for all $x, y \in X$.

Theorem 3. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN5**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN5**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_n) \\ &\quad + cS(x_{n+1}, x_{n+1}, x_{n-1}) + d \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= aS(x_{n-1}, x_{n-1}, x_n) + cS(x_{n+1}, x_{n+1}, x_{n-1}) \\ &\quad + d \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\}. \end{aligned} \tag{23}$$

Then using Lemma 1 and the conditions (11) and (23), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq aS(x_{n-1}, x_{n-1}, x_n) + 2cS(x_{n+1}, x_{n+1}, x_n) + cS(x_{n-1}, x_{n-1}, x_n) \\ &\quad + dS(x_n, x_n, x_{n-1}) + dS(x_{n+1}, x_{n+1}, x_n) \end{aligned}$$

and

$$(1 - 2c - d)S(x_n, x_n, x_{n+1}) \leq (a + c + d)S(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{a + c + d}{1 - 2c - d} S(x_{n-1}, x_{n-1}, x_n). \tag{24}$$

Let $p = \dfrac{a + c + d}{1 - 2c - d}$. Then we have $p < 1$ since $a + 3c + 2d < 1$.

Repeating this process in the condition (24), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{25}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (25), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1 - p} S(x_0, x_0, x_1).$$

Hence $\lim\limits_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim\limits_{n,m\to\infty} \dfrac{2p^n}{1 - p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) &= S(Tx_{n-1}, Tx_{n-1}, Tx) \leq aS(x_{n-1}, x_{n-1}, x) + bS(x_n, x_n, x) \\ &\quad + cS(Tx, Tx, x_{n-1}) + d \max\{S(x_n, x_n, x_{n-1}), S(Tx, Tx, x)\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(x, x, Tx) = S(Tx, Tx, x) \leq (c + d)S(Tx, Tx, x),$$

which is a contradiction since $0 \leq c + d < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition (**SN5**) and Lemma 1, we have

$$S(Tx, Tx, Ty) = S(x, x, y) \leq aS(x, x, y) + bS(x, x, y) + cS(y, y, x)$$
$$+d \max\{S(x, x, x), S(y, y, y)\} = (a + b + c)S(x, x, y),$$

which implies $x = y$ since $a + b + c < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n) \\ &+d \max\{S(Tx_n, Tx_n, x_n), S(Tx, Tx, x)\} \\ &= aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n) \\ &+dS(Tx_n, Tx_n, x_n). \end{aligned} \tag{26}$$

Then using the conditions (16), (26) and Lemma 1, we obtain

$$S(Tx_n, Tx_n, Tx) \leq aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n)$$
$$+2dS(Tx_n, Tx_n, x) + dS(x_n, x_n, x)$$

and

$$(1 - b - 2d)S(Tx_n, Tx_n, Tx) \leq (a + c + d)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a + c + d}{1 - b - 2d} S(x_n, x_n, x). \tag{27}$$

So using the condition (27), for $n \to \infty$ we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 3 is a generalization of Corollaries 3 and 4. Indeed, if we take $d = 0$ in Theorem 3, we obtain Corollary 4 and if we take $a = b = c = 0, d = h$ in Theorem 3, we obtain Corollary 3.

Notice that the condition (**SN1**) is the special case of the conditions (**SN3**) and (**SN5**) for $b = c = 0$ and $b = d = 0$, respectively. So we have obtained three generalizations of the Banach's contraction principle (1).

Now we give an example of a self-mapping satisfying the condition (**SN5**) such that the condition (4) is not satisfied.

Example 3. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{x}{2}.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned} S(Tx, Tx, Ty) &= |x - y|, \\ S(x, x, y) &= 2|x - y|, \\ S(Tx, Tx, y) &= 2\left|\frac{x}{2} - y\right|, \\ S(Ty, Ty, x) &= 2\left|\frac{y}{2} - x\right|, \\ S(Tx, Tx, x) &= |x|, \\ S(Ty, Ty, y) &= |y|. \end{aligned}$$

T satisfies the condition (**SN5**) for $a = \frac{1}{2}, b = c = 0$, and $d = \frac{1}{8}$. Then T has a unique fixed point $x = 0$. But T does not satisfy the condition (4). Indeed, for $x = 0, y \in [0, 1]$ we obtain

$$\begin{aligned} S(Tx, Tx, Ty) &= |y| \leq h \max\{S(Tx, Tx, x), S(Ty, Ty, y)\} \\ &= h \max\{|x|, |y|\} = h|y|, \end{aligned}$$

which is a contradiction since $h < 1$.

Definition 8. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN6**) There exist real numbers a, b, c, d satisfying $\max\{a + 3c + 2d, a + b + c, b + 2d\} < 1$ with $a, b, c, d \geq 0$ such that

$$\begin{aligned} S(T^m x, T^m x, T^m y) \leq\ & aS(x, x, y) + bS(T^m x, T^m x, y) + cS(T^m y, T^m y, x) \\ & + d \max\{S(T^m x, T^m x, x), S(T^m y, T^m y, y)\}, \end{aligned}$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 3.

Corollary 11. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN6**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 3 by the same method used in the proof of Corollary 9.

Definition 9. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN7**) There exist real numbers a, b, c, d, e, f satisfying $\max\{a + b + 3d + e + 3f, a + c + d + f, 2b + c + 2f\} < 1$ with $a, b, c, d, e, f \geq 0$ such that

$$\begin{aligned} S(Tx, Tx, Ty) \leq{}& aS(x, x, y) + bS(Tx, Tx, x) + cS(Tx, Tx, y) \\ &+ dS(Ty, Ty, x) + eS(Ty, Ty, y) + f \max\{S(x, x, y), \\ &S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, x), S(Ty, Ty, y)\}, \end{aligned}$$

for all $x, y \in X$.

Theorem 4. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN7**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN7**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) \\ &\quad + cS(x_n, x_n, x_n) + dS(x_{n+1}, x_{n+1}, x_{n-1}) + eS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n-1}), S(x_n, x_n, x_n), \\ &\qquad S(x_{n+1}, x_{n+1}, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) + dS(x_{n+1}, x_{n+1}, x_{n-1}) \\ &\quad + eS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n-1}), \\ &\qquad S(x_{n+1}, x_{n+1}, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\}. \end{aligned} \tag{28}$$

Then using Lemma 1 and the conditions (11) and (28), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq aS(x_{n-1}, x_{n-1}, x_n) + bS(x_n, x_n, x_{n-1}) + 2dS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + dS(x_{n-1}, x_{n-1}, x_n) + eS(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n-1}), \\ &\qquad 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= (a + b + d)S(x_{n-1}, x_{n-1}, x_n) + (2d + e)S(x_{n+1}, x_{n+1}, x_n) \\ &\quad + f\{2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)\} \\ &= (a + b + d + f)S(x_{n-1}, x_{n-1}, x_n) + (2d + e + 2f)S(x_{n+1}, x_{n+1}, x_n) \end{aligned}$$

and

$$(1-2d-e-2f)S(x_{n+1},x_{n+1},x_n) \leq (a+b+d+f)S(x_{n-1},x_{n-1},x_n),$$

which implies

$$S(x_n,x_n,x_{n+1}) \leq \frac{a+b+d+f}{1-2d-e-2f}S(x_{n-1},x_{n-1},x_n). \tag{29}$$

Let $p = \dfrac{a+b+d+f}{1-2d-e-2f}$. Then we have $p < 1$ since $a+b+3d+e+3f < 1$.

Repeating this process in the condition (29), we obtain

$$S(x_n,x_n,x_{n+1}) \leq p^n S(x_0,x_0,x_1). \tag{30}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (30), we have

$$S(x_n,x_n,x_m) \leq \frac{2p^n}{1-p}S(x_0,x_0,x_1).$$

Hence $\lim\limits_{n,m\to\infty} S(x_n,x_n,x_m) = 0$ since $\lim\limits_{n,m\to\infty} \dfrac{2p^n}{1-p}S(x_0,x_0,x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n,x_n,Tx) = S(Tx_{n-1},Tx_{n-1},Tx) &\leq aS(x_{n-1},x_{n-1},x) + bS(x_n,x_n,x_{n-1}) \\ &+cS(x_n,x_n,x) + dS(Tx,Tx,x_{n-1}) + eS(Tx,Tx,x) \\ &+f\max\{S(x_{n-1},x_{n-1},x), S(x_n,x_n,x_{n-1}), S(x_n,x_n,x), \\ &\quad S(Tx,Tx,x_{n-1}), S(Tx,Tx,x)\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$\begin{aligned} S(x,x,Tx) = S(Tx,Tx,x) &\leq dS(Tx,Tx,x) + eS(Tx,Tx,x) \\ &+f\max\{S(Tx,Tx,x), S(Tx,Tx,x)\} = (d+e+f)S(Tx,Tx,x), \end{aligned}$$

which is a contradiction since $0 \leq d+e+f < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN7)** and Lemma 1, we have

$$\begin{aligned} S(Tx,Tx,Ty) = S(x,x,y) &\leq aS(x,x,y) + bS(x,x,x) + cS(x,x,y) \\ &+dS(y,y,x) + eS(y,y,y) + f\max\{S(x,x,y), S(x,x,x), S(x,x,y), \\ &\quad S(y,y,x), S(y,y,y)\} = (a+c+d+f)S(x,x,y), \end{aligned}$$

which implies $x = y$ since $a+c+d+f < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) + cS(Tx_n, Tx_n, x) \\ &\quad + dS(Tx, Tx, x_n) + eS(Tx, Tx, x) + f \max\{S(x_n, x_n, x), S(Tx_n, Tx_n, x_n), \\ &\qquad S(Tx_n, Tx_n, x), S(Tx, Tx, x_n), S(Tx, Tx, x)\} \\ &= aS(x_n, x_n, x) + bS(Tx_n, Tx_n, x_n) + cS(Tx_n, Tx_n, x) + dS(Tx, Tx, x_n) \\ &\quad + f \max\{S(x_n, x_n, x), S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x)\}. \end{aligned} \tag{31}$$

Then using the conditions (16), (31) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &= S(Tx_n, Tx_n, x) \leq aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) \\ &\quad + bS(x_n, x_n, x) + cS(Tx_n, Tx_n, x) + dS(Tx, Tx, x_n) + f \max\{S(x_n, x_n, x) \\ &\quad + 2S(Tx_n, Tx_n, x) + S(x_n, x_n, x), S(Tx_n, Tx_n, x)\} \\ &= aS(x_n, x_n, x) + 2bS(Tx_n, Tx_n, x) + bS(x_n, x_n, x) + cS(Tx_n, Tx_n, x) \\ &\quad + dS(Tx, Tx, x_n) + 2fS(Tx_n, Tx_n, x) + fS(x_n, x_n, x) \\ &= (a + b + d + f)S(x_n, x_n, x) + (2b + c + 2f)S(Tx, Tx, x_n) \end{aligned}$$

and

$$(1 - 2b - c - 2f)S(Tx_n, Tx_n, Tx) \leq (a + b + d + f)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a + b + d + f}{1 - 2b - c - 2f} S(x_n, x_n, x). \tag{32}$$

So using the condition (32) for $n \to \infty$ we have

$$\lim_{n \to \infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 4 is a generalization of Corollaries 5 and 6. Indeed, if we take $f = 0$ in Theorem 4, we obtain Corollary 5 and if we take $a = b = c = d = e = 0, f = h$ in Theorem 4, we obtain Corollary 6. Also the condition $d + 2e < 1$ which is used in Corollary 5 is not necessary condition in Theorem 4.

Now we give an example of a self-mapping satisfying the condition (**SN7**) such that the condition (7) is not satisfied.

Example 4. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{x}{2} + \frac{1}{3}.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned} S(Tx, Tx, Ty) &= |x - y|, \\ S(x, x, y) &= 2|x - y|, \\ S(Tx, Tx, y) &= 2\left|\frac{x}{2} + \frac{1}{3} - y\right|, \\ S(Ty, Ty, x) &= 2\left|\frac{y}{2} + \frac{1}{3} - x\right|, \\ S(Tx, Tx, x) &= 2\left|\frac{-x}{2} + \frac{1}{3}\right|, \\ S(Ty, Ty, y) &= 2\left|\frac{-y}{2} + \frac{1}{3}\right|. \end{aligned}$$

T satisfies the condition (**SN7**) for $a = \frac{1}{2}, b = c = d = e = 0$ and $f = \frac{1}{7}$. Then T has a unique fixed point $x = \frac{2}{3}$. But T does not satisfy the condition (7). Indeed, for $x = 1, y = 0$ we obtain

$$S(Tx, Tx, Ty) = \frac{1}{2} \leq h \max\left\{\frac{5}{6}, 1, \frac{2}{3}, \frac{1}{6}, \frac{1}{3}\right\} = h,$$

which is a contradiction since $h < \frac{1}{3}$.

Definition 10. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN8**) There exist real numbers a, b, c, d, e, f satisfying $\max\{a + b + 3d + e + 3f, a + c + d + f, 2b + c + 2f\} < 1$ with $a, b, c, d, e, f \geq 0$ such that

$$\begin{aligned} &S(T^m x, T^m x, T^m y) \leq aS(x, x, y) + bS(T^m x, T^m x, x) + cS(T^m x, T^m x, y) \\ &+dS(T^m y, T^m y, x) + eS(T^m y, T^m y, y) + f \max\{S(x, x, y), \\ &\quad S(T^m x, T^m x, x), S(T^m x, T^m x, y), S(T^m y, T^m y, x), S(T^m y, T^m y, y)\}, \end{aligned}$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 4.

Corollary 12. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN8**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 4 by the same method used in the proof of Corollary 9.

Definition 11. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN9**) There exist real numbers a, b, c satisfying $3a + b + 2c < 1$ with $a, b, c \geq 0$ such that

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, y) + S(Ty, Ty, x)) + bS(x, x, y) \\ +c \max\{S(Tx, Tx, x), S(Ty, Ty, y)\},$$

for all $x, y \in X$.

Theorem 5. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN9**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN9**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq a(S(x_n, x_n, x_n) + S(x_{n+1}, x_{n+1}, x_{n-1})) \\ &\quad +bS(x_{n-1}, x_{n-1}, x_n) + c \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= aS(x_{n+1}, x_{n+1}, x_{n-1}) + bS(x_{n-1}, x_{n-1}, x_n) \\ &\quad +c \max\{S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\}. \end{aligned} \tag{33}$$

Then using Lemma 1 and the conditions (11) and (33), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq 2aS(x_{n+1}, x_{n+1}, x_n) + aS(x_{n-1}, x_{n-1}, x_n) + bS(x_{n-1}, x_{n-1}, x_n) \\ &\quad +c(S(x_n, x_n, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_n)) \\ &= 2aS(x_{n+1}, x_{n+1}, x_n) + (a + b)S(x_{n-1}, x_{n-1}, x_n) \\ &\quad +cS(x_n, x_n, x_{n-1}) + cS(x_{n+1}, x_{n+1}, x_n) \\ &= (2a + c)S(x_{n+1}, x_{n+1}, x_n) + (a + b + c)S(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

and

$$(1 - 2a - c)S(x_n, x_n, x_{n+1}) \leq (a + b + c)S(x_{n-1}, x_{n-1}, x_n),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{a + b + c}{1 - 2a - c} S(x_{n-1}, x_{n-1}, x_n). \tag{34}$$

Let $p = \dfrac{a + b + c}{1 - 2a - c}$. Then we have $p < 1$ since $3a + b + 2c < 1$.

Repeating this process in the condition (34), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{35}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (35), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1-p} S(x_0, x_0, x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) &\leq a(S(x_n, x_n, x) + S(Tx, Tx, x_{n-1})) \\ &+ bS(x_{n-1}, x_{n-1}, x) + c \max\{S(x_n, x_n, x_{n-1}), S(Tx, Tx, x)\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \leq (a + c)S(Tx, Tx, x),$$

which is a contradiction since $0 \leq a + c < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition (**SN9**) and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) &= S(x, x, y) \leq a(S(x, x, y) + S(y, y, x)) \\ &\quad + bS(x, x, y) + c \max\{S(x, x, x), S(y, y, y)\} \\ &= (2a + b)S(x, x, y), \end{aligned}$$

which implies $x = y$ since $2a + b < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq a(S(Tx_n, Tx_n, x) + S(Tx, Tx, x_n)) + bS(x_n, x_n, x) \\ &\quad + c \max\{S(Tx_n, Tx_n, x_n), S(Tx, Tx, x)\} \\ &= a(S(Tx_n, Tx_n, x) + S(Tx, Tx, x_n)) + bS(x_n, x_n, x) + cS(Tx_n, Tx_n, x_n). \end{aligned} \tag{36}$$

Then using the conditions (16), (36) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq aS(Tx_n, Tx_n, x) + aS(Tx, Tx, x_n) + bS(x_n, x_n, x) \\ &\quad + 2cS(Tx_n, Tx_n, x) + cS(x_n, x_n, x) \end{aligned}$$

and

$$(1 - a - 2c)S(Tx_n, Tx_n, Tx) \leq (a + b + c)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{a + b + c}{1 - a - 2c} S(x_n, x_n, x). \tag{37}$$

So using the condition (37), for $n \to \infty$ we have

$$\lim_{n \to \infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 5 is a generalization of Corollary 7. Indeed, if we take $b = c = 0$ in Theorem 5, we obtain Corollary 7.

Now we give an example of a self-mapping satisfying the condition **(SN9)** such that the condition (8) is not satisfied.

Example 5. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{2x}{3} + \frac{1}{4}.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$S(Tx, Tx, Ty) = \frac{4}{3}|x - y|,$$
$$S(x, x, y) = 2|x - y|,$$
$$S(Tx, Tx, y) = 2\left|\frac{2x}{3} + \frac{1}{4} - y\right|,$$
$$S(Ty, Ty, x) = 2\left|\frac{2y}{3} + \frac{1}{4} - x\right|,$$
$$S(Tx, Tx, x) = 2\left|\frac{-x}{3} + \frac{1}{4}\right|,$$
$$S(Ty, Ty, y) = 2\left|\frac{-y}{3} + \frac{1}{4}\right|.$$

T satisfies the condition (**SN9**) for $a = 0, b = \frac{2}{3}$ and $c = \frac{1}{7}$. Then T has a unique fixed point $x = \frac{3}{4}$. But T does not satisfy the condition (8). Indeed, for $x = 1, y = 0$ we obtain

$$S(Tx, Tx, Ty) = \frac{2}{3} \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)) = \frac{5a}{3},$$

which is a contradiction since $a < \frac{1}{3}$.

Definition 12. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN10**) There exist real numbers a, b, c satisfying $3a+b+2c < 1$ with $a, b, c \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq a(S(T^m x, T^m x, y) + S(T^m y, T^m y, x)) + bS(x, x, y) + c \max\{S(T^m x, T^m x, x), S(T^m y, T^m y, y)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 5.

Corollary 13. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN10**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 5 by the same method used in the proof of Corollary 9.

Definition 13. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN11**) There exist real numbers a, b, c satisfying $2a+b+3c < 1$ with $a, b, c \geq 0$ such that

$$S(Tx, Tx, Ty) \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)) + bS(x, x, y) + c \max\{S(Tx, Tx, y), S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 6. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN11**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN11**) we have

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &= S(Tx_{n-1},Tx_{n-1},Tx_n) \leq a(S(x_n,x_n,x_{n-1}) + S(x_{n+1},x_{n+1},x_n)) \\ &\quad + bS(x_{n-1},x_{n-1},x_n) + c\max\{S(x_n,x_n,x_n), S(x_{n+1},x_{n+1},x_{n-1})\} \\ &= aS(x_n,x_n,x_{n-1}) + aS(x_{n+1},x_{n+1},x_n) \\ &\quad + bS(x_{n-1},x_{n-1},x_n) + cS(x_{n+1},x_{n+1},x_{n-1}). \end{aligned} \tag{38}$$

Then using Lemma 1 and the conditions (11) and (38), we obtain

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &\leq aS(x_n,x_n,x_{n-1}) + aS(x_{n+1},x_{n+1},x_n) + bS(x_{n-1},x_{n-1},x_n) \\ &\quad + 2cS(x_{n+1},x_{n+1},x_n) + cS(x_{n-1},x_{n-1},x_n) \\ &= (a+2c)S(x_{n+1},x_{n+1},x_n) + (a+b+c)S(x_{n-1},x_{n-1},x_n) \end{aligned}$$

and

$$(1-a-2c)S(x_n,x_n,x_{n+1}) \leq (a+b+c)S(x_{n-1},x_{n-1},x_n),$$

which implies

$$S(x_n,x_n,x_{n+1}) \leq \frac{a+b+c}{1-a-2c}S(x_{n-1},x_{n-1},x_n). \tag{39}$$

Let $p = \dfrac{a+b+c}{1-a-2c}$. Then we have $p < 1$ since $2a+b+3c < 1$.

Repeating this process in the condition (39), we obtain

$$S(x_n,x_n,x_{n+1}) \leq p^n S(x_0,x_0,x_1). \tag{40}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (40), we have

$$S(x_n,x_n,x_m) \leq \frac{2p^n}{1-p}S(x_0,x_0,x_1).$$

Hence $\lim\limits_{n,m\to\infty} S(x_n,x_n,x_m) = 0$ since $\lim\limits_{n,m\to\infty} \dfrac{2p^n}{1-p}S(x_0,x_0,x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n,x_n,Tx) &= S(Tx_{n-1},Tx_{n-1},Tx) \leq a(S(x_n,x_n,x_{n-1}) + S(Tx,Tx,x)) \\ &\quad + bS(x_{n-1},x_{n-1},x) + c\max\{S(x_n,x_n,x), S(Tx,Tx,x_{n-1})\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \le (a + c)S(Tx, Tx, x),$$

which is a contradiction since $0 \le a + c < 1$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN11)** and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) &= S(x, x, y) \le a(S(x, x, x) + S(y, y, y)) \\ &\quad + bS(x, x, y) + c \max\{S(x, x, y), S(y, y, x)\} \\ &= (b + c)S(x, x, y), \end{aligned}$$

which implies $x = y$ since $b + c < 1$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\le a(S(Tx_n, Tx_n, x_n) + S(Tx, Tx, x)) + bS(x_n, x_n, x) \\ &\quad + c \max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\} \\ &= aS(Tx_n, Tx_n, x_n) + bS(x_n, x_n, x) \\ &\quad + c \max\{S(Tx_n, Tx_n, x), S(Tx, Tx, x_n)\}. \end{aligned} \tag{41}$$

Then using the conditions (16), (41) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\le 2aS(Tx_n, Tx_n, x) + aS(x_n, x_n, x) + bS(x_n, x_n, x) \\ &\quad + cS(Tx_n, Tx_n, x) + cS(Tx, Tx, x_n) \\ &= (2a + c)S(Tx_n, Tx_n, x) + (a + b + c)S(Tx, Tx, x_n) \end{aligned}$$

and

$$(1 - 2a - c)S(Tx_n, Tx_n, Tx) \le (a + b + c)S(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \le \frac{a + b + c}{1 - 2a - c} S(x_n, x_n, x). \tag{42}$$

So using the condition (42), for $n \to \infty$ we have

$$\lim_{n \to \infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

We note that Theorem 6 is a generalization of Corollary 8. Indeed, if we take $b = c = 0$ in Theorem 6, we obtain Corollary 8.

Now we give an example of a self-mapping satisfying the condition (**SN11**) such that the condition (9) is not satisfied.

Example 6. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let

$$Tx = \frac{3x}{4} + \frac{1}{5}.$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$\begin{aligned}
S(Tx, Tx, Ty) &= \frac{3}{2}|x - y|, \\
S(x, x, y) &= 2|x - y|, \\
S(Tx, Tx, y) &= 2\left|\frac{3x}{4} + \frac{1}{5} - y\right|, \\
S(Ty, Ty, x) &= 2\left|\frac{3y}{4} + \frac{1}{5} - x\right|, \\
S(Tx, Tx, x) &= 2\left|\frac{1}{5} - \frac{x}{4}\right|, \\
S(Ty, Ty, y) &= 2\left|\frac{1}{5} - \frac{y}{4}\right|.
\end{aligned}$$

T satisfies the condition (**SN11**) for $a = 0, b = \frac{3}{4}$, and $c = \frac{1}{13}$. Then T has a unique fixed point $x = \frac{4}{5}$. But T does not satisfy the condition (9). Indeed, for $x = 1, y = 0$, we obtain

$$S(Tx, Tx, Ty) = \frac{3}{2} \leq a(S(Tx, Tx, x) + S(Ty, Ty, y)) = \frac{a}{2},$$

which is a contradiction since $a < \frac{1}{2}$.

Definition 14. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN12**) There exist real numbers a, b, c satisfying $2a+b+3c < 1$ with $a, b, c \geq 0$ such that

$$S(T^m x, T^m x, T^m y) \leq a(S(T^m x, T^m x, x) + S(T^m y, T^m y, y)) + bS(x, x, y) + c \max\{S(T^m x, T^m x, y), S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 6.

Corollary 14. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN12**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 6 by the same method used in the proof of Corollary 9.

Definition 15. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN13**) There exist a real number h satisfying $0 \leq h < \frac{1}{4}$ such that

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, y)+S(Ty, Ty, y), S(Ty, Ty, x)+S(Tx, Tx, x)\},$$

for all $x, y \in X$.

Theorem 7. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN13**)*, then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN13**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq h \max\{S(x_n, x_n, x_n) + S(x_{n+1}, x_{n+1}, x_n), \\ &\quad S(x_{n+1}, x_{n+1}, x_{n-1}) + S(x_n, x_n, x_{n-1})\} \\ &= h \max\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, x_{n-1}) + S(x_n, x_n, x_{n-1})\}. \end{aligned} \tag{43}$$

Then using Lemma 1 and the conditions (11) and (43), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq h \max\{S(x_{n+1}, x_{n+1}, x_n), 2S(x_{n+1}, x_{n+1}, x_n) + 2S(x_{n-1}, x_{n-1}, x_n)\} \\ &= 2hS(x_{n+1}, x_{n+1}, x_n) + 2hS(x_n, x_n, x_{n-1}) \end{aligned}$$

and

$$(1 - 2h)S(x_n, x_n, x_{n+1}) \leq 2hS(x_n, x_n, x_{n-1}),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{2h}{1-2h} S(x_{n-1}, x_{n-1}, x_n). \tag{44}$$

Let $p = \frac{2h}{1-2h}$. Then we have $p < 1$ since $a < \frac{1}{4}$.

Repeating this process in the condition (44), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{45}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (45), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1-p} S(x_0, x_0, x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m\to\infty} \frac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$S(x_n, x_n, Tx) = S(Tx_{n-1}, Tx_{n-1}, Tx) \leq h \max\{S(x_n, x_n, x) + S(Tx, Tx, x), S(Tx, Tx, x_{n-1}) + S(x_n, x_n, x_{n-1})\}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \leq hS(Tx, Tx, x),$$

which is a contradiction since $0 \leq h < \frac{1}{4}$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN13)** and Lemma 1, we have

$$S(Tx, Tx, Ty) = S(x, x, y) \leq h \max\{S(x, x, y) + S(y, y, y), S(y, y, x) + S(x, x, x)\} = hS(x, x, y),$$

which implies $x = y$ since $h < \frac{1}{4}$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$S(Tx_n, Tx_n, Tx) \leq h \max\{S(Tx_n, Tx_n, x) + S(Tx, Tx, x), S(Tx, Tx, x_n) + S(Tx_n, Tx_n, x_n)\}. \tag{46}$$

Then using the conditions (16), (46) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq h \max\{S(Tx_n, Tx_n, x), 2S(x_n, x_n, x) + 2S(Tx_n, Tx_n, x)\} \\ &= 2hS(Tx_n, Tx_n, x) + 2hS(x_n, x_n, x) \end{aligned}$$

and

$$(1 - 2h)S(Tx_n, Tx_n, Tx) \leq 2hS(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{2h}{1 - 2h} S(x_n, x_n, x). \tag{47}$$

So using the condition (47), for $n \to \infty$ we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

Definition 16. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN14**) There exist a real number h satisfying $0 \leq h < \frac{1}{4}$ such that

$$\begin{aligned} S(T^m x, T^m x, T^m y) \leq h \max\{&S(T^m x, T^m x, y) + S(T^m y, T^m y, y), \\ &S(T^m y, T^m y, x) + S(T^m x, T^m x, x)\}, \end{aligned}$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 7.

Corollary 15. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN14**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 7 by the same method used in the proof of Corollary 9.

Definition 17. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN15**) There exist a real number h satisfying $0 \leq h < \frac{1}{3}$ such that

$$S(Tx, Tx, Ty) \leq h \max\{S(Tx, Tx, x) + S(Ty, Ty, y), S(Tx, Tx, y) + S(Ty, Ty, x)\},$$

for all $x, y \in X$.

Theorem 8. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN15**), *then T has a unique fixed point x in X and T is continuous at x.*

Proof. Let $x_0 \in X$ and let the sequence (x_n) be defined as in the proof of Theorem 1. Suppose that $x_n \neq x_{n+1}$ for all n. Using the condition (**SN15**) we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq h \max\{S(x_n, x_n, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_n), S(x_n, x_n, x_n) \\ &\quad +S(x_{n+1}, x_{n+1}, x_{n-1})\}. \end{aligned} \tag{48}$$

Then using Lemma 1 and the conditions (11) and (48), we obtain

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq h \max\{S(x_n, x_n, x_{n-1}) + S(x_{n+1}, x_{n+1}, x_n), \\ &\qquad 2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)\} \\ &= 2hS(x_{n+1}, x_{n+1}, x_n) + hS(x_n, x_n, x_{n-1}) \end{aligned}$$

and

$$(1 - 2h)S(x_n, x_n, x_{n+1}) \leq hS(x_n, x_n, x_{n-1}),$$

which implies

$$S(x_n, x_n, x_{n+1}) \leq \frac{h}{1-2h} S(x_{n-1}, x_{n-1}, x_n). \tag{49}$$

Let $p = \dfrac{h}{1-2h}$. Then we have $p < 1$ since $a < \dfrac{1}{3}$.

Repeating this process in the condition (49), we obtain

$$S(x_n, x_n, x_{n+1}) \leq p^n S(x_0, x_0, x_1). \tag{50}$$

Then for all $n, m \in \mathbb{N}$, $n < m$, using the conditions (14) and (50), we have

$$S(x_n, x_n, x_m) \leq \frac{2p^n}{1-p} S(x_0, x_0, x_1).$$

Hence $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$ since $\lim_{n,m\to\infty} \dfrac{2p^n}{1-p} S(x_0, x_0, x_1) = 0$. Therefore (x_n) is a Cauchy sequence. By the completeness hypothesis, there exists $x \in X$ such that (x_n) is convergent to x. Assume that $Tx \neq x$. Then we have

$$\begin{aligned} S(x_n, x_n, Tx) &= S(Tx_{n-1}, Tx_{n-1}, Tx) \leq h \max\{S(x_n, x_n, x_{n-1}) + S(Tx, Tx, x), \\ &\qquad S(x_n, x_n, x) + S(Tx, Tx, x_{n-1})\} \end{aligned}$$

and so taking the limit for $n \to \infty$, using the continuity of the function S and Lemma 1, we obtain

$$S(Tx, Tx, x) \leq hS(Tx, Tx, x),$$

which is a contradiction since $0 \leq h < \frac{1}{3}$. So we have $Tx = x$.

Now we show the uniqueness of x. Suppose that $x \neq y$ such that $Tx = x$ and $Ty = y$. Using the condition **(SN15)** and Lemma 1, we have

$$\begin{aligned} S(Tx, Tx, Ty) = S(x, x, y) &\leq h \max\{S(x, x, x) + S(y, y, y), \\ &S(x, x, y) + S(y, y, x)\} = 2hS(x, x, y), \end{aligned}$$

which implies $x = y$ since $h < \frac{1}{3}$.

Now we show that T is continuous at x. Let (x_n) be any sequence in X such that (x_n) is convergent to x. For $n \in \mathbb{N}$ we have

$$S(Tx_n, Tx_n, Tx) \leq h \max\{S(Tx_n, Tx_n, x_n)+S(Tx, Tx, x), S(Tx_n, Tx_n, x)+S(Tx, Tx, x_n)\}. \tag{51}$$

Then using the conditions (16), (51) and Lemma 1, we obtain

$$\begin{aligned} S(Tx_n, Tx_n, Tx) &\leq h \max\{2S(Tx_n, Tx_n, x) + S(x_n, x_n, x), \\ &S(Tx_n, Tx_n, x) + S(x_n, x_n, x)\} \\ &= 2hS(Tx_n, Tx_n, x) + hS(x_n, x_n, x) \end{aligned}$$

and

$$(1 - 2h)S(Tx_n, Tx_n, Tx) \leq hS(x_n, x_n, x),$$

which implies

$$S(Tx_n, Tx_n, Tx) \leq \frac{h}{1 - 2h} S(x_n, x_n, x). \tag{52}$$

So using the condition (52), for $n \to \infty$ we have

$$\lim_{n\to\infty} S(Tx_n, Tx_n, Tx) = 0.$$

Hence the sequence (Tx_n) is convergent to $Tx = x$ by Definition 2 (1). Consequently T is continuous at x by Lemma 3.

Definition 18. Let (X, S) be a complete S-metric space and T be a self-mapping of X.

(**SN16**) There exist a real number h satisfying $0 \leq h < \frac{1}{3}$ such that

$$S(T^m x, T^m x, T^m y) \leq h \max\{S(T^m x, T^m x, x) + S(T^m y, T^m y, y), S(T^m x, T^m x, y) + S(T^m y, T^m y, x)\},$$

for all $x, y \in X$ and some $m \in \mathbb{N}$.

We give the following corollary as a result of Theorem 8.

Corollary 16. *Let (X, S) be a complete S-metric space and T be a self-mapping of X. If T satisfies the condition* (**SN16**)*, then T has a unique fixed point x in X and T^m is continuous at x.*

Proof. It follows from Theorem 8 by the same method used in the proof of Corollary 9.

Notice that the condition (**SN15**) is the special case of the condition (**SN1**) for $a = 0, b = h$.

Example 7. Let $\mathbb{R}$ be the S-metric space with

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ [10]. Let us consider the following constant function:

$$Tx = k, k \in [0, 1].$$

Then T is a self-mapping on the complete S-metric space $[0, 1]$. We have

$$S(Tx, Tx, Ty) = 0,$$
$$S(Tx, Tx, y) = 2|k - y|,$$
$$S(Ty, Ty, x) = 2|k - x|,$$
$$S(Tx, Tx, x) = 2|k - x|,$$
$$S(Ty, Ty, y) = 2|k - y|.$$

T satisfies the conditions (**SN13**) and (**SN15**) for all $h \in [0, \frac{1}{3})$, respectively. Then T has a unique fixed point $x = k$.

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Functional Inequalities in Banach Spaces and Fuzzy Banach Spaces

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Dedicated to Professor Vladimir Arnold

Abstract This paper is a survey on the Hyers–Ulam stability of additive functional inequalities, quadratic functional inequalities, additive ρ-functional inequalities, and quadratic ρ-functional inequalities in Banach spaces and fuzzy Banach spaces.

Its content is divided into the following sections:

1. Introduction and Preliminaries.
2. Functional Inequalities in Banach Spaces.
3. Functional Inequalities in Fuzzy Banach Spaces.
4. ρ-Functional Inequalities in Banach Spaces.
5. ρ-Functional Inequalities in Fuzzy Banach Spaces.

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1 Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [46] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution

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of the Cauchy equation is said to be a *(Cauchy) additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [43] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th. M. Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called the *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The functional equation

$$2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

is called a *Jensen-type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [9, 13, 18–20, 22, 34–36, 44]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [45]. Fechner [11] and Gilányi [16] proved the Hyers–Ulam stability of the functional inequality (1). Park et al. [41] investigated the following three-variable additive functional inequalities

$$\|f(x) + f(y) + f(z)\| \leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \tag{2}$$

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\|, \tag{3}$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{4}$$

and proved the Hyers–Ulam stability of the functional inequalities (2)–(4) in Banach spaces.

Katsaras [23] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [12, 27, 47]. In particular, Bag and

Samanta [2], following Cheng and Mordeson [7], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [26]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 30, 32] to investigate a fuzzy version of the Hyers–Ulam stability for functional inequalities in the fuzzy normed vector space setting.

Definition 1 ([2, 30–32]). Let X be a real vector space. A function $N : X \times \mathbf{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbf{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N_5) $N(x, \cdot)$ is a nondecreasing function of $\mathbf{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbf{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [30, 32].

Definition 2 ([2, 30–32]). Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 3 ([2, 30, 32]). Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete*, and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ is converging to x_0 in X, and then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

In [28], Lee, Saadati, and Shin investigated the following functional inequalities

$$N(f(x) + f(y) + f(z), t) \geq N\left(f(x + y + z), \frac{t}{2}\right) \tag{5}$$

for all $x, y, z \in X$ and all $t > 0$, and

$$N(f(x) + f(y) + 2f(z), t) \geq N\left(2f\left(\frac{x + y}{2} + z\right), \frac{2}{3}t\right) \tag{6}$$

for all $x, y, z \in X$ and all $t > 0$, and proved the Hyers–Ulam stability of the functional inequalities (5) and (6) in fuzzy Banach spaces in the spirit of the Th. M. Rassias' stability approach.

In [39], Park defined and solved the additive ρ-functional inequalities

$$\|f(x+y)-f(x)-f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \tag{7}$$

and

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leq \|\rho\left(f(x+y)-f(x)-f(y)\right)\|, \tag{8}$$

where ρ is a fixed complex number with $|\rho| < 1.$, and proved the Hyers–Ulam stability of the additive ρ-functional inequalities (7) and (8) in complex Banach spaces by using the direct method.

In [40], Park defined and solved the quadratic ρ-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\|, \end{aligned} \tag{9}$$

where ρ is a fixed complex number with $|\rho| < 1$, and

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ &\leq \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|, \end{aligned} \tag{10}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

Using the direct method, Park [40] proved the Hyers–Ulam stability of the quadratic ρ-functional inequalities (9) and (10) in complex Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1 ([4, 10]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n, or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0;$
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$*;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

In 1996, Isac and Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 8, 30, 33, 37, 38, 42]).

In [25], Kim, Anastassiou, and Park solved the following additive ρ-functional inequalities

$$N(f(x+y) - f(x) - f(y), t) \leq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \tag{11}$$

and

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \leq N(\rho\,(f(x+y) - f(x) - f(y))\,, t) \tag{12}$$

in fuzzy normed spaces, where ρ is a fixed real number with $|\rho| < 1$, and proved the Hyers–Ulam stability of the additive ρ-functional inequalities (11) and (12) in fuzzy Banach spaces by using the fixed point method.

In [24], Kim and Park solved the quadratic ρ-functional inequalities

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\|, \end{aligned} \tag{13}$$

where ρ is a fixed complex number with $|\rho| < 1$, and

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right\| \\ &\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|, \end{aligned} \tag{14}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$, and proved the Hyers–Ulam stability of the quadratic ρ-functional inequalities (13) and (14) in complex Banach spaces by using the fixed point method.

This paper is organized as follows: In Sect. 2, we introduce and solve the three-variable additive functional inequalities (2)–(4) and prove the Hyers–Ulam stability of the three-variable additive functional inequalities (2)–(4) in Banach spaces by using the direct method.

In Sect. 3, we introduce and solve the additive functional inequalities (5) and (6) and prove the Hyers–Ulam stability of the additive functional inequalities (5) and (6) in fuzzy Banach spaces by using the direct method.

In Sect. 4, we introduce and solve the additive ρ-functional inequalities (7) and (8) and the quadratic ρ-functional inequalities (9) and (10) and prove the Hyers–Ulam stability of the additive ρ-functional inequalities (7) and (8) and the quadratic ρ-functional inequalities (9) and (10) in Banach spaces by using the direct method.

In Sect. 5, we introduce and solve the additive ρ-functional inequalities (11) and (12) and the quadratic ρ-functional inequalities (13) and (14) and prove the Hyers–Ulam stability of the additive ρ-functional inequalities (11) and (12) and the quadratic ρ-functional inequalities (13) and (14) in fuzzy Banach spaces by using the fixed point method.

2 Functional Inequalities in Banach Spaces

Throughout this section, assume that X is a normed space and Y is a Banach space.

This section contains the results given in [41].

Proposition 1 ([41, Proposition 2.1]). *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|2f\left(\frac{x+y+z}{2}\right)\| \tag{15}$$

for all $x, y, z \in X$. Then f is Cauchy additive.

Proof. Letting $x = y = z = 0$ in (15), we get $\|3f(0)\| \leq \|2f(0)\|$. So $f(0) = 0$.

Letting $z = 0$ and $y = -x$ in (15), we get

$$\|f(x) + f(-x)\| \leq \|2f(0)\| = 0$$

for all $x \in X$. Hence, $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ in (15), we get

$$\|f(x) + f(y) - f(x+y)\| = \|f(x) + f(y) + f(-x-y)\| \leq \|2f(0)\| = 0$$

for all $x, y \in X$. Thus,

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$, as desired.

Similarly, we can obtain the following propositions.

Proposition 2 ([41, Proposition 2.2]). *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\|$$

for all $x, y, z \in X$. Then f is Cauchy additive.

Proposition 3 ([41, Proposition 2.3]). *Let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + 2f(z)\| \le \left\| 2f\left(\frac{x+y}{2} + z\right)\right\|$$

for all $x, y, z \in X$. Then f is Cauchy additive.

We prove the Hyers–Ulam stability of a functional inequality associated with a Jordan-von Neumann type three-variable Jensen additive functional equation.

Theorem 2 ([41, Theorem 3.1]). *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \le \left\| 2f\left(\frac{x+y+z}{2}\right)\right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{16}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\left\|\frac{f(x) - f(-x)}{2} - h(x)\right\| \le \frac{2^r + 2}{2^r - 2}\theta\|x\|^r \tag{17}$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = -2x$ in (16), we get

$$\|2f(x) + f(-2x)\| \le (2 + 2^r)\theta\|x\|^r \tag{18}$$

for all $x \in X$. Replacing x by $-x$ in (18), we get

$$\|2f(-x) + f(2x)\| \le (2 + 2^r)\theta\|x\|^r \tag{19}$$

for all $x \in X$. Let $g(x) := \frac{f(x)-f(-x)}{2}$. It follows from (18) and (19) that

$$\|2g(x) - g(2x)\| \le (2 + 2^r)\theta\|x\|^r \tag{20}$$

for all $x \in X$. So

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| \le \frac{2 + 2^r}{2^r}\theta\|x\|^r$$

for all $x \in X$. Hence,

$$\begin{aligned}\left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j g\left(\frac{x}{2^j}\right) - 2^{j+1} g\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{2 + 2^r}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{21}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (21) that the sequence $\{2^n g(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n g(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n g(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (21), we get (17).

It follows from (16) that

$$\begin{aligned}
\|h(x) + h(y) + h(z)\| &= \lim_{n\to\infty} 2^n \|g(\frac{x}{2^n}) + g(\frac{y}{2^n}) + g(\frac{z}{2^n})\| \\
&= \lim_{n\to\infty} \frac{2^n}{2} \|f(\frac{x}{2^n}) + f(\frac{y}{2^n}) + (\frac{z}{2^n}) - f(\frac{-x}{2^n}) - f(\frac{-y}{2^n}) - (\frac{-z}{2^n})\| \\
&\le \lim_{n\to\infty} \frac{2^n}{2} \|2f(\frac{x+y+z}{2^n}) - 2f(\frac{x+y+z}{-2^n})\| + \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}} (\|x\|^r + \|y\|^r + \|z\|^r) \\
&= \|2h(\frac{x+y+z}{2})\|
\end{aligned}$$

for all $x, y, z \in X$. So

$$\|h(x) + h(y) + h(z)\|_Y \le \|2h(\frac{x+y+z}{2})\|_Y$$

for all $x, y, z \in X$. By Proposition 1, the mapping $h : X \to Y$ is Cauchy additive.

Now, let $T : X \to Y$ be another Cauchy additive mapping satisfying (17). Then we have

$$\begin{aligned}
\|h(x) - T(x)\| &= 2^n \|h(\frac{x}{2^n}) - T(\frac{x}{2^n})\| \\
&\le 2^n (\|h(\frac{x}{2^n}) - g(\frac{x}{2^n})\| + \|T(\frac{x}{2^n}) - g(\frac{x}{2^n})\|) \\
&\le \frac{2(2^r + 2)2^n}{(2^r - 2)2^{nr}} \theta \|x\|^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus, the mapping $h : X \to Y$ is a unique Cauchy additive mapping satisfying (17).

Similarly, we can obtain the following results.

Theorem 3 ([41, Theorem 3.2]). *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (16). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\|\frac{f(x) - f(-x)}{2} - h(x)\| \le \frac{2 + 2^r}{2 - 2^r} \theta \|x\|^r$$

for all $x \in X$.

We prove the Hyers–Ulam stability of a functional inequality associated with a Jordan-von Neumann type three-variable Cauchy additive functional equation.

Theorem 4 ([41, Theorem 4.1]). *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{22}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - h(x)\| \leq \frac{2^r + 2}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = -2x$ in (22), we get

$$\|2f(x) + f(-2x)\| \leq (2 + 2^r)\theta\|x\|^r \tag{23}$$

for all $x \in X$. Replacing x by $-x$ in (23), we get

$$\|2f(-x) + f(2x)\| \leq (2 + 2^r)\theta\|x\|^r \tag{24}$$

for all $x \in X$. Let $g(x) := \frac{f(x)-f(-x)}{2}$. It follows from (23) and (24) that

$$\|2g(x) - g(2x)\| \leq (2 + 2^r)\theta\|x\|^r$$

for all $x \in X$.

The rest of the proof is the same as in the proof of Theorem 2.

Theorem 5 ([41, Theorem 4.2]). *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (22). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\|\frac{f(x) - f(-x)}{2} - h(x)\| \leq \frac{2 + 2^r}{2 - 2^r}\theta\|x\|^r$$

for all $x \in X$.

We prove the Hyers–Ulam stability of a functional inequality associated with a Jordan-von Neumann type Cauchy-Jensen functional equation.

Theorem 6 ([41, Theorem 5.1]). *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x) + f(y) + 2f(z)\| \leq \|2f(\frac{x + y}{2} + z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{25}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - h(x)\| \leq \frac{2^r + 1}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. Replacing x by $2x$ and letting $y = 0$ and $z = -x$ in (25), we get

$$\|f(2x) + 2f(-x)\| \leq (1 + 2^r)\theta\|x\|^r \tag{26}$$

for all $x \in X$. Replacing x by $-x$ in (26), we get

$$\|f(-2x) + 2f(x)\| \leq (1 + 2^r)\theta\|x\|^r \tag{27}$$

for all $x \in X$. Let $g(x) := \frac{f(x)-f(-x)}{2}$. It follows from (26) and (27) that

$$\|2g(x) - g(2x)\| \leq (1 + 2^r)\theta\|x\|^r$$

for all $x \in X$. So

$$\|g(x) - 2g(\frac{x}{2})\| \leq \frac{1 + 2^r}{2^r}\theta\|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.

Theorem 7 ([41, Theorem 5.2]). *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (25). Then there exists a unique Cauchy additive mapping $h : X \to Y$ such that*

$$\|\frac{f(x) - f(-x)}{2} - h(x)\| \leq \frac{1 + 2^r}{2 - 2^r}\theta\|x\|^r$$

for all $x \in X$.

3 Functional Inequalities in Fuzzy Banach Spaces

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

In this section, we investigate the functional inequalities (5) and (6) in fuzzy normed vector spaces and prove the Hyers–Ulam stability of the functional inequalities (5) and (6) in fuzzy Banach spaces.

This section contains the results given in [28].

Lemma 1 ([28, Lemma 2.1]). *Let (Z, N) be a fuzzy normed vector space. Let $f : X \to Z$ be a mapping such that*

$$N(f(x) + f(y) + f(z), t) \geq N\left(f(x + y + z), \frac{t}{2}\right) \tag{28}$$

for all $x, y, z \in X$ and all $t > 0$. Then f is Cauchy additive, i.e., $f(x+y) = f(x)+f(y)$ for all $x, y \in X$.

Proof. Letting $x = y = z = 0$ in (28), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \geq N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(0) = 0$.

Letting $y = -x$ and $z = 0$ in (28), we get

$$N(f(x) + f(-x), t) \geq N\left(f(0), \frac{t}{2}\right) = N\left(0, \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from (N_2) that $f(x) + f(-x) = 0$ for all $x \in X$. So

$$f(-x) = -f(x)$$

for all $x \in X$.

Letting $z = -x - y$ in (28), we get

$$\begin{aligned} N(f(x) + f(y) - f(x + y), t) &= N(f(x) + f(y) + f(-x - y), t) \\ &\geq N\left(f(0), \frac{t}{2}\right) = N\left(0, \frac{t}{2}\right) = 1 \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By (N_2), $N(f(x) + f(y) - f(x + y), t) = 1$ for all $x, y \in X$ and all $t > 0$. It follows from (N_2) that

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$, as desired.

Similarly, we can obtain the following lemma.

Lemma 2 ([28, Lemma 2.2]). *Let (Z, N) be a fuzzy normed vector space. Let $f : X \to Z$ be a mapping such that*

$$N(f(x) + f(y) + 2f(z), t) \geq N\left(2f\left(\frac{x + y}{2} + z\right), \frac{2}{3}t\right) \tag{29}$$

for all $x, y, z \in X$ *and all* $t > 0$. *Then* f *is Cauchy additive, i.e.,* $f(x+y) = f(x)+f(y)$ *for all* $x, y \in X$.

Now we prove the Hyers–Ulam stability of the Cauchy functional inequality (3) in fuzzy Banach spaces.

Theorem 8 ([28, Theorem 2.3]). *Let* $\varphi : X^3 \to [0, \infty)$ *be a function such that*

$$\tilde{\varphi}(x, y, z) := \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n z) < \infty \tag{30}$$

for all $x, y, z \in X$. *Let* $f : X \to Y$ *be an odd mapping such that*

$$\lim_{t\to\infty} N(f(x) + f(y) + f(z), t\varphi(x, y, z)) = 1 \tag{31}$$

uniformly on X^3. *Then* $L(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for each* $x \in X$ *and defines a Cauchy additive mapping* $L : X \to Y$ *such that if for some* $\delta > 0, \alpha > 0$

$$N(f(x) + f(y) + f(z), \delta\varphi(x, y, z)) \geq \alpha \tag{32}$$

for all $x, y, z \in X$*, then*

$$N\left(f(x) - L(x), \frac{\delta}{2}\tilde{\varphi}(x, x, -2x)\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ *is a unique mapping such that*

$$\lim_{t\to\infty} N(f(x) - L(x), t\tilde{\varphi}(x, x, -2x)) = 1 \tag{33}$$

uniformly on X.

Proof. Since f is an odd mapping, $f(-x) = -f(x)$ for all $x \in X$ and $f(0) = 0$.

Given $\varepsilon > 0$, by (31), we can find some $t_0 > 0$ such that

$$N(f(x) + f(y) + f(z), t\varphi(x, y, z)) \geq 1 - \varepsilon \tag{34}$$

for all $t \geq t_0$. By induction on n, we show that

$$N\left(2^n f(x) - f(2^n x), t\sum_{k=0}^{n-1} 2^{n-k-1}\varphi(2^k x, 2^k x, -2^{k+1}x)\right) \geq 1 - \varepsilon \tag{35}$$

for all $t \geq t_0$, all $x \in X$, and all $n \in \mathbf{N}$.

Letting $y = x$ and $z = -2x$ in (34), we get

$$N(2f(x) - f(2x), t\varphi(x, x, -2x)) \geq 1 - \varepsilon$$

for all $x \in X$ and all $t \geq t_0$. So we get (35) for $n = 1$.

Assume that (35) holds for $n \in \mathbf{N}$. Then

$$\begin{aligned} &N\left(2^{n+1}f(x) - f(2^{n+1}x), t\sum_{k=0}^{n} 2^{n-k}\varphi(2^k x, 2^k x, -2^{k+1}x)\right) \\ &\geq \min\left\{N\left(2^{n+1}f(x) - 2f(2^n x), t_0\sum_{k=0}^{n-1} 2^{n-k}\varphi(2^n x, 2^n x, -2^{n+1}x)\right),\right. \\ &\qquad \left. N(2f(2^n x) - f(2^{n+1}x), t_0\varphi(2^n x, 2^n x, -2^{n+1}x))\right\} \\ &\geq \min\{1 - \varepsilon, 1 - \varepsilon\} = 1 - \varepsilon. \end{aligned}$$

This completes the induction argument. Letting $t = t_0$ and replacing n and x by p and $2^n x$ in (35), respectively, we get

$$N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^{n+p}x)}{2^{n+p}}, \frac{t_0}{2^{n+p}}\sum_{k=0}^{p-1} 2^{p-k-1}\varphi(2^{n+k}x, 2^{n+k}x, -2^{n+k+1}x)\right) \geq 1 - \varepsilon \quad (36)$$

for all integers $n \geq 0, p > 0$.

It follows from (30) and the equality

$$\sum_{k=0}^{p-1} 2^{-n-k-1}\varphi(2^{n+k}x, 2^{n+k}x, -2^{n+k+1}x) = \frac{1}{2}\sum_{k=n}^{n+p-1} 2^{-k}\varphi(2^k x, 2^k x, -2^{k+1}x)$$

that for a given $\delta > 0$ there is an $n_0 \in \mathbf{N}$ such that

$$\frac{t_0}{2}\sum_{k=n}^{n+p-1} 2^{-k}\varphi(2^k x, 2^k x, -2^{k+1}x) < \delta$$

for all $n \geq n_0$ and $p > 0$. Now we deduce from (36) that

$$\begin{aligned} &N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^{n+p}x)}{2^{n+p}}, \delta\right) \\ &\geq N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^{n+p}x)}{2^{n+p}}, \frac{t_0}{2^{n+p}}\sum_{k=0}^{p-1} 2^{p-k-1}\varphi(2^{n+k}x, 2^{n+k}x, -2^{n+k+1}x)\right) \\ &\geq 1 - \varepsilon \end{aligned}$$

for each $n \geq n_0$ and all $p > 0$. Thus, the sequence $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy in Y. Since Y is a fuzzy Banach space, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges to some $L(x) \in Y$. So we can define a mapping $L : X \to Y$ by $L(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, namely, for each $t > 0$ and $x \in X$, $\lim_{n\to\infty} N(\frac{f(2^n x)}{2^n} - L(x), t) = 1$.

Let $x, y, z \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since $\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n z) = 0$, there is an $n_1 > n_0$ such that $t_0\varphi(2^n x, 2^n y, 2^n z) < \frac{2^n t}{4}$ for all $n \geq n_1$. Hence, for each $n \geq n_1$, we have

$$\begin{aligned} N(L(x) + L(y) + L(z), t) \geq \min \Big\{ & N\left(L(x) - 2^{-n} f(2^n x), \frac{t}{16}\right), \\ & N\left(L(y) - 2^{-n} f(2^n y), \frac{t}{16}\right), N\left(L(z) - 2^{-n} f(2^n z), \frac{t}{16}\right), \\ & N\left(L(x+y+z) - 2^{-n} f(2^n(x+y+z)), \frac{t}{16}\right), \\ & N\left(f(2^n(x+y+z)) - f(2^n x) - f(2^n y) - f(2^n z), \frac{2^n t}{4}\right), \\ & N\left(L(x+y+z), \frac{t}{2}\right)\Big\}. \end{aligned}$$

The first four terms on the right-hand side of the above inequality tend to 1 as $n \to \infty$, and the fifth term is greater than

$$N(f(2^n(x+y+z)) - f(2^n x) - f(2^n y) - f(2^n z), t_0\varphi(2^n x, 2^n y, 2^n z)),$$

which is greater than or equal to $1 - \varepsilon$. Thus,

$$N(L(x) + L(y) + L(z), t) \geq \min\left\{N\left(L(x+y+z), \frac{t}{2}\right), 1 - \varepsilon\right\}$$

for all $t > 0$ and $0 < \varepsilon < 1$. So

$$N(L(x) + L(y) + L(z), t) \geq N\left(L(x+y+z), \frac{t}{2}\right)$$

for all $t > 0$, or

$$N(L(x) + L(y) + L(z), t) \geq 1 - \varepsilon$$

for all $t > 0$. For the former case, the mapping $L : X \to Y$ is Cauchy additive, by Lemma 1. For the latter case, $N(L(x) + L(y) + L(z), t) = 1$ for all $t > 0$. So $N(3L(x), t) = 1$ for all $t > 0$ and for all $x \in X$. By (N_2), $L(x) = 0$ for all $x \in X$. Thus, the mapping $L : X \to Y$ is Cauchy additive, i.e., $L(x + y) = L(x) + L(y)$ for all $x, y \in X$.

Now let for some positive δ and α (32) holds. Let

$$\varphi_n(x,y,z) := \sum_{k=0}^{n-1} 2^{-k-1}\varphi(2^k x, 2^k y, 2^k z)$$

for all $x, y, z \in X$. Let $x \in X$. By the same reasoning as in the beginning of the proof, one can deduce from (32) that

$$N\left(2^n f(x) - f(2^n x), \delta \sum_{k=0}^{n-1} 2^{n-k-1}\varphi(2^k x, 2^k x, -2^{k+1}x)\right) \geq \alpha \tag{37}$$

for all positive integers n. Let $t > 0$. We have

$$N(f(x) - L(x), \delta\varphi_n(x,x,-2x) + t) \geq \min\left\{N\left(f(x) - \frac{f(2^n x)}{2^n}, \delta\varphi_n(x,x,-2x)\right)\right.$$
$$\left. N\left(\frac{f(2^n x)}{2^n} - L(x), t\right)\right\} \tag{38}$$

Combining (37) and (38) and the fact that $\lim_{n\to\infty} N(\frac{f(2^n x)}{2^n} - L(x), t) = 1$, we observe that

$$N(f(x) - L(x), \delta\varphi_n(x,x,-2x) + t) \geq \alpha$$

for large enough $n \in \mathbf{N}$. Thanks to the continuity of the function $N(f(x) - L(x), \cdot)$, we see that $N(f(x) - L(x), \frac{\delta}{2}\tilde{\varphi}(x,x,-2x) + t) \geq \alpha$. Letting $t \to 0$, we conclude that

$$N\left(f(x) - L(x), \frac{\delta}{2}\tilde{\varphi}(x,x,-2x)\right) \geq \alpha.$$

To end the proof, it remains to prove the uniqueness assertion. Let T be another additive mapping satisfying (33). Fix $c > 0$. Given $\varepsilon > 0$, by (33) for L and T, we can find some $t_0 > 0$ such that

$$N\left(f(x) - L(x), \frac{t}{2}\tilde{\varphi}(x,x,-2x)\right) \geq 1 - \varepsilon,$$
$$N\left(f(x) - T(x), \frac{t}{2}\tilde{\varphi}(x,x,-2x)\right) \geq 1 - \varepsilon$$

for all $x \in X$ and all $t \geq t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$t_0 \sum_{k=n}^{\infty} 2^{-k}\varphi(2^k x, 2^k x, -2^{k+1}x) < \frac{c}{2}$$

for all $n \geq n_0$. Since

$$\sum_{k=n}^{\infty} 2^{-k}\varphi(2^k x, 2^k x, -2^{k+1}x) = \frac{1}{2^n}\sum_{k=n}^{\infty} 2^{-(k-n)}\varphi(2^{k-n}(2^n x), 2^{k-n}(2^n x), 2^{k-n}(-2^{n+1}x))$$
$$= \frac{1}{2^n}\sum_{m=0}^{\infty} 2^{-m}\varphi(2^m(2^n x), 2^m(2^n x), 2^m(-2^{n+1}x))$$
$$= \frac{1}{2^n}\tilde{\varphi}(2^n x, 2^n x, -2^{n+1}x),$$

we have

$$\begin{aligned}
&N(L(x) - T(x), c)\\
&\geq \min\left\{N\left(\frac{f(2^n x)}{2^n} - L(x), \frac{c}{2}\right), N\left(T(x) - \frac{f(2^n x)}{2^n}, \frac{c}{2}\right)\right\}\\
&= \min\{N(f(2^n x) - L(2^n x), 2^{n-1}c), N(T(2^n x) - f(2^n x), 2^{n-1}c)\}\\
&\geq \min\left\{N\left(f(2^n x) - L(2^n x), 2^n t_0 \sum_{k=n}^{\infty} 2^{-k}\varphi(2^k x, 2^k x, -2^{k+1}x)\right),\right.\\
&\quad \left. N\left(T(2^n x) - f(2^n x), 2^n t_0 \sum_{k=n}^{\infty} 2^{-k}\varphi(2^k x, 2^k x, -2^{k+1}x)\right)\right\}\\
&= \min\{N(f(2^n x) - L(2^n x), t_0\tilde{\varphi}(2^n x, 2^n x, -2^{n+1}x)),\\
&\quad N(T(2^n x) - f(2^n x), t_0\tilde{\varphi}(2^n x, 2^n x, -2^{n+1}x))\}\\
&\geq 1 - \varepsilon.
\end{aligned}$$

It follows that $N(L(x) - T(x), c) = 1$ for all $c > 0$. Thus, $L(x) = T(x)$ for all $x \in X$.

Corollary 1 ([28, Corollary 2.4]). *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping such that*

$$\lim_{t\to\infty} N(f(x) + f(y) + f(z), t\theta(\|x\|^p + \|y\|^p + \|z\|^p)) = 1 \tag{39}$$

uniformly on X^3. Then $L(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for each $x \in X$ and defines a Cauchy additive mapping $L : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N(f(x) + f(y) + f(z), \delta\theta(\|x\|^p + \|y\|^p + \|z\|^p)) \geq \alpha$$

for all $x, y, z \in X$, then

$$N\left(f(x) - L(x), \frac{2 + 2^p}{2 - 2^p}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - L(x), \frac{2+2^p}{2-2^p} 2t\theta\|x\|^p) = 1$$

uniformly on X.

Proof. Define $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and apply Theorem 8 to get the result.

Similarly, we can obtain the following. We will omit the proofs.

Theorem 9 ([28, Theorem 2.5]). *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y, z) := \sum_{n=1}^{\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) < \infty \tag{40}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an odd mapping satisfying (31). Then $L(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a Cauchy additive mapping $L : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N(f(x) + f(y) + f(z), \delta\varphi(x, y, z)) \geq \alpha$$

for all $x, y, z \in X$, then

$$N\left(f(x) - L(x), \frac{\delta}{2}\tilde{\varphi}(x, x, -2x)\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - L(x), t\tilde{\varphi}(x, x, -2x)) = 1$$

uniformly on X.

Corollary 2 ([28, Corollary 2.6]). *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying (39). Then $L(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a Cauchy additive mapping $L : X \to Y$ such that if for some $\delta > 0, \alpha > 0$*

$$N(f(x) + f(y) + f(z), \delta\theta(\|x\|^p + \|y\|^p + \|z\|^p)) \geq \alpha$$

for all $x, y, z \in X$, then

$$N\left(f(x) - L(x), \frac{2^p+2}{2^p-2}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ *is a unique mapping such that*

$$\lim_{t\to\infty} N(f(x) - L(x), \frac{2^p + 2}{2^p - 2} 2t\theta\|x\|^p) = 1$$

uniformly on X.

Proof. Define $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and apply Theorem 9 to get the result.

Finally, we prove the Hyers–Ulam stability of the Cauchy-Jensen functional inequality (4) in fuzzy Banach spaces.

Theorem 10 ([28, Theorem 2.7]). *Let* $\varphi : X^3 \to [0, \infty)$ *be a function satisfying* (30). *Let* $f : X \to Y$ *be an odd mapping such that*

$$\lim_{t\to\infty} N(f(x) + f(y) + 2f(z), t\varphi(x, y, z)) = 1 \tag{41}$$

uniformly on X^3. *Then* $L(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for each* $x \in X$ *and defines a Cauchy additive mapping* $L : X \to Y$ *such that if for some* $\delta > 0, \alpha > 0$

$$N(f(x) + f(y) + 2f(z), \delta\varphi(x, y, z)) \geq \alpha \tag{42}$$

for all $x, y, z \in X$*, then*

$$N\left(f(x) - L(x), \frac{\delta}{2}\tilde{\varphi}(0, -2x, x)\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ *is a unique mapping such that*

$$\lim_{t\to\infty} N(f(x) - L(x), t\tilde{\varphi}(0, -2x, x)) = 1 \tag{43}$$

uniformly on X.

Corollary 3 ([28, Corollary 2.8]). *Let* $\theta \geq 0$ *and let p be a real number with* $0 < p < 1$. *Let* $f : X \to Y$ *be an odd mapping such that*

$$\lim_{t\to\infty} N(f(x) + f(y) + 2f(z), t\theta(\|x\|^p + \|y\|^p + \|z\|^p)) = 1 \tag{44}$$

uniformly on X^3. *Then* $L(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for each* $x \in X$ *and defines a Cauchy additive mapping* $L : X \to Y$ *such that if for some* $\delta > 0, \alpha > 0$

$$N(f(x) + f(y) + 2f(z), \delta\theta(\|x\|^p + \|y\|^p + \|z\|^p)) \geq \alpha$$

for all $x, y, z \in X$*, then*

$$N\left(f(x) - L(x), \frac{1 + 2^p}{2 - 2^p}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N\left(f(x) - L(x), \frac{1+2^p}{2-2^p} 2t\theta\|x\|^p\right) = 1$$

uniformly on X.

Proof. Define $\varphi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and apply Theorem 10 to get the result.

Similarly, we can obtain the following results. We will omit the proofs.

Theorem 11 ([28, Theorem 2.9]). *Let $\varphi : X^3 \to [0,\infty)$ be a function satisfying (40). Let $f : X \to Y$ be an odd mapping satisfying (41). Then $L(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a Cauchy additive mapping $L : X \to Y$ such that if for some $\delta > 0, \alpha > 0$*

$$N(f(x) + f(y) + 2f(z), \delta\varphi(x,y,z)) \geq \alpha$$

for all $x, y, z \in X$, then

$$N\left(f(x) - L(x), \frac{\delta}{2}\tilde{\varphi}(0, -2x, x)\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - L(x), t\tilde{\varphi}(0, -2x, x)) = 1$$

uniformly on X.

Corollary 4 ([28, Corollary 2.10]). *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying (44). Then $L(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a Cauchy additive mapping $L : X \to Y$ such that if for some $\delta > 0, \alpha > 0$*

$$N(f(x) + f(y) + 2f(z), \delta\theta(\|x\|^p + \|y\|^p + \|z\|^p)) \geq \alpha$$

for all $x, y, z \in X$, then

$$N\left(f(x) - L(x), \frac{2^p+1}{2^p-2}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $L : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N\left(f(x) - L(x), \frac{2^p+1}{2^p-2} 2t\theta\|x\|^p\right) = 1$$

uniformly on X.

Proof. Define $\varphi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and apply Theorem 11 to get the result.

4 ρ-Functional Inequalities in Banach Spaces

Throughout this paper, assume that X is a complex normed space and that Y is a complex Banach space. Let ρ be a complex number with $|\rho| < 1$.

This section contains the results given in [39, 40].

We solve and investigate the additive ρ-functional inequality (7) in complex normed spaces.

Lemma 3 ([39, Lemma 2.1]). *A mapping $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \tag{45}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (45).

Letting $x = y = 0$ in (45), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = x$ in (45), we get

$$\|f(2x) - 2f(x)\| \leq 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus,

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{46}$$

for all $x \in X$.

It follows from (45) and (46) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \\ &= |\rho| \|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true.

We prove the Hyers–Ulam stability of the additive ρ-functional inequality (45) in complex Banach spaces.

Theorem 12 ([39, Theorem 2.3]). *Let* $\varphi : X^2 \to [0,\infty)$ *be a function and let* $f : X \to Y$ *be a mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{47}$$

$$\|f(x+y) - f(x) - f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \varphi(x,y) \tag{48}$$

for all $x, y \in X$*. Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{1}{2}\Psi(x,x) \tag{49}$$

for all $x \in X$*.*

Proof. Letting $y = x$ in (48), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x,x) \tag{50}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence,

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{51}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (51) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (51), we get (49).

Now, let $T : X \to Y$ be another additive mapping satisfying (49). Then we have

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (47) and (48) that

$$\begin{aligned}\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} \left\|2^n \rho \left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\quad + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y) - A(x) - A(y)\| \le \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 3, the mapping $A : X \to Y$ is additive.

Corollary 5 ([39, Corollary 2.4]). *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r) \tag{52}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 13 ([39, Theorem 2.5]). *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (48) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2}\Psi(x, x)$$

for all $x \in X$.

Corollary 6 ([39, Corollary 2.6]). *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (52). Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Remark 1. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

We solve and investigate the additive ρ-functional inequality (8) in complex normed spaces.

Lemma 4 ([39, Lemma 3.1]). *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \|\rho\left(f(x+y) - f(x) - f(y)\right)\| \tag{53}$$

for all $x, y \in X$ if and if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (53).

Letting $y = 0$ in (53), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{54}$$

and so $f\left(\frac{x}{2}\right) - \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (53) and (54) that

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \\ &\leq |\rho|\|f(x+y) - f(x) - f(y)\|\end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true.

Now, we prove the Hyers–Ulam stability of the additive ρ-functional inequality (53) in complex Banach spaces.

Theorem 14 ([39, Theorem 3.3]). *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \|\rho\left(f(x+y) - f(x) - f(y)\right)\| + \varphi(x,y) \quad (55)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x,0)$$

for all $x \in X$.

Corollary 7 ([39, Corollary 3.4]). *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \|\rho(f(x+y) - f(x) - f(y))\| + \theta(\|x\|^r + \|y\|^r) \quad (56)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 15 ([39, Theorem 3.5]). *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (55) and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \Psi(x,0)$$

for all $x \in X$.

Corollary 8 ([39, Corollary 3.6]). *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (56). Then there exists a unique* *additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Remark 2. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

Now, we solve and investigate the quadratic ρ-functional inequality (9) in complex normed spaces.

Lemma 5 ([40, Lemma 2.1]). *A mapping* $f : X \to Y$ *satisfies*

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\|\end{aligned} \tag{57}$$

for all $x, y \in X$ *if and only if* $f : X \to Y$ *is quadratic.*

Proof. Assume that $f : X \to Y$ satisfies (57).

Letting $x = y = 0$ in (57), we get $\|2f(0)\| \le |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (57), we get $\|f(2x) - 4f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus,

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{58}$$

for all $x \in X$.

It follows from (57) and (58) that

$$\begin{aligned}&\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &\quad = \frac{|\rho|}{2}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true.

We prove the Hyers–Ulam stability of the quadratic ρ-functional inequality (9) in complex Banach spaces.

Theorem 16 ([40, Theorem 2.3]). *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{59}$$

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| + \varphi(x, y) \end{aligned} \tag{60}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{1}{4}\Psi(x, x) \tag{61}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (60), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.
Letting $y = x$ in (60), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x, x) \tag{62}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence,

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{63}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (63) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (63), we get (61).

It follows from (59) and (60) that

$$\begin{aligned}
&\|h(x+y)+h(x-y)-2h(x)-2h(y)\| \\
&= \lim_{n\to\infty} 4^n \left\| f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 4^n|\rho| \left\| 2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\| \\
&\quad + \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) \\
&= |\rho| \left\| 2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\|h(x+y)+h(x-y)-2h(x)-2h(y)\| \\
&\le \left\| \rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 5, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (61). Then we have

$$\begin{aligned}
\|h(x)-T(x)\| &= \left\|4^q h\left(\frac{x}{2^q}\right)-4^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \left\|4^q h\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\| \\
&\le 2\cdot 4^{q-1}\Psi\left(\frac{x}{2^q},\frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus, the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (61).

Corollary 9 ([40, Corollary 2.4]). *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \qquad (64)\\
&\le \left\| \rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| + \theta(\|x\|^r+\|y\|^r)
\end{aligned}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \le \frac{2\theta}{2^r-4}\|x\|^r$$

for all $x \in X$.

Theorem 17 ([40, Theorem 2.5]). *Let $\varphi : X^2 \to [0, \infty)$ be a function with $\varphi(0, 0) = 0$ and let $f : X \to Y$ be a mapping satisfying (60) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{1}{4} \Psi(x, x)$$

for all $x \in X$.

Corollary 10 ([40, Corollary 2.6]). *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (64). Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{2\theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Remark 3. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

From now on, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

Now, we solve and investigate the quadratic ρ-functional inequality (10) in complex normed spaces.

Lemma 6 ([40, Lemma 3.1]). *A mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \\ &\quad \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \end{aligned} \tag{65}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (65).

Letting $x = y = 0$ in (65), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (65), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{66}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4} f(x)$ for all $x \in X$.

It follows from (65) and (66) that

$$\begin{aligned}
&\frac{1}{2}\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&= \left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\
&\leq |\rho|\|f(x+y)+f(x-y)-2f(x)-2f(y)\|
\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$.

The converse is obviously true.

We prove the Hyers–Ulam stability of the quadratic ρ-functional inequality (65) in complex Banach spaces.

Theorem 18 ([40, Theorem 3.3]). *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{67}$$

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\
&\leq \|\rho\,(f(x+y)+f(x-y)-2f(x)-2f(y))\|+\varphi(x,y)
\end{aligned} \tag{68}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \leq \Psi(x,0) \tag{69}$$

for all $x \in X$.

Proof. Letting $x=y=0$ in (68), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0)=0$.

Letting $y=0$ in (68), we get

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\| \leq \varphi(x,0) \tag{70}$$

for all $x \in X$. So

$$\begin{aligned}
\left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right)-4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j},0\right)
\end{aligned} \tag{71}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (71) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (71), we get (69).

The rest of the proof is similar to the proof of Theorem 16.

Corollary 11 ([40, Corollary 3.4]). *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\| \\ \le \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{72}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 19 ([40, Theorem 3.5]). *Let $\varphi : X^2 \to [0, \infty)$ be a function with $\varphi(0, 0) = 0$ and let $f : X \to Y$ be a mapping satisfying (68) and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \Psi(x, 0)$$

for all $x \in X$.

Corollary 12 ([40, Corollary 3.6]). *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying (72). Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2^r\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

Remark 4. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

5 ρ-Functional Inequalities in Fuzzy Banach Spaces

Throughout this section, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

In this section, we prove the Hyers–Ulam stability of the additive ρ-functional inequality (11) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < 1$.

This section contains the results given in [24, 25].

We need the following lemma to prove the main results.

Lemma 7 ([25, Lemma 2.1]). *Let (Y, N) be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping such that*

$$N(f(x+y) - f(x) - f(y), t) \geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \tag{73}$$

for all $x, y \in X$ and all $t > 0$. Then f is Cauchy additive.

Proof. Assume that $f : X \to Y$ satisfies (73).

Letting $x = y = 0$ in (73), we get $N(f(0), t) = N(0, t) = 1$. So $f(0) = 0$.

Letting $y = x$ in (73), we get $N(f(2x) - 2f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus,

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{74}$$

for all $x \in X$.

It follows from (73) and (74) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &\geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \\ &= N\left(\rho(f(x+y) - f(x) - f(y)), t\right) \\ &= N\left(f(x+y) - f(x) - f(y), \frac{t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) - f(x) - f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

Theorem 20 ([25, Theorem 2.2]). *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$N\left(f(x+y)-f(x)-f(y),t\right) \geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right),\frac{t}{t+\varphi(x,y)}\right\} \tag{75}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x),t\right) \geq \frac{(2-2L)t}{(2-2L)t+L\varphi(x,x)} \tag{76}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (75), we get

$$N\left(f(2x)-2f(x),t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{77}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbf{R}_+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\varphi(x,x)},\quad \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [29, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence,

$$N(Jg(x)-Jh(x),L\varepsilon t) = N\left(2g\left(\frac{x}{2}\right)-2h\left(\frac{x}{2}\right),L\varepsilon t\right)$$

$$= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right)$$

$$\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (77) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{78}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (78) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (76) holds.

By (75),

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right),2^n t\right)$$
$$\geq \min\left\{N\left(\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{y}{2^n}\right)\right),2^n t\right),\frac{t}{t+\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}\right\}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbf{N}$. So

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right),t\right)$$
$$\geq \min\left\{N\left(\rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right)-2^n f\left(\frac{x}{2^n}\right)-2^n f\left(\frac{y}{2^n}\right)\right),t\right),\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)}\right\}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbf{N}$. Since $\lim_{n\to\infty}\frac{\frac{t}{2^n}}{\frac{t}{2^n}+\frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(A(x+y)-A(x)-A(y),t) \geq N\left(\rho\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y)\right),t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 7, the mapping $A : X \to Y$ is Cauchy additive, as desired.

Corollary 13 ([25, Corollary 2.3]). *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$N(f(x+y)-f(x)-f(y),t)$$
$$\geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right),t\right),\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \geq \frac{(2^p-2)t}{(2^p-2)t+2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 20 by taking $\varphi(x,y) := \theta(\|x\|^p+\|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result.

Theorem 21. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x,y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (75). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(2-2L)t}{(2-2L)t + \varphi(x,x)} \tag{79}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 20.

It follows from (77) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \ge \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{1}{2}$. Hence,

$$d(f,A) \le \frac{1}{2-2L},$$

which implies that the inequality (79) holds.

The rest of the proof is similar to the proof of Theorem 20.

Corollary 14 ([25, Corollary 2.5]). *Let $\theta \ge 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned} &N(f(x+y) - f(x) - f(y), t) \\ &\ge \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right), \frac{t}{t+\theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x,y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 21 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x,y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result.

Now, we prove the Hyers–Ulam stability of the additive ρ-functional inequality (12) in fuzzy Banach spaces.

Lemma 8 ([25, Lemma 3.1]). *Let* (Y, N) *be a fuzzy normed vector space. A mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \geq N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right) \qquad (80)$$

for all $x, y \in X$ *and all* $t > 0$*. Then* f *is Cauchy additive, i.e.,* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (80).

Letting $y = 0$ in (80), we get $N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq N(0, t) = 1$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \qquad (81)$$

for all $x \in X$.

It follows from (80) and (81) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &= N\left(2f\left(\frac{x+y}{2} - f(x) - f(y)\right), t\right) \\ &\geq N\left(\rho(f(x+y) - f(x) - f(y)), t\right) \\ &= N\left(f(x+y) - f(x) - f(y), \frac{t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) - f(x) - f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

Theorem 22 ([25, Theorem 3.2]). *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$*. Let* $f : X \to Y$ *be an odd mapping satisfying*

$$\begin{aligned} &N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \qquad (82) \\ &\quad \geq \min\left\{N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right), \frac{t}{t + \varphi(x, y)}\right\} \end{aligned}$$

for all $x, y \in X$ *and all* $t > 0$. *Then* $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, 0)}$$

for all $x \in X$ *and all* $t > 0$.

Corollary 15 ([25, Corollary 3.3]). *Let* $\theta \geq 0$ *and let* p *be a real number with* $p > 1$. *Let* X *be a normed vector space with the norm* $\|\cdot\|$. *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right)$$
$$\geq \min\left\{N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\}$$

for all $x, y \in X$ *and all* $t > 0$. *Then* $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof follows from Theorem 22 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result.

Theorem 23 ([25, Theorem 3.4]). *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be an odd mapping satisfying (82). Then* $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + L\varphi(x, 0)}$$

for all $x \in X$ *and all* $t > 0$.

Corollary 16. *Let* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 1$. *Let* X *be a normed vector space with the norm* $\|\cdot\|$. *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right)$$
$$\geq \min\left\{N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\}$$

for all $x, y \in X$ *and all* $t > 0$*. Then* $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that*

$$N(f(x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + 2^p \theta \|x\|^p}$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 23 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$.

Then we can choose $L = 2^{p-1}$, and we get the desired result.

We prove the Hyers–Ulam stability of the quadratic ρ-functional inequality (13) in fuzzy Banach spaces. We need the following lemma to prove the main results.

Lemma 9 ([24, Lemma 2.1]). *Let* $f : X \to Y$ *be a mapping such that*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \end{aligned} \tag{83}$$

for all $x, y \in X$ *and all* $t > 0$*. Then* f *is quadratic.*

Proof. Assume that $f : X \to Y$ satisfies (83).

Letting $x = y = 0$ in (83), we get $N(2f(0), t) \geq N(\rho(2f(0)), t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(0) = 0$.

Letting $y = x$ in (83), we get $N(f(2x) - 4f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus,

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{84}$$

for all $x \in X$.

It follows from (83) and (84) that

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \\ &= N\left(\frac{1}{2}\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right) \\ &= N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), \frac{2t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y)+f(x-y)-2f(x)-2f(y),t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y)+f(x-y) = 2f(x)+2f(y)$ for all $x, y \in X$.

Theorem 24 ([24, Theorem 2.2]). *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \\ &\ge \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right), \frac{t}{t+\varphi(x,y)}\right\} \end{aligned} \tag{85}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x)-Q(x),t) \ge \frac{(4-4L)t}{(4-4L)t+L\varphi(x,x)} \tag{86}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 20.

Letting $y = x$ in (85), we get

$$N(f(2x)-4f(x),t) \ge \frac{t}{t+\varphi(x,x)} \tag{87}$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \ge \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\begin{aligned} N(Jg(x)-Jh(x),L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right)-4h\left(\frac{x}{2}\right),L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{4}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{4}}{\frac{Lt}{4}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)} \ge \frac{\frac{Lt}{4}}{\frac{Lt}{4}+\frac{L}{4}\varphi(x,x)} = \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (87) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \ge \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{L}{4}$.

By Theorem 1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{88}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (88) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\!-\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f,Q) \le \frac{L}{4-4L}.$$

This implies that the inequality (86) holds.

By (85),

$$\begin{aligned}
&N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \\
&\quad \ge \min\left\{N\left(\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\right),\right. \\
&\qquad \left.\frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbf{N}$. So

$$N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right),t\right)$$
$$\geq \min\left\{N\left(\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),t\right),\right.$$
$$\left.\frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)}\right\}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbf{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),t)$$
$$\geq N\left(\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right),t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 9, the mapping $Q : X \to Y$ is quadratic, as desired.

Corollary 17 ([24, Corollary 2.3]). *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N(f(x+y)+f(x-y)-2f(x)-2f(y),t)$$
$$\geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right),\right.$$
$$\left.\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x)-Q(x),t) \geq \frac{(2^p-4)t}{(2^p-4)t+2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 24 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result.

Theorem 25 ([24, Theorem 2.4]). *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (85). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(4-4L)t}{(4-4L)t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Corollary 18 ([24, Corollary 2.5]). *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\geq \min\left\{ N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right), \right. \\ &\left. \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(4-2^p)t}{(4-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 25 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result.

From now on, we prove the Hyers–Ulam stability of the quadratic ρ-functional inequality (14) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < \frac{1}{2}$. We need the following lemma to prove the main results.

Lemma 10 ([24, Lemma 3.1]). *Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\ &\geq N(\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)), t) \end{aligned} \tag{89}$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (89).

Letting $x = y = 0$ in (89), we get $N(2f(0), t) \geq N(\rho(2f(0)), t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$.

It follows from (N_2) that $f(0) = 0$.

Letting $y = 0$ in (89), we get $N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq N(0, t) = 1$ for all $t > 0$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{90}$$

for all $x \in X$.

It follows from (89) and (90) that

$$\begin{aligned}
&N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right)\\
&= N\left(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y), t\right)\\
&= N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), 2t\right)\\
&\geq N\left(\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)), t\right)\\
&= N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), \frac{t}{|\rho|}\right)
\end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$.

Theorem 26 ([24, Theorem 3.2]). *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying

$$\begin{aligned}
&N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\
&\geq \min\left\{N\left(\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right), \frac{t}{t + \varphi(x, y)}\right\}
\end{aligned} \tag{91}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, 0)} \tag{92}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = 0$ in (91), we get $N(2f(0), t) \geq N(\rho(2f(0)), t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. So $f(0) = 0$.

Letting $y = 0$ in (91), we get

$$N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{93}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 20. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (93) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq 1$.

By Theorem 1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{94}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (94) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N- \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{1 - L}.$$

This implies that the inequality (92) holds.

By (91),

$$N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 4^n t\right)$$
$$\geq \min\left\{N\left(\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\right),\right.$$
$$\left.\frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbf{N}$. So

$$N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right)$$
$$\geq \min\left\{N\left(\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), t\right),\right.$$
$$\left.\frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}\right\}$$

for all $x, y \in X$, all $t > 0$, and all $n \in \mathbf{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(2Q\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) - Q(x) - Q(y), t\right)$$
$$\geq N\left(\rho\left(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\right), t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 10, the mapping $Q : X \to Y$ is quadratic, as desired.

Corollary 19 ([24, Corollary 3.3]). *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right)$$
$$\geq \min\left\{N\left(\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 26 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result.

Theorem 27 ([24, Theorem 3.4]). *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (91). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(1 - L)t}{(1 - L)t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

Corollary 20 ([24, Corollary 3.5]). *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right)$$
$$\geq \min\left\{N\left(\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 27 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result.

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The Maslov Index in PDEs Geometry

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Abstract It is proved that the Maslov index naturally arises in the framework of PDEs geometry. The characterization of PDE solutions by means of Maslov index is given. With this respect, Maslov index for Lagrangian submanifolds is given on the ground of PDEs geometry. New formulas to calculate bordism groups of $(n-1)$-dimensional compact submanifolds bording via n-dimensional Lagrangian submanifolds of a fixed $2n$-dimensional symplectic manifold are obtained too. As a by-product, it is given a new proof of global smooth solutions existence, defined on all $\mathbb{R}^3$, for the Navier–Stokes PDE. Further, complementary results are given in Appendices concerning Navier–Stokes PDE and Legendrian submanifolds of contact manifolds.

1 Introduction

In 1965, V.P. Maslov introduced some integer cohomology classes useful to calculate phase shifts in semiclassical expressions for wave functions and in quantization conditions [31].[1] In the French translation, published in 1972 by the Gauthier-Villars, there is also a complementary article by V.I. Arnold, where new formulas for the calculation of these cohomology classes are given [4, 5, 8].[2] These studies emphasized the great importance of such invariants and hence stimulated a lot of mathematical work focused on characterization of Lagrangian Grassmannian, namely, the smooth manifold of Lagrangian subspaces of a symplectic space. After

[1]The Maslov index is the index of a closed curve in a Lagrangian submanifold of a $2n$-dimensional symplectic space V (coordinates (x, y)), calculated in a neighborhood of a caustic. (These are points of the Lagrangian manifold, where the projection on the x-plane has not constant rank n. Caustics are also called the projection on the x-plane of the set $\Sigma(V) \subset V$ of singular points of V, with respect to this projection.) See also [25].

[2]Further reformulations are given by Hormander [23], Leray [27], Lion and Vergue [28], Kashiwara [24], and Thomas [65].

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the suggestion by Floer to express the spectral flow of a curve of self-adjoint operators by the Maslov index of corresponding curves of Lagrangian subspaces (1988), interesting results have been obtained relating to the Maslov index and spectral flow. (See, e.g., Yoshida [70], Nicolaescu [30], and Cappell, Lee, and Miller [13].)

In 1980, V.I. Arnold introduced also the notion of Lagrangian cobordism in symplectic topology [6, 7, 9]. This new notion has been also studied by Y. Eliashberg and M. Audin in the framework of the algebraic topology [10, 16]. Next this approach has been generalized to higher-order PDEs by Prástaro [34].[3]

In this chapter, we give a general method to recognize the "Maslov index" in the framework of the PDEs geometry. Furthermore, we utilize our algebraic topology of PDEs to calculate suitable Lagrangian bordism groups in a $2n$-dimensional symplectic manifold.

As a by-product of our geometric methods in PDEs, we get another proof of existence of global smooth solutions, defined on all $\mathbb{R}^3$, for the Navier–Stokes PDE, (NS). This proof confirms one on the existence of global smooth solutions for (NS), given in some of our previous works [38–41, 45, 53].

Finally remark that we have written this work in an expository style, in order to be accessible at the most by a large audience of mathematicians and mathematical physicists.[4]

The main results are the following: Definitions 13 and 14 encoding Maslov cycles and Maslov indexes for solutions of PDEs that generalize usual ones; Theorem 16 giving a relation between Maslov cycles and Maslov indexes for solutions of PDEs; Theorem 17 recognizing Maslov index for any Lagrangian manifold, considered as a solution of suitable PDEs of first order; Theorem 18 giving G-singular Lagrangian bordism groups; and Theorem 19 characterizing closed weak Lagrangian bordism groups. In Appendix B are reproduced similar results for Legendrian submanifolds of a contact manifold. Theorem A 1 in Appendix A supports the method, given in Example 12, to build smooth global solutions of the Navier–Stokes PDEs, defined on all $\mathbb{R}^3$.

2 Maslov Index Overview

In this section, we give an algebraic approach to the Maslov index that is more useful to be recast in the framework of PDEs geometry. This approach essentially follows the one given by Arnold [4, 8], Kashiwara [24] and Thomas [65].

[3]See also [11] and references quoted therein.

[4]For general complementary information on algebraic topology and differential topology, see, e.g., [3, 15, 18, 22, 32, 56–64, 66–69].

Definition 1. Let (V, ω) be a symplectic $\mathbb{K}-$vector space over any field $\mathbb{K}$ (with characteristic $\neq 2$), where ω is a symplectic form. We denote by $L_{\rm agr}(V, \omega)$ the *set of Lagrangian subspaces*, defined in (1):

$$L_{\rm agr}(V, \omega) = \left\{L < V \,|\, L = L^{\perp}\right\} \tag{1}$$

with $E^{\perp} = \{v \in V \,|\, \omega(v, w) = 0, \ \forall w \in E\}$.

Example 1. Let us consider the simplest example of $L_{\rm agr}(V, \omega)$, with $V = \mathbb{R}^2$ and $\omega((x_1, y_1), (x_2, y_2)) = x_1 y_2 - y_1 x_2$. Then we get $L_{\rm agr}(V, \omega) \cong G_{1,2}(\mathbb{R}^2) \cong \mathbb{R}P^1$.[5] Therefore, $L_{\rm agr}(V, \omega)$ is a compact analytical manifold of dimension 1. If we consider oriented Lagrangian spaces, we get $L^{+}_{\rm agr}(V, \omega) \cong G^{+}_{1,2}(\mathbb{R}^2) \cong S^1$. Since $\mathbb{R}P^1 \cong S^1$, we get the commutative and exact diagram (2):

$$\begin{array}{ccc}
L^{+}_{agr}(V,\omega) & \underset{\det^2}{\overset{\backsim}{=\!=\!=\!=\!=}} & S^1 \\
\downarrow & & \| \wr \\
L_{agr}(V,\omega) & \overset{\backsim}{=\!=\!=\!=\!=} & \mathbb{R}P^1 \\
\downarrow & & \\
0 & &
\end{array} \tag{2}$$

In (2) $\det^2$ denotes the isomorphism $L(\theta) \mapsto e^{i2\theta}$, $\theta \in [0, \pi)$. One has the following cell decomposition into Schubert cells:

$$L_{\rm agr}(V, \omega) \cong \mathbb{R} \sqcup \{\infty\} = C_2 \sqcup C_1, \tag{3}$$

where C_2 is the cell of dimension 1 and C_1 is the cell of dimension 0. This allows us to calculate the (co)homology spaces of $L_{\rm agr}(V, \omega)$ as reported in (4):

$$H^k(L_{\rm agr}(V, \omega); \mathbb{Z}_2) \cong H_k(L_{\rm agr}(V, \omega); \mathbb{Z}_2) \cong \bigoplus_{N_k} \mathbb{Z}_2 = \left\{ \begin{array}{ll} \mathbb{Z}_2 \ , & 0 \leq k \leq 1 \\ 0 \ , & k > 1 \end{array} \right\}, \tag{4}$$

where N_k is the number of cells of dimension k. We get also the following fundamental homotopy group for $L_{\rm agr}(V, \omega)$:

$$\pi_1(L_{\rm agr}(V, \omega)) \cong \pi_1(S^1) \cong \mathbb{Z}. \tag{5}$$

[5] We use notations and results reported in [36] about Grassmann manifolds.

- The inverse diffeomorphism of $\det^2$ is the map $e^{i2\theta} \mapsto L(\theta)$ identifying the generator 1 of the isomorphism $\pi_1(L_{\mathrm{agr}}(V,\omega)) \cong \mathbb{Z}$.
- The degree of a loop $\gamma : S^1 \to L_{\mathrm{agr}}(V,\omega) \cong S^1$ is the number of elements $\gamma^{-1}(L)$ for a $L \in L_{\mathrm{agr}}(V,\omega)$.
- Let $\{e_1, e_2\} = \{(1,0),(0,1)\}$ be the canonical basis in $\mathbb{R}^2$. Then we call *real Lagrangian* as

$$\mathbb{R} = \{xe_1 \mid \forall x \in \mathbb{R}\} \subset \mathbb{R}^2$$

and *imaginary Lagrangian* as

$$i\mathbb{R} = \{ye_2 \mid \forall y \in \mathbb{R}\} \subset \mathbb{R}^2.$$

They are complementary: $\mathbb{R}^2 \cong \mathbb{R} \bigoplus i\mathbb{R}$.
- Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a symmetric bilinear form. One defines *graph* of $(\mathbb{R},\phi)$, by the following set:

$$\Gamma_{(\mathbb{R},\phi)} = \{(x, \phi_x(1)) \in \mathbb{R}^2\} \subset \mathbb{R}^2,$$

where $\phi_x : \mathbb{R} \to \mathbb{R}$ is the partial linear mapping, identified with a number via the canonical isomorphism $\mathbb{R}^* \cong \mathbb{R}$. $\Gamma_{(\mathbb{R},\phi)}$ is a Lagrangian space of $(\mathbb{R}^2,\omega)$. In fact, if $x' = \lambda x$, we get $\phi_{x'}(1) = \lambda\phi_x(1)$, for any $\lambda \in \mathbb{R}$.
- One has the identification of $L_{\mathrm{agr}}(V,\omega)$ with a symmetric space (and Einstein manifold), via the Grassmannian diffeomorphism reported in (6):

$$L_{\mathrm{agr}}(V,\omega) \cong G^+_{1,2}(\mathbb{R}^2) \cong SO(2)/SO(1) \times SO(1). \tag{6}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & SO(n) & \to & O(n) & \xrightarrow{\det} & O(1) = S^0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & SU(n) & \to & U(n) & \xrightarrow{\det} & U(1) = S^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L^+_{agr}(V,\omega) & \to & L_{agr}(V,\omega) & \xrightarrow{\det^2} & L_{agr}(\mathbb{R}^2,\omega') = S^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{7}$$

Example 2. The above considerations can be generalized to any dimension, namely, considering the symplectic space $(V, \omega) = (\mathbb{R}^{2n}, \omega)$, with

$$\omega((x, y), (x', y')) = \sum_{1 \leq j \leq n} x'_j y_j - y'_j x_j.$$

However, $L^+_{\mathrm{agr}}(V, \omega)$ does not coincide with the Grassmannian space $G^+_{n,2n}(\mathbb{R}^{2n}) \cong SO(2n)/SO(n) \times SO(n)$, but one has the isomorphism reported in (8).[6]

$$U(n)/O(n) \cong L_{\mathrm{agr}}(V, \omega),\ A \mapsto A(i\mathbb{R}^n). \tag{8}$$

Therefore, one has

$$\dim(L_{\mathrm{agr}}(V, \omega)) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}. \tag{9}$$

- The graph $\Gamma_{(\mathbb{R}^n, \phi)} = \{\phi^* = \phi \in M_n(\mathbb{R})\}$ defines a chart at $\mathbb{R}^n \in L_{\mathrm{agr}}(V, \omega)$.
- *Arnold* [4]. The square of the determinant function $\det^2 : L_{\mathrm{agr}}(V, \omega) \to S^1$, $L = A(i\mathbb{R}^n) \mapsto \det^2(A)$, induces the isomorphism

$$\left.\begin{array}{l} \det{}^2_* : \pi_1(L_{\mathrm{agr}}(V, \omega)) \cong \pi_1(S^1) \cong \mathbb{Z} \\ (\gamma : S^1 \to L_{\mathrm{agr}}(V, \omega)) \mapsto \mathrm{degree}(\ S^1 \xrightarrow{\gamma} L_{\mathrm{agr}}(V, \omega) \xrightarrow{\det^2} S^1\ ) \end{array}\right\}. \tag{10}$$

This is a consequence of the homotopy exact sequence of the exact commutative diagram (7) of fiber bundles. As a by-product, we get the first cohomology group of $L_{\mathrm{agr}}(V, \omega)$, with coefficients on $\mathbb{Z}$:

$$\left.\begin{array}{l} H^1(L_{\mathrm{agr}}(V, \omega); \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\pi_1(L_{\mathrm{agr}}(V, \omega)), \mathbb{Z}) \cong \mathbb{Z} \\ \alpha(\gamma) = \mathrm{degree}(\ S^1 \xrightarrow{\gamma} L_{\mathrm{agr}}(V, \omega) \xrightarrow{\det^2} S^1\ ) \in \mathbb{Z} \end{array}\right\}. \tag{11}$$

[6]To fix ideas and nomenclature, we have reported in Table 1 natural geometric structures that can be recognized on $\mathbb{R}^{2n}$, besides their corresponding symmetry groups. The complex structure i allows us to consider the isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$, $(x^j, y^j)_{1 \leq j \leq n} \mapsto (x^j + iy^j)_{1 \leq j \leq n} = (z^1, \cdots, z^n)$. Then the symmetry group of $(\mathbb{R}^{2n}, i) \cong \mathbb{C}^n$ is $\mathrm{GL}(n, \mathbb{C})$. Moreover, the symmetry group of $(\mathbb{R}^{2n}, i, \omega)$ is $Sp(n) \bigcap \mathrm{GL}(n, \mathbb{C}) = U(n)$. Therefore, the matrix A in (8) belongs to $U(n)$ and hence $\det^2(A) \in \mathbb{C}$. Furthermore, taking into account that A can be diagonalized with eigenvalues $\{e^{\pm i\theta_1}, \cdots, e^{\pm i\theta_1}\}$, it follows that $\det^2(A) = e^{i\lambda}$ for some $\lambda \in \mathbb{R}$. Therefore, $\det^2(A) \in S^1$.

Table 1 Natural geometric structures on $\mathbb{R}^{2n}$ and corresponding symmetry groups

Name	Structure	Symmetry group
Euclidean	$g:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$ $g(v,v')=\sum_{1\le j\le n}(x_jx'_j+y_jy'_j)$	$O(2n)=\{A=(a_{ij})\in M_{2n}(\mathbb{R})\,\vert\,\det A\neq 0,\ A^*A=I_{2n}\}$
Symplectic	$\omega:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$ $\omega(v,v')=\sum_{1\le j\le n}(x'_jy_j-x_jy'_j)$	$Sp(n)=\{A=(a_{ij})\in M_{2n}(\mathbb{R})\,\vert\,\det A\neq 0,\ A^*\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix}A=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\}$
Hermitian	$h:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{C}$ $h(v,v')=g(v,v')+i\omega(v,v')=\sum_{1\le j\le n}(x_j+iy_j)(x'_j+iy'_j)$	$U(n)=\{A=(a_{ij})\in M_n(\mathbb{C})\,\vert\,\det A\neq 0,\ AA^*=I_n\}$

$U(n)=Sp(n)\bigcap O(2n)$. $O(2n)$, $Sp(n)\subset \mathrm{GL}(2n,\mathbb{R})$, closed subgroups
$\mathrm{GL}(n,\mathbb{C})$ is the symmetry group of the complex structure. $O(2n)\bigcap \mathrm{GL}(n,\mathbb{C}) = \mathrm{GL}(n,\mathbb{C})\bigcap Sp(n)=U(n)$
$\mathbb{R}^{2n}=\{(x,y)=(x_1,\cdots,x_n,y_1,\cdots,y_n)\,|\,x_j,\ y_j\in\mathbb{R}\}$
$A^*=(\bar{a}_{ji})$, if $A=(a_{ij})$. In the real case $A^*=(a_{ji})$

Example 3. Let (V,σ) be a $2n$-dimensional real symplectic vector space, endowed with a complex structure $J:V\to V$, such that $g:V\times V\to\mathbb{R}$, $g(u,v)=\sigma(J(u),v)$, is an inner product. Then for any $L\in L_{\mathrm{agr}}(V,\sigma)$, the following propositions hold:

(i) One has the diffeomorphism $U(V)/O(L)\cong L_{\mathrm{agr}}(V,\sigma)$, $A\mapsto A(L)$, where

$$O(L)=\{A\in U(V)\,|\,A(L)=L\}.$$

(ii) One has the isomorphism $f_L:(\mathbb{R}^{2n},\omega,i)\cong(V,\sigma,J)$, $f_L(i\mathbb{R}^n)=L$.
(iii) One has the diffeomorphism

$$f_L:L_{\mathrm{agr}}(\mathbb{R}^{2n},\omega)\cong U(n)/O(n)\to L_{\mathrm{agr}}(V,\sigma)\cong U(V)/O(L),\ \lambda\mapsto f_L(\lambda).$$

(iv) If $L_1,L_2\in L_{\mathrm{agr}}(V,\sigma)$, there exists a *difference element* $\lambda[L_1,L_2]\in L_{\mathrm{agr}}(\mathbb{R}^{2n},\omega)$, such that $\lambda[L_1,L_2]\cong i\mathbb{R}^n\subset\mathbb{R}^{2n}$. Therefore, $L_{\mathrm{agr}}(V,\sigma)$ has a $L_{\mathrm{agr}}(\mathbb{R}^{2n},\omega)$-affine structure.

Definition 2. The *Witt group* of a field $\mathbb{K}$ is $W(\mathbb{K})=\pi_0(\mathcal{Q}_+)$, where $\mathcal{Q}_+$ is the category whose objects are *quadratic spaces*, namely, $\mathbb{K}-$vector spaces with nondegenerate, symmetric bilinear forms. We say that two quadratic spaces V_1, $V_2\in Ob(\mathcal{Q}_+)$ are *Witt-equivalent* if there exists a Lagrangian correspondence between them, more precisely a morphism $f\in\mathrm{Hom}_{\mathcal{Q}_+}(V_1,V_2):=L_{\mathrm{agr}}(V_1^o\oplus V_2)$, called the *space of Lagrangian correspondences*. There $(V,q)^o:=(V,-q)$, with q as the quadratic structure. Composition of morphisms is meant in the sense of composition of general correspondences. (For example, if $f : V_1 \to V_2$ is an isometry, then the graph $\Gamma_f \subset V_1^o \oplus V_2$ is Lagrangian. Think of composing functions $f : A \to B$ and $g : B \to C$ via the subsets of $A \times B$ and $B \times C$.)[7]

Proposition 1. *$W(\mathbb{K})$ is the group whose elements are Witt equivalence classes of quadratic spaces, with addition induced by direct sum, and the inverse $-(V,q)$ is given by $-(V,q) = (V,q)^o$.*

Example 4. Let us consider

$$(V,q) = \left(\mathbb{K}^2, \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

- One has the isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$ that is the index of q, namely, the number of positive eigenvalues minus the number of negative eigenvalues.
- One has the isomorphism $W(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ that is the dimension of $W(\mathbb{C})$.[8]

Theorem 1. *There exists a canonical mapping $\tau : L_{\rm agr}(V,\omega)^{\mathbb{Z}_r} \to W(\mathbb{K})$ that we call* Maslov index *and that factorizes as reported in the commutative diagram (12):*

$$\begin{array}{ccc} L_{agr}(V,\omega)^{\mathbb{Z}_r} & \longrightarrow & Ob(\mathcal{Q}_+) \\ & \searrow^{\tau} & \downarrow \\ & & W(\mathbb{K}) \end{array} \tag{12}$$

Proof. Given a r-tuple $L = (L_1, \cdots, L_r)$ of Lagrangian subspaces of (V,ω), we can identify a cochain complex (13):

$$\boxed{C_L = \bigoplus (L_i \bigcap L_{i+1})} \xrightarrow{\partial} \bigoplus_i L_i \xrightarrow{\Sigma} V \tag{13}$$

where Σ is the sum of the components and $\partial(a) = (a,-a) \in L_i \oplus L_{i+1}$, $\forall a \in L_i \bigcap L_{i+1}$. Then we get a quadratic space (T_L, q_L), with $T_L = \ker \sum / {\rm im}\, \partial$ and $q_L(a,b) = \sum_{i>j} \omega(a_i, b_j)$, (*Maslov form*), where a, $b \in T_L$ are lifted to the representative (a_i), $(b_i) \in \oplus_i L_i$. Then the *Maslov index* is defined by (14):

$$\tau(L) = \tau(L_1, \cdots, L_r) = (T_L, q_L) \in W(\mathbb{K}). \tag{14}$$

---

[7] If $f : V_1 \to V_2$ is an isomorphism, then the graph $\Gamma_f \subset V_1^o \bigoplus V_2$ is Lagrangian. The quadratic space (V,q) is equivalent to 0 iff it contains Lagrangian. (For more details on Witt group, see the following link Wikipedia-Witt-group and references therein.)

[8] In this chapter, we denote $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$.

One has the following properties:

(a) *Isometries*: $T(L_1,\cdots,L_r)=T(L_r,L_1,\cdots,L_{r-1})=T(L_1,\cdots,L_r)^o$.
(b) *Lagrangian correspondences*:
$T(L_1,\cdots,L_r)\oplus T(L_1,L_k,\cdots,L_r)\to T(L_1,\cdots,L_r)$, $k<r$.
By considering cell complex $C_L = C(L_1,\cdots,L_r)$, as *r-gon*, with the face labeled by V, edges labeled by L_i, and vertices labeled by $L_i\bigcap L_{i+1}$, property (b) allows us to reduce to the case of three Lagrangian subspaces. Furthermore, Lagrangian correspondences induce cobordism properties. For example, $C(L_1,L_2,L_3,L_4)$ cobords with $C(L_1,L_2,L_3)\bigcup C(L_1,L_3,L_4)$.
(c) *Cocycle property*:
$\tau(L_1,L_2,L_3)-\tau(L_1,L_2,L_4)+\tau(L_1,L_3,L_4)-\tau(L_2,L_3,L_4)=0$.

Theorem 2 (Leray's Function).

- (Case $\mathbb{K}=\mathbb{R}$).
Let $\pi:\widetilde{L_{\rm agr}(V,\omega)}\to L_{\rm agr}(V,\omega)$ *be the universal cover of the Lagrangian Grassmannian. Then there exists a function* (Leray's function)

$$m:\widetilde{L_{\rm agr}(V,\omega)}^2\to\mathbb{Z}\cong W(\mathbb{R})\cong\pi_1(L_{\rm agr}(V,\omega))$$

such that

$$\tau(\pi(\widetilde{L_1}),\cdots,\pi(\widetilde{L_r}))=\sum_{i\in\mathbb{Z}_r}m(\widetilde{L_i},\widetilde{L_{i+1}}).$$

- (Case $\mathbb{K}$ general ground field).
Let $L^+_{\rm agr}(V,\omega)$ *be the set of oriented Lagrangians. There exists a function*

$$m:L^+_{\rm agr}(V,\omega)\to W(\mathbb{K})$$

such that

$$\tau(L_1,\cdots,L_r)=\sum_i m(L_i,L_{i+1})\ mod\ I^2$$

where $I=\ker(\dim:W(\mathbb{K})\to\mathbb{Z}_2)$.[9]

Theorem 3 (Metaplectic Group). *The Maslov index allows to identify a central extension* $Mp(V)$ *of the group* $Sp(V)$ *that when* $\mathbb{K}=\mathbb{R}$ *is the unique double cover of* $Sp(V)$. *(*$Mp(V)$ *is called* metaplectic group.*)*

[9] $L_{\rm agr}(V,\omega)$ has a unique double cover $L^{(2)}{\rm agr}(V,\omega)$. For any pair $(\widetilde{L_1},\widetilde{L_2})$ with $\widetilde{L_1},\widetilde{L_2}\in L^{(2)}{\rm agr}(V,\omega)$, the number $m(\widetilde{L_1},\widetilde{L_2})$ is well-defined mod 4.

Proof. The cocycle property allows to equip $Mp_1(V) = W(\mathbb{K}) \times Sp(V)$, with the multiplication

$$(q,g).(q',g') = (q + q' + \tau(L, gL, gg'L), gg'). \tag{15}$$

Thus, $Mp_1(V)$ is a group and gives a central extension

$$0 \longrightarrow W(\mathbb{K}) \longrightarrow Mp_1(V) \longrightarrow Sp(V) \longrightarrow 1 \tag{16}$$

Moreover, set

$$Mp_2(V) = \left\{(m(g\widetilde{L}, \widetilde{L}) + q, g) \,|\, q \in I^2,\ g \in Sp(V)\right\} \subset Mp_1(V) \tag{17}$$

where $\widetilde{L} \in \Lambda$ over $L \in L_{\mathrm{agr}}(V)$. $Mp_2(V)$ is a subgroup, giving a central extension

$$0 \longrightarrow I^2 \longrightarrow Mp_2(V) \longrightarrow Sp(V) \longrightarrow 1 \tag{18}$$

By quotient I^2 by I^3, we define a central extension

$$0 \longrightarrow I^2/I^3 \longrightarrow Mp(V) \longrightarrow Sp(V) \longrightarrow 1 \tag{19}$$

defining $Mp(V)$, called *metaplectic group*.

When $\mathbb{K} = \mathbb{R}$, $I^2/I^3 \cong \mathbb{Z}_2$, so $Mp(V)$ is the unique double cover of $Sp(V)$. In this case, $Mp(V)$ has four connected components, among which $Mp_2(V)$ is the identity. $Mp_2(V)$ is the universal covering group of $Sp(V)$.[10]

Example 5 (Arnold's Maslov Index). The cohomology class of the Arnold's approach for Maslov index is $\alpha \in H^1(L_{\mathrm{agr}}(\mathbb{R}^{2n}, \omega); \mathbb{Z}) \cong \mathbb{Z}$, obtained as the pullback of the standard differential form $d\theta : S^1 \to T^*S^1$, via $\det^2 : L_{\mathrm{agr}}(\mathbb{R}^{2n}, \omega) \to S^1$. In (20) are summarized the Arnold's definitions of Maslov index for $L \in L_{\mathrm{agr}}(\mathbb{R}^2, \omega)$[11]:

$$\left.\begin{array}{l}
\tau(L(\theta)) = \begin{cases} 1 - \frac{2\theta}{\pi} & , 0 < \theta < \pi \\ 0 & , \theta = 0 \end{cases} \\
\tau(L_1, L_2) = -\tau(L_2, L_1) = \begin{cases} 1 - \frac{2(\theta_1 - \theta 2)}{\pi} & , 0 \leq \theta_1 < \theta_2 < \pi \\ 0 & , \theta_1 = \theta_2 \end{cases} \\
\tau(L_1, L_2, L_3) = \tau(L_1, L_2) + \tau(L_2, L_3) + \tau(L_3, L_1) \in \{-1, 0, 1\} \subset \mathbb{Z}
\end{array}\right\}. \tag{20}$$

[10] One can construct $Mp(V)$ also by observing that $Sp(V)$ embeds into $L_{\mathrm{agr}}(V^o \bigoplus V)$ by $g \mapsto \Gamma_g$, the graph of g. Then define multiplication on $Mp_2(V)$: $(q,g).(q',g') = (q + q' + \tau(\Gamma_1, \Gamma_g, \Gamma_{gg'}), gg')$. Moreover, Γ_g has a canonical orientation.

[11] In particular, if $0 \leq \theta_1 < \theta_2 < \theta_3 < \pi$, then $\tau(L_1, L_2, L_3) = 1$.

- Any couple (L_1, L_2) of Lagrangians in $L_{agr}(\mathbb{R}^2, \omega)$ determines a curve $\gamma_{12} : I = [0, 1] \to L_{agr}(\mathbb{R}^2, \omega)$, $\gamma_{12}(t) = L((1-t)\theta_1 + t\theta_2)$, connecting L_1 and L_2.
- A triple (L_1, L_2, L_3) of Lagrangians in $L_{agr}(\mathbb{R}^2, \omega)$ determines a loop $\gamma_{123} = \gamma_{12}\gamma_{23}\gamma_{31} : S^1 \to L_{agr}(\mathbb{R}^2, \omega)$, with homotopy class the Maslov index of the triple:

$$\gamma_{123} = \tau(L_1, L_2, L_3) \in \{-1, 0, 1\} \subset \pi_1(L_{agr}(\mathbb{R}^2, \omega)) \cong \mathbb{Z}.$$

In fact, for $0 \leq \theta_1 < \theta_2 < \theta_3 < \pi$, one has $\det{}^2\gamma_{123} = 1 : S^1 \to S^1$ and $\mathrm{degree}(\det{}^2\gamma_{123}) = 1 = \tau(L_1, L_2, L_3) \in \mathbb{Z}$.

In (21) are summarized the Arnold's definitions of Maslov index for $L \in L_{agr}(\mathbb{R}^{2n}, \omega)$, $n > 1$. There $\pm e^{i\theta_1}, \cdots, \pm e^{i\theta_n}$ denote the eigenvalues of the matrix $A \in U(n)$, such that $A(i\mathbb{R}^n) = L$:

$$\left.\begin{array}{l} \tau(L) = \sum_{1 \leq j \leq n}(1 - \frac{2\theta_j}{\pi}) \in \mathbb{R},\ 0 \leq \theta_j < \pi \\ \tau(L_1, L_2) = -\tau(L_2, L_1) = \left\{\begin{array}{ll} \sum_{1 \leq j \leq n}(1 - \frac{2(\theta_{1j} - \theta_{2j})}{\pi} & , 0 \leq \theta_{1j} < \theta_{2j} < \pi \\ 0 & , \theta_{1j} = \theta_{2j} \end{array}\right. \\ \tau(L_1, L_2, L_3) = \tau(L_1, L_2) + \tau(L_2, L_3) + \tau(L_3, L_1) \in \{-1, 0, 1\} \subset \mathbb{Z} \end{array}\right\}. \tag{21}$$

- *(Arnold)* [4]. The Poincaré dual $D\alpha$ of $\alpha \in H^1(L_{agr}(\mathbb{R}^{2n}, \omega); \mathbb{Z}) \cong \mathbb{Z}$ is called the *Maslov cycle*, and it results to

$$D\alpha = \{L \in L_{agr}(\mathbb{R}^{2n}, \omega) \mid L \bigcap i\mathbb{R}^n \neq \{0\}\} \tag{22}$$

with

$$[D\alpha] \in H_{\frac{(n+2)(n-1)}{2}}(L_{agr}(\mathbb{R}^{2n}, \omega); \mathbb{Z}). \tag{23}$$

Example 6 (The Wall Nonadditivity Invariant as Maslov Index). Let (V, ω) be a symplectic space and (L_1, L_2, L_3) a triple of Lagrangian subspaces. The *Wall nonadditivity invariant* $w(L_1, L_2, L_3) = \sigma(W, \psi)$, i.e., the signature of the non-singular symmetric form

$$\psi : W \times W \to \mathbb{R},\ \pi(x_1, x_2, x_3, y_1, y_2, y_3) = \omega(x_1, y_2)$$

with

$$W = \frac{\{(x_1, x_2, x_3) \in L_1 \oplus L_2 \oplus L_3 \mid x_1 + x_2 + x_3 = 0\}}{\mathrm{im}(L_1 \bigcap L_2 + L_2 \bigcap L_3 + L_3 \bigcap L_1)}.$$

- *(Wall* [67]) $w(L_1, L_2, L_3)$ can be identified with the defect of the Novikov additivity for the signature of the triple union of a $4k$-dimensional manifold with boundary $(X, \partial X)$:

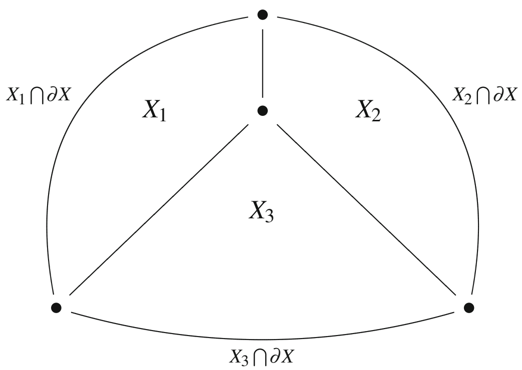
$$w(L_1, L_2, L_3) = \sigma(X_1) + \sigma(X_1) + \sigma(X_2) + \sigma(X_3) - \sigma(X) \in \mathbb{Z}$$

where $X = X_1 \bigcup X_2 \bigcup X_3$ and X_i, $i = 1, 2, 3$ are codimension 0 manifolds with boundary meeting transversely as pictured in (25). One has a non-singular symplectic intersection form on $H^{2k-1}(X_1 \bigcap X_2 \bigcap X_3; \mathbb{R})$ [12] and the following Lagrangian subspaces:

$$\left.\begin{array}{l} L_1 = \mathrm{im}(H^{2k-1}(X_2 \bigcap X_3; \mathbb{R}) \to H^{2k-1}(X_1 \bigcap X_2 \bigcap X_3; \mathbb{R})) \\ L_2 = \mathrm{im}(H^{2k-1}(X_1 \bigcap X_3; \mathbb{R}) \to H^{2k-1}(X_1 \bigcap X_2 \bigcap X_3; \mathbb{R})) \\ L_3 = \mathrm{im}(H^{2k-1}(X_1 \bigcap X_2; \mathbb{R}) \to H^{2k-1}(X_1 \bigcap X_2 \bigcap X_3; \mathbb{R})) \end{array}\right\}. \tag{24}$$

- *(Cappell et al.* [13]) The Maslov index of the triple (L_1, L_2, L_3) coincides with the Wall nonadditivity invariant of (L_1, L_2, L_3).[13]

$$\tau(L_1, L_2, L_3) = w(L_1, L_2, L_3, g).$$



(25)

3 Integral Bordism Groups in PDEs

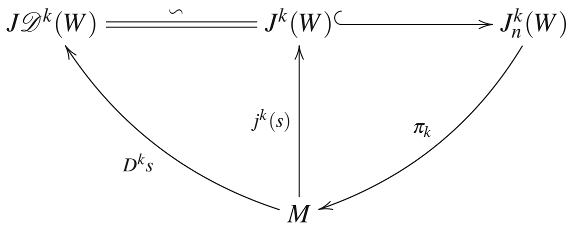
The definition of Maslov index can be recast in the framework of the PDEs geometry. In fact the *metasymplectic structure* of the Cartan distribution of k-jet-spaces $J^k_n(W)$ over a fiber bundle, $\pi : W \to M$, $\dim W = n + m$, $\dim M = n$, allows us to recognize the "Maslov index" associated to n-dimensional integral planes of the Cartan distribution of $J^k_n(W)$ and by restriction on any PDE $E_k \subset J^k_n(W)$. In

[12]The *intersection form* of a $2n$-dimensional topological manifold with boundary $(M, \partial M)$ is $(-1)^n$-symmetric form $\lambda : H^n(M, \partial M; \mathbb{Z})/Tor \times H^n(M, \partial M; \mathbb{Z})/Tor \to \mathbb{Z}$, $\lambda(x, y) = \langle x \bigcup y, [M] \rangle \in \mathbb{Z}$. The *signature* $\sigma(M)$ of $4k$-dimensional manifold $(M, \partial M)$ is $\sigma(M) = \sigma(\lambda) \in \mathbb{Z}$.

[13]A more recent different proof has been given by A. Ranicki (1997). (See in [56].)

the following, we shall give a short panorama on the geometric theory of PDEs and on the metasymplectic structure of the Cartan distribution and its relations with (singular) solutions of PDEs. (For more information, see also [29, 39].)[14]

Let W be a smooth manifold of dimension $m+n$. For any n-dimensional submanifold $N \subset W$, we denote by $[N]^k_a$ the *k-jet of N at the point* $a \in N$, i.e., the set of n-dimensional submanifolds of W that have in a a contact of order k. Set $J^k_n(W) \equiv \bigcup_{a\in W} J^k_n(W)_a$, $J^k_n(W)_a \equiv \{[N]^k_a | a \in W\}$. We call $J^k_n(W)$ the *space of all k-jets of submanifolds of dimension n* of W. $J^k_n(W)$ has the following natural structures of differential fiber bundles: $\pi_{k,s} : J^k_n(W) \to J^s_n(W)$, $s \leq k$, with affine fibers $J^k_n(W)_{\bar{q}}$, where $\bar{q} \equiv [N]^{k-1}_a \in J^{k-1}_n(W)$, $a \equiv \pi_{k,0}(\bar{q})$, with associated vector space $S^k(T^*_a N) \otimes \nu_a$, $\nu_a \equiv T_aW/T_aN$. For any n-dimensional submanifold $N \subset W$, one has the canonical embedding $j^k : N \to J^k_n(W)$, given by $j^k : a \mapsto j^k(a) \equiv [N]^k_a$. We call $j^k(N) \equiv N^{(k)}$ the *k-prolongation* of N. In the following, we shall also assume that there is a fiber bundle structure on W, $\pi : W \to M$, where $\dim M = n$. Then there exists a canonical open bundle submanifold $J^k(W)$ of $J^k_n(W)$ that is called the *k-jet space for sections* of π. $J^k(W)$ is diffeomorphic to the k-jet-derivative space of sections of π, $J\mathscr{D}^k(W)$ [33]. Then, for any section $s : M \to W$, one has the commutative diagram (26), where $D^k s$ is the k-derivative of s and $j^k(s)$ is the k-jet-derivative of s. If $s(M)^{(k)} \subset J^k_n(W)$ is the k-prolongation of $s(M) \subset W$, then one has $j^k(s)(M) \cong s(M)^{(k)} \cong s(M) \cong M$. Of course there are also n-dimensional submanifolds $N \subset W$ that are not representable as image of sections of π. As a consequence, in these cases, $N^{(k)} \cong N$ is not representable in the form $j^k(s)(M)$ for some section s of π. The condition that N is an image of some (local) section s of π is equivalent to the following local condition: $s^*\eta \equiv s^* dx^1 \wedge \cdots \wedge dx^n \neq 0$, where $(x^\alpha, y^j)_{1\leq\alpha\leq n, 1\leq j\leq m}$ are fibered coordinates on W, with y^j vertical coordinates. In other words, $N \subset W$ is locally representable by equations $y^j = y^j(x^1, \ldots, x^n)$. This is equivalent to saying that N is transversal to the fibers of π or that the tangent space TN identifies a horizontal distribution with respect to the vertical one $vTW|_N$ of the fiber bundle structure $\pi : W \to M$. Conversely, a completely integrable n-dimensional horizontal distribution on W determines a foliation of W by means of n-dimensional submanifolds that can be represented by images of sections of π. The *Cartan distribution* of $J^k_n(W)$ is the distribution $\mathbf{E}^k_n(W) \subset TJ^k_n(W)$ generated by tangent spaces to the k-prolongation $N^{(k)}$ of n-dimensional submanifolds N of W:



(26)

[14] For general information on PDEs geometry, see [12, 14, 19, 20, 26, 36].

Theorem 4 (Metasymplectic Structure of the Cartan Distribution). *There exists a canonical vector-fiber-valued* 2*-form on the Cartan distribution* $\mathbf{E}^k_n(W)$*, called* metasymplectic structure *of* $J^k_n(W)$.

Proof. The metasymplectic structure of the Cartan distribution $\mathbf{E}^k_n(W) \subset TJ^k_n(W)$ is a section

$$\Omega_k : J^k_n(W) \to [S^{k-1}(\tau^*) \otimes \nu] \bigotimes \Lambda^2(\mathbf{E}^k_n(W)^*),$$

where $S^{k-1}(\tau^*) \equiv \bigcup_{q\in J^k_n(W)} S^{k-1}(\tau^*)_q$, with $S^{k-1}(\tau^*)_q \equiv S^{k-1}(T^*_a N)$, $\nu \equiv \bigcup_{q\in J^k_n(W)} \nu_q$, with $\nu_q \equiv (T_a W/T_a N)$, $[N]^k_a = q$, such that the following diagram

$$\begin{array}{ccc} S^{k-1}(\tau^*)_q \otimes \nu_q & \overset{\backsim}{=\!=\!=} & T_q J^k_n(W)/\mathbf{E}^k_n(W)_q \\ \| \wr & & \| \wr \\ T_{\bar q} J^{k-1}_n(W)/L_q & \overset{\backsim}{=\!=\!=} & \pi^{-1}_{k,k-1*}(T_{\bar q} J^{k-1}_n(W))/\pi^{-1}_{k,k-1*}(L_q) \end{array}$$

is commutative, for all $q \in J^k_n(W)$, $\bar q \equiv \pi_{k,k-1}(q)$, $a \equiv \pi_{k,0}(q)$, where $L_q \subset T_{\bar q} J^{k-1}_n(W)$ is the integral vector space canonically identified by q. Then, for the *metasymplectic structure* $\Omega_{\mathbf{E}^k_n(W)}$ of $\mathbf{E}^k_n(W)$, we have

$$\begin{aligned} \Omega_k(q) \equiv \Omega_{\mathbf{E}^k_n(W)}(q) &\in [T_q J^k_n(W)/\mathbf{E}^k_n(W)_q] \bigotimes \Lambda^2(\mathbf{E}^k_n(W)^*_q) \\ &\cong [S^{k-1}(\tau^*)_q \otimes \nu_q] \bigotimes \Lambda^2(\mathbf{E}^k_n(W)^*_q). \end{aligned} \tag{27}$$

More precisely, $\Omega_k = d\omega_f|_{\mathbf{E}^k_n(W)}$, where $\omega_f = \langle \omega, f\rangle = \langle f, (\phi^k)^*\rangle \in \Omega^1(J^k_n(W))$, are the *Cartan forms* corresponding to smooth functions:

$$f : J^k_n(W) \to \nu^k := \bigcup_{q\in J^k_n(W)} \nu^k_q,\ \nu^k_q = T_{\bar q} J^{k-1}_n(W)/L_q.$$

ϕ^k is a canonical morphism of vector bundles over $J^k_n(W)$, defined by the exact sequence (28):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{E}^k_n(W) & \longrightarrow & TJ^k_n(W) & \xrightarrow{\phi^k} & \nu^k & \longrightarrow & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & J^k_n(W) & & & & \end{array} \tag{28}$$

For duality, one has also the exact sequence (29):

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbf{E}_n^k(W)^* & \longleftarrow & T^*J_n^k(W) & \xleftarrow{(\phi^k)^*} & (\nu^k)^* & \longleftarrow & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & J_n^k(W) & & & & \end{array} \quad (29)$$

Therefore, we get also a smooth section

$$\omega : J_n^k(W) \to \nu^k \bigotimes T^*J_n^k(W),$$

given by $\langle \omega, f\rangle = \langle f, (\phi^k)^*\rangle = f \circ \phi^k$, for any smooth section $f \in C^\infty((\nu^k)^*)$. It results

$$\mathbf{E}_n^k(W) = \bigcup_{f \in C^\infty((\nu^k)^*)} \ker(\omega_f). \quad (30)$$

Furthermore, for any $\tilde{q} \in \pi_{k+1,k}^{-1}(q) \subset J_n^{k+1}(W)$, $q = [N]_a^k \in J_n^k(W)$, one has the following splitting:

$$\mathbf{E}_n^k(W)_q \cong L_{\tilde{q}} \bigoplus [S^k(T_a^*N) \otimes \nu_a]. \quad (31)$$

The splitting (31) allows us to give the following evaluation of $\Omega_k(q)(\lambda)$, for any $q \in J_n^k(W)$ and $\lambda \in S^{k-1}(T_aN) \otimes \nu_a^*$:

$$\left.\begin{array}{ll} \Omega_k(q)(\lambda)(X,Y) = 0, & \forall X, Y \in L_{\tilde{q}},\ \pi_{k+1,k}(\tilde{q}) = q; \\ \Omega_k(q)(\lambda)(\theta_1,\theta_2) = 0, & \forall \theta_1, \theta_2 \in S^k(T_a^*N) \otimes \nu_a; \\ \Omega_k(q)(\lambda)(X,\theta) = \langle \lambda, X \rfloor \delta\theta\rangle, & \forall X \in L_{\tilde{q}},\ \theta \in S^k(T_a^*N) \otimes \nu_a \end{array}\right\}, \quad (32)$$

where δ is the morphism in the exact sequence (33).

If there is a fiber bundle structure $\alpha : W \to M$, $\dim M = n$, for the metasymplectic structure of $J\mathscr{D}^k(W)$, one has $\Omega_k(q) \in \Lambda^2(\mathbf{E}_n^k(W)_q^*) \bigotimes S^{k-1}(T_b^*M) \bigotimes vT_aW$ with $a \equiv \pi_{k,0}(q) \in W$, $b \equiv \pi_k(q) \in M$. If α is a trivial bundle $\alpha : W \equiv M \times F \to M$, then one has $\Omega_k(q) \in \Lambda^2(\mathbf{E}_n^k(W)_q^*) \bigotimes S^{k-1}(T_b^*M) \bigotimes T_fF$, $\forall a \equiv (b,f)$.

Definition 3. • We say that vectors $X, Y \in \mathbf{E}_n^k(W)_q$ are in *involution* if

$$\Omega_k(q)(\lambda)(X,Y) = 0,\ \forall \lambda \in S^{k-1}(T_aN) \bigotimes \nu_a^*.$$

- A subspace $P \subset \mathbf{E}_n^k(W)_q$ is called *isotropic* if any two vectors $X, Y \in P$ are in involution.
- We say that a subspace $P \subset \mathbf{E}_n^k(W)_q$ is a *maximal isotropic subspace* if P is not a proper subspace of any other isotropic subspace:

$$
\begin{array}{c}
0\\
\downarrow\\
[S^m(T^*_aN)\bigotimes\nu_a]\\
\downarrow\delta\\
T^*_aN\bigotimes[S^{m-1}(T^*_aN)\otimes\nu_a]\\
\downarrow\delta\\
\Lambda^2(T^*_aN)\bigotimes[S^{m-2}(T^*_aN)\otimes\nu_a]\\
\downarrow\delta\\
\cdots\\
\downarrow\delta\\
\Lambda^n(T^*_aN)\bigotimes[S^{m-n}(T^*_aN)\otimes\nu_a]\\
\downarrow\\
0
\end{array}
\tag{33}
$$

Theorem 5 (Structure of Maximal Isotropic Subspaces). *Any maximal isotropic subspace $P \subset \mathbf{E}^k_n(W)_q$ is one tangent at $q = [N]^k_a$ to a maximal integral manifold V of $\mathbf{E}^k_n(W)$. These are of dimension $m\binom{p+k-1}{k} + n - p$, such that $n - p = \dim(\pi_{k,0*}(T_qV)) \leq \dim T_aN = n$. Then one says that V is of* type *$n-p$. In particular, if $p = 0$, then $L_{\tilde{q}} \cong T_qV \cong T_aN$. In the* exceptional case*, i.e., $m = n = 1$, maximal integral manifolds are of dimension* 1 *having eventual subsets belonging to the fibers of $\pi_{k,k-1} : J^k_n(W) \to J^{k-1}_n(W)$.*

Proof. The degeneration subspace of $\Omega_k(q)(\lambda)$, for any $\lambda \in S^{k-1}(T_aN) \otimes \nu^*_a$, is the subspace $P \subset \mathbf{E}^k_n(W)_q$ given in (34):

$$
P \equiv \left\{ \langle x + \theta\rangle | x \in \mathrm{Ann}(\Xi) \subset T_aN,\ \theta \in S^k(\Xi)\bigotimes \nu_a \subset S^k(T^*_aN)\bigotimes \nu_a \right\}, \tag{34}
$$

where Ξ is a p-dimensional subspace of T^*_aN.

Let, now, $N \subset W$ be an n-dimensional submanifold of W and let $N_0 \subset N$ be a submanifold in N. Set

$$
N_0^{(k)}(N) \equiv \left\{ q \in J^k_n(W) \,|\, \pi_{k,k-1}(q) \in N_0^{(k-1)},\ L_q \supset T_{\pi_{k,k-1}(q)}N_0^{(k-1)} \right\}
$$

where $N_0^{(k-1)} \equiv \left\{ [N]^{k-1}_a \,|\, a \in N_0 \right\} \subset J^{k-1}_n(W)$. Then the tangent planes to $N_0^{(k)}(N)$ coincide with the maximal involutive subspaces described in (34). Therefore, $N_0^{(k)}(N)$ is a maximal integral manifold of the Cartan distribution:

$$
\begin{array}{c}
0\\
\downarrow\\
g_m(q)\\
\downarrow \delta\\
T_a^*N\bigotimes g_{m-1}(q)\\
\downarrow \delta\\
\Lambda^2(T_a^*N)\bigotimes g_{m-2}(q)\\
\downarrow \delta\\
\cdots\\
\downarrow \delta\\
\Lambda^n(T_a^*N)\bigotimes g_{m-n}(q)\\
\downarrow\\
0
\end{array}
\tag{35}
$$

Definition 4 (Partial Differential Equation for Submanifolds). A *partial differential equation* (PDE) for n-dimensional submanifolds of W is a submanifold $E_k \subset J^k_n(W)$.[15] A (*regular*) *solution* of E_k is a (regular) solution of $J^k_n(W)$ that is contained into E_k. In particular, if $E_k \subset J^k(W) \subset J^k_n(W)$, we can talk about PDE for sections of $\pi : W \to M$. The *prolongation of order* l of $E_k \subset J^k_n(W)$ is the subset $(E_k)_{+l} \subset J^{k+l}_n(W)$ defined by $(E_k)_{+l} \equiv J^l_n(E_k) \cap J^{k+l}_n(W)$. A PDE $E_k \subset J^k_n(W)$ is called *formally integrable* if for all $l \geq 0$ the prolongations $(E_k)_{+l}$ are smooth submanifolds and the projections $\pi_{k+l+1,k+l} : (E_k)_{+(l+1)} \to (E_k)_{+l}$, $\pi_{k,0} : E_k \to W$ are smooth bundles. The *symbol* of the PDE $E_k \subset J^k_n(W)$ at the point $q \equiv [N]^k_a \in E_k$ is defined to be the following subspaces: $g_k(q) \equiv T_q(E_k) \cap T_q(F_{\bar{q}})$, where $\bar{q} \equiv \pi_{k,k-1}(q)$ and $\pi^{-1}_{k,k-1}(\bar{q}) = F_{\bar{q}} \subset J^k_n(W)$. Using the affine structure on the fiber $F_{\bar{q}}$, we can identify the symbol $g_k(q)$ with a subspace in $S^k(T^*_aN) \otimes \nu_a$: $g_k(q) \subset S^k(T^*_aN) \otimes \nu_a$. Suppose that all prolongations $(E_k)_{+l}$ are smooth manifolds, then their symbols at points $\breve{q} \equiv [N]^{k+l}_a$ are lth prolongations of the symbol $g_k(q)$; hence, $g_{k+l}(\breve{q}) = g_{k+l}(q) \subset S^{k+l}(T^*_aN) \otimes \nu_a$ and $\delta(g_{k+l}(\breve{q})) \subset g_{k+(l-1)}(q) \otimes T^*_aN$, $l = 1, 2, \ldots$ where by $\delta : S^{k+l}(T^*_aN) \otimes \nu_a \to T^*_aN \otimes S^{k+l-1}(T^*_aN) \otimes \nu_a$, we denote δ-*Spencer operator*. Therefore, at each point $q \in E_k$, the δ-Spencer complex is defined, where $m \geq k$. We denote by $H^{m-j,j}(E_k, q)$ the cohomologies of this complex at the term $\Lambda^j(T^*_aN) \otimes g_{m-j}(q)$. They are called δ-*Spencer cohomologies* of PDE

[15] In this chapter, for the sake of simplicity, we shall consider only smooth PDEs. For information on the geometry of singular PDEs, see the following references [2, 41, 47, 49, 53].

at the point $q \in E_k$. We say that g_k is *involutive* if the sequences (35) are exact and that g_k is *r-acyclic* if $H^{m-j,j}(E_k, q) = 0$ for $m - j \geq k$, $0 \leq j \leq r$. If $E_k \subset J^k_n(W)$ is a 2-*acyclic* PDE , i.e., $H^{j,i}(E_k, q) = 0$, $\forall q \in E_k$, $0 \leq j \leq 2$, $m - j \geq k$, and $\pi_{k+1,k} : E^{(1)}_k \to E_k$, $\pi_{k,0} : E_k \to W$ are smooth bundles, then E_k is *formally integrable*.

Definition 5. We say that $E_k \subset J^k_n(W)$ is *completely integrable* if for any point $q \in E_k$, it passes a (local) solution of E_k and hence an n-dimensional manifold $V \subset E_k$, with $q \in V$ and $V = N^{(k)}$. This implies that the following sequence

$$(E_k)_{+r} \xrightarrow{\pi_{k+r,k+r-1}} E_{k+r-1} \longrightarrow 0$$

is exact for any $r \geq 1$. (This is equivalent to saying that $\pi_{k+r,k+r-1}|_{(E_k)_{+r}}$ is surjective.)

Proposition 2. *In the category of analytic manifolds (i.e., manifolds of class C^ω), the formal integrability implies the complete integrability.*

Definition 6. A *Cartan connection* on E_k is an n-dimensional subdistribution $\mathbf{H} \subset \mathbf{E}_k$ such that $T(\pi_{k,k-1})(\mathbf{H}_q) = L_q \equiv T_{\pi_{k,k-1}(q)}N^{(k-1)}$, $[N]^k_q \equiv q$, $\quad \forall q \in E_k$.[16] We call *curvature* of the Cartan connection $\mathbf{H}$ on $E_k \subset J^k_n(W)$ the field of geometric objects on E_k:

$$\begin{aligned} \Omega_{\mathbf{H}} : q \mapsto & \Lambda^2(\mathbf{H}^*_q) \bigotimes [S^{k-1}(T_a N) \otimes \nu^*_a / \mathrm{Ann}(g_{k-1})]^* \\ & \cong \Lambda^2(T^*_a N) \bigotimes [S^{k-1}(T_a N) \otimes \nu^*_a / \mathrm{Ann}(g_{k-1})]^* \end{aligned} \tag{36}$$

obtained by restriction on $\mathbf{H}$ of the metasymplectic structure on the distribution $\mathbf{E}^k_n$.

Proposition 3. *In any* flat Cartan connection $\mathbf{H} \subset \mathbf{E}_k$, *i.e., a Cartan connection having zero curvature: $\Omega_{\mathbf{H}} = 0$, any two vector $X, Y \in \mathbf{H}_q$, $q \in E_k$ are in involution.*

Definition 7. Let us assume that $(E_k)_{+1} \to E_k$ is a smooth subbundle of $J^{k+1}_n(W) \to J^k_n(W)$. Then any section $\rceil : E_k \to (E_k)_{+1}$ is called a *Bott connection*.

Theorem 6.

1) *A Cartan connection* $\mathbf{H}$ *is a Bott connection iff* $\Omega_{\mathbf{H}} = 0$.[17]
2) *A Cartan connection* $\mathbf{H}$ *gives a splitting of the Cartan distribution*

$$\mathbf{E}^k_n \cong g_k \bigoplus \mathbf{H}.$$

[16] As $\dim(L_q) = n = \dim \mathbf{H}_q$, then there exists an n-dimensional submanifold $X \subset W$ such that $T_q X^{(k)} = \mathbf{H}_q$, with $[X]^k_a = q$, $[X]^{k-1}_a = [N]^{k-1}_a$, $T_{\pi_{k,k-1}(q)} X^{(k-1)} = L_q$.

[17] If $(E_k)_{+1} \to E_k$ is a smooth subbundle of $J^{k+1}_n(W) \to J^k_n(W)$, then a flat Cartan connection is also an involutive distribution. On the other hand, a Bott connection identifies an involutive distribution iff it is a flat connection. (For more details on $(k+1)$-connections on W, see [39].)

Two Cartan connections $\mathbf{H}$, $\mathbf{H}'$ *on* E_k *identify a field of geometric objects* λ *on* E_k *called* soldering form*:* $\lambda \equiv \lambda_{\mathbf{H},\mathbf{H}'} : E_k \to \mathbf{H}^* \bigotimes g_k$, $\lambda(q) \in T^*_a N \otimes g_k(q)$. *One has:*

- $\Omega_{\mathbf{H}'} = \Omega_{\mathbf{H}} + \delta\lambda$.
- (Bianchi identity) $\delta\Omega_{\mathbf{H}} = 0$,

$$\Omega_{\mathbf{H}}(q)\, mod\, \delta(T^*_a N \otimes g_k(q)) \in H^{k-1,2}(E_k)_q.$$

We call such δ*-cohomology class of* $\Omega_{\mathbf{H}}$ *the* Weyl tensor *of* E_k *at* $q \in E_k$*:* $W_k(q) \equiv [\Omega_{\mathbf{H}}(q)]$*. Then, there exists a point* $u \in (E_k)_{+1}$ *over* $q \in E_k$ *iff* $W_k(q) = 0$.

3) *Suppose that* g_{k+1} *is a vector bundle over* $E_k \subset J^k_n(W)$*. Then, if the Weyl tensor* W_k *vanishes, the projection* $\pi_{k+1,k} : (E_k)_{+1} \to E_k$ *is a smooth affine bundle.*
4) *If* g_{k+l} *are vector bundles over* E_k *and* $W_{k+l} = 0$, $l \geq 0$*, then* E_k *is formally integrable.*
5) *If the system* E_k *is of finite type, i.e.,* $g_{k+l}(q) = 0$, $\forall q \in E_k$, $l \geq l_0$*, then* $W_{k+l} = 0$, $0 \leq l \leq l_0$ *is a sufficient condition for integrability.*

Theorem 7. *Given a Cartan connection* $\mathbf{H}$ *on* E_k*, for any regular solution* $N^{(k)} \subset E_k$*, we identify a section* ${}_{\mathbf{H}}\nabla \in C^\infty(T^*N \bigotimes g_k)$ *called* covariant differential *of* $\mathbf{H}$ *of the solution* N*. Furthermore, for any vector field* $\zeta : N \to TN$*, we get a section* ${}_{\mathbf{H}}\nabla\zeta \in C^\infty(g_k|_{N^{(k)}})$.

Theorem 8 (Characteristic Distribution of PDE). *Let* $E_k \subset J^k_n(W)$ *be a PDE such that* $(E_k)_{+1} \to E_k$ *is a smooth subbundle of* $J^{k+1}_n(W) \to J^k_n(W)$*. Then, for any* $\tilde{q} \in (E_k)_{+1}$*, the set* $\mathbf{Char}(E_k)_q$ *of vectors in the splitting* $(\mathbf{E}_k)_q \cong L_{\tilde{q}} \bigoplus (g_k)_q$, $\zeta = v + \theta$*, such that* $v \rfloor \delta(\theta) = 0$*, for any* $\theta \in (g_k)_q$*, is called the* space of characteristic vectors *at* $q \in E_k$. $\mathbf{Char}(E_k)$ *is an involutive subdistribution of the Cartan distribution* $\mathbf{E}_k$.

- $\mathbf{Char}(E_k) = \mathbf{E}_k \bigcap \mathfrak{s}(E_k)$*, where* $\mathfrak{s}(E_k)$*is the space of infinitesimal symmetries of* E_k*, namely, the set of vector field on* E_k *whose flows preserve the Cartan distribution.*

Proof. See [39].

Definition 8. We call a PDE $E_k \subset J^k_n(W)$ *degenerate* at the point $q \in E_k$ if there is a p-dimensional ($0 < p \leq n$), subspace $\Xi_q \subset T^*_a N$, such that

$$(g_k)_q \subset [S^k(\Xi_q) \bigotimes \nu_a].$$

Theorem 9. $\mathbf{Char}(E_k)_q \neq 0$ *iff* E_k *is a degenerate PDE at the point* $q \in E_k$*. The subspace*

$$\Xi_q = \mathrm{Ann}((\pi_{k,0})_*(\mathbf{Char}(E_k)_q))$$

is the subspace of degeneration of E_k *at the point* $q \in E_k$.

- *Let $E_k \subset J^k_n(W)$ be a PDE such that the following conditions hold:*
  (i) $\pi_{k+1,k} : (E_k)_{+1} \to E_k$ *and* $\pi_{k,k-1} : E_k \to J^{k-1}_n(W)$ *are smooth bundles.*
  (ii) $\Xi = \bigcup_{q \in E_k} \Xi_q$ *is a smooth vector bundle, where* Ξ_q *is a space of degeneration of* E_k *at the point* $q \in E_k$*. Then,* $\mathbf{Char}(E_k)$ *is a smooth distribution on* E_k *and solutions of* E_k *can be formulated by the method of characteristics.*[18]

In this section, we shall classify global singular solutions of PDEs by means of suitable bordism groups.

Definition 9 (Generalized Singular Solutions of PDE). Let $E_k \subset J^k_n(W)$ be a PDE. We call *bar singular chain complex*, *with coefficients into an abelian group* G, of E_k the chain complex:

$$\{\bar{C}_p(E_k; G), \bar{\partial}\},$$

where $\bar{C}_p(E_k; G)$ is the G-module of formal linear combinations, with coefficients in G, $\sum \lambda_i c_i$, where c_i is a singular p-chain $f : \triangle^p \to E_k$ that extends on a neighborhood $U \subset \mathbb{R}^{p+1}$, such that f on U is differentiable and $Tf(\triangle^p) \subset \mathbf{E}_k$. Denote by $\bar{H}_p(E_k; G)$ the corresponding homology (*bar singular homology with coefficients in* G) of E_k.

A *G-singular p-dimensional integral manifold* of $E_k \subset J^k_n(W)$ is a bar singular p-chain V with $p \leq n$ and coefficients into an abelian group G, such that $V \subset E_k$.

Set $\bar{B}_\bullet(E_k; G) \equiv \mathrm{im}\,(\bar{\partial})$, $\bar{Z}_\bullet(E_k; G) \equiv \ker(\bar{\partial})$. Therefore, one has the exact commutative diagram (37):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \to & \bar{B}_\bullet(E_k;G) & \to & \bar{Z}_\bullet(E_k;G) & \to & \bar{H}_\bullet(E_k;G) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \bar{C}_\bullet(E_k;G) & = & \bar{C}_\bullet(E_k;G) & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & {}^G\Omega^{E_k}_{\bullet,s} & \to & \bar{B}or_\bullet(E_k;G) & \to & \bar{C}yc_\bullet(E_k;G) & \to & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{37}$$

[18] In other words, the method of characteristics allows us to solve Cauchy problems in E_k, namely, to build a solution V containing a fixed $(n-1)$-dimensional integral manifold N_0: $N_0 \subset V$. In fact, if $\zeta : E_k \to TE_k$ is a characteristic vector field of E_k, transverse to N_0, then $V = \bigcup_t \phi_t(N_0)$ is a solution of E_k, if $\partial\phi = \zeta$.

Table 2 Legend for the commutative exact diagram (37)

Name	Definition	Properties
Bordism group	$\bar{B}or_\bullet(E_k;G)$	$b\in {}^G[a]_{E_k}\in \bar{B}or_\bullet(E_k;G)\Rightarrow \exists c\in \bar{C}_\bullet(E_k;G):\bar{\partial}c=a-b$
Cyclism group	$\bar{C}yc_\bullet(E_k;G)$	$b\in {}^G[a]_{E_k}\in \bar{C}yc_\bullet(E_k;G)\Rightarrow \bar{\partial}(a-b)=0$
Closed bordism group	${}^G\Omega^{E_k}_{\bullet,s}$	$b\in {}^G[a]_{E_k}\in {}^G\Omega^{E_k}_{\bullet,s}\Rightarrow \left\{\begin{array}{l}\bar{\partial}a=\bar{\partial}b=0\\ a-b=\bar{\partial}c\end{array}\right\}$

In Table 2 are given some more explicit properties about the symbols involved in (37).

Theorem 10 (Integral Singular Bordism Groups of PDE).

- *One has the following canonical isomorphism:*

$$ {}^G\Omega^{E_k}_{\bullet,s}\cong \bar{H}_\bullet(E_k;G). $$

- *If* ${}^G\Omega^{E_k}_{\bullet,s}=0$, *one has* $\bar{B}or_\bullet(E_k;G)\cong \bar{C}yc_\bullet(E_k;G)$.
- *If* $\bar{C}yc_\bullet(E_k;G)$ *is a free G-module, then the bottom horizontal exact sequence, in the above diagram, splits and one has the isomorphism*

$$ \bar{B}or_\bullet(E_k;G)\cong {}^G\Omega(E_k)_{\bullet,s}\bigoplus \bar{C}yc_\bullet(E_k;G). $$

Remark 1. By considering the dual complex

$$ \{\bar{C}^p(E_k;G)\equiv \mathrm{Hom}_{\mathbb{Z}}(\bar{C}_p(E_k;\mathbb{Z});G),\bar{\delta}\} \tag{38} $$

and $\bar{H}^p(E_k;G)$, the associated homology spaces (*bar singular cohomology*, *with coefficients into* G of E_k), we can talk also of singular cobordism groups with coefficients in G. These are important objects, but in this chapter, we will skip on these aspects.

Definition 10. A *G-singular p-dimensional quantum manifold* of E_k is a bar singular p-chain $V\subset J^k_n(W)$, with $p\le n$, and coefficients into an abelian group G, such that $\partial V\subset E_k$. Let us denote by ${}^G\Omega_{p,s}(E_k)$ the corresponding (closed) bordism groups in the singular case. Let us denote also by ${}^G[N]_{\overline{E_k}}$ the equivalence classes of quantum singular bordisms, respectively.[19]

Remark 2. Let us emphasize that a G-singular solution $V\subset E_k$ can be written as an n-chain $V=\sum_i a^i u_i$, where $a_i\in G$ and $u_i:\Delta^n\to E_k$, such that $u_i(\Delta^n)$ is

[19]These bordism groups can be called also G-singular p-dimensional integral bordism groups *relative* to $E_k\subset J^k_n(W)$. They play an important role in PDE algebraic topology. For more details, see [37, 39, 44, 46–49].

an integral manifold of E_k.[20] In particular a G-singular solution V of E_k can have tangent spaces T_qV in some points $q \in V$ such that T_qV is a *n-dimensional integral plane*, i.e., an n-dimensional subspace of $(\mathbf{E}_k)_q \subset T_qE_k$ of the type $L_{\tilde{q}}$, for some $\tilde{q} \in (E_k)_{+1}$, or admitting the splitting

$$T_qV = V_q^k \bigoplus V_q^0$$

where $V_q^k = T_qV \bigcap (g_k)_q \subset V_q \bigcap [S^k(T_a^*N) \bigotimes \nu_a]$ and $V_q^0 \subset L_{\tilde{q}}$, $V_q^0 \cong (\pi_{k,0}(V_q)) \subset T_aN$, $\dim V_q^0 = \mathrm{type}(V) = n - p$. $(g_k)_q$ is the unique maximal isotropic subspace of dimension equal to $m\binom{p+k-1}{k}$ (and type 0). Therefore, under the condition (39)

$$m\binom{p+k-1}{k} \geq n \tag{39}$$

a singular solution of E_k can contain pieces of type 0. We say that a singular solution is *completely degenerate* if it is an integral n-chain of type 0, namely, completely contained in the symbol $(g_k)_q$, for some $q \in E_k$. In general, a singular solution can contain completely degenerate pieces. When the set $\Sigma(V) \subset V$ of singular points of a singular solution $V \subset E_k$ is nowhere dense in V, therefore $\dim \Sigma(V) < n$, then we say that in V there are *Thom-Boardman singularities*. In such points $q \in V$, one has $\dim[T_qV \bigcap (g_k)_q] = p$, with $0 < p < n$. This is equivalent to state that $\dim[(\pi_{k,0})_*(T_qV)] = n - p$ or that q is a point of *Thom-Boardman-degeneration*. Finally when $\Sigma(V) = \varnothing$, and there are not completely degenerate points in V, we say that V is a *regular solution*. In such a case, V is diffeomorphic to its projection $X = \pi_{k,0}(V) \subset W$, or equivalently $\pi_{k,0}|_V : V \to W$ is an embedding.

Theorem 11 (Cauchy Problems in PDE). *If E_k is a completely integrable PDE, and $\dim(g_k)_{+1} \geq n$, given a $(n-1)$-dimensional regular integral manifold N, contained in E_k, there exists a solution $V \subset E_k$, such that $V \supset N$.*

Proof. In fact, since N is regular, it identifies a $(n-1)$-manifold in W, say $N_0 \subset W$. Let $Y \subset W$ be an n-dimensional manifold containing N_0. Then taking into account that E_k is completely integrable, we can assume that the $(k+1)$-prolongation $Y^{(k+1)} \subset J_n^{k+1}W$ of Y is such that $Y^{(k+1)} \bigcap (E_k)_{+1} = N_0^{(1)}$; namely, it coincides with an $(n-1)$-dimensional integral manifold that projects on E_k. We call $N_0^{(1)}$ the first prolongation of N_0. Now taking into account that $(E_k)_{+1}$ is the strong retract of $J_n^{k+1}(W)$, we can retract map $Y^{(k+1)}$ into $(E_k)_{+1}$, via the retraction, obtaining a solution $V' \subset (E_k)_{+1}$ of $(E_k)_{+1}$ passing for $N_0^{(1)}$. By projecting V' into E_k, we obtain a solution V containing N. Since $\dim(g_k)_{+1} \geq n$, the solution V' does not necessitate to be regular, but can have singular points.

[20]In such a category, they can be considered also as so-called *neck-pinching singular solutions* that are very important whether from a theoretical point of view as well as in applications. (See, e.g., [50, 51].)

Example 7. Let $E_2 \subset J\mathscr{D}^2(W)$, be an analytic dynamic equation of a rigid system with n-degree of freedoms. Let $\{t, q^i.\dot{q}^i, \dot{q}^i\}$ be local coordinates on $J\mathscr{D}^2(W)$. Such an equation is completely integrable. A Cauchy problem there is encoded by a point $q_0 \in E_2$ and hence for that point passes a unique solution V, i.e., an integral curve contained into E_2. Let us, however, try to apply the proceeding of the proof of Theorem 11. This is strictly impossible! In fact the symbol of such an equation is necessarily zero, $\dim(g_2)_q = 0$, for any $q \in E_2$.[21] On the other hand, we can consider a point $\tilde{q}_0$ belonging to $(E_2)_{+1}$ and such that $\pi_{3,2}(\tilde{q}_0) = q_0$ and $\pi_{2,0}(\tilde{q}_0) = a \in W$, and we can assume that there exists an integral curve $Y \subset J\mathscr{D}^3(W)$ passing for $\tilde{q}_0$, but when we retract such a curve into $(E_2)_{+1}$, we get the unique curve $\overline{\Gamma}$ passing for $\tilde{q}_0$ contained into $(E_2)_{+1}$. This curve does not necessarily pass for the point $\bar{q}_0 = V^{(1)} \bigcap \pi^{-1}_{3,2}(q_0)$, since the first prolongation $V^{(1)}$ of V does not necessarily coincide with $\overline{\Gamma}$. Thus, the proceeding considered in the proof of Theorem 11 does not apply to PDEs (or ODEs), having zero symbols $g_2 = 0$. In other words, for such PDEs, despite $\pi_{2,0}(q_0) = \pi_{2,0}(\overline{q}_0) = a \in W$, we cannot connect two regular solutions corresponding to two different initial conditions q_0 and $\overline{q}_0$, with a completely degenerate piece, or a Thom-Boardman-singular piece. However, a more general concept of solutions can be considered also when $g_k = 0$. In fact, *weak solutions* include solutions with discontinuity points.[22]

Remark 3. Weak solutions are of great importance and must be included in a geometric theory of PDEs too.

Definition 11. Let $\Omega^{E_k}_{n-1}$ (resp. $\Omega^{E_k}_{n-1,s}$, resp. $\Omega^{E_k}_{n-1,w}$) be the integral bordism group for $(n-1)$-dimensional smooth admissible regular integral manifolds contained in E_k, bounding smooth regular integral manifold solutions[23] (resp. piecewise-smooth or singular solutions, resp. singular-weak solutions) of E_k.

Theorem 12. *Let $\pi : W \to M$ be a fiber bundle with W and M smooth manifolds, respectively, of dimension $m+n$ and n. Let $E_k \subset J^k_n(W)$ be a PDE for n-dimensional submanifolds of W. One has the following exact commutative diagram relating the groups $\Omega^{E_k}_{n-1}$, $\Omega^{E_k}_{n-1,s}$, and $\Omega^{E_k}_{n-1,w}$:*

[21] In general, such dynamical equations have zero symbol since they are encoded by n analytic differential equations of the second order, where n is the degree of freedoms.

[22] It is worth to emphasize that weak solutions can be considered equivalent to solutions having completely degenerated pieces; in fact, their projections on the configuration space W are the same. However, a weak solution can exist also with trivial symbol $g_k = 0$, while solutions with completely degenerated pieces can exist only if $\dim g_k \geq n$. Furthermore, under this circumstance, namely, under condition (39), a *continuous weak solution*, i.e., a weak solution having completely degenerate pieces, can be deformed into solutions with Thom-Boardman singular points.

[23] This means that $N_1 \in [N_2] \in \Omega^{E_k}_{n-1}$, iff $N_1^{(\infty)} \in [N_2^{(\infty)}] \in \Omega^{E_\infty}_{n-1}$. (See [42, 53] for notations.)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K^{E_k}_{n-1,w/(s,w)} & \rightarrow & K^{E_k}_{n-1,w} & \rightarrow & K^{E_k}_{n-1,s,w} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K^{E_k}_{n-1,s} & \rightarrow & \Omega^{E_k}_{n-1} & \rightarrow & \Omega^{E_k}_{n-1,s} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \Omega^{E_k}_{n-1,w} & \rightarrow & \Omega^{E_k}_{n-1,w} & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{40}
$$

and the canonical isomorphisms reported in (41).

$$
\left.
\begin{array}{l}
K^{E_k}_{n-1,w/(s,w)} \cong K^{E_k}_{n-1,s} \\
\Omega^{E_k}_{n-1}/K^{E_k}_{n-1,s} \cong \Omega^{E_k}_{n-1,s} \\
\Omega^{E_k}_{n-1,s}/K^{E_k}_{n-1,s,w} \cong \Omega^{E_k}_{n-1,w} \\
\Omega^{E_k}_{n-1}/K^{E_k}_{n-1,w} \cong \Omega^{E_k}_{n-1,w}
\end{array}
\right\} . \tag{41}
$$

- *In particular, for $k = \infty$, one has the canonical isomorphisms reported in (42):*

$$
\left.
\begin{array}{l}
K^{E_\infty}_{n-1,w} \cong K^{E_\infty}_{n-1,s,w} \\
K^{E_\infty}_{n-1,w/(s,w)} \cong K^{E_\infty}_{n-1,s} \cong 0 \\
\Omega^{E_\infty}_{n-1} \cong \Omega^{E_\infty}_{n-1,s} \\
\Omega^{E_\infty}_{n-1}/K^{E_\infty}_{n-1,w} \cong \Omega^{E_\infty}_{n-1,s}/K^{E_\infty}_{n-1,s,w} \cong \Omega^{E_\infty}_{n-1,w}
\end{array}
\right\} . \tag{42}
$$

- *If E_k is formally integrable, then one has the isomorphisms reported in (43):*

$$
\Omega^{E_k}_{n-1} \cong \Omega^{E_\infty}_{n-1} \cong \Omega^{E_\infty}_{n-1,s}. \tag{43}
$$

Proof. The proof follows directly from the definitions and standard results of algebra. (For more details, see [39, 45].)

Theorem 13. *Let us assume that E_k is formally integrable and completely integrable and such that* $\dim E_k \geq 2n + 1$. *Then, one has the canonical isomorphisms reported in (44):*

$$
\Omega^{E_k}_{n-1,w} \cong \bigoplus_{r+s=n-1} H_r(W;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s \cong \Omega^{E_k}_{n-1}/K^{E_k}_{n-1,w} \cong \Omega^{E_k}_{n-1,s}/K^{E_k}_{n-1,s,w}. \tag{44}
$$

where Ω_s denotes the s-dimensional unoriented smooth bordism group.

- *Furthermore, if $E_k \subset J^k_n(W)$ has nonzero symbols, $g_{k+s} \neq 0$, $s \geq 0$ (this excludes those that can be $k = \infty$), then $K^{E_k}_{n-1,s,w} = 0$; hence, $\Omega^{E_k}_{n-1,s} \cong \Omega^{E_k}_{n-1,w}$.*

Proof. It follows from the above theorem and results in [42]. Furthermore, if $g_{k+s} \neq 0$, $s \geq 0$, we can always connect two branches of a weak solution with a singular solution of E_k. (For more details, see [42].)

4 Maslov Index in PDEs and Lagrangian Bordism Groups

In order to consider "Maslov index" canonically associated to PDEs, we follow a strategy to recast Arnold–Kashiwara–Thomas algebraic approach, resumed in Sect. 2, by substituting the Grassmannian of Lagrangian subspaces with the Grassmannian of n-dimensional integral planes, namely, n-dimensional isotropic subspaces of the Cartan distribution of a PDE. These are tangent to solutions of PDEs. In this way, we are able to generalize the "Maslov index" for Lagrangian submanifolds as introduced by V.I. Arnold, to any solution of PDEs. Really Lagrangian submanifolds of symplectic manifolds can be encoded as solutions of suitable first-order PDEs.

As a by-product, we get also a new proof for existence of the Navier–Stokes PDEs global smooth solutions, defined on all $\mathbb{R}^3$. (Example 12.)

In this section, we shall calculate also Lagrangian bordism groups in a $2n$-dimensional symplectic manifold (W, ω), where ω is a nondegenerate, close, differentiable 2-differential form on W. In [34], we have calculated the Lagrangian bordism groups in the case that ω is exact. This has been made by generalizing to higher-order PDE, a previous approach given by Arnold [4, 8] and Eliashberg [16]. Now we give completely new formulas, without assuming any restriction on ω and following our algebraic topology of PDEs. (See [33–37, 39, 41–54]. See also [1, 2, 29, 55].)

In this section, our main results are Theorems 16–19. The first is devoted to the relation between Maslov indexes and Maslov cycles for solutions of PDEs. The second characterizes such invariants for Lagrangian submanifolds of symplectic manifolds, by means of suitable formally integrable and completely integrable first-order PDEs. The other two theorems characterize Lagrangian bordism groups in such PDEs.

Theorem 14 (Grassmannian of n-Dimensional Integral Planes of $J^k_n(W)$). *Let $I_k(W)_q$ be the* Grassmannian of n-dimensional integral planes *at $q \in J^k_n(W)$, namely, the set of isotropic n-dimensional subspaces of the Cartan distribution $\mathbf{E}_k(W)_q$. One has the following properties:*

(i) *One has the natural fiber bundle structure $I_k(W) = \bigcup_{q \in J^k_n(W)} I_k(W)_q \to J^k_n(W)$.*

(ii) *In general an integral n-plane* $L \in I_k(W)_q$ *is projected via* $(\pi_{k,0})_*$ *onto an* $(n-l)$*-dimensional subspace of* T_aN, $q = [N]^k_a$.
(iii) *The set of n-integral planes such that* $\dim(\pi_{k,0})_*(L) = n = \dim(T_aN)$ *(namely, with* $l = 0$*) is identified with the affine fiber* $\pi^{-1}_{k+1,k}(q) \subset J^{k+1}_n(W)$*. These integral planes are called* regular integral planes.
(iv) *In general, an n-integral plane* $L \in I_k(W)_q$ *admits the following splitting:*

$$L \cong L_o \bigoplus L_v \tag{45}$$

where L_o (horizontal component) *is contained in some regular plane* $L_{\tilde{q}}$ *for some* $\tilde{q} \in \pi^{-1}_{k+1,k}(q) \subset J^{k+1}_n(W)$*. Furthermore,* L_v (vertical component) *is contained in the vector space* $T_q\pi^{-1}_{k,k-1}(\bar{q}) \cong S^k(T^*_aN) \bigotimes \nu_a$*, with* $\bar{q} = \pi_{k,k-1}(q) \in J^{k-1}_n(W)$.
(v) *Two different splittings,* $L \cong L_o \bigoplus L_v$ *and* $L \cong L'_o \bigoplus L'_v$*, of an n-integral plane* $L \subset \mathbf{E}_k(W)_q$, $q = [N]^k_a \in J^k_n(W)$ *are related by a fixed subspace* $V \subset S^k(T^*_aN) \bigotimes \nu_a$*. More precisely, one has*

$$L_o = L'_o \bigoplus V;\ L'_v = L_v \bigoplus V. \tag{46}$$

- (Cohomology ring $H^\bullet(I_k(W))$). *One has the following isomorphisms:*

$$\left.\begin{aligned} H^\bullet(I_k(W);\mathbb{Z}_2) &\cong H^\bullet(J^k_n(W);\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^\bullet(F_k(W);\mathbb{Z}_2) \\ &\cong H^\bullet(W;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^\bullet(F_k(W);\mathbb{Z}_2) \end{aligned}\right\}, \tag{47}$$

where $F_k(W)$ *is the fiber of* $I_k(W)$ *over* $J^k_n(W)$*. One has the following ring isomorphism:*

$$H^\bullet(F_k(W);\mathbb{Z}_2) \cong \mathbb{Z}_2[w^{(k)}_1, \cdots, w^{(k)}_n],$$

where $\deg(w^{(k)}_i) = i$*. Such generators coincide with Stiefel-Whitney classes of the tautological bundle* $E(\eta) \to I_k(W)$.

Proof. Let us only explicitly consider that the first part of the formula (47) follows from a direct application of some results about spectral sequences and their relations with fibration (*Leray-Hirsh theorem*). For more details, see Theorem 3 in [35].

Theorem 15 (Grassmannian of *n*-Dimensional Integral Planes of PDE). *Let*

$$I(E_k) = \bigcup_{q \in E_k} I(E_k)_q$$

be the Grassmannian of n-dimensional integral planes of E_k*. One has a natural fiber bundle structure* $I(E_k) \to E_k$*. Then each singular solution* $V \subset E_k$ *identifies a mapping* $i_V : V \to I(E_k)$*, given by* $i_V(q) = T_qV \in I(E_k)_q$*. Then one has an induced morphism:*

$$i_V^* : H^i(I(E_k) : \mathbb{Z}_2) \to H^i(V; \mathbb{Z}_2),\ \omega \mapsto i_V^*\omega. \tag{48}$$

$i_V^*\omega$ is the characteristic class *of V corresponding to ω.*

- *If E_k is a strong retract of $J^k_n(W)$, then $H^\bullet(I(E_k);\mathbb{Z}_2)$ is an algebra over $H^\bullet(E_k;\mathbb{Z}_2)$. More precisely, one has*

$$H^i(I(E_k);\mathbb{Z}_2) \cong \bigoplus_{r+s=i} H^r(E_k;\mathbb{Z}_2) \bigotimes_{\mathbb{Z}_2} H^s(F_k;\mathbb{Z}_2),$$

  where F_k is the fiber of $I(E_k)$ over E_k.
- *Furthermore, the ring $H^\bullet(F_k;\mathbb{Z}_2)$ is isomorphic up to n to the ring $\mathbb{Z}_2[\omega_1^{(k)},\cdots,\omega_n^{(k)})$ of polynomials in the generator $\omega_i^{(k)}$,* degree$(\omega_i^{(k)}) = i$. *These generators can be identified with the Stiefel-Whitney classes of the tautological bundle $E(\eta) \to I(E_k)$.*
- *If E_k is a formally integrable PDE, then*

$$\left.\begin{array}{rl} H^\bullet(I(E_{k+1});\mathbb{Z}_2) & \cong H^\bullet(I_{k+1}(W);\mathbb{Z}_2) \\ & \cong H^\bullet(W;\mathbb{Z}_2))\bigotimes_{\mathbb{Z}_2} H^\bullet(F_{k+1}(W);\mathbb{Z}_2) \\ & \cong H^\bullet(W;\mathbb{Z}_2))\bigotimes_{\mathbb{Z}_2} \mathbb{Z}_2[\omega_1^{(k+1)},\cdots,\omega_n^{(k+1)}) \end{array}\right\}. \tag{49}$$

- *If V is a non-singular solution of E_k, then all its characteristic classes are zero in dimension* ≥ 1.

Proof. After Theorem 14, let us only explicitly consider when E_k is a strong retract of $J^k_n(W)$. This fact implies the homotopy equivalence $E_k \backsimeq J^k_n(W)$. Then we can state also the homotopy equivalence between the corresponding integral planes fiber bundles $I(E_k) \backsimeq I_k(W)$. In fact, we use the following lemmas:

Lemma 1. *If $A \subset X$ is a strong retract, then the inclusion $i(A, x_0) \hookrightarrow (X, x_0)$ is a homotopy equivalence, and hence $i_*\pi_n(A, x_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 0$.*

Proof. This is a standard result. See, e.g., [41]. (This lemma is the inverse of the *Whitehead's theorem*.)

Lemma 2. *For a space B let $\mathscr{F}(B)$ be the set of fiber homotopy equivalence classes of fibrations $E \to B$. A map $f : B_1 \to B_2$ induces $f^* : \mathscr{F}(B_2) \to \mathscr{F}(B_1)$, depending only on the homotopy class of f. If f is a homotopy equivalence, then f^* becomes a bijection: $f^* : \mathscr{F}(B_2) \leftrightarrow \mathscr{F}(B_1)$.*

Proof. This is a standard result. See, e.g., [21].

From the above two lemmas, we can state that also $I(E_k)$ is a strong retract of $I_k(W)$; therefore, one has the following exact commutative diagram of homotopy equivalences:

$$\begin{array}{ccccc} 0 & \longrightarrow & I(E_k) & \overset{\sim}{\text{———}} & I_k(W) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_k & \overset{\sim}{\text{———}} & J^k_n(W) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array} \tag{50}$$

This induces the following commutative diagram of isomorphic cohomologies:

$$\begin{array}{ccc} H^\bullet(I(E_k);\mathbb{Z}_2) & \overset{\sim}{=\!=\!=} & H^\bullet(I_k(W);\mathbb{Z}_2) \\ \|\wr & & \|\wr \\ H^\bullet(E_k;\mathbb{Z}_2)\bigotimes_{\mathbb{Z}_2}H^\bullet(F_k;\mathbb{Z}_2) & \overset{\sim}{=\!=\!=} & H^\bullet(J^k_n(W);\mathbb{Z}_2)\bigotimes_{\mathbb{Z}_2}H^\bullet(F_k(W);\mathbb{Z}_2) \end{array} \tag{51}$$

Since

$$H^\bullet(E_k;\mathbb{Z}_2)\cong H^\bullet(J^k_n(W);\mathbb{Z}_2)\cong H^\bullet(W;\mathbb{Z}_2)$$

and

$$H^\bullet(F_k;\mathbb{Z}_2)\cong H^\bullet(F_k(W);\mathbb{Z}_2)\cong \mathbb{Z}_2[\omega_1^{(k)},\cdots,\omega_n^{(k)}],$$

we get

$$H^\bullet(I(E_k);\mathbb{Z}_2)\cong H^\bullet(W;\mathbb{Z}_2)\bigotimes_{\mathbb{Z}_2}\mathbb{Z}_2[\omega_1^{(k)},\cdots,\omega_n^{(k)}].$$

Therefore, $H^\bullet(I(E_k);\mathbb{Z}_2)$ is an algebra over $H^\bullet(W;\mathbb{Z}_2)$ isomorphic to $\mathbb{Z}_2$ $[\omega_1^{(k)},\cdots,\omega_n^{(k)}]$. Finally if E_k is formally integrable, then its r-prolongations $(E_k)_{+r}$ are strong retract of $J^{k+r}_n(W)$, for $r\geq 1$. Thus, we can repeat the above considerations by working on each $(E_k)_{+r}$ and obtain

$$H^\bullet(I((E_k)_{+r});\mathbb{Z}_2)\cong H^\bullet(W;\mathbb{Z}_2)\bigotimes_{\mathbb{Z}_2}\mathbb{Z}_2[\omega_1^{(k+r)},\cdots,\omega_n^{(k+r)}],$$

for $r\geq 1$.

Remark 4. It is worth to emphasize the comparison between metasymplectic structure on $J^k_n(W)$ and the symplectic structure in a symplectic vector space (V,ω). According to the definition given in the proof of Theorem 4, we can define *metasymplectic orthogonal* of a subspace $P\lhd \mathbf{E}_k(W)_q$, by the set

$$\left.\begin{array}{l} P^{\perp} = \{\zeta \in \mathbf{E}_k(W)_q \mid \Omega_k(\lambda)(\zeta,\xi) = 0,\ \forall \xi \in P,\ \forall \lambda \in S^{k-1}(T_aN) \otimes \nu_a^*\} \\ \quad = \bigcap_{\lambda \in S^{k-1}(T_aN)\otimes \nu_a^*} \ker(\Omega_k(\lambda)(\zeta, P)) \end{array}\right\} . \tag{52}$$

One has the following properties:

(a) $(P^{\perp})^{\perp} = P$;
(b) $P_1^{\perp} \bigcap P_2^{\perp} = (P_1 + P_2)^{\perp}$;
(c) $(P_1 \bigcap P_2)^{\perp} = (P_1)^{\perp} + (P_2)^{\perp}$.

Then one can define P *metasymplectic-isotropic* if $P \subset P^{\perp}$. Furthermore, we say that P is *metasymplectic-Lagrangian* if $P = P^{\perp}$. Maximal metasymplectic-isotropic spaces are metasymplectic-isotropic spaces that are not contained into larger ones. There any two vectors are an involutive couple. With respect to the above remarks, in Table 3, we have made a comparison between definitions related to the metasymplectic structure and symplectic structure. Let us underline that the metasymplectic structure considered is not a trivial extension of the canonical symplectic structure that can be recognized on any vector space E, of dimension n. In fact, it is well known that $V = E \bigoplus E^*$ has the canonical symplectic structure $\sigma((v,\alpha),(v',\alpha')) = \langle \alpha, v'\rangle - \langle \alpha', v\rangle$, called the *natural symplectic form* on E. Instead the metasymplectic structure arises from the differential of Cartan forms.

Definition 12 (Lagrangian Submanifolds of Symplectic Manifold). Let (W,ω) be a *symplectic manifold*, that is, W is a $2n$-dimensional manifold with symplectic 2-form $\omega : W \to \Lambda_2^0(W)$ (hence, ω is closed: $d\omega = 0$). We call a Lagrangian manifold an n-dimensional submanifold $V \subset W$, such that $\omega|_V = 0$.[24]

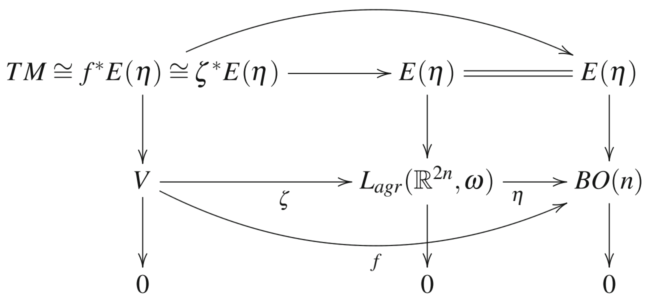
Example 8 ([4]). A Lagrangian submanifold of the symplectic space $(\mathbb{R}^{2n},\omega)$ is an n-dimensional submanifold $V \subset \mathbb{R}^{2n}$, such that for any $p \in V$, $T_pV \subset T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ is a Lagrangian subspace of $\mathbb{R}^{2n}$. This is equivalent to say that the symplectic 2-form $\sigma = \sum_{1\le r<s\le 2n} \sigma_{rs} d\xi^r \wedge d\xi^s$, with $\sigma_{rs} = \omega_{rs}$ and $(\xi^r)_{1\le r\le 2n} = (x^j, y^j)_{1\le j\le n}$, annihilates on V: $\sigma|_V = 0$. The tangent space TV, classified by the first classifying mapping $f : V \to BO(n)$, is the pullback of the tautological bundle $E(\eta)$ over $L_{\mathrm{agr}}(\mathbb{R}^{2n},\omega)$ or equivalently the pullback of $E(\eta)$ via the second classifying mapping $\zeta : V \to L_{\mathrm{agr}}(\mathbb{R}^{2n},\omega)$, $\zeta(p) = T_pV \cong \mathbb{R}^{2n}$. In fact, one has the exact commutative diagram (53):

[24] The tangent space T_pW, $\forall p \in W$, identifies a symplectic space via the 2-form $\omega(p) \in \Lambda^2(T_p^*W)$. Therefore, an n-dimensional submanifold V of a $2n$-symplectic manifold W is Lagrangian iff T_pV is a Lagrangian subspace of T_pW, $\forall p \in V$.

Table 3 Comparison between metasymplectic structure of $J^k_n(W)$ and symplectic structure of symplectic space (V, ω)

$(\mathbf{E}_k(W)_q\,,\ \Omega_k(\lambda))$ $\dim(\mathbf{E}_k(W)_q) = m\binom{k+n-1}{k} + n$, $\dim W = n + m$	(V, ω) $\dim V = 2n$
$P \lhd \mathbf{E}_k(W)_q$ $P^\perp = \{\zeta \in \lhd\mathbf{E}_k(W)_q \mid \Omega_k(\lambda)(\zeta, \xi) = 0,\ \forall \xi \in P,\ \forall \lambda \in S^{k-1}(T_aN) \otimes \nu_a^*\}$	$E \lhd V$ $E^\perp = \{v \in V \mid \omega(v, u) = 0,\ \forall u \in E\}$
P metasymplectic-isotropic iff $P \subset P^\perp$. $[\dim P \leq P^\perp]$	E is symplectic-isotropic iff $E \subset E^\perp$ $[\dim E \leq n]$
P is metasymplectic-Lagrangian iff $P = P^\perp$. $[\dim P = P^\perp]$	E is symplectic-Lagrangian iff $E = E^\perp$ $[\dim E = n]$
P is maximal metasymplectic-isotropic iff $P \not\subset Q \subset Q^\perp$. $[\dim P = m\binom{p+k-1}{k} + n - p, 0 \leq p \leq n]$ $[\text{type}(P) = n - p]$	E is symplectic-co-isotropic iff $E \supset E^\perp$ $[\dim E \geq n]$

A metasymplectic-isotropic space is metasymplectic-involutive
A metasymplectic-Lagrangian space is metasymplectic-involutive
A symplectic-Lagrangian space is maximally symplectic-isotropic
A symplectic-isotropic (or symplectic-co-isotropic) space E with $\dim E = n$ is symplectic-Lagrangian
A line (hyperplane) is symplectic-isotropic (symplectic-co-isotropic)
A maximal metasymplectic-isotropic space of type n has dimension n
A maximal metasymplectic-isotropic space of type 0 has dimension $m\binom{n+k-1}{k} = \dim[S^k(T_a^*N) \otimes \nu_a]$



(53)

- The *Maslov index class* of V is defined by $\tau(V) = \zeta^*(\alpha) \in H^1(V; \mathbb{Z})$, where $\alpha \in H^1(L_{\rm agr}(\mathbb{R}^{2n}, \omega); \mathbb{Z}) \cong \mathbb{Z}$ is the generator.
- The *Maslov cycle* of V is defined by

$$\Sigma(V) = \{p \in V \mid \dim(\ker(T(\pi)|_{T_pV})) > 0\},$$

where $\pi : \mathbb{R}^{2n} \cong \mathbb{R}^n \bigoplus i\mathbb{R}^n \to \mathbb{R}^n$. Therefore, $\Sigma(V) \cong T_pV \bigcap i\mathbb{R}^n \neq \{0\}$. The homology class $[\Sigma(V)] \in H_{n-1}(V;\mathbb{Z})$ is the Poincaré dual of the Maslov index class $\tau(V) \in H^1(V;\mathbb{Z})$:

$$[\Sigma(V)] = D\tau(V).$$

Therefore, one can state that $\tau(V)$ measures the failure of the morphism $\pi|_V : V \to \mathbb{R}^n$ to be a local diffeomorphism.

Example 9. $\mathbb{C}^n$ is a symplectic manifold. Any n-dimensional subspace is a Lagrangian submanifold.

Example 10. Any 1-dimensional submanifold of a 2-dimensional symplectic manifold is Lagrangian.[25]

Example 11. The cotangent space T^*M of an n-dimensional manifold M is a symplectic manifold, and each fiber T_p^*M of the fiber bundle $\pi : T^*M \to M$ is a Lagrangian submanifold.

- Let $V \subset T^*M$ be a Lagrangian submanifold of T^*M. Let us consider the fiber bundle

$$L_{\mathrm{agr}}(T^*M) = \bigcup_{q \in T^*M} L_{\mathrm{agr}}(T^*M)_q, \tag{54}$$

where $L_{\mathrm{agr}}(T^*M)_q$ is the set of Lagrangian subspaces of $T_q(T^*M)$. One has a canonical mapping

$$\zeta : V \to L_{\mathrm{agr}}(T^*M),\ q \mapsto T_qV.$$

Then, if $\alpha \in H^1(T^*M;\mathbb{Z}) \cong \mathbb{Z}$ is the generator, we get $\zeta^*\alpha \in H^1(V;\mathbb{Z})$ is the *Maslov index class* of V. The *Maslov cycle* of V is defined by the set

$$\Sigma(V) = \{q \in V \mid \dim(\ker(T(\pi)|_{T_qV} > 0,\ \pi : T^*M \to M\}.$$

Therefore, $\Sigma(V) \cong \{q \in V \mid T_qV \bigcap vT_q(T^*M) \neq \{0\}$. Here $vT_q(T^*M)$ denotes the vertical tangent space at $q \in T^*M$, with respect to the projection $\pi : T^*M \to M$. The homology class $[\Sigma(V)] \in H_{n-1}(V;\mathbb{Z})$ is the Poincaré dual of the Maslov index class $\tau(V) \in H^1(V;\mathbb{Z})$. $[\Sigma(V)] = D\tau(V)$. Therefore, $\tau(V)$ measures the failure of the mapping $\pi|_V : V \to M$ to be a local diffeomorphism.

[25] For example, any curve in S^2 is a Lagrangian submanifold.

Definition 13 (Maslov Cycles of PDE Solution). We call *i-Maslov cycle*, $1 \leq i \leq n-1$, of a solution $V \subset E_k \subset J^k_n(W)$, the set $\Sigma_i(V)$ of singular points $q \in V$, such that $\dim(\ker((\pi_{k,0})_*)|_{T_qV})) = n - i$.

Definition 14 (Maslov Index Classes of PDE Solution). We call *i-Maslov index class*, $1 \leq i \leq n-1$, of a solution $V \subset E_k \subset J^k_n(W)$,

$$\tau_i(V) = (i_V)^* \omega_i^{(k)} \in H^i(V; \mathbb{Z}_2),$$

where $\omega_i^{(k)}$ is the ith generators of the ring $\mathbb{Z}_2[\omega_1^{(k)}, \cdots, \omega_n^{(k)}] \cong H^\bullet(F_k, \mathbb{Z}_2)$ and $i_V : V \to I(E_k)$ is the canonical mapping, $i_V : q \mapsto T_qV$.

Theorem 16 (Maslov Indexes and Maslov Cycles Relations for Solution of PDE).

- *Let $E_k \subset J^k_n(W)$ be a strong retract of $J^k_n(W)$ and then the homology class, $[\Sigma_i(V)] \in H_{n-i}(V; \mathbb{Z})$, $1 \leq i \leq n-1$, is the Poincaré dual of the Maslov index class $\tau_i(V) \in H^i(V; \mathbb{Z})$. Formula (55) holds:*

$$[\Sigma_i(V)] = D\tau_i(V),\ 1 \leq i \leq n-1. \tag{55}$$

  Therefore, $\{\tau_i(V)\}_{1 \leq i \leq n-1}$, measure the failure of the mapping $\pi_{k,0} : V \to W$ to be a local embedding.
- *Let $E_k \subset J^k_n(W)$ be a formally integrable PDE. Then one can characterize each solution V on the first prolongations $(E_k)_{+1} \subset J^{k+1}_n(W)$, by means of i-Maslov indexes and i-Maslov cycles, as made in above point.*

Proof. Let us consider E_k a strong retract of $J^k_n(W)$. Then we can apply Theorem 15. In particular, we get the following isomorphisms:

$$\left.\begin{array}{ll} H^\bullet(E_k) & \cong H^\bullet(J^k_n(W)) \\ & \cong H^\bullet(W) \\ H^\bullet(I(E_k)) & \cong H^\bullet(I_k(W)) \end{array}\right\}. \tag{56}$$

Let us more explicitly calculate these cohomologies. Start with the case $i = 1$. One has the following isomorphisms:

$$\left.\begin{array}{ll} H^1(I(E_k); \mathbb{Z}_2) & \cong H^1(E_k; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^0(F_k; \mathbb{Z}_2) \bigoplus H^0(E_k; \mathbb{Z}) \otimes_{\mathbb{Z}_2} H^1(F_k; \mathbb{Z}_2) \\ & \cong H^1(E_k; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2 \bigoplus \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\omega_1^{(k)}] \\ & \cong H^1(E_k; \mathbb{Z}_2) \bigoplus \mathbb{Z}_2[\omega_1^{(k)}]. \end{array}\right\}. \tag{57}$$

Therefore, one has the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z}_2[\omega_1^{(k)}] & \longrightarrow & H^1(I(E_k);\mathbb{Z}_2) & \longrightarrow & H^1(E_k;\mathbb{Z}_2) & \longrightarrow 0 \\ & & \searrow^{(i_V)_*=(i_V)^*|_{\mathbb{Z}_2[\omega_1^{(k)}]}} & \downarrow{\scriptstyle (i_V)^*} & & & \\ & & & H^1(V;\mathbb{Z}_2) & & & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array} \tag{58}$$

Then the mapping $i_V : V \to I(E_k)$ induces the following morphism:

$$(i_V)_* = (i_V)^*|_{\mathbb{Z}_2[\omega_1^{(k)}]} : \mathbb{Z}_2[\omega_1^{(k)}] \to H^1(V;\mathbb{Z}_2).$$

Set $\beta_1(V) = (i_V)_*(\omega_1^{(k)})$. Here we suppose that V is compact (otherwise we shall consider cohomology with compact support). Now we get

$$\beta_1(V) \bigcap [V] = [\Sigma_1(V)]. \tag{59}$$

In (62) $[V]$ denotes the fundamental class of V that there exists also whether V is non-orientable. (For details, see, e.g. [41].)

We can pass to any degree, $1 \leq i \leq n-1$, by considering the following isomorphisms:

$$\left.\begin{array}{l} H^i(I(E_k);\mathbb{Z}_2) \cong H^i(E_k;\mathbb{Z}_2) \\ \qquad \bigoplus_{1\leq p\leq i-1} H^{i-p}(E_k;\mathbb{Z}_2) \otimes_{\mathbb{Z}} H^p(F_k;\mathbb{Z}_2) \\ \qquad \bigoplus \mathbb{Z}_2[\omega_1^{(k)}, \cdots, \omega_i^{(k)}] \end{array}\right\}. \tag{60}$$

One has the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z}_2[\omega_1^{(k)}, \cdots, \omega_i^{(k)}] & \longrightarrow & H^i(I(E_k);\mathbb{Z}_2) & \longrightarrow & \boxed{H^i(E_k;\mathbb{Z}_2)/\mathbb{Z}_2[\omega_1^{(k)}, \cdots, \omega_i^{(k)}]} & \longrightarrow 0 \\ & & \searrow^{(i_V)_*} & \downarrow{\scriptstyle (i_V)^*} & & & \\ & & & H^i(V;\mathbb{Z}_2) & & & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array} \tag{61}$$

Then the map $i_V : V \to I(E_k)$ induces the following morphism:

$$(i_V)_* : \mathbb{Z}_2[\omega_1^{(k)}, \cdots, \omega_i^{(k)}] \to H^i(V;\mathbb{Z}_2),\ 1 \leq i \leq n-1.$$

Set $\beta_i(V) = (i_V)_*(\omega_i^{(k)})$. We get

$$\beta_i(V) \bigcap [V] = [\Sigma_i(V)]. \tag{62}$$

For the case where E_k is formally integrable, we can repeat the above proceeding applied to the first prolongation $(E_k)_{+1}$ of E_k that is a strong retract of $J_n^{k+1}(W)$. In this way, we complete the proof.

Example 12 (Navier–Stokes PDEs and Global Space-Time Smooth Solutions). The non-isothermal Navier–Stokes equation can be encoded in a geometric way as a second-order PDE $\mathrm{(NS)} \subset \mathrm{J}_4^2(\mathrm{W})$, where $\pi : W = JD(M) \times_M T_0^0 M \times_M T_0^0 M \cong M \times \mathbf{I} \times \mathbb{R}^2 \to M$ is an affine bundle over the 4-dimensional affine Galilean space-time M. There $\mathbf{I} \subset \mathbf{M}$ represents a 3-dimensional affine subspace of the 4-dimensional vector space $\mathbf{M}$ of free vectors of M. A section $s : M \to W$ is a triplet $s = (v, p, \theta)$ representing the velocity field v, the isotropic pressure p, and the temperature θ. In [39] it is reported the explicit expression of (NS), formulated just in this geometric way. Then one can see there that (NS) is not formally integrable, but one can canonically recognize a sub-equation $\widehat{\mathrm{(NS)}} \subset \mathrm{(NS)} \subset \mathrm{J}_4^2(\mathrm{W})$ that is so and also completely integrable. Furthermore, (NS) is a strong deformed retract of $J_4^2(W)$, over a strong deformed retract (C) of $J_4^1(W)$. In other words, one has the following commutative diagram of homotopy equivalences:

$$\begin{array}{ccc}
(NS) & \overset{\sim}{\rule{3em}{0.4pt}} & J_4^2(W) \\
\downarrow & & \downarrow \\
(C) & \overset{\sim}{\rule{3em}{0.4pt}} & J_4^1(W) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{63}$$

Since $J_4^2(W)$ and $J_4^1(W)$ are affine spaces, they are topologically contractible to a point; hence, from (63), we are able to calculate the cohomology properties of (NS), as reported in (64):

$$\left.\begin{array}{l} H^0((\mathrm{NS}); \mathbb{Z}_2) = \mathbb{Z}_2 \\ H^r((\mathrm{NS}); \mathbb{Z}_2) = 0,\ \mathrm{r} > 0 \end{array}\right\} . \tag{64}$$

We get the cohomologies of $I(\mathrm{NS})$, as reported in (65):

$$\left.\begin{array}{ll}H^r(I(\mathrm{NS});\mathbb{Z}_2) & =\bigoplus_{p+q=r}H^p((\mathrm{NS});\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\mathrm{H^q}(\mathrm{F}_2;\mathbb{Z}_2)\\ & =H^0((\mathrm{NS});\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\mathrm{H^r}(\mathrm{F}_2;\mathbb{Z}_2)\\ & =\mathbb{Z}_2\otimes_{\mathbb{Z}_2}\mathbb{Z}_2[\omega_1^{(2)},\cdots\omega_r^{(2)}]\\ & =\mathbb{Z}_2[\omega_1^{(2)},\cdots\omega_r^{(2)}],\ 1\leq r\leq 4\end{array}\right\}. \tag{65}$$

Therefore, (66) are the conditions that $V\subset\widehat{(\mathrm{NS})}\subset(\mathrm{NS})$ must satisfy in order to be without singular points:

$$0=i_V^*\omega_i^{(2)}\in H^i(V;\mathbb{Z}_2),\ 1\leq i\leq 4. \tag{66}$$

In particular, if

$$V=D^2s(M)\subset\widehat{(\mathrm{NS})}\subset(\mathrm{NS})\subset \mathrm{J}D^2(\mathrm{W})\subset \mathrm{J}_4^2(\mathrm{W})$$

where $s:M\to W$ is a smooth global section, since $H^i(V;\mathbb{Z}_2)=0$, $\forall i>0$, we get that all its characteristic classes $i_V^*\omega_i^{(2)}$ are zero. Therefore, V cannot have singular points on V; namely, it is a global smooth solution on all the space time. Such global solutions certainly exist for (NS). In fact, a constant section $s:M\to W$ is surely a solution for (NS), localized on a equipotential space region. In fact, such solution satisfies $\widehat{(\mathrm{NS})}$ iff Eq. (67) are satisfied:

$$\left.\begin{array}{l}v^kG^j_{jk}=0\\ v^k(\partial x_\alpha.G^j_{jk})=0\\ v^sR^j_s+\rho(\partial x_i.f)g^{ij}=0\\ v^kv^pW_{kp}=0\end{array}\right\}. \tag{67}$$

We have adopted the same symbols used in [39]. Then, by using global Cartesian coordinates (this is possible for the affine structure of $J^2_4(W)$), we get that $g^{ij}=\delta^{ij}$, $R^j_s=0$ and $W_{kp}=0$. Therefore, Eq. (67) reduce to $\rho(\partial x_k.f)=0$. This means that such constant solutions exist iff they are localized in a equipotential space region.

Such constant global smooth solutions, even if very simple, can be used to build more complex ones, by using the linearized Navier–Stokes equation at such solutions. Let us denote by $\overline{(\mathrm{NS})}[s]\subset JD^2(s^*vTW)$ such a linearized PDE at the constant solution s. Similarly to the nonlinear case, we can associate to $\overline{(\mathrm{NS})}[s]$ a linear sub-PDE $\widehat{\overline{(\mathrm{NS})}}[s]\subset\overline{(\mathrm{NS})}[s]$ that is formally integrable and completely integrable. Then in a space-time neighborhood of a point $q\in\widehat{\overline{(\mathrm{NS})}}[s]$, we can build a smooth solution, say $\nu:M\to s^*vTW$. Since solutions of $\widehat{\overline{(\mathrm{NS})}}[s]$ locally transform solutions of $\widehat{(\mathrm{NS})}$ into other solutions of this last equation, we get that the original constant solution s can be transformed by means of the perturbation ν into another global solution $s':M\to W$, the perturbation being only localized into a

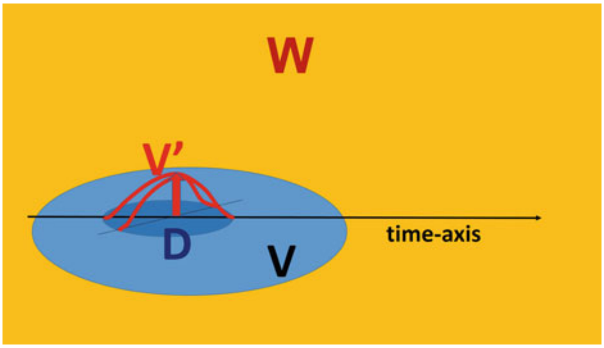


Fig. 1 Global space-time smooth solution representation $V' \subset \widehat{(\mathrm{NS})}$, obtained by means of a localized, space-time smooth perturbation of a constant global smooth solution $V \subset \widehat{(\mathrm{NS})}$. The perturbation, localized in the compact space-time region $D \subset V$ of the smooth global constant solution V, is a smooth solution of the linearized equation $\widehat{(\mathrm{NS})[s]}$. The vertical arrow denotes the local perturbation of the solution V, generating the non-constant global smooth solution $V' \subset J^2_4(W)$

local space-time region. In this way, we are able to obtain global space-time smooth solutions $V' \subset (\mathrm{NS})$. (See Fig. 1.) Since V and V' are both diffeomorphic to M, via the canonical projection $\pi_2 : J^2_4(W) \to M$, their characteristic classes are all zero: $i^*_V\omega_i^{(2)} = i^*_{V'}\omega_i^{(2)} = 0$, $i \in \{1, 2, 3, 4\}$. Really $H^i(V'; \mathbb{Z}_2) = 0 = H^i(V'; \mathbb{Z}_2)$, $\forall i > 0$. In the words of Theorem 16, we can say that in these global solutions V, one has

$$\Sigma_i(V)_K = \varnothing, \ \forall i \in \{1, 2, 3\} \tag{68}$$

for any compact domain $K \subset V$. In (68) $\Sigma_i(V)_K$ denotes the i-Maslov cycle of V inside the compact domain $K \subset V$. (For more details on the existence of such smooth solutions, built by means of perturbations of constant ones, see Appendix A.)[26]

[26]It is clear that whether we work with the constant solution with zero flow, we get a non-constant global solutions V' necessarily satisfying the following Clay–Navier–Stokes conditions:

1. $\mathbf{v}(x, t) \in \left[C^\infty(\mathbb{R}^3 \times [0, \infty))\right]^3$, $\quad p(x, t) \in C^\infty(\mathbb{R}^3 \times [0, \infty))$
2. There exists a constant $E \in (0, \infty)$ such that $\int_{\mathbb{R}^3} |\mathbf{v}(x, t)|^2 dx < E$. For more details on the Navier–Stokes Clay-problem, see the following reference: [17]. Therefore, this is another way to prove existence of global smooth solutions when one aims to obtain solutions defined on all the space $\mathbb{R}^3$. Really, by varying the localized perturbation, one can obtain different initial conditions and, as a by-product, global smooth solutions. Such global solutions do not necessitate to be stable at short times, since the symbol of the Navier–Stokes equation is not zero. However, by working on the infinity prolongation $\widehat{(\mathrm{NS})}_{+\infty} \subset J^\infty_4(W)$, all smooth solutions can be stabilized at finite times, since for $\widehat{(\mathrm{NS})}_{+\infty}$, one has $(g_2)_{+\infty} = 0$. Their average stability can be studied with the geometric methods given by A. Prástaro, also for

Theorem 17 (Maslov Index for Lagrangian Manifolds). *For n-Lagrangian submanifolds of a* $2n$*-dimensional symplectic manifold* (W,ω)*, we recognize i-Maslov indexes* $\beta_i(V)$ *and i-Maslov cycles* $\Sigma_i(V)$, $1\leq i\leq n-1$. *For* $i=1$*, we recover the Maslov index defined by Arnold.*

Proof. After recognizing the Maslov index for solutions of PDEs (Theorem 16), the first step to follow is to show that Lagrangian submanifolds of W are encoded by a suitable PDE. Let $\{x^\alpha,y^j\}_{1\leq\alpha,j\leq n}$ be local coordinates in a neighborhood of a point $a\in W$. In this way, an n-dimensional submanifold $N\subset W$, passing for a, can be endowed with local coordinates $\{x^\alpha\}_{1\leq\alpha\leq n}$. Let us represent ω in such a coordinate system:

$$\omega=\sum_{1\leq\alpha<\beta\leq n}\omega_{\alpha\beta}dx^\alpha\wedge dx^\beta+\sum_{1\leq\alpha,j\leq n}\bar{\omega}_{\alpha j}dx^\alpha\wedge dy^j+\sum_{1\leq i<j\leq n}\hat{\omega}_{ij}dy^i\wedge dy^j. \quad (69)$$

Then the restriction of ω on an n-dimensional submanifold $N\subset W$, with local coordinate $\{x^\alpha\}$, gives the formula (70).

$$\omega|_N=\sum_{1\leq\alpha<\beta\leq n}[\omega_{\alpha\beta}+\sum_{1\leq j\leq n}(\bar{\omega}_{\alpha j}y^j_\beta-\bar{\omega}_{\beta j}y^j_\alpha)+\sum_{1\leq i<j\leq n}\hat{\omega}_{ij}(y^i_\alpha y^j_\beta-y^i_\beta y^j_\alpha)]dx^\alpha\wedge dx^\beta. \quad (70)$$

Therefore, by imposing that they must be $\omega|_N=0$, we see that we can encode n-dimensional Lagrangian submanifolds $N\subset$W by means of solutions, of the PDE $\mathscr{L}_1$ reported in (71):

$$\mathscr{L}_1\subset J^1_n(W)\ :\ \left\{\begin{array}{l}\omega_{rs}(x)+\sum_{1\leq j\leq n}(\bar{\omega}_{rj}(x)y^j_s-\bar{\omega}_{sj}(x)y^j_r)\\ +\sum_{1\leq i<j\leq n}(y^i_ry^j_s-y^j_ry^i_s)\hat{\omega}_{ij}(x)=0\end{array}\right\}_{1\leq r<s\leq n}. \quad (71)$$

There ω_{rs} and $\hat{\omega}_{ij}$ are nondegenerate skew-symmetric $n\times n$ matrices and $\bar{\omega}_{rs}$ is a $n\times n$ matrix, all being analytic functions of $\{x^\alpha\}$.[27] One can prove that E_1 is a formally integrable and completely integrable PDE. In fact, one can see that $\pi_{r-1,r}:(E_1)_{+r}\to(E_1)_{+(r-1)}$ are subbundles of $\pi_{r+1,r}:J^{r+1}_n(W)\to J^r_n(W)$, $\forall r\geq 1$. Really, for $r=1$, we get

$$\left.\begin{array}{l}\dim(\mathscr{L}_1)_{+1}=n+n\frac{(+2)(n+1)}{2}-\frac{n(n-1)}{2}-\frac{n^2(n-1)}{2}\\ \dim(\mathscr{L}_1)=n+n(n+1)-\frac{n(n-1)}{2}\\ \dim(g_1)_{+1}=\frac{n^2(n+1)}{2}-\frac{n^2(n-1)}{2}\\ \dim(\mathscr{L}_1)_{+1}=\dim(\mathscr{L}_1)+\dim(g_1)_{+1}\end{array}\right\}. \quad (72)$$

global solutions defined on all the space, assuming perturbations with compact support. (See [45–49, 53].)

[27]Let us emphasize that the coefficients ω_{rs}, $\hat{\omega}_{ij}$, and $\bar{\omega}_{rs}$ are related by some first-order constraints coming from the condition that $d\omega=0$. However, for the formal integrability of Eq. (71), these constraints can be ruled out.

This is enough to state that the short sequence

$$(\mathscr{L}_1)_{+r} \xrightarrow{\pi_{r+1,r}} (\mathscr{L}_1)_{+(r-1)} \longrightarrow 0 \tag{73}$$

is exact for $r = 1$. Since this process can be iterated for any $r > 1$, we can state that one can arrive to a certain prolongation where the symbol is involutive. Hence, the PDE $\mathscr{L}_1$ is formally integrable. Since it is analytic, it is also completely integrable. Then we can apply Theorem 16 to $\mathscr{L}_1 \subset J^1_n(W)$, to state that there exists Maslov cycles and Maslov indexes for any solution $V \subset (\mathscr{L}_1)_{+1}$, on the first prolongation of $\mathscr{L}_1$. One has the following commutative diagram where all the vertical lines are surjectives:

$$\begin{array}{ccc}
(\mathscr{L}_1)_{+1} & \hookrightarrow & J^2_n(W) \\
\downarrow{\scriptstyle \pi_{2,1}|_{(\mathscr{L}_1)_{+1}}} & & \downarrow{\scriptstyle \pi_{2,1}} \\
\mathscr{L}_1 & \hookrightarrow & J^1_n(W) \\
\downarrow{\scriptstyle \pi_{1,0}|_{\mathscr{L}_1}} & & \downarrow{\scriptstyle \pi_{1,0}} \\
W & = & W \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\qquad \pi_{2,0}|_{(\mathscr{L}_1)_{+1}} : (\mathscr{L}_1)_{+1} \to W, \quad \pi_{2,0} : J^2_n(W) \to W \tag{74}$$

$(\mathscr{L}_1)_{+1}$ is a strong retract of $J^2_n(W)$; hence, one has the homotopy equivalence,

$$(\mathscr{L}_1)_{+1} \simeq J^2_n(W),$$

that induces isomorphisms on the corresponding cohomology spaces. Therefore, we can recognize i-Maslov index classes and i-Maslov cycle classes on each solution $V \subset (\mathscr{L}_1)_{+1}$.

As a by-product, we can apply these results to the symplectic space $(\mathbb{R}^{2n}, \omega)$, to recover the same results given by V.I Arnold. (See Example 8.) This justifies our Definitions 13 and 14 that can be recognized suitable generalizations, in PDEs geometry, of analogous definitions given by V.I. Arnold.

Theorem 18 (G-Singular Lagrangian Bordism Groups). *Let W be a symplectic $2n$-dimensional manifold. Let G be an abelian group. Then the G-singular bordism group of $(n-1)$-dimensional compact submanifolds of W, bording by means of n-dimensional Lagrangian submanifolds of W, is given in (75):*

$${}^G\Omega^{\mathscr{L}_1}_{\bullet,s} \cong \bar{H}_\bullet(\mathscr{L}_1; G). \tag{75}$$

- *If* ${}^G\Omega^{\mathscr{L}_1}_{\bullet,s}=0$*, one has* $\bar{B}or_\bullet(\mathscr{L}_1;G)\cong\bar{C}yc_\bullet(\mathscr{L}_1;G)$*.*
- *If* $\bar{C}yc_\bullet(\mathscr{L}_1;G)$ *is a free G-module, one has the isomorphism*

$$\bar{B}or_\bullet(\mathscr{L}_1;G)\cong{}^G\Omega(\mathscr{L}_1)_{\bullet,s}\bigoplus\bar{C}yc_\bullet(\mathscr{L}_1;G).$$

Proof. It is enough to applying Theorems 10 and 17 to get formula (75).

Theorem 19 (Closed Weak Lagrangian Bordism Groups). *Let W be a symplectic* $2n$*-dimensional manifold. Let G be an abelian group. Then the weak* $(n-1)$*-bordism group of closed compact* $(n-1)$*-dimensional submanifolds of W, bording by means of n-dimensional Lagrangian submanifolds of W, is given in (76).*

$$\Omega^{\mathscr{L}_1}_{n-1,w}\cong\bigoplus_{r+s=n-1}H_r(W;\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}\Omega_s\cong\Omega^{\mathscr{L}_1}_{n-1}/K^{\mathscr{L}_1}_{n-1,w}\cong\Omega^{\mathscr{L}_1}_{n-1,s}/K^{\mathscr{L}_1}_{n-1,s,w}. \tag{76}$$

Furthermore, since $\mathscr{L}_1\subset J^1_n(W)$ *has nonzero symbols,* $g_{1+s}\neq 0$ *and* $s\geq 0$*, then* $K^{\mathscr{L}_1}_{n-1,s,w}=0$*; hence,* $\Omega^{\mathscr{L}_1}_{n-1,s}\cong\Omega^{\mathscr{L}_1}_{n-1,w}$*.*

Proof. From the proof of Theorem 18 and by using Theorem 13, we get directly the proof.

Warning Lagrangian bordism considered in this paper, namely, Theorems 18 and 19, adopts a point of view that is directly related to one where compact (closed) manifolds bording by means of Lagrangian manifolds must be Lagrangian manifolds too. This is, for example, the Lagrangian bordism considered in [11]. Really these authors work on a manifold $W=\mathbb{R}^2\times M$, where M is a (compact) $2m$-dimensional symplectic manifold $(M,\hat{\omega})$, and $\mathbb{R}^2$ is endowed with the canonical symplectic form $\omega_{\mathbb{R}^2}=dx\wedge dy$. Thus, W is a $2(m+1)$-dimensional symplectic manifold with symplectic form $\omega=\hat{\omega}\oplus\omega_{\mathbb{R}^2}$. Therefore, one has a natural trivial fiber bundle structure $\pi:W\to\mathbb{R}^2$, with fiber the symplectic manifold M. Then one considers bordisms of (closed) compact Lagrangian m-dimensional submanifolds of M, bording by means of $(m+1)$-dimensional Lagrangian submanifolds of W. In such a situation, with respect to the framework considered in Theorems 18 and 19, one should specify that $n=m+1$ and that the $n-1=m$ compact manifolds bording with $(n=m+1)$-Lagrangian submanifolds of W must be Lagrangian submanifolds of M. In other words, the Lagrangian bordism groups considered in [11] are relative Lagrangian bordism groups, with respect to the submanifold $M\subset W$, in our formulation. However, since $(m+1)$-dimensional Lagrangian submanifolds V of W must necessarily be transverse to the fibers of $\pi:W\to\mathbb{R}^2$, except in the singular points, it follows that the compact (closed) m-dimensional manifolds N_1 and N_2 that they bord, namely, $\partial V=N_1\sqcup N_2$, must necessarily be submanifolds of M: $N_1,N_2\subset M$. Furthermore, by considering that $\omega|_V=0$, it follows that $\hat{\omega}|_{N_1}=\hat{\omega}|_{N_2}=0$; hence, N_1 and N_2 must necessarily be Lagrangian submanifolds of $(M,\hat{\omega})$, as considered in [11]. Therefore, our point of view is more general than the one adopted in [11] and recovers this last one when the structure of the symplectic manifold (W,ω) is of the type $(M\times\mathbb{R}^2,\hat{\omega}\oplus\omega_{\mathbb{R}^2})$.

Acknowledgements I would like thank my colleague Themistocles Rassias for inviting me to write my contribution for this Springer book in memory of V.I. Arnold. A similar version of this work was early posted on arXiv.1503.07851.

Appendices

Appendix A: On Global Smooth Solutions of the Navier–Stokes PDEs

In this appendix, we shall explicitly prove a theorem that one has implicitly used in Example 12.

Theorem A 1. *Any constant smooth solution s of the Navier–Stokes equation* $(\mathrm{NS}) \subset \mathrm{J}\mathscr{D}^2(\mathrm{W}) \subset \mathrm{J}^2_4(\mathrm{W})$ *admits perturbations that identify smooth, nonconstant solutions of* $(\mathrm{NS}) \subset \mathrm{J}\mathscr{D}^2(\mathrm{W}) \subset \mathrm{J}^2_4(\mathrm{W})$.

Proof. We shall use a surgery technique in order to prove this theorem. Let us divide the proof in some lemmas.

Lemma A 1. *Given a smooth constant solution s of* $(\mathrm{NS}) \subset \mathrm{J}\mathscr{D}^2(\mathrm{W})$*, we can identify a smooth solution with boundary diffeomorphic to* S^3 *and a compact smooth solution with boundary diffeomorphic to* S^3 *again, such that their canonical projections on M identify an annular domain in M.*

Proof. Let us consider a compact domain $D \subset M$ identified with a 4-dimensional disk D^4. Set $\partial D^4 = S^3$. By fixing a constant solution s of $(\mathrm{NS}) \subset \mathrm{J}\mathscr{D}^2(\mathrm{W})$, let us denote by N the image into $\widehat{(\mathrm{NS})}$ of S^3 by means of $D : N = D^2s(S^3) \subset \widehat{(\mathrm{NS})} \subset J^2_4(W)$. Set $V = D^2s(M) \subset \widehat{(\mathrm{NS})}$ and set

$$\widetilde{V} = (V \setminus D^2s(D^4)) \bigcup N \subset \widehat{(\mathrm{NS})}. \tag{77}$$

Then $\widetilde{V}$ is a smooth solution of $\widehat{(\mathrm{NS})}$ with boundary $\partial\widetilde{V} = N \cong S^3$.

Let $p_0 \in D^4$ be the center of the disk. Since $\widehat{(\mathrm{NS})} \subset J\mathscr{D}^2(W)$ is completely integrable, we can build a smooth (analytic) solution s_0 in a neighborhood $U_0 \subset D^4$ of p_0, such that $\widehat{V} = D^2s_0(U) \subset \widehat{(\mathrm{NS})}$. We can assume that s_0 does not coincide with s. (Otherwise, we could take a different constant value from s.) Let us consider in U_0 a disk D^4_0 centered on p_0. Set

$$N_0 = D^2s_0(\partial D^k_0) \subset \widehat{V} \subset \widehat{(\mathrm{NS})}. \tag{78}$$

Let us consider

$$\widetilde{\widehat{V}} = (\widehat{V} \setminus D^2s_0(D^4_0)) \bigcup N_0 \subset \widehat{(\mathrm{NS})}. \tag{79}$$

Then $\widetilde{\widehat{V}}$ is a smooth solution of $\widehat{(\mathrm{NS})}$ with boundary $\partial\widetilde{\widehat{V}} = N_0 \cong S^3$. Of course, the projections of N and N_0 on M via the canonical projection $\pi_2 : J\mathscr{D}^2(W) \to M$ identify an annular domain in M.

Lemma A 2. *The solutions* $\widetilde{V}$ *and* $\widetilde{\widehat{V}}$ *considered in Lemma 1 identify a connected smooth solution of* $(\mathrm{NS}) \subset \mathrm{J}_4^2(\mathrm{W})$.

Proof. Since both $\widetilde{V}$ and $\widetilde{\widehat{V}}$ are smooth solutions, we can consider their ∞-prolongations and look to them inside $\widehat{(\mathrm{NS})}_{+\infty}$. Now their boundaries are both diffeomorphic to S^3, and therefore there must exist a smooth solution $\overset{\infty}{V} \subset \widehat{(\mathrm{NS})}_{+\infty} \subset J_4^\infty(W)$ such that $\partial \overset{\infty}{V} = N_{+\infty} \bigcup (N_0)_{+\infty}$. In fact, from the commutative diagram (40) and Theorem 13, we get the exact commutative diagram (80).

$$
\begin{array}{ccccccccc}
0 & & 0 & & & & & & \\
& \searrow & \uparrow & & & & & & \\
& & \overline{K}_3^{\widehat{(NS)}} & & & & 0 & & \\
& & \uparrow & \searrow & & & \downarrow & & \\
0 & \longrightarrow & K_{3,s}^{\widehat{(NS)}} & \longrightarrow & \Omega_3^{\widehat{(NS)}} & \longrightarrow & \Omega_{3,s}^{\widehat{(NS)}} & \longrightarrow & 0\\
& & \uparrow & & & \searrow & \downarrow & & \\
& & 0 & = & = & = & \Omega_3 & & \\
& & & & & & \downarrow & \searrow & \\
& & & & & & 0 & & 0
\end{array}
\tag{80}
$$

where $\overline{K}_3^{\widehat{(\mathrm{NS})}} \cong K_{3,s}^{\widehat{(\mathrm{NS})}}$ distinguishes between non-diffeomorphic closed 3-dimensional integral smooth submanifolds of $\widehat{(\mathrm{NS})}$. In fact $\Omega_{3,s}^{\widehat{(\mathrm{NS})}} \cong \Omega_3 = 0$.[28] Since the Cartan distribution $\mathbf{E}_\infty \subset T\widehat{(\mathrm{NS})}_{+\infty}$ is 4-dimensional, it follows that $\overset{\infty}{V}$ smoothly solders with the solutions $\overset{\infty}{V}$, $\widetilde{V}$, and $\widetilde{\widehat{V}}$. In this way,

$$
X = \widetilde{V}_{+\infty} \bigcup_{N_{+\infty}} \overset{\infty}{V} \bigcup_{(N_0)_{+\infty}} \widetilde{\widehat{V}}_{+\infty} \tag{81}
$$

[28]This is related to the fact that the Navier–Stokes equation is an *extended* 0-*crystal PDE*. (See [46–49, 51, 53, 54].) In Table 4 are reported some unoriented smooth bordism groups Ω_n, $0 \leq n \leq 3$, useful to calculate $\Omega_{3,s}^{\widehat{(\mathrm{NS})}}$, according to Theorem 13.

Table 4 Unoriented smooth bordism groups Ω_n, $0 \leq n \leq 3$

n	Ω_n
0	$\mathbb{Z}_2$
1	0
2	$\mathbb{Z}_2$
3	0

is a smooth solution of $\widehat{(\mathrm{NS})}_{+\infty}$, hence of (NS).

To conclude the proof, let us assume that $\overset{\infty}{V}$ can be realized by means of a section s_∞ of $\pi : W \to M$, namely, $\overset{\infty}{V} = D^\infty(s_\infty)(A)$, where A is the annular domain above considered. Thus, we can say that the solution $X = D^\infty \bar{s}(M)$ for some smooth global section $\bar{s}$ of $\pi : W \to M$. Then taking into account of the affine structure of W, we can state that $\bar{s} = s + \nu$, where ν is a smooth perturbation of s on the disk D^4, such that $\nu|_{S^3} = 0$ and

$$\lim_{p \to S^3 \text{ (from inside)}} \nu(p) = 0, \ \nu|_{\complement D^4} = 0. \tag{82}$$

In other words, the perturbation is of the type pictured in Fig. 1.

So we have proved the following lemma.

Lemma A 3. *When perturbations of* $\widehat{(\mathrm{NS})}$ *are realized by means of smooth solutions of the corresponding linearized Navier–Stokes PDE,* $\widehat{\overline{(\mathrm{NS})}}[s] \subset J\mathscr{D}^2(s^* vTW)$*, the completely integrable part of* $\overline{(\mathrm{NS})}[s]$*, then the identified solutions of* $(\mathrm{NS}) \subset \mathrm{J}^2_4(\mathrm{W})$ *are also smooth solutions of* $(\mathrm{NS}) \subset \mathrm{J}\mathscr{D}^2(\mathrm{W}) \subset \mathrm{J}^2_4(\mathrm{W})$*; namely, they are identified with smooth sections of* $\pi : W \to M$*.*

Whether, instead, $\overset{\infty}{V}$ is a smooth solution that cannot be globally represented by means of a section of $\pi : W \to M$, then it means that there are in $\overset{\infty}{V}$, and hence in its projection into W, some pieces that climb on the fibers of $\pi : W \to M$. In such a case, we can continue to state that X is obtained by a perturbation of V inside the compact domain D, but the perturbation is a singular solution of the linearized Navier–Stokes PDE at the constant section s. Therefore, in such a case, it should not be possible to represent X as a smooth section of $\pi : W \to M$. This shows the necessity to realize the perturbation of s by means of a smooth solution ν of the linearized equation $\widehat{\overline{(\mathrm{NS})}}[s] \subset J\mathscr{D}^2(s^* vTW)$, such that $\nu|_{S^3} = 0$ and $\nu|_{\complement D^4} = 0$, in order that the perturbed solution X should be identified with a global nonconstant section of $\pi : W \to M$.

On the other hand, since $\widetilde{V}_{+\infty}$ and $\widetilde{\widehat{V}}_{+\infty}$ are both regular solutions with respect to the canonical projection $\pi_\infty : J\mathscr{D}^\infty(W) \to M$, and $(\mathrm{NS}) \subset \mathrm{J}\mathscr{D}^2(\mathrm{W})$ is an affine fiber bundle over its projection at the first order, with nonzero symbol, it follows that we can deform any eventual piece climbing on the fibers in such a way to obtain a regular solution with respect to the projection $\pi : W \to M$. Therefore, the projection

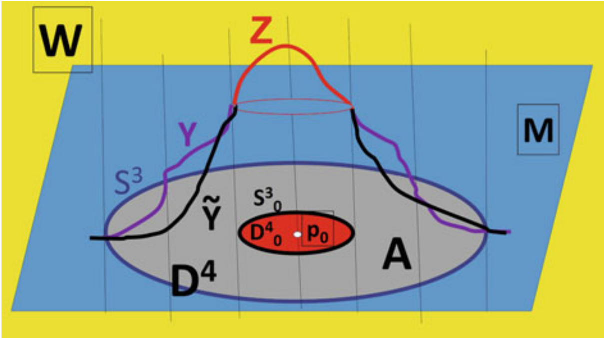


Fig. A.1 Deformation of a smooth solution (Y) of $\widehat{(\mathrm{NS})} \subset J^2_4(W)$, climbing along the fibers of $\pi_2 : \widehat{(\mathrm{NS})} \to M$, into a smooth solution $(\widetilde{Y})$ of $\widehat{(\mathrm{NS})} \subset J\mathscr{D}^2(W) \subset J^2_4(W)$. This is possible since the Navier–Stokes PDE is an affine fiber bundle over (C), and its symbol is not zero: $\dim(g_2)_q = 46$, $\forall q \in (\mathrm{NS})$, $\dim(\widehat{g_2})_q = 42$, $\forall q \in \widehat{(\mathrm{NS})}$. In the picture, $Z = \pi_{2,0}(\widehat{V}) \subset W$

$Y \subset \widehat{(\mathrm{NS})}$ of $\overset{\infty}{V}$ into $\widehat{(\mathrm{NS})}$ can be eventually deformed into a regular solution, $\widetilde{Y}$, with respect to the projection π_2. (See Fig. A.1.) In this way, the projection of $\widetilde{Y}$ into W smoothly relates regular smooth submanifolds that project on two domains of M that are outside the annular domain A, but that are disconnected each other. Thus, $\widetilde{Y}$ identifies a smooth 4-dimensional manifold transverse to the fibers of $\pi : W \to M$. By conclusion $\widetilde{Y}^{+\infty}$ is necessarily a regular solution of $\widehat{(\mathrm{NS})}_{+\infty} \subset J\mathscr{D}^\infty(W)$. Therefore, it can be obtained by a perturbation of the constant solution s, by means of a smooth solution of $\widehat{(\mathrm{NS})[s]} \subset J\mathscr{D}^2(s^*vTW)$. (See Fig. 1.)

Appendix B: On the Legendrian Bordism

Similarly to the way we considered Lagrangian bordism in this chapter, we can also formulate Legendrian bordism. Let us in this appendix recall some basic definitions and sketch only some steps. Really on a $(2n+1)$-dimensional manifold W, endowed with a *contact structure*, namely, a 1-differential form χ, such that $\chi(p) \wedge d\chi(p)^n \neq 0$, $\forall p \in W$, there exists a *characteristic vector field* $v : W \to TW$, i.e., the generator of the 1-dimensional annihilator of $d\chi$: $v \rfloor d\chi = 0$ and $v \rfloor \chi = 1$. Furthermore on W there exists also a *contact distribution*, namely, a $2n$-dimensional distribution $\mathbf{B} = \bigcup_{p \in W} \mathbf{B}_p$, $\mathbf{B}_p = \ker(\chi(p)) \subset T_pW$. One has the following properties.

Proposition B 1. *The following propositions hold.*

(bi) $d\chi(p)|_{\mathbf{B}_p}$, $\forall p \in W$, *is* nondegenerate, *i.e., if* $d\chi(\zeta, \xi) = 0$, $\forall \zeta \in \mathbf{B}_p$, *and* $\forall \xi \in \mathbf{B}_p$, *then* $\zeta = 0$.
(bii) $TW = \mathbf{B} \bigoplus \langle v \rangle$.

(biii) *(Darboux's theorem)* $\mathbf{B} \to W$ *is a symplectic vector bundle with symplectic form* $d\chi|_{\mathbf{B}}$.

(biv) *With respect to local coordinates* $\{x^\alpha, y_\alpha, z\}$ *on* W, χ *assumes the following form*[29]*:*

$$\chi = dz - y_\alpha dx^\alpha. \tag{83}$$

Proposition B 2. Integral manifold *of a contact structure* (W, χ) *is a submanifold* $N \subset W$*, such that* $\chi|_N = 0$ *(or equivalently* $T_pN \subset \mathbf{B}_p$, $\forall p \in N$*). One has*

$$\dim N < \frac{1}{2}(2n+1). \tag{84}$$

- Legendrian submanifolds *of* (W, χ) *are integral submanifolds* N *of maximal dimension:* $\dim N = n$.

Definition B 1. A *Legendrian bundle* $\pi : W \to M$ is a fiber bundle with $\dim W = 2n+1$, $\dim M = n+1$ and endowed with a contact structure (W, χ), such that each fiber W_p is a Legendrian submanifold, namely, $\chi|_{W_p} = 0$, $\forall p \in M$.

- If $L \subset W$ is a Legendrian submanifold of W, $(\chi|L = 0, \dim L = n)$, its *front* is $\pi(L) = X \subset M$. Singularities of $\pi|_L : L \to M$ are called *Legendrian singularities*. The front X of a Legendrian submanifold is an n-dimensional submanifold of M, with eventual singularities.

Similarly to the Lagrangian submanifolds of symplectic manifolds, we can characterize Legendrian submanifolds of a contact manifold by means of suitable PDEs. In fact we have the following:

Theorem B 1. *Given a contact structure on a* $(2n+1)$*-dimensional manifold* (W, χ)*, its Legendrian submanifolds are solutions of a first-order, involutive, formally integrable, and completely integrable PDE.*

i-Maslov indexes and i-Maslov cycles, $1 \leq i \leq n-1$*, can be recognized for such solutions.*

Proof. Let $\{x^\alpha, y_\alpha, z\}_{1\leq\alpha\leq n}$ be local coordinates on W. Then Legendrian submanifolds of W are the n-dimensional submanifolds of W that satisfy the PDE reported in (85):

$$\mathscr{L}eg \subset J^1_n(W) : \{z_\beta - y_\beta = 0\} \tag{85}$$

[29] All contact structure forms on W are locally diffeomorphic.

where $\{x^{\alpha}, y_{\alpha}, z, y_{\alpha\beta}, z_{\beta}\}_{1\leq\alpha,\beta\leq n}$ are local coordinates on $J^1_n(W)$. The first prolongation of $\mathscr{L}eg$ is given in (86).[30]

$$\mathscr{L}eg_{+1} \subset J^2_n(W) : \left\{ \begin{array}{l} z_{\beta} - y_{\beta} = 0 \\ z_{\beta\gamma} - y_{\beta\gamma} = 0 \end{array} \right\}. \tag{86}$$

Then one can see that

$$\left. \begin{array}{c} [\dim(\mathscr{L}eg_{+1}) = \frac{n+1}{2}(n^2 + 2n + 2)] = [\dim(\mathscr{L}eg) = (n+1)^2] \\ +[\dim((g_1)_{+1}) = \frac{(n+1)n^2}{2}] \end{array} \right\}. \tag{87}$$

Therefore, one has the surjectivity $\mathscr{L}eg_{+1} \to \mathscr{L}eg$. Furthermore, one can see that the symbol g_1 is involutive. In fact one has

$$\left. \begin{array}{rl} [\dim(((g_1)_{+1})) = \frac{n^2(n+1)}{2} & = [\dim(g_1) = n^2] + [\dim(g_1^{(1)}) = n^2 - n)] \\ & + [\dim(g_1^{(2)}) = n^2 - 2n)] \\ & + \cdots + [\dim(g_1^{(n-1)}) = n^2 - n(n-1)] \\ & = \frac{n^2(n+1)}{2} \end{array} \right\}. \tag{88}$$

We have used the formula $1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$. This is enough to state that $\mathscr{L}eg$ is formally integrable, and being analytic, it is also completely integrable.

Let us also remark that $\mathscr{L}eg$ is a strong retract of $J^1_n(W)$; therefore, one has the homotopic equivalence $J^1_n(W) \simeq \mathscr{L}eg$ that induces isomorphisms between the corresponding cohomology groups. Then by using Theorem 16, we can state that on each solution of $\mathscr{L}eg$ we are able to recognize i-Maslov indexes and i-Maslov cycles.

Definition B 2. Let W be a $(2n+1)$-dimensional contact manifold (W, χ). A Legendrian bordism is an n-dimensional Legendrian submanifold bording compact $(n-1)$-dimensional integral submanifolds of W.

Example B 1. Let M be an n-dimensional manifold. The derivative space

$$J\mathscr{D}(M, \mathbb{R}) \cong T^*M \times \mathbb{R}$$

has a canonical contact form $\chi = dy - y_{\alpha}dx^{\alpha}$, where x^{α} are local coordinates on M and y is a coordinate on $\mathbb{R}$. This is just the Cartan form on the derivative space $J\mathscr{D}(E)$, $E = M \times \mathbb{R}$, with respect to the fibration $\pi : E \to M$. The corresponding

[30] Let us note that the equations of second order in (86) are not all linearly independent. In fact, by considering that they must be $z_{\beta\gamma} = z_{\gamma\beta}$, we get, by difference of the equations, $z_{\beta\gamma} - y_{\beta\gamma} = 0$ and $z_{\gamma\beta} - y_{\gamma\beta} = 0$: $-y_{\beta\gamma} + y_{\gamma\beta} = 0$; namely, it must hold the symmetry under the exchange of indexes in $y_{\gamma\beta}$. Therefore, the number of independent equations for $\mathscr{L}eg_{+1}$ is $(n + \frac{n(n+1)}{2})$.

contact distribution coincides with the Cartan distribution $\mathbf{E}_1(E) \subset TJ\mathscr{D}(E)$. Every solution is a Legendrian submanifold. Therefore, in such a case, Legendrian bordism are identified with solutions bording $(n-1)$-dimensional integral submanifolds.

Remark B 1. From the above results, we can directly reproduce results similar to Theorems 18 and 19 also for singular Legendrian bordism groups. More precisely, one has the exact commutative diagram reported in (89), where the top horizontal line is an homotopy equivalence:

$$\begin{array}{ccc} \mathscr{L}eg & \xrightarrow{\sim} & J^1_n(W) \\ \downarrow & & \downarrow \\ W & = & W \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{89}$$

We get the following isomorphisms:

$$\left.\begin{array}{ll} H^1(I(\mathscr{L}eg);\mathbb{Z}_2) & \cong H^1(W;\mathbb{Z}_2)\bigoplus \mathbb{Z}_2[\omega_1^{(1)}] \\ H^i(I(\mathscr{L}eg);\mathbb{Z}_2) & \cong H^i(W;\mathbb{Z}_2) \\ & \quad \bigoplus_{1\le p\le i-1} H^{i-p}(W;\mathbb{Z}_2)\bigotimes_{\mathbb{Z}_2} H^p(F_1;\mathbb{Z}_2) \\ & \quad \bigoplus \mathbb{Z}_2[\omega_1^{(1)},\cdots,\omega_i^{(1)}] \end{array}\right\}. \tag{90}$$

Then the map $i_V : V \to I(\mathscr{L}eg)$ induces the following morphism:

$$(i_V)_* : \mathbb{Z}_2[\omega_1^{(1)},\cdots,\omega_i^{(1)}] \to H^i(V;\mathbb{Z}_2),\ 1\le i\le n-1. \tag{91}$$

Set $\beta_i(V) = (i_V)_*(\omega_i^{(1)})$ that is the i-Maslov index of the Legendrian manifold V. We get $\beta_i(V)\bigcap[V] = [\Sigma_i(V)]$ that relates the i-Maslov index of V with its i-Maslov cycle.

Theorem B 2 (G-Singular Legendrian Bordism Groups). *Let W be a contact $(2n+1)$-dimensional manifold. Let G be an abelian group. Then the G-singular bordism group of $(n-1)$-dimensional compact submanifolds of W, bording by means of n-dimensional Legendrian submanifolds of W, is given in (92):*

$${}^G\Omega_{\bullet,s}^{\mathscr{L}eg} \cong \bar{H}_\bullet(\mathscr{L}eg;G). \tag{92}$$

- *If ${}^G\Omega_{\bullet,s}^{\mathscr{L}eg} = 0$, one has: $\bar{B}or_\bullet(\mathscr{L}eg;G) \cong \bar{C}yc_\bullet(\mathscr{L}eg;G)$.*
- *If $\bar{C}yc_\bullet(\mathscr{L}eg;G)$ is a free G-module, one has the isomorphism:*

$$\bar{B}or_\bullet(\mathscr{L}eg;G) \cong {}^G\Omega(\mathscr{L}eg)_{\bullet,s}\bigoplus \bar{C}yc_\bullet(\mathscr{L}eg;G).$$

Theorem B 3 (Closed Weak Legendrian Bordism Groups). *Let W be a contact $(2n+1)$-dimensional manifold. Let G be an abelian group. Then the weak $(n-1)$-bordism group of closed compact $(n-1)$-dimensional submanifolds of W, bording by means of n-dimensional Legendrian submanifolds of W, is given in (93):*

$$\Omega^{\mathcal{L}eg}_{n-1,w} \cong \bigoplus_{r+s=n-1} H_r(W;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s \cong \Omega^{\mathcal{L}eg}_{n-1}/K^{\mathcal{L}eg}_{n-1,w} \cong \Omega^{\mathcal{L}eg}_{n-1,s}/K^{\mathcal{L}eg}_{n-1,s,w}. \tag{93}$$

Furthermore, since $\mathcal{L}eg \subset J^1_n(W)$ has nonzero symbols, then $K^{\mathcal{L}eg}_{n-1,s,w} = 0$; hence, $\Omega^{\mathcal{L}eg}_{n-1,s} \cong \Omega^{\mathcal{L}eg}_{n-1,w}$.

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On the Infimum of Certain Functionals

Biagio Ricceri

Abstract Here is a particular case of our main result: Let X be a real Banach space, $\varphi : X \to \mathbf{R}$ a nonzero continuous linear functional and $\psi : X \to \mathbf{R}$ a nonconstant Lipschitzian functional with Lipschitz constant equal to $\|\varphi\|_{X^*}$. Then, we have

$$\max\left\{\inf_{x\in X}(\varphi(x)+\psi(x)), \inf_{x\in X}(\varphi(x)-\psi(x))\right\}$$
$$=\inf_{x\in X}(\varphi(x)+|\psi(x)|)=\liminf_{\|x\|\to+\infty}(\varphi(x)+|\psi(x)|)$$

Here and in what follows, X is a real Banach space, $\varphi : X \to \mathbf{R}$ is a nonzero continuous linear functional, and $\psi : X \to \mathbf{R}$ is a nonconstant Lipschitzian functional with Lipschitz constant L. From Proposition 2.1 of [2], we know that, when $L < \|\varphi\|_{X^*}$, the functional $\varphi + \psi$ is unbounded below. When, to the contrary, $L \geq \|\varphi\|_{X^*}$, this is no longer true. That is, the same functional can be bounded below. The simplest examples are provided by taking $\psi(x) = |\varphi(x)|$ or $\psi(x) = \|\varphi\|_{X^*}\|x\|$. The aim of this very short paper is to study the infimum of that functional just when $L = \|\varphi\|_{X^*}$.

So, from now on, we assume that

$$L = \|\varphi\|_{X^*} \ .$$

Our basic result is as follows:

Theorem 1. *Let $[a, b]$ be a closed interval contained in $[-1, 1]$ and let $\gamma : [a, b] \to \mathbf{R}$ be a convex and lower semicontinuous function.*

Then, one has

$$\max\left\{\inf_{x\in X}(\varphi(x)+a\psi(x))-\gamma(a), \inf_{x\in X}(\varphi(x)+b\psi(x))-\gamma(b)\right\} = \inf_{x\in X}\sup_{\lambda\in[a,b]}(\varphi(x)+\lambda\psi(x)-\gamma(\lambda)) \ .$$

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Our proof of Theorem 1 is based on the use of the following result (Theorem 5.9 and Remark 5.10 of [3]):

Theorem A. *Let S be a topological space, $I \subset \mathbf{R}$ a compact interval, and $f : S \times I \to \mathbf{R}$ a function which is lower semicontinuous in X and upper semicontinuous and quasi-concave in I. Moreover, assume that there exists a set $D \subseteq I$ dense in I such that, for each $\lambda \in D$ and $r \in \mathbf{R}$, the set*

$$\{x \in S : f(x, \lambda) < r\}$$

is connected.

Then, one has

$$\sup_{\lambda \in I} \inf_{x \in S} f(x, \lambda) = \inf_{x \in S} \sup_{\lambda \in I} f(x, \lambda) .$$

To be able to use Theorem A, we first have to establish the following result:

Theorem 2. *For each $\lambda \in]-1, 1[$ and $r \in \mathbf{R}$, the set $(\varphi + \lambda\psi)^{-1}(]-\infty, r])$ is a retract of X.*

Proof. First, consider the multifunction $G : \mathbf{R} \to 2^X$ defined by

$$G(t) = \varphi^{-1}(]-\infty, t])$$

for all $t \in \mathbf{R}$. Let us check that

$$d_H(G(t), G(s)) \leq \frac{|t-s|}{\|\varphi\|_{X^*}} . \tag{1}$$

for all $t, s \in \mathbf{R}$, d_H being the usual Hausdorff distance. For instance, assume that $t < s$. Consequently

$$G(t) \subseteq G(s) . \tag{2}$$

Now, fix $x \in G(s) \setminus G(t)$. Consequently

$$t < \varphi(x) \leq s .$$

In view of the classical formula giving the distance of a point from a closed hyperplane, we have

$$\mathrm{dist}(x, G(t)) \leq \mathrm{dist}(x, \varphi^{-1}(t)) = \frac{\varphi(x) - t}{\|\varphi\|_{X^*}} \leq \frac{s-t}{\|\varphi\|_X^*} .$$

So

$$\sup_{x \in G(s)} \mathrm{dist}(x, G(t)) \leq \frac{s-t}{\|\varphi\|_X^*}$$

which together with (2) gives (1). Now, consider the multifunction $F : X \to 2^X$ defined by

$$F(x) = G(r - \lambda\psi(x))$$

for all $x \in X$. For each $x, y \in X$, we have

$$d_H(F(x), F(y)) \leq \frac{1}{\|\varphi\|_{X^*}}|\lambda||\psi(x) - \psi(y)| \leq |\lambda|\|x - y\| .$$

Hence, since $|\lambda| < 1$, F is a multivalued contraction with closed and convex values. Then, in view of [1], the set $Fix(F) := \{x \in X : x \in F(x)\}$ is a retract of X. To complete the proof, simply observe that $Fix(F) = (\varphi + \lambda\psi)^{-1}(] - \infty, r])$. △

Proof of Theorem 1. Consider the function $f : X \times [a, b] \to \mathbf{R}$ defined by

$$f(x, \lambda) = \varphi(x) + \lambda\psi(x) - \gamma(\lambda)$$

for all $(x, \lambda) \in X \times [a, b]$. Clearly, f is continuous in X, while it is upper semicontinuous and concave in $[a, b]$. Fix $\lambda \in]a, b[$ and $r \in \mathbf{R}$. Of course, we have

$$\{x \in X : f(x, \lambda) < r\} = \bigcup_{s<r}\{x \in X : f(x, \lambda) \leq s\} .$$

On the other hand, by Theorem 2, the sets of the family $\{\{x \in X : f(x, \lambda) \leq s\}\}_{s<r}$ are connected (being retracts of X) and pairwise non-disjoint. Consequently, the set $\{x \in X : f(x, \lambda) < r\}$ is connected too. Therefore, we can apply Theorem A. It ensures that

$$\sup_{\lambda\in[a,b]} \inf_{x\in X}(\varphi(x) + \lambda\psi(x) - \gamma(\lambda)) = \inf_{x\in X} \sup_{\lambda\in[a,b]} (\varphi(x) + \lambda\psi(x) - \gamma(\lambda)) .$$

Now, observe that, since $\inf_{x\in X}(\varphi(x) + \lambda\psi(x)) = -\infty$ for all $\lambda \in] - 1, 1[$, we have

$$\sup_{\lambda\in[a,b]} \inf_{x\in X}(\varphi(x) + \lambda\psi(x) - \gamma(\lambda)) = \max\left\{\inf_{x\in X}(\varphi(x) + a\psi(x)) - \gamma(a), \inf_{x\in X}(\varphi(x) + b\psi(x)) - \gamma(b)\right\}$$

and the conclusion follows. △

A consequence of Theorem 1 is as follows:

Theorem 3. *Let $[a, b]$ be a closed interval contained in $[-1, 1]$ and let $\gamma : [a, b] \to \mathbf{R}$ be a continuous function which is derivable in $]a, b[$. Assume that γ' is strictly increasing in $]a, b[$. Set*

$$A = \left\{x \in X : \psi(x) \leq \inf_{]a,b[} \gamma'\right\} ,$$

$$B = \left\{x \in X : \psi(x) \geq \sup_{]a,b[} \gamma'\right\}$$

and

$$C = \left\{ x \in X : \inf_{]a,b[} \gamma' < \psi(x) < \sup_{]a,b[} \gamma' \right\} .$$

Finally, denote by η the inverse of the function γ'.

Then, one has

$$\max \left\{ \inf_{x\in X}(\varphi(x) + a\psi(x)) - \gamma(a), \inf_{x\in X}(\varphi(x) + b\psi(x)) - \gamma(b) \right\}$$

$$= \min \left\{ \inf_{x\in A}(\varphi(x) + a\psi(x)) - \gamma(a), \inf_{x\in B}(\varphi(x) + b\psi(x)) \right.$$

$$\left. -\gamma(b), \inf_{x\in C}(\varphi(x) + \eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x)))) \right\} .$$

Proof. Let f be as in the proof of Theorem 1. Fix $x \in X$. Clearly, $f(x, \cdot)$ is concave in $[a, b]$. Moreover, according to the sign of its derivative, the function $f(x, \cdot)$ is nonincreasing (resp. nondecreasing) in $[a, b]$ if $x \in A$ (resp. $x \in B$). If $x \in C$, the derivative of $f(x, \cdot)$ vanishes at the point $\eta(\psi(x))$, and so, by concavity, such a point is the global maximum of $f(x, \cdot)$ in $[a, b]$. Summarizing, we have

$$\sup_{\lambda\in[a,b]} f(x, \lambda) = \begin{cases} \varphi(x) + a\psi(x) - \gamma(a) & \text{if } x \in A \\ \varphi(x) + b\psi(x) - \gamma(b) & \text{if } x \in B \\ \varphi(x) + \eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x))) & \text{if } x \in C , \end{cases}$$

and the conclusion clearly follows in view of Theorem 1. △

In turn, by applying Theorem 3, we obtain the following result:

Theorem 4. *We have*

$$\max \left\{ \inf_{x\in X}(\varphi(x) + \psi(x)), \inf_{x\in X}(\varphi(x) - \psi(x)) \right\} = \inf_{x\in X}(\varphi(x) + |\psi(x)|) \tag{3}$$

and

$$\liminf_{\|x\|\to+\infty} (\varphi(x) + |\psi(x)|) = \inf_{x\in X}(\varphi(x) + |\psi(x)|) . \tag{4}$$

Proof. First, we want to prove that

$$\max\left\{\inf_{x\in X}(\varphi(x)+\psi(x)), \inf_{x\in X}(\varphi(x)-\psi(x))\right\} = \inf_{x\in X}(\varphi(x)+|\psi(x)|+e^{-|\psi(x)|}) . \tag{5}$$

Consider the function $\gamma : [-1,1] \to \mathbf{R}$ defined by

$$\gamma(\lambda) = \begin{cases} (1-|\lambda|)\log(1-|\lambda|)+|\lambda| & \text{if } |\lambda|<1 \\ \\ 1 & \text{if } |\lambda|=1 . \end{cases}$$

Clearly, γ is continuous in $[-1,1]$, is derivable in $]-1,1[$, γ' is strictly increasing and $\gamma'(]-1,1[) = \mathbf{R}$. Moreover, η, the inverse of γ', is given by

$$\eta(\mu) = \begin{cases} \frac{|\mu|}{\mu}(1-e^{-|\mu|}) & \text{if } \mu \neq 0 \\ \\ 0 & \text{if } \mu = 0 . \end{cases}$$

So, for each $x \in X \setminus \psi^{-1}(0)$, we have

$$\begin{aligned} \eta(\psi(x))\psi(x) - \gamma(\eta(\psi(x))) &= |\psi(x)|(1-e^{-|\psi(x)|}) - (-e^{-|\psi(x)|}|\psi(x)| + 1 - e^{-|\psi(x)|}) \\ &= |\psi(x)| + e^{-|\psi(x)|} - 1 . \end{aligned}$$

Clearly, these equalities hold also if $\psi(x) = 0$. Consequently, by Theorem 2, after observing that $C = X$, we have

$$\begin{aligned} \max\left\{\inf_{x\in X}(\varphi(x)-\psi(x))-1, \inf_{x\in X}(\varphi(x)+\psi(x))-1\right\} &= \inf_{x\in X}(\varphi(x)+\eta(\psi(x))\psi(x) \\ &\quad -\gamma(\eta(\psi(x)))) \\ &= \inf_{x\in X}(\varphi(x)+|\psi(x)| \\ &\quad +e^{-|\psi(x)|})-1 \end{aligned}$$

which yields (5). Since

$$\begin{aligned} \max\left\{\inf_{x\in X}(\varphi(x)+\psi(x)), \inf_{x\in X}(\varphi(x)-\psi(x))\right\} &\leq \inf_{x\in X}(\varphi(x)+|\psi(x)|) \\ &\leq \inf_{x\in X}(\varphi(x)+|\psi(x)|+e^{-|\psi(x)|}) , \end{aligned}$$

from (5), we obtain both (3) and

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|)=\inf_{x\in X}(\varphi(x)+|\psi(x)|+e^{-|\psi(x)|})\ . \tag{6}$$

Finally, let us prove (4). Arguing by contradiction, assume that

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|)<\liminf_{\|x\|\to+\infty}(\varphi(x)+|\psi(x)|)\ .$$

Fix ξ satisfying

$$\inf_{x\in X}(\varphi(x)+|\psi(x)|)<\xi<\liminf_{\|x\|\to+\infty}(\varphi(x)+|\psi(x)|)\ . \tag{7}$$

So, there is some $\delta>0$ such that

$$\varphi(x)+|\psi(x)|>\xi \tag{8}$$

for all $x\in X$ satisfying $\|x\|>\delta$. Now, in view of (6), we can fix a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}(\varphi(x_n)+|\psi(x_n)|+e^{-|\psi(x_n)|})=\inf_{x\in X}(\varphi(x)+|\psi(x)|)\ . \tag{9}$$

Clearly

$$\lim_{n\to\infty}(\varphi(x_n)+|\psi(x_n)|)=\inf_{x\in X}(\varphi(x)+|\psi(x)|)\ . \tag{10}$$

In view of (7), there is $\nu\in\mathbf{N}$ such that

$$\varphi(x_n)+|\psi(x_n)|<\xi$$

for all $n>\nu$. Thus, by (8), we have

$$\sup_{n>\nu}\|x_n\|\le\delta\ .$$

Then, since ψ is Lipschitzian, the sequence $\{\psi(x_n)\}$ is bounded too. But, (9) and (10) imply that

$$\lim_{n\to\infty}e^{-|\psi(x_n)|}=0$$

which leads to a contradiction. The proof is complete. △

We conclude with a consequence of Theorem 4.

Proposition 1. *Assume that ψ is Gâteaux differentiable and that both φ and $-\varphi$ do not belong to $\psi'(X)$.*

Then, for every $r \in \mathbf{R}$, the functional $x \to \varphi(x) + |\psi(x) - r|$ has no global minima in X.

Proof. Arguing by contradiction, assume that there is $x_0 \in X$ such that

$$\varphi(x_0) + |\psi(x_0) - r| = \inf_{x \in X}(\varphi(x) + |\psi(x) - r|) .$$

Then, by Theorem 4 (applied to $\psi - r$) , x_0 would be a global minimum either of $\varphi + \psi$ or of $\varphi - \psi$. Accordingly, we would have either $\psi'(x_0) = -\varphi$ or $\psi'(x_0) = \varphi$, contrary our assumption. △

Remark 1. Of course, if $\|\psi'(x)\|_{X^*} < L$ for all $x \in X$, then both φ and $-\varphi$ do not belong to $\psi'(X)$.

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The Algebra of Gyrogroups: Cayley's Theorem, Lagrange's Theorem, and Isomorphism Theorems

Teerapong Suksumran

Abstract Using the Clifford algebra formalism, we show that the unit ball of a real inner product space equipped with Einstein addition forms a uniquely 2-divisible gyrocommutative gyrogroup or a B-loop in the loop literature. One notable result is a compact formula for Einstein addition in terms of Möbius addition. In the second part of this paper, we show that the symmetric group of a gyrogroup admits the gyrogroup structure, thus obtaining an analog of Cayley's theorem for gyrogroups. We examine subgyrogroups, gyrogroup homomorphisms, normal subgyrogroups, and quotient gyrogroups and prove the isomorphism theorems. We prove a version of Lagrange's theorem for gyrogroups and use this result to prove that gyrogroups of particular order have the Cauchy property.

1 Introduction

It is my pleasure to contribute a paper to this volume dedicated to the memory of Vladimir Arnold who made significant contributions in several fields, including dynamical systems theory, topology, algebraic geometry, differential equations, and classical mechanics. One of the remarkable results of Arnold's work is the theorem that bears his name, the Kolmogorov–Arnold–Moser theorem.

This expository paper grew from our three research papers [60–62]. Let c be a positive constant representing the speed of light in vacuum, and let $\mathbb{R}^3_c$ denote the c-ball of relativistically admissible velocities, $\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 \colon \|\mathbf{v}\| < c\}$. In [69], Einstein velocity addition $\oplus_E$ in the c-ball is given by the equation

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u} \right\},$$

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where $\gamma_{\mathbf{u}}$ is the Lorentz factor or gamma factor given by $\gamma_{\mathbf{u}} = \dfrac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}}$.

The system $(\mathbb{R}^3_c, \oplus_E)$ does not form a group since $\oplus_E$ is neither associative nor commutative. Nevertheless, Ungar showed that $(\mathbb{R}^3_c, \oplus_E)$ is rich in structure and encodes a group-like structure, namely, the gyrogroup structure. He introduced space rotations gyr[**u**, **v**], called *gyroautomorphisms*, to repair the lack of associativity in $(\mathbb{R}^3_c, \oplus_E)$:

$$\mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathbf{w}) = (\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$$
$$(\mathbf{u} \oplus_E \mathbf{v}) \oplus_E \mathbf{w} = \mathbf{u} \oplus_E (\mathbf{v} \oplus_E \mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w})$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3_c$. The resulting system forms a gyrocommutative gyrogroup, called the *Einstein gyrogroup*.

There are close connections between the Einstein gyrogroup and the Lorentz transformations, as described in [68] and [70, Chap. 11]. A Lorentz transformation without rotation is called a *Lorentz boost*. Let $L(\mathbf{u})$ and $L(\mathbf{v})$ denote the Lorentz boosts parameterized by **u** and **v** in $\mathbb{R}^3_c$. The composite of two Lorentz boosts is not a pure Lorentz boost, but a Lorentz boost followed by a space rotation:

$$L(\mathbf{u}) \circ L(\mathbf{v}) = L(\mathbf{u} \oplus_E \mathbf{v}) \circ \mathrm{Gyr}[\mathbf{u}, \mathbf{v}], \tag{1}$$

where Gyr[**u**, **v**] is a rotation of spacetime coordinates induced by the Einstein gyroautomorphism gyr[**u**, **v**].

Another example of a gyrogroup is the *Möbius gyrogroup*, which consists of the complex unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with Möbius addition

$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b} \tag{2}$$

for $a, b \in \mathbb{D}$. The Möbius gyroautomorphisms are given by

$$\mathrm{gyr}[a, b]z = \frac{1 + a\bar{b}}{1 + \bar{a}b} z, \qquad z \in \mathbb{D}. \tag{3}$$

The gyrogroup $(\mathbb{D}, \oplus_M)$ bears the name "Möbius" because for each $a \in \mathbb{D}$, the map $\tau_a : z \mapsto a \oplus_M z$ defines a Möbius transformation or conformal mapping of $\mathbb{D}$. The gyrogroup of qubit density matrices is presented in [37].

Einstein and Möbius gyrogroups play a major role in gyrogroup theory as they provide concrete models for the abstract theory. See, for instance, [1, 14, 18, 38, 58, 62, 69–71, 73].

Gyrogroup theory is related to various fields, including mathematical physics, non-Euclidean geometry, group theory, loop theory, and abstract algebra. For instance, the gyrogroup structure appears as an algebraic structure that encodes Einstein's velocity addition law [69, 72]. It is also an algebraic structure that underlies the qubit density matrices, which play an important role in quantum mechanics [37, 39, 66]. For a connection to Thomas precession, see [74].

Certain gyrogroups give rise to vector space-like structures, called *gyrovector spaces*, which form the algebraic setting for hyperbolic geometry of Bolyai and Lobachevsky just as vector spaces form the algebraic setting for Euclidean geometry [70, 73, 75]. More precisely, the Möbius gyrovector space is associated with the Poincaré model of conformal geometry on the open unit ball in n-dimensional Euclidean space $\mathbb{R}^n$ [11, 38], and the Einstein gyrovector space is associated with the Beltrami–Klein model of hyperbolic geometry on the unit ball in $\mathbb{R}^n$ [38, 52, 53, 59, 63].

As noted above, a gyrogroup is a group-like structure, but not a group since its binary operation is neither associative nor commutative, in general. However, gyrogroups share remarkable analogies with groups. Indeed, any group may be viewed as a gyrogroup with trivial gyroautomorphisms. Gyrogroups abound in group theory. For example, every gyrogroup is a twisted subgroup, and, under certain conditions, twisted subgroups are gyrocommutative gyrogroups [21]. Further, any group Γ can be turned into a *left* gyrogroup [22, Theorem 3.4], and this associated left gyrogroup forms a gyrogroup if and only if Γ is central by a 2-Engel group [22, Theorem 3.7]. The study of left gyrogroups in connection with near subgroups can be found in [13]. The study of right gyrogroups in connected Hausdorff topological groups can be found in [42]. The study of gyrogroups in connection with the Lorentz group is presented in [9, 58].

The study of gyrogroups is linked to that of Bol loops since every gyrogroup forms a left Bol loop with the A_ℓ-property and vice versa [55]. Furthermore, gyrocommutative gyrogroups and Bruck loops (also called K-loops) are equivalent [36, p. 72]. For links between gyrogroups and loops, see [34–36, 41, 44, 45]. For applications of gyrogroup theory in analysis and signal processing, see [15, 17].

In this paper, we study gyrogroups from the abstract point of view. The power of the abstract point of view lies in the fact that results for all concrete gyrogroups are obtained by proving a single result for the abstract gyrogroup. In the first part of this paper, we give an algebraic proof that an Einstein gyrogroup on the open unit ball in a real inner product space does form a uniquely 2-divisible gyrocommutative gyrogroup using the Clifford algebra formalism. This results in a compact formula for Einstein addition in terms of Möbius addition; see Eq. (43). A Clifford algebra approach to study Möbius and Einstein gyrogroups is very fruitful [14, 16, 18, 19, 44, 59, 62].

In the second part of this paper, we study gyrogroups from the algebraic point of view. In Sect. 5, we exhibit the gyrogroup of permutations and derive Cayley's theorem for gyrogroups. In Sect. 6, we provide the notion of subgyrogroups. In Sect. 7, we examine gyrogroup homomorphisms, normal subgyrogroups, and quotient gyrogroups and prove the isomorphism theorems. In Sect. 8, we prove an analog of Lagrange's theorem for gyrogroups using an important result in the theory of Bruck loops [3]. In Sect. 9, we apply Lagrange's theorem and results from loop theory to prove that gyrogroups of particular order have the Cauchy property.

2 Gyrogroups and Loops

In this section, we give the relevant definitions, summarize elementary properties of gyrogroups, and indicate certain connections between gyrogroups, quasigroups, and loops. Much of this section can be found in [67, 70, 73].

A pair $(G, \oplus)$ consisting of a nonempty set G and a binary operation $\oplus$ on G is called a *magma*. Let $(G, \oplus)$ be a magma. A bijection from G to itself is called an *automorphism* of G if $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$. The set of all automorphisms of G is denoted by $\mathrm{Aut}\,(G, \oplus)$. It is straightforward to check that $\mathrm{Aut}\,(G, \oplus)$ forms a group under function composition.

The formal definition of a gyrogroup, introduced by Ungar, is modeled on the key features of the relativistic ball in $\mathbb{R}^3$ with Einstein addition of relativistically admissible velocities and the complex unit disk with Möbius addition.

Definition 1 (Gyrogroups). A magma $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms:

(G1) $\exists 0 \in G\, \forall a \in G, 0 \oplus a = a$;
(G2) $\forall a \in G\, \exists b \in G, b \oplus a = 0$;
(G3) $\forall a, b \in G\, \exists \mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G, \oplus)\, \forall c \in G,$

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c;$$

(G4) $\forall a, b \in G, \mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$.

Let us denote by id_X the identity map on a set X and by $\ominus a$ the left inverse of an element a in a gyrogroup. The axioms in Definition 1 imply the right counterparts.

Theorem 1 ([70]). *A magma $(G, \oplus)$ forms a gyrogroup if and only if it satisfies the following properties:*

(g1) $\exists 0 \in G \forall a \in G, 0 \oplus a = a$ *and* $a \oplus 0 = a$; (two-sided identity)
(g2) $\forall a \in G \exists b \in G, b \oplus a = 0$ *and* $a \oplus b = 0$. (two-sided inverse)
For $a, b, c \in G$, define

$$\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)), \qquad \text{(gyrator identity)}$$

then
(g3) $\mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G, \oplus)$; (gyroautomorphism)
(g3a) $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$; (left gyroassociative law)
(g3b) $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$; (right gyroassociative law)
(g4a) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$; (left loop property)
(g4b) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$. (right loop property)

By the previous theorem, any gyrogroup contains the unique two-sided identity 0, and each element of the gyrogroup possesses a unique two-sided inverse. The map $\mathrm{gyr}[a, b]$ is called the *gyroautomorphism generated by a and b* and is completely determined by its generators via the *gyrator identity*. The gyroautomorphisms play a central role in the study of gyrogroups as they remedy the breakdown of associativity and commutativity in gyrogroups. The gyroautomorphisms of a gyrogroup G are *even*,

$$\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a, b], \tag{4}$$

and *inversive symmetric*,

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a] \tag{5}$$

for all $a, b \in G$.

In general, gyrogroups do not satisfy the associative law, but they satisfy the *left and right gyroassociative laws* (g3a) and (g3b) instead. Note that every group is a gyrogroup by defining the gyroautomorphisms to be the identity map. As we will see, gyrogroups are a natural generalization of groups. Gyrogroups that generalize abelian groups are given a name:

Definition 2 (Gyrocommutative Gyrogroups). A gyrogroup G with the additional property that

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \qquad \text{(gyrocommutative law)}$$

for all $a, b \in G$ is called a gyrocommutative gyrogroup.

It is known that a gyrogroup G is gyrocommutative if and only if G satisfies the *automorphic inverse property*, that is,

$$\ominus (a \oplus b) = \ominus a \ominus b \tag{6}$$

for all $a, b \in G$ [70, Theorem 3.2].

Gyrogroups share algebraic properties with groups, and many of group-theoretic theorems are extended to the case of gyrogroups with the aid of gyroautomorphisms [61, 67, 70]. For example, the gyrogroup counterpart of the group identity $(g^{-1}h)(h^{-1}k) = g^{-1}k$ is described by

$$(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c, \tag{7}$$

and the gyrogroup counterpart of the group identity $(gh)^{-1} = h^{-1}g^{-1}$ is described by

$$\ominus (a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a) \tag{8}$$

for all elements a, b, c of a gyrogroup.

Let G be a gyrogroup. To solve linear equations in G, Ungar introduced the *gyrogroup cooperation*, $\boxplus$, defined by

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b, \qquad a, b \in G. \tag{9}$$

Like groups, every linear equation in G possesses a unique solution in G.

Theorem 2 ([70]). *Let G be a gyrogroup and let $a, b \in G$. The unique solution to the equation $a \oplus x = b$ in the variable x is $x = \ominus a \oplus b$, and the unique solution to the equation $x \oplus a = b$ in the variable x is $x = b \boxplus (\ominus a)$.*

The following cancellation laws in gyrogroups are derived as a consequence of Theorem 2:

Theorem 3. *Let G be a gyrogroup. For all $a, b, c \in G$,*

1. *$a \oplus b = a \oplus c$ implies $b = c$;* (general left cancellation law)
2. *$\ominus a \oplus (a \oplus b) = b$;* (left cancellation law)
3. *$(b \ominus a) \boxplus a = b$;* (right cancellation law I)
4. *$(b \boxplus (\ominus a)) \oplus a = b$.* (right cancellation law II)

By Theorem 3, if a is an element of G, then left and right *gyrotranslations* by a,

$$L_a\colon x \mapsto a \oplus x \quad \text{and} \quad R_a\colon x \mapsto x \oplus a, \tag{10}$$

are permutations of G. Furthermore, one has the composition law

$$L_a \circ L_b = L_{a \oplus b} \circ \mathrm{gyr}[a, b] \tag{11}$$

for all $a, b \in G$. Accordingly, gyrogroups are special loops.

Quasigroups and Loops

This subsection gives a very brief account of quasigroups and loops. For complete accounts, the reader is referred to [5, 36, 51]. A magma $(L, \cdot)$ is called a *quasigroup* if for each $a \in L$, left multiplication by a, $L_a\colon x \mapsto a \cdot x$, and right multiplication by a, $R_a\colon x \mapsto x \cdot a$, are permutations of L. Equivalently, L is a quasigroup if and only if equations $a \cdot x = b$ and $x \cdot a = b$ for the unknown x have unique solutions in L for all $a, b \in L$. A magma L is said to be *uniquely 2-divisible* if the squaring map $x \mapsto x \cdot x$ is a bijection from L to itself. A quasigroup that has an identity element is called a *loop*.

Suppose that L is a loop and let a and b be arbitrary elements of L. The *left inner mapping* or *precession map generated by a and b* is defined by

$$\ell(a, b) = L_{a \cdot b}^{-1} \circ L_a \circ L_b. \tag{12}$$

We say that a loop L:

- Has the A_ℓ*-property* if $\ell(a, b)$ is an automorphism of L for all $a, b \in L$.
- Has the *automorphic inverse property* if every element of L has a unique inverse and

$$(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$$

for all $a, b \in L$.

- Is a *left Bol loop* if

$$a \cdot (b \cdot (a \cdot c)) = (a \cdot (b \cdot a)) \cdot c \qquad \text{(left Bol identity)}$$

for all $a, b, c \in L$.

- Is a *right Bol loop* if

$$((a \cdot b) \cdot c) \cdot b = a \cdot ((b \cdot c) \cdot b) \qquad \text{(right Bol identity)}$$

for all $a, b, c \in L$.

- Is a *K-loop* or *Bruck loop* if it is a left Bol loop satisfying the automorphic inverse property.
- Is a *B-loop* if it is a uniquely 2-divisible K-loop.
- Is a *Moufang loop* if

$$(a \cdot b) \cdot (c \cdot a) = (a \cdot (b \cdot c)) \cdot a \qquad \text{(Moufang identity)}$$

for all $a, b, c \in L$.

It follows from (10) and (11) that every gyrogroup is a loop with the A_ℓ-property, where the gyroautomorphisms correspond to left inner mappings. Sabinin et al. [55] showed that the left loop property (G4) is equivalent to the left Bol identity, so every gyrogroup is a left Bol-A_ℓ-loop under the same operation and vice versa. From this the results involving gyrogroups can be recast in the framework of left Bol loops with the A_ℓ-property. By (6), every gyrocommutative gyrogroup is a K-loop. In fact, it is known in the loop literature that gyrocommutative gyrogroups and K-loops are equivalent. The term "K-loop" was coined by Ungar in 1989 [64], as explained in [36, pp. 169–170], and the early history of K-loops is described in [56, pp. 141–142]. Table 1 summarizes transitions between terminology in gyrogroup theory and loop theory.

We end this section with the duality between left Bol loops and right Bol loops. Given a loop L with multiplication $\cdot$, the *dual loop* of L consists of the underlying set L with the dual operation $a * b := b \cdot a$ for $a, b \in L$. Note that L and the dual

Table 1 Terminology in gyrogroup theory and loop theory

Gyrogroup theory	Loop theory
Gyrogroup	Left Bol loop with the A_ℓ-property
Gyrocommutative gyrogroup	K-loop, Bruck loop
Uniquely 2-divisible gyrocommutative gyrogroup	B-loop
Gyroautomorphism	Left inner mapping, precession map

of L share the same identity. Furthermore, b is an inverse of an element a of L with respect to $\cdot$ if and only if b is an inverse of a with respect to $*$. Note also that the double dual of L is itself. It is routine to check that L is a right Bol loop if and only if the dual of L is a left Bol loop and that L has the automorphic inverse property if and only if the dual of L has the automorphic inverse property. Actually, left Bol and right Bol loops share the same algebraic properties, and a left Bol loop can be obtained from a right Bol loop by taking the dual of the right Bol loop. In what follows, by Bol loops we mean left Bol or right Bol loops.

3 Quadratic Spaces and Clifford Algebras

In this section, we review the basic theory of quadratic spaces and Clifford algebras. For complete accounts of the theory, we refer the reader to [10, 27, 31, 47]. As we will see in the next section, Clifford algebras prove useful in the study of Möbius and Einstein gyrogroups.

3.1 Quadratic Spaces

Let V be a vector space over a field $\mathbb{F}$ of characteristic different from 2. A *quadratic form* on V is a map $Q\colon V \to \mathbb{F}$ such that:

1. $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in \mathbb{F}$, $v \in V$, and
2. the map $B\colon V \times V \to \mathbb{F}$ defined by

$$B(u, v) = \frac{1}{2}\Big(Q(u + v) - Q(u) - Q(v)\Big)$$

   is a symmetric bilinear form on V.

Note that any symmetric bilinear form B on V gives rise to a quadratic form Q by defining

$$Q(v) = B(v, v), \qquad v \in V. \tag{13}$$

Thus, if the characteristic of $\mathbb{F}$ is not equal to 2, then the notions of quadratic forms and symmetric bilinear forms are equivalent.

A symmetric bilinear form B on V is said to be *nondegenerate* if for each $u \in V$, $B(u, v) = 0$ for all $v \in V$ implies $u = 0$. A *quadratic space* is a vector space together with a quadratic form on which the associated bilinear form is nondegenerate. Let (V, Q) be a quadratic space with the corresponding bilinear form B. By $u \perp v$ we mean $B(u, v) = 0$ and by $v \perp V$ we mean $v \perp w$ for all $w \in V$. In the case where V is of finite dimension, a basis $\{e_1, e_2, \ldots, e_n\}$ of V is said to be *orthogonal* if $e_i \perp e_j$ for $i \neq j$. Let $\{v_1, v_2 \ldots, v_n\}$ be a basis for V. Set $b_{ij} = B(v_i, v_j)$ for $1 \leq i, j \leq n$ and define the matrix of B relative to $\{v_i\}$ as $\widehat{B} := [b_{ij}]$. It is proved in [31, Chap. 2] that any two representing matrices of B have the same rank. Hence, the rank of B, written $\operatorname{rank}(B)$, can be defined as the rank of a representing matrix of B. The following theorem guarantees the existence of an orthogonal basis:

Theorem 4 ([31]). *If B is a symmetric bilinear form on a finite-dimensional vector space V, then V has an orthogonal basis $\{e_1, e_2, \ldots, e_n\}$, relative to which B has diagonal representing matrix*

$$\begin{bmatrix} b_1 & & & & & \\ & \ddots & & & 0 & \\ & & b_r & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

with all $b_i \neq 0, r = \operatorname{rank}(B)$, and $\{e_{r+1}, \ldots, e_n\}$ a basis for $\operatorname{rad}(V) = \{v \in V \colon v \perp V\}$.

Note in particular that if V is a finite-dimensional quadratic space, then V has an orthogonal basis with respect to its symmetric bilinear form. In this case, $\operatorname{rad}(V)$ is trivial, whence $r = n$ and $Q(e_i) = b_i \neq 0$ for $i = 1, 2, \ldots, n$.

3.2 Clifford Algebras

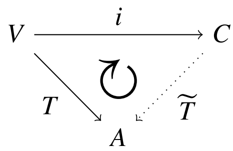
A Clifford algebra is defined for a vector space with a symmetric bilinear form. It contains both the underlying vector space and the base field. Any rotation of the vector space can be expressed in a simple form by using Clifford algebra operations. Further, conjugation and norm remain valid in the Clifford algebra. Clifford algebras have a wide application both in mathematics and in physics [27, 28, 32, 47].

There are several definitions of a Clifford algebra such as Clifford's original definition, definition by generators and relations, definition by the universal property, and so on [47, Chap. 14]. We follow [31] for the formal definition of a Clifford algebra by the universal property.

Let (V, Q) be a quadratic space over a field $\mathbb{F}$ of characteristic different from 2. An associative $\mathbb{F}$-algebra A with unity 1_A is said to be *compatible* with Q if there is a linear transformation $T \colon V \to A$ such that

$$(Tv)^2 = Q(v)1_A, \qquad v \in V. \tag{14}$$

Definition 3 (Clifford Algebras). A Clifford algebra for a quadratic space (V, Q) is a pair (C, i) of a unital associative algebra C and a linear transformation $i: V \to C$ such that C is compatible with Q via i and such that the following universal property is satisfied. Given a unital associative algebra A compatible with Q via a linear transformation $T: V \to A$, there exists a unique algebra homomorphism $\widetilde{T}: C \to A$ making the following diagram commutative:



A Clifford algebra for a quadratic space (V, Q) is unique up to an $\mathbb{F}$-algebra isomorphism and always exists. Set $V_0 = \mathbb{F}$ and let V_k denote the k-fold tensor product of V, that is, $V_k = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ copies}}$. Let $T(V)$ denote the direct sum of $V_0, V_1, V_2, \ldots$, that is,

$$T(V) = \bigoplus_{k=0}^{\infty} V_k.$$

In [31], Grove showed that $T(V)$ forms a unital associative algebra, called the *tensor algebra of* V. This algebra is used to construct a Clifford algebra for V.

Theorem 5 ([31]). *Suppose that* (V, Q) *is a quadratic space. Let* $T(V)$ *be the tensor algebra of* V *and* I *the two-sided ideal of* $T(V)$ *generated by the set*

$$\{v \otimes v - Q(v)1_{T(V)}: v \in V\}.$$

Then the quotient algebra $T(V)/I$ *along with the canonical projection* $v \mapsto v + I$, $v \in V$, *forms a Clifford algebra for* V.

Accordingly, it makes sense to speak of *the* Clifford algebra of (V, Q), which will be denoted by $C\ell(V, Q)$. The underlying vector space V is naturally embedded into its Clifford algebra by $v \mapsto v + I$. Henceforth, we make the identification $v \leftrightarrow v + I$ and regard V as a subspace of $C\ell(V, Q)$. By compatibility (14), one has the following fundamental relations in $C\ell(V, Q)$:

$$uv + vu = 2B(u, v)1 \quad \text{and} \quad v^2 = Q(v)1 \tag{15}$$

for all $u, v \in V$. Here, 1 stands for the unity of $C\ell(V, Q)$.

3.2.1 Bases and Dimension

Given a finite-dimensional quadratic space (V, Q), one can find an orthogonal basis $\{e_1, \ldots, e_n\}$ for V (Theorem 4). This basis is used to construct a natural basis of the Clifford algebra of V. For convenience, we use the following notation: for $I = \{i_1, i_2, \ldots, i_k \in \mathbb{N}: k \in \mathbb{N}, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$, define $e_I = e_{i_1}e_{i_2}\cdots e_{i_k}$, where the product on the right-hand side is taken in $\mathrm{C}\ell\,(V, Q)$. We also define $e_\emptyset = 1$, the unity of $\mathrm{C}\ell\,(V, Q)$. The following theorem is based on relations $e_i^2 = Q(e_i)1$ and $e_ie_j = -e_je_i$ for $i \neq j$ in $\mathrm{C}\ell\,(V, Q)$:

Theorem 6 ([31]). *If $\{e_1, e_2, \ldots, e_n\}$ is an orthogonal basis for a quadratic space (V, Q), then the set*

$$\{e_I: I = \emptyset \text{ or } I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}\}$$

is a spanning set for $\mathrm{C}\ell\,(V, Q)$ as a vector space. Thus, $\dim \mathrm{C}\ell\,(V, Q) \leq 2^n$.

The dimension of $\mathrm{C}\ell\,(V, Q)$ is indeed 2^n [31, Theorem 8.12], so the spanning set given in Theorem 6 becomes a basis for $\mathrm{C}\ell\,(V, Q)$ as a vector space. We remark that the *finiteness* of the dimension of V is a crucial hypothesis.

3.2.2 Three Standard Maps of $\mathrm{C}\ell\,(V, Q)$

Let A and B be unital associative algebras. A map $\varphi: A \to B$ is called an *algebra antihomomorphism* if φ is linear and preserves unity, and $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in A$. A bijective algebra antihomomorphism from A to itself is called an *algebra antiautomorphism* of A. A map $\varphi: X \to X$, where X is a nonempty set, is said to be *involutive* if $\varphi^2 := \varphi \circ \varphi = \mathrm{id}_X$.

There are three standard maps of the Clifford algebra of a quadratic space. One is an involutive algebra automorphism, and the others are involutive algebra antiautomorphisms. The proof of the following two theorems relies on the universal property of a Clifford algebra; see, for instance, [31]:

Theorem 7. *There is a unique algebra antiautomorphism of $\mathrm{C}\ell\,(V, Q)$, denoted by ρ, such that $\rho(v) = v$ for every $v \in V$ and $\rho^2 = \mathrm{id}_{\mathrm{C}\ell\,(V,Q)}$. In other words, ρ is an involutive algebra antiautomorphism of $\mathrm{C}\ell\,(V, Q)$.*

The map ρ is referred to as the *reversion* of $\mathrm{C}\ell\,(V, Q)$. The image of an element a of $\mathrm{C}\ell\,(V, Q)$ under ρ will be sometimes denoted by $\tilde{a}$ for convenience. In this notation, Theorem 7 says that

$$\tilde{v} = v, \qquad \widetilde{ab} = \tilde{b}\tilde{a}, \qquad \tilde{\tilde{a}} = a \tag{16}$$

for all $v \in V$ and $a, b \in \mathrm{C}\ell\,(V, Q)$.

Theorem 8. *There is a unique algebra automorphism of* $\mathrm{C}\ell(V, Q)$*, denoted by* τ*, such that* $\tau(v) = -v$ *for every* $v \in V$ *and* $\tau^2 = \mathrm{id}_{\mathrm{C}\ell(V,Q)}$*. In other words,* τ *is an involutive algebra automorphism of* $\mathrm{C}\ell(V, Q)$*.*

The map τ is referred to as the *grade involution* of $\mathrm{C}\ell(V, Q)$. The image of an element a of $\mathrm{C}\ell(V, Q)$ under τ will be sometimes denoted by $\hat{a}$. In this notation, Theorem 8 says that

$$\hat{v} = -v, \qquad \widehat{ab} = \hat{a}\hat{b}, \qquad \hat{\hat{a}} = a \tag{17}$$

for all $v \in V$ and $a, b \in \mathrm{C}\ell(V, Q)$.

The composite $\kappa = \rho \circ \tau$ is an algebra antiautomorphism of $\mathrm{C}\ell(V, Q)$. Hence, κ^2 is an algebra automorphism of $\mathrm{C}\ell(V, Q)$ that leaves V fixed pointwise. It follows from the universal property that κ^2 must be the identity map, and we have the following proposition:

Proposition 1. κ *is an algebra antiautomorphism of* $\mathrm{C}\ell(V, Q)$ *such that* $\kappa(v) = -v$ *for every* $v \in V$ *and* $\kappa^2 = \mathrm{id}_{\mathrm{C}\ell(V,Q)}$*. In other words,* κ *is an involutive algebra antiautomorphism of* $\mathrm{C}\ell(V, Q)$*.*

Corollary 1. *The maps* ρ *and* τ *commute, that is,* $\rho \circ \tau = \tau \circ \rho$*.*

Proof. $\rho \circ \tau = \kappa = \kappa^{-1} = (\rho \circ \tau)^{-1} = \tau^{-1} \circ \rho^{-1} = \tau \circ \rho$. □

Corollary 2. *Any two maps of* $\rho, \tau,$ *and* κ *commute.*

Proof. A direct computation gives $\rho \circ \kappa = \rho \circ (\rho \circ \tau) = \tau$ and $\kappa \circ \rho = (\tau \circ \rho) \circ \rho = \tau$. Hence, $\rho \circ \kappa = \kappa \circ \rho$. One obtains similarly that $\tau \circ \kappa = \rho = \kappa \circ \tau$. □

The map κ is referred to as the *Clifford conjugation* or just *conjugation*. The image of an element a of $\mathrm{C}\ell(V, Q)$ under κ will be sometimes denoted by $\bar{a}$. In this notation, Proposition 1 and Corollary 1 say that

$$\bar{v} = -v, \qquad \overline{ab} = \bar{b}\bar{a}, \qquad \bar{\bar{a}} = a, \qquad \bar{a} = \hat{\tilde{a}} = \tilde{\hat{a}} \tag{18}$$

for all $v \in V$ and $a, b \in \mathrm{C}\ell(V, Q)$.

Table 2 lists basic properties of three standard maps of a Clifford algebra.

Table 2 Three standard maps of $\mathrm{C}\ell(V, Q)$

Map	Type	On V
Reversion $\rho(a) = \tilde{a}$	Antiautomorphism	id_V
Grade involution $\tau(a) = \hat{a}$	Automorphism	$-\mathrm{id}_V$
Clifford conjugation $\kappa(a) = \bar{a}$	Antiautomorphism	$-\mathrm{id}_V$

3.2.3 Clifford–Lipschitz Groups

In this subsection, we introduce an important subgroup of the group $C\ell^{\times}(V, Q)$ of units of $C\ell(V, Q)$. This subgroup is known as the *Clifford group* or *Lipschitz group* and is used for studying orthogonal transformations of a quadratic space.

The Clifford group of $C\ell(V, Q)$, denoted by $\Gamma(V, Q)$, is defined via the grade involution as

$$\Gamma(V, Q) = \{g \in C\ell^{\times}(V, Q) \colon \forall v \in V,\ \hat{g}vg^{-1} \in V\}. \tag{19}$$

Each element g of $\Gamma(V, Q)$ induces an injective linear transformation $T_g \colon v \mapsto \hat{g}vg^{-1}$, which will become a linear automorphism of V if V is finite dimensional. The map π that sends g to T_g defines a group homomorphism from $\Gamma(V, Q)$ to $GL(V)$, so π is a linear representation of $\Gamma(V, Q)$, known as the *twisted adjoint representation*. In addition, the grade involution descends to a group automorphism of $\Gamma(V, Q)$, while the reversion and Clifford conjugation descend to group antiautomorphisms of $\Gamma(V, Q)$. Henceforth, we assume that (V, Q) is a finite-dimensional quadratic space unless otherwise stated.

Proposition 2. *For each $g \in \Gamma(V, Q)$, the map T_g defined by*

$$T_g v = \hat{g}vg^{-1}, \qquad v \in V,$$

is a linear automorphism of V.

Proof. T_g is linear since multiplication of $C\ell(V, Q)$ is bilinear. Since g is invertible and the grade involution is an algebra automorphism, $\hat{g}$ is invertible. It follows that $T_g v = \hat{g}vg^{-1} = 0$ implies $v = 0$, which proves that T_g is injective. Since $\dim V < \infty$, T_g is also surjective and hence is a linear automorphism of V. □

Proposition 3. *$\Gamma(V, Q)$ is a subgroup of the group of units of the Clifford algebra.*

Proof. For all $g, h \in \Gamma(V, Q), v \in V$, we have $\widehat{gh}v(gh)^{-1} = \hat{g}(\hat{h}vh^{-1})g^{-1} \in V$ since $\hat{h}vh^{-1} \in V$. This proves $gh \in \Gamma(V, Q)$.

Let $g \in \Gamma(V, Q)$ and $v \in V$. As T_g is surjective, $T_g w = v$ for some $w \in V$. Hence, $v = \hat{g}wg^{-1}$. It follows that $\widehat{g^{-1}}v(g^{-1})^{-1} = (\hat{g})^{-1}vg = w \in V$. Since v is arbitrary, $g^{-1} \in \Gamma(V, Q)$. □

Proposition 4. *Let $\Gamma(V, Q)$ be the Clifford group of* $C\ell(V, Q)$. *Then $T_1 = \mathrm{id}_V$ and $T_g \circ T_h = T_{gh}$ for all $g, h \in \Gamma(V, Q)$.*

Proof. This follows directly from the definition of $\Gamma(V, Q)$. □

Corollary 3. *The map $\pi \colon g \mapsto T_g$ is a group homomorphism from $\Gamma(V, Q)$ to* $GL(V)$. *In other words, π is a linear representation of $\Gamma(V, Q)$.*

Proposition 5. *The Clifford group is invariant under the reversion, grade involution, and Clifford conjugation.*

Proof. Let $g \in \Gamma(V, Q)$. For each $v \in V$, we have

$$\begin{aligned}\tau(\tau(g))v\tau(g)^{-1} &= \tau(\tau(g))\tau(-v)\tau(g^{-1})\\ &= \tau(\tau(g)(-v)g^{-1})\\ &= -(\tau(g)(-v)g^{-1})\\ &= \tau(g)vg^{-1}.\end{aligned}$$

We obtain the third equation since $\tau(g)(-v)g^{-1} \in V$ and $\tau(w) = -w$ if $w \in V$. This proves that $\tau(g) \in \Gamma(V, Q)$. Hence, $\tau(\Gamma(V, Q)) \subseteq \Gamma(V, Q)$. We apply the same arguments again, with ρ in place of τ, to obtain $\rho(\Gamma(V, Q)) \subseteq \Gamma(V, Q)$. We also have $\kappa(\Gamma(V, Q)) \subseteq \Gamma(V, Q)$ since $\kappa = \rho \circ \tau$. □

The following theorem says that the grade involution descends to a group automorphism of $\Gamma(V, Q)$ and that the reversion and Clifford conjugation descend to group antiautomorphisms of $\Gamma(V, Q)$:

Theorem 9. *The restriction of τ to $\Gamma(V, Q)$ is an involutive group automorphism. The restriction of ρ to $\Gamma(V, Q)$ is an involutive group antiautomorphism.*

Proof. By Proposition 5, the restriction $\tau\big|_{\Gamma(V,Q)}$ is an injective group homomorphism of $\Gamma(V, Q)$. For each $h \in \Gamma(V, Q)$, $\tau(h) \in \Gamma(V, Q)$ and $\tau(\tau(h)) = h$. Hence, the restriction is surjective. That $\tau\big|_{\Gamma(V,Q)}$ is involutive is clear. The same reasoning applies to the case of $\rho\big|_{\Gamma(V,Q)}$. □

Corollary 4. *The restriction of κ to $\Gamma(V, Q)$ is an involutive group antiautomorphism.*

Proof. From Proposition 5, we have $\kappa\big|_{\Gamma(V,Q)} = \rho\big|_{\Gamma(V,Q)} \circ \tau\big|_{\Gamma(V,Q)}$. □

Next, we will see that the twisted adjoint representation maps the Clifford group onto the orthogonal group of the quadratic space. Let B be the associated symmetric bilinear form of (V, Q). The *orthogonal group*, $\mathrm{O}(V, B)$, consists of all the linear automorphisms of V that preserve B:

$$\mathrm{O}(V, B) = \{T \in \mathrm{GL}(V) \colon \forall u, v \in V,\ B(Tu, Tv) = B(u, v)\}. \tag{20}$$

If $T \in \mathrm{O}(V, B)$, then $Q(Tv) = B(Tv, Tv) = B(v, v) = Q(v)$ for all $v \in V$. Conversely, if T is a linear automorphism of V and if $Q(Tv) = Q(v)$ for all $v \in V$, then the definition of B and the linearity of T together imply $B(Tu, Tv) = B(u, v)$ for all $u, v \in V$. This proves that

$$\mathrm{O}(V, B) = \{T \in \mathrm{GL}(V) \colon \forall v \in V,\ Q(Tv) = Q(v)\}. \tag{21}$$

Note that the grade involution of $C\ell(V, Q)$ fixes $\mathbb{F}1 := \{\lambda 1 \colon \lambda \in \mathbb{F}\}$ pointwise. In view of the twisted adjoint representation π, any nonzero scalar multiple of unity acts trivially on V. In other words, $\mathbb{F}^{\times}1 \subseteq \ker \pi$. The following proposition shows that the kernel of the twisted adjoint representation is indeed $\mathbb{F}^{\times}1$; its proof can be found in [46, p. 14]:

Proposition 6. *In a finite-dimensional quadratic space, the kernel of the twisted adjoint representation is* $\mathbb{F}^{\times}1 = \{\lambda 1 : \lambda \in \mathbb{F}, \lambda \neq 0\}$.

The Clifford conjugation is used to define a group homomorphism of $\Gamma(V, Q)$, which is an analog of the complex modulus. Let η be the map defined by

$$\eta(a) = a\bar{a}, \qquad a \in \mathrm{C}\ell(V, Q). \tag{22}$$

Note that $\eta(\Gamma(V, Q)) \subseteq \Gamma(V, Q)$ because $\Gamma(V, Q)$ is invariant under the Clifford conjugation. Note also that if v is a vector in V, then, by (15),

$$\eta(v) = v\bar{v} = -v^2 = -Q(v)1. \tag{23}$$

Proposition 7. η *maps the Clifford group to* $\mathbb{F}^{\times}1$*, that is,* $\eta(\Gamma(V, Q)) \subseteq \mathbb{F}^{\times}1$.

Proof. The key idea of the proof is to show that if g is in $\Gamma(V, Q)$, then π sends $\eta(g)$ to the identity transformation of V. Thus, $\eta(g)$ belongs to $\ker \pi = \mathbb{F}^{\times}1$. Let $g \in \Gamma(V, Q)$. For each $v \in V$, $\tilde{g}v\bar{g}^{-1} = \hat{\bar{g}}v\bar{g}^{-1} \in V$ since $\bar{g} \in \Gamma(V, Q)$. Hence

$$\tilde{g}v\bar{g}^{-1} = \rho(\hat{\bar{g}}v\bar{g}^{-1}) = \rho(\bar{g})^{-1}\rho(v)\rho(\tilde{g}) = \hat{g}^{-1}vg.$$

It follows that $\tau(\eta(g))v\eta(g)^{-1} = \tau(g\bar{g})v(g\bar{g})^{-1} = \hat{g}\hat{\bar{g}}v\bar{g}^{-1}g^{-1} = \hat{g}(\tilde{g}v\bar{g}^{-1})g^{-1} = v$, whence $\pi(\eta(g)) = T_{\eta(g)} = \mathrm{id}_V$. □

In the case where the base field $\mathbb{F}$ is an ordered field in which every positive element has a square root (e.g., $\mathbb{F} = \mathbb{R}$), we adopt the following notation:

$$|a| = \sqrt{\eta(a)} = \sqrt{a\bar{a}} \tag{24}$$

whenever $\eta(a)$ is in $\mathbb{F}1$ and $\eta(a) \geq 0$.

Let (V, Q) be a quadratic space. A nonzero vector v in V is said to be *nonisotropic* or *anisotropic* if $Q(v) \neq 0$. A nonzero vector v in V is *isotropic* if it is not nonisotropic, that is, $Q(v) = 0$. Since $\eta(v) = -Q(v)1$ for all $v \in V$, a nonzero vector v is nonisotropic if and only if $\eta(v) \neq 0$.

Proposition 8. *A nonzero vector* v *in* V *is nonisotropic if and only if* $v \in \Gamma(V, Q)$.

Proof. If $Q(v) \neq 0$, then $v\left(\dfrac{v}{Q(v)}\right) = \dfrac{v^2}{Q(v)} = 1 = \left(\dfrac{v}{Q(v)}\right)v$. Hence, $v/Q(v)$ is an inverse of v and so $v \in \mathrm{C}\ell^{\times}(V, Q)$.

From (15), we have

$$-vuv^{-1} = u - (uv + vu)v^{-1} = u - (2B(u, v)1)\left(\frac{v}{Q(v)}\right) = u - \frac{2B(u, v)}{Q(v)}v,$$

which implies $\hat{v}uv^{-1} = -vuv^{-1} \in V$ for all $u \in V$. This proves $v \in \Gamma(V, Q)$.

Conversely, if $v \in \Gamma(V, Q)$, then v^{-1} exists. This implies $Q(v) \neq 0$; otherwise, we would have $v^2 = Q(v)1 = 0$ and have $v = v^2(v^{-1}) = 0$, a contradiction. □

In light of the proof of Proposition 8, if v is a nonisotropic vector, then T_v is nothing but the *reflection about the hyperplane orthogonal to* v, given by

$$T_v(u) = u - \frac{2B(u, v)}{Q(v)} v, \qquad u \in V. \tag{25}$$

Note that $T_v(v) = -v$ and if $u \perp v$, then $T_v(u) = u$.

Proposition 9. *The restriction* $\eta\colon \Gamma(V, Q) \to \mathbb{F}^{\times}1$ *is a group homomorphism. The map* η *is multiplicative over the set of products of vectors in* V*:*

$$\eta(v_1 v_2 \cdots v_k) = \eta(v_1)\eta(v_2)\cdots\eta(v_k)$$

for all $v_1, v_2, \ldots, v_k \in V$.

Proof. The first statement is a consequence of Proposition 7. For $v_1, v_2, \ldots, v_k \in V$, we compute

$$\eta(v_1 v_2 \cdots v_k) = v_1 v_2 \cdots v_k \overline{v_1 v_2 \cdots v_k} = (v_1 v_2 \cdots v_k)(\overline{v_k}\,\overline{v_{k-1}} \cdots \overline{v_1}) = \prod_{i=1}^{k} \eta(v_i).$$

We obtain the last equation because $v\bar{v} = \eta(v) \in \mathbb{F}1$ for all $v \in V$. □

Note that $\eta(\hat{g}) = \eta(g)$ and $\eta(\bar{g}) = \eta(\tilde{g})$ for all $g \in \Gamma(V, Q)$. Also, if $\alpha \in \mathbb{F}^{\times}1$ and $g \in \Gamma(V, Q)$, then $\eta(\alpha g) = \alpha^2\eta(g)$. This implies that

$$\eta(\bar{g}) = \eta(\eta(g)g^{-1}) = \eta(g)^2\eta(g)^{-1} = \eta(g).$$

We have proved the following proposition:

Proposition 10. *For each* $g \in \Gamma(V, Q)$, $\eta(\hat{g}) = \eta(\bar{g}) = \eta(\tilde{g}) = \eta(g)$.

Corollary 5. *If* $g \in \Gamma(V, Q)$*, then* $g^{-1} = \dfrac{\bar{g}}{\eta(g)}$.

Proof. This is because $g\left(\dfrac{\bar{g}}{\eta(g)}\right) = 1$ and $\left(\dfrac{\bar{g}}{\eta(g)}\right) g = \dfrac{\eta(\bar{g})}{\eta(g)} = 1$. □

To show that Einstein gyroautomorphisms represent rotations of the open unit ball in a real inner product space, we will make use of the following theorem:

Theorem 10. *The twisted adjoint representation maps the Clifford group to the orthogonal group of* V*. In particular,* T_g *is an orthogonal transformation of* V *for all* $g \in \Gamma(V, Q)$.

Proof. Suppose that $g \in \Gamma(V, Q)$ and let $v \in V$. We will show that $Q(T_g v) = Q(v)$. This is clear if $v = 0$, so we may assume that $v \neq 0$.

In the case $Q(v) \neq 0$, v belongs to $\Gamma(V, Q)$ by Proposition 8. It follows that

$$Q(T_g v)1 = Q(\hat{g}vg^{-1})1 = -\eta(\hat{g}vg^{-1}) = -\eta(g)\eta(v)\eta(g)^{-1} = -\eta(v) = Q(v)1,$$

whence $Q(T_g v) = Q(v)$. We obtain the second equation since $\hat{g}vg^{-1} \in V$, the third equation since η is a group homomorphism, and the fourth equation since $\eta(g) \in \mathbb{F}^{\times}1$.

In the case $Q(v) = 0$, we will prove the contrapositive: $Q(T_g v) \neq 0 \Rightarrow Q(v) \neq 0$. Suppose that $Q(T_g v) \neq 0$. Thus, $T_g v \neq 0$ and hence $T_g v$ is a nonisotropic vector in V. This implies that $\hat{g}vg^{-1} = T_g v$ belongs to $\Gamma(V, Q)$, say $\hat{g}vg^{-1} = h$. It follows that $v = \hat{g}^{-1}hg \in \Gamma(V, Q)$, so $Q(v) \neq 0$. This completes the proof. □

By the Cartan–Dieudonné theorem (see Theorem 2.7 in [46]), any orthogonal transformation T of V can be expressed as a product of reflections

$$T = T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_k},$$

where $v_1, v_2, \ldots, v_k$ are nonisotropic vectors and $k \leq \dim V$. Thus, $\pi(v_1 v_2 \cdots v_k) = T$ and so the twisted adjoint representation is surjective.

3.2.4 Elements of the Form $1 + uv$

In this subsection, we provide a necessary and sufficient condition for invertibility of elements of the form $1 + uv$, where u and v are vectors in a quadratic space. Let (V, Q) be a quadratic space with the corresponding bilinear form B. From now on, the term "vector" is reserved for the elements of V.

Lemma 1. *If u, v and w are vectors, then so are uvu and $uvw + wvu$.*

Proof. According to (15), if $u, v \in V$, then $uv + vu = 2B(u, v)1$. A direct computation gives $uvu = [2B(u, v)1 - vu]u = 2B(u, v)u - vu^2 = 2B(u, v)u - Q(u)v$, which implies $uvu \in V$. Similarly, one can check that

$$uvw + wvu = 2B(u, v)w + 2B(w, v)u - 2B(w, u)v,$$

so $uvw + wvu \in V$. □

Proposition 11. *If u and v are vectors, then either:*

1. *$1 + uv$ is a product of vectors or*
2. *$1 + uv$ belongs to $\Gamma(V, Q)$ and $\eta(1 + uv) = 1$.*

Proof. Recall that if w is a nonisotropic vector, then w is invertible, and $w^{-1} = w/Q(w)$ is again a vector.

Case 1. u is nonisotropic. Then $1 + uv = u(u^{-1} + v)$ is a product of vectors.
Case 2. u is isotropic. In this case, $Q(u) = 0$:

Subcase 2.1. v is nonisotropic. Then $1 + uv = (v^{-1} + u)v$ is a product of vectors.

Subcase 2.2. v is isotropic. If $u \not\perp v$, then $Q(u + v) = 2B(u, v) \neq 0$. Hence, $u + v$ is nonisotropic and we have $(1 + uv)(u + v) = u + v + 2B(u, v)u$. This implies $1+uv = (u+v+2B(u, v)u)(u+v)^{-1}$ is a product of vectors. If $u \perp v$, then $\eta(1 + uv) = 1 + 2B(u, v)1 + Q(u)Q(v)1 = 1$. By the lemma,

$$\tau(1+uv)w(1+uv)^{-1} = (1+\hat{u}\hat{v})w(1+vu) = w+wvu+uvw+uvwvu$$

belongs to V for all $w \in V$. Hence, $1 + uv \in \Gamma(V, Q)$. □

Proposition 12. *For all $u, v \in V$, $1 + uv \in \Gamma(V, Q)$ if and only if $\eta(1 + uv) \neq 0$.*

Proof. ($\Rightarrow$) If $1+uv \in \Gamma(V, Q)$, then $\eta(1+uv) \in \mathbb{F}^{\times}1$ by Proposition 7. Hence, $\eta(1 + uv) \neq 0$.

($\Leftarrow$) Suppose that $\eta(1 + uv) \neq 0$. By Proposition 11, either $1 + uv$ already belongs to $\Gamma(V, Q)$ or $1 + uv$ is a product of vectors. In the latter case, $1 + uv = w_1w_2 \cdots w_k$ for some $w_1, w_2, \ldots, w_k \in V$. By Proposition 9,

$$0 \neq \eta(1 + uv) = \eta(w_1w_2 \cdots w_k) = \eta(w_1)\eta(w_2) \cdots \eta(w_k),$$

which implies that none of $\eta(w_i)$ are zeros. Thus, $w_1, w_2, \ldots, w_k$ are all nonisotropic vectors and hence $1 + uv$ belongs to $\Gamma(V, Q)$. □

3.2.5 Negative and Paravector Spaces

In this subsection, we develop the structure that we will use in the next section in connecting Möbius and Einstein gyrogroups on the open unit ball of a real inner product space.

Let V be an n-dimensional real inner product space with a positive inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. The *negative space* of V consists of the vector space V together with the nondegenerate symmetric bilinear form

$$B(\mathbf{u}, \mathbf{v}) = -\langle \mathbf{u}, \mathbf{v} \rangle, \qquad \mathbf{u}, \mathbf{v} \in V. \tag{26}$$

Its associated quadratic form is given by the equation

$$Q(\mathbf{v}) = -\|\mathbf{v}\|^2, \qquad \mathbf{v} \in V. \tag{27}$$

It is clear that (V, Q) forms a quadratic space and that B is *negative* definite.

For simplicity, let $C\ell_{0,n}$ denote the Clifford algebra of negative space of V, and let $\Gamma_{0,n}$ denote the Clifford group of $C\ell_{0,n}$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of V, that is, $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ for $i \neq j$ and $\|\mathbf{e}_i\| = 1$ for $i = 1, 2, \ldots, n$. From now on, we identify elements of $\mathbb{R}1$ with real numbers, $r1 \leftrightarrow r$ for $r \in \mathbb{R}$.

Proposition 13. *In the Clifford algebra* $C\ell_{0,n}$*, the following hold:*

1. $\mathbf{uv} + \mathbf{vu} = -2\langle \mathbf{u}, \mathbf{v} \rangle$ *for all* $\mathbf{u}, \mathbf{v} \in V$*;*
2. $\mathbf{v}^2 = -\|\mathbf{v}\|^2$ *for all* $\mathbf{v} \in V$*;*
3. $\mathbf{e}_i^2 = -1$*,* $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ *for* $1 \leq i, j \leq n$ *and* $i \neq j$*;*
4. $1 - \mathbf{uv} \in \Gamma_{0,n}$ *and*

$$(1 - \mathbf{uv})^{-1} = \frac{1 - \mathbf{vu}}{\eta(1 - \mathbf{uv})}$$

for all $\mathbf{u}, \mathbf{v} \in V$ *with* $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$*;*

5. $\eta(\mathbf{w}(1 - \mathbf{uv})^{-1}) = \dfrac{\eta(\mathbf{w})}{\eta(1 - \mathbf{uv})}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ *with* $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$.

Proof. Item 1 follows from (15) and (26). Setting $\mathbf{u} = \mathbf{v}$ in Item 1, we obtain Item 2. Since $\{\mathbf{e}_i\}$ is an orthonormal basis of V, we obtain Item 3.

The Cauchy–Schwarz inequality gives

$$\begin{aligned}
\eta(1 - \mathbf{uv}) &= (1 - \mathbf{uv})\overline{(1 - \mathbf{uv})} \\
&= 1 - (\mathbf{uv} + \mathbf{vu}) + \mathbf{uv}^2\mathbf{u} \\
&= 1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \\
&\geq 1 - 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \\
&= (1 - \|\mathbf{u}\|\|\mathbf{v}\|)^2.
\end{aligned}$$

It follows that if $\mathbf{u}, \mathbf{v} \in V$ with $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$, then $\eta(1 - \mathbf{uv}) > 0$, and hence $1 - \mathbf{uv}$ belongs to $\Gamma_{0,n}$ by Proposition 12. Since $1 - \mathbf{uv} \in \Gamma_{0,n}$, we have from Corollary 5 that $(1 - \mathbf{uv})^{-1} = \dfrac{\overline{1 - \mathbf{uv}}}{\eta(1 - \mathbf{uv})} = \dfrac{1 - \mathbf{vu}}{\eta(1 - \mathbf{uv})}$. This proves Item 4.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ with $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$. If $\mathbf{w} = \mathbf{0}$, equality holds trivially, so we may assume $\mathbf{w} \neq \mathbf{0}$. Hence, $\mathbf{w} \in \Gamma_{0,n}$. By Item 4, $1 - \mathbf{uv} \in \Gamma_{0,n}$, and so

$$\eta(\mathbf{w}(1 - \mathbf{uv})^{-1}) = \eta(\mathbf{w})\eta((1 - \mathbf{uv})^{-1}) = \frac{\eta(\mathbf{w})}{\eta(1 - \mathbf{uv})}$$

since η is a group homomorphism of $\Gamma_{0,n}$. This proves Item 5. □

In light of the proof of Proposition 13 (4), if $\mathbf{u}$ and $\mathbf{v}$ are vectors in V, then $\eta(1 - \mathbf{uv})$ represents a real number, and $\eta(1 - \mathbf{uv}) \geq 0$ so that the notation $|1 - \mathbf{uv}|$ is unambiguous. Note that $|\mathbf{v}| = \sqrt{\eta(\mathbf{v})} = \|\mathbf{v}\|$ for all vectors $\mathbf{v}$ in V.

Paravector Spaces

In the Clifford algebra $C\ell_{0,n}$, one can construct the subspace $W = \mathbb{R} \oplus V$ of paravectors, called the *paravector space* of V. A typical element of W is of the form $w = w_0 + \mathbf{w}$ with w_0 in $\mathbb{R}$ and $\mathbf{w}$ in V. By definition of η, if $w = w_0 + \mathbf{w}$, then

$$\eta(w) = w_0^2 + \|\mathbf{w}\|^2. \tag{28}$$

Thus, $\eta(w) \geq 0$ and the notation $|w|$ is unambiguous whenever w is a paravector in W. Further, if w is a nonzero paravector, then w is invertible and $w^{-1} = \dfrac{\bar{w}}{\eta(w)}$ since $\eta(w) = \eta(\bar{w})$.

Let $u = u_0 + \mathbf{u}$ and $v = v_0 + \mathbf{v}$ be paravectors. After a direct calculation, one finds

$$\frac{1}{2}\Big(\eta(u+v) - \eta(u) - \eta(v)\Big) = u_0v_0 + \langle \mathbf{u}, \mathbf{v}\rangle. \tag{29}$$

Hence, $B(u, v) = 1/2\,(\eta(u+v) - \eta(u) - \eta(v))$ defines a nondegenerate symmetric bilinear form on W, and (W, η) is a quadratic space. Indeed, W is a real inner product space with inner product defined by (29).

Proposition 14. *Let W be the paravector space of V. Then W forms a real inner product space with inner product*

$$\langle u, v\rangle = \frac{1}{2}\Big(u\bar{v} + v\bar{u}\Big), \qquad u, v \in W. \tag{30}$$

Proof. Write $u = u_0 + \mathbf{u}$ and $v = v_0 + \mathbf{v}$. A direct computation gives

$$\frac{1}{2}\Big(u\bar{v} + v\bar{u}\Big) = u_0v_0 + \langle \mathbf{u}, \mathbf{v}\rangle.$$

From this it is clear that (30) defines a positive definite inner product on W. □

Note that W is an extension of V in the sense that (30) reduces to the inner product of V if u and v are paravectors with a zero scalar component.

4 Möbius and Einstein Gyrogroups

In this section, we extend the results of Suksumran and Wiboonton [62] to an arbitrary real inner product space. Let V be an n-dimensional real inner product space with a positive definite inner product $\langle \cdot, \cdot\rangle$ and the corresponding norm $\|\cdot\|$. Let V_t denote the open t-ball of V:

$$V_t = \{\mathbf{v} \in V : \|\mathbf{v}\| < t\}.$$

In [71], Möbius addition (2) on the complex unit disk is extended to the t-ball of V:

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + \frac{2}{t^2}\langle \mathbf{u}, \mathbf{v}\rangle + \frac{1}{t^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{t^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 + \frac{2}{t^2}\langle \mathbf{u}, \mathbf{v}\rangle + \frac{1}{t^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}. \tag{31}$$

With the help of computer algebra, Ungar asserted that $(V_t, \oplus_M)$ forms a gyrocommutative gyrogroup, called a *Möbius gyrogroup*. Later, Lawson [44] and Ferreira and Ren [18] proved this result using the Clifford algebra formalism.

Let $\mathbb{B}_t$ denote the open ball of radius t in the paravector space $W = \mathbb{R} \oplus V$:

$$\mathbb{B}_t = \{w \in W : \|w\| < t\}.$$

In [18, Sect. 4], Ferreira and Ren established that

$$\frac{(1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^2}\|b\|^2)a + (1 - \frac{1}{t^2}\|a\|^2)b}{1 + \frac{2}{t^2}\langle a, b\rangle + \frac{1}{t^4}\|a\|^2\|b\|^2} = (a+b)\left(1 + \frac{\bar{a}b}{t^2}\right)^{-1} \tag{32}$$

for all $a, b \in \mathbb{B}_t$, where the product and inverse on the right-hand side are performed in the Clifford algebra $C\ell_{0,n}$. Further, they proved that $\mathbb{B}_t$ with operation defined by

$$a \oplus_M b = (a+b)\left(1 + \frac{\bar{a}b}{t^2}\right)^{-1} \tag{33}$$

forms a gyrocommutative gyrogroup.

Theorem 11 ([18]). *$\mathbb{B}_t$ with Möbius addition defined by (33) is a gyrocommutative gyrogroup. Its gyroautomorphisms are given by*

$$\mathrm{gyr}[a, b]c = qc\tilde{q}, \qquad q = \frac{t^2 + a\bar{b}}{|t^2 + a\bar{b}|}.$$

A Euclidean version of Theorem 11 was proved by Lawson in [44].

For convenience, we will consider the case $t = 1$. Let $\mathbb{B}$ denote the open unit ball of V, $\mathbb{B} = \{\mathbf{v} \in V : \|\mathbf{v}\| < 1\}$. Viewing any vector in $\mathbb{B}$ as a paravector in $\mathbb{B}_1$ with a zero scalar component, we have $\mathbb{B} \subseteq \mathbb{B}_1$ so that Eq. (31) reduces to

$$\mathbf{u} \oplus_M \mathbf{v} = (\mathbf{u} + \mathbf{v})(1 - \mathbf{u}\mathbf{v})^{-1} \tag{34}$$

since $\bar{\mathbf{v}} = -\mathbf{v}$ for all $\mathbf{v} \in V$. By Proposition 13 (5), Eq. (34) implies

$$\eta(\mathbf{u} \oplus_M \mathbf{v}) = \frac{\eta(\mathbf{u} + \mathbf{v})}{\eta(1 - \mathbf{u}\mathbf{v})} \tag{35}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$.

The next theorem shows that $(\mathbb{B}, \oplus_M)$ forms a *subgyrogroup* of $(\mathbb{B}_1, \oplus_M)$. For the precise definition of a subgyrogroup, see Sect. 6.

Theorem 12. *The Möbius gyrogroup* $(\mathbb{B}, \oplus_M)$ *forms a subgyrogroup of* $(\mathbb{B}_1, \oplus_M)$. *In particular,* $(\mathbb{B}, \oplus_M)$ *is a gyrocommutative gyrogroup.*

Proof. If $\mathbf{v} \in \mathbb{B}$, then $\ominus_M \mathbf{v} \in \mathbb{B}$ since $\ominus_M \mathbf{v} = -\mathbf{v}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. From (32), we have

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2},$$

so $\mathbf{u} \oplus_M \mathbf{v} \in V$. Note that $0 < (1+Q(\mathbf{u}))(1+Q(\mathbf{v})) = 1+Q(\mathbf{u})+Q(\mathbf{v})+Q(\mathbf{u})Q(\mathbf{v})$. Thus, $-Q(\mathbf{u}+\mathbf{v}) = -2B(\mathbf{u}, \mathbf{v}) - Q(\mathbf{u}) - Q(\mathbf{v}) < 1 - 2B(\mathbf{u}, \mathbf{v}) + Q(\mathbf{u})Q(\mathbf{v})$. It follows that $\eta(\mathbf{u} + \mathbf{v}) = -Q(\mathbf{u} + \mathbf{v}) < 1 - 2B(\mathbf{u}, \mathbf{v}) + Q(\mathbf{u})Q(\mathbf{v}) = \eta(1 - \mathbf{uv})$, whence

$$0 \leq \eta(\mathbf{u} \oplus_M \mathbf{v}) = \frac{\eta(\mathbf{u} + \mathbf{v})}{\eta(1 - \mathbf{uv})} < 1.$$

Since $\eta(\mathbf{u} \oplus_M \mathbf{v}) = \|\mathbf{u} \oplus_M \mathbf{v}\|^2$, we also have $\|\mathbf{u} \oplus_M \mathbf{v}\| < 1$. Hence, $\mathbf{u} \oplus_M \mathbf{v} \in \mathbb{B}$. □

In [69], Ungar extended Einstein addition of relativistically admissible velocities to the case of a real inner product space:

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{t^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{t^2}\frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u} \right\}, \quad \mathbf{u}, \mathbf{v} \in V_t, \tag{36}$$

where $\gamma_{\mathbf{u}}$ is the Lorentz factor or gamma factor given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{t^2}}}. \tag{37}$$

Ungar declared that $(V_t, \oplus_E)$ forms a gyrocommutative gyrogroup, called an *Einstein gyrogroup*, where the gyrogroup axioms can be checked using computer algebra. In this section, we will give an algebraic proof that $(\mathbb{B}, \oplus_E)$ does form a uniquely 2-divisible gyrocommutative gyrogroup, which is sometimes referred to as a B-loop. In order to do so, we employ the following theorem, which enables us to impose the gyrogroup structure on any set that has the same cardinality as a gyrogroup:

Theorem 13 ([16]). *Let* $(G, \oplus)$ *be a gyrogroup, let* X *be a set, and let* $\phi: X \to G$ *be a bijection. Then* X *endowed with the induced operation*

$$a \oplus_X b := \phi^{-1}(\phi(a) \oplus \phi(b)), \qquad a, b \in X,$$

becomes a gyrogroup.

According to the proof of this theorem in [16], the identity of the induced gyrogroup X is indeed $\phi^{-1}(0)$, and the inverse of an element a of X is indeed $\phi^{-1}(\ominus\phi(a))$. The induced gyroautomorphisms of X are given by

$$\mathrm{gyr}_X[a,b] = \phi^{-1} \circ \mathrm{gyr}[\phi(a), \phi(b)] \circ \phi. \tag{38}$$

By construction, $\phi(a \oplus_X b) = \phi(\phi^{-1}(\phi(a) \oplus \phi(b))) = \phi(a) \oplus \phi(b)$ for all $a, b \in X$. Hence, ϕ acts as a *gyrogroup isomorphism* so that G and X are *isomorphic* gyrogroups. From an algebraic point of view, G and X have the same properties.

We say that a gyrogroup G is *uniquely 2-divisible* if for each $a \in G$, there exists a unique $b \in G$ for which $b \oplus b = a$. This definition is equivalent to saying that the doubling map $a \mapsto a \oplus a$ is a bijection from G to itself.

Proposition 15. *Let G and X be as in Theorem 13. If G is gyrocommutative, then so is X. If G is uniquely 2-divisible, then so is X.*

Proof. Suppose that G is gyrocommutative. Let $a, b \in X$. We have

$$\begin{aligned}
\mathrm{gyr}_X[a,b](b \oplus_X a) &= \phi^{-1}(\mathrm{gyr}[\phi(a), \phi(b)]\phi(b \oplus_X a)) \\
&= \phi^{-1}(\mathrm{gyr}[\phi(a), \phi(b)](\phi(b) \oplus \phi(a))) \\
&= \phi^{-1}(\phi(a) \oplus \phi(b)) \\
&= a \oplus_X b.
\end{aligned}$$

Hence, X is gyrocommutative.

Let D_G and D_X denote the doubling maps of G and X, respectively. Suppose that G is uniquely 2-divisible, that is, D_G is bijective. For all $a \in X$,

$$D_X(a) = a \oplus_X a = \phi^{-1}(\phi(a) \oplus \phi(a)) = \phi^{-1}(D_G(\phi(a))) = (\phi^{-1} \circ D_G \circ \phi)(a),$$

whence $D_X = \phi^{-1} \circ D_G \circ \phi$. It follows that D_X is bijective and hence X is uniquely 2-divisible. □

For the remainder of this section, we work in the Clifford algebra of negative space of V, $C\ell_{0,n}$. In light of Theorem 13, we express Einstein addition via Möbius addition to deduce that $(\mathbb{B}, \oplus_E)$ forms a uniquely 2-divisible gyrocommutative gyrogroup.

For each $\mathbf{v} \in \mathbb{B}$, set

$$r_{\mathbf{v}} = \frac{1}{1 + \sqrt{1 - \|\mathbf{v}\|^2}}. \tag{39}$$

It is not hard to see that $r_{\mathbf{v}} = \dfrac{1 - \sqrt{1 - \|\mathbf{v}\|^2}}{\|\mathbf{v}\|^2}$ and $r_{\mathbf{v}} = \dfrac{1}{1 + \sqrt{1 + \mathbf{v}^2}}$ in $C\ell_{0,n}$. In terms of the Lorentz factor (37) normalized to $t = 1$, $r_{\mathbf{v}}$ can be rewritten as

$$r_{\mathbf{v}} = \frac{1}{1+\gamma_{\mathbf{v}}^{-1}} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}.$$

It follows that $0 < r_{\mathbf{v}} < 1$. Further, it is straightforward to check that $r_{\mathbf{v}}$ is a solution to the quadratic equation $\|\mathbf{v}\|^2 x^2 - 2x + 1 = 0$ in the variable x. Since $\mathbf{v}^2 = -\|\mathbf{v}\|^2$, we have

$$\frac{2r_{\mathbf{v}}}{1 - r_{\mathbf{v}}^2\mathbf{v}^2} = 1. \tag{40}$$

Let Ψ be the map defined by

$$\Psi(\mathbf{v}) = r_{\mathbf{v}}\mathbf{v}, \qquad \mathbf{v} \in \mathbb{B}. \tag{41}$$

Since $0 < r_{\mathbf{v}} < 1$, we obtain $\|\Psi(\mathbf{v})\| = \|r_{\mathbf{v}}\mathbf{v}\| = r_{\mathbf{v}}\|\mathbf{v}\| < 1$. Hence, Ψ maps $\mathbb{B}$ to $\mathbb{B}$. Let Φ denote the doubling map of Möbius gyrogroup $(\mathbb{B}, \oplus_M)$, that is,

$$\Phi(\mathbf{v}) = \mathbf{v} \oplus_M \mathbf{v}. \tag{42}$$

From Eq. (34), we have

$$\Phi(\mathbf{v}) = \frac{2\mathbf{v}}{1-\mathbf{v}^2} = \frac{2\mathbf{v}}{1+\|\mathbf{v}\|^2}.$$

If $\|\mathbf{v}\| < 1$, then $\|\Phi(\mathbf{v})\| = \left\|\dfrac{2\mathbf{v}}{1+\|\mathbf{v}\|^2}\right\| = \dfrac{2}{\frac{1}{\|\mathbf{v}\|} + \|\mathbf{v}\|} < 1$ since $\dfrac{1}{\|\mathbf{v}\|} + \|\mathbf{v}\| > 2$. Hence, Φ maps $\mathbb{B}$ to $\mathbb{B}$.

The doubling map Φ is of importance for the study of Möbius and Einstein gyrogroups. For instance, Φ is used in proving that the Poincaré metric on the unit ball in $\mathbb{R}^n$ is twice the rapidity metric of the Möbius gyrogroup [38, Theorem 3.7] and in proving that the Cayley–Klein metric on the unit ball in $\mathbb{R}^n$ agrees with the rapidity metric of the Einstein gyrogroup [38, Theorem 3.9].

Proposition 16. *The maps Ψ and Φ are bijections from $\mathbb{B}$ onto itself and are inverses of each other.*

Proof. Let $\mathbf{v} \in \mathbb{B}$. Since $1 - \|\Phi(\mathbf{v})\|^2 = 1 - \dfrac{4\|\mathbf{v}\|^2}{(1+\|\mathbf{v}\|^2)^2} = \left(\dfrac{1-\|\mathbf{v}\|^2}{1+\|\mathbf{v}\|^2}\right)^2$, we have

$$1 + \sqrt{1+\Phi(\mathbf{v})^2} = 1 + \sqrt{1 - \|\Phi(\mathbf{v})\|^2} = 1 + \frac{1-\|\mathbf{v}\|^2}{1+\|\mathbf{v}\|^2} = \frac{2}{1+\|\mathbf{v}\|^2} = \frac{2}{1-\mathbf{v}^2}.$$

This implies $(\Psi \circ \Phi)(\mathbf{v}) = \Psi(\Phi(\mathbf{v})) = r_{\Phi(\mathbf{v})}\Phi(\mathbf{v}) = \dfrac{1}{1+\sqrt{1+\Phi(\mathbf{v})^2}}\dfrac{2\mathbf{v}}{1-\mathbf{v}^2} = \mathbf{v}$. From (40), we have $(\Phi \circ \Psi)(\mathbf{v}) = \Phi(\Psi(\mathbf{v})) = \dfrac{2\Psi(\mathbf{v})}{1-\Psi(\mathbf{v})^2} = \dfrac{2r_{\mathbf{v}}}{1-r_{\mathbf{v}}^2\mathbf{v}^2}\mathbf{v} = \mathbf{v}$. This proves that $\Psi \circ \Phi = \mathrm{id}_{\mathbb{B}}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathbb{B}}$. Hence, Φ and Ψ are bijections from $\mathbb{B}$ onto $\mathbb{B}$, $\Phi^{-1} = \Psi$, and $\Psi^{-1} = \Phi$. □

Corollary 6. *The Möbius gyrogroup* $(\mathbb{B}, \oplus_M)$ *is uniquely 2-divisible.*

Proof. Since the doubling map Φ is bijective, the corollary follows. □

Proposition 17. *The unit ball* $\mathbb{B}$ *with the induced addition given by*

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \Psi^{-1}(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})), \qquad \mathbf{u}, \mathbf{v} \in \mathbb{B},$$

forms a uniquely 2-divisible gyrocommutative gyrogroup.

Proof. Applying Theorem 13 with $(\mathbb{B}, \oplus_M)$ in place of G and Ψ in place of ϕ, we obtain that $(\mathbb{B}, \oplus_{\mathbb{B}})$ forms a gyrogroup. By Theorem 12 and Corollary 6, $(\mathbb{B}, \oplus_M)$ is a uniquely 2-divisible gyrocommutative gyrogroup, and by Proposition 15, $(\mathbb{B}, \oplus_{\mathbb{B}})$ is a uniquely 2-divisible gyrocommutative gyrogroup. □

The induced addition $\oplus_{\mathbb{B}}$ in Proposition 17 is nothing but Einstein addition, as shown in the following theorem:

Theorem 14. *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{B}$, $\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \mathbf{u} \oplus_E \mathbf{v}$. *In particular,* $(\mathbb{B}, \oplus_E)$ *forms a uniquely 2-divisible gyrocommutative gyrogroup. In terms of the Clifford algebra* $C\ell_{0,n}$, *Einstein addition can be rewritten as*

$$\mathbf{u} \oplus_E \mathbf{v} = 2(r_{\mathbf{u}}\mathbf{u} \oplus_M r_{\mathbf{v}}\mathbf{v})\left(1-(r_{\mathbf{u}}\mathbf{u} \oplus_M r_{\mathbf{v}}\mathbf{v})^2\right)^{-1} \tag{43}$$

and the Einstein gyroautomorphisms are given by

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = q\mathbf{w}\tilde{q}, \qquad q = \frac{1-r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}}{|1-r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}|}.$$

Proof. Since $\Psi^{-1} = \Phi$, we have

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \Phi(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})) = \frac{2\,(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}))}{1-(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}))^2}.$$

Equations (34) and (35) and Proposition 13 together imply

$$\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} = \frac{2\,(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}))}{1+\eta(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v}))}$$

$$= \frac{2\left((\Psi(\mathbf{u}) + \Psi(\mathbf{v}))\,(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))^{-1}\right)}{1 + \frac{\eta(\Psi(\mathbf{u})+\Psi(\mathbf{v}))}{\eta(1-\Psi(\mathbf{u})\Psi(\mathbf{v}))}}$$

$$= \frac{2\,(\Psi(\mathbf{u}) + \Psi(\mathbf{v}))\,(1 - \Psi(\mathbf{v})\Psi(\mathbf{u}))}{\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v})) + \eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))}. \tag{44}$$

Since $1 - r_{\mathbf{u}}^2\mathbf{u}^2 - r_{\mathbf{v}}^2\mathbf{v}^2 + r_{\mathbf{u}}^2\mathbf{u}^2 r_{\mathbf{v}}^2\mathbf{v}^2 = (1 - r_{\mathbf{u}}^2\mathbf{u}^2)(1 - r_{\mathbf{v}}^2\mathbf{v}^2) = (2r_{\mathbf{u}})(2r_{\mathbf{v}}) = 4r_{\mathbf{u}}r_{\mathbf{v}}$, we have

$$\begin{aligned}
&\eta(\Psi(\mathbf{u}) + \Psi(\mathbf{v})) + \eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))\\
&\quad= -(\Psi(\mathbf{u}) + \Psi(\mathbf{v}))^2 + (1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))(1 - \Psi(\mathbf{v})\Psi(\mathbf{u}))\\
&\quad= 1 - \Psi(\mathbf{u})^2 - \Psi(\mathbf{v})^2 + \Psi(\mathbf{u})^2\Psi(\mathbf{u})^2 - 2(\Psi(\mathbf{u})\Psi(\mathbf{v}) + \Psi(\mathbf{v})\Psi(\mathbf{u}))\\
&\quad= 1 - r_{\mathbf{u}}^2\mathbf{u}^2 - r_{\mathbf{v}}^2\mathbf{v}^2 + r_{\mathbf{u}}^2\mathbf{u}^2 r_{\mathbf{v}}^2\mathbf{v}^2 + 4r_{\mathbf{u}}r_{\mathbf{v}}\langle\mathbf{u},\mathbf{v}\rangle\\
&\quad= 4r_{\mathbf{u}}r_{\mathbf{v}}(1 + \langle\mathbf{u},\mathbf{v}\rangle).
\end{aligned} \tag{45}$$

We also have

$$\begin{aligned}
&\frac{1}{2r_{\mathbf{u}}r_{\mathbf{v}}}\Big(\Psi(\mathbf{u}) + \Psi(\mathbf{v})\Big)\Big(1 - \Psi(\mathbf{v})\Psi(\mathbf{u})\Big)\\
&\quad= \frac{1}{2r_{\mathbf{u}}r_{\mathbf{v}}}\Big(r_{\mathbf{u}}\mathbf{u} + r_{\mathbf{v}}\mathbf{v}\Big)\Big(1 - r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{v}\mathbf{u}\Big)\\
&\quad= \frac{1}{2r_{\mathbf{u}}r_{\mathbf{v}}}\Big(r_{\mathbf{u}}\mathbf{u} - r_{\mathbf{u}}^2 r_{\mathbf{v}}\mathbf{u}\mathbf{v}\mathbf{u} + r_{\mathbf{v}}\mathbf{v} - r_{\mathbf{u}}r_{\mathbf{v}}^2\mathbf{v}^2\mathbf{u}\Big)\\
&\quad= \frac{\mathbf{u}}{2r_{\mathbf{v}}} - \frac{r_{\mathbf{u}}}{2}\mathbf{u}\mathbf{v}\mathbf{u} + \frac{\mathbf{v}}{2r_{\mathbf{u}}} - \frac{r_{\mathbf{v}}}{2}\mathbf{v}^2\mathbf{u}\\
&\quad= \frac{1}{2}\left(\frac{1}{r_{\mathbf{v}}} - r_{\mathbf{v}}\mathbf{v}^2\right)\mathbf{u} - \frac{r_{\mathbf{u}}}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})\mathbf{u} + \frac{1}{2}\left(r_{\mathbf{u}}\mathbf{u}^2 + \frac{1}{r_{\mathbf{u}}}\right)\mathbf{v}\\
&\quad= \mathbf{u} + \frac{\gamma_{\mathbf{u}}}{1 + \gamma_{\mathbf{u}}}\langle\mathbf{u},\mathbf{v}\rangle\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}.
\end{aligned} \tag{46}$$

We obtain the fifth equation because $\dfrac{1}{r_{\mathbf{w}}} = 1 + \sqrt{1 - \|\mathbf{w}\|^2} = 1 + \dfrac{1}{\gamma_{\mathbf{w}}}$ and

$$r_{\mathbf{w}}\mathbf{w}^2 = \frac{1 - \sqrt{1 - \|\mathbf{w}\|^2}}{\|\mathbf{w}\|^2}(-\|\mathbf{w}\|^2) = \sqrt{1 - \|\mathbf{w}\|^2} - 1 = \frac{1}{\gamma_{\mathbf{w}}} - 1$$

for all $\mathbf{w} \in \mathbb{B}$.

Combining Eqs. (44)–(46) gives

$$\begin{aligned}\mathbf{u} \oplus_{\mathbb{B}} \mathbf{v} &= \frac{4r_{\mathbf{u}}r_{\mathbf{v}}\left\{\mathbf{u} + \frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}\right\}}{4r_{\mathbf{u}}r_{\mathbf{v}}(1 + \langle \mathbf{u}, \mathbf{v}\rangle)} \\ &= \frac{1}{1 + \langle \mathbf{u}, \mathbf{v}\rangle}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}\right\} \\ &= \mathbf{u} \oplus_E \mathbf{v}.\end{aligned}$$

Let us temporarily denote by $\mathrm{gyr}_E[\mathbf{u}, \mathbf{v}]$ an Einstein gyroautomorphism and by $\mathrm{gyr}_M[\mathbf{u}, \mathbf{v}]$ a Möbius gyroautomorphism. Since $\oplus_{\mathbb{B}}$ and $\oplus_E$ agree on $\mathbb{B}$, we have from the gyrator identity (see Theorem 1) that $\mathrm{gyr}_E[\mathbf{u}, \mathbf{v}]$ coincides with the induced gyroautomorphism $\mathrm{gyr}_{\mathbb{B}}[\mathbf{u}, \mathbf{v}]$. It follows from Eq. (38) that

$$\mathrm{gyr}_E[\mathbf{u}, \mathbf{v}] = \Phi \circ \mathrm{gyr}_M[\Psi(\mathbf{u}), \Psi(\mathbf{v})] \circ \Phi^{-1}.$$

Because Φ is the doubling map of $(\mathbb{B}, \oplus_M)$ and $\mathrm{gyr}_M[\Psi(\mathbf{u}), \Psi(\mathbf{v})]$ preserves $\oplus_M$, Φ and $\mathrm{gyr}_M[\Psi(\mathbf{u}), \Psi(\mathbf{v})]$ commute. Hence,

$$\mathrm{gyr}_E[\mathbf{u}, \mathbf{v}] = \mathrm{gyr}_M[\Psi(\mathbf{u}), \Psi(\mathbf{v})].$$

By Theorems 11 and 12, $\mathrm{gyr}_M[\Psi(\mathbf{u}), \Psi(\mathbf{v})]\mathbf{w} = q\mathbf{w}\tilde{q}$, where

$$q = \frac{1 + \Psi(\mathbf{u})\overline{\Psi(\mathbf{v})}}{|1 + \Psi(\mathbf{u})\overline{\Psi(\mathbf{v})}|} = \frac{1 - \Psi(\mathbf{u})\Psi(\mathbf{v})}{|1 - \Psi(\mathbf{u})\Psi(\mathbf{v})|} = \frac{1 - r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}}{|1 - r_{\mathbf{u}}r_{\mathbf{v}}\mathbf{u}\mathbf{v}|}.$$

□

Theorem 15. *Let q be as in Theorem 14. Then $q \in \Gamma_{0,n}$, and the restriction of T_q to $\mathbb{B}$ equals the Einstein gyroautomorphism generated by* **u** *and* **v**. *Consequently, any Einstein gyroautomorphism represents a rotation of the unit ball.*

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. Since $\Psi(\mathbf{u}), \Psi(\mathbf{v}) \in \mathbb{B}$, Proposition 13 (1) implies that q belongs to $\Gamma_{0,n}$. Furthermore,

$$\eta(q) = \eta\left(\frac{1 - \Psi(\mathbf{u})\Psi(\mathbf{v})}{|1 - \Psi(\mathbf{u})\Psi(\mathbf{v})|}\right) = \frac{\eta(1 - \Psi(\mathbf{u})\Psi(\mathbf{v}))}{|1 - \Psi(\mathbf{u})\Psi(\mathbf{v})|^2} = 1.$$

From (16)–(18), we have $\hat{q} = q$ and $\bar{q} = \tilde{q}$, which gives

$$T_q\mathbf{w} = \hat{q}\mathbf{w}q^{-1} = q\mathbf{w}\frac{\bar{q}}{\eta(q)} = q\mathbf{w}\tilde{q}$$

for all $\mathbf{w} \in V$. Hence, T_q and $\mathrm{gyr}_E[\mathbf{u}, \mathbf{v}]$ agree on $\mathbb{B}$. By Theorem 10, T_q defines an orthogonal transformation of (V, B). Furthermore, T_q preserves the inner product of V since $B(\mathbf{u}, \mathbf{v}) = -\langle \mathbf{u}, \mathbf{v}\rangle$. This completes the proof. □

From Proposition 17 and Theorem 14, we have a strong connection between Einstein addition and Möbius addition:

$$\mathbf{u} \oplus_E \mathbf{v} = \Psi^{-1}(\Psi(\mathbf{u}) \oplus_M \Psi(\mathbf{v})).$$

Using the principle of special relativity, Friedman and Scarr established this relation as well; see [24, Chap. 2].

Equation (41) shows a close relationship between elements of Einstein and Möbius gyrogroups. This result is stated by Ungar [70, Eq. (6.297)] and by Ferreira [16, Proposition 6]. In terms of *Einstein scalar multiplication*, defined by

$$\begin{aligned} r \otimes_E \mathbf{v} &= \frac{(1 + \|\mathbf{v}\|)^r - (1 - \|\mathbf{v}\|)^r}{(1 + \|\mathbf{v}\|)^r + (1 - \|\mathbf{v}\|)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= \tanh\left(r \tanh^{-1}(\|\mathbf{v}\|)\right)\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right), \end{aligned}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{B}$, $\mathbf{v} \neq \mathbf{0}$, and $r \otimes_E \mathbf{0} = \mathbf{0}$ (see [67, p. 194]), Eqs. (41) and (42) can be rewritten as

$$\Psi(\mathbf{v}) = \frac{1}{2} \otimes_E \mathbf{v} \quad \text{and} \quad \Phi(\mathbf{v}) = 2 \otimes_E \mathbf{v},$$

which reflects the fact that Ψ and Φ are inverses of each other.

In terms of the Clifford algebra $C\ell_{0,n}$, the Einstein gyroautomorphism generated by $\mathbf{u}$ and $\mathbf{v}$ has a compact formula:

$$\mathrm{gyr}_E[\mathbf{u}, \mathbf{v}]\mathbf{w} = q\mathbf{w}\tilde{q}, \qquad q = \frac{1 - r_{\mathbf{u}} r_{\mathbf{v}} \mathbf{u}\mathbf{v}}{|1 - r_{\mathbf{u}} r_{\mathbf{v}} \mathbf{u}\mathbf{v}|}. \tag{47}$$

An Einstein gyroautomorphism is known in mathematical physics as a *Thomas gyration*, which is the mathematical abstraction of the relativistic effect known as *Thomas precession* [48, 65, 74]. Geometrically, it represents a rotation of the unit ball. For a deep discussion of Einstein gyroautomorphisms, see [70, Sect. 10.3].

As noted in the introduction to this paper, gyrogroups can be studied from the abstract point of view. The remainder of this paper is devoted to the study of gyrogroups in a general setting. Following the modern treatment of abstract algebra, we study subgyrogroups, normal subgyrogroups, quotient gyrogroups, and gyrogroup homomorphisms. Further, we show that several well-known results in group theory continue to hold for gyrogroups. In particular, we extend the results of Suksumran and Wiboonton [60, 61] by including more details and adding some new results.

5 Cayley's Theorem

Recall that any Möbius transformation of $\mathbb{C}$ is a map that can be expressed in the form

$$z \mapsto \frac{az+b}{cz+d}, \tag{48}$$

where z is a complex variable and a, b, c and d are complex constants with $ad - bc \neq 0$. Note that one needs *the one-point compactification* of $\mathbb{C}$, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, to solve the problem of division by zero in (48).

In view of (48), any Möbius transformation of $\mathbb{C}$ can be associated with a 2×2 invertible matrix in the general linear group over $\mathbb{C}$, $\mathrm{GL}_2(\mathbb{C})$, by defining

$$\Upsilon\colon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto f, \qquad f(z) = \frac{az+b}{cz+d}. \tag{49}$$

In fact, (49) defines a group homomorphism from the general linear group over $\mathbb{C}$ to the group of Möbius transformations of $\mathbb{C}$.

Theorem 16 ([4]). *The map Υ is a surjective group homomorphism from* $\mathrm{GL}_2(\mathbb{C})$ *onto the group of Möbius transformations of* $\mathbb{C}$ *whose kernel is*

$$\left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} : \lambda \in \mathbb{C},\ \lambda \neq 0 \right\}.$$

Denote the open unit disk of $\mathbb{C}$ by $\mathbb{D}$ and denote the unit circle of $\mathbb{C}$ by S^1:

$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}, \qquad S^1 = \{z \in \mathbb{C} \colon |z| = 1\}.$$

There are two important types of Möbius transformations preserving the complex unit disk:

1. *Möbius Translations.* For $a \in \mathbb{D}$, let τ_a be the Möbius transformation induced by

$$\begin{bmatrix} 1 & a \\ \bar{a} & 1 \end{bmatrix}, \qquad \tau_a(z) = \frac{z+a}{1+\bar{a}z}.$$

   By Lemma 6.2.2 of [29], τ_a is indeed a conformal self-map of $\mathbb{D}$. Also, τ_a is a left gyrotranslation of the complex Möbius gyrogroup, $\tau_a(z) = a \oplus_M z$.

2. *Rotations.* For $\omega \in S^1$, let ρ_ω be the Möbius transformation induced by

$$\begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho_\omega(z) = \omega z.$$

It is clear that ρ_ω represents a rotation of the complex unit disk.

The Möbius transformations of the complex unit disk are completely characterized by τ_a and ρ_ω, with a in $\mathbb{D}$ and ω in S^1, as shown in the following theorem:

Theorem 17 ([29]). *A holomorphic function $f\colon \mathbb{D} \to \mathbb{D}$ is a conformal self-map of $\mathbb{D}$ if and only if there are complex numbers $a \in \mathbb{D}$ and $\omega \in S^1$ such that $f = \rho_\omega \circ \tau_a$.*

Note that $\rho_\omega^{-1} = \rho_{\overline{\omega}}$ and $\rho_{\omega_1} \circ \rho_{\omega_2} = \rho_{\omega_1\omega_2}$ for all $\omega, \omega_1, \omega_2 \in S^1$. This means that the set of rotations, $\{\rho_\omega\colon \omega \in S^1\}$, forms a group under function composition, and this group is isomorphic to the group of unit complex numbers via $\omega \mapsto \rho_\omega$. In contrast, the set of Möbius translations, $\{\tau_a\colon a \in \mathbb{D}\}$, does not form a group under function composition.

Let $a, b \in \mathbb{D}$. A direct computation gives

$$\begin{bmatrix} 1 & a \\ \bar{a} & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ \bar{b} & 1 \end{bmatrix} = \begin{bmatrix} 1+\bar{a}b & 0 \\ 0 & 1+\bar{a}b \end{bmatrix}\begin{bmatrix} 1 & \frac{a+b}{1+\bar{a}b} \\ \frac{\bar{a}+\bar{b}}{1+a\bar{b}} & 1 \end{bmatrix}\begin{bmatrix} \frac{1+a\bar{b}}{1+\bar{a}b} & 0 \\ 0 & 1 \end{bmatrix}. \tag{50}$$

By Theorem 16, the matrix equation (50) reads

$$\tau_a \circ \tau_b = \tau_{a\oplus_M b} \circ \rho_\omega, \tag{51}$$

where $a \oplus_M b = \dfrac{a+b}{1+\bar{a}b}$ and $\omega = \dfrac{1+a\bar{b}}{1+\bar{a}b}$. In view of (51), the set of Möbius translations is not closed under function composition and hence cannot be a group. However, the set of Möbius translations is a gyrogroup under the operation $\tau_a \oplus \tau_b = \tau_{a\oplus_M b}$. This gyrogroup is isomorphic to the complex Möbius gyrogroup via $a \mapsto \tau_a$. From (3) and (51), we derive the composition law of Möbius translations:

$$\tau_a \circ \tau_b = \tau_{a\oplus_M b} \circ \mathrm{gyr}[a, b]. \tag{52}$$

In this section, we abstract the composition law of Möbius translations. Further, we prove that if G is an arbitrary gyrogroup, then the set of left gyrotranslations of G admits the gyrogroup structure induced by G. The gyrogroup of left gyrotranslations is isomorphic to the underlying gyrogroup G. This results in a version of *Cayley's theorem* for gyrogroups [61].

Theorem 18. *Let G be an arbitrary gyrogroup.*

1. *For each $a \in G$, the left gyrotranslation, $L_a\colon x \mapsto a \oplus x$, is a permutation of G.*
2. *Denote the set of all left gyrotranslations of G by $\overline{G}$. The map $\psi\colon G \to \overline{G}$ defined by $\psi(a) = L_a$ is bijective. The inverse map $\phi := \psi^{-1}$ fulfills the condition in Theorem 13. In this case, the induced operation $\oplus_{\overline{G}}$ is given by*

$$L_a \oplus_{\overline{G}} L_b = L_{a\oplus b}$$

for all $a, b \in G$.

3. *For all* $a, b, c \in G$,

$$L_a \circ L_b = L_{a \oplus b} \circ \operatorname{gyr}[a, b] \tag{53}$$

and

$$\operatorname{gyr}_{\overline{G}}[L_a, L_b]L_c = L_{\operatorname{gyr}[a,b]c}. \tag{54}$$

Proof.

(1) That L_a is injective follows from the general left cancellation law. That L_a is surjective follows from Theorem 2.

(2) Clearly, ψ is surjective. If $\psi(a) = \psi(b)$, then $a = L_a(0) = L_b(0) = b$. Hence, ψ is injective. By Theorem 13, the induced operation is given by

$$L_a \oplus_{\overline{G}} L_b = \phi^{-1}(\phi(L_a) \oplus \phi(L_b)) = \psi(\psi^{-1}(L_a) \oplus \psi^{-1}(L_b)) = \psi(a \oplus b) = L_{a \oplus b}.$$

Note that the identity element of $\overline{G}$ is id_G since $\phi^{-1}(0) = L_0 = \mathrm{id}_G$ and that the inverse of L_a in $\overline{G}$ is $L_{\ominus a}$ since $L_a \oplus_{\overline{G}} L_{\ominus a} = L_{a \oplus (\ominus a)} = L_0 = L_{(\ominus a) \oplus a} = L_{\ominus a} \oplus_{\overline{G}} L_a$.

(3) By the left cancellation law, $L_a \circ L_{\ominus a} = \mathrm{id}_G = L_{\ominus a} \circ L_a$. Hence, the inverse map of L_a with respect to $\circ$ is indeed $L_{\ominus a}$. In other words, $L_a^{-1} = L_{\ominus a}$ for all $a \in G$. By the gyrator identity, $\operatorname{gyr}[a, b] = L_{\ominus(a \oplus b)} \circ L_a \circ L_b$ and hence $\operatorname{gyr}[a, b] = L_{a \oplus b}^{-1} \circ L_a \circ L_b$. It follows that $L_a \circ L_b = L_{a \oplus b} \circ \operatorname{gyr}[a, b]$. Applying the gyrator identity, we obtain

$$\begin{aligned} \operatorname{gyr}_{\overline{G}}[L_a, L_b]L_c &= \ominus_{\overline{G}}(L_a \oplus_{\overline{G}} L_b) \oplus_{\overline{G}} (L_a \oplus_{\overline{G}} (L_b \oplus_{\overline{G}} L_c)) \\ &= L_{\ominus(a \oplus b) \oplus (a \oplus (b \oplus c))} \\ &= L_{\operatorname{gyr}[a,b]c}. \end{aligned}$$

□

Remark 1. For simplicity, we will not distinguish between the notation for induced and usual gyroautomorphisms. Hence, Eq. (54) reads $\operatorname{gyr}[L_a, L_b]L_c = L_{\operatorname{gyr}[a,b]c}$ for all $a, b, c \in G$.

Note that the composition law (53) is an abstract version of the composition law (52) of Möbius translations. Also, (53) is in some sense related to the composition law (1) of Lorentz boosts. The importance of the composition law lies in the fact that it relates the group operation $\circ$ and the gyrogroup operation $\oplus$ in a natural way.

Let G be a gyrogroup. Let $\operatorname{Stab}(0)$ denote the set of permutations of G leaving the gyrogroup identity fixed:

$$\operatorname{Stab}(0) = \{\rho \in \operatorname{Sym}(G) \colon \rho(0) = 0\}.$$

It is clear that Stab(0) is a subgroup of the symmetric group, Sym (G), and we have the following inclusions:

$$\{\mathrm{gyr}[a,b]\colon a,b\in G\}\subseteq \mathrm{Aut}\,(G)\leqslant \mathrm{Stab}(0)\leqslant \mathrm{Sym}\,(G).$$

The next two propositions show that the induced gyrogroup $\overline{G}$ is a twisted subgroup of Sym (G) and that $\overline{G}$ is a transversal of Stab(0) in Sym (G). Recall that a subset K of a group Γ is a *twisted subgroup* of Γ [21] if

1. $1_\Gamma \in K$, 1_Γ being the identity element of Γ, and
2. $x, y \in K$ implies $xyx \in K$.

A subset B of Γ is a *(left) transversal* of a subgroup Ξ of Γ if every $g\in\Gamma$ can be written uniquely as $g = bh$, where $b\in B$ and $h\in\Xi$ [21, p. 30].

Proposition 18. *Let G be a gyrogroup. Then $\overline{G}$ is a twisted subgroup of* Sym (G).

Proof. The first condition for a twisted subgroup holds: $\mathrm{id}_G = L_0 \in \overline{G}$.

Let $a, b \in G$. By Theorem 3 (4), $c = (a\oplus b)\boxplus a$ is such that $c\ominus a = a\oplus b$. Applying the left and right loop properties, we have

$$\begin{aligned}\mathrm{gyr}[a,b] &= \mathrm{gyr}[a\oplus b, b]\\ &= \mathrm{gyr}[c\ominus a, b]\\ &= \mathrm{gyr}[c\ominus a, \ominus a\oplus(c\ominus a)]\\ &= \mathrm{gyr}[c\ominus a, \ominus a]\\ &= \mathrm{gyr}[c, \ominus a].\end{aligned}$$

It follows that $L_a\circ L_b = L_{a\oplus b}\circ \mathrm{gyr}[a,b] = L_{c\ominus a}\circ\mathrm{gyr}[c,\ominus a] = L_c\circ L_{\ominus a} = L_c\circ L_a^{-1}$, which implies $L_a\circ L_b\circ L_a = L_c$ belongs to $\overline{G}$. Hence, the second condition for a twisted subgroup holds. □

Proposition 19. *Let G be a gyrogroup. For each $\sigma\in$* Sym (G)*, σ can be written uniquely as $\sigma = L_a\circ\rho$, where $a\in G$ and $\rho\in$* Stab(0)*. In other words, $\overline{G}$ is a transversal of* Stab(0) *in* Sym (G).

Proof. Suppose that $L_a\circ\rho = L_b\circ\eta$, where $a,b\in G$ and $\rho,\eta\in\mathrm{Stab}(0)$. Then $a = (L_a\circ\rho)(0) = (L_b\circ\eta)(0) = b$, which implies $L_a = L_b$. This in turn implies $\rho = \eta$. Hence, the factorization, when it exists, is unique.

Let σ be an arbitrary permutation of G. Choose $a = \sigma(0)$ and set $\rho = L_{\ominus a}\circ\sigma$. Note that $\rho(0) = L_{\ominus a}(a) = \ominus a\oplus a = 0$. Hence, ρ lies in Stab(0). Since $L_{\ominus a} = L_a^{-1}$, we have $\sigma = L_a\circ\rho$. This proves the existence of factorization. □

The normalizer of $\overline{G}$ in Stab(0) is equal to the automorphism group of G, as shown in the following proposition. For $\rho\in\mathrm{Sym}\,(G)$, let $\rho\overline{G}\rho^{-1}$ denote the set $\{\rho\circ L_a\circ\rho^{-1}\colon a\in G\}$.

Proposition 20. *Let* $\rho \in \mathrm{Stab}(0)$. *Then* ρ *normalizes* $\overline{G}$*, that is,* $\rho\overline{G}\rho^{-1} = \overline{G}$*, if and only if* $\rho \in \mathrm{Aut}\,(G)$.

Proof. Note that $\rho^{-1}(0) = 0$.

To prove the "only if" part, we need only assume that $\rho\overline{G}\rho^{-1} \subseteq \overline{G}$. Let $a, b \in G$. Since $\rho \circ L_a \circ \rho^{-1} \in \rho\overline{G}\rho^{-1}$, we have $\rho \circ L_a \circ \rho^{-1} = L_c$ for some $c \in G$. In fact, $c = L_c(0) = (\rho \circ L_a \circ \rho^{-1})(0) = \rho(a)$. Hence, $\rho \circ L_a = L_{\rho(a)} \circ \rho$. It follows that $\rho(a \oplus b) = (\rho \circ L_a)(b) = (L_{\rho(a)} \circ \rho)(b) = \rho(a) \oplus \rho(b)$, which proves ρ is an automorphism of G.

Suppose conversely that $\rho \in \mathrm{Aut}\,(G)$. For $a, x \in G$, we have

$$(\rho \circ L_a \circ \rho^{-1})(x) = \rho(a \oplus \rho^{-1}(x)) = \rho(a) \oplus x = L_{\rho(a)}(x).$$

Hence, $\rho \circ L_a \circ \rho^{-1} = L_{\rho(a)}$. This implies $\rho\overline{G}\rho^{-1} \subseteq \overline{G}$ since a is arbitrary. For each $b \in G$, there is an element a of G such that $\rho(a) = b$. Since $L_b = L_{\rho(a)} = \rho \circ L_a \circ \rho^{-1}$, we have $L_b \in \rho\overline{G}\rho^{-1}$. This proves $\overline{G} \subseteq \rho\overline{G}\rho^{-1}$ and so equality holds. □

In light of the proof of Proposition 20, one has

$$\rho \circ L_a = L_{\rho(a)} \circ \rho \tag{55}$$

whenever ρ is an automorphism of G. The *commutation relation* (55) determines how to commute a left gyrotranslation and an automorphism of G. It is worth pointing out that Proposition 19 has an analogous result in loop theory; see, for instance, [40, Sect. 2]. We next see how the gyrogroup structure appears in the symmetric group of a gyrogroup.

Gyrogroups of Permutations

Let G and H be gyrogroups. A map $\varphi\colon G \to H$ such that $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$ is called a *gyrogroup homomorphism*. A bijective gyrogroup homomorphism is called a *gyrogroup isomorphism*. We say that G and H are *isomorphic gyrogroups* if there is a gyrogroup isomorphism between them.

Let G be a gyrogroup. Proposition 19 enables us to introduce a binary operation $\oplus$ on the symmetric group of G so that $\mathrm{Sym}\,(G)$ equipped with $\oplus$ becomes a gyrogroup containing an isomorphic copy of G. This results in Cayley's theorem for gyrogroups [61].

Let σ and τ be arbitrary permutations of G. By Proposition 19, σ and τ have factorizations $\sigma = L_a \circ \gamma$ and $\tau = L_b \circ \delta$, where $a, b \in G$ and $\gamma, \delta \in \mathrm{Stab}(0)$. Define an operation $\oplus$ on $\mathrm{Sym}\,(G)$ by

$$\sigma \oplus \tau = L_{a \oplus b} \circ (\gamma \circ \delta). \tag{56}$$

Because of the uniqueness of factorization, $\oplus$ is a binary operation on Sym (G). In fact, (Sym $(G), \oplus$) forms a gyrogroup.

Theorem 19. *Let G be a gyrogroup. Then* Sym (G) *with operation defined by (56) is a gyrogroup and*

$$L_a \oplus L_b = L_{a\oplus b} = L_a \oplus_{\overline{G}} L_b$$

for all $a, b \in G$. In particular, the map $a \mapsto L_a$ defines an injective gyrogroup homomorphism from G into Sym (G).

Proof. Suppose that $\sigma = L_a \circ \gamma$, $\tau = L_b \circ \delta$, and $\rho = L_c \circ \lambda$, where $a, b, c \in G$ and $\gamma, \delta, \lambda \in \mathrm{Stab}(0)$.

(G1) id_G acts as a left identity of Sym (G) with respect to $\oplus$:

$$\mathrm{id}_G \oplus \sigma = L_{0\oplus a} \circ (\mathrm{id}_G \circ \gamma) = L_a \circ \gamma = \sigma.$$

(G2) $L_{\ominus a}\circ\gamma^{-1}$ is a left inverse of σ: $(L_{\ominus a}\circ\gamma^{-1})\oplus\sigma = (L_{\ominus a\oplus a})\circ(\gamma^{-1}\circ\gamma) = \mathrm{id}_G$.

(G3) Define a map $\mathrm{gyr}[\sigma, \tau]\colon \mathrm{Sym}\,(G) \to \mathrm{Sym}\,(G)$ by

$$\mathrm{gyr}[\sigma, \tau]\rho = (\mathrm{gyr}[L_a, L_b]L_c) \circ \lambda = L_{\mathrm{gyr}[a,b]c} \circ \lambda. \tag{57}$$

Suppose $\rho' = L_{c'} \circ \lambda'$ is such that $\mathrm{gyr}[\sigma, \tau]\rho = \mathrm{gyr}[\sigma, \tau]\rho'$. Then $L_{\mathrm{gyr}[a,b]c} \circ \lambda = L_{\mathrm{gyr}[a,b]c'} \circ \lambda'$ and, by Proposition 19, $L_{\mathrm{gyr}[a,b]c} = L_{\mathrm{gyr}[a,b]c'}$ and $\lambda = \lambda'$. Thus, $c = c'$ and hence $\rho = \rho'$. This proves that $\mathrm{gyr}[\sigma, \tau]$ is injective.
For $L_y \circ \mu \in \mathrm{Sym}\,(G)$, we can choose L_x in $\overline{G}$ such that $\mathrm{gyr}[L_a, L_b]L_x = L_y$ since $\mathrm{gyr}[L_a, L_b]$ is surjective. Because $\mathrm{gyr}[\sigma, \tau](L_x \circ \mu) = (\mathrm{gyr}[L_a, L_b]L_x) \circ \mu = L_y \circ \mu$, $\mathrm{gyr}[\sigma, \tau]$ is surjective.
For any $\rho = L_c \circ \lambda$, $\varsigma = L_d \circ \nu \in \mathrm{Sym}\,(G)$, we have

$$\begin{aligned}(\mathrm{gyr}[\sigma, \tau]\rho) \oplus (\mathrm{gyr}[\sigma, \tau]\varsigma) &= (L_{\mathrm{gyr}[a,b]c} \circ \lambda) \oplus (L_{\mathrm{gyr}[a,b]d} \circ \nu)\\ &= L_{\mathrm{gyr}[a,b]c\oplus\mathrm{gyr}[a,b]d} \circ (\lambda \circ \nu)\\ &= L_{\mathrm{gyr}[a,b](c\oplus d)} \circ (\lambda \circ \nu)\\ &= \mathrm{gyr}[\sigma, \tau](\rho \oplus \varsigma).\end{aligned}$$

This proves that $\mathrm{gyr}[\sigma, \tau]$ defines an automorphism of (Sym $(G), \oplus$). To prove that the left gyroassociative law holds in Sym (G), we compute

$$\begin{aligned}\sigma \oplus (\tau \oplus \rho) &= L_{a\oplus(b\oplus c)} \circ (\gamma \circ (\delta \circ \lambda))\\ &= L_{(a\oplus b)\oplus\mathrm{gyr}[a,b]c} \circ ((\gamma \circ \delta) \circ \lambda)\\ &= (\sigma \oplus \tau) \oplus (L_{\mathrm{gyr}[a,b]c} \circ \lambda)\\ &= (\sigma \oplus \tau) \oplus (\mathrm{gyr}[\sigma, \tau]\rho).\end{aligned}$$

(G4) To prove that the left loop property holds in Sym (G), we compute

$$\mathrm{gyr}[\sigma \oplus \tau, \tau]\rho = (\mathrm{gyr}[L_a \oplus_{\overline{G}} L_b, L_b]L_c) \circ \lambda = (\mathrm{gyr}[L_a, L_b]L_c) \circ \lambda = \mathrm{gyr}[\sigma, \tau]\rho.$$

Since ρ is arbitrary, we have $\mathrm{gyr}[\sigma \oplus \tau, \tau] = \mathrm{gyr}[\sigma, \tau]$. □

By Theorem 19, the following version of Cayley's theorem for gyrogroups is immediate:

Corollary 7 (Cayley's Theorem). *Every gyrogroup is isomorphic to a subgyrogroup of the gyrogroup of permutations.*

Proof. The map $a \mapsto L_a$ defines a gyrogroup isomorphism from G onto $\overline{G}$, and $\overline{G}$ is a subgyrogroup of Sym (G). □

6 Subgyrogroups

The following definition of a subgyrogroup first appeared in [14, Sect. 4] with the term "gyro-subgroup":

Definition 4 (Subgyrogroups). Let G be a gyrogroup. A nonempty subset H of G is a subgyrogroup, written $H \leqslant G$, if H is a gyrogroup under the operation inherited from G and the restriction of $\mathrm{gyr}[a, b]$ to H becomes an automorphism of H for all $a, b \in H$.

Remark 2. Let G be a gyrogroup and let H be a subgyrogroup of G.

1. If 0_H is an identity element of H, then 0_H must equal the identity element of G. This is because $0_H \oplus 0_H = 0_H = 0_H \oplus 0$ and hence $0_H = 0$.
2. For each $a \in H$, if b is an inverse of a in H, then b must equal the inverse of a in G. This is because $a \oplus b = 0_H = 0 = a \ominus a$, whence $b = \ominus a$.

Proposition 21 (The Subgyrogroup Criterion). *A nonempty subset H of G is a subgyrogroup if and only if $a \in H$ implies $\ominus a \in H$ and $a, b \in H$ implies $a \oplus b \in H$.*

Proof. The "only if" part follows from the definition of a subgyrogroup and Remark 2. To prove the "if" part, suppose that the two conditions hold. Since $H \neq \emptyset$, there is an element $a \in H$ so that $0 = \ominus a \oplus a \in H$, and axiom (G1) holds. Axiom (G2) holds by the first condition. Axiom (G4) holds trivially.

Let $a, b \in H$. By the gyrator identity, $\mathrm{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$, and by the second condition, $\mathrm{gyr}[a, b]c \in H$ for all $c \in H$. Hence, $\mathrm{gyr}[a, b](H) \subseteq H$. Likewise, $\mathrm{gyr}[b, a](H) \subseteq H$. For each $d \in H$, choose $c \in G$ such that $\mathrm{gyr}[a, b]c = d$. Since $c = \mathrm{gyr}^{-1}[a, b]d = \mathrm{gyr}[b, a]d \in H$, we have $d = \mathrm{gyr}[a, b]c \in \mathrm{gyr}[a, b](H)$. This proves $H \subseteq \mathrm{gyr}[a, b](H)$ and so equality holds. It follows that the restriction of $\mathrm{gyr}[a, b]$ to H is an automorphism of H and hence axiom (G3) holds. □

If H is a *finite* nonempty subset of G, then it suffices to check that H is closed under the gyrogroup operation.

Proposition 22. *A nonempty finite subset H of a gyrogroup G is a subgyrogroup if and only if $a \oplus b \in H$ for all $a, b \in H$.*

Proof. Suppose that $H \neq \emptyset$. To complete the proof, we need only check that $\ominus a \in H$ for all $a \in H$. For each $a \in H$, define recursively the following sequence:

$$a_0 = 0, \quad a_n = a \oplus (a_{n-1}), \qquad n \geq 1.$$

By induction, $a_n \in H$ for all $n \geq 0$ since H is closed under $\oplus$. Because H is finite, there must be repetitions among $a_0, a_1, a_2, \ldots$, say $a_m = a_n$ with $m < n$. Applying the general left cancellation law repeatedly, we obtain $a_{n-m} = 0$. Hence, $a \ominus a = 0 = a \oplus a_{n-m-1}$, which implies $\ominus a = a_{n-m-1} \in H$. □

As proved in Theorem 12, the Möbius gyrogroup $(\mathbb{B}, \oplus_M)$ forms a subgyrogroup of $(\mathbb{B}_1, \oplus_M)$. Other examples of subgyrogroups are given in the following example:

Example 1. By using the duality between right Bol and left Bol loops, the right Bol loop 8.3.2.1 satisfying the automorphic inverse property in [49] can be turned into a gyrocommutative gyrogroup, called $G_8 = \{0, 1, \ldots, 7\}$, whose addition table is presented in Table 3. In G_8, there is only one nonidentity gyroautomorphism, denoted by A, whose transformation is given by (58):

$$\begin{array}{ll} 0 \mapsto 0 & 4 \mapsto 6 \\ 1 \mapsto 1 & 5 \mapsto 7 \\ 2 \mapsto 2 & 6 \mapsto 4 \\ 3 \mapsto 3 & 7 \mapsto 5 \end{array} \tag{58}$$

The gyration table for G_8 is presented in Table 4. By Proposition 22, $H = \{0, 7\}$, $H_1 = \{0, 3\}$, $H_2 = \{0, 1, 2, 3\}$, and $H_3 = \{0, 3, 4, 6\}$ are easily seen to be subgyrogroups of G_8.

Table 3 Addition table for the gyrocommutative gyrogroup G_8

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	3	0	2	7	4	5	6
2	2	0	3	1	5	6	7	4
3	3	2	1	0	6	7	4	5
4	4	5	7	6	3	2	0	1
5	5	6	4	7	2	0	1	3
6	6	7	5	4	0	1	3	2
7	7	4	6	5	1	3	2	0

Table 4 Gyration table for G_8. There are two gyroautomorphisms of G_8. One is the identity automorphism I and the other is the automorphism A given by (58)

gyr	0	1	2	3	4	5	6	7
0	I	I	I	I	I	I	I	I
1	I	I	I	I	A	A	A	A
2	I	I	I	I	A	A	A	A
3	I	I	I	I	I	I	I	I
4	I	A	A	I	I	A	I	A
5	I	A	A	I	A	I	A	I
6	I	A	A	I	I	A	I	A
7	I	A	A	I	A	I	A	I

6.1 Subgroups of a Gyrogroup

Subgyrogroups that arise as groups with respect to the gyrogroup operation are of great importance in the study of gyrogroups. Such subgyrogroups are called *subgroups* [21].

Definition 5 (Subgroups). A nonempty subset X of a gyrogroup G is a subgroup if it is a group under the operation on G restricted to X.

Indeed, any subgroup of a gyrogroup is simply a subgyrogroup with trivial gyroautomorphisms.

Proposition 23. *A nonempty subset X of a gyrogroup G is a subgroup if and only if it is a subgyrogroup of G and the restriction of* $\mathrm{gyr}[a, b]$ *to X equals the identity map on X for all $a, b \in X$.*

Proof. ($\Rightarrow$) Suppose that X is a subgroup of G. The general left cancellation law implies that the group identity of X and the gyrogroup identity of G coincide. Also, if b is a group inverse of a in X, then $b = \ominus a$. Hence, $X \leqslant G$. For all $a, b, x \in X$, we have $(a \oplus b) \oplus x = a \oplus (b \oplus x) = (a \oplus b) \oplus \mathrm{gyr}[a, b]x$, whence $\mathrm{gyr}[a, b]x = x$. This proves $\mathrm{gyr}[a, b]\big|_X = \mathrm{id}_X$.

($\Leftarrow$) Since $X \neq \emptyset$, there exists an element a of X so that $0 = \ominus a \oplus a \in X$, and 0 acts as a group identity of X. For each $x \in X$, $\ominus x$ acts as a group inverse of x in X. By hypothesis, $a \oplus (b \oplus x) = (a \oplus b) \oplus \mathrm{gyr}[a, b]x = (a \oplus b) \oplus x$ for all $a, b, x \in X$. Hence, the associative law holds in X, and X becomes a group. □

Example 2. Let $\mathbb{B}$ denote the open unit ball of a real Hilbert space H. Recall that $\mathbb{B}$, equipped with the Möbius addition defined by (31) with $t = 1$, is a gyrogroup. According to Eq. (12) of [14], the Möbius gyroautomorphisms are given by $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} + \mathbf{w}$, where

$$\alpha = \frac{2(\langle \mathbf{v}, \mathbf{w}\rangle(1 + 2\langle \mathbf{u}, \mathbf{v}\rangle) - \langle \mathbf{u}, \mathbf{w}\rangle\|\mathbf{v}\|^2)}{1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \quad \text{and} \quad \beta = -\frac{2(\langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle\|\mathbf{u}\|^2)}{1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}.$$

For a fixed $\omega \in \{\mathbf{v} \in H : \|\mathbf{v}\| = 1\}$, Proposition 6 of [14] states that

$$L_\omega = \{t\omega : -1 < t < 1\} \quad \text{and} \quad D_\omega = \{\mathbf{v} \in \mathbb{B} : \langle \mathbf{v}, \omega \rangle = 0\}$$

are subgyrogroups of $\mathbb{B}$. Note that if s and t are real numbers with $-1 < s, t < 1$, then $\dfrac{1 + s\omega\overline{t\omega}}{|1 + s\omega\overline{t\omega}|} = \dfrac{1 + st}{|1 + st|} = 1$. By Theorem 11, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] = \mathrm{id}_{\mathbb{B}}$ for all $\mathbf{u}, \mathbf{v} \in L_\omega$. Hence, by Proposition 23, L_ω is a subgroup of $\mathbb{B}$.

Foguel and Ungar [21] formulated the definition of a *normal subgroup* of a gyrogroup. They showed that any normal subgroup gives rise to a quotient gyrogroup of left cosets and that every gyrogroup contains a normal subgroup. For more information about quotient gyrogroups, see Sect. 7.

Definition 6 (Normal Subgroups). A subgroup X of a gyrogroup G is normal if

1. $\mathrm{gyr}[a, x] = \mathrm{id}_G$ for all $x \in X$, $a \in G$;
2. $\mathrm{gyr}[a, b](X) \subseteq X$ for all $a, b \in G$;
3. $a \oplus X = X \oplus a$ for all $a \in G$.

We remark that condition 2 in Definition 6 implies the reverse inclusion, as shown in the following proposition:

Proposition 24. *Let G be a gyrogroup and let $X \subseteq G$. The following are equivalent:*

1. $\mathrm{gyr}[a, b](X) \subseteq X$ *for all* $a, b \in G$;
2. $\mathrm{gyr}[a, b](X) = X$ *for all* $a, b \in G$.

Proof. Suppose that Item 1 holds. Let $a, b \in G$ and let $d \in X$. Choose $c \in G$ such that $\mathrm{gyr}[a, b]c = d$. By (5), $c = \mathrm{gyr}^{-1}[a, b]d = \mathrm{gyr}[b, a]d$ and, by assumption, $c \in X$. Hence, $d = \mathrm{gyr}[a, b]c \in \mathrm{gyr}[a, b](X)$, and we have $X \subseteq \mathrm{gyr}[a, b](X)$. □

Example 3. By Proposition 23, the subgyrogroups H_1 and H_2 in Example 1 are easily seen to be subgroups of G_8. Furthermore, one can check by inspection that H_1 forms a normal subgroup of G_8.

Let H be a subgyrogroup of a gyrogroup G. For $a \in G$, define

$$a \oplus H = \{a \oplus h : h \in H\} \quad \text{and} \quad H \oplus a = \{h \oplus a : h \in H\},$$

called a *left coset* and a *right coset* of H in G, respectively. The set of left cosets of H in G is denoted by G/H and is referred to as the *coset space*.

Theorem 20 ([21]). *If X is a normal subgroup of a gyrogroup G, then the coset space G/X is a gyrogroup with operation defined by*

$$(a \oplus X) \oplus (b \oplus X) = (a \oplus b) \oplus X.$$

In Sect. 7, we weaken the hypothesis of Theorem 20 by replacing a normal subgroup by a normal subgyrogroup. In Sect. 8, we will make use of the following theorem, due to Foguel and Ungar, to prove an analog of Lagrange's theorem for gyrogroups:

Theorem 21 ([21]). *If G is a gyrogroup, then G has a normal subgroup N such that G/N is a gyrocommutative gyrogroup.*

6.2 Subgyrogroups Generated by Subsets of a Gyrogroup

In full analogy with groups, we examine the subgyrogroup generated by a subset of a gyrogroup, in particular, the cyclic subgyrogroup generated by one element. Many of the results in this subsection are proved by techniques similar to those used in the theory of groups, where the gyroautomorphisms play a major role and the associative law is replaced by the gyroassociative law.

Proposition 25. *Let G be a gyrogroup and let $\mathcal{A}$ be a nonempty collection of subgyrogroups of G. Then the intersection $\bigcap_{H\in\mathcal{A}} H$ forms a subgyrogroup of G.*

Proof. This follows directly from the subgyrogroup criterion. □

Proposition 26. *Let A be a nonempty subset of a gyrogroup G. There exists a unique subgyrogroup of G, denoted by $\langle A\rangle$, such that*

1. *$A \subseteq \langle A\rangle$, and*
2. *if $H \leqslant G$ and $A \subseteq H$, then $\langle A\rangle \subseteq H$.*

Proof. Set $\mathcal{A} = \{H : H \leqslant G \text{ and } A \subseteq H\}$. Then $\langle A\rangle := \bigcap_{H\in\mathcal{A}} H$ is a subgyrogroup of G satisfying the two conditions. The uniqueness of $\langle A\rangle$ follows from condition 2. □

The unique subgyrogroup in Proposition 26 is called the *subgyrogroup generated by A*. It is the smallest subgyrogroup of G containing A in the sense of being the minimal element of the set of all subgyrogroups of G containing A, partially ordered by inclusion. The subgyrogroup generated by one-element set $\{a\}$ is called the *cyclic subgyrogroup generated by a*, which will be denoted by $\langle a\rangle$ instead of the more cumbersome $\langle\{a\}\rangle$. Also, the subgyrogroup generated by $\{a_1, a_2, \ldots, a_n\}$ will be denoted by $\langle a_1, a_2, \ldots, a_n\rangle$.

Next, we will give an explicit description of $\langle a\rangle$. Let G be a gyrogroup and let a be an element of G. For $m \in \mathbb{Z}$, define recursively the following notation:

$$0\cdot a = 0, \quad m\cdot a = a \oplus ((m-1)\cdot a),\ m \geq 1, \quad m\cdot a = (-m)\cdot(\ominus a),\ m < 0. \tag{59}$$

We also define the right counterparts:

$$a\cdot 0 = 0,\quad a\cdot m = (a\cdot(m-1))\oplus a,\ m\geq 1,\quad a\cdot m = (\ominus a)\cdot(-m),\ m<0. \tag{60}$$

Since $\mathrm{gyr}[a,0]=\mathrm{id}_G$ and $\mathrm{gyr}^{-1}[a,b]=\mathrm{gyr}[b,a]$ for all $a,b\in G$, one can prove by induction that $\mathrm{gyr}[a,a\cdot m]=\mathrm{id}_G$ and $\mathrm{gyr}[a\cdot m,a]=\mathrm{id}_G$ for all $a\in G$ and all $m\geq 0$. Combining this with the right gyroassociative law gives $a\cdot m = m\cdot a$ for all $m\in\mathbb{Z}$. If $m<0$, a direct computation gives

$$\begin{aligned}\mathrm{gyr}[a,a\cdot m] &= \mathrm{gyr}[a,(-m)\cdot(\ominus a)]\\ &= \mathrm{gyr}[a\oplus((-m)\cdot(\ominus a)),(-m)\cdot(\ominus a)]\\ &= \mathrm{gyr}[(-m-1)\cdot(\ominus a),\ominus a\oplus(-m-1)\cdot(\ominus a)]\\ &= \mathrm{gyr}[(-m-1)\cdot(\ominus a),\ominus a]\\ &= \mathrm{id}_G.\end{aligned}$$

We have the second and fourth equations from the left and right loop properties and the third equation from the left cancellation law. Thus, we have proved the following lemma:

Lemma 2. *Let G be a gyrogroup. For any element a of G,*

$$\mathrm{gyr}[a\cdot m,a]=\mathrm{gyr}[m\cdot a,a]=\mathrm{gyr}[a,m\cdot a]=\mathrm{gyr}[a,a\cdot m]=\mathrm{id}_G$$

for all $m\in\mathbb{Z}$.

Using Lemma 2, one can prove that

$$(m\cdot a)\oplus(k\cdot a)=(m+k)\cdot a \tag{61}$$

for all $a\in G$ and all $m,k\geq 0$. In fact, (61) holds for all integers m and k.

Theorem 22. *Let a be an element of a gyrogroup. For all $m,k\in\mathbb{Z}$,*

$$(m\cdot a)\oplus(k\cdot a)=(m+k)\cdot a.$$

Proof. In the case $k=0$, the statement is clear, so assume that $k\neq 0$. If $m\geq 0$ and $k>0$, the statement holds by (61). Furthermore, if $m<0$ and $k<0$, then

$$(m\cdot a)\oplus(k\cdot a)=((-m)\cdot(\ominus a))\oplus((-k)\cdot(\ominus a))=(-m-k)\cdot(\ominus a)=(m+k)\cdot a.$$

Note that $a\oplus(n\cdot a)=(n+1)\cdot a$ for all $n\in\mathbb{Z}$.

If $m \geq 0$ and $k < 0$, we proceed by induction on m. Assume inductively that the statement is true for $m > 0$. Then

$$\begin{aligned}((m+1)\cdot a) \oplus (k\cdot a) &= (a \oplus (m\cdot a)) \oplus (k\cdot a)\\ &= a \oplus ((m\cdot a) \oplus \mathrm{gyr}[m\cdot a, a](k\cdot a))\\ &= a \oplus ((m\cdot a) \oplus (k\cdot a))\\ &= a \oplus ((m+k)\cdot a)\\ &= (m+1+k)\cdot a,\end{aligned}$$

which completes the induction.

If $m < 0$ and $k > 0$, then

$$(m\cdot a) \oplus (k\cdot a) = ((-m)\cdot(\ominus a)) \oplus ((-k)\cdot(\ominus a)) = (-m-k)\cdot(\ominus a) = (m+k)\cdot a$$

since $-m > 0$ and $-k < 0$. □

The following theorem gives an explicit description of the cyclic subgyrogroup generated by one element of a gyrogroup:

Theorem 23. *Let G be a gyrogroup and let $a \in G$. Then*

$$\langle a \rangle = \{m\cdot a : m \in \mathbb{Z}\}.$$

In particular, $\langle a \rangle$ is a subgroup of G.

Proof. Set $H = \{m\cdot a : m \in \mathbb{Z}\}$. For each $m \in \mathbb{Z}$, $(-m)\cdot a$ is such that

$$((-m)\cdot a) \oplus (m\cdot a) = 0\cdot a = 0 = (m\cdot a) \oplus ((-m)\cdot a).$$

Hence, $\ominus(m\cdot a) = (-m)\cdot a \in H$. By Theorem 22, $(m\cdot a)\oplus(k\cdot a) = (m+k)\cdot a \in H$ for all $m, k \in \mathbb{Z}$. This proves $H \leqslant G$. Since $a \in H$, we have $\langle a \rangle \subseteq H$ by the minimality of $\langle a \rangle$. By the closure property of subgyrogroups, $H \subseteq \langle a \rangle$ and so equality holds.

Note that $(m\cdot a) \oplus ((n\cdot a) \oplus (k\cdot a)) = (m+n+k)\cdot a = ((m\cdot a) \oplus (n\cdot a)) \oplus (k\cdot a)$ for all $m, n, k \in \mathbb{Z}$. Hence, $\mathrm{gyr}[m\cdot a, n\cdot a]\big|_{\langle a \rangle} = \mathrm{id}_{\langle a \rangle}$ for all $m, n \in \mathbb{Z}$. By Proposition 23, $\langle a \rangle$ is a subgroup of G. □

Theorem 24. *Let G be a gyrogroup. For any element a of G,*

$$\mathrm{gyr}[m\cdot a, n\cdot a] = \mathrm{id}_G$$

for all $m, n \in \mathbb{Z}$.

Proof. First, we will show that $L_{m\cdot a} = L_a^m$ for all $a \in G$ and all $m \geq 0$. This is clear if m is zero. Assume inductively that $L_{m\cdot a} = L_a^m$ for $m \in \mathbb{N}$. Lemma 2 implies

$$L_a^{m+1} = L_a \circ L_a^m = L_a \circ L_{m\cdot a} = L_{a\oplus m\cdot a} \circ \mathrm{gyr}[a, m\cdot a] = L_{(m+1)\cdot a},$$

which completes the induction.

As proved in Theorem 18 (2), $L_a^{-1} = L_{\ominus a}$ for all $a \in G$. Hence, if $m < 0$, then

$$L_{m\cdot a} = L_{(-m)\cdot(\ominus a)} = L_{\ominus a}^{-m} = (L_a^{-1})^{-m} = L_a^m.$$

This proves that $L_{m\cdot a} = L_a^m$ for all $a \in G$ and all $m \in \mathbb{Z}$.

From the composition law (53), we have

$$\mathrm{gyr}[m\cdot a, n\cdot a] = L_{-(m+n)\cdot a} \circ L_{m\cdot a} \circ L_{n\cdot a} = L_a^{-(m+n)} \circ L_a^m \circ L_a^n = \mathrm{id}_G$$

for all $m, n \in \mathbb{Z}$. □

Recall that a loop L is said to be *left power alternative* if every element a in L has a unique inverse and if $L_a^m = L_{a^m}$ for all $m \in \mathbb{Z}$, where the notation a^m is defined by (59) with operation written multiplicatively [36, p. 65]. In light of the proof of Theorem 24, if G is a gyrogroup, then $L_a^m = L_{m\cdot a}$ for all $a \in G$ and all $m \in \mathbb{Z}$. Hence, we obtain the following corollary:

Corollary 8. *Gyrogroups are left power alternative.*

Proposition 27. *If a is an element of a gyrogroup, then $\langle a\rangle$ is a cyclic group with generator a under the gyrogroup operation.*

Proof. By Theorem 23, $\langle a\rangle$ is a group under the gyrogroup operation. By induction, $m\cdot a = a^m$ for all $m \geq 0$, where the notation a^m is used as in group theory. If $m < 0$, one obtains similarly that $m\cdot a = a^m$. Hence, $\langle a\rangle$ is a cyclic group with generator a. □

Corollary 9. *Any gyrogroup generated by one element is a cyclic group.*

Recall that a loop L is said to be *power associative* if every element of L is contained in a cyclic subgroup of L [36, p. 67]. In light of Proposition 27, we obtain the following corollary:

Corollary 10. *Gyrogroups are power associative.*

Theorem 23 prompts us to define the *order* of an element in a gyrogroup.

Definition 7 (Order of an Element). Let G be a gyrogroup and let $a \in G$. The order of a, denoted by $|a|$, is defined to be the cardinality of $\langle a\rangle$ if $\langle a\rangle$ is finite. In this case, we will write $|a| < \infty$. If $\langle a\rangle$ is infinite, the order of a is defined to be infinity, and we will write $|a| = \infty$.

Because the *gyrogroup* order of a coincides with the *group* order of a, we obtain the following results:

Theorem 25. *Let G be a gyrogroup and let $a \in G$.*

1. *If $|a| < \infty$, then $|a|$ is the smallest positive integer such that $|a| \cdot a = 0$.*
2. *If $|a| = \infty$, then $m \cdot a \neq 0$ for all $m \neq 0$ and $m \cdot a \neq k \cdot a$ for all $m \neq k$ in $\mathbb{Z}$.*

Proof.

(1) Suppose that $|a| < \infty$. Let n be the smallest positive integer such that $n \cdot a = 0$. Since $\langle a \rangle$ is finite, such an integer n exists by the well-ordering principle. We claim that $0 \cdot a, 1 \cdot a, \ldots, (n-1) \cdot a$ are all distinct. In fact, if $m \cdot a = k \cdot a$ for some m, k with $0 \leq m < k < n$, then $(k-m) \cdot a = (k \cdot a) \ominus (m \cdot a) = 0$, contrary to the minimality of n. Note that $\{0 \cdot a, 1 \cdot a, \ldots, (n-1) \cdot a\} \subseteq \langle a \rangle$.

For any $m \in \mathbb{Z}$, write $m = nk + r$ for some $k \in \mathbb{Z}$ and $0 \leq r < n$. It follows that $m \cdot a = ((nk) \cdot a) \oplus (r \cdot a)$. By the choice of n, $(nk) \cdot a = 0$. Hence, $m \cdot a = r \cdot a$, which implies $\langle a \rangle \subseteq \{0 \cdot a, 1 \cdot a, \ldots, (n-1) \cdot a\}$ and so equality holds. This proves $|a| = |\langle a \rangle| = n$ and $|a| \cdot a = 0$.

(2) Suppose that $|a| = \infty$. If there were a nonzero integer m such that $m \cdot a = 0$, then we would find a positive integer n such that $n \cdot a = 0$ and would have $|a| < \infty$, contrary to assumption. Hence, $m \cdot a \neq 0$ for all $m \neq 0$. If there were integers $m \neq k$ such that $m \cdot a = k \cdot a$, then $(m-k) \cdot a = 0$, contradicting what we have just proved. □

Corollary 11. *Let a be an element of a gyrogroup. If a is of finite order n, then*

$$\langle a \rangle = \{0 \cdot a, 1 \cdot a, \ldots, (n-1) \cdot a\}.$$

Proof. This is an immediate consequence of Theorem 25 (1). □

Proposition 28. *Let a be an element of a gyrogroup and let $m \in \mathbb{Z} \setminus \{0\}$.*

1. *If $|a| = \infty$, then $|m \cdot a| = \infty$.*
2. *If $|a| < \infty$, then $|m \cdot a| = \dfrac{|a|}{\gcd(|a|, m)}$.*

Proof. Item 1 follows from Theorem 25 (2). As $\langle a \rangle$ is a cyclic group with generator a and the gyrogroup order of a equals the group order of a, we have Item 2. □

6.3 L-Subgyrogroups

Recall that if Γ is a (finite) group and Ξ is a subgroup of Γ, then the relation $\sim$ defined by $g \sim h$ if and only if $g^{-1}h \in \Xi$ is an equivalence relation on Γ. The equivalence class of g is indeed the left coset $g\Xi$ so that the coset space $\{g\Xi : g \in \Gamma\}$ forms a disjoint partition of Γ. Because any left coset of Ξ has the same cardinality as Ξ, the order of Ξ divides the order of Γ. This is *Lagrange's theorem* familiar from abstract algebra.

Let G be a gyrogroup and let H be a subgyrogroup of G. In contrast to groups, the relation

$$a \sim b \quad \text{if and only if} \quad \ominus a \oplus b \in H \tag{62}$$

does not, in general, define an equivalence relation on G. To see this, consider the gyrogroup G_8 and its subgyrogroup $H = \{0, 7\}$ (see Example 1). Note that $\ominus 6 \oplus 2 = 7$ belongs to H, whereas $\ominus 2 \oplus 6 = 5$ does not. In other words, $6 \sim 2$ but $2 \not\sim 6$. This means that the relation $\sim$ is not symmetric and hence cannot be an equivalence relation on G_8. Nevertheless, we can modify (62) to obtain an equivalence relation on G. From this point of view, any subgyrogroup of G partitions G. This leads to the introduction of *L-subgyrogroups*.

Let G be a gyrogroup and let H be a subgyrogroup of G. Define a relation $\sim_H$ on G by letting

$$a \sim_H b \quad \text{if and only if} \quad \ominus a \oplus b \in H \text{ and } \mathrm{gyr}[\ominus a, b](H) = H. \tag{63}$$

In order to prove that (63) defines an equivalence relation on G, we need the following lemma:

Lemma 3. *Let A be a nonempty set, let $B \subseteq A$, and let f be a bijection from A to itself. If $f(B) = B$, then $f^{-1}(B) = B$, where f^{-1} denotes the inverse map of f.*

Proof. The proof is straightforward. □

Theorem 26. *The relation $\sim_H$ defined by (63) is an equivalence relation on G.*

Proof. Reflexive. Let $a \in G$. Then $\ominus a \oplus a = 0 \in H$. By the left cancellation law, $\mathrm{gyr}[\ominus a, a]$ equals id_G. Hence, $\mathrm{gyr}[\ominus a, a](H) = H$, which proves $a \sim_H a$.

Symmetric. Let $a, b \in G$. Suppose that $a \sim_H b$. By (8),

$$\mathrm{gyr}[\ominus a, b](\ominus b \oplus a) = \ominus(\ominus a \oplus b).$$

It follows that $\ominus b \oplus a = \mathrm{gyr}^{-1}[\ominus a, b](\ominus(\ominus a \oplus b))$, which implies $\ominus b \oplus a \in H$ since $\mathrm{gyr}^{-1}[\ominus a, b](H) = H$ by Lemma 3. According to (4) and (5), we have

$$\mathrm{gyr}[\ominus a, b] = \mathrm{gyr}[\ominus a, \ominus(\ominus b)] = \mathrm{gyr}[a, \ominus b] = \mathrm{gyr}^{-1}[\ominus b, a].$$

Hence, $\mathrm{gyr}[\ominus b, a] = \mathrm{gyr}^{-1}[\ominus a, b]$. By the same lemma, $\mathrm{gyr}[\ominus b, a](H) = H$, which proves $b \sim_H a$.

Transitive. Let $a, b, c \in G$. Suppose that $a \sim_H b$ and $b \sim_H c$. By (7),

$$\ominus a \oplus c = (\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b](\ominus b \oplus c)$$

and so $\ominus a \oplus c \in H$. By the composition law (53), $\mathrm{gyr}[\ominus b, c] = L^{-1}_{\ominus b \oplus c} \circ L_{\ominus b} \circ L_c$. Hence, $L_c = L_b \circ L_{\ominus b \oplus c} \circ \mathrm{gyr}[\ominus b, c]$. To complete the proof, we compute

$$\begin{aligned}
&\mathrm{gyr}[\ominus a, c] \\
&\quad = L_{\ominus(\ominus a \oplus c)} \circ L_{\ominus a} \circ L_c \\
&\quad = L^{-1}_{\ominus a \oplus c} \circ L_{\ominus a} \circ L_b \circ L_{\ominus b \oplus c} \circ \mathrm{gyr}[\ominus b, c] \\
&\quad = L^{-1}_{\ominus a \oplus c} \circ L_{\ominus a \oplus b} \circ \mathrm{gyr}[\ominus a, b] \circ L_{\ominus b \oplus c} \circ \mathrm{gyr}[\ominus b, c] \\
&\quad = L^{-1}_{\ominus a \oplus c} \circ L_{\ominus a \oplus b} \circ L_{\mathrm{gyr}[\ominus a, b](\ominus b \oplus c)} \circ \mathrm{gyr}[\ominus a, b] \circ \mathrm{gyr}[\ominus b, c] \\
&\quad = L^{-1}_{\ominus a \oplus c} \circ L_{\ominus a \oplus c} \circ \mathrm{gyr}[\ominus a \oplus b, \mathrm{gyr}[\ominus a, b](\ominus b \oplus c)] \circ \mathrm{gyr}[\ominus a, b] \circ \mathrm{gyr}[\ominus b, c] \\
&\quad = \mathrm{gyr}[\ominus a \oplus b, \mathrm{gyr}[\ominus a, b](\ominus b \oplus c)] \circ \mathrm{gyr}[\ominus a, b] \circ \mathrm{gyr}[\ominus b, c]. \qquad (64)
\end{aligned}$$

We obtain the third and fifth equations from (53) and the fourth equation from (55). It follows from (64) that $\mathrm{gyr}[\ominus a, c](H) = H$, which proves $a \sim_H c$. □

Let H be a subgyrogroup of a gyrogroup G. For each $a \in G$, let $[a]$ denote the equivalence class of a determined by $\sim_H$, that is,

$$[a] = \{x \in G : x \sim_H a\}. \qquad (65)$$

Since any equivalence relation on G induces a partition of G, Theorem 26 implies that the set of equivalence classes, $\{[a] : a \in G\}$, is a partition of G. Furthermore, the equivalence class with representative a is contained in the left coset $a \oplus H$.

Proposition 29. *For each $a \in G$, $[a] \subseteq a \oplus H$.*

Proof. Assume that $x \in [a]$. Then $a \sim_H x$ and by (63), $\ominus a \oplus x \in H$. It follows that $x = a \oplus (\ominus a \oplus x) \in a \oplus H$. This proves $[a] \subseteq a \oplus H$. □

Remark 3. To see that the inclusion in Proposition 29 may be proper, let G and H be as in Example 1. Note that $\ominus 4 \oplus x$ is in H if and only if $x = 4$ or 1. Since $\mathrm{gyr}[\ominus 4, 1](H) = \{0, 5\} \neq H$, we have $4 \not\sim_H 1$. Thus, 1 does not belong to [4] and hence $[4] = \{4\} \subset 4 \oplus H$.

Proposition 29 suggests the following definition of an L-subgyrogroup:

Definition 8 (L-subgyrogroups). A subgyrogroup H of a gyrogroup G is said to be an L-subgyrogroup, denoted by $H \leqslant_L G$, if $\mathrm{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$.

We remark that the definition of an L-subgyrogroup is *symmetric* in the sense that if $\mathrm{gyr}[a, h](H) = H$ with a in G and h in H, then $\mathrm{gyr}[h, a](H) = H$ as well. This is because $\mathrm{gyr}[h, a] = \mathrm{gyr}^{-1}[a, h]$ and so Lemma 3 applies. The importance of an L-subgyrogroup lies in its property: if H is an L-subgyrogroup of G, then $[a] = a \oplus H$ for all $a \in G$. As a consequence, if G is a *finite* gyrogroup, then the order of H divides the order of G.

Example 4. According to Example 1, the gyroautomorphism A of G_8 has cycle decomposition $(4\ 6)(5\ 7)$. From this it is clear that H_1 and H_2 are L-subgyrogroups of G_8, whereas H is not.

Example 5. According to Theorem 19, if $\sigma = L_a \circ \gamma$ and $\tau = L_b \circ \delta$, then

$$\mathrm{gyr}[\sigma, \tau]L_c = L_{\mathrm{gyr}[a,b]c}$$

for all $c \in G$. Hence, $\mathrm{gyr}[\sigma, \tau](\overline{G}) \subseteq \overline{G}$. By Proposition 24, $\mathrm{gyr}[\sigma, \tau](\overline{G}) = \overline{G}$, which implies that $\overline{G}$ is an L-subgyrogroup of $\mathrm{Sym}\,(G)$.

Example 6. Recall that L_ω and D_ω in Example 2 are subgyrogroups of the Möbius gyrogroup. However, D_ω is not an L-subgyrogroup. Specifically, if $\mathbf{0} \neq \mathbf{v} = \mathbf{w} \in D_\omega$ and $\mathbf{u} = \omega$, then, since $D_\omega = L_\omega^{\perp}$,

$$\langle \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}, \omega\rangle = \langle \alpha\mathbf{u} + \beta\mathbf{v} + \mathbf{w}, \omega\rangle = \alpha = \frac{2\|\mathbf{v}\|^2}{1 + \|\mathbf{v}\|^2} \neq 0.$$

Thus, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} \notin D_\omega$ and hence $\mathrm{gyr}[\mathbf{u}, \mathbf{v}](D_\omega) \neq D_\omega$.

Proposition 30. *Let G be a gyrogroup. Then*

$$H = \{x \in G\colon \forall a, b \in G,\ \mathrm{gyr}[a, b]x = x\}$$

forms an L-subgyrogroup of G.

Proof. $H \neq \emptyset$ since $0 \in H$. If $x \in H$, then $\mathrm{gyr}[a, b](\ominus x) = \ominus\mathrm{gyr}[a, b]x = \ominus x$ for all $a, b \in G$. Hence, $\ominus x \in H$. If $x, y \in H$, then $\mathrm{gyr}[a, b](x \oplus y) = (\mathrm{gyr}[a, b]x) \oplus (\mathrm{gyr}[a, b]y) = x \oplus y$ for all $a, b \in G$. Hence, $x \oplus y \in H$. This proves $H \leqslant G$. Since $\mathrm{gyr}[a, b](H) \subseteq H$ for all $a, b \in G$, H forms an L-subgyrogroup of G. □

Proposition 31. *Let G be a gyrogroup and let $a \in G$. If $H \leqslant_L G$, then $[a] = a \oplus H$.*

Proof. Assume that $H \leqslant_L G$. By Proposition 29, $[a] \subseteq a \oplus H$. If $x = a \oplus h$ for some $h \in H$, then $\ominus a \oplus x = h$ is in H. The left and right loop properties together imply $\mathrm{gyr}[\ominus a, x] = \mathrm{gyr}[\ominus a \oplus x, x] = \mathrm{gyr}[h, a \oplus h] = \mathrm{gyr}[h, a] = \mathrm{gyr}^{-1}[a, h]$. By assumption, $\mathrm{gyr}[a, h](H) = H$ and by Lemma 3, $\mathrm{gyr}[\ominus a, x](H) = \mathrm{gyr}^{-1}[a, h](H) = H$. Hence, $a \sim_H x$ or, equivalently, $x \in [a]$ and we have the reverse inclusion. □

Theorem 27. *If H is an L-subgyrogroup of a gyrogroup G, then the coset space $\{a \oplus H\colon a \in G\}$ is a disjoint partition of G.*

Proof. Since $\sim_H$ is an equivalence relation on G, the set $\{[a]\colon a \in G\}$ is a partition of G. By Proposition 31, $[a] = a \oplus H$, which completes the proof. □

Recall that a loop L is said to have a *left coset expansion* modulo its subloop Y provided that the left cosets of Y in L partition L [5, p. 92]. Theorem 27 says that any (finite or infinite) gyrogroup has a left coset expansion modulo its L-subgyrogroup. Further, we have a version of Lagrange's theorem for L-subgyrogroups.

Lemma 4. *Let G be a gyrogroup and let $a \in G$. If $H \leqslant G$, then H and $a \oplus H$ have the same cardinality.*

Proof. The restriction of L_a to H is a bijection from H to $a \oplus H$. □

Theorem 28 (Lagrange's Theorem for L-subgyrogroups). *In a finite gyrogroup G, if $H \leqslant_L G$, then $|H|$ divides $|G|$.*

Proof. Being a finite gyrogroup, G has a finite number of left cosets, namely, $a_1 \oplus H$, $a_2 \oplus H, \ldots, a_n \oplus H$. By Lemma 4, $|a_i \oplus H| = |H|$ for $i = 1, 2, \ldots, n$. It follows that

$$|G| = \left|\bigcup_{i=1}^{n} a_i \oplus H\right| = \sum_{i=1}^{n} |a_i \oplus H| = n|H|,$$

which completes the proof. □

We will prove in Sect. 8 that Lagrange's theorem holds for finite gyrogroups, that is, if H is an *arbitrary* subgyrogroup of a finite gyrogroup G, then the order of G is divisible by the order of H.

Let H be a subgyrogroup of a gyrogroup G. As in group theory, the *index* of H in G, written $[G\colon H]$, is defined to be the cardinality of the coset space G/H if G/H has a finite number of left cosets. If G/H has infinitely many left cosets, the index $[G\colon H]$ is defined to be infinity, and we will write $[G\colon H] = \infty$.

Corollary 12. *In a finite gyrogroup G, if $H \leqslant_L G$, then $|G| = [G\colon H]|H|$.*

Proof. See the calculation in the proof of Theorem 28. □

For a *non*-L-subgyrogroup K of a gyrogroup G, it is no longer true that the left cosets of K partition G. To see this, consider the gyrogroup G_8 in Example 1. One can check by inspection that $1 \oplus H = \{1, 6\}$ and $4 \oplus H = \{1, 4\}$. Hence, $1 \oplus H$ and $4 \oplus H$ have nonempty intersection. Moreover, the formula $|G| = [G\colon K]|K|$ is not true, in general.

7 Gyrogroup Homomorphisms and Normal Subgyrogroups

In this section, we examine gyrogroup homomorphisms, normal subgyrogroups, and quotient gyrogroups, in full analogy with group theory. The main goal of this section is to prove the isomorphism theorems for gyrogroups [61].

7.1 Gyrogroup Homomorphisms

Let G and H be gyrogroups. A map $\varphi\colon G \to H$ is called a *gyrogroup homomorphism* if $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$. A bijective gyrogroup homomorphism is called a *gyrogroup isomorphism*. We say that G and H are *isomorphic* gyrogroups, written $G \cong H$, if there exists a gyrogroup isomorphism from G to H. Of course, isomorphic gyrogroups are essentially the same, differing only in the notation for the elements.

Proposition 32. *Let $\varphi\colon G \to H$ be a homomorphism of gyrogroups.*

1. $\varphi(0) = 0$.
2. $\varphi(\ominus a) = \ominus\varphi(a)$ *for all* $a \in G$.
3. $\varphi(\mathrm{gyr}[a, b]c) = \mathrm{gyr}[\varphi(a), \varphi(b)]\varphi(c)$ *for all* $a, b, c \in G$.
4. $\varphi(a \boxplus b) = \varphi(a) \boxplus \varphi(b)$ *for all* $a, b \in G$.

Proof. Item 1 follows from the left cancellation law. Item 2 follows from the uniqueness of an inverse in a gyrogroup. Item 3 follows from the gyrator identity. To prove Item 4, we compute

$$\varphi(a \boxplus b) = \varphi(a \oplus \mathrm{gyr}[a, \ominus b]b) = \varphi(a) \oplus \mathrm{gyr}[\varphi(a), \ominus\varphi(b)]\varphi(b) = \varphi(a) \boxplus \varphi(b).$$

□

The proof of the following two propositions is routine, using the subgyrogroup criterion and the definition of an L-subgyrogroup:

Proposition 33. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. If $K \leqslant G$, then $\varphi(K) \leqslant H$. If $K \leqslant_L G$ and if φ is surjective, then $\varphi(K) \leqslant_L H$.*

Proposition 34. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism. If $K \leqslant H$, then $\varphi^{-1}(K) \leqslant G$. If $K \leqslant_L H$, then $\varphi^{-1}(K) \leqslant_L G$.*

Let G and H be gyrogroups and let $\varphi\colon G \to H$ be a gyrogroup homomorphism. Define the *kernel* of φ to be the set

$$\{a \in G\colon \varphi(a) = 0\}.$$

In other words, $\ker\varphi = \varphi^{-1}(\{0\})$. By Proposition 34, the kernel of φ is a subgyrogroup of G. Indeed, it is an L-subgyrogroup of G.

Proposition 35. *If $\varphi\colon G \to H$ is a gyrogroup homomorphism, then the kernel of φ is an L-subgyrogroup of G.*

Proof. For all $a, b \in G$, $c \in \ker\varphi$,

$$\varphi(\mathrm{gyr}[a, b]c) = \mathrm{gyr}[\varphi(a), \varphi(b)]\varphi(c) = \mathrm{gyr}[\varphi(a), \varphi(b)]0 = 0.$$

Hence, $\mathrm{gyr}[a,b](\ker\varphi) \subseteq \ker\varphi$. By Proposition 24, $\mathrm{gyr}[a,b](\ker\varphi) = \ker\varphi$. This proves $\ker\varphi \leqslant_L G$. □

In light of the proof of Proposition 35, the kernel of φ is invariant under all the gyroautomorphisms of G. Hence, the relation (63) becomes

$$a \sim_{\ker\varphi} b \quad \text{if and only if} \quad \ominus a \oplus b \in \ker\varphi \tag{66}$$

for all $a, b \in G$. More precisely, we have the following proposition:

Proposition 36. *Let $\varphi: G \to H$ be a homomorphism of gyrogroups. For all $a, b \in G$, the following are equivalent:*

1. $a \sim_{\ker\varphi} b$;
2. $\ominus a \oplus b \in \ker\varphi$;
3. $\varphi(a) = \varphi(b)$;
4. $a \oplus \ker\varphi = b \oplus \ker\varphi$.

Proof. By Proposition 32 (2), $\ominus a \oplus b \in \ker\varphi$ if and only if $\varphi(\ominus a \oplus b) = 0$ if and only if $\varphi(a) = \varphi(b)$. Since $\sim_{\ker\varphi}$ is an equivalence relation on G and $\ker\varphi \leqslant_L G$, $a \sim_{\ker\varphi} b$ if and only if $[a] = [b]$ if and only if $a \oplus \ker\varphi = b \oplus \ker\varphi$. □

Proposition 36 allows us to define an operation on the coset space $G/\ker\varphi$ in the following natural way:

$$(a \oplus \ker\varphi) \oplus (b \oplus \ker\varphi) = (a \oplus b) \oplus \ker\varphi, \qquad a, b \in G. \tag{67}$$

This operation is independent of the choice of representatives for the left cosets, that is, it is a well-defined operation. Specifically, let $c \in a \oplus \ker\varphi$ and $d \in b \oplus \ker\varphi$. Then $c = a \oplus k_1$ and $d = b \oplus k_2$ for some $k_1, k_2 \in \ker\varphi$. Since

$$\varphi(c \oplus d) = \varphi(a \oplus k_1) \oplus \varphi(b \oplus k_2) = \varphi(a) \oplus \varphi(b) = \varphi(a \oplus b),$$

it follows from Proposition 36 that $(a \oplus b) \oplus \ker\varphi = (c \oplus d) \oplus \ker\varphi$. In fact, the coset space $G/\ker\varphi$ forms a gyrogroup, called a *quotient gyrogroup*.

Theorem 29. *If $\varphi: G \to H$ is a gyrogroup homomorphism, then $G/\ker\varphi$ with operation defined by (67) is a gyrogroup.*

Proof. Set $K = \ker\varphi$.

(G1) The coset $0 \oplus K$ is a left identity: $(0 \oplus K) \oplus (a \oplus K) = (0 \oplus a) \oplus K = a \oplus K$.

(G2) Let $a \oplus K \in G/K$. The coset $(\ominus a) \oplus K$ is a left inverse of $a \oplus K$:

$$((\ominus a) \oplus K) \oplus (a \oplus K) = (\ominus a \oplus a) \oplus K = 0 \oplus K.$$

(G3) For $X = a \oplus K, Y = b \oplus K \in G/K$, define

$$\mathrm{gyr}[X, Y](c \oplus K) = (\mathrm{gyr}[a,b]c) \oplus K, \qquad c \oplus K \in G/K.$$

If $d \in c \oplus K$, then $d = c \oplus k$ with k in K. Note that

$$\varphi(\text{gyr}[a,b]d) = \varphi(\text{gyr}[a,b]c) \oplus \varphi(\text{gyr}[a,b]k) = \varphi(\text{gyr}[a,b]c).$$

Hence, $(\text{gyr}[a,b]d) \oplus K = (\text{gyr}[a,b]c) \oplus K$, which proves $\text{gyr}[X,Y]$ is well defined.
Let $d \oplus K$ be an arbitrary coset of G/K. Choose $c \in G$ such that $\text{gyr}[a,b]c = d$. Since $\text{gyr}[X,Y](c \oplus K) = (\text{gyr}[a,b]c) \oplus K = d \oplus K$, $\text{gyr}[X,Y]$ is surjective. Since

$$\begin{aligned}
\text{gyr}[X,Y](c \oplus K) = \text{gyr}[X,Y](d \oplus K) &\Rightarrow (\text{gyr}[a,b]c) \oplus K = (\text{gyr}[a,b]d) \oplus K \\
&\Rightarrow \varphi(\text{gyr}[a,b]c) = \varphi(\text{gyr}[a,b]d) \\
&\Rightarrow \text{gyr}[\varphi(a),\varphi(b)]\varphi(c) = \text{gyr}[\varphi(a),\varphi(b)]\varphi(d) \\
&\Rightarrow \varphi(c) = \varphi(d) \\
&\Rightarrow c \oplus K = d \oplus K,
\end{aligned}$$

$\text{gyr}[X,Y]$ is injective. Furthermore, $\text{gyr}[X,Y]$ preserves $\oplus$:

$$\begin{aligned}
\text{gyr}[X,Y]((c \oplus K) \oplus (d \oplus K)) &= (\text{gyr}[a,b](c \oplus d)) \oplus K \\
&= (\text{gyr}[a,b]c \oplus K) \oplus (\text{gyr}[a,b]d \oplus K) \\
&= \text{gyr}[X,Y](c \oplus K) \oplus \text{gyr}[X,Y](d \oplus K).
\end{aligned}$$

This proves $\text{gyr}[X,Y]$ is an automorphism of G/K.
(G4) For $X = a \oplus K, Y = b \oplus K, Z = c \oplus K \in G/K$,

$$\begin{aligned}
\text{gyr}[X \oplus Y, Y]Z &= (\text{gyr}[(a \oplus b) \oplus K, b \oplus K]Z \\
&= (\text{gyr}[a \oplus b, b]c) \oplus K \\
&= (\text{gyr}[a,b]c) \oplus K \\
&= \text{gyr}[X,Y]Z.
\end{aligned}$$

Hence, $\text{gyr}[X \oplus Y, Y] = \text{gyr}[X,Y]$, and the left loop property holds. □

The map $\Pi\colon G \to G/\ker\varphi$ given by $\Pi(a) = a \oplus \ker\varphi$ defines a surjective gyrogroup homomorphism, which will be referred to as the *canonical projection*. Since $a \oplus \ker\varphi = 0 \oplus \ker\varphi$ if and only if $a \in \ker\varphi$, we have $\ker\Pi = \ker\varphi$.

As the coset space $G/\ker\varphi$ forms a gyrogroup, it is reasonable to speculate that the isomorphism theorems hold for gyrogroups. The following theorem shows that this is the case:

Theorem 30 (The First Isomorphism Theorem). *If $\varphi\colon G \to H$ is a gyrogroup homomorphism, then $G/\ker\varphi \cong \varphi(G)$ as gyrogroups.*

Proof. Set $K = \ker\varphi$. Define $\phi\colon G/K \to \varphi(G)$ by $\phi(a \oplus K) = \varphi(a)$. By Proposition 36, ϕ is well defined and injective. A direct computation gives

$$\phi((a \oplus K) \oplus (b \oplus K)) = \phi(a \oplus K) \oplus \phi(b \oplus K)$$

for all $a, b \in G$. Hence, ϕ is a gyrogroup isomorphism from G/K onto $\varphi(G)$. □

Proposition 37. *Let $\varphi\colon G \to H$ be a gyrogroup homomorphism.*

1. *φ is injective if and only if* $\ker\varphi = \{0\}$.
2. $[G\colon \ker\varphi] = |\varphi(G)|$.

Proof. Item 1 follows from Propositions 32 (1) and 36. Item 2 is an immediate consequence of the first isomorphism theorem. □

Before stating the remaining isomorphism theorems for gyrogroups, we need to introduce the notion of normal subgyrogroups. Recall that a subgroup Ξ of a group Γ is normal if and only if $g\Xi g^{-1} = \Xi$ for all $g \in \Gamma$. A normal subgyrogroup cannot be defined in this way because of the lack of associativity in a gyrogroup. However, one can define a normal subgyrogroup using another characterization of a normal subgroup.

7.2 Normal Subgyrogroups

It is known in the literature that a subgroup of a group is normal if and only if it is the kernel of some group homomorphism; see, for instance, [12, Proposition 7]. This characterization of a normal subgroup allows us to define a normal subgyrogroup in a similar fashion, as follows.

Definition 9 (Normal Subgyrogroups). A subgyrogroup N of a gyrogroup G is normal in G, written $N \trianglelefteq G$, if it is the kernel of a gyrogroup homomorphism of G.

Let G be a gyrogroup. By Theorem 29, any normal subgyrogroup N of G gives rise to the quotient gyrogroup G/N. Furthermore, the relation (63) becomes

$$a \sim_N b \quad \Leftrightarrow \quad \ominus a \oplus b \in N \quad \Leftrightarrow \quad a \oplus N = b \oplus N \tag{68}$$

for all $a, b \in G$. By Proposition 35, N is an L-subgyrogroup of G so that if G is finite, then $|G| = [G\colon N]|N|$. This index formula will be useful for proving that finite gyrogroups satisfy the Lagrange property in the next section.

By Theorem 20, if X is a normal subgroup of a gyrogroup G, then G/X is a gyrogroup. Since X equals the kernel of the canonical projection $\Pi\colon G \to G/X$, X is also a normal subgyrogroup of G. Therefore, Theorem 29 is an extension of Theorem 20.

The following theorem gives a characterization of normal subgyrogroups, in full analogy with its group counterpart:

Theorem 31. *Let H be a subgyrogroup of a gyrogroup G. Then $H \trianglelefteq G$ if and only if the operation on the coset space G/H given by*

$$(a \oplus H) \oplus (b \oplus H) = (a \oplus b) \oplus H \tag{69}$$

is well defined.

Proof. The "only if" part is clear. In order to prove the "if" part, we must define the gyroautomorphisms of G/H.

Note that $a \in H$ if and only if $a \oplus H = 0 \oplus H$. We first prove $\text{gyr}[a, b](H) \subseteq H$ for all $a, b \in G$. By Proposition 24, this implies $\text{gyr}[a, b](H) = H$ for all $a, b \in G$. Let $a, b \in G$ and let $h \in H$. A direct computation gives

$$\begin{aligned}(\ominus(a \oplus b) \oplus (a \oplus (b \oplus h))) \oplus H &= (\ominus(a \oplus b) \oplus H) \oplus ((a \oplus H) \oplus ((b \oplus h) \oplus H))\\ &= (\ominus(a \oplus b) \oplus H) \oplus ((a \oplus b) \oplus H)\\ &= 0 \oplus H.\end{aligned}$$

According to the gyrator identity, $\text{gyr}[a, b]h = \ominus(a \oplus b) \oplus (a \oplus (b \oplus h)) \in H$. Since h is arbitrary, $\text{gyr}[a, b](H) \subseteq H$.

For $X = a \oplus H, Y = b \oplus H$, let us define

$$\text{gyr}[X, Y](c \oplus H) = (\text{gyr}[a, b]c) \oplus H, \qquad c \oplus H \in G/H.$$

To prove that $\text{gyr}[X, Y]$ is well defined, suppose that $c \oplus H = d \oplus H$. Hence, $d = c \oplus h$ for some $h \in H$. It follows that

$$(\text{gyr}[a, b]d) \oplus H = ((\text{gyr}[a, b]c) \oplus H) \oplus ((\text{gyr}[a, b]h) \oplus H) = (\text{gyr}[a, b]c) \oplus H,$$

noting that $\text{gyr}[a, b]h \in H$ and so $(\text{gyr}[a, b]h) \oplus H = 0 \oplus H$.

Suppose that $\text{gyr}[X, Y](c \oplus H) = \text{gyr}[X, Y](d \oplus H)$. By definition of $\text{gyr}[X, Y]$,

$$(\text{gyr}[a, b](\ominus c \oplus d)) \oplus H = 0 \oplus H,$$

which implies $\text{gyr}[a, b](\ominus c \oplus d) \in H$. Since $\text{gyr}[a, b](H) = H$, we have $\ominus c \oplus d = h_1$ for some $h_1 \in H$. Thus, $d = c \oplus h_1$. If $x \in d \oplus H$, then $x = d \oplus h_2$, with $h_2 \in H$. Hence, $x = (c \oplus h_1) \oplus h_2 = c \oplus (h_1 \oplus \text{gyr}[h_1, c]h_2) \in c \oplus H$. This proves $d \oplus H \subseteq c \oplus H$. Likewise, $c = d \oplus h_3$ for some $h_3 \in H$, which implies $c \oplus H \subseteq d \oplus H$, and so equality holds. This proves $\text{gyr}[X, Y]$ is injective.

The rest of the proof that G/H forms a gyrogroup runs as in the proof of Theorem 29, with K replaced by H. Hence, $\Pi\colon a \mapsto a \oplus H$ defines a surjective gyrogroup homomorphism from G to G/H with kernel H, so $H \trianglelefteq G$. □

The following proposition provides a sufficient condition for normality of a subgyrogroup:

Proposition 38. *Let G be a gyrogroup. If H is a subgyrogroup of G such that*

1. $\mathrm{gyr}[h, a] = \mathrm{id}_G$ *for all* $h \in H, a \in G$*;*
2. $\mathrm{gyr}[a, b](H) \subseteq H$ *for all* $a, b \in G$*; and*
3. $a \oplus H = H \oplus a$ *for all* $a \in G$*,*

then $H \trianglelefteq G$*.*

Proof. To complete the proof, we must verify that the operation given by (69) is well defined. By condition 1, H is an L-subgyrogroup of G. From (63) and the left loop property, we have $a \oplus H = b \oplus H$ if and only if $\ominus a \oplus b \in H$ for all $a, b \in G$.

Suppose that $a_1 \in a \oplus H$ and $b_1 \in b \oplus H$. Then $a_1 = a \oplus h_1$ and $b_1 = b \oplus k_1$ for some $h_1, k_1 \in H$. By condition 3, $h_1 \oplus b = b \oplus h_2$ with $h_2 \in H$. Set $k_2 = \mathrm{gyr}[a \oplus h_1, b]k_1$ and $k_3 = \mathrm{gyr}[\mathrm{gyr}[a, b]h_2, a \oplus b]k_2$. By condition 2, $k_2, k_3 \in H$. The left and right loop properties imply $a_1 \oplus b_1 = (a \oplus b) \oplus ((\mathrm{gyr}[a, b]h_2) \oplus k_3)$. Hence, $\ominus(a \oplus b) \oplus (a_1 \oplus b_1) = (\mathrm{gyr}[a, b]h_2) \oplus k_3 \in H$ and so $(a \oplus b) \oplus H = (a_1 \oplus b_1) \oplus H$. □

Proposition 39. *Let G be a gyrogroup. If* $N \trianglelefteq G$*, then* $a \oplus N = N \oplus a$ *for all* $a \in G$*.*

Proof. By assumption, $N = \ker \varphi$, where φ is a gyrogroup homomorphism of G. Let $a \in G$ and let $y \in N \oplus a$. Then $y = n \oplus a$ with $n \in N$. By Theorem 2, $n_1 = \ominus a \oplus (n \oplus a)$ is a solution to the equation $n \oplus a = a \oplus x$ in the variable x. Since

$$\varphi(n_1) = \varphi(\ominus a \oplus (n \oplus a)) = \varphi(\ominus a) \oplus \varphi(a) = 0,$$

n_1 belongs to N, and hence $y = n \oplus a = a \oplus n_1 \in a \oplus N$. This proves $N \oplus a \subseteq a \oplus N$. Similarly, $a \oplus N \subseteq N \oplus a$ and so equality holds. □

As an example of the use of Proposition 39, we find that the cyclic subgyrogroup $\langle 5 \rangle$ is not normal in G_8 since $2 \oplus \langle 5 \rangle = \{2, 6\}$, whereas $\langle 5 \rangle \oplus 2 = \{2, 4\}$. This also shows that subgyrogroups of a gyrocommutative gyrogroup need not be normal in that gyrogroup.

Proposition 40. *Let G be a gyrogroup. If* $N \trianglelefteq G$*, then*

$$(a \oplus b) \oplus N = a \oplus (N \oplus b) = N \oplus (a \oplus b) = (a \oplus N) \oplus b \tag{70}$$

for all $a, b \in G$*.*

Proof. Let $N = \ker \varphi$. The verification of the first two equalities follows the same steps as in the proof of Proposition 39. Next, we prove that $a \oplus (N \oplus b) = (a \oplus N) \oplus b$. For each $n \in N$, by Theorem 2, $x = (a \oplus (n \oplus b)) \boxplus (\ominus b)$ is such that $a \oplus (n \oplus b) = x \oplus b$. A direct computation gives

$$\varphi(x) = \varphi((a \oplus (n \oplus b)) \boxplus (\ominus b)) = (\varphi(a) \oplus \varphi(b)) \boxplus (\ominus \varphi(b)) = \varphi(a).$$

Hence, $\ominus a \oplus x \in N$ and so $x \in a \oplus N$. This implies $a \oplus (n \oplus b) = x \oplus b \in (a \oplus N) \oplus b$ and so $a \oplus (N \oplus b) \subseteq (a \oplus N) \oplus b$. Conversely, $y = \ominus a \oplus ((a \oplus n) \oplus b)$ is such that $(a \oplus n) \oplus b = a \oplus y$. Since $y = (\ominus a \oplus (a \oplus n)) \oplus \mathrm{gyr}[\ominus a, a \oplus n]b = n \oplus \mathrm{gyr}[\ominus a, a \oplus n]b$, we have $\varphi(y) = \mathrm{gyr}[\ominus \varphi(a), \varphi(a)]\varphi(b) = \varphi(b)$. Hence, $y \in b \oplus N = N \oplus b$. It follows that $(a \oplus n) \oplus b = a \oplus y \in a \oplus (N \oplus b)$, and the reverse inclusion holds. □

It is worth pointing out that (70) is a characteristic property of normal subgyrogroups, as shown in the following theorem:

Theorem 32. *Let H be a subgyrogroup of a gyrogroup G. Then $H \trianglelefteq G$ if and only if*

$$a \oplus (H \oplus b) = (a \oplus b) \oplus H = (a \oplus H) \oplus b$$

for all $a, b \in G$.

Proof. The "only if" part follows from Proposition 40. To prove the "if" part, we show that the operation given by (69) is well defined. Setting $b = 0$ in the hypothesis, we find that $a \oplus H = H \oplus a$ for all $a \in G$. Suppose that $a \oplus H = a_1 \oplus H$ and $b \oplus H = b_1 \oplus H$. Then

$$\begin{aligned}(a_1 \oplus b_1) \oplus H &= a_1 \oplus (H \oplus b_1)\\ &= a_1 \oplus (H \oplus b)\\ &= (a_1 \oplus H) \oplus b\\ &= (a \oplus H) \oplus b\\ &= (a \oplus b) \oplus H.\end{aligned}$$

This proves the operation (69) is well defined and by Theorem 31, $H \trianglelefteq G$. □

7.3 Isomorphism Theorems

In Sect. 7.1, we proved the first isomorphism theorem for gyrogroups (Theorem 30). In this subsection, we prove the remaining isomorphism theorems. The isomorphism theorems are extremely useful in the study of the structure of a gyrogroup. For instance, the second isomorphism theorem and the lattice isomorphism theorem are used in proving that finite gyrogroups have the Lagrange property. For more information about this, see the next section.

Lemma 5. *Let G be a gyrogroup. If $A \leqslant G$ and $B \trianglelefteq G$, then*

$$A \oplus B := \{a \oplus b : a \in A, b \in B\}$$

forms a subgyrogroup of G.

Proof. By assumption, $B = \ker\phi$, where ϕ is a gyrogroup homomorphism of G. From Proposition 39, we have $B \oplus a = a \oplus B$ for all $a \in G$.

Let $x = a \oplus b$, with $a \in A$, $b \in B$. Since $\phi(\mathrm{gyr}[a,b]\ominus a) = \mathrm{gyr}[\phi(a),0]\phi(\ominus a) = \phi(\ominus a)$, we have $\mathrm{gyr}[a,b]\ominus a = \ominus a \oplus b_1$ for some $b_1 \in B$. Set $b_2 = \mathrm{gyr}[a,b]\ominus b$. Since $b_2 \in B$ and $B \oplus (\ominus a) = (\ominus a) \oplus B$, there is an element $b_3 \in B$ for which $b_2 \ominus a = \ominus a \oplus b_3$. It follows that

$$\begin{aligned}
\ominus x &= \ominus(a \oplus b)\\
&= \mathrm{gyr}[a,b](\ominus b \ominus a)\\
&= b_2 \oplus (\ominus a \oplus b_1)\\
&= (b_2 \ominus a) \oplus \mathrm{gyr}[b_2, \ominus a]b_1\\
&= (\ominus a \oplus b_3) \oplus \mathrm{gyr}[b_2, \ominus a]b_1\\
&= \ominus a \oplus (b_3 \oplus \mathrm{gyr}[b_3, \ominus a](\mathrm{gyr}[b_2, \ominus a]b_1))
\end{aligned}$$

belongs to $A \oplus B$.

For $x, y \in A \oplus B$, we have $x = a \oplus b$ and $y = c \oplus d$ for some $a, c \in A$, $b, d \in B$. Since $\phi(b \oplus \mathrm{gyr}[b,a](c \oplus d)) = \phi(b) \oplus \mathrm{gyr}[\phi(b), \phi(a)](\phi(c) \oplus \phi(d)) = \phi(c)$, we have $b \oplus \mathrm{gyr}[b,a](c \oplus d) = c \oplus b_1$ for some $b_1 \in B$. It follows that

$$x \oplus y = a \oplus (b \oplus \mathrm{gyr}[b,a](c \oplus d)) = a \oplus (c \oplus b_1) = (a \oplus c) \oplus \mathrm{gyr}[a,c]b_1 \tag{71}$$

belongs to $A \oplus B$. Hence, $A \oplus B \leqslant G$. ◻

Corollary 13. *Let G be a gyrogroup. If $A \leqslant G$ and $B \trianglelefteq G$, then $A \oplus B = B \oplus A$ and $B \oplus A$ forms a subgyrogroup of G.*

Proof. Since $B \trianglelefteq G$, $B \oplus a = a \oplus B$ for all $a \in G$. This implies $B \oplus A = A \oplus B$. Hence, $B \oplus A \leqslant G$.

Theorem 33 (The Second Isomorphism Theorem). *Let G be a gyrogroup and let $A, B \leqslant G$. If $B \trianglelefteq G$, then $A \cap B \trianglelefteq A$ and $(A \oplus B)/B \cong A/(A \cap B)$ as gyrogroups.*

Proof. As in Lemma 5, $B = \ker\phi$. Note that $A \cap B \trianglelefteq A$ since $\ker\phi\big|_A = A \cap B$. Hence, $A/(A \cap B)$ admits the quotient gyrogroup structure.

Define $\varphi\colon A \oplus B \to A/(A \cap B)$ by $\varphi(a \oplus b) = a \oplus (A \cap B)$ for $a \in A$ and $b \in B$. To see that φ is well defined, suppose that $a \oplus b = a_1 \oplus b_1$, where $a, a_1 \in A$ and $b, b_1 \in B$. Note that $b_1 = \ominus a_1 \oplus (a \oplus b) = (\ominus a_1 \oplus a) \oplus \mathrm{gyr}[\ominus a_1, a]b$. Set $b_2 = \ominus\mathrm{gyr}[\ominus a_1, a]b$. Then $b_2 \in B$ and $b_1 = (\ominus a_1 \oplus a) \ominus b_2$. The right cancellation law I gives

$$\ominus a_1 \oplus a = b_1 \boxplus b_2 = b_1 \oplus \mathrm{gyr}[b_1, \ominus b_2]b_2,$$

which implies $\ominus a_1 \oplus a \in A \cap B$. By Proposition 36, $a_1 \oplus (A \cap B) = a \oplus (A \cap B)$.

As we computed in (71), if $a, c \in A$ and $b, d \in B$, then

$$(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus \text{gyr}[a, c]\tilde{b}$$

for some $\tilde{b} \in B$. Hence,

$$\begin{aligned}\varphi((a \oplus b) \oplus (c \oplus d)) &= \varphi((a \oplus c) \oplus \text{gyr}[a, c]\tilde{b})\\ &= (a \oplus c) \oplus A \cap B\\ &= (a \oplus (A \cap B)) \oplus (c \oplus (A \cap B))\\ &= \varphi(a \oplus b) \oplus \varphi(c \oplus d).\end{aligned}$$

This proves that $\varphi: A \oplus B \to A/(A \cap B)$ is a surjective gyrogroup homomorphism whose kernel is $\{a \oplus b: a \in A, b \in B, a \in A \cap B\} = B$. Thus, $B \trianglelefteq A \oplus B$ and by the first isomorphism theorem, $(A \oplus B)/B \cong A/(A \cap B)$. □

Corollary 14. *Let A and B be finite subgyrogroups of G. If $B \trianglelefteq G$, then*

$$|A \oplus B| = \frac{|A||B|}{|A \cap B|}.$$

Proof. Since $B \trianglelefteq A \oplus B$, $|A \oplus B| = [A \oplus B: B]|B|$. Similarly, $|A| = [A: A \cap B]|A \cap B|$. Combining these with the fact that $(A \oplus B)/B \cong A/(A \cap B)$ gives the desired equality. □

Theorem 34 (The Third Isomorphism Theorem). *Let G be a gyrogroup and let H, K be normal subgyrogroups of G such that $H \subseteq K$. Then $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong G/K$ as gyrogroups.*

Proof. Let ϕ and ψ be gyrogroup homomorphisms of G such that $\ker \phi = H$ and $\ker \psi = K$. Define $\varphi: G/H \to G/K$ by $\varphi(a \oplus H) = a \oplus K$ for $a \in G$. To see that φ is well defined, suppose that $a \oplus H = b \oplus H$. Hence, $\ominus a \oplus b \in H$. Since $H \subseteq K$, we have $\ominus a \oplus b \in K$ and so $a \oplus K = b \oplus K$. That φ preserves the gyrogroup operations is clear. Thus, $\varphi: G/H \to G/K$ is a surjective gyrogroup homomorphism whose kernel is $\{a \oplus H: a \in G, a \oplus K = 0 \oplus K\} = \{a \oplus H: a \in K\} = K/H$. Hence, $K/H \trianglelefteq G/H$ and by the first isomorphism theorem, $(G/H)/(K/H) \cong G/K$. □

Theorem 35 (The Lattice Isomorphism Theorem). *Let G be a gyrogroup and let $N \trianglelefteq G$. Then there is a bijection Φ from the set of subgyrogroups of G containing N onto the set of subgyrogroups of G/N. The bijection Φ has the following properties:*

1. *$A \subseteq B$ if and only if $\Phi(A) \subseteq \Phi(B)$;*
2. *$A \leqslant_L G$ if and only if $\Phi(A) \leqslant_L G/N$;*
3. *$A \trianglelefteq G$ if and only if $\Phi(A) \trianglelefteq G/N$*

for all subgyrogroups A and B of G containing N.

Proof. Set $\mathcal{S} = \{K \subseteq G\colon K \leqslant G \text{ and } N \subseteq K\}$. Let $\mathcal{T}$ denote the set of subgyrogroups of G/N. Define a map Φ by $\Phi(K) = K/N$ for $K \in \mathcal{S}$. By Proposition 33, $\Phi(K) = K/N = \Pi(K)$ is a subgyrogroup of G/N, where $\Pi\colon G \to G/N$ is the canonical projection. Hence, Φ maps $\mathcal{S}$ to $\mathcal{T}$.

Assume that $K_1/N = K_2/N$, with K_1, K_2 in $\mathcal{S}$. For $a \in K_1$, $a \oplus N \in K_2/N$ implies $a \oplus N = b \oplus N$ for some $b \in K_2$. Hence, $\ominus b \oplus a \in N$. Since $N \subseteq K_2$, $\ominus b \oplus a \in K_2$, which implies $a = b \oplus (\ominus b \oplus a) \in K_2$. This proves $K_1 \subseteq K_2$. By interchanging the roles of K_1 and K_2, one obtains similarly that $K_2 \subseteq K_1$. Hence, $K_1 = K_2$ and Φ is injective.

Let Y be an arbitrary subgyrogroup of G/N. By Proposition 34,

$$\Pi^{-1}(Y) = \{a \in G\colon a \oplus N \in Y\}$$

is a subgyrogroup of G containing N for $a \in N$ implies $a \oplus N = 0 \oplus N \in Y$. Because $\Phi(\Pi^{-1}(Y)) = Y$, Φ is surjective. Hence, Φ defines a bijection from $\mathcal{S}$ onto $\mathcal{T}$.

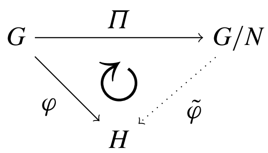
The proof of Item 1 is straightforward. From Propositions 33 and 34, we have Item 2. Specifically, if $A \leqslant_L G$, then $\Phi(A) = A/N = \Pi(A) \leqslant_L G/N$. Conversely, if $\Phi(A) = A/N \leqslant_L G/N$, then $\Pi^{-1}(A/N) \leqslant_L G$. Since $\Phi(\Pi^{-1}(A/N)) = A/N = \Phi(A)$, we obtain $\Pi^{-1}(A/N) = A$. Hence, $A \leqslant_L G$.

To prove Item 3, suppose that $A \trianglelefteq G$. Then $A = \ker \psi$, where $\psi\colon G \to H$ is a gyrogroup homomorphism. Define $\varphi\colon G/N \to H$ by $\varphi(a \oplus N) = \psi(a)$. To see that φ is well defined, suppose that $a \oplus N = b \oplus N$. Then $\ominus a \oplus b \in N$. Since $N \subseteq A$, $\ominus a \oplus b \in A$ and so $\psi(a) = \psi(b)$. Since ψ is a gyrogroup homomorphism, so is φ. Since $\ker \varphi = A/N$, we have $A/N \trianglelefteq G/N$.

Suppose conversely that $\Phi(A) \trianglelefteq G/N$. Then $A/N = \ker \phi$, where ϕ is a gyrogroup homomorphism of G/N. Set $\varphi = \phi \circ \Pi$. Thus, φ is a gyrogroup homomorphism of G with kernel A. This proves $A \trianglelefteq G$. □

The lattice isomorphism theorem gives a one-to-one correspondence between the subgyrogroups of G containing N and the subgyrogroups of G/N. In particular, it says that any subgyrogroup of G/N is of the form H/N, where H is a subgyrogroup of G containing N. The proof of the lattice isomorphism theorem motivates the following result, which determines when a gyrogroup homomorphism of G *factors through* N:

Theorem 36 (Factorization of Homomorphisms). *Let G be a gyrogroup and let $N \trianglelefteq G$. Given a gyrogroup homomorphism $\varphi\colon G \to H$, there is a gyrogroup homomorphism $\tilde{\varphi}\colon G/N \to H$ making the following diagram commutative if and only if $N \subseteq \ker \varphi$:*



Proof.

($\Rightarrow$) Suppose that $\tilde{\varphi}$ is such that $\tilde{\varphi} \circ \Pi = \varphi$. If $x \in N$, then $x \oplus N = 0 \oplus N$. By Proposition 32 (1), $\varphi(x) = \tilde{\varphi}(x \oplus N) = \tilde{\varphi}(0 \oplus N) = 0$. Hence, $x \in \ker \varphi$ and we have $N \subseteq \ker \varphi$.

($\Leftarrow$) Suppose that $N \subseteq \ker \varphi$. Define $\tilde{\varphi}$ by $\tilde{\varphi}(a \oplus N) = \varphi(a)$. The same argument as in the proof of the lattice isomorphism theorem shows that $\tilde{\varphi}$ is well defined. That $\tilde{\varphi}$ preserves the gyrogroup operations is clear. By construction, $\tilde{\varphi} \circ \Pi = \varphi$, which completes the proof. □

8 The Lagrange Property

Lagrange's theorem (that the order of any subgroup of a finite group Γ divides the order of Γ) is well known in group theory and has impact on several branches of mathematics, especially finite group theory, number theory, and combinatorics. Lagrange's theorem proves useful for unraveling mathematical structures. For instance, it implies that any finite field must have prime power order [33, Theorem 6.12]. Certain classification theorems of finite groups arise as an application of Lagrange's theorem [25, 26, 43, 57]. Further, Fermat's little theorem and Euler's theorem may be viewed as a consequence of this theorem. Also relevant are the orbit-stabilizer theorem familiar from abstract algebra and the Cauchy–Frobenius lemma (or Burnside's lemma) familiar from combinatorics. A history of Lagrange's theorem on groups can be found in [54].

In loop theory, the Lagrange property becomes a nontrivial issue. For example, whether Lagrange's theorem holds for Moufang loops was an open problem in the theory of Moufang loops for more than four decades [8, p. 43]. This problem was answered in the affirmative by Grishkov and Zavarnitsine [30]. In fact, not every loop satisfies the Lagrange property as one can construct a loop of order 5 containing a subloop of order 2 (see Example 7). Nevertheless, certain loops satisfy the Lagrange property.

Baumeister and Stein [3] proved a version of Lagrange's theorem for Bruck loops by studying in detail the structure of a finite Bruck loop. Foguel et al. [23] proved that left Bol loops of *odd* order satisfy the strong Lagrange property. It is, however, still an open problem whether or not finite Bol loops satisfy the Lagrange property [20, p. 592]. In Sect. 6.3, we proved that the order of a finite gyrogroup is divisible by the order of an *L-subgyrogroup*. In this section, we extend this result by proving that the order of a gyrogroup is divisible by the order of any subgyrogroup.

Once Lagrange's theorem for gyrogroups is established [60], certain structure theorems for finite gyrogroups arise naturally as a consequence of this theorem. For instance, Lagrange's theorem implies that any gyrogroup of prime order must be a cyclic group. Furthermore, any gyrogroup of order pq, where p and q are distinct primes, is generated by two elements; one is of order p and the other is of order q. One of the remarkable consequences of Lagrange's theorem is that some finite gyrogroups have the *Cauchy property*.

Table 5 Multiplication table for the loop L in Example 7

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	2	5	1
4	4	5	1	2	3
5	5	3	4	1	2

Example 7. Let $L = \{1, 2, \ldots, 5\}$ be a loop with multiplication table presented in Table 5. Since $Y = \{1, 2\}$ is closed under loop multiplication of L, Y is a subloop of L of order 2. This example shows that Lagrange's theorem fails for general loops.

Recall that a finite loop L has the *weak Lagrange property* if for each subloop Y of L, the order of Y divides the order of L. It has the *strong Lagrange property* if every subloop of L has the weak Lagrange property [8, p. 74]. In gyrogroup theory, one need not distinguish between the "weak" and "strong" Lagrange property since the weak Lagrange property implies the strong Lagrange property. More precisely, if *every* gyrogroup has the weak Lagrange property, then any gyrogroup has the strong Lagrange property since any subgyrogroup of a gyrogroup is again a gyrogroup.

Throughout this section, all gyrogroups are finite.

Definition 10 (The Lagrange Property). A finite gyrogroup G is said to have the Lagrange property if for each subgyrogroup H of G, the order of H divides the order of G.

The next proposition shows that the Lagrange property is an invariant property of gyrogroups.

Proposition 41. *Let G and H be gyrogroups. If $G \cong H$, then G has the Lagrange property if and only if H has the Lagrange property.*

Proof. Let $\phi\colon G \to H$ be a gyrogroup isomorphism. It suffices to prove that if G has the Lagrange property, then so has H. This is because the inverse map $\phi^{-1}\colon H \to G$ is also a gyrogroup isomorphism. Suppose that G has the Lagrange property. Let $B \leqslant H$. Since $\phi^{-1}(B) \leqslant G$, $|\phi^{-1}(B)|$ divides $|G|$. Since $|\phi^{-1}(B)| = |B|$ and $|H| = |G|$, $|B|$ divides $|H|$. This proves that H has the Lagrange property. □

A version of the following proposition for loops was proved by Bruck in [5]. Its proof emphasizes the importance of isomorphism theorems.

Proposition 42. *Let H be a subgyrogroup of a gyrogroup G and let B be a normal subgyrogroup of H. If B and H/B have the Lagrange property, then so has H.*

Proof. Suppose that $A \leqslant H$. By the second isomorphism theorem, $A \cap B \trianglelefteq A$ and $A \oplus B/B \cong A/A \cap B$. Assume that $|A \oplus B/B| = |A/A \cap B| = m$ and $|A \cap B| = n$. By the lattice isomorphism theorem, $B \leqslant A \oplus B \leqslant H$ implies $A \oplus B/B \leqslant H/B$. Since H/B has the Lagrange property, $|A \oplus B/B|$ divides $|H/B|$. Thus, $|H/B| = ms$ for some $s \in \mathbb{N}$. Since $A \cap B \trianglelefteq A$, we have $|A| = [A\colon A \cap B]|A \cap B| = mn$. Since $A \cap B \leqslant B$ and B has the Lagrange property, $|B| = nt$ for some $t \in \mathbb{N}$. Since

$$|H| = [H:B]|B| = (ms)(nt) = |A|(st),$$

it follows that $|A|$ divides $|H|$, which completes the proof. □

The following corollary indicates that the structure of a gyrogroup G is reflected in the structure of the quotient gyrogroups and the subgyrogroups of G:

Corollary 15. *Let N be a normal subgyrogroup of a gyrogroup G. If N and G/N have the Lagrange property, then so has G.*

Proof. Applying the proposition to the case where H is the entire gyrogroup G, we obtain the corollary. □

Recall that any gyrogroup G has a normal subgroup N such that G/N is a gyrocommutative gyrogroup (Theorem 21). If *all* gyrocommutative gyrogroups have the Lagrange property, then Lagrange's theorem for gyrogroups will follow immediately from Corollary 15. This is the case, as shown by Baumeister and Stein in [3, Theorem 3] in the language of Bruck loops.

Proposition 43. *Let X be a subgroup of a gyrogroup G. If $H \leqslant X$, then $|H|$ divides $|X|$. In other words, every subgroup of G has the Lagrange property.*

Proof. Suppose that $H \leqslant X$. Since $\text{gyr}[a,b]\big|_H = \text{id}_H$ for all $a, b \in H$, H forms a subgroup of G. By definition, X is a group and hence H becomes a subgroup of X. By Lagrange's theorem for groups, $|H|$ divides $|X|$. □

Theorem 37. *In a gyrocommutative gyrogroup G, if $H \leqslant G$, then $|H|$ divides $|G|$.*

Proof. Let $H \leqslant G$. By using the equivalence of gyrocommutative gyrogroups and Bruck loops, G is a Bruck loop, and H becomes a subloop of G. By Theorem 3 of [3], $|H|$ divides $|G|$. □

We are now in a position to prove an analog of Lagrange's theorem for gyrogroups.

Theorem 38 (Lagrange's Theorem). *If H is a subgyrogroup of a gyrogroup G, then $|H|$ divides $|G|$. In other words, every gyrogroup has the Lagrange property.*

Proof. Let G be a gyrogroup. By Theorem 21, G has a normal subgroup N such that G/N is gyrocommutative. As noted in Sect. 7.2, N is a normal subgyrogroup of G. By Proposition 43 and Theorem 37, N and G/N have the Lagrange property. By Corollary 15, G has the Lagrange property. □

We now give some applications of Lagrange's theorem. Other applications to proving theorems about the Cauchy property appear in the next section.

Proposition 44. *If G is a gyrogroup and $a \in G$, then $|a|$ divides $|G|$. In particular, $|G| \cdot a = 0$.*

Proof. By definition, $|a| = |\langle a\rangle|$. By Lagrange's theorem, $|a|$ divides $|G|$. Write $|G| = |a|k$ with $k \in \mathbb{N}$. We have $|G| \cdot a = (|a|k) \cdot a = \underbrace{|a| \cdot a \oplus \cdots \oplus |a| \cdot a}_{k \text{ copies}} = 0$. □

By a result of Burn [6, Corollary 2], every left Bol loop of prime order is a cyclic group. Although this result implies that every gyrogroup of prime order is a cyclic group, we present the following theorem as a consequence of Lagrange's theorem:

Theorem 39. *If G is a gyrogroup of prime order p, then G is a cyclic group of order p under the gyrogroup operation.*

Proof. Let a be a nonidentity element of G. Then $|a| \neq 1$ and $|a|$ divides p. Thus, $|a| = p$ and hence $G = \langle a\rangle$ since G is finite. By Proposition 27, $\langle a\rangle$ is a cyclic group of order p, which completes the proof. □

9 The Cauchy Property

It is well known that finite groups have the Cauchy property, that is, if p is a prime dividing the order of a group Γ, then Γ has an element of order p. In contrast, general loops fail to have the Cauchy property. Foguel et al. proved that left Bol loops of odd order have the Cauchy property [23, Theorem 6.2]. However, not every Bol loop has the Cauchy property as Nagy proves the existence of a simple right Bol loop of exponent 2 and of order 96 [50, Corollary 3.7]. This example of a loop also ends the speculation on whether all gyrogroups satisfy the Cauchy property.

Recall that a loop L has exponent 2 if 2 is the smallest positive integer such that $a^2 = 1$ for all $a \in L$. Note that if L is a Bol loop of exponent 2, then $a = a^{-1}$ for all $a \in L$. Hence, $(a \cdot b)^{-1} = a \cdot b = a^{-1} \cdot b^{-1}$ for all $a, b \in L$. This shows in particular that Bol loops of exponent 2 have the automorphic inverse property. Thus, every left Bol loop of exponent 2 is a Bruck loop or, equivalently, a gyrocommutative gyrogroup. As we described in the end of Sect. 2, the dual loop of Nagy's loop mentioned above forms a gyrocommutative gyrogroup of order 96 in which every nonidentity element has order 2. This means that Cauchy's theorem does not hold for finite gyrogroups.

Using Lagrange's theorem and results from loop theory, we show that gyrogroups of particular order satisfy the Cauchy property. Throughout this section, all gyrogroups are finite.

Definition 11 (The Weak Cauchy Property, WCP). A gyrogroup G is said to have the weak Cauchy property if for every prime p dividing $|G|$, G has an element of order p.

Definition 12 (The Strong Cauchy Property, SCP). A gyrogroup G is said to have the strong Cauchy property if every subgyrogroup of G has the weak Cauchy property.

It is clear that the strong Cauchy property implies the weak Cauchy property. The Cauchy property is an invariant property of gyrogroups, as shown in the following proposition:

Proposition 45. *Let $\phi\colon G \to H$ be a gyrogroup isomorphism.*

1. *If G has the weak Cauchy property, then so has H.*
2. *If G has the strong Cauchy property, then so has H.*

Proof. To prove Item 1, it suffices to prove that $|\phi(a)| = |a|$ for all $a \in G$. By induction, $\phi(n \cdot a) = n \cdot \phi(a)$ for all $a \in G$ and $n \in \mathbb{N}$. Let $a \in G$ and let $|a| = n$. Since $n \cdot a = 0$, we have $n \cdot \phi(a) = \phi(n \cdot a) = \phi(0) = 0$. If there were a positive integer $m < n$ such that $m \cdot \phi(a) = 0$, then m would satisfy $\phi(m \cdot a) = 0$, and so $m \cdot a = 0$, contrary to the minimality of n. This proves n is the smallest positive integer such that $n \cdot \phi(a) = 0$. By Theorem 25 (1), $|\phi(a)| = n = |a|$.

To prove Item 2, let $B \leqslant H$. Set $A = \phi^{-1}(B)$. Hence, $A \leqslant G$ and by assumption, A has the WCP. Since $\phi\big|_A$ is a gyrogroup isomorphism from A to B, B has the WCP by Item 1. □

Remark 4. In proving that $|\phi(a)| = |a|$, we used only the fact that ϕ was an injective gyrogroup homomorphism.

Corollary 16. *If G and H are isomorphic gyrogroups, then G has the weak (resp. strong) Cauchy property if and only if H has the weak (resp. strong) Cauchy property.*

Proof. This is because if $\phi\colon G \to H$ is a gyrogroup isomorphism, then so is its inverse $\phi^{-1}\colon H \to G$. □

Theorem 40. *Let H be a subgyrogroup of a gyrogroup G and let B be a normal subgyrogroup of H.*

1. *If B and H/B have the weak Cauchy property, then so has H.*
2. *If B and H/B have the strong Cauchy property, then so has H.*

Proof. Suppose that p is a prime dividing $|H|$. Since $|H| = [H\colon B]|B|$, p divides $|B|$ or $|H/B|$. If p divides $|B|$, by assumption, B has an element of order p and we are done. We may therefore assume $p \nmid |B|$. Hence, p divides $|H/B|$. By assumption, H/B has an element of order p, say $a \oplus B$. By induction, $n \cdot (a \oplus B) = (n \cdot a) \oplus B$ for all $n \geq 0$. Hence, by Theorem 25 (1), p is the smallest positive integer such that $p \cdot a \in B$. In particular, $a \notin B$. Note that $\gcd(|a|, p) = 1$ or p. If $\gcd(|a|, p) = 1$ were true, we would have $|p \cdot a| = \dfrac{|a|}{\gcd(|a|, p)} = |a|$ and would have $a \in \langle a \rangle = \langle p \cdot a \rangle \leqslant B$, a contradiction. Hence, $\gcd(|a|, p) = p$, which implies p divides $|a|$. Write $|a| = mp$. We have $|m \cdot a| = \dfrac{|a|}{\gcd(|a|, m)} = p$, which finishes the proof of Item 1.

Suppose that B and H/B have the SCP. Let $A \leqslant H$. By assumption, $A \cap B$ has the WCP. Since $A \oplus B/B \leqslant H/B$, $A \oplus B/B$ has the WCP. Since $A/A \cap B \cong A \oplus B/B$, $A/A \cap B$ has the WCP. By Item 1, A has the WCP. □

The following corollary shows that information about a gyrogroup G can be obtained from information on a normal subgyrogroup N and on the quotient G/N:

Corollary 17. *Let N be a normal subgyrogroup of a gyrogroup G. If N and G/N have the weak (strong) Cauchy property, then so has G.*

Proof. Taking $H = G$ in the theorem, the corollary follows easily. □

It is proved in Theorem 6.2 of [23] that every left Bol loop of odd order has the weak Cauchy property. Further, it is proved in Theorem 4 of [6] that a right Bol loop of order $2p$, with p a prime, is a group. Using the duality between left Bol and right Bol loops, we deduce that every left Bol loop of order $2p$, with p a prime, is a group. We use these results to prove that if p and q are primes, then gyrogroups of order pq have the strong Cauchy property.

Theorem 41. *Let p and q be primes. Every gyrogroup of order pq has the strong Cauchy property.*

Proof. If pq is odd, by Theorem 6.2 of [23], G has the WCP. By Lagrange's theorem, any subgyrogroup of G is of order $1, p, q$ or pq. Hence, every subgyrogroup of G also has the WCP. This proves that G has the SCP. If pq is even, at least one of p or q must be 2. Hence, G is of order $2\tilde{p}$, with $\tilde{p}$ a prime. As noted above, G is a group and hence has the SCP. □

Theorem 42. *Let p and q be primes and let G be a gyrogroup of order pq. If $p = q$, then G is a group. If $p \neq q$, then G is generated by two elements; one has order p and the other has order q.*

Proof. In the case $p = q$, G is a left Bol loop of order p^2 and hence must be a group by a result of Burn [6, Theorem 5].

Suppose that $p \neq q$. By Theorem 41, there exist elements a and b of G of order p and q, respectively. We claim that $\langle a\rangle \cap \langle b\rangle = \{0\}$. In fact, if $x \in \langle a\rangle \cap \langle b\rangle$, then $|x|$ divides both p and q. Thus, $|x| = 1$ and hence $x = 0$. For all $m, n, s, t \in \mathbb{Z}$, if $(m \cdot a) \oplus (n \cdot b) = (s \cdot a) \oplus (t \cdot b)$, then

$$\ominus(s \oplus a) \oplus (m \cdot a) = (t \cdot b) \boxplus (\ominus(n \cdot b)) = (t \cdot b) \ominus (n \cdot b)$$

belongs to $\langle a\rangle \cap \langle b\rangle$. It follows that $\ominus(s \oplus a) \oplus (m \cdot a) = 0$ and that $(t \cdot b) \ominus (n \cdot b) = 0$. Hence, $m \cdot a = s \cdot a$ and $n \cdot b = t \cdot b$. This proves that the set

$$\{(m \cdot a) \oplus (n \cdot b) \colon 0 \leq m < p, 0 \leq n < q\}$$

contains pq distinct elements of G. Since G has order pq, it follows that

$$G = \{(m \cdot a) \oplus (n \cdot b) \colon 0 \leq m < p, 0 \leq n < q\} = \langle a, b\rangle.$$

□

Table 6 Addition table for the gyrocommutative gyrogroup G_{15}

$\oplus$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	0	4	6	11	3	14	13	7	8	12	5	10	9
2	2	0	1	6	3	12	4	9	10	14	13	5	11	8	7
3	3	4	5	7	8	9	13	0	1	2	12	6	14	11	10
4	4	10	8	11	13	1	5	6	14	0	7	2	9	12	3
5	5	14	12	9	7	8	2	11	0	10	3	4	6	1	13
6	6	11	4	13	10	3	14	8	12	1	2	9	7	5	0
7	7	8	9	0	1	2	11	3	4	5	14	13	10	6	12
8	8	13	6	10	11	0	12	4	5	3	9	7	2	14	1
9	9	5	11	14	0	6	7	10	2	12	1	3	13	4	8
10	10	3	13	12	5	14	8	2	9	6	11	0	1	7	4
11	11	12	7	1	14	4	9	13	6	8	0	10	3	2	5
12	12	6	3	8	9	7	10	1	11	13	5	14	4	0	2
13	13	7	14	2	12	10	1	5	3	4	6	8	0	9	11
14	14	9	10	5	2	13	0	12	7	11	4	1	8	3	6

In general, a gyrogroup of order pq, where p and q are distinct primes not equal to 2, need not be a group. This is a situation where gyrogroups are different from Moufang loops. Recall that a loop L is *dissociative* if the subloop generated by a and b is an associative subloop (hence, a group) for all $a, b \in L$ [51, p. 11]. As Moufang loops are dissociative [7, p. 33], any Moufang loop generated by two elements must be a group. This implies that if p and q are primes, then every Moufang loop of order pq is a group [7, Proposition 3]. The following example shows that a gyrogroup of order pq, where $p \neq q$ and $p, q \neq 2$, need not be a group:

Example 8. By translating from a right Bol loop to a left Bol loop, the loop 15.10.1.1 in [49] can be turned into a gyrocommutative gyrogroup of order 15, called $G_{15} = \{0, 1, 2, \ldots, 14\}$, whose addition table is presented in Table 6. The gyration table for G_{15} is presented in Table 7. In cyclic notation, four nonidentity gyroautomorphisms of G_{15} can be expressed as in (72):

$$\begin{aligned} A &= (1\ 7\ 5\ 10\ 6)(2\ 3\ 8\ 11\ 14) \\ B &= (1\ 6\ 10\ 5\ 7)(2\ 14\ 11\ 8\ 3) \\ C &= (1\ 10\ 7\ 6\ 5)(2\ 11\ 3\ 14\ 8) \\ D &= (1\ 5\ 6\ 7\ 10)(2\ 8\ 14\ 3\ 11). \end{aligned} \tag{72}$$

The gyrogroup G_{15} is not a group since it has four nonidentity gyroautomorphisms. Note that, by Theorem 42, $G_{15} = \langle 1, 4 \rangle$ because $|1| = 3$ and $|4| = 5$.

Let G be a finite *nongyrocommutative* gyrogroup. By Theorem 21, G has a normal subgroup N such that G/N is gyrocommutative. Because G is nongyro-

Table 7 Gyration table for G_{15}. Here, I denotes the identity automorphism of G_{15}; A, B, C, and D are given by (72)

gyr	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	I	I	I	I	I	I	I	I	I	I	I	I	I	I	I
1	I	I	I	A	A	B	C	D	D	B	A	C	C	D	B
2	I	I	I	D	B	D	B	A	B	A	C	A	D	C	C
3	I	B	C	I	B	A	C	I	D	A	D	B	D	C	A
4	I	B	A	A	I	B	B	B	A	I	B	A	I	I	A
5	I	A	C	B	A	I	B	C	I	B	D	A	C	D	D
6	I	D	A	D	A	A	I	B	C	B	C	B	C	D	I
7	I	C	B	I	A	D	A	I	A	B	B	D	C	D	C
8	I	C	A	C	B	I	D	B	I	A	A	D	D	C	B
9	I	A	B	B	I	A	A	A	B	I	A	B	I	I	B
10	I	B	D	C	A	C	D	A	B	B	I	I	C	D	A
11	I	D	B	A	B	B	A	C	C	A	I	I	D	C	D
12	I	D	C	C	I	D	D	D	C	I	D	C	I	I	C
13	I	C	D	D	I	C	C	C	D	I	C	D	I	I	D
14	I	A	D	B	B	C	I	D	A	A	B	C	D	C	I

commutative, we have N is nontrivial, since otherwise $\Pi\colon G \to G/N$ would be a gyrogroup isomorphism and G and G/N would be isomorphic gyrogroups. From this we can deduce the following results:

Theorem 43. *Let p be a prime. Every nongyrocommutative gyrogroup of order p^3 has the strong Cauchy property.*

Proof. Let G be a nongyrocommutative gyrogroup of order p^3. As noted above, G has a nontrivial normal subgroup N. By Lagrange's theorem, $|N| = p, p^2$, or p^3. If $|N| = p^3$, then $G = N$ is a group and has the SCP. If $|N| = p$, then $|G/N| = p^2$ and if $|N| = p^2$, then $|G/N| = p$. In any case, N and G/N form groups. Hence, N and G/N have the SCP. By Corollary 17, G has the SCP. □

Theorem 44. *Let p, q and r be primes. Every nongyrocommutative gyrogroup of order pqr has the strong Cauchy property.*

Proof. Let G be a nongyrocommutative gyrogroup of order pqr. As in the proof of Theorem 43, $|N| = p, q, r, pq, pr, qr$ or pqr. If $|N| = pqr$, then $G = N$ is a group and has the SCP. If $|N| \in \{p, q, r, pq, pr, qr\}$, then $|G/N| \in \{p, q, r, pq, pr, qr\}$. In all cases, N and G/N have the SCP, which implies that G has the SCP. □

In [2], Aschbacher formulated the definition of a *solvable loop*. Here, we modify the definition as follows.

Definition 13 (Solvable Gyrogroups). A (finite or infinite) gyrogroup G is called solvable if there exists a series

$$\{0\} = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$$

of subgyrogroups with $G_i \trianglelefteq G_{i+1}$ and G_{i+1}/G_i an abelian group.

In Example 3, we showed that $H_1 = \{0, 3\}$ is a normal subgroup of G_8. Since G_8/H_1 has order 4, it must be a group. As any group of order 4 is abelian, G_8/H_1 is an abelian group. Hence, the series $\{0\} \leqslant H_1 \leqslant G_8$ satisfies the condition in Definition 13, and so G_8 is solvable. Finite solvable gyrogroups have the strong Cauchy property, as shown in the following proposition:

Proposition 46. *Every solvable gyrogroup has the strong Cauchy property.*

Proof. We proceed by induction on the number n of subgyrogroups in a subnormal series. For $n = 1$, $G_0 = \{0\}, G_1 = G$ and $G \cong G/G_0$ are an abelian group. Hence, G has the SCP. Assume inductively that if G is a solvable gyrogroup with series $\{0\} = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$, then G has the SCP. Let G be a solvable gyrogroup with series $\{0\} = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n \leqslant G_{n+1} = G$. Since G_n is a solvable gyrogroup with series $\{0\} = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n$, G_n has the SCP. Since G/G_n is an abelian group, G/G_n has the SCP. It follows from Corollary 17 that G has the SCP, which completes the induction. □

The next result gives a connection between *simple* gyrogroups and gyrogroups satisfying the Cauchy property. As in group theory, we define a simple gyrogroup as follows.

Definition 14 (Simple Gyrogroups). A (finite or infinite) gyrogroup G is called simple if $|G| > 1$ and the only normal subgyrogroups of G are $\{0\}$ and G.

Theorem 45. *Let $\mathcal{F}$ be a family of finite gyrogroups such that*

1. *$G \in \mathcal{F}$ and $N \trianglelefteq G$ implies $N \in \mathcal{F}$;*
2. *$G \in \mathcal{F}$ and $N \trianglelefteq G$ implies $G/N \in \mathcal{F}$;*
3. *every simple gyrogroup in $\mathcal{F}$ has the weak Cauchy property.*

Then $G \in \mathcal{F}$ implies G has the weak Cauchy property.

Proof. We proceed by induction on the order of G. In the case $|G| = 1$, the statement is clear. Assume that the statement holds for every gyrogroup of order less than n. Let $G \in \mathcal{F}$ be a gyrogroup of order n. If G is simple, by condition 3, G has the WCP. If G is not simple, there exists a normal subgyrogroup N such that $\{0\} < N < G$. By condition 1, $N \in \mathcal{F}$ and by inductive hypothesis, N has the WCP. By condition 2, $G/N \in \mathcal{F}$. By inductive hypothesis, $|G/N| = |G|/|N| < |G|$ implies G/N has the WCP. Since N and G/N have the WCP, so has G. This completes the induction. □

Corollary 18. *Let $\mathcal{F}$ be a family of finite gyrogroups such that*

1. *$G \in \mathcal{F}$ and $H \leqslant G$ implies $H \in \mathcal{F}$;*
2. *$G \in \mathcal{F}$ and $N \trianglelefteq G$ implies $G/N \in \mathcal{F}$;*
3. *every simple gyrogroup in $\mathcal{F}$ has the weak Cauchy property.*

Then every gyrogroup in $\mathcal{F}$ has the strong Cauchy property.

Proof. By assumption, every gyrogroup in $\mathcal{F}$ has the WCP. Let $G \in \mathcal{F}$. If $H \leqslant G$, by condition 1, $H \in \mathcal{F}$. Hence, H has the WCP. This proves G has the SCP. □

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Mild Continuity Properties of Relations and Relators in Relator Spaces

Árpád Száz and Amr Zakaria

Abstract In this paper, we establish several useful consequences of the following, and some other closely related, basic definitions introduced in some former papers by the first author.

A family $\mathscr{R}$ of relations on one set X to another Y is called a relator on X to Y. Moreover, the ordered pair $(X, Y)(\mathscr{R}) = \big((X, Y), \mathscr{R}\big)$ is called a relator space.

A function $\Box$ of the class of all relator spaces to the class of all relators is called a direct unary operation for relators if, for any relator $\mathscr{R}$ on X to Y, the value $\mathscr{R}^{\Box} = \mathscr{R}^{\Box_{XY}} = \Box\big((X, Y)(\mathscr{R})\big)$ is also relator on X to Y.

If $(X, Y)(\mathscr{R})$ and $(Z, W)(\mathscr{S})$ are relator spaces and $\Box$ is a direct unary operation for relators, then a pair $(\mathscr{F}, \mathscr{G})$ of relators $\mathscr{F}$ on X to Z and $\mathscr{G}$ on Y to W is called mildly $\Box$–continuous if, under the elementwise inversion and compositions of relators, we have $\big((\mathscr{G}^{\Box})^{-1} \circ \mathscr{S}^{\Box} \circ \mathscr{F}^{\Box}\big)^{\Box} \subseteq \mathscr{R}^{\Box\Box}$.

1 Introduction

In this paper, we continue the investigations initiated by the first author in [28, 31, 33, 41, 42, 50, 51] on the basic continuity properties of a single relation, and also of a pair of relations, on one relator (generalized uniform) space to another.

Meantime, we have observed that, much more generally, the corresponding, and some other, continuity properties of pairs of relators (families of relations) can also be naturally investigated [55, 57].

Here, a family $\mathscr{R}$ of relations on one set X to another Y is called a *relator* on X to Y, and the ordered pair $(X, Y)(\mathscr{R}) = \big((X, Y), \mathscr{R}\big)$ is called a *relator space*. Thus, relator spaces are common generalizations of *ordered sets* [3], *formal contexts* [10], and *uniform spaces* [9].

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Note that even on the real line $\mathbb{R}$, we already have two natural relators. Namely, $\mathscr{R} = \{\leq\}$ where $\leq$ is the usual ordering on $\mathbb{R}$, and $\mathscr{S} = \{B_r : \ r > 0\}$ where $B_r = \{x \in \mathbb{R}^2 : \ |x_1 - x_2| < r\}$. Thus, *birelator spaces* should also be studied.

In a recent paper [57], to motivate the definitions of the corresponding continuity properties of pairs of relators on one relator space to another, the first author offered the following motivating arguments.

Example 1. Suppose that $X = X(\leq_X)$ and $Y = Y(\leq_Y)$ are *generalized ordered sets* in the sense that $\leq_X$ and $\leq_Y$ are arbitrary relations on the sets X and Y, respectively.

Then, a function f of X to Y may be naturally called *increasing*, with respect to the inequalities $\leq_X$ and $\leq_Y$, if for every $u, v \in X$

$$u \leq_X v \quad \Longrightarrow \quad f(u) \leq_Y f(v)\,.$$

Now, by using the more convenient notations $R = \leq_X$ and $S = \leq_Y$, the above implication can be reformulated in the form that

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v)\,,$$

or equivalently

$$(u, v) \in R \quad \Longrightarrow \quad \big(f(u), f(v)\big) \in S\,.$$

Example 2. Suppose that $X = X(d_X)$ and $Y = Y(d_Y)$ are *generalized metric spaces* in the sense that d_X and d_Y are arbitrary functions of X^2 and Y^2 to $[0, +\infty]$, respectively.

Then, a function f of X to Y may be naturally called *uniformly continuous*, with respect to the distance functions d_X and d_Y, if for each $s > 0$ there exists $r > 0$ such that for every $u, v \in X$

$$d_X(u, v) < r \quad \Longrightarrow \quad d_Y\big(f(u)\ f(v)\big) < s\,.$$

Now, by using the surroundings

$$R = B_r^{d_X} = \big\{x \in X^2 : \quad d_X(x_1, x_2) < r\big\}$$

and

$$S = B_s^{d_Y} = \big\{y \in Y^2 : \quad d_X(y_1, y_2) < s\big\},$$

the above implication can be reformulated in the form that

$$(u, v) \in R \quad \Longrightarrow \quad \big(f(u), f(v)\big) \in S\,.$$

The above examples clearly reveal that the seemingly quite different algebraic and topological notions such as "increasingness" and "uniform continuity" are essentially equivalent.

Moreover, they naturally lead us to the following simple unifying

Definition 1. Assume that $X = X(R)$ and $Y = Y(S)$ are *relational spaces* in the sense that R and S are arbitrary relations on X and Y, respectively.

Then, a function f of X to Y will be called *increasing* or *continuous*, with respect to the relations R and S, if

$$(u, v) \in R \quad \Longrightarrow \quad \big(f(u), f(v)\big) \in S$$

for all $u, v \in X$. That is, the function f is, in a certain sense, relation preserving.

Now, by using this definition, we can easily prove the following theorem of [57] presented partly also in [56].

Theorem 1. *For any function f of one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f is increasing (continuous),
(2) $f \circ R \subseteq S \circ f$,
(3) $R \subseteq f^{-1} \circ S \circ f$,
(4) $f \circ R \circ f^{-1} \subseteq S$,
(5) $R \circ f^{-1} \subseteq f^{-1} \circ S$.

Proof. By the corresponding definitions, it is clear that the following assertions are equivalent:

(a) $f \circ R \subseteq S \circ f$,
(b) $\forall\ u \in X: \qquad (f \circ R)(u) \subseteq (S \circ f)(u)$,
(c) $\forall\ u \in X: \qquad f\,[R(u)] \subseteq S\big(f(u)\big)$,
(d) $\forall\ u \in X: \qquad \forall\ v \in R(u): \ f(v) \in S\big(f(u)\big)$,
(e) $\forall\ u, v \in X: \quad \big((u, v) \in R \implies \big(f(u), f(v)\big) \in S\big)$.

Therefore, assertions (2) and (1) are equivalent.

The proofs of the remaining equivalences depend on the increasingness and associativity of composition, and the inclusions

$$\Delta_X \subseteq f^{-1} \circ f \qquad \text{and} \qquad f \circ f^{-1} \subseteq \Delta_Y,$$

where Δ_X and Δ_Y are the identity functions of X and Y, respectively.

Remark 1. The latter inclusions indicate that assertions (2)–(5) need not be equivalent for an arbitrary relation f on $X(R)$ to $Y(S)$. Therefore, they can be used to define different increasingness or continuity properties of relations.

Remark 2. In [53], having in mind set-valued functions, a relation F on a generalized ordered set $X(\leq)$ to a set Y has been called *increasing* if $u \leq v$ implies $F(u) \subseteq F(v)$ for all $u, v \in X$.

Thus, it can be easily shown that the relation F is increasing if and only if its inverse F^{-1} is *ascending-valued* in the sense that $F^{-1}(y)$ is an ascending subset of $X(\leq)$ for all $y \in Y$.

By using the more convenient notation $R = \leq$, the latter statement can be reformulated in the form that $R\,[\,F^{-1}(y)\,] \subseteq F^{-1}(y)$ for all $y \in Y$. That is, $R \circ F^{-1} \subseteq F^{-1}$.

The latter inclusion can be reformulated in the form that $R \circ F^{-1} \subseteq F^{-1} \circ \Delta_Y$. This shows that the $R = \Delta_X$ and $S = \Delta_Y$ particular cases of Theorem 1 may also be of some interest.

In [57], it has been proved that, under the notations of Definition 1, we have $F \circ R \circ F^{-1} \subseteq S$ if and only if, for every $u, v \in X$, we have $F(u)\, S\, F(v)$ in the sense that $y\, S\, z$ for all $y \in F(u)$ and $z \in F(v)$.

However, it is now more important to note that, by using the corresponding particular cases of the plausible operations, defined by $\mathscr{R}^{-1} = \{ R^{-1} : \ R \in \mathscr{R} \}$,

$$\mathscr{R}^{*} = \big\{ S \subseteq X \times Y : \ \ \exists\ R \in \mathscr{R} : \ \ R \subseteq S \big\},$$

and $\mathscr{S} \circ \mathscr{R} = \big\{ S \circ R : \ \ R \in \mathscr{R}, \ \ S \in \mathscr{S} \big\}$ for any relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, Theorem 1 can be reformulated in the following more instructive form.

Theorem 2. *If f is a function of one relational space $X(R)$ to another $Y(S)$, then under the notations*

$$\mathscr{F} = \{f\}, \qquad \mathscr{R} = \{R\} \qquad \textit{and} \qquad \mathscr{S} = \{S\},$$

the following assertions are equivalent:

(1) f is increasing (continuous),
(2) $\big(\mathscr{S}^{*} \circ \mathscr{F}^{*}\big)^{*} \subseteq \big(\mathscr{F}^{*} \circ \mathscr{R}^{*}\big)^{*}$,
(3) $\Big(\big(\mathscr{F}^{*}\big)^{-1} \circ \mathscr{S}^{*} \circ \mathscr{F}^{*}\Big)^{*} \subseteq \mathscr{R}^{**}$,
(4) $\mathscr{S}^{**} \subseteq \Big(\mathscr{F}^{*} \circ \mathscr{R}^{*} \circ \big(\mathscr{F}^{*}\big)^{-1}\Big)^{*}$,
(5) $\Big(\big(\mathscr{F}^{*}\big)^{-1} \circ \mathscr{S}^{*}\Big)^{*} \subseteq \Big(\mathscr{R}^{*} \circ \big(\mathscr{F}^{*}\big)^{-1}\Big)^{*}$.

Hint. The proof of the equivalences of the assertions (2)–(5) of this theorem to those of Theorem 1 depends on the fact that $*$ is an *inversion and composition compatible closure operation for relators* in the sense that:

(a) $\big(\mathscr{R}^{*}\big)^{-1} = \big(\mathscr{R}^{-1}\big)^{*}$ for any relator $\mathscr{R}$ on X to Y,
(b) $\mathscr{R}^{*} \subseteq \mathscr{S}^{*}$ is equivalent to $\mathscr{R} \subseteq S^{*}$ for any relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y,
(c) $\big(\mathscr{S} \circ \mathscr{R}\big)^{*} = \big(\mathscr{S} \circ \mathscr{R}^{*}\big)^{*} = \big(\mathscr{S}^{*} \circ \mathscr{R}\big)^{*}$ for any relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z.

Now, the Pexiderizations of the inclusions in Theorem 2, and an abstraction of the operation $*$, naturally lead us to the following straightforward extension of [42, Definition 4.1].

Definition 2. Suppose that $(X, Y)(\mathscr{R})$ and $(Z, W)(\mathscr{S})$ are relator spaces, $\mathscr{F}$ is a relator on X to Z, and $\mathscr{G}$ is a relator on Y to W.

Moreover, assume that $\Box$ is a *direct unary operation for relators* in the sense that it is function of the class of all relator spaces to the class of all relators such that, for any relator $\mathscr{R}$ on X to Y, the value $\mathscr{R}^{\Box} = \mathscr{R}^{\Box_{XY}} = \Box\big((X, Y)(\mathscr{R})\big)$ is also a relator on X to Y.

Then, we say that the ordered pair

(1) $(\mathscr{F}, \mathscr{G})$ is *upper* $\Box$*–continuous* with respect to the relators $\mathscr{R}$ and $\mathscr{S}$ if

$$\big(\mathscr{S}^{\Box} \circ \mathscr{F}^{\Box}\big)^{\Box} \subseteq \big(\mathscr{G}^{\Box} \circ \mathscr{R}^{\Box}\big)^{\Box},$$

(2) $(\mathscr{F}, \mathscr{G})$ is *mildly* $\Box$*–continuous* with respect to the relators $\mathscr{R}$ and $\mathscr{S}$ if

$$\Big(\big(\mathscr{G}^{\Box}\big)^{-1} \circ \mathscr{S}^{\Box} \circ \mathscr{F}^{\Box}\Big)^{\Box} \subseteq \mathscr{R}^{\Box\Box},$$

(3) $(\mathscr{F}, \mathscr{G})$ is *vaguely* $\Box$*–continuous* with respect to the relators $\mathscr{R}$ and $\mathscr{S}$ if

$$\mathscr{S}^{\Box\Box} \subseteq \Big(\mathscr{G}^{\Box} \circ \mathscr{R}^{\Box} \circ \big(\mathscr{F}^{\Box}\big)^{-1}\Big)^{\Box},$$

(4) $(\mathscr{F}, \mathscr{G})$ is *lower* $\Box$*–continuous* with respect to the relators $\mathscr{R}$ and $\mathscr{S}$ if

$$\Big(\big(\mathscr{G}^{\Box}\big)^{-1} \circ \mathscr{S}^{\Box}\Big)^{\Box} \subseteq \Big(\mathscr{R}^{\Box} \circ \big(\mathscr{F}^{\Box}\big)^{-1}\Big)^{\Box}.$$

Remark 3. To keep in mind the above assumptions, for any $R \in \mathscr{R}$, $S \in \mathscr{S}$, $F \in \mathscr{F}$ and $G \in \mathscr{G}$, we can use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\;F\;} & Z \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ Y & \xrightarrow{\;G\;} & W \end{array}$$

Moreover, to clarify the notion of a direct unary operation for relators, we can note that $*$ is a direct, but -1 is a non-direct unary operation for relators. Of course, if we restrict ourself to relator spaces of the simpler type $X(\mathscr{R}) = (X, X)(\mathscr{R})$, then the latter inconvenience does not occur.

Now, since there is a great number of important direct unary operations for relators, the investigation of the above continuity properties and their relationships to each other offer an exhausting work for hundreds of mathematicians.

In this paper, to let the reader feel the main directions in the abovementioned investigations, we shall only point out some basic facts concerning the various mild continuities of relators, relations, and functions.

For instance, we shall prove the following two basic theorems.

Theorem 3. *If in particular $\square$ is an inversion and composition compatible closure operation for relators, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly $\square$–continuous with respect to the relators $\mathscr{R}$ and $\mathscr{S}$,*
(2) $(\mathscr{F}, \mathscr{G})$ *is properly mildly continuous with respect to the relators $\mathscr{R}^{\square}$ and $\mathscr{S}$ in the sense that* $\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F} \subseteq \mathscr{R}^{\square}$,
(3) $(\mathscr{F}, \mathscr{G})$ *is elementwise mildly $\square$–continuous with respect to the relators $\mathscr{R}$ and $\mathscr{S}$ in the sense that, for any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the pair (F, G), i.e., $\big(\{F\}, \{G\}\big)$, is mildly $\square$–continuous with respect to the relators $\mathscr{R}$ and $\mathscr{S}$.*

Theorem 4. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) $G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{*}$,
(2) (F, G) *is mildly $*$–continuous,*
(3) for each $S \in \mathscr{S}$ there exists $R \in \mathscr{R}$ such that $R \subseteq G^{-1} \circ S \circ F$,
(4) for each $S \in \mathscr{S}$ there exists $R \in \mathscr{R}$ such that for every $x \in X$ and $y \in R(x)$ we have $G(y) \cap S\,[F(x)\,] \neq \emptyset$,
(5) for each $S \in \mathscr{S}$ there exists $R \in \mathscr{R}$ such that for every $x \in X$ and $y \in R(x)$ there exist $z \in F(x)$ and $w \in G(y)$ such that $w \in S(z)$.

In this respect, it is also worth noticing that, by using the notations

$$\mathscr{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \ \exists\ R \in \mathscr{R} : \quad R\,[A\,] \subseteq S\,[A\,] \big\}$$

for any relator $\mathscr{R}$ on X to Y, and

$$\operatorname{Int}_{\mathscr{R}}(B) = \big\{ A \subseteq X : \quad \exists\ R \in \mathscr{R} : \quad R\,[A\,] \subseteq B \big\}$$

and

$$\operatorname{Cl}_{\mathscr{R}}(B) = \big\{ A \subseteq X : \quad \forall\ R \in \mathscr{R} : \quad R\,[A\,] \cap B \neq \emptyset \big\}$$

for any $B \subseteq Y$, we can also prove the following.

Theorem 5. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) $G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{\#}$,
(2) (F, G) *is mildly #–continuous,*
(3) $A \in \operatorname{Cl}_{\mathscr{R}}(B)$ *implies* $F\,[A\,] \in \operatorname{Cl}_{\mathscr{S}}(G\,[B\,])$,
(4) $F\,[A\,] \in \operatorname{Int}_{\mathscr{S}}(D)$ *implies* $A \in \operatorname{Int}_{\mathscr{R}}\big(G^{-1}\,[D\,]\big)$.

Remark 4. Because of Example 2 and Theorem 5, the pair (F, G) may be naturally called *uniformly (proximally) mildly continuous* if it is mildly $*$–continuous (#–continuous).

Unfortunately, the uniform continuity of (F, G) can only be characterized in terms of the *convergence (adherence)* of one preordered net of sets to another, which is already a rather difficult notion.

In the subsequent preparatory sections, we shall list some basic facts on relations and relators, and structures and unary operations for relators, which are possibly unfamiliar to the reader. The proofs will be frequently omitted.

2 A Few Basic Facts on Relations

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. Thus, the *empty relation* $\emptyset$ is the smallest and the *universal relation* $X \times Y$ is the largest relation on X to Y. Moreover, $\mathscr{P}(X \times Y)$ is the family of all relations on X to Y.

If in particular $F \subseteq X^2$, with $X^2 = X \times X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation* and its complement $\Delta_X^c = X^2 \setminus \Delta_X$ is called the *diversity relation* on X.

If F is a relation on X to Y, then by the above definitions we can at once see that F is also a relation on $X \cup Y$. However, for our subsequent purposes, the latter view of the relation F would be quite unnatural.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$, the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images* of x and A under F. If $(x, y) \in F$, then we may also write $x\,F\,y$.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain* and *range* of F. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *total relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. Moreover, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. Thus, we also have $F^c = X \times Y \setminus F$. Moreover, it noteworthy that $F^c[A]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$. (See [52].)

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$. Thus, we have $F^{-1}[B] = \{x \in X : F(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$, and hence in particular $D_F = F^{-1}[Y]$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product relation* $F \boxtimes G$ can be naturally defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$. (See [52].)

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for an arbitrary family of relations too.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection* of F if $f \subseteq F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, by the axiom of choice, every relation has a selection. Moreover, it is the union of its selections.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^{\vartriangleright}$ of X to $\mathscr{P}(Y)$ and $F^{\blacktriangleright}$ of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$, such that $F^{\vartriangleright}(x) = F(x)$ for all $x \in X$ and $F^{\blacktriangleright}(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathscr{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ are more general objects than relations on X to Y. They were briefly called *corelations* on X to Y in [54].

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$ and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$ and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. Moreover, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For instance, for $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder relation on X. (See [17] and [47].) While, for a *pseudo-metric* d on X and $r > 0$, the *surrounding* $B_r^d = \{x \in X^2 : \ d(x_1, x_2) < r\}$ is a tolerance relation on X.

Moreover, we may recall that if $\mathscr{A}$ is a *partition* of X, i.e., a family of pairwise disjoint, nonvoid subsets of X such that $X = \bigcup \mathscr{A}$, then $S_{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} A^2$ is an equivalence relation on X, which can, to some extent, be identified with $\mathscr{A}$.

According to algebra, for any relation R on X, we may naturally define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also naturally define $R^{\infty} = \bigcup_{n=0}^{\infty} R^n$. Thus, R^{∞} is the smallest preorder relation containing R [11].

Now, in contrast to $(F^c)^c = F$ and $(F^{-1})^{-1} = F$, we have $(R^{\infty})^{\infty} = R^{\infty}$. Moreover, analogously to $(F^c)^{-1} = (F^{-1})^c$, we also have $(R^{\infty})^{-1} = (R^{-1})^{\infty}$. Thus, in particular R^{-1} is also a preorder on X if R is a preorder on X.

3 A Few Basic Facts on Relators

A family $\mathscr{R}$ of relations on one set X to another Y is called a *relator on X to Y*. Moreover, the ordered pair $(X, Y)(\mathscr{R}) = \big((X, Y), \mathscr{R}\big)$ is called a *relator space*. (For the origins, see [31, 37, 43, 45] and the references in [31].)

If in particular $\mathscr{R}$ is a relator on X to itself, then we may simply say that $\mathscr{R}$ is a *relator on X*. Moreover, by identifying singletons with their elements, we may naturally write $X(\mathscr{R})$ in place of $(X, X)(\mathscr{R})$, since $(X, X) = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [3] and *uniform spaces* [9]. However, they are insufficient for some important purposes. (See, for instance, [10] and [42].)

A relator $\mathscr{R}$ on X to Y, or a relator space $(X, Y)(\mathscr{R})$, is called *simple* if there exists a relation R on X to Y such that $\mathscr{R} = \{R\}$ [24]. In this case, by identifying singletons with their elements, we may write $(X, Y)(R)$ in place of $(X, Y)\big(\{R\}\big)$.

According to Száz [44], a simple relator space $X(R)$ may be called a *goset* (generalized ordered set). Moreover, by Ganter and Wille [10, p. 17], a simple relator space $\big(X, Y\big)(R)$ may be called a *formal context* or *context space*.

A relator $\mathscr{R}$ on X, or a relator space $X(\mathscr{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathscr{R}$ is reflexive. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators*.

For instance, for any family $\mathscr{A}$ of subsets of X, the family $\mathscr{R}_{\mathscr{A}} = \{R_A : A \in \mathscr{A}\}$, where $R_A = A^2 \cup A^c \times X$, is a preorder relator on X. Such relators were first used by Davis [4] and Pervin [27].

While, for any family $\mathscr{D}$ of pseudo-metrics on X, the family $\mathscr{R}_{\mathscr{D}} = \{B_r^d : r > 0, \ d \in \mathscr{D}\}$, where $B_r^d = \big\{x \in X^2 : \ d(x_1, x_2) < r\big\}$, is a tolerance relator on X. Such relators were first considered by Weil [59].

Moreover, if $\mathfrak{S}$ is a family of partitions of X, then the family $\mathscr{R}_{\mathfrak{S}} = \{S_{\mathscr{A}} : \mathscr{A} \in \mathfrak{S}\}$, where $S_{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} A^2$, is an equivalence relator on X. Such practically important relators were first investigated by Levine [16].

A function $\square$ of the class of all relator spaces to the class of all relators is called a *direct (indirect) unary operation for relators* if, for any relator $\mathscr{R}$ on X to Y, the value $\mathscr{R}^{\square} = \mathscr{R}^{\square_{XY}} = \square\big((X, Y)(\mathscr{R})\big)$ is a relator on X to Y (on Y to X).

More generally, a function $\mathfrak{F}$ of the class of all relator spaces to some other class is called a *structure for relators* if, for any relator $\mathscr{R}$ on X to Y, the value $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{R}}^{XY} = \mathfrak{F}\big((X, Y)(\mathscr{R})\big)$ is in a power set depending only on X and Y.

In accordance with [54], for a structure $\mathfrak{F}$ for relators, we say that:

(1) $\mathfrak{F}$ is *quasi-increasing* if $\mathfrak{F}_{\{R\}} \subseteq \mathfrak{F}_{\mathscr{R}}$ for any relator $\mathscr{R}$ on X to Y and $R \in \mathscr{R}$,
(2) $\mathfrak{F}$ is *increasing* if $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$ for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y with $\mathscr{R} \subseteq \mathscr{S}$,
(3) $\mathfrak{F}$ is *union preserving* if $\mathfrak{F}_{\bigcup_{i \in I} \mathscr{R}_i} = \bigcup_{i \in I} \mathfrak{F}_{\mathscr{R}_i}$ for any family $\big(\mathscr{R}_i\big)_{i \in I}$ of relators on X to Y.

Thus, "union preserving" $\Rightarrow$ "increasing" $\Rightarrow$ "quasi-increasing". Moreover, it can be shown that $\mathfrak{F}$ is union preserving if and only if $\mathfrak{F}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathfrak{F}_{\{R\}}$ for any relator $\mathscr{R}$ on X to Y. Here, we may again write R instead of $\{R\}$.

A unary operation $\square$ for relators is called *extensive, intensive, involutive*, and *idempotent* if for any relator $\mathscr{R}$ on X to Y we have $\mathscr{R} \subseteq \mathscr{R}^{\square}$, $\mathscr{R}^{\square} \subseteq \mathscr{R}$, $\mathscr{R}^{\square\square} = \mathscr{R}$, and $\mathscr{R}^{\square\square} = \mathscr{R}^{\square}$, respectively.

In particular, an increasing idempotent operation for relators is called a *modification operation* [21]. While, an extensive (intensive) modification operation for relators is called a *closure (interior) operation*.

Moreover, an increasing extensive (intensive) operation is called a *preclosure (preinterior) operation*. While, an extensive (intensive) idempotent operation is called a *semiclosure (semiinterior)* operation.

For instance, the functions c and -1, defined by

$$\mathscr{R}^{\,c} = \left\{ R^{\,c} : \quad R \in \mathscr{R} \right\} \qquad \text{and} \qquad \mathscr{R}^{-1} = \left\{ R^{-1} : \quad R \in \mathscr{R} \right\}$$

for any relator $\mathscr{R}$ on X to Y, are increasing involution operations for relators such that $\left(\mathscr{R}^{\,c}\right)^{-1} = \left(\mathscr{R}^{-1}\right)^{c}$. Thus, the operation c is *inversion compatible*.

Moreover, the functions ∞ and ∂, defined by

$$\mathscr{R}^{\,\infty} = \left\{ R^{\,\infty} : \quad R \in \mathscr{R} \right\} \qquad \text{and} \qquad \mathscr{R}^{\,\partial} = \left\{ S \subseteq X^2 : \quad S^{\,\infty} \in \mathscr{R} \right\}$$

for any relator $\mathscr{R}$ on X, are modification operations for relators such that, for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X, we have

$$\mathscr{R}^{\,\infty} \subseteq \mathscr{S} \quad \Longleftrightarrow \quad \mathscr{R} \subseteq \mathscr{S}^{\,\partial}.$$

Therefore, the operations ∞ and ∂ form a Galois connection [3, p. 155]. Thus, in particular $\infty\,\partial$ is a closure operation for relators such that $\infty = \infty\,\partial\,\infty$.

To investigate inclusions between generalized topologies derived from relations and relators, the operations ∞ and ∂ were first introduced by Mala [19] and Pataki [25], respectively. Moreover, by using several more powerful structures derived from relators, Száz [38] and Pataki [25] defined a great abundance of important closure operations for relators. Some of them were already considered by Kenyon [14] and H. Nakano and K. Nakano [22].

Beside the abovementioned unary operations, we may also naturally introduce several important binary operations for relators. For instance, for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, we may naturally define

$$\mathscr{S} \circ \mathscr{R} = \left\{ S \circ R : \quad R \in \mathscr{R}\,, \quad S \in \mathscr{S} \right\}.$$

Hence, by using that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ for all $R \in \mathscr{R}$ and $S \in \mathscr{S}$, we can easily see that $(\mathscr{S} \circ \mathscr{R})^{-1} = \mathscr{R}^{-1} \circ \mathscr{S}^{-1}$. Moreover, it can also be easily seen that the composition of relators is also associative.

4 Some Important Structures for Relators

If $\mathscr{R}$ is a relator on X to Y, then for any $A \subseteq X$, $B \subseteq Y$ and $x \in X$ we write:

(1) $A \in \operatorname{Int}_{\mathscr{R}}(B)$ if $R[A] \subseteq B$ for some $R \in \mathscr{R}$,
(2) $A \in \operatorname{Cl}_{\mathscr{R}}(B)$ if $R[A] \cap B \neq \emptyset$ for all $R \in \mathscr{R}$,
(3) $x \in \operatorname{int}_{\mathscr{R}}(B)$ if $\{x\} \in \operatorname{Int}_{\mathscr{R}}(B)$,

(4) $x \in \operatorname{cl}_{\mathscr{R}}(B)$ if $\{x\} \in \operatorname{Cl}_{\mathscr{R}}(B)$,
(5) $B \in \mathscr{E}_{\mathscr{R}}$ if $\operatorname{int}_{\mathscr{R}}(B) \neq \emptyset$,
(6) $B \in \mathscr{D}_{\mathscr{R}}$ if $\operatorname{cl}_{\mathscr{R}}(B) = X$.

Moreover, if in particular $\mathscr{R}$ is a relator on X, then for any $A \subseteq X$ we also write:

(7) $A \in \tau_{\mathscr{R}}$ if $A \in \operatorname{Int}_{\mathscr{R}}(A)$,
(8) $A \in \tau\!\!\!-_{\mathscr{R}}$ if $A^c \notin \operatorname{Cl}_{\mathscr{R}}(A)$,
(9) $A \in \mathscr{T}_{\mathscr{R}}$ if $A \subseteq \operatorname{int}_{\mathscr{R}}(A)$,
(10) $A \in \mathscr{F}_{\mathscr{R}}$ if $\operatorname{cl}_{\mathscr{R}}(A) \subseteq A$.

The relations $\operatorname{Int}_{\mathscr{R}}$ and $\operatorname{int}_{\mathscr{R}}$ are called *the proximal and topological interiors* induced by $\mathscr{R}$, respectively. While, the members of the families, $\tau_{\mathscr{R}}$, $\mathscr{T}_{\mathscr{R}}$ and $\mathscr{E}_{\mathscr{R}}$ are called the *proximally open, topologically open, and fat subsets* of the relator spaces $X(\mathscr{R})$ and $(X, Y)(\mathscr{R})$, respectively.

The origins of the relations Cl_R and $\operatorname{Int}_{\mathscr{R}}$ go back to Efremović's proximity δ [6] and Smirnov's strong inclusion $\Subset$ [29], respectively. The families $\tau_{\mathscr{R}}$ and $\mathscr{E}_{\mathscr{R}}$ were first explicitly used by the first author [37]. In particular, the practical notation $\tau\!\!\!-_{\mathscr{R}}$ has been suggested by János Kurdics [15].

Because of the above definitions, for any relator $\mathscr{R}$ on X to Y and $B \subseteq Y$, we have

$$\operatorname{Cl}_{\mathscr{R}}(B) = \mathscr{P}(X) \setminus \operatorname{Int}_{\mathscr{R}}\big(B^c\big) \qquad \text{and} \qquad \operatorname{cl}_{\mathscr{R}}(B) = X \setminus \operatorname{int}_{\mathscr{R}}\big(B^c\big),$$

and

$$\mathscr{D}_{\mathscr{R}} = \big\{ D \subseteq Y : \quad D^c \notin \mathscr{E}_{\mathscr{R}} \big\} = \big\{ D \subseteq Y : \quad \forall\ E \in \mathscr{E}_{\mathscr{R}} : \quad E \cap D \neq \emptyset \big\}.$$

Moreover, if in particular, $\mathscr{R}$ is a relator on X, then we also have

$$\tau\!\!\!-_{\mathscr{R}} = \big\{ A \subseteq X : \quad A^c \in \tau_{\mathscr{R}} \big\} \quad \text{and} \qquad \mathscr{F}_{\mathscr{R}} = \big\{ A \subseteq X : \quad A^c \in \mathscr{T}_{\mathscr{R}} \big\}.$$

In this respect, it is also worth mentioning that, for any relator $\mathscr{R}$ on X to Y, we have

$$\operatorname{Cl}_{\mathscr{R}^{-1}} = \operatorname{Cl}_{\mathscr{R}}^{-1} \qquad \text{and} \qquad \operatorname{Int}_{\mathscr{R}^{-1}} = \mathscr{C}_Y \circ \operatorname{Int}_{\mathscr{R}}^{-1} \circ \mathscr{C}_X,$$

where $\mathscr{C}_X(A) = X \setminus A$ for all $A \subseteq X$. Moreover, in particular, for any relator $\mathscr{R}$ on X, we have $\tau\!\!\!-_R = \tau_{R^{-1}}$. Therefore, the proximal closures and proximally open sets are usually more convenient tools than the topological closures (proximal interiors) and topologically open sets, respectively.

The fat sets are frequently also more convenient tools than the topologically open sets [35]. For instance, if $\leq$ is a certain order relation on X, then $\mathscr{T}_{\leq}$ and $\mathscr{E}_{\leq}$ are just the families of all *ascending and residual subsets* of the ordered set $X(\leq)$, respectively.

To clarify the advantage of fat sets over the open ones, we can also note that if in particular $X = \mathbb{R}$, and R is a relation on X such that

$$R(x) = \{ x - 1 \} \cup [x \, , \, +\infty [$$

for all $x \in X$, then $\mathscr{T}_R = \{ \emptyset \, , \, X \}$, but $\mathscr{E}_R$ is quite large family. Namely, the supersets of each $R(x)$, with $x \in X$, are also in $\mathscr{E}_R$.

If $\mathscr{R}$ is a relator on X to Y, and Φ and Ψ are relations on a relator space $\Gamma(\mathscr{U})$ to X and Y, respectively, then by using the relation $(\Phi \otimes \Psi)$, defined such that

$$(\Phi \otimes \Psi)(\gamma) = \Phi(\gamma) \times \Psi(\gamma)$$

for all $\gamma \in \Gamma$, we may also define

(11) $\Phi \in \operatorname{Lim}_{\mathscr{R}}(\Psi)$ if $(\Phi \otimes \Psi)^{-1}[R] \in \mathscr{E}_{\mathscr{U}}$ for all $R \in \mathscr{R}$,
(12) $\Phi \in \operatorname{Adh}_{\mathscr{R}}(\Psi)$ if $(\Phi \otimes \Psi)^{-1}[R] \in \mathscr{D}_{\mathscr{U}}$ for all $R \in \mathscr{R}$.

Now, for any $A \subseteq X$, we may also naturally write:

(13) $A \in \lim_{\mathscr{R}}(\Psi)$ if $A_\Gamma \in \operatorname{Lim}_{\mathscr{R}}(\Psi)$,
(14) $A \in \operatorname{adh}_{\mathscr{R}}(\Psi)$ if $A_\Gamma \in \operatorname{Adh}_{\mathscr{R}}(\Psi)$,

where A_Γ is a relation on Γ to X such that $A_\Gamma(\gamma) = A$ for all $\gamma \in \Gamma$.

The *big limit relation* $\operatorname{Lim}_{\mathscr{R}}$, suggested by Efremović and Švarc [7], is, in general, a much stronger tool in the relator space $(X, Y)(\mathscr{R})$ than the *big closure and interior relations* $\operatorname{Cl}_{\mathscr{R}}$ and $\operatorname{Int}_{\mathscr{R}}$ suggested by Efremović [6] and Smirnov [29].

Namely, it can be shown that, for any $A \subseteq X$ and $B \subseteq Y$, we have $A \in \operatorname{Cl}_{\mathscr{R}}(B)$ if and only if there exist a preordered set $\Gamma(\leq)$ and functions φ and ψ of Γ to A and B, respectively, such that $\varphi \in \operatorname{Lim}_{\mathscr{R}}(\psi)$ $\big(\varphi \in \operatorname{Adh}_{\mathscr{R}}(\psi)\big)$.

To check the less obvious part of this statement, note that if $A \in \operatorname{Cl}_{\mathscr{R}}(B)$, then for each $R \in \mathscr{R}$, we have $R[A] \cap B \neq \emptyset$. Therefore, there exist $x_R \in A$ and $y_R \in B$ such that $y_R \in R(x_R)$.

Now, by defining $\varphi(R) = x_R$ and $\psi(R) = y_R$ for all $R \in \mathscr{R}$, and moreover $R_1 \leq R_2$ if $R_1, R_2 \in \mathscr{R}$ such that $R_2 \subseteq R_1$, we can easily see that $\mathscr{R}(\leq)$ is a partially ordered set, and in addition to $\varphi(R) = x_R \in A$ and $\psi(R) = y_R \in B$, we also have

$$(\varphi \otimes \psi)(R) = \big(\varphi(R), \, \psi(R)\big) = (x_R, \, y_R) \in R \, ,$$

and thus $R \in (\varphi \otimes \psi)^{-1}[R]$ for all $R \in \mathscr{R}$.

Therefore, if $R \in \mathscr{R}$, then for every $S \in \mathscr{R}$, with $S \geq R$, i.e. $S \subseteq R$, we have

$$S \in (\varphi \otimes \psi)^{-1}[S] \subseteq (\varphi \otimes \psi)^{-1}[R] \, ,$$

and thus $[R \, , \, +\infty [\subseteq (\varphi \otimes \psi)^{-1}[R]$. This shows that $(\varphi \otimes \psi)^{-1}[R]$ is a residual, i.e., a fat subset of $\mathscr{R}(\leq)$. Thus, by the definition of the relation $\operatorname{Lim}_{\mathscr{R}}$, we have $\varphi \in \operatorname{Lim}_{\mathscr{R}}(\psi)$.

Note that, to prove the corresponding statement for the relation $\operatorname{Adh}_{\mathscr{R}}$, we have to define $R_1 \leq R_1$ for all $R_1, R_2 \in \mathscr{R}$. Therefore, for our present purposes, partially ordered sets are not, in general, sufficient.

Finally, we note that if $\mathscr{R}$ is a relator on X to Y, then according to [43] for any $A \subseteq X$, $B \subseteq Y$, $x \in X$, and $y \in Y$, we may also naturally write:

(a) $B \in \operatorname{Ub}_{\mathscr{R}}(A)$ and $A \in \operatorname{Lb}_{\mathscr{R}}(B)$ if $A \times B \subseteq R$ for some $R \in \mathscr{R}$,
(b) $y \in \operatorname{ub}_{\mathscr{R}}(A)$ if $\{y\} \in \operatorname{Ub}_{\mathscr{R}}(B)$,
(c) $x \in \operatorname{lb}_{\mathscr{R}}(B)$ if $\{x\} \in \operatorname{Lb}_{\mathscr{R}}(A)$,
(d) $A \in \mathscr{U}_{\mathscr{R}}$ if $\operatorname{ub}_{\mathscr{R}}(A) \neq \emptyset$,
(e) $B \in \mathscr{L}_{\mathscr{R}}$ if $\operatorname{lb}_{\mathscr{R}}(B) \neq \emptyset$.

Moreover, in particular $\mathscr{R}$ is a relator on X, and then for any $A \subseteq X$, we may also naturally define:

(f) $\max_{\mathscr{R}}(A) = A \cap \operatorname{ub}_{\mathscr{R}}(A)$,
(g) $\min_{\mathscr{R}}(A) = A \cap \operatorname{lb}_{\mathscr{R}}(A)$,
(h) $\operatorname{Max}_{\mathscr{R}}(A) = \mathscr{P}(A) \cap \operatorname{Ub}_{\mathscr{R}}(A)$,
(i) $\operatorname{Min}_{\mathscr{R}}(A) = \mathscr{P}(A) \cap \operatorname{Lb}_{\mathscr{R}}(A)$,

and thus also

(j) $\sup_{\mathscr{R}}(A) = \min_{\mathscr{R}}\big(\operatorname{ub}_{\mathscr{R}}(A)\big)$,
(k) $\inf_{\mathscr{R}}(A) = \max_{\mathscr{R}}\big(\operatorname{lb}_{\mathscr{R}}(A)\big)$.
(l) $\operatorname{Sup}_{\mathscr{R}}(A) = \operatorname{Min}_{\mathscr{R}}\big[\operatorname{Ub}_{\mathscr{R}}(A)\big]$,
(m) $\operatorname{Inf}_{\mathscr{R}}(A) = \operatorname{Max}_{\mathscr{R}}\big[\operatorname{Lb}_{\mathscr{R}}(A)\big]$.

Now, analogously to the families $\tau_{\mathscr{R}}$ and $\mathscr{T}_{\mathscr{R}}$, we may also naturally define:

(n) $A \in \mathfrak{u}_{\mathscr{R}}$ if $A \in \operatorname{Ub}_{\mathscr{R}}(A)$,
(o) $A \in \mathfrak{U}_{\mathscr{R}}$ if $A \subseteq \operatorname{ub}_{\mathscr{R}}(A)$,
(p) $A \in \mathfrak{L}_{\mathscr{R}}$ if $A \subseteq \operatorname{lb}_{\mathscr{R}}(A)$.

Thus, for instance, it can be shown that

$$A \in \mathfrak{u}_{\mathscr{R}} \iff A \in \operatorname{Lb}_{\mathscr{R}}(A) \iff A \in \operatorname{Min}_{\mathscr{R}}(A) \iff A \in \operatorname{Inf}_{\mathscr{R}}(A),$$

and $\mathfrak{u}_{\mathscr{R}} = \operatorname{Min}_{\mathscr{R}}\big[\mathscr{P}(X)\big] = \operatorname{Max}_{\mathscr{R}}\big[\mathscr{P}(X)\big]$. Moreover, $\operatorname{Lb}_{\mathscr{R}} = \operatorname{Ub}_{\mathscr{R}^{-1}} = \operatorname{Ub}_{\mathscr{R}}^{-1}$.

However, the above algebraic structures are not independent of the former topological ones. Namely, if R is a relation on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$ we have

$$A \times B \subseteq R \iff \forall\, a \in A: \; B \subseteq R(a) \iff \forall\, a \in A: \; R(a)^{c} \subseteq B^{c}$$
$$\iff \forall\, a \in A: \; R^{c}(a) \subseteq B^{c} \iff R^{c}[A] \subseteq B^{c}.$$

Therefore, if $\mathscr{R}$ is a relator on X to Y, then by the corresponding definitions, for any $A \subseteq X$ and $B \subseteq Y$, we also have

$$A \in \operatorname{Lb}_{\mathscr{R}}(B) \iff A \in \operatorname{Int}_{\mathscr{R}^c}(B^c) \iff A \in \left(\operatorname{Int}_{\mathscr{R}^c} \circ \mathscr{C}_Y\right)(B) .$$

Hence, we can already infer that

$$\operatorname{Lb}_{\mathscr{R}} = \operatorname{Int}_{\mathscr{R}^c} \circ \mathscr{C}_Y, \qquad \text{and} \qquad \operatorname{Int}_{\mathscr{R}} = \operatorname{Lb}_{\mathscr{R}^c} \circ \mathscr{C}_Y .$$

Therefore, in contrast to a common belief, some algebraic and topological structures are just as closely related to each other by the above equalities, and their particular cases

$$\operatorname{lb}_{\mathscr{R}} = \operatorname{int}_{\mathscr{R}^c} \circ \mathscr{C}_Y, \qquad \text{and} \qquad \operatorname{int}_{\mathscr{R}} = \operatorname{lb}_{\mathscr{R}^c} \circ \mathscr{C}_Y,$$

as the exponential and the trigonometric functions are by the celebrated Euler formulas [30, p. 227].

5 Increasingly Regular Structures for Relators

According to [55], we shall also use the following.

Definition 3. If $\mathfrak{F}$ is a structure and $\square$ is a direct unary operation for relators, then, we say that:

(1) $\mathfrak{F}$ is *increasingly upper* $\square$*–regular* if $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$ implies $\mathscr{R} \subseteq \mathscr{S}^{\square}$ for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y,

(2) $\mathfrak{F}$ is *increasingly lower* $\square$*–regular* if $\mathscr{R} \subseteq \mathscr{S}^{\square}$ implies $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$ for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y.

Remark 5. Now, the structure $\mathfrak{F}$ may be naturally called *increasingly* $\square$*–regular* if it is increasingly both upper and lower $\square$–regular.

Moreover, for instance, $\mathfrak{F}$ may also be naturally called *increasingly regular* if it is increasingly $\square$–regular for some operation $\square$ for relators.

Remark 6. If $\mathfrak{F}$ is an increasingly $\square$–regular structure for relators, then because of the fundamental work of Pataki [25], we may also say that the pair $(\mathfrak{F}, \square)$ is an *increasing Pataki connection for relators*.

In the theory of relators, increasing Pataki connections can also be most naturally obtained from the increasing Galois ones according to [46].

Definition 4. For any structure $\mathfrak{F}$ for relators, we define a unary operation $\square_{\mathfrak{F}}$ for relators such that

$$\mathscr{R}^{\square_{\mathfrak{F}}} = \left\{ S \subseteq X \times Y : \quad \mathfrak{F}_S \subseteq \mathfrak{F}_{\mathscr{R}} \right\}$$

for any relator $\mathscr{R}$ on X to Y.

Remark 7. Note that if in particular the structure $\mathfrak{F}$ is increasing, then by the above definition the operation $\Box_{\mathfrak{F}}$ is also increasing.

The appropriateness of Definition 4 is also apparent from the following extensions and supplements of the corresponding results of Pataki [25], which were mainly proved in [55]. The proofs will only be included here for the reader's convenience.

Theorem 6. *If $\mathfrak{F}$ is a structure and $\Box$ is an operation for relators such that $\mathfrak{F}$ is increasingly $\Box$–regular, then $\Box = \Box_{\mathfrak{F}}$.*

Proof. By the corresponding definitions, we have

$$S \in \mathscr{R}^{\Box} \iff \{S\} \subseteq \mathscr{R}^{\Box} \iff \mathfrak{F}_{\{S\}} \subseteq \mathfrak{F}_{\mathscr{R}} \iff \mathfrak{F}_S \subseteq \mathfrak{F}_{\mathscr{R}} \iff S \in \mathscr{R}^{\Box_{\mathfrak{F}}}$$

for any relator $\mathscr{R}$ and relation S on X to Y.

From this theorem, we can immediately derive the following three corollaries.

Corollary 1. *If $\mathfrak{F}$ is a structure for relators, then there exists at most one operation $\Box$ for relators such that $\mathfrak{F}$ is increasingly $\Box$–regular.*

Corollary 2. *If $\mathfrak{F}$ is an increasingly regular structure for relators, then $\Box = \Box_{\mathfrak{F}}$ is the unique operation for relators such that $\mathfrak{F}$ is increasingly $\Box$–regular.*

Corollary 3. *A structure $\mathfrak{F}$ for relators is increasingly regular if and only if it is increasingly $\Box_{\mathfrak{F}}$–regular.*

Theorem 7. *If $\mathfrak{F}$ is a quasi-increasing structure for relators, then*

(1) $\mathfrak{F}$ is increasingly upper $\Box_{\mathfrak{F}}$–regular,
(2) $\Box_{\mathfrak{F}}$ is extensive.

Proof. If $\mathscr{R}$ and $\mathscr{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$, then by the quasi-increasingness of $\mathfrak{F}$, for any $R \in \mathscr{R}$, we also have $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathscr{S}}$. Hence, by Definition 4, we can already infer that $R \in \mathscr{S}^{\Box_{\mathfrak{F}}}$. Therefore, $\mathscr{R} \subseteq \mathscr{S}^{\Box_{\mathfrak{F}}}$. Thus, by Definition 3, assertion (1) is true.

Now, from the inclusion $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{R}}$, by using (1), we can infer that $\mathscr{R} \subseteq \mathscr{R}^{\Box_{\mathfrak{F}}}$. Therefore, (2) is also true.

From this theorem, by using Corollary 3 and Remark 7, we can immediately derive the following two corollaries.

Corollary 4. *A quasi-increasing structure $\mathfrak{F}$ for relators is increasingly regular if and only if it is increasingly lower $\Box_{\mathfrak{F}}$–regular.*

Corollary 5. *If in particular $\mathfrak{F}$ is an increasing structure for relators, then $\Box_{\mathfrak{F}}$ is already preclosure operation.*

Theorem 8. *If $\mathfrak{F}$ is an union-preserving structure for relators, then*

(1) $\mathfrak{F}$ *is increasingly* $\square_{\mathfrak{F}}$*–regular,*
(2) $\square_{\mathfrak{F}}$ *is a closure operation.*

Proof. Suppose that $\mathscr{R}$ and $\mathscr{S}$ are relators on X to Y such that $\mathscr{R} \subseteq \mathscr{S}^{\square_{\mathfrak{F}}}$, and $\Omega \in \mathfrak{F}_{\mathscr{R}}$. Then, since $\mathfrak{F}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathfrak{F}_R$, there exists $R \in \mathscr{R}$ such that $\Omega \in \mathfrak{F}_R$. Now, since $\mathscr{R} \subseteq \mathscr{S}^{\square_{\mathfrak{F}}}$, we also have $R \in \mathscr{S}^{\square_{\mathfrak{F}}}$. Hence, by Definition 4, we can infer that $\mathfrak{F}_R \subseteq \mathfrak{F}_{\mathscr{S}}$. Therefore, we also have $\Omega \in \mathfrak{F}_{\mathscr{S}}$. Consequently, $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$. This shows that $\mathfrak{F}$ is increasingly lower $\square_{\mathfrak{F}}$–semiregular. Hence, by Theorem 7, we can see that (1) is true.

Now, assertion (2) will follow from (1) by the forthcoming Theorem 10.

Thus, in particular, we also have the following.

Corollary 6. *Every union-preserving structure* $\mathfrak{F}$ *for relators is increasingly regular.*

In [55], by using the arguments of [48], we have also proved the following three theorems.

Theorem 9. *If $\mathfrak{F}$ is an increasingly $\square$–regular structure for relators, then*

(1) $\square$ *is extensive,*
(2) $\mathfrak{F}$ *is increasing,*
(3) $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{R}^{\square}}$ *for any relator $\mathscr{R}$ on X to Y.*

Proof. If $\mathscr{R}$ is a relator on X to Y, then from the inclusion $\mathscr{R}^{\square} \subseteq \mathscr{R}^{\square}$, by using the increasing lower $\square$–regularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathscr{R}^{\square}} \subseteq \mathfrak{F}_{\mathscr{R}}$.

On the other hand, from the inclusion $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{R}}$, by using the increasing upper $\square$–regularity of $\mathfrak{F}$, we can infer that $\mathscr{R} \subseteq \mathscr{R}^{\square}$. Therefore, (1) is true.

Now, if $\mathscr{S}$ is a relator on X to Y such that $\mathscr{R} \subseteq \mathscr{S}$, then by using (1) we can see that $\mathscr{R} \subseteq \mathscr{S}^{\square}$ also holds. Hence, by using the increasing lower $\square$–regularity of $\mathfrak{F}$, we can infer that $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$. Therefore, assertion (2) is also true.

Now, from the inclusion $\mathscr{R} \subseteq \mathscr{R}^{\square}$, by using (2), we can infer that $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{R}^{\square}}$. Therefore, assertion (3) is also true.

From this theorem, by using Theorem 8, we can immediately derive

Corollary 7. *If $\mathfrak{F}$ is a union-preserving structure for relators, then $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{R}^{\square_{\mathfrak{F}}}}$ for any relator $\mathscr{R}$ on X to Y.*

Theorem 10. *For an operation $\square$ for relators, the following assertions are equivalent:*

(1) $\square$ *is a closure operation*
(2) $\square$ *is increasingly* $\square$*–regular,*
(3) *there exists an increasingly $\square$–regular structure $\mathfrak{F}$ for relators.*

Proof. To prove the implication (3) $\Longrightarrow$ (1), note that if (3) holds, then by Theorem 9 the operation $\square$ is extensive. Moreover, for any relator $\mathscr{R}$ on X to Y, we have $\mathfrak{F}_{\mathscr{R}^\square} = \mathfrak{F}_{\mathscr{R}}$. Hence, by taking $\mathscr{R}^\square$ in place of $\mathscr{R}$, we can see that $\mathfrak{F}_{\mathscr{R}^{\square\square}} = \mathfrak{F}_{\mathscr{R}^\square}$, and thus $\mathfrak{F}_{\mathscr{R}^{\square\square}} = \mathfrak{F}_{\mathscr{R}}$ also holds. Hence, by using the increasing upper $\square$–regularity of $\mathfrak{F}$, we can already infer that $\square$ is increasingly upper semiidempotent in the sense that $\mathscr{R}^{\square\square} \subseteq \mathscr{R}^\square$. Now, by the extensivity of $\square$, it is clear that the corresponding equality is also true. That is, $\square$ is idempotent.

Thus, to obtain (1), it remains only to show that $\square$ is also increasing. For this, note that if $\mathscr{R}$ and $\mathscr{S}$ are relators on X to Y such that $\mathscr{R} \subseteq \mathscr{S}$, then by Theorem 9 we also have $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$. Moreover, we have $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{R}^\square}$, and thus also $\mathfrak{F}_{\mathscr{R}^\square} \subseteq \mathfrak{F}_{\mathscr{S}}$. Hence, by using the increasing upper $\square$–regularity of $\mathfrak{F}$, we can already infer that $\mathscr{R}^\square \subseteq \mathscr{S}^\square$.

From this theorem, by Theorem 6, it is clear that in particular we also have

Corollary 8. *If $\lozenge$ is a closure operation for relators, then $\lozenge = \square_\lozenge$.*

Moreover, from Theorem 10, by using Definition 3, we can immediately derive

Corollary 9. *For a structure $\mathfrak{F}$ and an operation $\square$ for relators, the following assertions are equivalent:*

(1) $\mathfrak{F}$ is increasingly $\square$–regular,
(2) $\square$ is a closure operation, and for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y, we have $\mathfrak{F}_{\mathscr{R}} \subseteq \mathfrak{F}_{\mathscr{S}}$ if and only if $\mathscr{R}^\square \subseteq \mathscr{S}^\square$.

Theorem 11. *For a structure $\mathfrak{F}$ and an operation $\square$ for relators, the following assertions are equivalent:*

(1) $\mathfrak{F}$ is increasingly $\square$–regular,
(2) $\mathfrak{F}$ is increasing, and for every relator $\mathscr{R}$ on X to Y, $\mathscr{S} = \mathscr{R}^\square$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathscr{S}} \subseteq \mathfrak{F}_{\mathscr{R}}$.

Proof. If (1) holds, then by Theorem 9 the structure $\mathfrak{F}$ is increasing, and for any relator $\mathscr{R}$ on X to Y, we have $\mathfrak{F}_{\mathscr{R}^\square} = \mathfrak{F}_{\mathscr{R}}$. Moreover, if $\mathscr{S}$ is a relator on X to Y such that $\mathfrak{F}_{\mathscr{S}} \subseteq \mathfrak{F}_{\mathscr{R}}$, then by using the upper $\square$–regularity of $\mathfrak{F}$ we can see that $\mathscr{S} \subseteq \mathscr{R}^\square$. Thus, in particular, (2) also holds.

On the other hand, if (2) holds, and $\mathscr{R}$ and $\mathscr{S}$ are relators on X to Y such that $\mathfrak{F}_{\mathscr{S}} \subseteq \mathfrak{F}_{\mathscr{R}}$, then from the assumed maximality property of $\mathscr{R}^\square$ we can see that $\mathscr{S} \subseteq \mathscr{R}^\square$. Therefore, $\mathfrak{F}$ is upper $\square$–regular.

Conversely, if $\mathscr{R}$ and $\mathscr{S}$ are relators on X to Y such that $\mathscr{S} \subseteq \mathscr{R}^\square$, then by using the assumed increasingness of $\mathfrak{F}$ we can see that $\mathfrak{F}_{\mathscr{S}} \subseteq \mathfrak{F}_{\mathscr{R}^\square}$. Hence, by the assumed inclusion $\mathfrak{F}_{\mathscr{R}^\square} \subseteq \mathfrak{F}_{\mathscr{R}}$, it follows that $\mathfrak{F}_{\mathscr{S}} \subseteq \mathfrak{F}_{\mathscr{R}}$. Therefore, $\mathfrak{F}$ is also lower $\square$–regular, and thus (1) also holds.

From this theorem, by Theorem 9, it is clear that we also have

Corollary 10. *If $\mathfrak{F}$ is a $\square$–regular structure for relators, then for any relator $\mathscr{R}$ on X to Y, $\mathscr{S} = \mathscr{R}^\square$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathscr{S}} = \mathfrak{F}_{\mathscr{R}}$.*

Now, by Theorem 6, it is clear that in particular we also have

Corollary 11. *If $\mathfrak{F}$ is a regular structure for relators, then for any relator $\mathscr{R}$ on X to Y, $\mathscr{S} = \mathscr{R}^{\,\Box_{\mathfrak{F}}}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathscr{S}} \subseteq \mathfrak{F}_{\mathscr{R}}$ $\big(\mathfrak{F}_{\mathscr{S}} = \mathfrak{F}_{\mathscr{R}}\big)$.*

Hence, by Theorem 8, it is clear that more specially we also have

Corollary 12. *If $\mathfrak{F}$ is a union-preserving structure for relators, then for any relator $\mathscr{R}$ on X to Y, $S = \mathscr{R}^{\,\Box_{\mathfrak{F}}}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathscr{S}} \subseteq \mathfrak{F}_{\mathscr{R}}$ $\big(\mathfrak{F}_{\mathscr{S}} = \mathfrak{F}_{\mathscr{R}}\big)$.*

Finally, we note that, analogously to [25, Theorem 1.5], the following theorem is also true.

Theorem 12. *For a direct unary operation $\Box$ for relators, the following assertions are equivalent:*

(1) $\Box$ is a semiclosure ,
(2) for every relator $\mathscr{R}$ on X to Y, $\mathscr{S} = \mathscr{R}^{\,\Box}$ is the largest relator on X to Y such that $\mathscr{R}^{\,\Box} = \mathscr{S}^{\,\Box}$,
(3) there exists a structure $\mathfrak{F}$ for relators such that, for every relator $\mathscr{R}$ on X to Y, $\mathscr{S} = \mathscr{R}^{\,\Box}$ is the largest relator on X to Y such that $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{S}}$.

Remark 8. If $\mathfrak{F}$ is a structure for relators, then two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y are called $\mathfrak{F}$–*equivalent* if $\mathfrak{F}_{\mathscr{R}} = \mathfrak{F}_{\mathscr{S}}$.

Moreover, the relator $\mathscr{R}$ is called $\mathfrak{F}$-*simple* if it is $\mathfrak{F}$–equivalent to a singleton relator. And, in particular, $\mathscr{R}$ is called *properly simple* if it is $\mathfrak{F}$-simple with $\mathfrak{F}$ being the identity operation for relators.

6 Some Important Unary Operations for Relators

Definition 5. For any relator $\mathscr{R}$ on X to Y, the relators

$$\mathscr{R}^{\,*} = \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathscr{R} : \quad R \subseteq S \big\},$$
$$\mathscr{R}^{\,\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathscr{R} : \quad R[A] \subseteq S[A] \big\},$$
$$\mathscr{R}^{\,\wedge} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathscr{R} : \quad R(x) \subseteq S(x) \big\},$$

and

$$\mathscr{R}^{\,\triangle} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \ \exists\ u \in X : \ \exists\ R \in \mathscr{R} : \quad R(u) \subseteq S(x) \big\}$$

are called the *uniform, proximal, topological, and paratopological closures (refinements)* of the relator $\mathscr{R}$, respectively.

Remark 9. Thus, we evidently have

$$\mathscr{R} \subseteq \mathscr{R}^{\,*} \subseteq \mathscr{R}^{\,\#} \subseteq \mathscr{R}^{\,\wedge} \subseteq \mathscr{R}^{\,\triangle}.$$

Moreover, if in particular $\mathscr{R}$ is a relator on X, then we can easily see that

$$\mathscr{R}^{\infty} \subseteq \mathscr{R}^{*\infty} \subseteq \mathscr{R}^{\infty *} \subseteq \mathscr{R}^{*} .$$

Remark 10. However, it is now more important to note that, because of the corresponding definitions of Sect. 4, we also have

$$\begin{aligned}
\mathscr{R}^{\#} &= \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_R\big(S[A]\big) \big\} , \\
\mathscr{R}^{\wedge} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad x \in \operatorname{int}_R\big(S(x)\big) \big\} , \\
\mathscr{R}^{\Delta} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad S(x) \in \mathscr{E}_{\mathscr{R}} \big\} .
\end{aligned}$$

Now, by using this remark and Definition 4, we can easily prove the following

Theorem 13. *For any relator $\mathscr{R}$ on X to Y, we have*

(1) $\mathscr{R}^{\#} = \mathscr{R}^{\square_{\operatorname{Int}}}$,
(2) $\mathscr{R}^{\wedge} = \mathscr{R}^{\square_{\operatorname{int}}}$,
(3) $\mathscr{R}^{\Delta} = \mathscr{R}^{\square_{\mathscr{E}}}$,

Proof. We shall only prove that $\mathscr{R}^{\square_{\operatorname{Int}}} \subseteq \mathscr{R}^{\#}$. The proof of the converse inclusion, and those of (2) and (3), will be left to the reader.

For this, we can note that if $S \in \mathscr{R}^{\square_{\operatorname{Int}}}$, then by Definition 4 S is a relation on X to Y such that $\operatorname{Int}_S \subseteq \operatorname{Int}_{\mathscr{R}}$, and so $\operatorname{Int}_S(B) \subseteq \operatorname{Int}_{\mathscr{R}}(B)$ for all $B \subseteq Y$.

Thus, in particular, for any $A \subseteq X$, we have $\operatorname{Int}_S\big(S[A]\big) \subseteq \operatorname{Int}_{\mathscr{R}}\big(S[A]\big)$. Hence, by using that $A \in \operatorname{Int}_S\big(S[A]\big)$, we can already infer that $A \in \operatorname{Int}_{\mathscr{R}}\big(S[A]\big)$. Therefore, by Remark 10, $S \in \mathscr{R}^{\#}$ also holds.

From this theorem, by using Theorem 8, we can immediately derive

Theorem 14. $\#$, $\wedge$, *and* Δ *are closure operations for relators.*

Proof. By the corresponding definitions, it is clear that

$$\operatorname{Int}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \operatorname{Int}_R , \qquad \operatorname{int}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \operatorname{int}_R , \qquad \text{and} \qquad \mathscr{E}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathscr{E}_R$$

for any relator $\mathscr{R}$ on X to Y.

Therefore, the structures Int, int, and $\mathscr{E}$ are union preserving. Thus, by Theorem 8, the operations $\square_{\operatorname{Int}}$, $\square_{\operatorname{int}}$, and $\square_{\mathscr{E}}$ are closures. Therefore, by Theorem 13, the required assertions are also true.

Remark 11. By using the definition of the operation $*$, we can easily see that $*$ is also a closure operation for relators.

It can actually be derived from the structures Lim and Adh. While, the structures lim and adh lead only to the operation $\wedge$.

Now, by using Remark 9 and Theorem 14, we can also easily prove

Theorem 15. *For any relator $\mathscr{R}$ on X to Y, we have*

(1) $\mathscr{R}^{\#} = \left(\mathscr{R}^{*}\right)^{\#} = \left(\mathscr{R}^{\#}\right)^{*}$,
(2) $\mathscr{R}^{\wedge} = \left(\mathscr{R}^{\Diamond}\right)^{\wedge} = \left(\mathscr{R}^{\wedge}\right)^{\Diamond}$ *with* $\Diamond = *$ *or* $\#$,
(3) $\mathscr{R}^{\Delta} = \left(\mathscr{R}^{\Diamond}\right)^{\Delta} = \left(\mathscr{R}^{\Delta}\right)^{\Diamond}$ *with* $\Diamond = *, \#,$ *or* $\wedge$.

Proof. To prove (1), note that, by Remark 9 and Theorem 14, we have

$$\mathscr{R}^{\#} \subseteq \left(\mathscr{R}^{\#}\right)^{*} \subseteq \mathscr{R}^{\#\#} = \mathscr{R}^{\#} \qquad \text{and} \qquad \mathscr{R}^{\#} \subseteq \mathscr{R}^{*\#} \subseteq \mathscr{R}^{\#\#} = \mathscr{R}^{\#}.$$

Therefore, the corresponding equalities are also true.

Remark 12. By using Remark 9 and we can also easily prove that

(1) $\mathscr{R}^{*\infty} = \mathscr{R}^{\infty*\infty}$,
(2) $\mathscr{R}^{\infty*} = \mathscr{R}^{*\infty*}$.

However, it is now more important to note that, by using Theorems 13, 8, and 10, and Corollary 12, we can also easily establish the following two theorems.

Theorem 16. *For any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y, we have*

(1) $\mathscr{S} \subseteq \mathscr{R}^{\#} \iff \mathscr{S}^{\#} \subseteq \mathscr{R}^{\#} \iff \operatorname{Int}_{\mathscr{S}} \subseteq \operatorname{Int}_{\mathscr{R}} \iff \operatorname{Cl}_{\mathscr{R}} \subseteq \operatorname{Cl}_{\mathscr{S}}$,
(2) $\mathscr{S} \subseteq \mathscr{R}^{\wedge} \iff \mathscr{S}^{\wedge} \subseteq \mathscr{R}^{\wedge} \iff \operatorname{int}_{\mathscr{S}} \subseteq \operatorname{int}_{\mathscr{R}} \iff \operatorname{cl}_{\mathscr{R}} \subseteq \operatorname{cl}_{\mathscr{S}}$,
(3) $\mathscr{S} \subseteq \mathscr{R}^{\Delta} \iff \mathscr{S}^{\Delta} \subseteq \mathscr{R}^{\Delta} \iff \mathscr{E}_{\mathscr{S}} \subseteq \mathscr{E}_{\mathscr{R}} \iff \mathscr{D}_{\mathscr{R}} \subseteq \mathscr{D}_{\mathscr{S}}$.

Corollary 13. *For any two relators $\mathscr{R}$ and $\mathscr{S}$ on X,*

(1) $\mathscr{S} \subseteq \mathscr{R}^{\#} \implies \tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}} \iff \tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}}$,
(2) $\mathscr{S} \subseteq \mathscr{R}^{\wedge} \implies \mathscr{T}_{\mathscr{S}} \subseteq \mathscr{T}_{\mathscr{R}} \iff \mathscr{F}_{\mathscr{S}} \subseteq \mathscr{F}_{\mathscr{R}}$,

Theorem 17. *For any relator $\mathscr{R}$ on X to Y,*

(1) $\mathscr{S} = \mathscr{R}^{\#}$ *is the largest relator on X to Y such that* $\operatorname{Int}_{\mathscr{S}} \subseteq \operatorname{Int}_{\mathscr{R}}$ $\left(\operatorname{Int}_{\mathscr{S}} = \operatorname{Int}_{\mathscr{R}}\right)$, *or equivalently* $\operatorname{Cl}_{\mathscr{R}} \subseteq \operatorname{Cl}_{\mathscr{S}}$ $\left(\operatorname{Cl}_{\mathscr{R}} = \operatorname{Cl}_{\mathscr{S}}\right)$,
(2) $\mathscr{S} = \mathscr{R}^{\wedge}$ *is the largest relator on X to Y such that* $\operatorname{int}_{\mathscr{S}} \subseteq \operatorname{int}_{\mathscr{R}}$ $\left(\operatorname{int}_{\mathscr{S}} = \operatorname{int}_{\mathscr{R}}\right)$, *or equivalently* $\operatorname{cl}_{\mathscr{R}} \subseteq \operatorname{cl}_{\mathscr{S}}$ $\left(\operatorname{cl}_{\mathscr{R}} = \operatorname{cl}_{\mathscr{S}}\right)$,
(3) $\mathscr{S} = \mathscr{R}^{\Delta}$ *is the largest relator on X to Y such that* $\mathscr{E}_{\mathscr{S}} \subseteq \mathscr{E}_{\mathscr{R}}$ $\left(\mathscr{E}_{\mathscr{S}} = \mathscr{E}_{\mathscr{R}}\right)$, *or equivalently* $\mathscr{D}_{\mathscr{R}} \subseteq \mathscr{D}_{\mathscr{S}}$ $\left(\mathscr{D}_{\mathscr{R}} = \mathscr{D}_{\mathscr{S}}\right)$.

Corollary 14. *For any relator $\mathscr{R}$ on X, we have*

(1) $\tau_{\mathscr{R}} = \tau_{\mathscr{R}^{\#}}$,
(2) $\mathscr{T}_{\mathscr{R}} = \mathscr{T}_{\mathscr{R}^{\wedge}}$.

Remark 13. To prove the above two theorems and their corollaries, recall that

$$\operatorname{Cl}_{\mathscr{R}} = \left(\operatorname{Int}_{\mathscr{R}} \circ \mathscr{C}_Y\right)^{c}, \quad \operatorname{cl}_{\mathscr{R}} = \left(\operatorname{int}_{\mathscr{R}} \circ \mathscr{C}_Y\right)^{c}, \quad \text{and} \quad \mathscr{D}_{\mathscr{R}} = \left\{B \subseteq Y : \ B^{c} \notin \mathscr{E}_{\mathscr{R}}\right\}$$

for any relator $\mathscr{R}$ on X to Y, and in particular

$$\tau_{\mathscr{R}} = \{A \subseteq X : \quad A^c \in \tau_{\mathscr{R}}\} \qquad \text{and} \qquad \mathscr{F}_{\mathscr{R}} = \{A \subseteq X : \quad A^c \in \mathscr{T}_{\mathscr{R}}\}$$

for any relator $\mathscr{R}$ on X.

Concerning the operations $\wedge$ and Δ, we can also prove the following straightforward extensions of [31, Theorem 6.7] and [26, Theorem 5.16].

Theorem 18. *If $\mathscr{R}$ is a nonvoid relator on X to Y and $B \subseteq Y$, then*

(1) $\operatorname{Int}_{\mathscr{R}^\wedge}(B) = \mathscr{P}\big(\operatorname{int}_{\mathscr{R}}(B)\big)$,
(2) $\operatorname{Cl}_{\mathscr{R}^\wedge}(B) = \mathscr{P}\big(\operatorname{cl}_{\mathscr{R}}(B)^c\big)^c$.

Proof. To prove the less obvious part of (1), note that if $A \in \mathscr{P}\big(\operatorname{int}_{\mathscr{R}}(B)\big)$, i.e., $A \subseteq \operatorname{int}_{\mathscr{R}}(B)$, then for each $x \in A$ there exists $R_x \in \mathscr{R}$ such that $R_x(x) \subseteq B$. Hence, by defining $S(x) = R_x(x)$ if $x \in A$, and $S(x) = Y$ if $x \in A^c$, we can see that $S \in \mathscr{R}^\wedge$ and $S[A] \subseteq B$. Therefore, $A \in \operatorname{Int}_{\mathscr{R}^\wedge}(B)$ also holds.

Note that if $\mathscr{R}$ is not supposed to be nonvoid, then instead of (1) we can only prove that $\mathscr{P}\big(\operatorname{int}_{\mathscr{R}}(B)\big) = \operatorname{Int}_{\mathscr{R}^\wedge}(B) \cup \{\emptyset\}$.

Corollary 15. *If $\mathscr{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathscr{R}^\wedge} = \mathscr{T}_{\mathscr{R}}$,
(2) $\tau_{\mathscr{R}^\wedge} = \mathscr{F}_{\mathscr{R}}$.

Corollary 16. *If $\mathscr{R}$ is a nonvoid relator on X, then*

(1) $\mathscr{T}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}^\wedge} \mathscr{T}_R$,
(2) $\mathscr{F}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}^\wedge} \mathscr{F}_R$.

Remark 14. Note that if in particular $\mathscr{R}$ is a relator on X such that $\mathscr{R} = \emptyset$, then by the definition of $\mathscr{T}_{\mathscr{R}}$ we have $\mathscr{T}_{\mathscr{R}} = \{\emptyset\}$.

Moreover, if in addition $X \neq \emptyset$, then by the definition of $\mathscr{R}^\wedge$ we also have $\mathscr{R}^\wedge = \emptyset$. Thus, $\bigcup_{R \in \mathscr{R}^\wedge} \mathscr{T}_R = \emptyset$.

Therefore, if $\mathscr{R} = \emptyset$, but $X \neq \emptyset$, then the equalities stated in Corollary 16, and thus also those stated in Corollary 15 and Theorem 18, do not hold.

Theorem 19. *If $\mathscr{R}$ is a nonvoid relator on X to Y and $B \subseteq Y$, then*

(1) $\operatorname{Int}_{\mathscr{R}^\Delta}(B) = \{\emptyset\}$ *if* $B \notin \mathscr{E}_{\mathscr{R}}$ *and* $\operatorname{Int}_{\mathscr{R}^\Delta}(B) = \mathscr{P}(X)$ *if* $B \in \mathscr{E}_{\mathscr{R}}$,
(2) $\operatorname{Cl}_{\mathscr{R}^\Delta}(B) = \emptyset$ *if* $B \notin \mathscr{D}_{\mathscr{R}}$ *and* $\operatorname{Cl}_{\mathscr{R}^\Delta}(B) = \mathscr{P}(X) \setminus \{\emptyset\}$ *if* $B \in \mathscr{D}_{\mathscr{R}}$.

Proof. If $A \in \operatorname{Int}_{\mathscr{R}^\Delta}(B)$, then there exists $S \in \mathscr{R}^\Delta$ such that $S[A] \subseteq B$. Therefore, if $A \neq \emptyset$, then there exists $x \in X$ such that $S(x) \subseteq B$. Hence, since $S(x) \in \mathscr{E}_{\mathscr{R}}$, it is clear that $B \in \mathscr{E}_{\mathscr{R}}$. Therefore, the first part of (1) is true.

To prove the second part of (1), it is enough to note only that if $B \in \mathscr{E}_{\mathscr{R}}$, then $R = X \times B \in \mathscr{R}^\Delta$ such that $R[A] \subseteq B$, and thus $A \in \operatorname{Int}_{\mathscr{R}^\Delta}(B)$ for all $A \subseteq X$.

Corollary 17. *If $\mathscr{R}$ is a nonvoid relator on X to Y and $B \subseteq Y$, then*

(1) $\operatorname{cl}_{\mathscr{R}^\triangle}(B) = \emptyset$ *if* $B \notin \mathscr{D}_{\mathscr{R}}$ *and* $\operatorname{Cl}_{\mathscr{R}^\triangle}(B) = X$ *if* $B \in \mathscr{D}_{\mathscr{R}}$,
(2) $\operatorname{int}_{\mathscr{R}^\triangle}(B) = \emptyset$ *if* $B \notin \mathscr{E}_{\mathscr{R}}$ *and* $\operatorname{int}_{\mathscr{R}^\triangle}(B) = X$ *if* $B \in \mathscr{E}_{\mathscr{R}}$.

Corollary 18. *If $\mathscr{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathscr{R}^\triangle} = \mathscr{T}_{\mathscr{R}^\triangle} = \mathscr{E}_{\mathscr{R}} \cup \{\emptyset\}$,
(2) $\bar{\tau}_{\mathscr{R}^\triangle} = \mathscr{F}_{\mathscr{R}^\triangle} = \big(\mathscr{P}(X) \setminus \mathscr{D}_{\mathscr{R}}\big) \cup \{X\}$.

Proof. To check the first part of (1), because of Corollary 15 and Theorem 15, we can also note that $\tau_{\mathscr{R}^\triangle} = \tau_{\mathscr{R}^{\triangle\wedge}} = \mathscr{T}_{\mathscr{R}^\triangle}$.

Remark 15. Note that if in particular $\mathscr{R}$ is a relator on X such that $\mathscr{R} = \emptyset$, then by the definition of $\mathscr{E}_{\mathscr{R}}$ we have $\mathscr{E}_{\mathscr{R}} = \emptyset$. Moreover, if in addition $X \neq \emptyset$, then we also have $\mathscr{R}^{\Delta} = \emptyset$. Hence, by the definition of $\tau_{\mathscr{R}}$, we can again see that $\tau_{\mathscr{R}^\wedge} = \emptyset$.

Therefore, if $\mathscr{R} = \emptyset$, but $X \neq \emptyset$, then the assertions (1) and (2) of Corollary 18, and thus also those of Corollary 17 and Theorem 19, do not hold.

However, the second equalities in the assertions (1) and (2) of Corollary 18 do not require relator $\mathscr{R}$ to be nonvoid. Moreover, we can also prove the following.

Theorem 20. *If $\mathscr{R}$ is a total relator on X, then*

(1) $\mathscr{E}_{\mathscr{R}} = \mathscr{T}_{\mathscr{R}^\triangle} \setminus \{\emptyset\}$,
(2) $\mathscr{D}_{\mathscr{R}} = \mathscr{F}^{\,c}_{\mathscr{R}^\triangle} \cup \{X\}$.

Remark 16. A relator $\mathscr{R}$ on X to Y is called *total* if each member R of $\mathscr{R}$ is a total relation in the sense that the whole X is the domain of R.

By using the corresponding definitions, it can be easily seen that the relator $\mathscr{R}$ is total if and only if $\emptyset \notin \mathscr{E}_{\mathscr{R}}$ $(Y \in \mathscr{D}_{\mathscr{R}})$, or equivalently $\mathscr{D}_{\mathscr{R}} \neq \emptyset$ $(\mathscr{E}_{\mathscr{R}} \neq \mathscr{P}(Y))$.

In this respect, it is also noteworthy that conversely we have $\emptyset \notin \mathscr{D}_{\mathscr{R}}$ $(Y \in \mathscr{E}_{\mathscr{R}})$, or equivalently $\mathscr{E}_{\mathscr{R}} \neq \emptyset$ $(\mathscr{D}_{\mathscr{R}} \neq \mathscr{P}(Y))$, if and only if $X \neq \emptyset$ and $\mathscr{R} \neq \emptyset$.

7 Some Further Important Unary Operations for Relators

In addition to Theorem 17, we can also prove the following.

Theorem 21. *If R is a relation on X, then $S = R^{\infty}$ is the largest relation on X such that $\tau_R \subseteq \tau_S$ $\big(\tau_R = \tau_S\big)$, or equivalently $\bar{\tau}_R \subseteq \bar{\tau}_S$ $\big(\bar{\tau}_R = \bar{\tau}_S\big)$.*

Proof. If $A \in \tau_{R^\infty}$, then by the corresponding definitions we have $R^{\infty}[A] \subseteq A$. Hence, by using that $R \subseteq R^{\infty}$, and thus $R[A] \subseteq R^{\infty}[A]$, we can already infer that $R[A] \subseteq A$. Therefore, $A \in \tau_R$ also holds.

While, if $A \in \tau_R$ holds, then we have $R[A] \subseteq A$. Hence, by induction, we can see that $R^n[A] \subseteq A$ for all $n \in \mathbb{N}$. Now, since $R^0[A] = \Delta_X[A] = A$, we can already state that

$$R^{\infty}[A] = \left(\bigcup_{n=0}^{\infty} R^{n}\right)[A] = \bigcup_{n=0}^{\infty} R^{n}[A] \subseteq \bigcup_{n=0}^{\infty} A = A\,.$$

Therefore, $A \in \tau_{R^{\infty}}$ also holds.

The above arguments show that $\tau_{\mathscr{R}} = \tau_{\mathscr{R}^{\infty}}$. Therefore, to complete the proof of the first statement of the theorem, it remains to show only that if S is a relation on X such that $\tau_R \subseteq \tau_S$, then we necessarily have $S \subseteq R^{\infty}$.

For this, note that if $x \in X$, then because of the inclusion $R \subseteq R^{\infty}$ and the transitivity of R^{∞} we have

$$R[R^{\infty}(x)] \subseteq R^{\infty}[R^{\infty}(x)] = \big(R^{\infty} \circ R^{\infty}\big)(x) \subseteq R^{\infty}(x)\,.$$

Therefore, $R^{\infty}(x) \in \tau_{\mathscr{R}}$. Hence, by using the assumption $\tau_R \subseteq \tau_S$, we can already infer that $R^{\infty}(x) \in \tau_S$, and thus $S[R^{\infty}(x)] \subseteq R^{\infty}(x)$. Now, by using the reflexivity of R^{∞}, we can see that $S(x) \subseteq R^{\infty}(x)$ also holds.

Remark 17. This theorem, and the fact that

$$R^{\infty}(x) = \bigcap \big\{A \in \tau_R : \quad x \in A\big\}$$

for all $x \in X$, was first proved by Mala [19].

Hence, we can immediately infer that

$$R^{\infty} = \bigcap \big\{R_A : \quad A \in \tau_R\big\}, \qquad \text{where} \qquad R_A = A^2 \cup A^c \times X.$$

Now, as an immediate consequence of Theorem 21, we can also state

Corollary 19. *For any relator $\mathscr{R}$ on X, we have*

(1) $\tau_{\mathscr{R}} = \tau_{\mathscr{R}^{\infty}}$,
(2) $\bar{\tau}_{\mathscr{R}} = \bar{\tau}_{\mathscr{R}^{\infty}}$.

Proof. By the corresponding definitions, we have $\mathscr{R}^{\infty} = \{R^{\infty} : \; R \in \mathscr{R}\}$,

$$\tau_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \tau_R\,, \qquad \text{and} \qquad \bar{\tau}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \bar{\tau}_R\,.$$

for any relator $\mathscr{R}$ on X. Thus, Theorem 21 can be applied to get the required equalities.

However, it now more important to note that, in addition to Theorem 13, we can also prove the following.

Theorem 22. *For any relator $\mathscr{R}$ on X to Y, we have*

$$\mathscr{R}^{\square_\tau} = \mathscr{R}^{\#\partial}\,.$$

Proof. If $S \in \mathscr{R}^{\#\partial}$, then by the definition of the operation ∂ we have $S^{\infty} \in \mathscr{R}^{\#}$. Hence, by using Theorem 21 and Corollary 13, we can already see that $\tau_S = \tau_{S^\infty} \subseteq \tau_{\mathscr{R}}$. Therefore, by Definition 4, $S \in \mathscr{R}^{\Box_\tau}$ also holds.

Conversely, if $S \in \mathscr{R}^{\Box_\tau}$, then Definition 4 S is a relation on X to Y such that $\tau_S \subseteq \tau_{\mathscr{R}}$. Therefore, $A \in \tau_S$ implies $A \in \tau_{\mathscr{R}}$.

On the other hand, if $A \subseteq X$, then, by using that $S \subseteq S^{\infty}$ and S^{∞} is transitive, we can note that

$$S\big[S^{\infty}[A]\big] \subseteq S^{\infty}\big[S^{\infty}[A]\big] = \big(\mathscr{S}^{\infty} \circ S^{\infty}\big)[A] \subseteq S^{\infty}[A],$$

and thus $S^{\infty}[A] \in \tau_S$.

Therefore, by the inclusion $\tau_S \subseteq \tau_{\mathscr{R}}$, for any $A \subseteq X$, we also have $S^{\infty}[A] \in \tau_{\mathscr{R}}$, and thus $\operatorname{Int}_{\mathscr{R}}\big(S^{\infty}[A]\big)$. Hence, by using that $A \subseteq \mathscr{S}^{\infty}[A]$, we can infer that $A \in \operatorname{Int}_{\mathscr{R}}\big(S^{\infty}[A]\big)$ also holds. Therefore, by Remark 10, $S^{\infty} \in \mathscr{R}^{\#}$, and thus $S \in \mathscr{R}^{\#\partial}$ also holds.

Now, by using Theorems 22, 8, and 10, and Corollary 12, we can easily establish the following counterparts of Theorems 14, 16, and 17.

Theorem 23. *The following assertions are true:*

(1) $\#\,\partial$ *is a closure operation for relators,*
(2) for any two relators $\mathscr{R}$ *and* $\mathscr{S}$ *on X, we have*

$$\mathscr{S} \subseteq \mathscr{R}^{\#\partial} \iff \mathscr{S}^{\#\partial} \subseteq \mathscr{R}^{\#\partial} \iff \tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}} \iff \bar{\tau}_{\mathscr{S}} \subseteq \bar{\tau}_{\mathscr{R}},$$

(3) for any relator $\mathscr{R}$ *on X,* $\mathscr{S} = \mathscr{R}^{\#\partial}$ *is the largest relator on X such that* $\tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}}$ $\big(\tau_{\mathscr{S}} = \tau_{\mathscr{R}}\big)$, *or equivalently* $\bar{\tau}_{\mathscr{S}} \subseteq \bar{\tau}_{\mathscr{R}}$ $\big(\bar{\tau}_{\mathscr{S}} = \bar{\tau}_{\mathscr{R}}\big)$.

Remark 18. The above two theorems and the next theorem were first proved by Pataki [25] and Mala [19], respectively, in somewhat different forms.

Theorem 24. *The following assertions are true:*

(1) $\#\,\infty$ *is a modification operation for relators,*
(2) for any two relators $\mathscr{R}$ *and* $\mathscr{S}$ *on X, we have*

$$\tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}} \iff \bar{\tau}_{\mathscr{S}} \subseteq \bar{\tau}_{\mathscr{R}} \iff \mathscr{S}^{\infty} \subseteq \mathscr{R}^{\#} \iff \mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#}$$
$$\iff \mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#\infty}.$$

(3) for any relator $\mathscr{R}$ *on X,* $\mathscr{S} = \mathscr{R}^{\#\infty}$ *is the largest preorder relator on X such that* $\tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}}$ $\big(\tau_{\mathscr{S}} = \tau_{\mathscr{R}}\big)$ *or equivalently* $\bar{\tau}_{\mathscr{S}} \subseteq \bar{\tau}_{\mathscr{R}}$ $\big(\bar{\tau}_{\mathscr{S}} = \bar{\tau}_{\mathscr{R}}\big)$.

Proof. If $\mathscr{R}$ and $\mathscr{S}$ are relators on X, then by Theorem 23 and the definition of the operation ∂, we have

$$\tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}} \iff \mathscr{S} \subseteq \mathscr{R}^{\#\partial} \iff \mathscr{S}^{\infty} \subseteq \mathscr{R}^{\#}.$$

Moreover, from Corollary 14, we can see that $\tau_{\mathscr{S}} = \tau_{\mathscr{S}^{\#}}$. Therefore, we also have

$$\tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}} \iff \tau_{\mathscr{S}^{\#}} \subseteq \tau_{\mathscr{R}} \iff \mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#}.$$

Furthermore, since ∞ is modification operation, we can note that

$$\mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#} \implies \mathscr{S}^{\#\infty\infty} \subseteq \mathscr{R}^{\#\infty} \implies \mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#\infty}.$$

Moreover, by using Remark 9 and Theorem 15, we can also easily that

$$\mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#\infty} \implies \mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#*} \implies \mathscr{S}^{\#\infty} \subseteq \mathscr{R}^{\#}.$$

Therefore, assertion (2) is true.

On the other hand, if $\mathscr{R}$ is a relator on X and $\mathscr{S} = \mathscr{R}^{\#\infty}$, then from Corollaries 19 and 14 we can see that $\tau_{\mathscr{S}} = \tau_{\mathscr{R}}$. Hence, by using assertion (2), we can infer that $\mathscr{S}^{\#\infty} = \mathscr{R}^{\#\infty}$, and thus $\left(\mathscr{R}^{\#\infty}\right)^{\#\infty} = \mathscr{R}^{\#\infty}$. Thus, since the operation # and ∞ are increasing, assertion (1) is also true.

Now, to prove the first part assertion (3), it remains only to note only that $\mathscr{R}$ is an arbitrary and $\mathscr{S}$ is a preorder relator on relator on X such that $\tau_{\mathscr{S}} \subseteq \tau_{\mathscr{R}}$, then by assertion (2) we have $\mathscr{S} = \mathscr{S}^{\infty} \subseteq \mathscr{R}^{\#}$, and thus also $\mathscr{S} = \mathscr{S}^{\infty} \subseteq \mathscr{R}^{\#\infty}$.

Remark 19. In this respect, it is worth noticing that, for any relator $\mathscr{R}$ on X, the following assertions are also equivalent:

(1) $S^{\#\partial} \subseteq \mathscr{R}^{\#\partial}$,
(2) $S^{\#\infty} \subseteq \mathscr{R}^{\#\infty}$,
(3) $\mathscr{S}^{\infty\#} \subseteq \mathscr{R}^{\infty\#}$.

The advantage of the modification operations $\#\infty$ and $\infty\#$ over the closure operation $\#\partial$ lies mainly in the fact that, in contrast to $\#\partial$, they are stable in the sense that they leave the relator $\{X^2\}$ fixed for any set X.

Now, in addition to Theorem 22, we can also prove the following

Theorem 25. *For any relator $\mathscr{R}$ on X to Y, we have*

$$\mathscr{R}^{\square_{\mathscr{T}}} = \mathscr{R}^{\wedge\partial}.$$

Proof. If $\mathscr{R} \neq \emptyset$, then by the corresponding definitions, Corollary 15 and Theorems 23 and 15, it is clear that

$$S \in \mathscr{R}^{\square_{\mathscr{T}}} \iff \mathscr{T}_S \subseteq \mathscr{T}_{\mathscr{R}} \iff \tau_S \subseteq \tau_{\mathscr{R}^{\wedge}} \iff S \in \mathscr{R}^{\wedge\#\partial} \iff S \in \mathscr{R}^{\wedge\partial}.$$

While, if $\mathscr{R} = \emptyset$, then by using the corresponding definitions we can see that

$$\mathscr{R}^{\,\square\,\mathscr{T}} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathscr{R}^{\,\square\,\mathscr{T}} = \{\emptyset\} \quad \text{if} \quad X = \emptyset$$

and

$$\mathscr{R}^{\,\wedge\,\partial} = \emptyset \quad \text{if} \quad X \neq \emptyset \qquad \text{and} \qquad \mathscr{R}^{\,\wedge\,\partial} = \{\emptyset\} \quad \text{if} \quad X = \emptyset\,.$$

Therefore, the required equality is again true.

Unfortunately, by the following example, the structures $\mathscr{T}$ and $\mathscr{F}$ are not union preserving.

Example 3. For any set X, with $\operatorname{card}(X) > 2$, there exists an equivalence relator $\mathscr{R} = \{R_1, R_2\}$ on X such that $\mathscr{T}_{\mathscr{R}} \neq \mathscr{T}_{R_1} \cup \mathscr{T}_{R_2}$ and $\mathscr{F}_{\mathscr{R}} \neq \mathscr{F}_{R_1} \cup \mathscr{F}_{R_2}$.

Namely, if $x_1 \in X$ and $x_2 \in X \setminus \{x_1\}$, then by defining

$$R_i = \{x_i\}^2 \cup \big(X \setminus \{x_i\}\big)^2$$

for all $i = 1, 2$ we can see that $\{x_1, x_2\} \in \mathscr{T}_{\mathscr{R}} \setminus \big(\mathscr{T}_{R_1} \cup \mathscr{T}_{R_2}\big)$.

Therefore, because of Theorems 25 and 7 and Corollary 5, we can only state the following

Theorem 26. *The following assertions are true:*

(1) $\wedge\,\partial$ *is a preclosure operation for relators,*

(2) for any two relators $\mathscr{R}$ *and* $\mathscr{S}$ *on X, we have*

$$\mathscr{T}_{\mathscr{S}} \subseteq \mathscr{T}_{\mathscr{R}} \iff \mathscr{F}_{\mathscr{S}} \subseteq \mathscr{F}_{\mathscr{R}} \implies \mathscr{S}^{\,\wedge} \subseteq \mathscr{R}^{\,\wedge\,\partial} \implies \mathscr{S}^{\,\wedge\,\partial} \subseteq \mathscr{R}^{\,\wedge\,\partial}.$$

Remark 20. If X is a set with $\operatorname{card}(X) > 2$, then by using the equivalence relator $\mathscr{R} = \{X^2\}$, considered first by Mala [19, Example 5.3], it can be shown that the operation $\wedge\,\partial$ is not idempotent [25, Example 7.2].

Therefore, by Theorems 10 and 6, the structure $\mathscr{T}$ is not regular. Moreover, by Theorem 12, there does not exist a structure $\mathfrak{F}$ for relators such that, for every relator $\mathscr{R}$ on X, $\mathscr{S} = \mathscr{R}^{\,\wedge\,\partial}$ is the largest relator on X such that $\mathfrak{F}_{\mathscr{S}} = \mathfrak{F}_{\mathscr{R}}$.

However, from Theorem 24, by using Corollary 15, we can easily derive the following theorem of Mala [19].

Theorem 27. *The following assertions are true:*

(1) $\wedge\,\infty$ *is a modification operation for relators,*

(2) for any two relators $\mathscr{R}$ *and* $\mathscr{S}$ *on X, we have*

$$\mathscr{T}_{\mathscr{S}} \subseteq \mathscr{T}_{\mathscr{R}} \iff \mathscr{F}_{\mathscr{S}} \subseteq \mathscr{F}_{\mathscr{R}} \iff \mathscr{S}^{\,\wedge\,\infty} \subseteq \mathscr{R}^{\,\wedge} \iff \mathscr{S}^{\,\wedge\,\infty} \subseteq \mathscr{R}^{\,\wedge\,\infty}.$$

(3) for any relator $\mathscr{R}$ *on X,* $\mathscr{S} = \mathscr{R}^{\,\wedge\,\infty}$ *is the largest preorder relator on X such that* $\mathscr{T}_{\mathscr{S}} \subseteq \mathscr{T}_{\mathscr{R}}$ $\big(\mathscr{T}_{\mathscr{S}} = \mathscr{T}_{\mathscr{R}}\big)$*, or equivalently* $\mathscr{F}_{\mathscr{S}} \subseteq \mathscr{F}_{\mathscr{R}}$ $\big(\mathscr{F}_{\mathscr{S}} = \mathscr{F}_{\mathscr{R}}\big)$.

Remark 21. Note that if X and $\mathscr{R}$ are as in Remark 20, then by [19, Example 5.3] there does not exist a largest relator S on X such that $\mathscr{T}_{\mathscr{S}} = \mathscr{T}_{\mathscr{R}}$.

In the light of the several disadvantages of the structure $\mathscr{T}$, it is rather curious that most of the works in topology and analysis are based on open sets suggested by Tietze [58] and standardized by Bourbaki [2] and Kelley [13].

Moreover, it also a striking fact that, despite the results of Pervin [27], Fletcher and Lindgren [9], and Száz [47], generalized topologies and minimal structures are still intensively investigated by a great number of mathematicians.

8 Some Further Results on Unary Operations for Relators

In the sequel, we shall also use the following terminology of Pataki [25].

Definition 6. For any two unary operations $\square$ and $\Diamond$ for relators, we say that $\square$ is *$\Diamond$–dominating, $\Diamond$–invariant, $\Diamond$–absorbing, and $\Diamond$–compatible* if, for any relator $\mathscr{R}$ on X to Y, we have

$$\mathscr{R}^{\Diamond} \subseteq \mathscr{R}^{\square}, \qquad \mathscr{R}^{\square} = \mathscr{R}^{\square\Diamond}, \qquad \mathscr{R}^{\square} = \mathscr{R}^{\Diamond\square}, \qquad \text{and} \qquad \mathscr{R}^{\square\Diamond} = \mathscr{R}^{\Diamond\square},$$

respectively.

Remark 22. Thus, the operation $\square$ is extensive if and only if it dominates the identity operation for relators. Moreover, $\square$ is idempotent if and only if it is $\square$–invariant ($\square$–absorbing).

In this respect, it is also worth mentioning that the operation $\square$ is $\Diamond$–invariant ($\square$–absorbing) if and only if $\mathscr{R}^{\square}$ is $\Diamond$–invariant ($\mathscr{R}$ and $\mathscr{R}^{\Diamond}$ are $\square$–equivalent) for every relator $\mathscr{R}$ on X to Y.

Remark 23. From Theorem 15, we can see that if $\Diamond, \square \in \{*, \#, \wedge, \Delta\}$ such that $\Diamond$ precedes $\square$ in the above list, then $\square$ is both $\Diamond$–invariant and $\Diamond$–absorbing. Thus, in particular it is also $\Diamond$–compatible.

By using Definition 6, somewhat more generally, we can also state the following

Theorem 28. *If $\Diamond$ is an extensive and $\square$ is a $\Diamond$–dominating idempotent operation for relators, then $\square$ is $\Diamond$–invariant. Moreover, if in addition $\square$ is increasing, then $\square$ is $\Diamond$–absorbing and $\Diamond$–compatible.*

Remark 24. In this respect, it is also worth mentioning that if $\Diamond$ is an extensive and $\square$ is a $\Diamond$–dominating operation for relators, then $\square$ is also extensive.

Moreover, if $\Diamond$ is an increasing and $\square$ is an extensive operation for relators such that $\mathscr{R}^{\square\Diamond} \subseteq \mathscr{R}^{\square}$ for every relator $\mathscr{R}$ on X to Y, then $\square$ is $\Diamond$–dominating.

The importance of the compatibility property of operations lies mainly in

Theorem 29. *If* $\square$ *and* $\lozenge$ *are compatible closure (modification) operations for relators, then* $\square\,\lozenge$ *is also a closure (modification) operation for relators.*

Proof. By using the associativity of composition, and the idempotency and compatibility of the operations $\square$ and $\lozenge$, we can see that

$$\square\,\lozenge\,\square\,\lozenge = \lozenge\,\square\,\square\,\lozenge = \lozenge\,\square\,\lozenge = \square\,\lozenge\,\lozenge = \square\,\lozenge\,.$$

Therefore, the operation $\square\,\lozenge$ is also idempotent.

Remark 25. In this respect, it is also worth noticing that the composition of two union-preserving operations is also union preserving.

It can be easily seen that the operations c, -1, ∞, ∂, and $*$ are union preserving. Thus, their compositions are also union preserving.

However, the important closure operations $\#$, $\wedge$, and Δ are not union preserving. Concerning them, we can only prove the following.

Theorem 30. *If* $\square$ *is a closure operation for relators, then for any family* $\big(\mathscr{R}_i\big)_{i\in I}$ *of relators on X to Y, we have*

$$\Big(\bigcup_{i\in I}\mathscr{R}_i\Big)^{\square} = \Big(\bigcup_{i\in I}\mathscr{R}_i^{\,\square}\Big)^{\square}.$$

Proof. If $\mathscr{R} = \bigcup_{i\in I}\mathscr{R}_i$, then for each $i \in I$ we have $\mathscr{R}_i \subseteq \mathscr{R}$. Hence, by using the increasingness of $\square$, we can infer that $\mathscr{R}_i^{\,\square} \subseteq \mathscr{R}^{\,\square}$. Therefore, we have $\bigcup_{i\in I}\mathscr{R}_i^{\,\square} \subseteq \mathscr{R}^{\,\square}$. Hence, by using the increasingness and the idempotency of $\square$, we can already infer that $\big(\bigcup_{i\in I}\mathscr{R}_i^{\,\square}\big)^{\square} \subseteq \mathscr{R}^{\,\square\,\square} = \mathscr{R}^{\,\square}$.

On the other hand, by the extensivity of $\square$, for each $i \in I$ we have $\mathscr{R}_i \subseteq \mathscr{R}_i^{\,\square}$, and hence also $\mathscr{R}_i \subseteq \bigcup_{i\in I}\mathscr{R}_i^{\,\square}$. Therefore, $\mathscr{R} = \bigcup_{i\in I}\mathscr{R}_i \subseteq \bigcup_{i\in I}\mathscr{R}_i^{\,\square}$. Hence, by using the increasingness of $\square$, we can already infer that $\mathscr{R}^{\,\square} \subseteq \big(\bigcup_{i\in I}\mathscr{R}_i^{\,\square}\big)^{\square}$. Therefore, the required equality is also true.

Remark 26. Hence, we can see that $\big(\bigcup_{i\in I}\mathscr{R}_i\big)^{\square} = \bigcup_{i\in I}\mathscr{R}_i^{\,\square}$ if and only the relator $\bigcup_{i\in I}\mathscr{R}_i^{\,\square}$ is $\square$–invariant.

Now, analogously to Theorem 30, we can also prove the following.

Theorem 31. *If* $\square$ *is a closure operation for relators, then for any family* $\big(\mathscr{R}_i\big)_{i\in I}$ *of relators on X to Y, we have*

$$\bigcap_{i\in I}\mathscr{R}_i^{\,\square} = \Big(\bigcap_{i\in I}\mathscr{R}_i^{\,\square}\Big)^{\square}.$$

Proof. Now, we evidently have $\left(\bigcap_{i\in I}\mathscr{R}_i\right)^{\square}\subseteq\bigcap_{i\in I}\mathscr{R}_i^{\square}$. Hence, by taking $\mathscr{R}_i^{\square}$ in place of $\mathscr{R}_i$, we can easily see that the required equality is also true.

Remark 27. Hence, we can see that the relator $\bigcap_{i\in I}\mathscr{R}_i^{\square}$ is always $\square$–invariant. Moreover, if each $\mathscr{R}_i$ is $\square$-invariant, then the relator $\bigcap_{i\in I}\mathscr{R}_i$ is also $\square$–invariant.

Remark 28. Note that the proofs of the above three theorems also yield some useful statements for preclosure, semiclosure, and modification operations.

Analogously to the equivalence of the assertions (1) and (2) in Theorem 10, we can also prove the following.

Theorem 32. *For a unary operation $\square$ for relators, the following assertions are equivalent:*

(1) $\square$ is an increasing involution,
(2) for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y, we have

$$\mathscr{R}^{\square}\subseteq\mathscr{S}\iff\mathscr{R}\subseteq\mathscr{S}^{\square}.$$

Proof. If (1) holds, then for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y

$$\mathscr{R}^{\square}\subseteq\mathscr{S}\implies\mathscr{R}^{\square\square}\subseteq\mathscr{S}^{\square}\implies\mathscr{R}\subseteq\mathscr{S}^{\square}\implies\mathscr{R}^{\square}\subseteq\mathscr{S}^{\square\square}$$
$$\implies\mathscr{R}^{\square}\subseteq\mathscr{S}.$$

Therefore, (2) also holds.

Conversely, if (2) holds, then for any relator $\mathscr{R}$ on X to Y

$$\mathscr{R}^{\square}\subseteq\mathscr{R}^{\square}\implies\mathscr{R}\subseteq\mathscr{R}^{\square\square},\ \mathscr{R}^{\square\square}\subseteq\mathscr{R}\implies\mathscr{R}=\mathscr{R}^{\square\square}.$$

Therefore, $\square$ is involutive. Thus, for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y

$$\mathscr{R}\subseteq\mathscr{S}\implies\mathscr{R}^{\square\square}\subseteq\mathscr{S}^{\square\square}\implies\mathscr{R}^{\square}\subseteq\mathscr{S}^{\square\square\square}\implies\mathscr{R}^{\square}\subseteq\mathscr{S}^{\square}.$$

Therefore, $\square$ is increasing, and thus (1) also holds.

Now, in addition Theorem 29, we can also prove the following.

Theorem 33. *If $\square$ is a closure (modification) and $\diamond$ is an increasing involution operation for relators, then $\Diamond=\diamond\square\diamond$ is also a closure (modification) operation for relators.*

Proof. By using the associativity of composition, the involutiveness of $\diamond$, and the idempontency of $\square$, we can see that

$$\Diamond\,\Diamond=(\diamond\,\square\,\diamond)(\diamond\,\square\,\diamond)=(\diamond\,\square)\big((\diamond\,\diamond)(\square\,\diamond)\big)$$
$$=(\diamond\,\square)\big(\Delta(\square\,\diamond)\big)=(\diamond\,\square)\big(\square\,\diamond\big)=\diamond\big((\square\,\square)\diamond\big)=\diamond(\square\,\diamond)=\Diamond,$$

where Δ is the identity operation for relators. Therefore, $\Diamond$ is also idempotent.

Because of this theorem, we may also naturally introduce the following.

Definition 7. For any unary operation $\square$ for relators, we write

$$\circledcirc = c \,\square\, c \qquad \text{and} \qquad \boxdot = -1 \,\square\, -1 \,.$$

Remark 29. Thus, by Theorem 33, for instance, $\circledast$ and $\boxast$ are also closure operations for relators.

However, this also quite obvious from the fact that, by the corresponding definitions, for any relator $\mathscr{R}$ on X to Y, we have

(1) $\mathscr{R}^{\boxast} = \mathscr{R}^{*}$,
(2) $\mathscr{R}^{\circledast} = \bigcup_{R \in \mathscr{R}} \mathscr{P}(R)$.

Namely, if, for instance, $S \in \mathscr{R}^{\circledast}$, then $S \in \mathscr{R}^{c*c}$, and thus $S^c \in \mathscr{R}^{c*}$. Therefore, there exists $R \in \mathscr{R}$ such that $R^c \subseteq S^c$. Hence, it follows that $S \subseteq R$, and thus $S \in \mathscr{P}(R)$. Therefore, $S \in \bigcup_{R \in \mathscr{R}} \mathscr{P}(R)$ also holds.

Now, to clear up the importance of Theorem 33 and Definition 7, we can also prove the following.

Theorem 34. *For any relator $\mathscr{R}$ on X to Y, we have*

(1) $\mathscr{R}^{\textcircled{\#}} = \mathscr{R}^{\square_{\mathrm{Lb}}}$,
(2) $\mathscr{R}^{\textcircled{\wedge}} = \mathscr{R}^{\square_{\mathrm{lb}}}$.

Proof. By using the corresponding definitions, Theorem 13, and the equality $\mathrm{Lb}_{\mathscr{R}} = \mathrm{Int}_{\mathscr{R}^c} \circ \mathscr{C}$, we can easily that, for any relation S on X to Y, we have

$$\begin{aligned} S \in \mathscr{R}^{\square_{\mathrm{Lb}}} &\iff \mathrm{Lb}_S \subseteq \mathrm{Lb}_{\mathscr{R}} \iff \mathrm{Int}_{S^c} \circ \mathscr{C} \subseteq \mathrm{Int}_{\mathscr{R}^c} \circ \mathscr{C} \\ &\iff \mathrm{Int}_{S^c} \subset \mathrm{Int}_{\mathscr{R}^c} \iff S^c \subset \mathscr{S}^{c\#} \iff \mathscr{R} \subset \mathscr{S}^{c\#c} \\ &\iff \mathscr{R} \subset \mathscr{S}^{\textcircled{\#}} . \end{aligned}$$

Therefore, assertion (1) is true. The proof of assertion (2) is quite similar.

By the corresponding definitions, it is clear that $\mathrm{Lb}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathrm{Lb}_R$ and $\mathrm{lb}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \mathrm{lb}_R$ for any relator $\mathscr{R}$ on X to Y.

Therefore, analogously to Theorems 14, 16, and 17, we can also easily establish the following three theorems.

Theorem 35. $\textcircled{\#}$ *and* $\textcircled{\wedge}$ *are closure operations for relators.*

Theorem 36. *For any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y, we have*

(1) $\mathscr{S} \subseteq \mathscr{R}^{\textcircled{\#}} \iff \mathscr{S}^{\textcircled{\#}} \subseteq \mathscr{R}^{\textcircled{\#}} \iff \mathrm{Lb}_{\mathscr{S}} \subseteq \mathrm{Lb}_{\mathscr{R}}$,
(2) $\mathscr{S} \subseteq \mathscr{R}^{\textcircled{\wedge}} \iff \mathscr{S}^{\textcircled{\wedge}} \subseteq \mathscr{R}^{\textcircled{\wedge}} \iff \mathrm{lb}_{\mathscr{S}} \subseteq \mathrm{lb}_{\mathscr{R}}$.

Theorem 37. *For any relator $\mathscr{R}$ on X to Y,*

(1) $\mathscr{S} = \mathscr{R}^{\textcircled{\#}}$ *is the largest relator on X to Y such that* $\mathrm{Lb}_{\mathscr{S}} \subseteq \mathrm{Lb}_{\mathscr{R}}$, $\big(\mathrm{Lb}_{\mathscr{S}} = \mathrm{Lb}_{\mathscr{R}}\big)$,
(2) $\mathscr{S} = \mathscr{R}^{\textcircled{\wedge}}$ *is the largest relator on X to Y such that* $\mathrm{lb}_{\mathscr{S}} \subseteq \mathrm{lb}_{\mathscr{R}}$ $\big(\mathrm{lb}_{\mathscr{S}} = \mathrm{lb}_{\mathscr{R}}\big)$.

Remark 30. If $\mathscr{R}$ is a relator on X to Y, then in addition to Theorem 34, we can also state that

(1) $\mathscr{R}^{\square_{\mathrm{Ub}}} = \mathscr{R}^{\textcircled{\#}}$,
(2) $\mathscr{R}^{\square_{\mathrm{ub}}} = \mathscr{R}^{\boxed{\textcircled{\wedge}}}$.

Namely, if S is a relation on X to Y, then by using the equalities

$$\mathrm{Ub}_{\mathscr{R}} = \mathrm{Lb}_{\mathscr{R}^{-1}} = \mathrm{Lb}_{\mathscr{R}}^{-1} \qquad \text{and} \qquad \mathrm{ub}_{\mathscr{R}} = \mathrm{lb}_{\mathscr{R}^{-1}}\,,$$

and Theorem 36, we can easily see that

$$\begin{aligned} S \in \mathscr{R}^{\square_{\mathrm{Ub}}} &\iff \mathrm{Ub}_S \subseteq \mathrm{Ub}_{\mathscr{R}} \iff \mathrm{Lb}_S^{-1} \subseteq \mathrm{Lb}_{\mathscr{R}}^{-1} \\ &\iff \mathrm{Lb}_S \subseteq \mathrm{Lb}_{\mathscr{R}} \iff S \in \mathscr{R}^{\textcircled{\#}} \end{aligned}$$

and

$$\begin{aligned} S \in \mathscr{R}^{\square_{\mathrm{ub}}} &\iff \mathrm{ub}_S \subseteq \mathrm{ub}_{\mathscr{R}} \iff \mathrm{lb}_{S^{-1}} \subseteq \mathrm{lb}_{\mathscr{R}^{-1}} \\ &\iff S^{-1} \in \mathscr{R}^{-1\,\textcircled{\wedge}} \iff S \in \mathscr{R}^{-1\,\textcircled{\wedge}\,-1} \iff S \in \mathscr{R}^{\boxed{\textcircled{\wedge}}}\,. \end{aligned}$$

In this respect, it is also worth noticing that, by the associativity of composition and the inversion compatibility of c, we also have

$$\boxed{\textcircled{\wedge}} = -1\,\textcircled{\wedge}\,-1 = -1\;c\;\wedge\;c\;-1 = c\;-1\;\wedge\;-1\;c = c\,\boxed{\wedge}\,c = \textcircled{\boxed{\wedge}}\,.$$

9 Inversion Compatible Operations for Relators

According to Definition 6, we may naturally have the following.

Definition 8. A unary operation $\square$ for relators is called *inversion compatible* if for any relator $\mathscr{R}$ on X to Y we have

$$\big(\mathscr{R}^{\square}\big)^{-1} = \big(\mathscr{R}^{-1}\big)^{\square}.$$

Now, by using this definition, we can easily prove the following.

Theorem 38. *For a unary operation $\square$ on relators, the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*
(2) $\big(\mathscr{R}^{\,\square}\big)^{-1} \subseteq \big(\mathscr{R}^{-1}\big)^{\square}$ *for any relator $\mathscr{R}$ on X to Y,*
(3) $\big(\mathscr{R}^{-1}\big)^{\square} \subseteq \big(\mathscr{R}^{\,\square}\big)^{-1}$ *for any relator $\mathscr{R}$ on X to Y.*

Proof. Note that if, for instance, (2) holds, then for any relator $\mathscr{R}$ on X to Y we also have $\Big(\big(\mathscr{R}^{-1}\big)^{\square}\Big)^{-1} \subseteq \mathscr{R}^{\,\square}$, and hence also $\big(\mathscr{R}^{-1}\big)^{\square} \subseteq \big(\mathscr{R}^{\,\square}\big)^{-1}$.

Hence, by using Definition 7, we can immediately derive the following.

Theorem 39. *For a unary operation $\square$ for relators, the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*
(2) $\mathscr{R}^{\,\square} = \mathscr{R}^{\,\boxdot}$ *for any relator $\mathscr{R}$ on X to Y.*
(3) $\mathscr{R}^{\,\square} \subseteq \mathscr{R}^{\,\boxdot}$ $\big(\mathscr{R}^{\,\boxdot} \subseteq \mathscr{R}^{\,\square}\big)$ *for any relator $\mathscr{R}$ on X to Y.*

However, it is now more important to note that, by using the corresponding definitions, we can also easily prove the following.

Theorem 40. *If $\square$ is a union-preserving operation for relators, then the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*
(2) $\big(\{R\}^{\square}\big)^{-1} = \big\{R^{-1}\big\}^{\square}$ *for any relation R on X to Y.*

Proof. To prove that (2) also implies (1), note that the operation -1 is union preserving. Therefore, for any relator $\mathscr{R}$ on X to Y, we have

$$\big(\mathscr{R}^{\,\square}\big)^{-1} = \Big(\bigcup_{R\in\mathscr{R}} \{R\}^{\square}\Big)^{-1} = \bigcup_{R\in\mathscr{R}} \big(\{R\}^{\square}\big)^{-1}$$
$$= \bigcup_{R\in\mathscr{R}} \big\{R^{-1}\big\}^{\square} = \Big(\bigcup_{R\in\mathscr{R}} \big\{R^{-1}\big\}\Big)^{\square} = \big(\mathscr{R}^{-1}\big)^{\square}.$$

Now, analogously to Theorem 39, we can also state the following.

Corollary 20. *If $\square$ is a union-preserving operation for relators, then the following assertions are equivalent:*

(1) $\square$ *is inversion compatible,*
(2) $\big(\{R\}^{\square}\big)^{-1} \subseteq \big\{R^{-1}\big\}^{\square}$ *for any relator $\mathscr{R}$ on X to Y,*
(3) $\big\{R^{-1}\big\}^{\square} \subseteq \big(\{R\}^{\square}\big)^{-1}$ *for any relator $\mathscr{R}$ on X to Y.*

However, this corollary cannot actually be used to simplify the proof of

Theorem 41. *The operations c, ∞, ∂, and $*$ are inversion compatible.*

Proof. By the corresponding definitions, it is clear that ∞ is a union-preserving operation for relators. Moreover, for any relation R on X, we have

$$\left(R^{\infty}\right)^{-1} = \left(\bigcup_{n=0}^{\infty} R^{n}\right)^{-1} = \bigcup_{n=0}^{\infty} \left(R^{n}\right)^{-1} = \bigcup_{n=0}^{\infty} \left(R^{-1}\right)^{n} = \left(R^{-1}\right)^{\infty}.$$

Therefore, by Theorem 40, the operation ∞ is inversion compatible.

Now, to prove the inversion-compatibility of the operation ∂, it is enough to note only that, for any relator $\mathscr{R}$ and relation S on X, we have

$$S \in \left(\mathscr{R}^{-1}\right)^{\partial} \iff S^{\infty} \in \mathscr{R}^{-1} \iff \left(S^{\infty}\right)^{-1} \in \mathscr{R}$$
$$\iff \left(S^{-1}\right)^{\infty} \in \mathscr{R} \iff S^{-1} \in \mathscr{R}^{\partial} \iff S \in \left(\mathscr{R}^{\partial}\right)^{-1}.$$

Now, by using Theorem 38 and our former results on the structure Int, we can also prove the following.

Theorem 42. *The operation # is also inversion compatible.*

Proof. If $\mathscr{R}$ is a relator on X to Y, then by [36, Theorem 1.2] and Theorem 17 we have

$$\mathrm{Cl}_{(\mathscr{R}^{\#})^{-1}} = \left(\mathrm{Cl}_{\mathscr{R}^{\#}}\right)^{-1} = \left(\mathrm{Cl}_{\mathscr{R}}\right)^{-1} = \mathrm{Cl}_{\mathscr{R}^{-1}}\ .$$

Hence, by using Theorem 16, we can already infer that $\left(\mathscr{R}^{\#}\right)^{-1} \subseteq \left(\mathscr{R}^{-1}\right)^{\#}$. Therefore, by Theorem 38, the corresponding equality is also true.

Remark 31. Unfortunately, the operations $\wedge$ and Δ are not inversion compatible. Therefore, we have also to consider the operations $\vee$ and ∇ defined by $\mathscr{R}^{\vee} = \left(\mathscr{R}^{\wedge}\right)^{-1}$ and $\mathscr{R}^{\nabla} = \left(\mathscr{R}^{\Delta}\right)^{-1}$ for every relator $\mathscr{R}$ on X to Y.

However, these operations already have some very curious properties [20]. For instance, the operations $\vee\vee$ and $\nabla\nabla$ already coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined for any relator $\mathscr{R}$ on X to Y such that

$$\mathscr{R}^{\bullet} = \left\{\delta_{\mathscr{R}}\right\}^{*}, \qquad \text{where} \qquad \delta_{\mathscr{R}} = \bigcap \mathscr{R},$$

and

$$\mathscr{R}^{\blacklozenge} = \mathscr{R} \quad \text{if} \quad \mathscr{R} = \left\{X \times Y\right\} \qquad \text{and} \qquad \mathscr{R}^{\blacklozenge} = \mathscr{P}(X \times Y) \quad \text{if} \quad \mathscr{R} \neq \left\{X \times Y\right\}.$$

Note that $\blacklozenge$ is the ultimate stable unary operation for relators.

The usefulness of inversion compatible operations is apparent from the following simple theorems of [55].

Theorem 43. *If $\square$ is an inversion compatible operation for relators, then for any relator $\mathscr{R}$ on X to Y the following assertions are equivalent:*

(1) $\mathscr{R}$ *is $\square$–invariant,*
(2) $\mathscr{R}^{-1}$ *is $\square$–invariant.*

Definition 9. If $\square$ is a unary operation for relators, then a relator $\mathscr{R}$ on X is called $\square$*–symmetric* if

$$\left(\mathscr{R}^{\square}\right)^{-1}=\mathscr{R}^{\square}.$$

Remark 32. Now, the relator $\mathscr{R}$ may, for instance, be naturally called *properly, uniformly, proximally, topologically, and paratopologically symmetric* if it is $\square$–symmetric with $\square = \varDelta$, $*$, $\#$, $\wedge$, and Δ, respectively.

Theorem 44. *If $\mathscr{R}$ is a properly symmetric relator on X, then $\mathscr{R}$ is $\square$–symmetric for every inversion compatible operation $\square$ for relators.*

Theorem 45. *If $\square$ is an inversion compatible operation for relators, then for any relator $\mathscr{R}$ on X the following assertions are equivalent:*

(1) $\mathscr{R}$ *or* $\mathscr{R}^{-1}$ *is $\square$–symmetric,*
(2) $\mathscr{R}$ *and* $\mathscr{R}^{-1}$ *are $\square$–equivalent.*

Remark 33. In this respect, it is also worth noticing that if $\square$ is an unary operation for relators and $\mathscr{R}$ is a $\square$–symmetric relator on X to Y such that $\mathscr{R}$ and $\mathscr{R}^{-1}$ are $\square$–equivalent, then $\left(\mathscr{R}^{\square}\right)^{-1}=\mathscr{R}^{\square}=\left(\mathscr{R}^{-1}\right)^{\square}$.

However, it is now more important to note that, in addition to Theorem 45, we can also prove the following.

Theorem 46. *If $\square$ is an inversion compatible closure operation for relators, then for any relator $\mathscr{R}$ on X the following assertions are equivalent:*

(1) $\mathscr{R}$ *is $\square$–symmetric,*
(2) $\mathscr{R}^{-1}\subseteq\mathscr{R}^{\square}$*;*
(3) $\mathscr{R}\subseteq\left(\mathscr{R}^{-1}\right)^{\square}$*,*
(4) $\mathscr{R}$ *is $\square$-equivalent to a properly symmetric relator $\mathscr{S}$ on X.*

Proof. If (1) holds, then by the extensivity of $\square$, it is clear that $\mathscr{R}^{-1}\subseteq\left(\mathscr{R}^{\square}\right)^{-1}=\mathscr{R}^{\square}$. Therefore, (2) also holds.

Moreover, if (2) holds, then we can see that $\mathscr{R}\subseteq\left(\mathscr{R}^{\square}\right)^{-1}=\left(\mathscr{R}^{-1}\right)^{\square}$. Therefore, (3) also holds.

While, if (3) holds, then we can quite similarly see that (2) also holds. From (2) and (3), by using Theorem 10, we can infer that $\left(\mathscr{R}^{-1}\right)^{\square}\subseteq\mathscr{R}^{\square}\subset\left(\mathscr{R}^{-1}\right)^{\square}$, and thus $\mathscr{R}^{\square}=\left(\mathscr{R}^{-1}\right)^{\square}$. Therefore, by Theorem 45, (1) also holds.

On the other hand, if (1) holds, then $\mathscr{R}^{\square}$ is properly symmetric. Hence, since $\mathscr{R}^{\square} = \left(\mathscr{R}^{\square}\right)^{\square}$, we can already see that (4) holds with $\mathscr{S} = \mathscr{R}^{\square}$.

Conversely, if (4) holds, then it is clear that $\left(\mathscr{R}^{\square}\right)^{-1} = \left(\mathscr{S}^{\square}\right)^{-1} = \left(\mathscr{S}^{-1}\right)^{\square} = \mathscr{S}^{\square} = \mathscr{R}^{\square}$. Therefore, (1) also holds.

From this theorem, by using Theorem 8, we can immediately derive

Corollary 21. *If $\mathfrak{F}$ is a union-preserving structure for relators such that the induced operation $\square_{\mathfrak{F}}$ is inversion compatible, then for any relator $\mathscr{R}$ on X the following assertions are equivalent:*

(1) $\mathscr{R}$ *is* $\square_{\mathfrak{F}}$*–symmetric,*
(2) $\mathfrak{F}_{\mathscr{R}^{-1}} \subset \mathfrak{F}_{\mathscr{R}}$*;*
(3) $\mathfrak{F}_{\mathscr{R}} \subset \mathfrak{F}_{\mathscr{R}^{-1}}$.

Remark 34. Note that the theorems proved in this section can be generalized by using an arbitrary increasing involution operation $\diamond$ for relators instead of the inversion -1.

10 Composition Compatible Operations for Relators

Composition compatibility properties of operations for relators have formerly been considered only in [50] and [55].

Definition 10. For a direct unary operation $\square$ for relators, we say that:

(1) $\square$ is left composition compatible if $\left(\mathscr{S} \circ \mathscr{R}\right)^{\square} = \left(\mathscr{S} \circ \mathscr{R}^{\square}\right)^{\square}$ for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z,
(2) $\square$ is right composition compatible if $\left(\mathscr{S} \circ \mathscr{R}\right)^{\square} = \left(\mathscr{S}^{\square} \circ \mathscr{R}\right)^{\square}$ for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z.

Remark 35. Now, the operation $\square$ may be naturally called composition compatible if it is both left and right composition compatible.

Note that, this is also very weak composition compatibility property. However, because of the following theorems, it will be sufficient for our subsequent purposes.

Theorem 47. *If $\square$ is a left (right) composition compatible unary operation for relators, then $\square$ is, in particular, idempotent.*

Proof. If $\square$ is left composition compatible, then for any relator $\mathscr{R}$ on X to Y

$$\mathscr{R}^{\square\square} = \left(\mathscr{R}^{\square}\right)^{\square} = \left(\{\Delta_Y\} \circ \mathscr{R}^{\square}\right)^{\square} = \left(\{\Delta_Y\} \circ \mathscr{R}\right)^{\square} = \mathscr{R}^{\square}.$$

Theorem 48. *If $\square$ is a composition compatible unary operation for relators, then for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z we have*

$$(\mathscr{S}\circ\mathscr{R})^{\square}=(\mathscr{S}^{\square}\circ\mathscr{R}^{\square})^{\square}.$$

Proof. Namely, we have $(\mathscr{S}\circ\mathscr{R})^{\square}=(\mathscr{S}\circ\mathscr{R}^{\square})^{\square}=(\mathscr{S}^{\square}\circ\mathscr{R}^{\square})^{\square}$.

Remark 36. Note that, in this case, we also have $(\mathscr{S}^{\square}\circ\mathscr{R})^{\square}=(\mathscr{S}\circ\mathscr{R}^{\square})^{\square}$.

From Theorem 48, by using the associativity of composition, we can derive

Corollary 22. *If $\square$ is a composition compatible unary operation for relators, then for any three relators $\mathscr{R}$ on X to Y, $\mathscr{S}$ on Y to Z, and $\mathscr{T}$ on Z to W we have*

$$(\mathscr{T}\circ\mathscr{S}\circ\mathscr{R})^{\square}=(\mathscr{T}^{\square}\circ\mathscr{S}^{\square}\circ\mathscr{R}^{\square})^{\square}.$$

Proof. By using Theorem 48, we can see that

$$(\mathscr{T}\circ(\mathscr{S}\circ\mathscr{R}))^{\square}=\left(\mathscr{T}^{\square}\circ(\mathscr{S}\circ\mathscr{R})^{\square}\right)^{\square}=\left(\mathscr{T}^{\square}\circ(\mathscr{S}^{\square}\circ\mathscr{R}^{\square})\right)^{\square}.$$

Remark 37. In this case, by using Definition 10, we can also prove that

$$(\mathscr{T}\circ\mathscr{S}\circ\mathscr{R})^{\square}=(\mathscr{T}^{\square}\circ\mathscr{S}\circ\mathscr{R})^{\square}=(\mathscr{T}\circ\mathscr{S}^{\square}\circ\mathscr{R})^{\square}=(\mathscr{T}\circ\mathscr{S}\circ\mathscr{R}^{\square})^{\square}.$$

However, it is now more important to note that, by using the corresponding definitions, we can also easily prove the following.

Theorem 49. *If $\square$ is a preclosure operation for relators, then for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z we have*

(1) $(\mathscr{S}\circ\mathscr{R})^{\square}\subseteq(\mathscr{S}\circ\mathscr{R}^{\square})^{\square}\subseteq(\mathscr{S}^{\square}\circ\mathscr{R}^{\square})^{\square}$,

(2) $(\mathscr{S}\circ\mathscr{R})^{\square}\subseteq(\mathscr{S}^{\square}\circ\mathscr{R})^{\square}\subseteq(\mathscr{S}^{\square}\circ\mathscr{R}^{\square})^{\square}$.

Proof. By the extensivity of $\square$, we have $\mathscr{R}\subseteq\mathscr{R}^{\square}$. Hence, by the increasingness of the elementwise composition of relators, we can see that $\mathscr{S}\circ\mathscr{R}\subseteq\mathscr{S}\circ\mathscr{R}^{\square}$. Thus, by the increasingness of $\square$, we also have $(\mathscr{S}\circ\mathscr{R})^{\square}\subseteq(\mathscr{S}\circ\mathscr{R}^{\square})^{\square}$. Hence, by writing $\mathscr{S}^{\square}$ in place of $\mathscr{S}$, we can see that $(\mathscr{S}^{\square}\circ\mathscr{R})^{\square}\subseteq(\mathscr{S}^{\square}\circ\mathscr{R}^{\square})^{\square}$. Therefore, the first part of (1) and the second part of (2) are true.

From this theorem, by using Definition 10, we can immediately derive

Corollary 23. *If $\square$ is a preclosure operation for relators, then*

(1) $\square$ *is left composition compatible if and only if* $(\mathscr{S}\circ\mathscr{R}^{\square})^{\square}\subseteq(\mathscr{S}\circ\mathscr{R})^{\square}$ *for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z,*

(2) $\square$ *is right composition compatible if and only if* $(\mathscr{S}^{\square}\circ\mathscr{R})^{\square}\subseteq(\mathscr{S}\circ\mathscr{R})^{\square}$ *for any two relators R on X to Y and $\mathscr{S}$ on Y to Z.*

Hence, by Theorem 10, it is clear that in particular we also have

Corollary 24. *If $\square$ is a closure operation for relators, then*

(1) $\square$ is left composition compatible if and only if $\mathscr{S} \circ \mathscr{R}^{\square} \subseteq \big(\mathscr{S} \circ \mathscr{R}\big)^{\square}$ for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z,

(2) $\square$ is right composition compatible if and only if $\mathscr{S}^{\square} \circ \mathscr{R} \subseteq \big(\mathscr{S} \circ \mathscr{R}\big)^{\square}$ for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z.

Remark 38. In addition to the above results, it is also worth noticing that an involution operation $\square$ for relators is left composition compatible if and only if $\mathscr{S} \circ \mathscr{R} = \mathscr{S} \circ \mathscr{R}^{\square}$ for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z.

Moreover, since $\mathscr{S} \circ \mathscr{R} = \bigcup_{S \in \mathscr{S}} S \circ \mathscr{R}$, we can also at once state that an involution operation $\square$ for relators is left composition compatible if and only if $S \circ \mathscr{R} = S \circ \mathscr{R}^{\square}$ for any relator $\mathscr{R}$ on X to Y and relation S on Y to Z.

Now, by using Corollary 24 and Theorem 30, we can also prove the following.

Theorem 50. *If $\square$ is a closure operation for relators, then*

(1) $\square$ is left composition compatible if and only if $S \circ \mathscr{R}^{\square} \subseteq \big(S \circ \mathscr{R}\big)^{\square}$ for any relator $\mathscr{R}$ on X to Y and relation S on Y to Z,

(2) $\square$ is right composition compatible if and only if $\mathscr{S}^{\square} \circ R \subseteq \big(\mathscr{S} \circ R\big)^{\square}$ for any relation R on X to Y and relator $\mathscr{S}$ on Y to Z.

Proof. If $\square$ is left composition compatible, then by Corollary 24, for any relator $\mathscr{R}$ and relation S on Y to Z, we have $\{S\} \circ \mathscr{R}^{\square} \subseteq \big(\{S\} \circ \mathscr{R}\big)^{\square}$, and thus $S \circ \mathscr{R}^{\square} \subseteq \big(S \circ \mathscr{R}\big)^{\square}$. Therefore, the "only if part" of (1) is true.

Conversely, if $\mathscr{R}$ is a relator on X to Y and $\mathscr{S}$ is a relator on Y to Z, and the inclusion $S \circ \mathscr{R}^{\square} \subseteq \big(S \circ \mathscr{R}\big)^{\square}$ holds for any relation S on Y to Z, then by using the corresponding definitions and Theorem 30 we can see that

$$\mathscr{S} \circ \mathscr{R}^{\square} = \bigcup_{S \in \mathscr{S}} S \circ \mathscr{R}^{\square} \subset \bigcup_{S \in \mathscr{S}} \big(S \circ \mathscr{R}\big)^{\square}$$

$$\subseteq \Big(\bigcup_{S \in \mathscr{S}} \big(S \circ \mathscr{R}\big)^{\square}\Big)^{\square} = \Big(\bigcup_{S \in \mathscr{S}} S \circ \mathscr{R}\Big)^{\square} = \big(\mathscr{S} \circ \mathscr{R}\big)^{\square}.$$

Therefore, by Corollary 24, the "if part" of (1) is also true.

By using this theorem, we can somewhat more easily establish the composition compatibility properties of the basic closure operations considered in Sect. 6.

Theorem 51. *The operations $*$ and $\#$ are composition compatible.*

Proof. To prove right composition compatibility of #, by Theorem 50, it is enough to prove only that, for any relation R on X to Y and relator $\mathscr{S}$ on Y to Z, we have $\mathscr{S}^{\#} \circ R \subseteq \big(\mathscr{S} \circ R\big)^{\#}$.

For this, suppose that $W \in \mathscr{S}^{\#} \circ R$ and $A \subset X$. Then, there exists $V \in \mathscr{S}^{\#}$ such that $W = V \circ R$. Moreover, there exists $S \in \mathscr{S}$ such that $S\big[R[A] \big] \subseteq V\big[R[A] \big]$, and thus $(S \circ R)[A] \subseteq (V \circ R)[A] = W[A]$. Hence, by taking $U = S \circ R$, we can see that $U \in \mathscr{S} \circ R$ such that $U[A] \subseteq W[A]$. Therefore, $W \in \big(\mathscr{S} \circ R\big)^{\#}$ also holds.

Theorem 52. *The operations $\wedge$ and Δ are left composition compatible.*

Proof. To prove left composition compatibility of Δ, by Theorem 50, it is enough to prove only that, for any relator $\mathscr{R}$ on X to Y and relation S on Y to Z, we have $S \circ \mathscr{R}^{\Delta} \subseteq \big(S \circ \mathscr{R}\big)^{\Delta}$.

For this, suppose that $W \in S \circ \mathscr{R}^{\Delta}$ and $x \in X$. Then, there exists $V \in \mathscr{R}^{\Delta}$ such that $W = S \circ V$. Moreover, there exist $u \in X$ and $R \in \mathscr{R}$ such that $R(u) \subseteq V(x)$. Hence, we can infer that

$$(S \circ R)(u) = S\big[R(u) \big] \subseteq S\big[V(x) \big] = (S \circ V)(x) = W(x)\,.$$

Now, by taking $U = S \circ R$, we can see that $U \in S \circ \mathscr{R}$ such that $U(u) \subseteq W(x)$. Therefore, $W \in \big(S \circ R\big)^{\Delta}$ also holds.

Instead of the right composition compatibility of the operations $\wedge$ and Δ, we can only prove the following.

Theorem 53. *For any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, we have*

(1) $\big(\mathscr{S} \circ \mathscr{R}\big)^{\wedge} = \big(\mathscr{S}^{\#} \circ \mathscr{R}\big)^{\wedge}$, *(2)* $\big(\mathscr{S} \circ \mathscr{R}\big)^{\Delta} = \big(\mathscr{S}^{\#} \circ \mathscr{R}\big)^{\Delta}$.

Proof. By the extensivity of #, we have $\mathscr{S} \subseteq \mathscr{S}^{\#}$. Hence, by the elementwise definition of composition of relators, we can see that $\mathscr{S} \circ \mathscr{R} \subseteq \mathscr{S}^{\#} \circ \mathscr{R}$. Thus, by the increasingness of $\wedge$, we also have $\big(\mathscr{S} \circ \mathscr{R}\big)^{\wedge} \subseteq \big(\mathscr{S}^{\#} \circ \mathscr{R}\big)^{\wedge}$.

To get the converse inclusion, by Theorem 16, it is now enough to prove only that $S^{\#} \circ \mathscr{R} \subseteq \big(\mathscr{S} \circ \mathscr{R}\big)^{\wedge}$. For this, suppose that $W \in S^{\#} \circ \mathscr{R}$ and $x \in X$. Then, there exists $V \in \mathscr{S}^{\#}$ and $R \in \mathscr{R}$ such that $W = V \circ R$. Moreover, there exists $S \in \mathscr{S}$, such that $S\big[R(x) \big] \subseteq V\big[R(x) \big]$, and thus $(S \circ R)(x) \subseteq (V \circ R)(x) = W(x)$. Hence, by taking $U = S \circ R$, we can see that $U \in \mathscr{S} \circ \mathscr{R}$ such that $U(x) \subseteq W(x)$. Therefore, $W \in \big(\mathscr{S} \circ \mathscr{R}\big)^{\wedge}$ also holds.

Thus, we have proved (1). Assertion (2) can now be immediately derived from (1) by using that $\mathscr{U}^{\wedge\Delta} = \mathscr{U}^{\Delta}$ for any relator $\mathscr{U}$ on X to Z.

From this theorem, by using Theorem 52, we can immediately derive

Corollary 25. *For any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, we have*

(1) $\big(\mathscr{S} \circ \mathscr{R}\big)^{\wedge} = \big(\mathscr{S}^{\#} \circ \mathscr{R}^{\wedge}\big)^{\wedge}$,

(2) $\big(\mathscr{S} \circ \mathscr{R}\big)^{\Delta} = \big(\mathscr{S}^{\#} \circ \mathscr{R}^{\Delta}\big)^{\Delta}$.

Remark 39. By using Theorem 50, we can also somewhat more easily prove that the operation $\circledast$, considered in Remark 29, is also composition compatible.

11 Reflexive, Topological, and Proximal Relators

The subsequent definitions and theorems have been mainly taken from [34]. (For some closely related results, see [32] and [39].)

Definition 11. A relator $\mathscr{R}$ on X is called *reflexive* if each member R of $\mathscr{R}$ is a reflexive relation on X.

Remark 40. Thus, for a relator $\mathscr{R}$ on X, the following assertions are equivalent:

(1) $\mathscr{R}$ is reflexive,
(2) $x \in R(x)$ for all $x \in X$ and $R \in \mathscr{R}$,
(3) $A \subseteq R[A]$ for all $A \subseteq X$ and $R \in \mathscr{R}$.

The importance of reflexive relators is also apparent from the following two obvious theorems.

Theorem 54. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ *is reflexive,*
(2) $A \subseteq \operatorname{cl}_{\mathscr{R}}(A)$ *for all* $A \subseteq X$,
(3) $\operatorname{int}_{\mathscr{R}}(A) \subseteq A$ *for all* $A \subseteq X$.

Hint. To prove the implication (3) $\Longrightarrow$ (1), note that, for any $x \in X$ and $R \in \mathscr{R}$, we have $R(x) \subseteq R(x)$, and thus $x \in \operatorname{int}_{\mathscr{R}}\big(R(x)\big)$.

Remark 41. In addition to Remark 40 and Theorem 54, it is also worth mentioning that the relator $\mathscr{R}$ is reflexive if and only if the relation $\delta_{\mathscr{R}} = \bigcap \mathscr{R}$ is reflexive.

Namely, if $\mathscr{R}$ is a relator on X to Y, then by using the closure formula $\operatorname{cl}_{\mathscr{R}}(B) = \bigcap_{R \in \mathscr{R}^{-1}} [B]$, it can be easily seen that $\operatorname{cl}_{\mathscr{R}}(y) = \operatorname{cl}_{\mathscr{R}}\big(\{y\}\big) = \delta_{\mathscr{R}}^{-1}(y)$.

Theorem 55. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ *is reflexive,*
(2) $A \in \operatorname{Int}_{\mathscr{R}}(B)$ *implies* $A \subseteq B$ *for all* $A, B \subseteq X$,
(3) $A \cap B \neq \emptyset$ *implies* $A \in \operatorname{Cl}_{\mathscr{R}}(B)$ *for all* $A, B \subseteq X$.

Remark 42. In addition to the above two theorems, it is also worth mentioning that if $\mathscr{R}$ is a reflexive relator on X, then

(1) $\operatorname{Int}_{\mathscr{R}}$ is a transitive relation on $\mathscr{P}(X)$,
(2) $\operatorname{int}_{\mathscr{R}}\big(A \setminus \operatorname{int}_{\mathscr{R}}(A)\big) = \emptyset = \operatorname{int}_{\mathscr{R}}\big(\operatorname{cl}_{\mathscr{R}}(A) \setminus A\big)$ for all $A \subseteq X$.

Thus, for instance, for any $A \subseteq X$ we have $A \in \mathscr{F}_{\mathscr{R}}$ if and only if $\operatorname{cl}_{\mathscr{R}}(A) \setminus A \in \mathscr{T}_{\mathscr{R}}$.

Definition 12. For a relator $\mathscr{R}$ on X, we say that:

(1) $\mathscr{R}$ is *quasi-topological* if $x \in \operatorname{int}_{\mathscr{R}}\big(\operatorname{int}_{\mathscr{R}}\big(R(x)\big)\big)$ for all $x \in X$ and $R \in \mathscr{R}$,
(2) $\mathscr{R}$ is *topological* if for any $x \in X$ and $R \in \mathscr{R}$ there exists $V \in \mathscr{T}_{\mathscr{R}}$ such that $x \in V \subseteq R(x)$.

The appropriateness of these definitions is already quite obvious from the following three theorems.

Theorem 56. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ *is quasi-topological,*
(2) $\operatorname{int}_{\mathscr{R}}\big(R(x)\big) \in \mathscr{T}_{\mathscr{R}}$ *for all* $x \in X$ *and* $R \in \mathscr{R}$,
(3) $\operatorname{int}_{\mathscr{R}}(A) \in \mathscr{T}_{\mathscr{R}}$ $\big(\operatorname{cl}_{\mathscr{R}}(A) \in \mathscr{F}_{\mathscr{R}}\big)$ *for all* $A \subseteq X$.

Theorem 57. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ *is topological,*
(2) $\mathscr{R}$ *is reflexive and quasi-topological.*

Theorem 58. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ *is topological,*
(3) $\operatorname{int}_{\mathscr{R}}(A) = \bigcup \mathscr{T}_{\mathscr{R}} \cap \mathscr{P}(A)$ *for all* $A \subseteq X$,
(3) $\operatorname{cl}_{\mathscr{R}}(A) = \bigcap \mathscr{F}_{\mathscr{R}} \cap \mathscr{P}^{-1}(A)$ *for all* $A \subseteq X$.

Remark 43. By Theorem 56, the relator $\mathscr{R}$ may be called *weakly (strongly) quasi-topological* if $\operatorname{cl}_{\mathscr{R}}(x) \in \mathscr{F}_{\mathscr{R}}$ $\big(R(x) \in \mathscr{T}_{\mathscr{R}}\big)$ for all $x \in X$ and $R \in \mathscr{R}$.

Moreover, by Theorem 57, the relator $\mathscr{R}$ may be called *weakly (strongly) topological* if it is reflexive and weakly (strongly) quasi-topological.

However, it is now more important to note that, as a immediate consequence of the above theorems, we can also state

Corollary 26. *If $\mathscr{R}$ is a topological relator on X, then for any* $A \subset X$

(1) $A \in \mathscr{E}_{\mathscr{R}}$ *if and only if there exists* $V \in \mathscr{T}_{\mathscr{R}} \setminus \{\emptyset\}$ *such that* $V \subseteq A$,
(2) $A \in \mathscr{D}_{\mathscr{R}}$ *if and only if for all* $W \in \mathscr{F}_{\mathscr{R}} \setminus \{X\}$ *we have* $A \setminus W \neq \emptyset$.

Theorem 59. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ *is topological,*
(2) $\mathscr{R}$ *is topologically equivalent to* $\mathscr{R}^{\wedge\infty}$,
(3) $\mathscr{R}$ *is topologically equivalent to a preorder relator on* X.

Proof. For instance, we shall show that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2). Namely, (2) trivially implies (3). Moreover, the proof of the implication (3) $\Longrightarrow$ (1) is quite straightforward.

For this, note that if (1) holds, then by Definition 12, for any $x \in X$ and $R \in \mathscr{R}$, there exists $V \in \mathscr{T}_{\mathscr{R}}$ such that $x \in V \subseteq R(x)$.

Hence, by considering the Pervin relator

$$\mathscr{S} = \mathscr{R}_{\mathscr{T}_{\mathscr{R}}} = \big\{ R_V : \ V \in \mathscr{T}_{\mathscr{R}} \big\}, \qquad \text{where} \qquad R_V = V^2 \cup V^c \times X,$$

we can note that $\mathscr{R} \subseteq \mathscr{S}^{\wedge}$, and thus $\mathscr{R}^{\wedge} \subseteq \mathscr{S}^{\wedge\wedge} = \mathscr{S}^{\wedge}$.

Moreover, since

$$R_V(x) = V \quad \text{if} \quad x \in V \qquad \text{and} \qquad R_V(x) = X \quad \text{if} \quad x \in V^c,$$

we can also note that $\mathscr{S} \subseteq \mathscr{R}^{\wedge}$, and thus $\mathscr{S}^{\wedge} \subseteq \mathscr{R}^{\wedge\wedge} = \mathscr{R}^{\wedge}$.

Therefore, we actually have $\mathscr{R}^{\wedge} = \mathscr{S}^{\wedge}$, and thus $\mathscr{R}$ is topologically equivalent to $\mathscr{S}$. Hence, since $\mathscr{S}$ is a preorder relator on X, we can already see that (3) also holds.

On the other hand, if $\mathscr{S}$ is a preorder relator on X such that $\mathscr{R}^{\wedge} = \mathscr{S}^{\wedge}$, then we can easily see $\mathscr{S} \subseteq \mathscr{R}^{\wedge}$ and thus $\mathscr{S} = \mathscr{S}^{\infty} \subseteq \mathscr{R}^{\wedge\infty}$. Therefore, we also have $\mathscr{R}^{\wedge} = \mathscr{S}^{\wedge} \subseteq \left(\mathscr{R}^{\wedge\infty}\right)^{\wedge}$.

Moreover, by using Remark 9 and Theorem 15, we can can also easily see that $\mathscr{R}^{\wedge\infty} \subseteq \mathscr{R}^{\wedge *} = \mathscr{R}^{\wedge}$, and thus $\left(\mathscr{R}^{\wedge\infty}\right)^{\wedge} \subseteq \mathscr{R}^{\wedge\wedge} = \mathscr{R}^{\wedge}$. Therefore, we actually have $\mathscr{R}^{\wedge} = \left(\mathscr{R}^{\wedge\infty}\right)^{\wedge}$, and thus (2) also holds.

Definition 13. For any relator $\mathscr{R}$ on X, we say that:

(1) $\mathscr{R}$ is *quasi-proximal* if $A \in \operatorname{Int}_{\mathscr{R}}\left[\tau_{\mathscr{R}} \cap \operatorname{Int}_{\mathscr{R}}\left(R[A]\right)\right]$ for all $A \subseteq X$ and $R \in \mathscr{R}$,

(2) $\mathscr{R}$ is *proximal* if for any $A \subseteq X$ and $R \in \mathscr{R}$ there exists $V \in \tau_{\mathscr{R}}$ such that $A \subseteq V \subseteq R[A]$.

Remark 44. Note that thus, for any relator $\mathscr{R}$ on X, the following assertions are equivalent:

(1) $\mathscr{R}$ is quasi-proximal ,

(2) for any $A \subseteq X$ and $R \in \mathscr{R}$, there exists $V \in \tau_{\mathscr{R}}$ such that $A \in \operatorname{Int}_{\mathscr{R}}(V)$ and $V \in \operatorname{Int}_{\mathscr{R}}\left(R[A]\right)$.

The appropriateness of the above definitions is already quite obvious from the following analogues of Theorems 57, 58, and 59.

Theorem 60. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ is proximal,

(2) $\mathscr{R}$ is reflexive and quasi-proximal.

Theorem 61. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ is proximal ,

(2) $\operatorname{Int}_{\mathscr{R}}(A) = \mathscr{P}\left[\tau_{\mathscr{R}} \cap \mathscr{P}(A)\right]$ for all $A \subseteq X$,

(3) for any $B \in \operatorname{Int}_{\mathscr{R}}(A)$, there exists $V \in \tau_{\mathscr{R}}$ such that $B \subseteq V \subseteq A$.

Theorem 62. *If $\mathscr{R}$ is relator on X, then the following assertions are equivalent:*

(1) $\mathscr{R}$ is proximal ,

(2) $\mathscr{R}$ is proximally equivalent to $\mathscr{R}^{\infty}$ or $\mathscr{R}^{\#\infty}$,

(3) $\mathscr{R}$ is proximally equivalent to a preorder relator on X.

In principle, each theorem on topological and quasi-topological relators can be immediately derived from a corresponding theorem on proximal and quasi-proximal relators by using the following two theorems.

Theorem 63. *For a nonvoid relator $\mathscr{R}$ on X, the following assertions are equivalent:*

(1) $\mathscr{R}$ is quasi-topological,
(2) $\mathscr{R}^{\wedge}$ is quasi-proximal.

Proof. Note that if (2) holds, then in particular, for any $x \in X$ and $R \in \mathscr{R}$, we have

$$\{x\} \in \operatorname{Int}_{\mathscr{R}^{\wedge}}\big[\,\tau_{\mathscr{R}^{\wedge}} \cap \operatorname{Int}_{\mathscr{R}^{\wedge}}\big(R\,[\,\{x\}\,]\big)\,\big] \subseteq \operatorname{Int}_{\mathscr{R}^{\wedge}}\big[\operatorname{Int}_{\mathscr{R}^{\wedge}}\big(R(x)\big)\,\big].$$

Therefore, there exists $V \in \operatorname{Int}_{\mathscr{R}^{\wedge}}\big(R(x)\big)$ such that $\{x\} \in \operatorname{Int}_{\mathscr{R}^{\wedge}}(V)$. Hence, by using Theorem 18, we can infer that $x \in \operatorname{int}_{\mathscr{R}}(V)$ and $V \subseteq \operatorname{int}_{\mathscr{R}}\big(R(x)\big)$. Therefore, $x \in \operatorname{int}_{\mathscr{R}}\big(\operatorname{int}_{\mathscr{R}}\big(R(x)\big)\big)$ also holds, and thus $\mathscr{R}$ is quasi-topological.

To prove the converse implication, assume now that (1) holds and $A \subseteq X$ and $S \in \mathscr{R}^{\wedge}$. Define $V = \operatorname{int}_{\mathscr{R}}\big(S\,[\,A\,]\big)$. Then, by Theorem 56 and Corollary 15, we have $V \in \mathscr{T}_{\mathscr{R}} = \tau_{\mathscr{R}^{\wedge}}$. Moreover, since $V \subseteq \operatorname{int}_{\mathscr{R}}\big(S\,[\,A\,]\big)$, by Theorem 18 we also have $V \in \operatorname{Int}_{\mathscr{R}^{\wedge}}\big(S\,[\,A\,]\big)$. Therefore, $V \in \tau_{\mathscr{R}^{\wedge}} \cap \operatorname{Int}_{\mathscr{R}^{\wedge}}\big(S\,[\,A\,]\big)$.

On the other hand, since $S \in \mathscr{R}^{\wedge}$ and $S\,[\,A\,] \subseteq S[\,A\,]$, we can also note that $A \in \operatorname{Int}_{\mathscr{R}^{\wedge}}\big(S\,[\,A\,]\big)$. Hence, by using Theorem 18, we can infer that $A \subseteq \operatorname{int}_{\mathscr{R}}\big(S\,[\,A\,]\big) = V$. Moreover, since $V \in \tau_{\mathscr{R}^{\wedge}}$, we can also note that $V \in \operatorname{Int}_{\mathscr{R}^{\wedge}}(V)$. Hence, since $A \subseteq V$, we can infer that $A \in \operatorname{Int}_{\mathscr{R}^{\wedge}}(V)$. Therefore, since $V \in \tau_{\mathscr{R}^{\wedge}} \cap \operatorname{Int}_{\mathscr{R}^{\wedge}}\big(S\,[\,A\,]\big)$, we also have $A \in \operatorname{Int}_{\mathscr{R}}\big[\,\tau_{\mathscr{R}} \cap \operatorname{Int}_{\mathscr{R}}\big(R\,[\,A\,]\big)\,\big]$. This shows that (2) also holds.

Remark 45. From the above proof, we can see that, for a relator $\mathscr{R}$ on X, the following assertions are also equivalent:

(1) $\mathscr{R}$ is quasi-proximal,
(2) $\{x\} \in \operatorname{Int}_{\mathscr{R}^{\wedge}}\big[\operatorname{Int}_{\mathscr{R}^{\wedge}}\big(R(x)\big)\,\big]$ for all $x \in X$ and $R \in \mathscr{R}$,
(3) $A \in \operatorname{Int}_{\mathscr{R}^{\wedge}}\big[\operatorname{Int}_{\mathscr{R}^{\wedge}}\big(S\,[\,A\,]\big)\,\big]$ for all $A \subseteq X$ and $S \in \mathscr{R}^{\wedge}$.

Theorem 64. *For any relator $\mathscr{R}$ on X, the following assertions are equivalent:*

(1) $\mathscr{R}$ is topological,
(2) $\mathscr{R}^{\wedge}$ is proximal.

Proof. If (1) holds, then by Theorem 57 the relator $\mathscr{R}$ is reflexive and quasi-topological. Hence, by the corresponding definitions, it is clear that the relator $\mathscr{R}^{\wedge}$ is also reflexive. Moreover, if $\mathscr{R} \neq \emptyset$, then from Theorem 62 we can see that $\mathscr{R}^{\wedge}$ is quasi-proximal. Thus, by Theorem 60, assertion (2) also holds.

Quite similarly, we can also see that (2) implies (1) whenever $\mathscr{R} \neq \emptyset$. The case $\mathscr{R} = \emptyset$ has to be treated separately by using that $\mathscr{R}^{\wedge} = \emptyset$ if $\mathscr{R} = \emptyset$ and $X \neq \emptyset$.

12 The Main Definitions of Mild Continuities

Notation 1. In the sequel, we shall assume that:

(1) $\square$ *is a direct unary operation for relators,*
(2) $(X, Y)(\mathscr{R})$ *and* $(Z, W)(\mathscr{S})$ *are relator spaces,*
(3) $\mathscr{F}$ *is a relator on* X *to* Z *and* $\mathscr{G}$ *is a relator on* Y *to* W*.*

Remark 46. Now, to keep in mind the above assumptions, for any $R \in \mathscr{R}$, $S \in \mathscr{S}$, $F \in \mathscr{F}$ and $G \in \mathscr{G}$, we can use the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ F\ } & Z \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ Y & \xrightarrow{\ G\ } & W \end{array}$$

Moreover, by Definition 2, we may naturally consider the following.

Definition 14. Under the above assumptions, we say that the pair $(\mathscr{F}, \mathscr{G})$ of relators is *mildly* $\square$*–continuous*, with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, if

$$\Big(\big(\mathscr{G}^{\square}\big)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square}\Big)^{\square} \subseteq \mathscr{R}^{\square\square}.$$

Remark 47. Thus, the pair $(\mathscr{F}, \mathscr{G})$ may be naturally called *properly mildly continuous* if it is mildly $\square$–continuous with $\square$ being the identity operation for relators. That is, $\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F} \subseteq \mathscr{R}$.

Remark 48. Moreover, the pair $(\mathscr{F}, \mathscr{G})$ may, for instance, be naturally called *uniformly, proximally, topologically, and paratopologically mildly continuous* if it is mildly $\square$–continuous with $\square = *$, $\#$, $\wedge$, and Δ, respectively.

Remark 49. And, the pair $(\mathscr{F}, \mathscr{G})$ may, for instance, be naturally called *quasi-topologically and ultra-topologically* mildly continuous if it is mildly $\square$–continuous with $\square = \wedge\infty$ and $\wedge\,\partial$, respectively.

Remark 50. Moreover, the pair $(\mathscr{F}, \mathscr{G})$ may, for instance, be naturally called *infinitesimally and ultimately mildly continuous* if it is $\square$–mildly continuous with $\square = \bullet$ and $\blacklozenge$, respectively.

Now, by specializing Definition 14, we may also naturally have the following.

Definition 15. For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the pair (F, G) of relations is called *mildly* $\square$*–continuous*, with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, if the pair $\big(\{F\}, \{G\}\big)$ of relators has the same property.

Remark 51. To apply this definition, note that if in particular $\square = \#$ or $\wedge$, then for any $F \in \mathscr{F}$ and $G \in \mathscr{G}$ we have

$$\{F\}^{\square} = \{F\}^{*} \qquad \text{and} \qquad \left(\{G\}^{\square}\right)^{-1} = \left(\{G\}^{*}\right)^{-1} = \left\{G^{-1}\right\}^{*}.$$

However, in contrast to the above equalities, for instance, we already have

$$\{F\}^{\Delta} = \left(F \circ X^{X}\right)^{*} \quad \text{and} \quad \left(\{G\}^{\Delta}\right)^{-1} = \left(\left(G \circ Y^{Y}\right)^{*}\right)^{-1} = \left(\left(Y^{Y}\right)^{-1} \circ G^{-1}\right)^{*}.$$

Now, by using Definition 15, we may also naturally introduce the following.

Definition 16. Under the assumptions of Notation 1, we say that the pair $(\mathscr{F}, \mathscr{G})$ of relators is *elementwise mildly* $\square$*–continuous*, with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, if for any $F \in \mathscr{F}$ and $G \in \mathscr{G}$ the pair (F, G) of relations is mildly $\square$–continuous with respect to the relators $\mathscr{R}$ and $\mathscr{S}$.

Remark 52. Thus, the pair $(\mathscr{F}, \mathscr{G})$ may, for instance, be naturally called *elementwise topologically mildly continuous* if it is elementwise mildly $\square$–continuous with $\square = \wedge$.

Now, as a natural extension of [42, Definition 4.6], we may also naturally have

Definition 17. Under the assumptions of Notation 1, we say that the pair

(1) $(\mathscr{F}, \mathscr{G})$ is *lower selectionally mildly* $\square$*–continuous* if for any $F \in \mathscr{F}$ and $G \in \mathscr{G}$ and any selection f of F the pair (f, G) is mildly $\square$–continuous,
(2) $(\mathscr{F}, \mathscr{G})$ is *upper selectionally mildly* $\square$*–continuous* if for any $F \in \mathscr{F}$ and $G \in \mathscr{G}$ and any selection g of G the pair (F, g) is mildly $\square$–continuous.

Remark 53. Now, the pair $(\mathscr{F}, \mathscr{G})$ may also be naturally called *selectionally mildly* $\square$*–continuous* if it is both lower and upper selectionally mildly $\square$–continuous.

Remark 54. Moreover, the pair $(\mathscr{F}, \mathscr{G})$ may also be naturally called *doubly selectionally mildly* $\square$*–continuous* if for any $F \in \mathscr{F}$ and $G \in \mathscr{G}$ and for any selections f of F and g of G, the pair (f, g) is mildly $\square$–continuous.

Remark 55. Finally, we note that, in the $X = Y$ and $Z = W$ particular case, the relator $\mathscr{F}$ and a relation $F \in \mathscr{F}$ may, for instance, be naturally called mildly $\square$–continuous if the pairs $(\mathscr{F}, \mathscr{F})$ and (F, F), respectively, have the same property.

13 Reduction Theorems for Mild Continuities

Remark 56. If in particular the operation $\square$ is idempotent, then by the corresponding definitions it is clear that the following assertions are equivalent:

(1) $(\mathscr{F}, \mathscr{G})$ is mildly $\square$–continuous,
(2) $\left(\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square}\right)^{\square} \subseteq \mathscr{R}^{\square}$.

Moreover, by using Theorems 10 and 32, we can easily prove the following two theorems.

Theorem 65. *If in particular* $\square$ *is a closure operation, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous,*
(2) $\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square} \subseteq \mathscr{R}^{\square}$.

Proof. Note that now, by the idempotency of $\square$ and Theorem 10, for any two relators $\mathscr{U}$ and $\mathscr{V}$ on X to Y we have

$$\mathscr{V}^{\square} \subseteq \mathscr{U}^{\square\square} \iff \mathscr{V}^{\square} \subseteq \mathscr{U}^{\square} \iff \mathscr{V} \subseteq \mathscr{U}^{\square}.$$

Theorem 66. *If in particular* $\square$ *is an increasing involution, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous,*
(2) $\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square} \subseteq \mathscr{R}^{\square}$,
(3) $\left(\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square}\right)^{\square} \subseteq \mathscr{R}$.

Proof. Note that now, by the involutiveness of $\square$ and Theorem 32, for any two relators $\mathscr{U}$ and $\mathscr{V}$ on X to Y we have

$$\mathscr{V}^{\square} \subseteq \mathscr{U}^{\square\square} \iff \mathscr{V}^{\square} \subseteq \mathscr{U} \iff \mathscr{V} \subseteq \mathscr{U}^{\square}.$$

Now, as an immediate consequence of the above two theorems and Remark 47, we can also state

Corollary 27. *If in particular* $\square$ *is either a closure operation or an increasing involution, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $\left(\mathscr{F}^{\square}, \mathscr{G}^{\square}\right)$ *is properly mildly continuous with respect to* $\mathscr{R}^{\square}$ *and* $\mathscr{S}^{\square}$.

However, it is now more important to note that, by using Theorem 47 and Corollary 22, we can easily prove the following.

Theorem 67. *If in particular* $\square$ *is inversion and composition compatible, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous,*
(2) $\left(\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F}\right)^{\square} \subseteq \mathscr{R}^{\square}$.

Proof. Note that now, by the assumed compatibilities of $\square$ and Corollary 22,

$$\left(\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square}\right)^{\square} = \left(\left(\mathscr{G}^{-1}\right)^{\square} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square}\right)^{\square} = \left(\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F}\right)^{\square}.$$

Moreover, by Theorem 47, we now also have $\mathscr{R}^{\square\square} = \mathscr{R}^{\square}$.

From this theorem, by Theorem 10, it is clear that in particular we also have

Theorem 68. *If in particular* $\square$ *is an inversion and composition compatible closure operation, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous,*
(2) $\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F} \subseteq \mathscr{R}^{\square}$.

Hence, by using Remark 47, we can immediately derive the following.

Corollary 28. *Under the above assumptions on* $\square$*, the following assertions are also equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $(\mathscr{F}, \mathscr{G})$ *is properly mildly continuous with respect to* $\mathscr{R}^{\square}$ *and* $\mathscr{S}$.

However, it is now more important to note that, by using Theorem 68, we can also easily prove the following two theorems.

Theorem 69. *If in particular* $\square$ *is an inversion and composition compatible closure operation, dominating another such operation* $\Diamond$ *for relators, then the mild* $\Diamond$*–continuity of* $(\mathscr{F}, \mathscr{G})$ *implies the mild* $\square$*–continuity of* $(\mathscr{F}, \mathscr{G})$.

Proof. If the pair $(\mathscr{F}, \mathscr{G})$ is mildly $\Diamond$–continuous, then by Theorem 68 we have $\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F} \subseteq \mathscr{R}^{\Diamond}$. Hence, by using the inclusion $\mathscr{R}^{\Diamond} \subseteq \mathscr{R}^{\square}$, we can infer that $G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{\square}$. Therefore, by Theorem 68, the pair $(\mathscr{F}, \mathscr{G})$ is also mildly $\square$–continuous.

Remark 57. From this theorem, by Theorems 14, 41, and 51, it is clear that, for instance, the "proper mild continuity of $(\mathscr{F}, \mathscr{G})$" $\Rightarrow$ the "uniform mild continuity of $(\mathscr{F}, \mathscr{G})$" $\Rightarrow$ the "proximal mild continuity of $(\mathscr{F}, \mathscr{G})$."

Theorem 70. *If in particular* $\square$ *is an inversion and composition compatible closure operation, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous,*
(2) $(\mathscr{F}, \mathscr{G})$ *is elementwise mildly* $\square$*–continuous.*

Proof. By Theorem 68 and the corresponding definitions, it is clear that

$$(1) \iff \mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F} \subseteq \mathscr{R}^{\square}$$
$$\iff \forall\, F \in \mathscr{F} : \forall\, G \in \mathscr{G} : \ G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{\square} \iff (2)\,.$$

Remark 58. Unfortunately, this theorem cannot also be applied to the operations $\wedge$ and $\triangle$.

Therefore, it is worth noticing that the implication $(1) \Longrightarrow (2)$ is already true if $\square$ is only increasing.

For this, it is convenient to prove first a more general theorem.

Theorem 71. *If in particular* $\square$ *is increasing and* $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous, then for any* $\mathscr{F}_1 \subseteq \mathscr{F}$ *and* $\mathscr{G}_1 \subseteq \mathscr{G}$ *the pair* $(\mathscr{F}_1, \mathscr{G}_1)$ *is also mildly* $\square$*–continuous.*

Proof. Because of the assumed increasingness of $\square$, we have

$$\mathscr{F}_1^{\,\square} \subseteq \mathscr{F}^{\,\square}, \qquad \text{and thus also} \qquad \big(\mathscr{G}_1^{\,\square}\big)^{-1} \subseteq \big(\mathscr{G}^{\,\square}\big)^{-1}.$$

Hence, by using the increasingness of composition of relators, we can infer that

$$\big(\mathscr{G}_1^{\,\square}\big)^{-1} \circ \mathscr{S}^{\,\square} \circ \mathscr{F}_1^{\,\square} \subseteq \big(\mathscr{G}^{\,\square}\big)^{-1} \circ \mathscr{S}^{\,\square} \circ \mathscr{F}^{\,\square}.$$

Thus, again by the increasingness $\square$, we also have

$$\Big(\big(\mathscr{G}_1^{\,\square}\big)^{-1} \circ \mathscr{S}^{\,\square} \circ \mathscr{F}_1^{\,\square}\Big)^{\square} \subseteq \Big(\big(\mathscr{G}^{\,\square}\big)^{-1} \circ \mathscr{S}^{\,\square} \circ \mathscr{F}^{\,\square}\Big)^{\square}.$$

Therefore, by Definition 14, the mild $\square$–continuity of $(\mathscr{F}, \mathscr{G})$ implies that of $\big(\mathscr{F}_1, \mathscr{G}_1\big)$.

Hence, by letting $\mathscr{F}_1$ and $\mathscr{G}_1$ to be singletons, we can immediately derive

Corollary 29. *If in particular* $\square$ *is increasing and* $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous, then* $(\mathscr{F}, \mathscr{G})$ *is elementwise mildly* $\square$*–continuous.*

14 Some Further Theorems on Mild Continuities

By using the corresponding definitions, we can also easily prove the following.

Theorem 72. *If in particular* $\square$ *is* $\Diamond$*–absorbing, for some direct unary operation* $\Diamond$ *for relators, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*–continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $\big(\mathscr{F}^{\,\Diamond}, \mathscr{G}^{\,\Diamond}\big)$ *is mildly* $\square$*–continuous with respect to* $\mathscr{R}^{\,\Diamond}$ *and* $\mathscr{S}^{\,\Diamond}$*.*

Proof. By the corresponding definitions, it is clear that

$$(1) \iff \Big(\big(\mathscr{G}^{\,\square}\big)^{-1} \circ \mathscr{S}^{\,\square} \circ \mathscr{F}^{\,\square}\Big)^{\square} \subseteq \mathscr{R}^{\,\square\square}$$

$$\iff \Big(\Big(\big(\mathscr{G}^{\,\Diamond}\big)^{\square}\Big)^{-1} \circ \big(\mathscr{S}^{\,\Diamond}\big)^{\square} \circ \big(\mathscr{F}^{\,\Diamond}\big)^{\square}\Big)^{\square} \subseteq \big(\mathscr{R}^{\,\Diamond}\big)^{\square\square} \iff (2).$$

From this theorem, by letting $\Diamond = \square$, we can immediately derive

Corollary 30. *If in particular $\Box$ is idempotent, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly $\Box$–continuous with respect to $\mathscr{R}$ and $\mathscr{S}$,*
(2) $\big(\mathscr{F}^{\,\Box}, \mathscr{G}^{\,\Box}\big)$ *is mildly $\Box$–continuous with respect to $\mathscr{R}^{\,\Box}$ and $\mathscr{S}^{\,\Box}$.*

Now, as an extension of Theorem 71, we can also easily prove the following

Theorem 73. *If in particular $\Box$ is increasing and $\Diamond$–absorbing, for some direct unary operation $\Diamond$ for relators, and the pair $(\mathscr{F}, \mathscr{G})$ is mildly $\Box$–continuous, then for any $\mathscr{F}_1 \subseteq \mathscr{F}^{\,\Diamond}$ and $\mathscr{G}_1 \subseteq \mathscr{G}^{\,\Diamond}$ the pair $\big(\mathscr{F}_1, \mathscr{G}_1\big)$ is also mildly $\Box$–continuous.*

Proof. Because of the above assumptions, we have

$$\mathscr{F}_1^{\,\Box} \subseteq \mathscr{F}^{\,\Diamond\,\Box} = \mathscr{F}^{\,\Box}, \qquad \text{and thus also} \qquad \big(\mathscr{G}_1^{\,\Box}\big)^{-1} \subseteq \big(\mathscr{G}^{\,\Box}\big)^{-1}.$$

Hence, as in the proof of Theorem 71, we can already infer that

$$\Big(\big(G_1^{\,\Box}\big)^{-1} \circ \mathscr{S}^{\,\Box} \circ \mathscr{F}_1^{\,\Box}\Big)^{\Box} \subseteq \Big(\big(\mathscr{G}^{\,\Box}\big)^{-1} \circ \mathscr{S}^{\,\Box} \circ \mathscr{F}^{\,\Box}\Big)^{\Box}.$$

Therefore, by Definition 14, the mild $\Box$–continuity of $(\mathscr{F}, \mathscr{G})$ implies that of $\big(\mathscr{F}_1, \mathscr{G}_1\big)$.

Remark 59. In this theorem, instead of the $\Diamond$–absorbingness of $\Box$, it is enough to assume only that $\mathscr{U}^{\,\Diamond\,\Box} \subseteq \mathscr{U}^{\,\Box}$ for every relator $\mathscr{U}$.

However, if in particular $\Diamond$ is extensive, then because of the assumed increasingness of $\Box$ the corresponding equality is also true.

A simple application of the $\Diamond = *$ particular cease of Theorem 73 to singleton relators gives the following.

Corollary 31. *If in particular $\Box$ is an increasing, $*$–absorbing operation, and $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that the pair (F, G) is mildly $\Box$–continuous, then for any $F_1 \in \mathscr{F}$ and $G_1 \in \mathscr{G}$, with $F \subseteq F_1$ and $G \subseteq G_1$, the pair (F_1, G_1) is also mildly $\Box$–continuous.*

Proof. By the above assumptions on F_1 and G_1, and the definition of $*$, we have $\{F_1\} \subseteq \{F\}^*$ and $\{G_1\} \subseteq \{G\}^*$. Therefore, by Theorem 73, the mild $\Box$–continuity of $\big(\{F\}, \{G\}\big)$ implies that of $\big(\{F_1\}, \{G_1\}\big)$. Thus, by Definition 15, the mild $\Box$–continuity of (F, G) also implies that of (F_1, G_1).

By using Theorem 65, we can also easily prove the following.

Theorem 74. *If in particular $\Box$ is a $\Diamond$–invariant closure operation for some closure operation $\Diamond$ for relators, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly $\Box$–continuous with respect to $\mathscr{R}$ and $\mathscr{S}$,*
(2) $\big(\mathscr{G}^{\,\Box}, \mathscr{F}^{\,\Box}\big)$ *is mildly $\Diamond$–continuous with respect to $\mathscr{R}^{\,\Box}$ and $\mathscr{S}^{\,\Box}$.*

Proof. By Theorem 65 and the corresponding definitions, it is clear that

$$(1) \iff \left(\mathscr{G}^{\,\Box}\right)^{-1} \circ \mathscr{S}^{\,\Box} \circ \mathscr{F}^{\,\Box} \subseteq \mathscr{R}^{\,\Box}$$
$$\iff \left(\left(\mathscr{G}^{\,\Box}\right)^{\Diamond}\right)^{-1} \circ \left(\mathscr{S}^{\,\Box}\right)^{\Diamond} \circ \left(\mathscr{F}^{\,\Box}\right)^{\Diamond} \subseteq \left(\mathscr{R}^{\,\Box}\right)^{\Diamond} \iff (2)\,.$$

Now, in additions to Theorems 74 and 72, we can also easily prove

Theorem 75. *If in particular $\Box$ is a $\Diamond$–compatible closure operation, for some closure operation $\Diamond$ for relators, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly $\Box\,\Diamond$–continuous with respect to $\mathscr{R}$ and $\mathscr{S}$,*
(2) $\left(\mathscr{F}^{\,\Box}, \mathscr{G}^{\,\Box}\right)$ *is mildly $\Diamond$–continuous with respect to $\mathscr{R}^{\,\Box}$ and $\mathscr{S}^{\,\Box}$,*
(3) $\left(\mathscr{G}^{\,\Diamond}, \mathscr{F}^{\,\Diamond}\right)$ *is mildly $\Box$–continuous with respect to $\mathscr{R}^{\,\Diamond}$ and $\mathscr{S}^{\,\Diamond}$.*

Proof. From Theorem 29, we know that $\Box\,\Diamond$ is also a closure operation for relators. Hence, by Theorem 65 and the corresponding definitions, it is clear that

$$(1) \iff \left(\mathscr{G}^{\,\Box\Diamond}\right)^{-1} \circ \mathscr{S}^{\,\Box\Diamond} \circ \mathscr{F}^{\,\Box\Diamond} \subseteq \mathscr{R}^{\,\Box\Diamond}$$
$$\iff \left(\left(\mathscr{G}^{\,\Box}\right)^{\Diamond}\right)^{-1} \circ \left(\mathscr{S}^{\,\Box}\right)^{\Diamond} \circ \left(\mathscr{F}^{\,\Box}\right)^{\Diamond} \subseteq \left(\mathscr{R}^{\,\Box}\right)^{\Diamond} \iff (2)\,.$$

Now, since $\Box\,\Diamond = \Diamond\,\Box$, it is clear that assertions (1) and (3) are also equivalent.

Remark 60. From the latter two theorems, by letting $\Diamond$ to be the identity operation for relators, we can also immediately derive the "closure operation part" of Corollary 31.

However, it is now more important to note that by using the corresponding definitions, we can also easily prove the following

Theorem 76. *If in particular $\Box$ is inversion compatible, then the following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly $\Box$–continuous with respect to $\mathscr{R}$ and $\mathscr{S}$,*
(2) $(\mathscr{G}, \mathscr{F})$ *is mildly $\Box$–continuous with respect to $\mathscr{R}^{-1}$ and $\mathscr{S}^{-1}$.*

Proof. By Definition 14 and an inversion property of composition, it is clear that

$$(1) \iff \left(\left(\mathscr{G}^{\,\Box}\right)^{-1} \circ \mathscr{S}^{\,\Box} \circ \mathscr{F}^{\,\Box}\right)^{\Box} \subseteq \left(\mathscr{R}^{\,\Box}\right)^{\Box}$$
$$\iff \left(\left(\left(\mathscr{G}^{\,\Box}\right)^{-1} \circ \mathscr{S}^{\,\Box} \circ \mathscr{F}^{\,\Box}\right)^{\Box}\right)^{-1} \subseteq \left(\left(\mathscr{R}^{\,\Box}\right)^{\Box}\right)^{-1}$$
$$\iff \left(\left(\left(\mathscr{G}^{\,\Box}\right)^{-1} \circ \mathscr{S}^{\,\Box} \circ \mathscr{F}^{\,\Box}\right)^{-1}\right)^{\Box} \subseteq \left(\left(\mathscr{R}^{\,\Box}\right)^{-1}\right)^{\Box}$$

$$\iff \Big(\big(\mathscr{F}^{\,\square}\big)^{-1} \circ \big(\mathscr{S}^{\,\square}\big)^{-1} \circ \mathscr{G}^{\,\square} \Big)^{\square} \subseteq \Big(\big(\mathscr{R}^{\,\square}\big)^{-1} \Big)^{\square}$$

$$\iff \Big(\big(\mathscr{F}^{\,\square}\big)^{-1} \circ \big(\mathscr{S}^{\,-1}\big)^{\square} \circ \mathscr{G}^{\,\square} \Big)^{\square} \subseteq \Big(\big(\mathscr{R}^{\,-1}\big)^{\square} \Big)^{\square} \iff (2)\,.$$

Remark 61. Unfortunately, concerning the elementwise complementation of relators, we cannot prove a similar theorem.

15 Detailed Reformulations of Proper, Uniform, and Proximal Mild Continuities

Recall that if in particular $\square$ is an inversion and composition compatible closure operation, then by Theorem 70, instead of the mild $\square$–continuity of the pair $(\mathscr{F}, \mathscr{G})$, it is enough to investigate only that of the pair (F, G) for all $F \in \mathscr{F}$ and $G \in \mathscr{G}$.

Therefore, concerning proper, uniform, and proximal mild continuities, we shall only prove here some very particular theorems.

Theorem 77. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) (F, G) *is properly mildly continuous,*
(2) $G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}$,
(3) $\operatorname{cl}_{F \boxtimes G}(\mathscr{S}) \subseteq \mathscr{R}$.

Proof. From Remark 47, we know that (1) and (2) are equivalent. Moreover, by [52, Theorem 4.3], for any $F \in \mathscr{F}$, $G \in \mathscr{G}$, and $S \in \mathscr{S}$ we have

$$G^{-1} \circ S \circ F = \big(F^{-1} \boxtimes G^{-1}\big)[S] = (F \boxtimes G)^{-1}[S] = \operatorname{cl}_{F \boxtimes G}(S)\,.$$

Therefore, because of the plausible notation

$$\operatorname{cl}_{F \boxtimes G}(\mathscr{S}) = \big\{ \operatorname{cl}_{F \boxtimes G}(S) : \quad S \in \mathscr{S} \big\}\,,$$

assertions (2) and (3) are also equivalent.

Now, by using this theorem, we can also easily prove the following.

Theorem 78. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) (F, G) *is properly mildly continuous,*
(2) for each $S \in \mathscr{S}$ there exists $R \in \mathscr{R}$ such that $R = G^{-1} \circ S \circ F$,
(3) for each $S \in \mathscr{S}$ there exists $R \in \mathscr{R}$ such that for every $x \in X$ and $y \in Y$ we have $y \in R(x)$ if and only if $G(y) \cap S[F(x)] \neq \emptyset$,

(4) *for each* $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that for every* $x \in X$ *and* $y \in Y$ *we have* $y \in R(x)$ *if and only if there exist* $z \in F(x)$ *and* $w \in G(y)$ *such that* $w \in S(z)$.

Proof. By Theorem 77 and the corresponding definitions, it is clear that

$$(1) \iff G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R} \iff \forall\ S \in \mathscr{S} :\ G^{-1} \circ S \circ F \in \mathscr{R}$$
$$\iff \forall\ S \in \mathscr{S} :\ \exists\ R \in \mathscr{R} :\ R = G^{-1} \circ S \circ F\,.$$

Moreover, we can note that

$$R = G^{-1} \circ S \circ F \iff \forall\ x \in X :\ \forall\ y \in Y :$$
$$\Big(y \in R(x) \iff y \in \big(G^{-1} \circ S \circ F\big)\,(x)\Big)\,.$$

Furthermore, by using some basic facts on relations, we can also easily see that

$$y \in \big(G^{-1} \circ S \circ F\big)\,(x) \iff y \in G^{-1}\big[\,S\,[\,F(x)\,]\,\big] \iff G(y) \cap S\,[\,F(x)\,] \neq \emptyset$$
$$\iff \exists\ w \in G(y) :\ w \in S\,[\,F(x)\,]$$
$$\iff \exists\ w \in G(y) :\ \exists\ z \in F(x) :\ w \in S(z)\,.$$

Therefore, assertions (1)–(4) are also equivalent.

Analogously to Theorem 77, we can also easily prove the following.

Theorem 79. *For any* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$*, the following assertions are equivalent:*

(1) $(F\,,\ G)$ *is uniformly mildly continuous,*
(2) $G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{*}$*,*
(3) $\mathrm{cl}_{F \boxtimes G}(\mathscr{S}) \subseteq \mathscr{R}^{*}$.

Proof. By Remark 11, Theorems 41 and 51, $*$ is an inversion and composition compatible closure operation for relators. Therefore, by Theorem 68 and Remark 48, (1) and (2) are equivalent. Moreover, from the proof of Theorem 77, it is clear that (2) and (3) are also equivalent.

From this theorem, by Theorem 77, it is clear that we also have

Corollary 32. *For any* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$*, the following assertions are equivalent:*

(1) $(F\,,\ G)$ *is uniformly mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $(F\,,\ G)$ *is properly mildly continuous with respect to* $\mathscr{R}^{*}$ *and* $\mathscr{S}$.

Moreover, by using Theorem 79, we can also easily prove the following.

Theorem 80. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) (F, G) *is uniformly mildly continuous,*
(2) for each $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that* $R \subseteq G^{-1} \circ S \circ F$,
(3) for each $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that for every* $x \in X$ *and* $y \in R(x)$ *we have* $G(y) \cap S[F(x)] \neq \emptyset$,
(4) for each $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that for every* $x \in X$ *and* $y \in R(x)$ *there exist* $z \in F(x)$ *and* $w \in G(y)$ *such that* $w \in S(z)$.

Proof. By Theorem 79 and the corresponding definitions, it is clear that

$$(1) \iff G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{*} \iff \forall\ S \in \mathscr{S}:\ G^{-1} \circ S \circ F \in \mathscr{R}^{*}$$
$$\iff \forall\ S \in \mathscr{S}:\ \exists\ R \in \mathscr{R}:\ R \subseteq G^{-1} \circ S \circ F .$$

Moreover, as in the proof of Theorem 78, we can note that

$$R \subseteq G^{-1} \circ S \circ F \iff \forall\ x \in X:\ R(x) \subseteq \big(G^{-1} \circ S \circ F\big)(x)$$
$$\iff \forall\ x \in X:\ \forall\ y \in R(x):\ y \in \big(G^{-1} \circ S \circ F\big)(x) .$$

Furthermore, from the proof of Theorem 78, we know that

$$y \in \big(G^{-1} \circ S \circ F\big)(x) \iff G(y) \cap S[F(x)] \neq \emptyset$$
$$\iff \exists\ w \in G(y):\ \exists\ z \in F(x):\ w \in S(z) .$$

Therefore, assertions (1)–(4) are also equivalent.

Analogously to Theorem 79, we can also easily prove the following.

Theorem 81. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) (F, G) *is proximally mildly continuous,*
(2) $G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{\#}$,
(3) $\mathrm{cl}_{F \boxtimes G}(\mathscr{S}) \subseteq \mathscr{R}^{\#}$.

Remark 62. From Theorems 77, 79, and 81, by the inclusions $\mathscr{R} \subseteq \mathscr{R}^{*} \subseteq \mathscr{R}^{\#}$, it is also clear that "the proper mild continuity of (F, G)" $\Rightarrow$ "the uniform mild continuity of (F, G)" $\Rightarrow$ "the proximal mild continuity of (F, G)."

Moreover, from the abovementioned theorems and the equality $\mathscr{R}^{\#} = \big(\mathscr{R}^{\#}\big)^{*}$, it is clear that we also have

Corollary 33. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) (F, G) *is proximally mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$,
(2) (F, G) *is properly or uniformly mildly continuous with respect to* $\mathscr{R}^{\#}$ *and* $\mathscr{S}$.

Now, by using Theorem 81, we can also easily prove the following.

Theorem 82. *For any* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$, *the following assertions are equivalent:*

(1) (F, G) *is proximally mildly continuous*,
(2) *for each* $A \subseteq X$ *and* $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that* $R[A] \subseteq G^{-1}\big[S\big[F[A]\big]\big]$,
(3) *for each* $A \subseteq X$ *and* $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that for every* $x \in A$ *and* $y \in R(x)$ *we have* $G(y) \cap S\big[F[A]\big] \neq \emptyset$,
(4) *for each* $A \subseteq X$ *and* $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that for every* $x \in X$ *and* $y \in R(x)$ *there exist* $u \in A$ *and* $z \in F(u)$ *and* $w \in G(y)$ *such that* $w \in S(z)$.

Proof. By Theorem 81 and the corresponding definitions, it is clear that

$$\begin{aligned}(1) \iff\ & G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{\#} \iff \forall\ S \in \mathscr{S}:\ G^{-1} \circ S \circ F \in \mathscr{R}^{\#} \\ \iff\ & \forall\ S \in \mathscr{S}:\ \forall\ A \subseteq X:\ \exists\ R \in \mathscr{R}:\ \ R[A] \subseteq \big(G^{-1} \circ S \circ F\big)[A].\end{aligned}$$

Moreover, since $R[A] = \bigcup_{x \in A} R(x)$, we can also note that

$$\begin{aligned}R[A] \subseteq \big(G^{-1} \circ S \circ F\big)[A] \iff\ & \forall\ x \in A:\ R(x) \subseteq \big(G^{-1} \circ S \circ F\big)[A] \\ \iff\ & \forall\ x \in A:\ \forall\ y \in R(x):\ \ y \in \big(G^{-1} \circ S \circ F\big)[A].\end{aligned}$$

Furthermore, by using some basic facts on relations, we can also easily see that

$$\begin{aligned}y \in \big(G^{-1} \circ S \circ F\big)[A] \iff\ & y \in G^{-1}\big[S\big[F[A]\big]\big] \\ \iff\ & G(y) \cap S\big[F[A]\big] \neq \emptyset \\ \iff\ & \exists\ w \in G(y):\ \ w \in S\big[F[A]\big] \\ \iff\ & \exists\ w \in G(y):\ \exists\ z \in F[A]:\ \ w \in S(z) \\ \iff\ & \exists\ w \in G(y):\ \exists\ u \in A:\ \exists\ z \in F(u):\ \ w \in S(z).\end{aligned}$$

Therefore, assertions (1)–(4) are also equivalent.

16 Detailed Reformulations of Topological Mild Continuity

By using Theorem 68, we can easily establish the following.

Theorem 83. *The following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is topologically mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $(\mathscr{F}^{\wedge}, \mathscr{G}^{\wedge})$ *is properly, uniformly, or proximally mildly continuous with respect to* $\mathscr{R}^{\wedge}$ *and* $\mathscr{S}^{\wedge}$*.*

Proof. From Remark 11 and Theorems 14, 15, and 51, we know that $*$, $\#$, and $\wedge$ are closure operations for relators such that $\wedge$ is both $*-$ and $\#$–invariant. Therefore, by Theorem 68, assertion (1) is equivalent to both "the uniformly and proximally part" of (2). Moreover, from Corollary 27, we know that (1) is also equivalent to "the properly part" of (2).

From this theorem, by using Theorem 70, we can immediately derive

Corollary 34. *The following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is topologically mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $(\mathscr{F}^{\wedge}, \mathscr{G}^{\wedge})$ *is elementwise properly, uniformly, or proximally mildly continuous with respect to* $\mathscr{R}^{\wedge}$ *and* $\mathscr{S}^{\wedge}$*.*

Remark 63. This corollary shows that, instead of the topological mild continuity of the pair $(\mathscr{F}, \mathscr{G})$ with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, it is enough to investigate only the proper, uniform, or proximal mild continuity of the pair (F, G) with respect to the relators $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$ for all $F \in \mathscr{F}^{\wedge}$ and $G \in \mathscr{G}^{\wedge}$.

Thus, since $\mathscr{F}$ and $\mathscr{G}$ were quite arbitrary relators in Notation 1, it is actually enough to prove the following.

Theorem 84. *For any* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$*, the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) (F, G) *is properly, uniformly, or proximally mildly continuous with respect to* $\mathscr{R}^{\wedge}$ *and* $\mathscr{S}^{\wedge}$*.*

Proof. By Definition 15, assertion (1) is equivalent to the statement that:

(a) $\big(\{F\}, \{G\}\big)$ is topologically mildly continuous with respect to $\mathscr{R}$ and $\mathscr{S}$.
Moreover, by Theorem 83, assertion (a) is equivalent to the statement that:

(b) $\big(\{F\}^{\wedge}, \{G\}^{\wedge}\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$.
Now, by Remark 51, we can see that assertion (b) is equivalent to the statement that:

(c) $\big(\{F\}^{*}, \{G\}^{*}\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$.
Moreover, since $\wedge$ is $*$–invariant, we can see that assertion (c) is equivalent to the statement that:

(d) $\big(\{F\}^{*}, \{G\}^{*}\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\big(\mathscr{R}^{\wedge}\big)^{*}$ and $\big(\mathscr{S}^{\wedge}\big)^{*}$.

Now, we may recall that the operations $*$ and $\#$ are $*$–absorbing, Therefore, by Theorem 72, "the uniformly or proximally part" of assertion (d) is equivalent to the statement that:

(e) $\big(\{F\},\ \{G\}\big)$ is uniformly or proximally mildly continuous with respect to $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$.

Moreover, we can note that $*$ is an inversion and composition compatible closure operation. Therefore, by Corollary 28, "the uniformly part" of assertion (e) is equivalent to the statement that:

(f) $\big(\{F\},\ \{G\}\big)$ is properly mildly continuous with respect to $\big(\mathscr{R}^{\wedge}\big)^{*}$ and $\mathscr{S}^{\wedge}$.

However, since $\big(\mathscr{R}^{\wedge}\big)^{*}=\mathscr{R}^{\wedge}$, assertion (f) is equivalent to the statement that:

(g) $\big(\{F\},\ \{G\}\big)$ is properly mildly continuous with respect to $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$.

Now, by Definition 15, we can see that assertion (e) is equivalent to the statement that:

(h) $(F,\ G)$ is uniformly or proximally mildly continuous with respect to $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$.

Moreover, assertion (g) is equivalent to the statement that:

(i) $(F,\ G)$ is properly mildly continuous with respect to $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$.

Hence, it is clear that assertions (1) and (2) are also equivalent.

Now, from the "properly part" of this theorem, by using Theorem 77, we can immediately derive the following.

Theorem 85. *For any $F\in\mathscr{F}$ and $G\in\mathscr{G}$, the following assertions are equivalent:*

(1) $(F,\ G)$ *is topologically mildly continuous,*

(2) $G^{-1}\circ\mathscr{S}^{\wedge}\circ F\subseteq\mathscr{R}^{\wedge}$,

(3) $\operatorname{cl}_{F\boxtimes G}(\mathscr{S}^{\wedge})\subseteq\mathscr{R}^{\wedge}$.

Moreover, from the "uniform part" of Theorem 84, by using Theorem 80, we can easily derive the following.

Theorem 86. *For any $F\in\mathscr{F}$ and $G\in\mathscr{G}$, the following assertions are equivalent:*

(1) $(F,\ G)$ *is topologically mildly continuous,*

(2) for each $x\in X$ and $V\in\mathscr{S}^{\wedge}$ there exists $R\in\mathscr{R}$ such that $R(x)\subseteq G^{-1}\big[\,V\,[\,F(x)\,]\,\big]$,

(3) for each $x\in X$ and $V\in\mathscr{S}^{\wedge}$ there exists $R\in\mathscr{R}$ such that for every $y\in R(x)$ we have $G(y)\cap V\,[\,F(x)\,]\neq\emptyset$,

(4) for each $x\in X$ and $V\in\mathscr{S}^{\wedge}$ there exists $R\in\mathscr{R}$ such that for every $y\in R(x)$ there exist $z\in F(u)$ and $w\in G(y)$ such that $w\in V(z)$.

Proof. By the "uniform part" of Theorem 84, assertion (1) is equivalent to the statement that:

(a) $\big(F\,,\ G\big)$ is uniformly mildly continuous with respect to $\mathscr{R}^{\wedge}$ and $\mathscr{S}^{\wedge}$.

Moreover, by Theorem 80, assertion (a) is equivalent to the statements that:

(b) for each $V \in \mathscr{S}^{\wedge}$ there exists $U \in \mathscr{R}^{\wedge}$ such that $U \subseteq G^{-1} \circ V \circ F$,

(c) for each $V \in \mathscr{S}^{\wedge}$ there exists $U \in \mathscr{R}^{\wedge}$ such that for every $x \in X$ and $y \in U(x)$ we have $G(y) \cap V\,[\,F(x)\,] \neq \emptyset$,

(d) for each $V \in \mathscr{S}^{\wedge}$ there exists $U \in \mathscr{R}^{\wedge}$ such that for every $x \in X$ and $y \in U(x)$ there exist $z \in F(x)$ and $w \in G(y)$ such that $w \in V(z)$.

Therefore, to complete the proof, we need to only show that each of assertions (b)–(d) is equivalent to the corresponding assertion of the theorem. For this, for instance, we shall show that assertions (c) and (3) are equivalent.

Note that if (c) holds, then for each $V \in \mathscr{S}^{\wedge}$ there exists $U \in \mathscr{R}^{\wedge}$ such that $U \subseteq G^{-1} \circ V \circ F$, and thus also $U(x) \subseteq G^{-1}\big[\,V\,[\,F(x)\,]\,\big]$ for all $x \in X$. Moreover, by the definition of $\mathscr{R}^{\wedge}$, for each $x \in X$ there exists $R \in \mathscr{R}$ such that $R(x) \subseteq U(x)$, and thus also $R(x) \subseteq G^{-1}\big[\,V\,[\,F(x)\,]\,\big]$. Therefore, (3) also holds.

Conversely, if (3) holds, then for each $x \in X$ and $V \in \mathscr{S}^{\wedge}$ there exists $R_x \in \mathscr{R}$ such that $R_x(x) \subseteq G^{-1}\big[\,V\,[\,F(x)\,]\,\big]$. Hence, by defining a relation U on X to Y such that $U(x) = R_x(x)$ for all $x \in X$, we can at once see that $U \in \mathscr{R}^{\wedge}$ such that $U(x) = R_x(x) \subseteq G^{-1}\big[\,V\,[\,F(x)\,]\,\big]$ for all $x \in X$. Therefore, $U \subset G^{-1} \circ V \circ F$, and thus (c) also holds.

Now, from Theorem 85, by letting F to be a function, we can easily derive

Theorem 87. *If $f \in \mathscr{F}$ and $G \in \mathscr{G}$ such that f is a function, then the following assertions are equivalent:*

(1) $(f\,,\ G)$ *is topologically mildly continuous,*

(2) $G^{-1} \circ \mathscr{S} \circ f \subseteq \mathscr{R}^{\wedge}$,

(3) $\operatorname{cl}_{f \boxtimes G}(\mathscr{S}) \subseteq \mathscr{R}^{\wedge}$.

Proof. From Theorem 85, we know that assertion (1) is equivalent to the statement that: (a) $G^{-1} \circ \mathscr{S}^{\wedge} \circ f \subseteq \mathscr{R}^{\wedge}$.

Moreover, by the inclusion $\mathscr{S} \subseteq \mathscr{S}^{\wedge}$ and the increasingness of composition, it is clear that (a) implies (2). Therefore, to prove the equivalence of (1) and (2), we need only show that (2) also implies (a).

For this, we can note that if $x \in X$ and $V \in \mathscr{S}^{\wedge}$, then by the assumption that $\operatorname{card}\big(f(x)\big) \leq 1$ and the definition of $\mathscr{S}^{\wedge}$ there exists $S \in \mathscr{S}$ such that $S\,[\,f(x)\,] \subseteq V\,[\,f(x)\,]$. Hence, we can already infer that

$$\big(G^{-1} \circ S \circ f\big)(x) = G^{-1}\big[\,S\,[\,f(x)\,]\,\big] \subseteq G^{-1}\big[\,V\,[\,f(x)\,]\,\big] = \big(G^{-1} \circ V \circ f\big)(x)\,.$$

Moreover, if (2) holds, then $G^{-1} \circ S \circ f \in \mathscr{R}^{\wedge}$. Therefore, there exists $R \in \mathscr{R}$ such that $R(x) \subseteq \big(G^{-1} \circ S \circ f\big)(x)$, and thus $R(x) \subseteq \big(G^{-1} \circ V \circ f\big)(x)$. This shows that $G^{-1} \circ V \circ f \in \mathscr{R}^{\wedge}$, and thus (a) also holds.

Now, to complete the proof, it remains to note only that, by Theorem 77, assertions (2) and (3) are also equivalent.

Remark 64. From this theorem, by using the inclusion $\mathscr{R}^{\#} \subseteq \mathscr{R}^{\wedge}$ and Theorem 74, we can at once see that "the proximal mild continuity of (f, G)" implies "the topological mild continuity of (f, G)".

Moreover, from Theorem 87, by using Theorems 77, 79, and 81, we can also easily derive

Corollary 35. *Under the assumptions of Theorem 87, the following assertions are also equivalent:*

(1) (f, G) *is topologically mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) (f, G) *is properly, uniformly or proximally mildly continuous with respect to* $\mathscr{R}^{\wedge}$ *and* $\mathscr{S}$*.*

On the other hand, from the proof of Theorem 87, it is clear that, as a consequence of Theorem 86, we can also state the following.

Theorem 88. *If* $f \in \mathscr{F}$ *and* $G \in \mathscr{G}$ *such that* f *is a function, then the following assertions are equivalent:*

(1) (f, G) *is topologically mildly continuous,*
(2) for each $x \in X$ *and* $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that* $R(x) \subseteq G^{-1}\big[\, S\,[f(x)\,]\,\big]$*,*
(3) for each $x \in X$ *and* $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that for every* $y \in R(x)$ *we have* $G(y) \cap S[f(x)\,] \neq \emptyset$*,*
(4) for each $x \in X$ *and* $S \in \mathscr{S}$ *there exists* $R \in \mathscr{R}$ *such that for every* $y \in R(x)$ *there exists* $w \in G(y)$ *such that* $w \in S\,[f(x)\,]$*.*

Remark 65. Note that if in particular the whole X is the domain of f, then in the above assertions we may write $S\big(f(x)\big)$ instead of $S\,[f(x)\,]$.

17 Detailed Reformulations of Paratopological Mild Continuity

By using Theorem 74 and Corollary 27, analogously to Theorem 83, we can also easily prove the following.

Theorem 89. *The following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is paratopologically mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $(\mathscr{F}^{\Delta}, \mathscr{G}^{\Delta})$ *is properly, uniformly, proximally, or topologically mildly continuous with respect to* $\mathscr{R}^{\Delta}$ *and* $\mathscr{S}^{\Delta}$*.*

Hence, by using Theorem 70, we can only derive the following.

Corollary 36. *The following assertions are equivalent:*

(1) $(\mathscr{F}, \mathscr{G})$ *is paratopologically mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $(\mathscr{F}^{\Delta}, \mathscr{G}^{\Delta})$ *is elementwise properly, uniformly, or proximally mildly continuous with respect to* $\mathscr{R}^{\Delta}$ *and* $\mathscr{S}^{\Delta}$*.*

Remark 66. This corollary shows that, instead of the paratopological mild continuity of the pair $(\mathscr{F}, \mathscr{G})$ with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, it is enough to investigate only the proper, uniform, or proximal mild continuity of the pair (F, G) with respect to the relators $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$ for all $F \in \mathscr{F}^{\Delta}$ and $G \in \mathscr{G}^{\Delta}$.

However, instead of an analogue of Theorem 84, we can now only prove

Theorem 90. *For any* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$*, the following assertions are equivalent:*

(1) (F, G) *is paratopologically mildly continuous with respect to* $\mathscr{R}$ *and* $\mathscr{S}$*,*
(2) $\big(F \circ X^X, \ G \circ Y^Y\big)$ *is properly, uniformly, proximally, or topologically mildly continuous with respect to* $\mathscr{R}^{\Delta}$ *and* $\mathscr{S}^{\Delta}$*.*

Proof. By Definition 15, assertion (1) is equivalent to the statement that:

(a) $\big(\{F\}, \{G\}\big)$ is paratopologically mildly continuous with respect to $\mathscr{R}$ and $\mathscr{S}$.
Moreover, by Theorem 89, assertion (a) is equivalent to the statement that:

(b) $\big(\{F\}^{\Delta}, \{G\}^{\Delta}\big)$ is properly, uniformly, proximally, or topologically mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$.
Now, by Remark 51, we can see that assertion (b) is equivalent to the statement that:

(c) $\big(\big(F \circ X^X\big)^*, \ \big(G \circ Y^Y\big)^*\big)$ is properly, uniformly, proximally, or topologically mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$.
Moreover, since Δ is $*$–invariant, we can see that assertion (c) is equivalent to the statement that:

(d) $\big(\big(F \circ X^X\big)^*, \ \big(G \circ Y^Y\big)^*\big)$ is properly, uniformly, proximally, or topologically mildly continuous with respect to $\big(\mathscr{R}^{\Delta}\big)^*$ and $\big(\mathscr{S}^{\Delta}\big)^*$.
Now, we may recall that the operations $*$, $\#$, and $\wedge$ are $*$–absorbing. Therefore, by Theorem 72, the "uniformly, proximally, or topologically part" of assertion (d) is equivalent to the statement that:

(e) $\big(F \circ X^X, \ G \circ Y^Y\big)$ is uniformly, proximally, or topologically mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$.
Moreover, we may recall that $*$ is an inversion and composition compatible closure operation. Therefore, by Corollary 28, "the uniformly part" of assertion (e) is equivalent to the statement that:

(f) $\big(F \circ X^X, \ G \circ Y^Y\big)$ is properly mildly continuous with respect to $\big(\mathscr{R}^{\Delta}\big)^*$ and $\mathscr{S}^{\Delta}$.

However, since $(\mathscr{R}^{\Delta})^{*} = \mathscr{R}^{\Delta}$, assertion (f) is equivalent to the statement that:

(g) $\big(F \circ X^X ,\ G \circ Y^Y\big)$ is properly mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$.

Hence, it is clear that assertions (1) and (2) are also equivalent.

From this theorem, by using Theorem 70, we can immediately derive

Corollary 37. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) $(F ,\ G)$ is paratopologically mildly continuous with respect to $\mathscr{R}$ and $\mathscr{S}$,
(2) $\big(F \circ \varphi ,\ G \circ \psi\big)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

Remark 67. Hence, it is clear that if in particular $X = Z$ and $Y = W$, then the following assertions are equivalent:

(1) $(\Delta_X ,\ \Delta_Y)$ is paratopologically mildly continuous with respect to $\mathscr{R}$ and $\mathscr{S}$,
(2) $(\varphi ,\ \psi)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

Moreover, from Corollary 37, by using Theorem 68, we can also easily derive

Theorem 91. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) $(F ,\ G)$ is paratopologically mildly continuous with respect to $\mathscr{R}$ and $\mathscr{S}$,
(2) $(\varphi ,\ \psi)$ is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $G^{-1} \circ \mathscr{S}^{\Delta} \circ F$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

Proof. By " the properly part" of Corollary 37, assertion (1) is equivalent the statement that:

(a) $(G \circ \psi)^{-1} \circ \mathscr{S}^{\Delta} \circ (F \circ \varphi) \subseteq \mathscr{R}^{\Delta}$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

However, by the corresponding properties of composition, we have

$$\psi^{-1} \circ \big(G^{-1} \circ \mathscr{S}^{\Delta} \circ F\big) \circ \varphi = (G \circ \psi)^{-1} \circ \mathscr{S}^{\Delta} \circ (F \circ \varphi)$$

for all $\varphi \in X^X$ and $\psi \in Y^Y$.

Therefore, assertion (a) is equivalent to the statement that:

(b) $\psi^{-1} \circ \big(G^{-1} \circ \mathscr{S}^{\Delta} \circ F\big) \circ \varphi \subseteq \mathscr{R}^{\Delta}$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

However, by the corresponding definitions, this means only that:

(c) $(\varphi ,\ \psi)$ is properly mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $G^{-1} \circ \mathscr{S}^{\Delta} \circ F$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

Moreover, we can note that $\mathscr{R}^{\Delta} = \big(\mathscr{R}^{\Delta}\big)^{\Diamond}$ for $\Diamond = *$ and #. Therefore, assertion (b) is equivalent to the statement that:

(d) $\psi^{-1} \circ \big(G^{-1} \circ \mathscr{S}^{\Delta} \circ F\big) \circ \varphi \subseteq \big(\mathscr{R}^{\Delta}\big)^{\Diamond}$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

However, by Theorem 68, assertion (d) is equivalent to the statement that:

(e) (φ, ψ) is mildly $\Diamond$–continuous with respect to $\mathscr{R}^{\Delta}$ and $G^{-1} \circ \mathscr{S}^{\Delta} \circ F$ for all $\varphi \in X^X$ and $\psi \in Y^Y$.

Hence, it is clear that assertions (1) and (2) are also equivalent.

Because of Corollary 37, it is also worth proving here the following analogue of [42, Theorem 6.1].

Theorem 92. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) (F, G) is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}$,
(2) $G^{-1} \circ \mathscr{S} \circ F \subseteq \mathscr{R}^{\Delta}$,
(3) $G^{-1}\big[S[F(x)] \big] \in \mathscr{E}_{\mathscr{R}}$ for all $x \in X$ and $S \in \mathscr{S}$,
(4) for each $x \in X$ and $S \in \mathscr{S}$ there exist $u \in X$ and $R \in \mathscr{R}$ such that $R(u) \subseteq G^{-1}\big[S[F(x)] \big]$,
(5) for each $x \in X$ and $S \in \mathscr{S}$ there exist $u \in X$ and $R \in \mathscr{R}$ such that for any $y \in R(u)$ we have $G(y) \cap S[F(x)] \neq \emptyset$,
(6) for each $x \in X$ and $S \in \mathscr{S}$ there exist $u \in X$ and $R \in \mathscr{R}$ such that for any $y \in R(u)$ there exist $z \in F(x)$ and $w \in G(y)$ such that $w \in S(z)$.

Proof. By Remark 47, it is clear that the "properly part" of (1) is equivalent to (2). Moreover, we can note that $\mathscr{R}^{\Delta} = \big(\mathscr{R}^{\Delta}\big)^{\Diamond}$ for $\Diamond = *$ and #. Therefore, by Theorem 68, "the uniformly or proximally parts" of (1) are also equivalent to (2).

Furthermore, if (1) holds, by the corresponding definitions, for each $S \in \mathscr{S}$, we have $G^{-1} \circ S \circ F \in \mathscr{R}^{\Delta}$. Thus, by the definition of $\mathscr{R}^{\Delta}$, for each $x \in X$ there exist $u \in X$ and $R \in \mathscr{R}$ such that $R(u) \subseteq \big(G^{-1} \circ S \circ F\big)(x)$, and thus also $R(u) \subseteq G^{-1}\big[S[F(x)] \big]$. Therefore, (4), and thus (3) also holds. Now, by using the latter argument, we can also easily see that (3) also implies (1).

Moreover, from the proofs of Theorems 78 and 80, it is clear that the equivalences (4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) are also true.

Remark 68. Note that, by Theorem 80, assertion (1) is, for instance, also equivalent to the statement that:

(3') for each $S \in \mathscr{S}$ there exists $U \in \mathscr{R}^{\Delta}$ such that $U \subseteq G^{-1} \circ \mathscr{S} \circ F$.

However, it is now, more important to note that, analogously to [42, Theorem 6.3], now we can now also prove the following

Theorem 93. *If f is a function on X onto Z and $G \in \mathscr{G}$, then under the addition assumption $\mathscr{S} \neq \emptyset$ the following assertions are equivalent:*

(1) (f, G) is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}$,
(2) (f, G) is properly, uniformly, or proximally mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$.

Proof. From Theorem 92, we can see that assertion (1) is equivalent to the statement that:

(a) for each $x \in X$ and $S \in \mathscr{S}$ there exist $u \in X$ and $R \in \mathscr{R}$ such that $R(u) \subseteq G^{-1}\big[\, S\, [f(x)\,]\,\big]$.

Moreover, assertion (2) is equivalent to the statement that:

(b) for each $x \in X$ and $V \in \mathscr{S}^{\Delta}$ there exist $u \in X$ and $R \in \mathscr{R}$ such that for $R(u) \subseteq G^{-1}\big[\, V\, [f(x)\,]\,\big]$.

Hence, since $\mathscr{S} \subseteq \mathscr{S}^{\Delta}$, it is clear that (b) implies (a), and thus (2) implies (1). Therefore, we need only show that (a) also implies (b), and thus (1) also implies (2) .

For this, assume that (a) holds, and $x \in X$ and $V \in \mathscr{S}^{\Delta}$. Now, if in particular $x \in f^{-1}[Z]$, then since f is a function we can see that $f(x) \in Z$. Therefore, by the definition of $\mathscr{S}^{\Delta}$, there exists $z \in Z$ and $S \in \mathscr{S}$ such that $S(z) \subseteq V\big(f(x)\big)$. Moreover, since $Z = f[X]$, there exists $t \in X$ such that $z = f(t)$. Thus, we also have $S\big(f(t)\big) \subseteq V\big(f(x)\big)$. Hence, we can see that $S\,[f(t)\,] \subseteq V\,[f(x)\,]$, and thus also $G^{-1}\big[\, S\,[f(t)\,]\,\big] \subseteq G^{-1}\big[\, V\,[f(x)\,]\,\big]$. Moreover, by using (a), we can see that there exist $u \in X$ and $R \in \mathscr{R}$ such that $R(u) \subseteq G^{-1}\big[\, S\,[f(t)\,]\,\big]$. Thus, we also have $R(u) \subseteq G^{-1}\big[\, V\,[f(t)]\,\big]$.

On the other hand, if $x \in X \setminus f^{-1}[Z]$, then by taking $S \in \mathscr{S}$ and using (a) we can see that there exist $u \in X$ and $R \in \mathscr{R}$ such that $R(u) \subseteq G^{-1}[\, S\,[f(x)\,]\,\big]$. Hence, since $f(x) = \emptyset$, we can infer that $R(u) \subseteq \emptyset$. Therefore, $R(u) \subseteq G^{-1}\big[\, V\,[f(x)\,]\,\big]$ also holds.

18 Characterizations of Proximal Mild Continuity

The subsequent theorems have also been mainly taken from [31, 34].

Theorem 94. *For any $F \in \mathscr{F}$ and $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) $(F,\, G)$ *is proximally mildly continuous ,*

(2) $A \in \operatorname{Cl}_{\mathscr{R}}(B)$ *implies* $F\,[A\,] \in \operatorname{Cl}_{\mathscr{S}}\,(G\,[B\,])$ *for all* $A \subseteq X$ *and* $B \subseteq Y$,

(3) $F\,[A\,] \in \operatorname{Int}_{\mathscr{S}}\big(D\big)$ *implies* $A \in \operatorname{Int}_{\mathscr{R}}\big(G^{-1}\,[D\,]\big)$ *for all* $A \subseteq X$ *and* $D \subseteq W$.

Proof. Define $\mathscr{U} = G^{-1} \circ \mathscr{S} \circ F$. Then, by Theorems 81 and 16, we have

$$(1) \iff \mathscr{U} \subseteq \mathscr{R}^{\#} \iff \operatorname{Cl}_{\mathscr{R}} \subseteq \operatorname{Cl}_{\mathscr{U}}$$
$$\iff \big(A \in \operatorname{Cl}_{\mathscr{R}}(B) \implies A \in \operatorname{Cl}_{\mathscr{U}}(B)\big).$$

Therefore, to prove the equivalence of (1) and (2), we need only show that, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$A \in \operatorname{Cl}_{\mathscr{U}}(B) \iff F\,[A\,] \in \operatorname{Cl}_{\mathscr{S}}\,(G\,[B\,])\ .$$

For this, by the corresponding definitions and some basic theorems on relations, it is enough to note only that, for any $S \in \mathscr{S}$, we have

$$\big(G^{-1} \circ S \circ F\big)\,[A] \cap B \neq \emptyset$$
$$\iff \quad G^{-1}\big[S[F[A]]\big] \cap B \neq \emptyset \quad \iff \quad S\big[F[A]\big] \cap G[B] \neq \emptyset\,.$$

Finally, to complete the proof, we note that the equivalence of (2) and (3) can be proved by using the relationships between the structures Cl and Int and the inclusions

$$G^{-1}\big[G[B]^c\big] \subseteq B^c \qquad \text{and} \qquad G\big[G^{-1}[D]^c\big] \subseteq D^c$$

with $B \subseteq Y$ and $D \subseteq W$. (To check the latter one, instead of a direct proof, one can note that $G^{-1}[D]^c = \mathrm{cl}_G(D)^c = \mathrm{int}_G(D^c)$ and $G[\mathrm{int}_G(D^c)] \subseteq D^c$.)

Namely, if (2) holds, then, by the abovementioned results, it is clear that for any $A \subseteq X$ and $D \subseteq W$

$$F[A] \in \mathrm{Int}_{\mathscr{S}}(D) \implies F[A] \notin \mathrm{Cl}_{\mathscr{S}}(D^c) \implies F[A] \notin \mathrm{Cl}_{\mathscr{S}}\big(G\big[G^{-1}[D]^c\big]\big)$$
$$\implies A \notin \mathrm{Cl}_{\mathscr{R}}\big(G^{-1}[D]^c\big) \implies A \in \mathrm{Int}_{\mathscr{R}}\big(G^{-1}[D]\big)\,.$$

Thus, (3) also holds.

While, if (3) holds, then again by the above mentioned results, it is clear that for any $A \subseteq X$ and $B \subseteq Y$

$$F[A] \notin \mathrm{Cl}_{\mathscr{S}}\big(G[B]\big) \implies F[A] \in \mathrm{Int}_{\mathscr{S}}\big(G[B]^c\big)$$
$$\implies A \in \mathrm{Int}_{\mathscr{R}}\big(G^{-1}\big[G[B^c]\big]\big)$$
$$\implies A \in \mathrm{Int}_{\mathscr{R}}\big(B^c\big) \implies A \notin \mathrm{Cl}_{\mathscr{R}}(B)\,.$$

Therefore, (2) also holds.

Now, by using the above theorem, we can also easily prove the following.

Theorem 95. *If $F \in \mathscr{F}$ and $g \in \mathscr{G}$ such that g is a function and $Y = g^{-1}[W]$, then the following assertions are equivalent:*

(1) (F, g) is proximally mildly continuous,
(2) $A \in \mathrm{Cl}_{\mathscr{R}}\big(g^{-1}[D]\big)$ implies $F[A] \in \mathrm{Cl}_{\mathscr{S}}(D)$ for all $A \subseteq X$ and $D \subseteq W$.

Proof. If (1) holds and $A \subseteq X$ and $D \subseteq W$, then by Theorem 89 and the inclusion $g\big[g^{-1}[D]\big] \subseteq D$ it is clear that

$$A \in \mathrm{Cl}_{\mathscr{R}}\big(g^{-1}[D]\big) \implies F[A] \in \mathrm{Cl}_{\mathscr{S}}\big(g\big[g^{-1}[D]\big]\big) \implies F[A] \in \mathrm{Cl}_{\mathscr{S}}(D)\,.$$

Therefore, (2) also holds.

While, if (2) holds, and $A \subseteq X$ and $B \subseteq Y$, then by using the inclusion $B \subseteq g^{-1}\big[\, g\,[\, B\,]\,\big]$ and assertion (2) we can see that

$$A \in \mathrm{Cl}_{\mathscr{R}}(B) \implies A \in \mathrm{Cl}_{\mathscr{R}}\left(g^{-1}\big[\, g\,[\, B\,]\,\big]\right) \implies F\,[\, A\,] \in \mathrm{Cl}_{\mathscr{S}}\big(g\,[\, B\,]\big)\,.$$

Therefore, by Theorem 94, assertion (1) also holds.

Remark 69. More exactly, we can also state that (1) implies (2) if g is a function, and (2) implies (1) if $Y = g^{-1}\,[\, W\,]$.

Moreover, by using Theorem 94, we can also easily prove the following two theorems.

Theorem 96. *If $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that G^{-1} is a function and $G\,[\, Y\,] = W$, then the following assertions are equivalent:*

(1) $(F\,,\, G)$ *is proximally mildly continuous ,*
(2) $F\,[\, A\,] \in \mathrm{Int}_{\mathscr{S}}\,(G\,[\, B\,]\,)$ *implies* $A \in Int_{\mathscr{R}}(B)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$.

Remark 70. More exactly, we can also state that (1) implies (2) if G^{-1} is a function, and (2) implies (1) if $G\,[\, Y\,] = W$.

Theorem 97. *If $f \in \mathscr{F}$ and $G \in \mathscr{G}$ such that f is a function and $X = f^{-1}[\, Z\,]$, then the following assertions are equivalent:*

(1) $(f\,,\, G)$ *is proximally mildly continuous ,*
(2) $C \in \mathrm{Int}_{\mathscr{S}}(D)$ *implies* $f^{-1}\,[\, C\,] \in \mathrm{Int}_{\mathscr{R}}\left(G^{-1}\,[\, D\,]\,\right)$ *for all* $C \subseteq Z$ *and* $D \subseteq W$.

Remark 71. More exactly, we can also state that (1) implies (2) if f is a function, and (2) implies (1) if $X = f^{-1}\,[\, Y\,]$.

Now, by using Theorem 97, we can also easily prove the following.

Theorem 98. *If $f \in \mathscr{F}$ and $g \in \mathscr{G}$ such that f and g are functions and $X = f^{-1}\,[\, Z\,]$ and $Y = g^{-1}\,[\, W\,]$, then the following assertions are equivalent:*

(1) $(f\,,\, g)$ *is proximally mildly continuous ,*
(2) $f^{-1}\,[\, C\,] \in \mathrm{Cl}_{\mathscr{R}}\left(g^{-1}\,[\, D\,]\right)$ *implies* $C \in \mathrm{Cl}_{\mathscr{S}}(D)$ *for all* $C \subseteq Z$ *and* $D \subseteq W$.

Remark 72. More exactly, we can also state that (1) implies (2) if f and g are functions, and (2) implies (1) if $X = f^{-1}\,[\, Z\,]$ and $Y = g^{-1}\,[\, W\,]$.

Moreover, by using Theorems 96 and 97, we can also easily prove

Theorem 99. *If $f \in \mathscr{F}$ and $G \in \mathscr{G}$ such that f and G^{-1} are functions and $X = f^{-1}[\, Z\,]$ and $G\,[\, Y\,] = W$, then the following assertions are equivalent:*

(1) $(f\,,\, G)$ *is proximally mildly continuous ,*
(2) $C \in \mathrm{Int}_{\mathscr{S}}\,(G\,[\, B\,]\,)$ *implies* $f^{-1}\,[\, C\,] \in \mathrm{Int}_{\mathscr{R}}(B)$ *for all* $B \subseteq Y$ *and* $C \subseteq Z$.

Remark 73. More exactly, we can also state that (1) implies (2) if f and G^{-1} are functions, and (2) implies (1) if $X = f^{-1}\,[\, Z\,]$ and $G\,[\, Y\,] = W$.

19 Characterizations of Topological Mild Continuity

From the results of Sect. 18, by using Theorems 84 and 18, we can easily derive several criteria for topological mild continuity.

However, for this, we have to assume tacitly throughout this section that the relators $\mathscr{R}$ and $\mathscr{S}$, considered in Notation 1, are nonvoid.

Theorem 100. *If $F \in \mathscr{F}$ and $G \in \mathscr{G}$, then the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous*,
(2) $F[A] \subseteq \operatorname{int}_{\mathscr{S}}(D)$ *implies* $A \subseteq \operatorname{int}_{\mathscr{R}}\big(G^{-1}[D]\big)$ *for all* $A \subseteq X$ *and* $D \subseteq W$,
(3) $A \cap \operatorname{cl}_{\mathscr{R}}(B) \neq \emptyset$ *implies* $F[A] \cap \operatorname{cl}_{\mathscr{S}}(G[B]) \neq \emptyset$ *for all* $A \subseteq X$ *and* $B \subseteq Y$.

Proof. From Theorem 84, we can see that assertion (1) is equivalent to the statement that:

(a) (F, G) is proximally mildly continuous with respect to the relators $\mathscr{R}^\wedge$ and $\mathscr{S}^\wedge$.
   Moreover, from Theorem 94, we can see that assertion (a) is equivalent to the statement that:
(b) $F[A] \in \operatorname{Int}_{\mathscr{S}^\wedge}(D)$ implies $A \in \operatorname{Int}_{\mathscr{R}^\wedge}\big(G^{-1}[D]\big)$ for all $A \subseteq X$ and $D \subseteq W$.
   Furthermore, from Theorem 18, we can see that assertion (b) is equivalent to assertion (2).

Finally, we note that the equivalence of assertions (2) and (3) can be proved by using the relationship between the structures cl and int.

Remark 74. Note that if (1) holds, then by Theorem 85 we have

$$G^{-1} \circ \mathscr{S}^\wedge \circ F \subseteq \mathscr{R}^\wedge .$$

Moreover, if, for instance, $\mathscr{R} = \emptyset$, but $X \neq \emptyset$, then by the definition of $\mathscr{R}^\wedge$ we have $\mathscr{R}^\wedge = \emptyset$. Therefore, we necessarily have $\mathscr{S}^\wedge = \emptyset$, and thus also $\mathscr{S} = \emptyset$.

Hence, by the definition of int, we can see that $\operatorname{int}_{\mathscr{S}}(D) = \emptyset$ and quite similarly $\operatorname{int}_{\mathscr{R}}\big(G^{-1}[D]\big) = \emptyset$ for all $D \subseteq W$. Therefore, assertion (2) fails to hold if $X \neq F^{-1}[Z]$.

However, it is now more important to note that the above theorem can be reformulated in the following more concise form.

Corollary 38. *Under the conditions of Theorem 100, the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous*,
(2) $\operatorname{cl}_{\mathscr{R}}(B) \subseteq F^{-1}\big[\operatorname{cl}_{\mathscr{S}}(G[B])\big]$ *for all* $B \subseteq Y$,
(3) $\operatorname{int}_{\mathscr{R}}\big(G^{-1}[D]\big)^c \subseteq F^{-1}\big[\operatorname{int}_{\mathscr{S}}(D)^c\big]$ *for all* $D \subseteq W$.

Proof. By using that $F[A] = \bigcup_{x \in A} F(x)$ for all $A \subseteq X$, we can see that assertion (2) of Theorem 100 is equivalent to the statement that:

(a) $F(x) \subseteq \operatorname{int}_{\mathscr{S}}(D)$ implies $x \in \operatorname{int}_{\mathscr{R}}\left(G^{-1}[D]\right)$ for all $x \in X$ and $D \subseteq W$.
Moreover, we can note that, for any $x \in X$, we have

$$F(x) \subseteq \operatorname{int}_{\mathscr{S}}(D) \iff F(x) \cap \operatorname{int}_{\mathscr{S}}(D)^c = \emptyset$$
$$\iff x \notin F^{-1}\left[\operatorname{int}_{\mathscr{S}}(D)^c\right] \iff x \in F^{-1}\left[\operatorname{int}_{\mathscr{S}}(D)^c\right]^c.$$

Therefore, the assertion (a) is equivalent to the statement that:
(b) $F^{-1}\left[\operatorname{int}_{\mathscr{S}}(D)^c\right]^c \subseteq \operatorname{int}_{\mathscr{R}}\left(G^{-1}[D]\right)$ for all $D \subseteq W$.

And, this is evidently equivalent to the assertion (3) of the present corollary.

Now analogously to Theorem 100 and Corollary 38, we can also easily prove the following theorems.

Theorem 101. *If $F \in \mathscr{F}$ and $g \in \mathscr{G}$ such that g is a function and $Y = g^{-1}[W]$, then the following assertions are equivalent:*

(1) (F, g) *is topologically mildly continuous,*
(2) $\operatorname{cl}_{\mathscr{R}}\left(g^{-1}[D]\right) \subseteq F^{-1}\left[\operatorname{cl}_{\mathscr{S}}(D)\right]$ *for all* $D \subset W$,
(3) $A \cap \operatorname{cl}_{\mathscr{R}}\left(g^{-1}[D]\right) \neq \emptyset$ *implies* $F[A] \cap \operatorname{cl}_{\mathscr{S}}(D) \neq \emptyset$ *for all* $A \subseteq X$ *and* $D \subseteq W$.

Theorem 102. *If $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that G^{-1} is a function and $G[Y] = W$, then the following assertions are equivalent:*

(1) (F, G) *is topologically mildly continuous,*
(2) $\operatorname{int}_{\mathscr{R}}(B)^c \subseteq F^{-1}\left[\operatorname{int}_{\mathscr{S}}(G[B])^c\right]$ *for all* $B \subseteq Y$,
(3) $F[A] \subseteq \operatorname{int}_{\mathscr{S}}(G[B])$ *implies* $A \subseteq \operatorname{int}_{\mathscr{R}}(B)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$.

Theorem 103. *If $f \in \mathscr{F}$ and $G \in \mathscr{G}$ such that f is a function and $X = f^{-1}[Z]$, then the following assertions are equivalent:*

(1) (f, G) *is topologically mildly continuous,*
(2) $f^{-1}[\operatorname{int}_{\mathscr{S}}(D)] \subseteq \operatorname{int}_{\mathscr{R}}\left(G^{-1}[D]\right)$ *for all* $D \subseteq W$,
(3) $C \subseteq \operatorname{int}_{\mathscr{S}}(D)$ *implies* $f^{-1}[C] \subseteq \operatorname{int}_{\mathscr{R}}\left(G^{-1}[D]\right)$ *for all* $C \subseteq Z$ *and* $D \subseteq W$.

Theorem 104. *If $f \in \mathscr{F}$ and $g \in \mathscr{G}$ such that f and g are functions and $f^{-1}[Z] = X$ and $Y = g^{-1}[W]$, then the following assertions are equivalent:*

(1) (f, g) *is topologically mildly continuous,*
(2) $f\left[\operatorname{Cl}_{\mathscr{R}}\left(g^{-1}[D]\right)\right] \subseteq \operatorname{Cl}_{\mathscr{S}}(D)$ *for all* $D \subseteq W$,
(3) $f^{-1}[C] \cap \operatorname{Cl}_{\mathscr{R}}\left(g^{-1}[D]\right) \neq \emptyset$ *implies* $C \subseteq \operatorname{Cl}_{\mathscr{S}}(D)$ *for all* $C \subseteq Z$ *and* $D \subseteq W$.

Theorem 105. *If $f \in \mathscr{F}$ and $G \in \mathscr{G}$ such that f and G^{-1} are functions and $X = f^{-1}[Z]$ and $G[Y] = W$, then the following assertions are equivalent:*

(1) (f, G) *is topologically mildly continuous,*
(2) $f^{-1}[\operatorname{int}_{\mathscr{S}}(G[B])] \subseteq \operatorname{int}_{\mathscr{R}}(B)$ *for all* $B \subseteq Y$,
(3) $C \subseteq \operatorname{int}_{\mathscr{S}}(G[B])$ *implies* $f^{-1}[C]) \subseteq \operatorname{int}_{\mathscr{R}}(B)$ *for all* $C \subseteq Z$ *and* $B \subseteq Y$.

20 Fatness and Denseness Preserving and Reversing Relations

The following definition has been first introduced in [42].

Definition 18. For any $G \in \mathscr{G}$, we say that the relation

(1) G is *fatness preserving*, with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, if $E \in \mathscr{E}_{\mathscr{R}}$ implies $G[E] \in \mathscr{E}_{\mathscr{S}}$,
(2) G is *denseness preserving*, with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, if $D \in \mathscr{D}_{\mathscr{R}}$ implies $G[D] \in \mathscr{D}_{\mathscr{S}}$.

Remark 75. Recall that, by the corresponding definitions and some basic theorems on fat and dense sets, we have

(1) $\mathscr{D}_{\mathscr{R}} = \{D \subseteq Y : \quad \forall\ R \in \mathscr{R} : \quad X = R^{-1}[D]\}$,
(2) $\mathscr{E}_{\mathscr{R}} = \{E \subseteq Y : \quad \exists\ x \in X : \quad \exists\ R \in \mathscr{R} : \quad R(x) \subseteq E\}$,
(3) $\mathscr{D}_{\mathscr{R}} = \{D \subseteq Y : \quad D^c \notin \mathscr{E}_{\mathscr{R}}\} = \{D \subseteq Y : \quad \forall\ E \in \mathscr{E}_{\mathscr{R}} : \quad D \cap E \neq \emptyset\}$,
(4) $\mathscr{E}_{\mathscr{R}} = \{E \subseteq Y : \quad E^c \notin \mathscr{D}_{\mathscr{R}}\} = \{E \subseteq Y : \quad \forall\ D \in \mathscr{D}_{\mathscr{R}} : \quad D \cap E \neq \emptyset\}$.

Now, by using Definition 18 and Remark 75, we can easily prove the following theorem of [42].

Theorem 106. *For any $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) G *is denseness preserving,*
(2) G^{-1} *is fatness preserving.*

Proof. Suppose that (1) holds and $E \in \mathscr{E}_{\mathscr{S}}$. Then, by Definition 18, for any $D \in \mathscr{D}_{\mathscr{R}}$ we have $G[D] \in \mathscr{D}_{\mathscr{S}}$. Moreover, by Remark 75, we can state that $E \cap G[D] \neq \emptyset$, and thus $G^{-1}(E) \cap D \neq \emptyset$. Hence, by Remark 75, we can already infer that $G^{-1}[E] \in \mathscr{E}_{\mathscr{R}}$. Thus, (2) also holds.

The converse implication (2) $\Longrightarrow$ (1) can be proved quite similarly.

Moreover, by using this theorem, we can also easily prove the following reformulation of [51, Theorem 11.3].

Theorem 107. *For any $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) G is denseness preserving ,

(2) $G^{-1}[S(z)] \in \mathscr{E}_{\mathscr{R}}$ for all $z \in Z$ and $S \in \mathscr{S}$,

(3) for any $z \in Z$ and $S \in \mathscr{S}$ there exist $x \in X$ and $R \in \mathscr{R}$ such that $R(x) \subseteq G^{-1}[S(z)]$,

(4) for any $z \in Z$ and $S \in \mathscr{S}$ there exist $x \in X$ and $R \in \mathscr{R}$ such that for any $y \in R(x)$ we have $G(y) \cap S(z) \neq \emptyset$.

Proof. If $z \in Z$ and $S \in \mathscr{S}$, then $S(z) \in \mathscr{E}_{\mathscr{S}}$. Hence, if (1) holds, by using Theorem 106, we can infer that $G^{-1}[S(z)] \in \mathscr{E}_{\mathscr{R}}$. Therefore, there exist $x \in X$ and $R \in \mathscr{R}$ such that $R(x) \subseteq G^{-1}[S(z)]$. Thus, for any $y \in R(x)$, we have $y \in G^{-1}[S(z)]$, and so $G(y) \cap S(z) \neq \emptyset$.

Hence, it is clear that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). The converse implications can be proved quite similarly.

From the above theorem, it is clear that more specially we also have

Corollary 39. *If in particular $X = Z$, then the following assertions are equivalent:*

(1) G is denseness preserving with respect to $\mathscr{R}$ and $\mathscr{S}$,

(2) (Δ_X, G) is properly mildly continuous with respect to $\mathscr{R}^{\Delta}$ and $\mathscr{S}$.

Proof. To check this, note that now, by Remark 10, the assertion (2) of Theorem 107 can be written in the shorter form that $G^{-1} \circ \mathscr{S} \circ \Delta_X = G^{-1} \circ \mathscr{S} \subseteq \mathscr{R}^{\Delta}$.

However, it now more important to note that, by using Theorems 106 and 92, we can also easily prove the following improvement of [42, Theorems 6.5].

Theorem 108. *If $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that F is total and G is denseness preserving, then the pair (F, G) is properly, uniformly, and proximally mildly continuous with respect to the relators $\mathscr{R}^{\Delta}$ and $\mathscr{S}$.*

Proof. If $x \in X$ and $S \in \mathscr{S}$, then since $F(x) \neq \emptyset$ we have $S[F(x)] \in \mathscr{E}_{\mathscr{S}}$. Hence, by using Theorem 106, we can infer that $G^{-1}\big[S[F(x)]\big] \in \mathscr{E}_{\mathscr{R}}$. Therefore, by Theorem 92, the required assertion is also true.

From this theorem, by using Theorem 93, we can immediately derive

Corollary 40. *If f is a function of X onto Z and $G \in \mathscr{G}$ such that G is denseness preserving, and $\mathscr{S} \neq \emptyset$, then the pair (f, G) is properly, uniformly, and proximally mildly continuous with respect to the relators $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$.*

Moreover, by using Theorems 106 and 92, we can also easily prove the following improvement of [42, Theorems 6.4].

Theorem 109. *If $G \in \mathscr{G}$ such that there exists a function f on X onto Z such that the pair (f, G) is properly, uniformly, or proximally mildly continuous with respect to the relators $\mathscr{R}^{\Delta}$ and $\mathscr{S}$, then G is denseness preserving.*

Proof. By Theorem 106, it suffices to show that G^{-1} is fatness preserving. For this, note that if $E \in \mathcal{E}_{\mathcal{S}}$, then by Remark 75 there exist $z \in Z$ and $S \in \mathcal{S}$ such that $S(z) \subseteq E$. Moreover, since $Z = f[X]$, there exists $x \in X$ such that $z = f(x)$. Therefore, we also have $S\big(f(x)\big) \subseteq E$, and hence $G^{-1}\big[S\big(f(x)\big)\big] \subseteq G^{-1}[E]$. Moreover, by Theorem 92, we have $G^{-1}\big[S\big(f(x)\big)\big] \in \mathcal{E}_{\mathcal{R}}$. Therefore, $G^{-1}[E] \in \mathcal{E}_{\mathcal{R}}$ also holds.

Now, as an immediate consequence of Theorems 108 and 109, we can also state

Corollary 41. *For any function f of X onto Y and $G \in \mathcal{G}$, the following assertions are equivalent:*

(1) G is denseness preserving,

(2) (f, G) is properly, uniformly, or proximally mildly continuous with respect to $\mathcal{R}^{\Delta}$ and $\mathcal{S}$.

Moreover, in addition Theorem 109, we can also prove the following improvement of [26, Theorem 9.17].

Theorem 110. *If $G \in \mathcal{G}$ such that there exists $F \in \mathcal{F}$ such that the pair (F, G) is properly, uniformly, or proximally mildly continuous with respect to the relators $\mathcal{R}^{\Delta}$ and $\mathcal{S}^{\Delta}$ and moreover either $F \neq \emptyset$ or $X \neq \emptyset$ and $\mathcal{R}$ is total, then G is denseness preserving.*

Proof. By Theorem 106, it suffices to show that G^{-1} is fatness preserving. For this, note that if $E \in \mathcal{E}_{\mathcal{S}}$, then by Remark 10 the relation $V = X \times E$ is in $\mathcal{S}^{\Delta}$. Therefore, if the pair (F, G) is properly mildly continuous with respect to the relators $\mathcal{R}^{\Delta}$ and $\mathcal{S}^{\Delta}$, then the relation $U = G^{-1} \circ V \circ F$ is in $\mathcal{R}^{\Delta}$. Thus, by Remark 10, for any $x \in X$, we have $U(x) \in \mathcal{E}_{\mathcal{R}}$.

Moreover, we can note that $U(x) = G^{-1}\big[\,V\,[\,F(x)\,]\,\big]$, and thus

$$U(x) = G^{-1}\,[E] \quad \text{if} \quad x \in D_F \qquad \text{and} \qquad U(x) = \emptyset \quad \text{if} \quad x \in D_F^c.$$

Hence, if $F \neq \emptyset$, and thus $D_F \neq \emptyset$, by choosing $x \in D_F$, we can see that $G^{-1}[E] = U(x) \in \mathcal{E}_{\mathcal{R}}$. If $\mathcal{R}$ is total, then by Remark 16 we have $\emptyset \notin \mathcal{E}_R$. Hence, we can see that $D_F = X$, and thus F is also total. Therefore, $D_F \neq \emptyset$, and thus $F \neq \emptyset$, whenever $X \neq \emptyset$ also holds. Thus, by the former case, $G^{-1}[E] \in \mathcal{E}_{\mathcal{R}}$ again holds.

Now, because of Corollary 40 and Theorem 110, we can also state

Corollary 42. *If $X \neq \emptyset$ and $\mathcal{S} \neq \emptyset$, then for any function f of X onto Y and $G \in \mathcal{G}$ the following assertions are equivalent:*

(1) G is denseness preserving ,

(2) (f, G) is properly, uniformly, or proximally mildly continuous with respect to the relators $\mathcal{R}^{\Delta}$ and $\mathcal{S}^{\Delta}$.

Moreover, by using Theorem 110 and Corollary 37, we can also easily prove

Theorem 111. *If $G \in \mathscr{G}$ such that there exists $F \in \mathscr{F}$ such that the pair (F, G) is paratopologically mildly continuous, and moreover $X \neq \emptyset$ and $\mathscr{R}$ is total, then the relation $G \circ \psi$ is denseness preserving for all $\psi \in Y^Y$.*

Proof. Now, by Corollary 37, the pair $\big(F \circ \varphi\,,\ G \circ \psi\big)$ is properly mildly continuous with respect to the relators $\mathscr{R}^{\Delta}$ and $\mathscr{S}^{\Delta}$ for all $\varphi \in X^X$ and $\psi \in Y^Y$. Hence, by Theorem 110, we can already see that the required assertion is also true.

Remark 76. To see the usefulness of denseness preserving relations, we can also note that if $\mathscr{R}$ is total, then $Y \in \mathscr{D}_{\mathscr{R}}$. Therefore, if $G \in \mathscr{G}$ such that G is denseness preserving, then $G[\,Y\,] \in \mathscr{E}_{\mathscr{S}}$.

Hence, we can infer that $W \in \mathscr{D}_{\mathscr{S}}$, and thus $\mathscr{S}$ is also total. Moreover, if in particular $Z = W$ and $G[\,Y\,] \in \mathscr{F}_{\mathscr{S}}$, then $W = Z = \mathrm{cl}_{\mathscr{S}}\big(G[\,Y\,]\big) \subseteq G[\,Y\,] \subseteq W$, and thus $G[\,Y\,] = W$ also holds.

In [42], having in mind the definition of contra continuous functions [5, 23], the first author also introduced following

Definition 19. For any $G \in \mathscr{G}$, we say that the relation

(1) G is *fatness reversing*, with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, if $E \in \mathscr{E}_{\mathscr{R}}$ implies $G[\,E\,] \in \mathscr{D}_{\mathscr{S}}$,
(2) G is *denseness reversing*, with respect to the relators $\mathscr{R}$ and $\mathscr{S}$, if $D \in \mathscr{D}_{\mathscr{R}}$ implies $G[\,D\,] \in \mathscr{E}_{\mathscr{S}}$.

Now, by using some similar arguments as above, we can also easily prove the following two theorems of [42].

Theorem 112. *For any $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) G is fatness (denseness) reversing,
(2) G^{-1} is fatness (denseness) reversing.

Remark 77. Note that the implication (2) $\Longrightarrow$ (1) can now be derived from the converse implication by using the fact that $G = \big(G^{-1}\big)^{-1}$.

Moreover, it is also worth noticing that "the fatness reversing part" of the above theorem can be more easily proved with the help of the following theorem.

Theorem 113. *Under the notation $F = X \times Z$, for any $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) G is fatness reversing,
(2) $\mathscr{S}^{-1} \circ \{G\} \circ \mathscr{R} \subseteq \{F\}$,
(3) $Z = S^{-1}\big[\,G\big[\,R(x)\,\big]\,\big]$ for all $x \in X$, $R \in \mathscr{R}$, and $S \in \mathscr{S}$.

Proof. If $x \in X$ and $R \in \mathscr{R}$, then by Remark 75 we have $R(x) \in \mathscr{E}_{\mathscr{R}}$. Hence, if assertion (1) holds, we can infer that $G[\,R(x)\,] \in \mathscr{D}_{\mathscr{S}}$. Therefore, by Remark 75, we have $Z = S^{-1}\big[\,G\big[\,R(x)\,\big]\,\big]$ for all $S \in \mathscr{S}$, and thus assertion (3) also holds.

While, if $E \in \mathscr{E}_{\mathscr{R}}$, then by Remark 75, there exist $x \in X$ and $R \in \mathscr{R}$ such that $R(x) \subseteq E$, and thus $S^{-1}\big[\, G\,[\,R(x)\,]\,\big] \subseteq S^{-1}\big[\, G\,[\,E\,]\,\big]$ for all $S \in \mathscr{S}$. Hence, if assertion (3) holds, we can infer that $Z = S^{-1}\big[\, G\,[\,E\,]\,\big]$ for all $S \in \mathscr{S}$. Therefore, by Remark 75, we also have $G\,[\,E\,] \in \mathscr{D}_{\mathscr{S}}$, and thus assertion (1) also holds.

The equivalence of assertions (2) and (3) is immediate from the corresponding definitions.

Remark 78. Note that if in particular the relators $\mathscr{R}$ and $\mathscr{S}$ are nonvoid, then instead of (2) we may also write that $\{F\} = \mathscr{S}^{-1} \circ \{G\} \circ \mathscr{R}$.

Moreover, it is also worth noticing that

$$\mathrm{cl}_S\big(G\,[\,R(x)\,]\big) = S^{-1}\big[\, G\,\big[\,R(x)\,\big]\,\big] = \big(S^{-1} \circ G \circ R\big)(x) = \mathrm{cl}_{R \boxtimes S}\big(\{G\}\big)(x)$$

for all $x \in X$, $R \in \mathscr{R}$ and $S \in \mathscr{S}$.

Now, as a detailed reformulation of Theorem 113, we can also state

Corollary 43. *For any $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) G is fatness reversing,
(2) $S(z) \cap G\,[\,R(x)\,] \neq \emptyset$ for all $x \in X$, $z \in Z$, $R \in \mathscr{R}$ and $S \in \mathscr{S}$,
(3) for any $x \in X$, $z \in Z$, $R \in \mathscr{R}$, and $S \in \mathscr{S}$, there exist $y \in R(x)$ and $w \in S(z)$ such that $w \in G(y)$.

However, it is now more important to note that, as an immediate consequence of Theorems 50 and 68, and the fact that, under the notation $F = X \times Z$, we have $\{F\} = \{F\}^{\lozenge}$ for any stable unary operation $\lozenge$ for relators, we can also state

Theorem 114. *If in particular $\square$ is a stable, inversion and composition compatible closure operation, then under the notation $F = X \times Z$, for any $G \in \mathscr{G}$, the following assertions are equivalent:*

(1) G is fatness reversing with respect to $\mathscr{R}$ and $\mathscr{S}$,
(2) $(\mathscr{R}, \mathscr{S})$ is mildly $\square$–continuous with respect to $\{F\}$ and $\{G\}$.

Remark 79. Note that now, under the notation $F = X \times Z$, for any $R \in \mathscr{R}$ and $S \in \mathscr{S}$, we have to consider the diagram:

$$\begin{CD} X @>R>> Y \\ @VFVV @VVGV \\ Z @>S>> W \end{CD}$$

Remark 80. In addition to the fatness and denseness preserving and reversing relations, the proximal and topological openness and closedness preserving and reversing relations should also be investigated.

However, by the results of [51], these relations are rather connected with the proximally and topologically lower and upper continuous relations than with the proximally and topologically mildly continuous ones.

Surprisingly enough, in the context of topological spaces and their obvious generalizations, functions and relations with topological openness and closedness reversing inverses, under the name "contra-continuous functions and upper and lower contra-continuous multifunctions," have also been intensively investigated by a great number of mathematicians. (See, for instance, [1, 5, 8, 12, 18, 23].)

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Contraction Maps in Pseudometric Structures

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Abstract In Sect. 1, an extension to semigroup couple metric spaces is given for the fixed point result in Matkowski (Diss Math 127:1–68, 1975). In Sect. 2, we show that the simulation-type contractive maps in quasi-metric spaces introduced by Alsulami et al. (Discrete Dyn Nat Soc 2014, Article ID 269286, 2014) are in fact Meir–Keeler maps. Finally, in Sect. 3, the Brezis–Browder ordering principle (Adv Math 21:355–364, 1976) is used to get a proof, in the reduced axiomatic system (ZF-AC+DC), of a fixed point result [in the complete axiomatic system (ZF)] over Cantor complete ultrametric spaces due to Petalas and Vidalis (Proc Am Math Soc 118:819–821, 1993). The methodological approach we chose consisted in treating each section from a self-contained perspective; so, ultimately, these are independent units of the present exposition.

1 Matkowski-Type Contractions in Semigroup Couple Metric Spaces

1.1 Introduction

Let X be a nonempty set. Call the subset Y of X, *almost singleton* (in short: *asingleton*), provided [$y_1, y_2 \in Y$ implies $y_1 = y_2$], and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$. Further, let $d : X \times X \to R_+ := [0, \infty[$ be a *metric* over it; the couple (X, d) will be termed a *metric space*. Finally, let $T \in \mathscr{F}(X)$ be a *selfmap* of X. [Here, for each couple A, B of nonempty sets, $\mathscr{F}(A, B)$ stands for the class of all *functions* from A to B; when $A = B$, we write $\mathscr{F}(A)$ in place of $\mathscr{F}(A, A)$.] Denote $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. In the metrical fixed point theory, such points are to be determined according to the context below, comparable with the one in Rus [59, Chap. 2, Sect. 2.2]:

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(pic-1) We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent; and a *globally Picard operator* (modulo d) if, in addition, $\mathrm{Fix}(T)$ is an asingleton

(pic-2) We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent with $\lim_n (T^n x) \in \mathrm{Fix}(T)$; and a *globally strong Picard operator* (modulo d) if, in addition, $\mathrm{Fix}(T)$ is an asingleton (hence, a singleton).

Sufficient conditions for such properties will be stated in the class of "functional" metric contractions. Call T, $(d; \varphi)$*-contractive* (for some $\varphi \in \mathscr{F}(R_+)$), when

(a01) $d(Tx, Ty) \leq \varphi(d(x, y))$, for all $x, y \in X$.

The functions to be considered here are as follows. Let us say that $\varphi \in \mathscr{F}(R_+)$ is *increasing*, in case $[t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)]$; the class of all these will be denoted as $\mathscr{F}(in)(R_+)$. Further, call $\varphi \in \mathscr{F}(in)(R_+)$, *regressive* in case $\varphi(0) = 0$ and $[\varphi(t) < t, \forall t > 0]$; the subclass of all these will be denoted as $\mathscr{F}(in, re)(R_+)$. Finally, we shall say that $\varphi \in \mathscr{F}(in, re)(R_+)$ is *Matkowski admissible*, provided

(a02) $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \in R_+$.

[Here, for each $n \geq 0$, φ^n stands for the n-th *iterate* of φ.] The following fixed point result in Matkowski [44] is then available.

Theorem 1. *Suppose that T is $(d; \varphi)$-contractive, for some Matkowski admissible function $\varphi \in \mathscr{F}(in, re)(R_+)$. In addition, let X be d-complete. Then, T is globally strong Picard (modulo d); precisely,*

$$\mathrm{Fix}(T) = \{z\} \text{ and } T^n x \xrightarrow{d} z, \text{ for each } x \in X.$$

Note that, when φ is *linear*, i.e.,

(a03) $\varphi(t) = \alpha t$, $t \in R_+$, for some $\alpha \in [0, 1[$,

then all regularity properties above hold, and the contractive condition becomes

(a04) $d(Tx, Ty) \leq \alpha d(x, y)$, for all $x, y \in X$

[referred to as T is $(d; \alpha)$*-contractive*]; the corresponding version of Theorem 1 is just the 1922 Banach contraction mapping principle [4]. On the other hand, Theorem 1 has certain overlaps with the 1969 fixed point result in Boyd and Wong [8]; however, a complete identification of these is not possible. A similar conclusion is to be derived with respect to the related statement in Leader [43], based on contractive conditions like

(a05) $\psi(d(Tx, Ty)) \leq d(x, y)$, for all $x, y \in X$;

where $\psi \in \mathscr{F}(R_+)$ is endowed with certain regularity properties; we do not give further details.

By the discussed relationships and its simple construction, Matkowski's fixed point result found a multitude of applications in operator equations theory; so, it was the subject of various extensions. From the perspective of this exposition, the following ones are of interest.

(I) Contractive extensions: the contractive property is to be taken as

(a06) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0$,
for all $x, y \in X$ with $x\mathscr{R}y$,

where $F : R_+^6 \to R$ is a function and $\mathscr{R} \subseteq X \times X$ is a *relation* over X. When $\mathscr{R} = X \times X$ (the *trivial relation* over X), a large list of such metrical contractions may be found in the 1977 survey paper by Rhoades [57]. On the other hand, if

(a07) $\mathscr{R}$ is *reflexive, transitive, antisymmetric* (hence, a *(partial) order*),

an extension of Matkowski's theorem was carried out in the 1986 paper by Turinici [67]. (Note that, two decades later, these results have been rediscovered—at the level of Banach contractive maps—by Ran and Reurings [56]; see also Nieto and Rodriguez-Lopez [51].) Finally, a functional extension of the linear-type results we just quoted was performed in Agarwal et al. [1]; and, since then, the number of such papers increased rapidly.

(II) Conical extension: the co-domain R_+ of $d(.,.)$ is to be taken as a *(convex) cone* P of a topological vector space $(Y, \mathscr{T})$. The pioneering results in this direction are contained in the 2007 paper due to Huang and Zhang [31] (where a vectorial generalization of Banach's theorem is being obtained) and the 2008 contribution in Di Bari and Vetro [19] (devoted to a corresponding extension of Matkowski's theorem). This line of research was then developed in the next years by many authors; for a large list of their contributions, we refer to the 2011 survey paper by Janković et al. [35]. However, according to this exposition—as well as the related ones in Du [21] and Khamsi [39]—most of these vector statements are in fact reducible to their "scalar" versions. So, we may ask whether this conclusion remains true beyond the topological vector space setting. It is our aim in the following to give a (partial) negative answer to this, in the realm of (ordered) semigroup couple metric spaces. Further aspects will be delineated in a separate paper.

1.2 Preliminaries

Throughout our exposition, the ambient axiomatic system is Zermelo–Fraenkel's (abbreviated: ZF). In fact, the *reduced* system (ZF-AC+DC) will suffice; here, (AC) stands for the *Axiom of Choice*, and (DC) is the *Dependent Choice Principle*. The notations and basic facts to be used in this reduced system are standard. Some important ones are described below.

(A) Let X be a nonempty set. By a *relation* over X, we mean any nonempty part $\mathscr{R}$ of $X \times X$. For simplicity, we sometimes write $(x, y) \in \mathscr{R}$ as $x\mathscr{R}y$. Note that $\mathscr{R}$ may be regarded as a mapping between X and 2^X (= the class of all subsets in X). In fact, denote for $x \in X$:

$X(x, \mathscr{R}) = \{y \in X; x\mathscr{R}y\}$ (the *section* of $\mathscr{R}$ through x);

then, the desired mapping representation is $(\mathscr{R}(x) = X(x, \mathscr{R}); x \in X)$. A basic example of such object is

$\mathscr{I} = \{(x, x); x \in X\}$ [the *identical relation* over X].

Given the relations $\mathscr{R}$, $\mathscr{S}$ over X, define their *product* $\mathscr{R} \circ \mathscr{S}$ as

$(x, z) \in \mathscr{R} \circ \mathscr{S}$, if there exists $y \in X$ with $(x, y) \in \mathscr{R}$, $(y, z) \in \mathscr{S}$.

Also, for each relation $\mathscr{R}$ in X, denote

$\mathscr{R}^{-1} = \{(x, y) \in X \times X; (y, x) \in \mathscr{R}\}$ (the *inverse* of $\mathscr{R}$).

Finally, given the relations $\mathscr{R}$ and $\mathscr{S}$ on X, let us say that $\mathscr{R}$ is *coarser* than $\mathscr{S}$ (or equivalently: $\mathscr{S}$ is *finer* than $\mathscr{R}$), provided

$\mathscr{R} \subseteq \mathscr{S}$; i.e., $x\mathscr{R}y$ implies $x\mathscr{S}y$.

Given a relation $\mathscr{R}$ on X, the following properties are to be discussed here:

(P1) $\mathscr{R}$ is reflexive: $\mathscr{I} \subseteq \mathscr{R}$.
(P2) $\mathscr{R}$ is irreflexive: $\mathscr{R} \cap \mathscr{I} = \emptyset$.
(P3) $\mathscr{R}$ is transitive: $\mathscr{R} \circ \mathscr{R} \subseteq \mathscr{R}$.
(P4) $\mathscr{R}$ is symmetric: $\mathscr{R}^{-1} = \mathscr{R}$.
(P5) $\mathscr{R}$ is antisymmetric: $\mathscr{R}^{-1} \cap \mathscr{R} \subseteq \mathscr{I}$.

This yields the classes of relations to be used; the following ones are important for our developments:

(C0) $\mathscr{R}$ is *trivial* (i.e., $\mathscr{R} = X \times X$).
(C1) $\mathscr{R}$ is a *(partial) order* (reflexive, transitive, antisymmetric).
(C2) $\mathscr{R}$ is a *strict order* (irreflexive and transitive).
(C3) $\mathscr{R}$ is a *quasi-order* (reflexive and transitive).
(C4) $\mathscr{R}$ is an *equivalence* (reflexive, transitive, symmetric).

A basic ordered structure is $(N, \leq)$; here, $N = \{0, 1, \ldots\}$ is the set of *natural numbers* and $(\leq)$ is defined as

$m \leq n$ iff $m + p = n$, for some $p \in N$.

In fact, $(N, \leq)$ is well ordered; i.e., any (nonempty) subset of N has a first element. By a *sequence* in X, we mean any mapping $x : N \to X$. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$, or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with

$(i(n); n \geq 0)$ is *divergent* (in the sense: $i(n) \to \infty$ as $n \to \infty$)

will be referred to as a *subsequence* of $(x_n; n \geq 0)$.

(B) Let P be a nonempty set and $(\leq)$ be a *(partial) order* (i.e., reflexive, transitive, antisymmetric relation) over it, fulfilling

(f-ele) P has a *first element*, θ (i.e., $\theta \leq t,\ \forall t \in P$);

note that θ is uniquely determined in this way. Let also $(<)$ denote the relation over X attached to $(\leq)$, according to

$(<) = (\leq) \setminus \mathscr{I}$ (or equivalently: $x < y$ iff $x \leq y$ and $x \neq y$);

it is irreflexive and transitive—hence, a strict order. Further, take a binary operation $(t, s) \mapsto t + s$ over P; the following conditions are to be considered

(bop-1) (*zero element*): $t = \theta + t = t + \theta,\ \forall t \in P$ (hence, $\theta + \theta = \theta$)
(bop-2) (*monotone*): $t_1 \leq t_2, s_1 \leq s_2 \implies t_1 + s_1 \leq t_2 + s_2$.

Finally, let us introduce a strict order $(\ll)$ over P and a (nonempty) subset Q of $P_0 := P \setminus \{\theta\}$, subjected to the conditions

(osc-1) $(\ll)$ is *coarser* than $(\leq)$: $x \ll y$ implies $x \leq y$.
(osc-2) $(\leq, \ll)$ is *transitive*: $x \leq y, y \ll z$ imply $x \ll z$.
(osc-3) Q is *small* ($\forall q \in Q, \exists r \in Q$: $r + r \ll q$), hence, $(\theta \ll q, \forall q \in Q)$.
(osc-4) $(\ll)$ is *right translative*: $r, q \in Q, t \in P, r \ll q \implies r + t \ll q + t$.
(osc-5) Q is *uniformly-dense*: $t, s \in P$, $[t \ll q + s, \forall q \in Q] \implies t \leq s$.
(Hence, in particular, Q is *zero-dense*: $t \in P$, $[t \ll q, \forall q \in Q] \implies t = \theta$.)

When all these conditions (f-ele), (bop-1)+(bop-2), and (osc-1)-(osc-5) are holding, then $(P; \leq, \ll; +; Q)$ will be called an *ordered semigroup couple*.

Suppose in the following that we introduced such a structure. Define a zero (sequential) *convergence* $(\to)$ over P as: we say that the sequence $(t_n; n \geq 0)$ in P, *converges* to $\theta \in P$ (and we write $t_n \to \theta$), iff

(z-conv) $\forall q \in Q, \exists n(q) \geq 0$: $n \geq n(q) \implies t_n \ll q$.

Some basic properties of this convergence are collected in

Lemma 1. *Under these conventions, we have*

(zc-1) *the constant sequence* $(t_n = \theta; n \geq 0)$ *fulfills* $t_n \to \theta$
(zc-2) *if* $t_n \to \theta$*, then* $s_n \to \theta$*, for each subsequence* (s_n) *of* (t_n)
(zc-3) $(s_n \leq t_n, \forall n)$ *and* $t_n \to \theta$ *imply* $s_n \to \theta$
(zc-4) $s_n \to \theta$*,* $t_n \to \theta$ *imply* $s_n + t_n \to \theta$.

Proof. (zc-1): Let $q \in Q$ be arbitrary fixed. By the small property of Q, there exists (an associated element) $r \in Q$, with

$$(\theta \leq) r + r \ll q, \quad \text{hence } \theta \ll q.$$

(zc-2): Evident.

(zc-3): Let $q \in Q$ be arbitrary fixed. As $t_n \to \theta$, there exists $n(q) \geq 0$, such that $n \geq n(q)$ implies $t_n \ll q$. This, along with (hypothesis and) the transitive property, gives $s_n \ll q$, $\forall n \geq n(q)$, and the conclusion follows.

(zc-4): Let $q \in Q$ be arbitrary fixed and $r \in Q$ be given by the small property of Q. By the imposed hypotheses, there must be some $n(r) \geq 0$ such that (taking the coarser property into account)

$$n \geq n(r) \text{ implies } s_n \ll r, t_n \ll r \text{ (hence, } s_n \leq r, t_n \leq r).$$

Combining with the monotone property of our binary operation, we get (for the same n): $s_n + t_n \leq r + r$. This, along with $r + r \ll q$, yields (again by means of the transitive property)

$$s_n + t_n \ll q, \ \forall n \geq n(r),$$

and the assertion follows.

Finally, note that all these facts are holding as well over *partially ordered semigroups* taken as in Bourbaki [7, Chap. 3, Sect. 2]. Further aspects may be found in the 1968 contribution due to Popa [55].

(C) Let $(P, \leq, \ll; +; Q)$ be an ordered semigroup couple and X be a nonempty set. We say that the map $d : X \times X \to P$ is a *P-metric* on X, provided

(b01) d is *reflexive sufficient*: $x = y$ iff $d(x, y) = \theta$
(b02) d is *symmetric*: $d(x, y) = d(y, x)$, $\forall x, y \in X$
(b03) d is *triangular*: $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$.

In this case, (X, d) will be referred to as a *P-metric space*.

Given this space, we may introduce a (sequential) d-convergence and d-Cauchy structure on X, as below. Let us say that the sequence $(x_n; n \geq 0)$ in X, *d-converges* toward the point $x \in X$ (and write: $x_n \xrightarrow{d} x$), if

$d(x_n, x) \to \theta$ as $n \to \infty$;

or equivalently (by definition)

(b04) $\forall q \in Q, \exists n(q) \geq 0$: $n \geq n(q) \implies d(x_n, x) \ll q$.

The set of all such x will be denoted as $\lim_n(x_n)$. Some basic facts about this concept are being collected in

Lemma 2. *The introduced convergence* ($\xrightarrow{d}$) *has the properties*

(conv-1) (reflexive)
$(\forall u \in X)$: $(x_n = u; n \geq 0)$ *fulfills* $x_n \xrightarrow{d} u$

(conv-2) (hereditary)
if $x_n \xrightarrow{d} x$, *then* $y_n \xrightarrow{d} x$, *for each subsequence* (y_n) *of* (x_n);

so, it fulfills the general requirements in Kasahara [38]. Moreover, the convergence in question has the property

(conv-3) *(separated)*
$\lim_n(x_n)$ *is an asingleton, for each sequence* $(x_n; n \geq 0)$ *in* X*.*

[This will be also referred to as d is separated.]

Proof. (conv-1), (conv-2): Evident.

(conv-3): Suppose that the sequence (x_n) fulfills $x_n \xrightarrow{d} u$, $x_n \xrightarrow{d} v$, and let $q \in Q$ be arbitrary fixed. By the small property of Q, there exists $r \in Q$ such that

$$(r \leq) r + r \ll q, \text{ hence, } r \ll q.$$

On the other hand, from the convergence hypothesis above, there must be some rank $n(r) \geq 0$ such that

$$n \geq n(r) \Longrightarrow d(x_n, u), d(x_n, v) \ll r.$$

This, along with the triangular inequality, gives

$$d(u, v) \leq d(x_n, u) + d(x_n, v) \leq r + r(\ll q),$$

whence (by the transitive property) $d(u, v) \ll q$. As $q \in Q$ was arbitrarily taken, one must have (as Q is zero-dense) $d(u, v) = \theta$, hence $u = v$.

Let $(x_n; n \geq 0)$ be a sequence in X. If $\lim_n(x_n)$ is nonempty, the underlying sequence will be referred to as *d-convergent*. By the (last part of the) preceding result, this is equivalent with $\{z\} = \lim_n(x_n)$ (for some $z \in X$); as usually, this is to be written as $z = \lim_n(x_n)$.

Further, let us say that $(x_n; n \geq 0)$ is *d-Cauchy*, provided

$d(x_n, x_m) \to \theta$ as $n, m \to \infty, n \leq m$,

or equivalently (by definition)

(b05) $\forall q \in Q, \exists n(q) \geq 0$: $n(q) \leq n \leq m \Longrightarrow d(x_n, x_m) \ll q$.

Note that, by the small property of Q, each d-convergent sequence is d-Cauchy. The reciprocal is not in general true, as simple examples show. Concerning this aspect, call the sequence $(x_n; n \geq 0)$ in X, *d-semi-Cauchy*, provided

$d(x_n, x_{n+1}) \to \theta$ as $n \to \infty$,

or equivalently (by definition)

(b06) $\forall q \in Q, \exists n(q) \geq 0$: $n(q) \leq n \Longrightarrow d(x_n, x_{n+1}) \ll q$.

Clearly, each d-Cauchy sequence is d-semi-Cauchy too; the reciprocal inclusion is not in general true.

1.3 Statement of the Problem

Let $(P; \leq, \ll; +; Q)$ be an ordered semigroup couple (see above). Further, let X be a nonempty set. Take a P-metric $d : X \times X \to P$ on it, and let $(\preceq)$ be a *quasi-order* (i.e., reflexive and transitive relation) over X; the triple $(X, d, \preceq)$ will be referred to as a *quasi-ordered P-metric space*. Call the subset Y of X, $(\preceq)$-*almost-singleton* (in short: $(\preceq)$-*asingleton*), provided $[y_1, y_2 \in Y, y_1 \preceq y_2 \implies y_1 = y_2]$; and $(\preceq)$-*singleton* when, in addition, Y is nonempty. Finally, let T be a selfmap of X. We have to determine circumstances under which $\text{Fix}(T) := \{x \in X; x = Tx\}$ is nonempty; and, if this holds, to establish whether T is *fix-*$(\preceq)$*-asingleton* (i.e., $\text{Fix}(T)$ is $(\preceq)$-asingleton) or equivalently T is *fix-*$(\preceq)$*-singleton* (in the sense: $\text{Fix}(T)$ is $(\preceq)$-singleton). To do this, we start from the hypotheses

(c01) T is $(\preceq)$-*semi-progressive* $[X(T, \preceq) := \{x \in X; x \preceq Tx\} \neq \emptyset]$.
(c02) T is $(\preceq)$-*increasing* $[x \preceq y$ implies $Tx \preceq Ty]$.

In this setting, the basic directions under which the investigations be conducted are described by the list below, comparable with the one in Turinici [71]:

(spic-1) We say that T is a *Picard operator* (modulo $(d, \preceq)$) if, for each $x \in X(T, \preceq)$, $(T^n x; n \geq 0)$ is d-convergent; and a *globally Picard operator* (modulo $(d, \preceq)$) if, in addition, T is fix-$(\preceq)$-asingleton
(spic-2) We say that T is a *strong Picard operator* (modulo $(d, \preceq)$) when, for each $x \in X(T, \preceq)$, $(T^n x; n \geq 0)$ is d-convergent with $\lim_n(T^n x)$ belonging to $\text{Fix}(T)$; and a *globally strong Picard operator* (modulo $(d, \preceq)$) when, in addition, T is fix-$(\preceq)$-asingleton (hence, fix-$(\preceq)$-singleton)
(spic-3) We say that T is a *Bellman Picard operator* (modulo $(d, \preceq)$) when, for each $x \in X(T, \preceq)$, $(T^n x; n \geq 0)$ is d-convergent with $T^n x \preceq \lim_n(T^n x) \in \text{Fix}(T)$, for all n; and a *globally Bellman Picard operator* (modulo $(d, \preceq)$) when, in addition, T is fix-$(\preceq)$-asingleton (hence, fix-$(\preceq)$-singleton).

The sufficient (regularity) conditions for such properties are being founded on *ascending orbital* concepts (in short: *(a-o)-concepts*). Namely, call the sequence $(z_n; n \geq 0)$ in X, $(\preceq)$-*ascending* if $z_i \preceq z_j$, for $i \leq j$; and T-*orbital* when it is a subsequence of $(T^n x; n \geq 0)$, for some $x \in X$; the intersection of these is just the precise notion.

(sreg-1) Call X, *(a-o,d)-complete* provided (for each (a-o)-sequence): d-Cauchy $\implies$ d-convergent.
(sreg-2) We say that T is *(a-o,d)-continuous*, if $[(z_n) =$ (a-o)-sequence and $z_n \xrightarrow{d} z]$ imply $Tz_n \xrightarrow{d} Tz$.
(sreg-3) Call $(\preceq)$, *(a-o,d)-selfclosed* when $[(z_n) =$ (a-o)-sequence, $z_n \xrightarrow{d} z]$ imply $[z_n \preceq z, \forall n]$; or, in other words: the d-limit of each d-convergent (a-o)-sequence in X is an upper bound of it.

When $(\preceq)$ is the trivial relation over X, these conventions are comparable with the ones in Rus [59, Chap. 2, Sect. 2.2], because, in this case, $X(T, \preceq) = X$.

As a completion of these, we must now describe the contractive conditions to be used. Denote, for $x, y \in X$

$$M_1(x,y) = d(x,y), M_2(x,y) = d(x,Tx),\ M_3(x,y) = d(y,Ty),$$
$$M_4(x,y) = d(Tx,y),\ \mathscr{M} = \{M_1, M_2, M_3, M_4\}.$$

Note that each nonempty subset $\mathscr{H}$ of $\mathscr{M}$ may be written as

$$\mathscr{H} = \{M_i; i \in I\}, \text{ where } \emptyset \neq I \subseteq \{1,2,3,4\};$$

in this case, denote

$$\mathscr{H}(x,y) = \{M_i(x,y); i \in I\}, x, y \in X.$$

Let $(\prec)$ stand for the *strict quasi-order* attached to $(\preceq)$, as

$$x \prec y \text{ iff } x \preceq y \text{ and } x \neq y.$$

[Clearly, the underlying relation is irreflexive ($x \prec x$ is impossible, $\forall x \in X$), but not in general transitive.] Given a nonempty subset $\mathscr{H}$ of $\mathscr{M}$ and a mapping $\varphi \in \mathscr{F}(P)$, call $T \in \mathscr{F}(X)$, $(d, \preceq; \varphi, \mathscr{H})$*-contractive* provided

(c03) $\quad d(Tx,Ty) \leq \varphi(\mathscr{H}(x,y)), \forall x, y \in X, x \prec y.$

(Here, given the couple $U, V \in 2^X$, we put

$$U \leq V \text{ iff } \forall u \in U, \exists v \in V\colon u \leq v;$$

this relation is reflexive and transitive, but not in general antisymmetric.) The functions $\varphi \in \mathscr{F}(P)$ appearing here are subjected to regularity conditions like

(rc-1) φ is *increasing* (on P): $t_1 \leq t_2 \implies \varphi(t_1) \leq \varphi(t_2)$.
(rc-2) φ is *regressive* (on P): $\varphi(\theta) = \theta$ and $\varphi(t) < t$, $\forall t \in P_0 = P \setminus \{\theta\}$.
(rc-3) φ is *Matkowski* (on P): $\varphi^n(t) \to \theta$, for each $t \in P$.
(rc-4) φ is *strongly regressive* on Q ($\forall q \in Q$, $\exists r \in Q$: $r + \varphi(q) \ll q$).

Note that the last condition above has a strong connection with the small property of Q; we do not give details.

1.4 Main Result

Let $(P; \leq, \ll; +; Q)$ be an ordered semigroup couple; this, by definition, means

(osc-1) $(\leq)$ is a (partial) order on P with respect to which P has a first element, θ; in addition, the binary operation $(+)$ has θ as null element and is monotone.
(osc-2) $(\ll)$ is a strict order on P and Q is a (nonempty) subset of $P_0 := P \setminus \{\theta\}$, with: $(\ll)$ is coarser than $(\leq)$, $(\leq, \ll)$ is transitive, and Q is small.
(osc-3) $(\ll)$ is right translative and Q is uniformly dense (hence, zero-dense).

Further, let $(X, d, \preceq)$ be a quasi-ordered P-metric space, and take a selfmap T of X, supposed to be $(\preceq)$-semi-progressive and $(\preceq)$-increasing. The basic directions and regularity conditions under which the problem of determining fixed points of T is to be solved were already listed, and the contractive-type framework was settled.

The first main result of this exposition is

Theorem 2. *Suppose that T is $(d, \preceq; \varphi, \mathscr{H})$-contractive, where $\emptyset \neq \mathscr{H} \subseteq \mathscr{M}$ and $\varphi \in \mathscr{F}(P)$ is increasing, regressive, Matkowski (on P), and strongly regressive on Q. In addition, let the ambient space X be (a-o,d)-complete. Then,*

I-a) *T is a strong Picard operator (modulo $(d, \preceq)$) when, in addition, T is assumed to be (a-o,d)-continuous.*

I-b) *T is a Bellman Picard operator (modulo $(d, \preceq)$) when, in addition, $(\preceq)$ is taken as (a-o,d)-selfclosed.*

Proof. Clearly, without loss, one may assume (via φ = increasing) that $\mathscr{H} = \mathscr{M}$. There are some general and specific steps to be passed.

Step-gen 1. Let us prove the $(\preceq)$-asingleton property of $\mathrm{Fix}(T)$. Take a couple of points $z_1, z_2 \in \mathrm{Fix}(T)$ with $z_1 \preceq z_2$, and assume by contradiction that $z_1 \neq z_2$; hence, $z_1 \prec z_2$, $t := d(z_1, z_2) > \theta$. By definition,

$$d(z_1, z_2) = t,\ d(z_1, Tz_1) = d(z_2, Tz_2) = \theta \leq t,\ d(Tz_1, z_2) = d(z_1, z_2) = t.$$

In this case, the contractive condition yields (as φ = increasing)

$t = d(Tz_1, Tz_2) \leq \varphi(t)$; impossible, as φ is regressive,

wherefrom the claim follows.

Step-gen 2. We now establish the useful evaluation

$$d(Tx, T^2x) \leq \varphi(d(x, Tx)),\ \text{whenever } x \prec Tx, Tx \prec T^2x. \qquad (1)$$

In fact, let $x \in X$ be as in the premise above; hence, in particular, $d(x, Tx) > \theta$, $d(Tx, T^2x) > \theta$. By definition,

$$\mathscr{M}(x, Tx) = \{d(x, Tx), d(Tx, T^2x), \theta\};$$

so, from the contractive condition, the alternatives below hold:

$d(Tx, T^2x) \leq \varphi(d(x, Tx))$; i.e., the desired conclusion
$d(Tx, T^2x) \leq \varphi(d(Tx, T^2x))$; impossible, via φ = regressive
$d(Tx, T^2x) \leq \varphi(\theta) = \theta$; impossible, via $d(Tx, T^2x) > \theta$,

hence the conclusion.

It remains now to establish the strong/Bellman Picard property (modulo $(d, \preceq)$). Fix $x_0 \in X(T, \preceq)$ and put $(x_n = T^n x_0; n \geq 0)$; this is an ascending orbital sequence in X. If $x_k = x_{k+1}$, for some $k \geq 0$, we are done (in view of $x_k \in \mathrm{Fix}(T)$); so, it remains to discuss the opposite case; i.e., for all $n \geq 0$,

(d01) $x_n \neq x_{n+1}$, hence, $x_n \prec x_{n+1}$, $\rho_n := d(x_n, x_{n+1}) > \theta$.

There are several (specific) steps to be passed.

Step 1. From the preceding (general) step, it follows that

$$\rho_{n+1} \leq \varphi(\rho_n) < \rho_n, \ \forall n. \tag{2}$$

This firstly tells us that $(\rho_n; n \geq 0)$ is strictly descending in $P_0 = P \setminus \{\theta\}$. Secondly, as φ is increasing, one gets (by a repeated application of φ to the above evaluation)

$$\rho_n \leq \varphi^n(\rho_0), \text{ for all } n. \tag{3}$$

This, combined with the Matkowski property (over P), gives

$$\rho_n := d(x_n, x_{n+1}) \to \theta \text{ as } n \to \infty, \tag{4}$$

i.e., $(x_n; n \geq 0)$ is a d-semi-Cauchy sequence in X.

Step 2. Suppose that

(d02) there exist $i, j \in N$ such that $i < j$, $x_i = x_j$.

This yields $x_{i+1} = x_{j+1}$; so that, $\rho_i = \rho_j$. On the other hand, by the strict descending property above, one has $\rho_i > \rho_j$; in contradiction with the previous relation. Hence, our working hypothesis cannot hold, wherefrom

$$\text{for all } i, j \in N\text{: } i < j \text{ implies } x_i \prec x_j\text{; hence, } d(x_i, x_j) > \theta. \tag{5}$$

Step 3. We now show that $(x_n; n \geq 0)$ is d-Cauchy. Let $q \in Q$ be arbitrary fixed. As φ is strictly regressive on Q, there exists $r \in Q$ such that

$$(r \leq) r + \varphi(q) \ll q, \text{ hence, } r \ll q. \tag{6}$$

By (4), there exists for this r, some rank $n(r) \geq 0$, in such a way that

$$d(x_n, x_{n+1}) \ll r, \ \forall n \geq n(r). \tag{7}$$

We now claim that the following property holds

$$(\forall i \geq 0): \ d(x_n, x_{n+i}) \ll q, \ \forall n \geq n(r), \tag{8}$$

wherefrom the d-Cauchy property of $(x_n; n \geq 0)$ follows. A verification of this assertion is to be made via (ordinary) induction. The case $i = 0$ is clear, via

(Q = small), and the case $i = 1$ follows via (6)+(7). Assume that (8) holds for all $i \in \{1, \ldots, j\}$, where $j \geq 1$; we have to verify that it holds as well for $i = j + 1$. So, let $n \geq n(r)$ be arbitrary fixed. By the inductive hypothesis and (7),

$$d(x_n, x_{n+j}), d(x_{n+1}, x_{n+j}) \ll q,$$
$$d(x_n, x_{n+1}), d(x_{n+j}, x_{n+j+1}) \ll r(\ll q).$$

By the preceding step, the contractive condition applies to (x_n, x_{n+j}) and yields (via φ = increasing)

$$d(x_{n+1}, x_{n+j+1}) = d(Tx_n, Tx_{n+j}) \leq \varphi(d(x_n, x_{n+j})) \leq \varphi(q).$$

This, finally combined with (6), yields (by the triangular inequality)

$$d(x_n, x_{n+j+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+j+1}) \leq r + \varphi(q)(\ll q),$$

and conclusion follows (by the transitive property).

Step 4. As X is (a-o,d)-complete, $x_n \xrightarrow{d} z$, for some (uniquely determined) $z \in X$. If there exists a sequence of ranks $(i(n); n \geq 0)$ with $i(n) \to \infty$ as $n \to \infty$ such that

$(x_{i(n)} = z, \forall n)$, hence, $(x_{i(n)+1} = Tz, \forall n)$

then, as $(x_{i(n)+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$, one gets $x_{i(n)+1} \xrightarrow{d} z$; whence (as d = sufficient), $z = Tz$. So, in the following, we may assume that the opposite alternative is true:

(d03) $\exists h = h(z) \geq 0$: $n \geq h \implies x_n \neq z$.

There are two cases to be discussed.

Case 1. Suppose that T is (a-o,d)-continuous. Then

$y_n := Tx_n \xrightarrow{d} Tz$ as $n \to \infty$.

On the other hand, $(y_n = x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$, whence (by the hereditary property), $y_n \xrightarrow{d} z$; and this yields (as d = separated), $z = Tz$.

Case 2. Suppose that $(\preceq)$ is (a-o,d)-selfclosed; note that, as a direct consequence of the convergence property above, $x_n \preceq z$, $\forall n$. We show that $b := d(z, Tz) > \theta$ yields a contradiction. Let $q \in Q$ be arbitrary fixed and $r \in Q$ be such that

$$(r \leq) r + \varphi(q) \ll q; \text{ hence } r \ll q.$$

Given this r, there must be some rank $n(r) \geq h$, such that (by the convergence property and previous facts) we have, for all $n \geq n(r)$

$$M_1(x_n, z) = d(x_n, z) \leq r,\ M_2(x_n, z) = d(x_n, Tx_n) = d(x_n, x_{n+1}) \leq r,$$
$$M_3(x_n, z) = d(z, Tz) = b,\ M_4(x_n, z) = d(Tx_n, z) = d(x_{n+1}, z) \leq r.$$

Remember that, in view of φ =increasing, one may take $\mathscr{H} = \mathscr{M}$. From the contractive property, one derives (for the same ranks n)

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq \varphi(r), \text{ or } d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq \varphi(b).$$

The former case gives, by the triangular inequality,

$$b \leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \leq r + \varphi(r) \leq r + \varphi(q) \leq (r + \varphi(q)) + \varphi(b),$$
$$\text{hence, } b \leq (r + \varphi(q)) + \varphi(b).$$

Since $r + \varphi(q) \ll q$, the right translative property of $(\ll)$ yields

$$(r + \varphi(q)) + \varphi(b) \ll q + \varphi(b);$$

so that (taking the previous relation into account)

$$b \ll q + \varphi(b) \text{ (for all } q \in Q). \tag{9}$$

On the other hand, the latter case gives (again by the triangular inequality)

$$b \leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \leq r + \varphi(b);\ \text{hence, } b \leq r + \varphi(b).$$

Since $r \ll q$, the right translative property of $(\ll)$ yields

$$r + \varphi(b) \ll q + \varphi(b);$$

so that (taking the previous relation into account) (9) is again valid. But, the underlying relation yields (via Q =uniformly dense), $b \leq \varphi(b)$, in contradiction with (the working upon b condition and) regressiveness of φ. Hence, necessarily, $b = \theta$, i.e., $z = Tz$. The proof is complete.

Note that further extensions of these facts are possible, in the realm of *triangular P-symmetric spaces*, taken as in Hicks and Rhoades [29], or in the setting of *partial P-metric spaces*, introduced under the lines in Matthews [46]. We shall discuss all these elsewhere.

1.5 Particular Aspects (Conical Metrics)

Let Y be a (real) *vector space*. In particular, this shows that $(Y, +)$ is an *Abelian group*; so that, if θ stands for the zero element of Y, we have

$$\begin{aligned} & y = y + \theta = \theta + y,\ \forall y \in Y; \\ & (+) \text{ is commutative and associative}; \\ & \forall y \in Y,\ \exists(-y) \in Y:\ y + (-y) = (-y) + y = \theta. \end{aligned} \tag{10}$$

(A) Call the (nonempty) subset P of Y a *(convex) cone* of it, when

(e01) $\alpha P + \beta P \subseteq P$, for all $\alpha, \beta \in R_+$.

Fix in the following such an object, P; endowed with the properties

(e02) P is *proper* ($P \neq Y$) and *nondegenerate* ($P \neq \{0\}$).
(e03) P is *pointed* ($P \cap (-P) = \{0\}$).

In this case, the relation (over Y)

$y_1 \leq_P y_2$ iff $y_2 - y_1 \in P$

(denoted simply as ($\leq$), when P is understood) is reflexive, transitive, and antisymmetric—hence, a (partial) order—on Y. Moreover, ($\leq$) is *compatible* with the linear space structure (of Y):

$$u \leq v \Longrightarrow u + z \leq v + z,\ \lambda u \leq \lambda v,\ \forall z \in Y,\ \forall \lambda \in R_+. \tag{11}$$

(B) We say that $c \in P$ is an *algebraic interior* (or *internal*) point of P, when

(e04) $\forall u \in Y, \exists \rho = \rho(u) \in]0, 1[$, with $c + \rho(u - c) \in P$.

Note that, by the convexity of P, this may be also written as

$\forall u \in Y, \exists \rho = \rho(u) \in]0, 1[$, with $c + \lambda(u - c) \in P$, $\forall \lambda \in [0, \rho]$.

And this, passing to the c-symmetric point $v = 2c - u$, is equivalent with the usual definition of the concept in question

(e05) $\forall u \in Y, \exists \rho = \rho(u) \in]0, 1[$, such that $c + \mu(u - c) \in P$, $\forall \mu \in [-\rho, \rho]$.

The class of all such points will be denoted aint(P) (the *algebraic interior* of P). Clearly, $\theta \in Y$ is not an element of aint(P); for, otherwise, as P is cone, the former definition above yields $P = Y$ contradiction. Further properties of this set are contained in

Lemma 3. *Under these notations, we have*

i) $\text{aint}(P) = \lambda \text{aint}(P)$, *for each* $\lambda > 0$.
ii) $\text{aint}(P) + P \subseteq \text{aint}(P)$, *hence,* $\text{aint}(P) + \text{aint}(P) \subseteq \text{aint}(P)$.

Proof. i): Let $\lambda > 0$ and $c \in \text{aint}(P)$ be arbitrary fixed. For each $v \in Y$, there must be some point $\rho > 0$ such that

$$c + \rho(u - c) \in P, \text{ where } u = (1/\lambda)v,$$

and this (via $P =$ cone) yields

$$\lambda c + \rho(v - \lambda c) = \lambda c + \rho(\lambda u - \lambda c) \in P.$$

As $v \in Y$ was arbitrarily fixed, one gets $\lambda c \in \text{aint}(P)$; wherefrom the right to left inclusion follows. The converse inclusion is now a consequence of the first part (applied to $\mu := 1/\lambda$) and

$$\text{aint}(P) = \lambda[\mu\text{aint}(P)] \subseteq \lambda\text{aint}(P).$$

ii): Let $c \in \text{aint}(P)$ and $b \in P$ be arbitrary fixed. Again by definition, for each $u \in Y$, there exists $\rho > 0$ with $c + \rho(u - b - c) \in P$, wherefrom (as $P =$ cone)

$$c + b + \rho(u - (c + b)) \in P,$$

which tells us that $c + b \in \text{aint}(P)$. This proves the first half of our conclusion. The second one is immediate, in view of $\text{aint}(P) \subseteq P$.

(C) In the following, two specific additional hypotheses involving our (convex) cone P are formulated. The former of these writes

(e06) P is *algebraically solid* [$\text{aint}(P) \neq \emptyset$].

Note that, as a consequence of the above developments, the relation ($\ll$) over Y introduced as

(e07) $y_1 \ll y_2$ iff $y_2 - y_1 \in \text{aint}(P)$

is *irreflexive* ($y \ll y$ is false, for each $y \in Y$) and *transitive*; hence, it is a *strict order* on Y. On the other hand, ($\ll$) is *coarser* than ($\leq$) (since $\text{aint}(P) \subseteq P$) and *compatible* with the linear space operations:

$$x \ll y \text{ implies } x + z \ll y + z, \lambda x \ll \lambda y, \forall z \in Y, \forall \lambda > 0. \tag{12}$$

The latter specific condition to be imposed upon our (convex) cone may be expressed as

(e08) P is *Archimedean*: $[y \in Y, c \in P, y \leq \sigma c, \forall \sigma > 0] \Longrightarrow y \leq \theta$;

cf. Cristescu [17, Chap. 5, Sect. 1]. Technically speaking, these additional conditions allow us to conclude that $(P; \leq, \ll; +; Q)$ (where $Q := \text{aint}(P)$) is an ordered semigroup couple (according to our general convention). The following density statement involving these data is a basic step toward this assertion.

Proposition 1. *Let the general conditions upon P be accepted, as well as the specific ones (P is algebraically solid and Archimedean). Then,*

(den-1) $y \in Y$, $[y \leq q, \forall q \in Q]$ *imply* $y \leq \theta$.
(den-2) $y \in Y$, $[y \ll q, \forall q \in Q]$ *imply* $y \leq \theta$.
(den-3) $y \in P$, $[y \leq q, \forall q \in Q]$ *imply* $y = \theta$.
(den-4) $y \in P$, $[y \ll q, \forall q \in Q]$ *imply* $y = \theta$.
(den-5) $y, z \in P$, $y \leq q + z$, $\forall q \in Q$ *imply* $y \leq z$.
(den-6) $y, z \in P$, $y \ll q + z$, $\forall q \in Q$ *imply* $y \leq z$.

Proof. (den-1): Let $y \in Y$ be as in the premise above and fix $r \in Q$. By a previous fact, $\{\lambda r; \lambda > 0\} \subseteq Q$; so that, by the underlying premise, $y \leq \lambda r$, for each $\lambda > 0$. The obtained relation yields (as $P =$ Archimedean), $y \leq 0$; and we are done.
(den-2): Evident, via $[y \ll q \implies y \leq q]$.
(den-3), (den-4): Evident, via $[y \in P, y \leq \theta \implies y = \theta]$.
(den-5), (den-6): Evident, by a simple translation.

Summing up these developments, we arrived at the following synthetic result.

Proposition 2. *Let the (convex) cone P of Y be endowed with the properties: proper, nondegenerate, pointed, algebraically solid, and Archimedean. Further, let* $(\leq)$ *be the (partial) order on Y induced by P and* $(\ll)$ *stand for the strict order on Y induced by its algebraic interior* $Q :=$ aint(P). *Then,* $(P; \leq, \ll; +; Q)$ *is an ordered semigroup couple (see above).*

Proof. The desired conclusion is ultimately obtainable from the preceding facts. However, for completeness, we shall provide an appropriate argument for it.

Step 1. By definition, θ (= the null element of Y) is the first element of $(P, \leq)$. Moreover, the binary operation $(+)$ is monotone, in the sense

$$y_1 \leq y_2 \text{ and } z_1 \leq z_2 \text{ imply } y_1 + z_1 \leq y_2 + z_2. \tag{13}$$

In fact, by the compatible relation, we have

$$y_1 + z_1 \leq y_1 + z_2 \leq y_2 + z_2,$$

and, from this, all is clear.

Step 2. As already precise, $Q :=$ aint(P) is a subset of $P_0 := P \setminus \{\theta\}$; in addition, $Q \neq \emptyset$ (as P is algebraically solid). On the other hand, its associated strict order $(\ll)$ is coarser than $(\leq)$, and (by a previous auxiliary fact)

$$[x \leq y, y \ll z \text{ or } [x \ll y, y \leq z] \text{ imply } x \ll z;$$

hence, both $(\leq, \ll)$ and $(\ll, \leq)$ are transitive. Further, we claim that Q is small. Indeed, let $q \in Q$ be arbitrary fixed and put $r = \delta q$, where $0 < \delta < 1/3$. Then (as $0 < 2\delta < 1$), $r + r = 2\delta q \ll q$ and the claim follows.

Step 3. By the compatible relations between $(\ll)$ and linear space operations,

$$x \ll y, z \in Y \text{ implies } (z + x =) x + z \ll y + z (= z + y);$$

hence, $(\ll)$ is right (and left) translative. Finally, by the above density result, Q is uniformly dense (hence, zero-dense).
Putting these together yields all conclusions in the statement.

(D) Let again P be a (convex) cone of Y, supposed to be proper, nondegenerate, pointed, algebraically solid, and Archimedean. By the above result, $(P, \leq, \ll; +; Q)$ is an ordered semigroup couple, where $Q = \text{aint}(P)$ and $(\ll)$ is the attached strict order. Further, let X be a nonempty set. We say that the map $d : X \times X \to P$ is a *P-metric* on X, provided

(pm-1) d is *reflexive sufficient*: $x = y$ iff $d(x, y) = \theta$
(pm-2) d is *symmetric*: $d(x, y) = d(y, x)$, $\forall x, y \in X$
(pm-3) d is *triangular*: $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$.

In this case, (X, d) will be referred to as a *P-metric space*. [Clearly, this concept is just the one we already introduced in a previous place.] Let also $(\preceq)$ be a quasi-order (i.e., reflexive and transitive relation) over X; the triple $(X, d, \preceq)$ will be referred to as a *quasi-ordered P-metric space*. Remember that the subset U of X is $(\preceq)$-*almost-singleton* (in short: $(\preceq)$-*asingleton*), provided $[u_1, u_2 \in U, u_1 \preceq u_2 \implies u_1 = u_2]$, and $(\preceq)$-*singleton* when, in addition, U is nonempty. Letting T be a selfmap of X, we have to determine circumstances under which $\text{Fix}(T) := \{x \in X; x = Tx\}$ is nonempty; and, if this holds, to establish whether T is *fix-*$(\preceq)$*-asingleton* (i.e., $\text{Fix}(T)$ is $(\preceq)$-asingleton) or equivalently T is *fix-*$(\preceq)$*-singleton* (in the sense: $\text{Fix}(T)$ is $(\preceq)$-singleton). To do this, we start from the hypotheses

(e09) T is $(\preceq)$-semi-progressive $[X(T, \preceq) := \{x \in X; x \preceq Tx\} \neq \emptyset]$
(e10) T is $(\preceq)$-increasing $[x \preceq y$ implies $Tx \preceq Ty]$.

The Picard-type concepts and regularity conditions to be used here are the ones we just defined in our general setting. As a completion of these, we must now describe the contractive conditions to be used. Denote, for $x, y \in X$

$$M_1(x, y) = d(x, y), M_2(x, y) = d(x, Tx), \ M_3(x, y) = d(y, Ty),$$
$$M_4(x, y) = d(Tx, y), \ \mathscr{M} = \{M_1, M_2, M_3, M_4\}.$$

Note that each nonempty subset $\mathscr{H}$ of $\mathscr{M}$ may be written as

$$\mathscr{H} = \{M_i; i \in I\}, \text{ where } \emptyset \neq I \subseteq \{1, 2, 3, 4\};$$

in this case, denote

$$\mathscr{H}(x, y) = \{M_i(x, y); i \in I\}, x, y \in X.$$

Let $(\prec)$ stand for the *strict quasi-order* attached to $(\preceq)$, as

$$x \prec y \text{ iff } x \preceq y \text{ and } x \neq y.$$

[Clearly, the underlying relation is irreflexive ($x \prec x$ is impossible, $\forall x \in X$), but not in general transitive.] Given a nonempty subset $\mathscr{H}$ of $\mathscr{M}$ and a mapping $\varphi \in \mathscr{F}(P)$, call $T \in \mathscr{F}(X)$, $(d, \preceq; \varphi, \mathscr{H})$-*contractive* provided

(e11) $\quad d(Tx, Ty) \leq \varphi(\mathscr{H}(x, y))$, $\forall x, y \in X, x \prec y$.

According to the same general theory, the functions $\varphi \in \mathscr{F}(P)$ to be considered here are subjected to regularity conditions like

(rc-1) $\quad \varphi$ is increasing (on P): $t_1 \leq t_2 \implies \varphi(t_1) \leq \varphi(t_2)$
(rc-2) $\quad \varphi$ is regressive (on P): $\varphi(\theta) = \theta$ and $\varphi(t) < t$, $\forall t \in P_0 = P \setminus \{\theta\}$
(rc-3) $\quad \varphi$ is Matkowski (on P): $\varphi^n(t) \to \theta$, for each $t \in P$
(rc-4) $\quad \varphi$ is strongly regressive on Q ($\forall q \in Q$, $\exists r \in Q$: $r + \varphi(q) \ll q$).

The natural question to be posed here is that of giving a characterization (imposed by our specific framework) of the strong regressive property listed above. As we shall see, this is realizable under the (apparently weaker) condition

(rc-5) $\quad \varphi$ is *nearly regressive* on Q: $\varphi(q) \ll q$, for each $q \in Q$.

In fact, these conditions are equivalent to each other; as results from

Lemma 4. *Let $\varphi \in \mathscr{F}(P)$ be increasing, regressive, and Matkowski (on P). Then,*

(j) *If φ is nearly regressive on Q, then φ is strongly regressive on Q.*
(jj) *If φ is strongly regressive on Q, then φ is nearly regressive on Q.*
(jjj) *[φ is strongly regressive on Q] iff [φ is nearly regressive on Q].*

Proof. (j): Suppose that φ is nearly regressive on Q, and let $q \in Q$ be arbitrary fixed. By definition, we have

$$\varphi(q) \ll q; \text{ hence, } s := q - \varphi(q) \in Q.$$

By a previous result, $r := (1/2)s \in Q$; we claim that this is our desired element so as to satisfy the strong regressive property. In fact, $r \ll s$; so that, by the right translative property, $r + \varphi(q) \ll s + \varphi(q) = q$, hence the claim.

(jj): Suppose that φ is strongly regressive on Q, and let $q \in Q$ be arbitrary fixed. By hypothesis. there must be some $r \in Q$ with $r + \varphi(q) \ll q$. Combining with

$$\varphi(q) = \theta + \varphi(q) \leq r + \varphi(q),$$

one gets (as $(\leq, \ll)$ is transitive), $\varphi(q) \ll q$, and the assertion follows.

(jjj): Evident, by the above.

Having these precise, we may now pass to the second main result of this exposition. Let P be a convex cone of the (real) linear space Y, supposed to be proper, nondegenerate, pointed, algebraically solid, and Archimedean (see above); and $(\leq)$ stand for the associated (partial) order (on Y). Denote $Q = \text{aint}(P)$ (hence, $Q \neq \emptyset$), and let $(\ll)$ stand for its associated strict order (on Y). Further, let X be a nonempty set, $d : X \times X \to P$ be a P-metric on it and $(\preceq)$ stand for a quasi-order on X. Finally, let $T \in \mathscr{F}(X)$ be a selfmap of X, supposed to be $(\preceq)$-semi-progressive and $(\preceq)$-increasing.

Theorem 3. *Suppose that T is $(d, \preceq; \varphi, \mathscr{H})$-contractive, where $\emptyset \neq \mathscr{H} \subseteq \mathscr{M}$ and $\varphi \in \mathscr{F}(P)$ is increasing, regressive, Matkowski (on P), and strongly regressive on Q. In addition, let the ambient space X be (a-o,d)-complete. Then,*

II-a) *T is a strong Picard operator (modulo $(d, \preceq)$) when, in addition, T is (a-o,d)-continuous.*
II-b) *T is a Bellman Picard operator (modulo $(d, \preceq)$) when, in addition, $(\preceq)$ is (a-o,d)-selfclosed.*

The following topological version of this result is useful in practice. Let $(Y, \mathscr{T})$ be a (real) topological vector space; note that its (linear) topological properties (modulo $\mathscr{T}$) are ultimately characterized via

$\mathscr{V}(\theta) =$ the neighborhood filter of (the null element) $\theta \in Y$.

Further, let P be a (convex) cone in Y; endowed with the properties

(e12) P is proper, nondegenerate, pointed
(e13) P is *solid* ($\text{int}(P) \neq \emptyset$) and *closed* ($P = \text{cl}(P)$).

[Here, int and cl are the *interior* and *closure (= adherence)* operator, (modulo $\mathscr{T}$), respectively.] Note that, from the closed property, it easily follows that P is Archimedean; we do not give details. It remains to show that, from the solid property, P is algebraically solid. In fact, a more precise conclusion is to be derived; but, prior to this, some preliminary facts are needed.

Let A be some part of Y with

(e14) A is absorbent ($\theta \in aint(A)$) and convex.

The attached Minkowski functional

(e15) $M_A(y) = \inf\{\sigma > 0; y \in \sigma A\},\ y \in Y$

is well defined (from Y to R_+), and

$$M_A(y_1 + y_2) \leq M_A(y_1) + M_A(y_2),\ \ \forall y_1, y_2 \in Y \tag{14}$$

$$M_A(\lambda y) = \lambda M_A(y),\ \forall \lambda \in R_+, \forall y \in Y; \tag{15}$$

(i.e., $y \mapsto M_A(y)$ is *sublinear*). Moreover, the following inclusions hold

$$\text{aint}(A) \subseteq \{y \in Y; M_A(y) < 1\} \subseteq A \subseteq \{y \in Y; M_A(y) \leq 1\}. \tag{16}$$

The verification is immediate; see, for instance, Zălinescu [79, Chap. 1, Sect. 1.1].

Lemma 5. *Let the precise conditions about P be admitted. Then*

$$\text{int}(P) = \text{aint}(P);\ \textit{hence}\ \text{aint}(P) \neq \emptyset. \tag{17}$$

Proof. Fix some $a \in \text{int}(P)$ (nonempty, by hypothesis), and put $G = P - a$. Hence, $\theta \in \text{int}(G)$; we claim that, in such a case, $\text{aint}(G) \subseteq \text{int}(G)$. For the moment, G is absorbent convex and contains $\theta \in Y$ (as interior point). Let $b \in \text{aint}(G)$ be arbitrary fixed. By (16), we therefore have

$$M_G(b) < 1; \text{hence } M_G(\mu b) < 1, \text{ for some } \mu > 1,$$

which, again by (16), tells us that $c := \mu b$ is an element of G. But then, $b = (1/\mu)c \in \text{int}(G)$ (cf. Cristescu [17, Chap. 1, Sect. 2.4]); and the written inclusion follows. The proof is complete.

Summing up, the second main result is applicable to this topological setting. Note that the linear topology $\mathscr{T}$ on Y endowed with such properties of the (convex) cone P is normable; see, for instance, the survey paper by Janković et al. [35]. This topological version of the quoted result includes a related statement in Di Bari and Vetro [19], proved via different methods. In particular, when

φ is *linear*: $\varphi(y) = \lambda y$, $y \in P$, for some $\lambda \in [0, 1[$

the corresponding version of our second main result includes a related contribution due to Huang and Zhang [31]. Further aspects may be found in Choudhury and Metiya [13]; see also Pathak and Shahzad [53].

2 Meir–Keeler Maps in Quasi-Metric Spaces

2.1 Introduction

Let X be a nonempty set. Call the subset Y of X, *almost singleton* (in short: *asingleton*), provided [$y_1, y_2 \in Y$ implies $y_1 = y_2$], and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$. Take a metric $d : X \times X \to R_+ := [0, \infty[$ over X; the couple (X, d) will be then referred to as a *metric space*. Finally, let $T \in \mathscr{F}(X)$ be a *selfmap* of X. [Here, for each couple A, B of nonempty sets, $\mathscr{F}(A, B)$ denotes the class of all functions from A to B; when $A = B$, we write $\mathscr{F}(A)$ in place of $\mathscr{F}(A, A)$.] Denote $\text{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. The determination of such points is to be performed in the context below, comparable with the one in Rus [59, Chap. 2, Sect. 2.2]:

(pic-1) We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent; and a *globally Picard operator* (modulo d) if, in addition, $\text{Fix}(T)$ is an asingleton.

(pic-2) We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent with $\lim_n(T^n x) \in \text{Fix}(T)$; and a *globally strong Picard operator* (modulo d) if, in addition, $\text{Fix}(T)$ is an asingleton (hence, a singleton).

The basic result in this area is the 1922 one due to Banach [4]. Call the selfmap T, $(d;\alpha)$-*contractive* (where $\alpha \geq 0$), if

(a01) $d(Tx, Ty) \leq \alpha d(x, y)$, for all $x, y \in X$.

Theorem 4. *Assume that T is $(d;\alpha)$-contractive, for some $\alpha \in [0, 1[$. In addition, let (X, d) be complete. Then, T is globally strong Picard (modulo d).*

This result (referred to as *Banach's fixed point theorem*) found some basic applications to the operator equations theory. As a consequence, a multitude of extensions for it were proposed. From the perspective of this exposition, the implicit ones are of interest. These, roughly speaking, may be written as

(a02) $(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \in \mathscr{M}$, for all $x, y \in X$,

where $\mathscr{M} \subseteq R^6_+$ is a (nonempty) subset. In particular, when $\mathscr{M}$ is the zero-section of a certain function $F : R^6_+ \to R$; i.e.,

(a03) $\mathscr{M} = \{(t_1, \ldots, t_6) \in R^6_+; F(t_1, \ldots, t_6) \leq 0\}$,

the implicit contractive condition above has the familiar form:

(a04) $F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \leq 0$, for all $x, y \in X$.

For the explicit case of it, characterized as

(a05) $d(Tx, Ty) \leq G(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y))$, for all $x, y \in X$

(where $G : R^5_+ \to R_+$ is a function), some consistent lists of such contractions may be found in the survey papers by Rhoades [57], Collaco and E Silva [16], Kincses and Totik [42], as well as the references therein. And, for the implicit setting above, certain technical aspects have been considered by Leader [43] and Turinici [65].

A basic particular case of the implicit contractive property above is

(a06) $(d(Tx, Ty), d(x, y)) \in \mathscr{M}$, for all $x, y \in X$,

where $\mathscr{M} \subseteq R^2_+$ is a (nonempty) subset. The classical example in this direction is due to Meir and Keeler [47]. Further refinements of the method were proposed by Matkowski [45]; see also Ćirić [14] and Jachymski [34].

Recently, an interesting contractive condition of the type (a06) was introduced by Khojasteh et al. [41]. The so-called *simulation contractive methods* proposed there were appreciated as interesting enough to be used in various fixed point or coincidence point problems involving univalued and multivalued maps; see, for example, Du and Khojasteh [22]. On the other hand, certain efforts have been made toward a (technical) extension of these results; an outstanding contribution is the quasi-metric one due to Alsulami et al. [2]. Having these precise, it is natural to ask about the effectiveness of such methods with respect to the old ones we just sketched. A partial negative answer to this was given, in the metric framework, by Găvruţa et al. [27], where it has been shown that the simulation-type contractions over metrical spaces are in fact Meir–Keeler maps. In this exposition, we bring the discussion a bit further, by establishing that the simulation-type contractions over quasi-metric spaces are also Meir–Keller maps over metric spaces. Further aspects will be delineated in a separate paper.

2.2 Meir–Keeler Contractions

In the following, the concept of *Meir–Keeler contraction* is introduced, and a fixed point theorem involving such maps is given.

Let X be a nonempty set. By a *sequence* in X, we mean any mapping $x : N \to X$, where $N := \{0, 1, \ldots\}$ is the set of *natural numbers*. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$ or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with $(i(n); n \geq 0)$ being *divergent* [$i(n) \to \infty$ as $n \to \infty$] will be referred to as a *subsequence* of $(x_n; n \geq 0)$.

Let $d(.,.)$ be a *metric* over X; the couple (X, d) will be then referred to as a *metric space*. Define a sequential *d-convergence* $(\stackrel{d}{\longrightarrow})$ on X, according to: for each sequence $(x_n; n \geq 0)$ in X and each $x \in X$, $x_n \stackrel{d}{\longrightarrow} x$ iff $d(x_n, x) \to 0$; i.e.,

$$\forall \varepsilon > 0, \exists i(\varepsilon), \text{ such that } i(\varepsilon) \leq n \implies d(x_n, x) \leq \varepsilon;$$

referred to as: x is the *d-limit* of $(x_n; n \geq 0)$. Denote by $\lim_n(x_n)$ the set of all such elements; if it is nonempty, then $(x_n; n \geq 0)$ is called *d-convergent*.

The basic properties of our convergence structure are

(cv-1) $(\stackrel{d}{\longrightarrow})$ is *reflexive*:
$\forall u \in X$, the constant sequence $(x_n = u; n \geq 0)$ fulfills $x_n \stackrel{d}{\longrightarrow} u$.

(cv-2) $(\stackrel{d}{\longrightarrow})$ is *hereditary*:
$x_n \stackrel{d}{\longrightarrow} x$ implies $y_n \stackrel{d}{\longrightarrow} x$, for each subsequence (y_n) of (x_n).

(cv-3) $(\stackrel{d}{\longrightarrow})$ is *separated* (also referred to as d is *separated*):
$x_n \stackrel{d}{\longrightarrow} x$, $x_n \stackrel{d}{\longrightarrow} y$ imply $x = y$.

Note that, by the first and second properties above, $(\stackrel{d}{\longrightarrow})$ fulfills all general requirements (for a sequential convergence) imposed by Kasahara [38]. On the other hand, from the third property, $\lim_n(x_n)$ is an asingleton, for each sequence (x_n) in X. In particular, when (x_n) is d-convergent, we have

$$\lim_n(x_n) \text{ is a singleton, } \{z\} \text{ (where } z \in X);$$

for simplicity, we write this as $\lim_n(x_n) = z$.

Further, call the sequence $(x_n; \geq 0)$ in X, *d-Cauchy*, when $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$, i.e.,

$$\forall \varepsilon > 0, \exists j(\varepsilon), \text{ such that } j(\varepsilon) \leq m < n \implies d(x_m, x_n) \leq \varepsilon.$$

The class of all these will be indicated as Cauchy(X, d); some basic properties of it are described below:

(Cau-1) (inclusion of constant sequences):
$\forall u \in X$, the constant sequence $(x_n = u; n \geq 0)$ is d-Cauchy.
(Cau-2) (the hereditary property):
$(x_n; n \geq 0)$ is d-Cauchy implies $(y_n; n \geq 0)$ is d-Cauchy,
for each subsequence $(y_n; n \geq 0)$ of $(x_n; n \geq 0)$.

A weaker form of this concept is the following: call $(x_n; n \geq 0)$, *d-semi-Cauchy* when $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. In fact,

$$(\forall \text{ sequence}): d\text{-Cauchy} \implies d\text{-semi-Cauchy};$$

but the converse relation is not in general true.

Concerning the relationships between these two concepts, note that (by the properties of d), we have

$$(X, d) \text{ is } \textit{regular}: \text{any } d\text{-convergent sequence is } d\text{-Cauchy}.$$

The reciprocal is not in general true; but, if it holds too [any d-Cauchy sequence is d-convergent], then (X, d) is called *complete*.

Having these precise, take some $T \in \mathscr{F}(X)$. We say that T is *Meir–Keeler d-contractive* if

(b01) $x \neq y$ implies $d(Tx, Ty) < d(x, y)$.
expressed as: T is *strictly nonexpansive* (modulo d)
(b02) $\forall \varepsilon > 0, \exists \delta > 0$: $[\varepsilon < d(x, y) < \varepsilon + \delta] \Longrightarrow d(Tx, Ty) \leq \varepsilon$;
expressed as: T has the *Meir–Keeler property* (modulo d).

Note that, by the former of these, the Meir–Keeler property may be written as

(b03) $\forall \varepsilon > 0, \exists \delta > 0$: $[0 < d(x, y) < \varepsilon + \delta] \Longrightarrow d(Tx, Ty) \leq \varepsilon$.

Moreover, by the same property, T is *d-nonexpansive*:

(b04) $d(Tx, Ty) \leq d(x, y)$, $\forall x, y \in X$,

whence T is d-continuous on X. This, along with the obtained conclusion, tells us that the Meir–Keeler property may be also written, in a *complete form*, as

(b05) $\forall \varepsilon > 0, \exists \delta > 0$: $d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) \leq \varepsilon$.

The following technical aspects of the introduced convention must be noted.

(I) Call $T \in \mathscr{F}(X)$, *original Meir–Keeler d-contractive*, provided (b06) $\forall \varepsilon > 0, \exists \delta > 0$: $[\varepsilon \le d(x, y) < \varepsilon + \delta] \Longrightarrow d(Tx, Ty) < \varepsilon$.

The motivation of this convention comes from the fact that it is *exactly* the contractive condition in the 1969 paper by Meir and Keller [47]. The relationship with the Meir–Keller contractive condition we just introduced is characterized by the following auxiliary fact.

Proposition 3. *Under these conventions, the generic inclusion is valid (for all selfmaps $T \in \mathscr{F}(X)$):*

$$\textit{original Meir–Keeler d-contractive} \Longrightarrow \textit{Meir–Keeler d-contractive.} \tag{18}$$

Proof. Suppose that $T \in \mathscr{F}(X)$ is original Meir–Keeler d-contractive. Then,

(MK-1) If $x, y \in X$ are such that $d(x, y) > 0$, then with $\varepsilon = d(x, y)$ (and $\delta > 0$ taken by the underlying property),

$$\varepsilon \le d(x, y) < \varepsilon + \delta, \text{ whence, } d(Tx, Ty) < \varepsilon = d(x, y),$$

which tells us that T is strictly nonexpansive (modulo d).

(MK-2) Let $\varepsilon > 0$ be given, and $\delta > 0$ be assured by the original Meir–Keeler d-contractive property. If $x, y \in X$ are such that $\varepsilon < d(x, y) < \varepsilon + \delta$ then (as $\varepsilon \le d(x, y) < \varepsilon + \delta$) we have (by the underlying condition) $d(Tx, Ty) < \varepsilon$; hence $d(Tx, Ty) \le \varepsilon$; and this tells us that T is Meir–Keeler contractive (modulo d).

Putting these together, it results that our assertion is true.

(II) Call $T \in \mathscr{F}(X)$, *Cirić d-contractive* [14], provided

(b07) $\forall \varepsilon > 0, \exists \delta > 0$: $[\varepsilon < d(x, y) < \varepsilon + \delta] \Longrightarrow d(Tx, Ty) \le \varepsilon$.

Clearly, this is nothing else than the Meir–Keeler property (modulo d) from our Meir–Keeler d-contractive condition. Hence, we have the generic inclusion

$$(\forall T \in \mathscr{F}(X)): \text{Meir–Keeler } d\text{-contractive} \Longrightarrow \text{Cirić } d\text{-contractive.} \tag{19}$$

Moreover, according to Cirić [14], this contractive condition imposed by him will suffice for deducing a (globally strong) Picard property for the considered class of selfmaps. As a consequence, the strict nonexpansive property (modulo d) we added in our convention seems to be superfluous for getting the same conclusion. However, as proved in Jachymski [34], the situation is exactly opposite; a verification of this fact is ultimately assured by the developments below.

Remember that $\text{Fix}(T) = \{x \in X; x = Tx\}$ denotes the class of *fixed points* of T in X. The determination of such elements is to be performed in the Picard context we just exposed. We may now ask whether the introduced Meir–Keeler d-contractive condition may give us conclusions like in Banach's fixed point principle. The answer is positive; in fact, the following result (referred to as *Meir–Keeler fixed point theorem*) is holding (under the described setting).

Theorem 5. *Suppose that T is Meir–Keeler d-contractive. In addition, let (X,d) be complete. Then, T is a globally strong Picard operator (modulo d).*

Proof. By the strict nonexpansive condition, $\mathrm{Fix}(T)$ is an asingleton; so, it remains to establish that T is a strong Picard operator (modulo d). Fix some $x_0 \in X$, and put $(x_n = T^n x_0; n \geq 0)$. If $x_k = x_{k+1}$ for some $k \geq 0$, we are done; so, without loss, one may assume that

$x_n \neq x_{n+1}$ (i.e., $\rho_n := d(x_n, x_{n+1}) > 0$), for all n.

The argument will be divided into several steps.

Part 1. Again by the strict nonexpansive condition, $\rho_{n+1} < \rho_n$, for all $n \geq 0$, wherefrom $(\rho_n; n \geq 0)$ is a strictly descending sequence in $R^0_+ :=]0,\infty[$. As a consequence, $\rho := \lim_n \rho_n$ exists in R_+, and $\rho_n > \rho$, $\forall n$. Assume that $\rho > 0$ and let $\sigma > 0$ be the number given by the Meir–Keeler property. By definition, there exists a rank $n(\sigma)$ such that

$$n \geq n(\sigma) \Longrightarrow \rho < \rho_n = d(x_n, x_{n+1}) < \rho + \sigma.$$

This, by the quoted condition, yields (for the same n)

$$(\rho <)\ \rho_{n+1} = d(Tx_n, Tx_{n+1}) \leq \rho$$

contradiction. Hence, $\rho = 0$; so that, $(x_n; n \geq 0)$ is a d-semi-Cauchy sequence.

Part 2. Let $\varepsilon > 0$ be arbitrary fixed and $\delta > 0$ be the number associated by the Meir–Keeler property; without loss, one may assume that $\delta < \varepsilon$. By the obtained d-semi-Cauchy property, there exists a rank $n(\delta) \geq 0$ such that

$$d(x_n, x_{n+1}) < \delta/2, \text{ for all } n \geq n(\delta). \tag{20}$$

We claim that

$$\forall i \geq 1 : \ [d(x_n, x_{n+i}) < \varepsilon + \delta/2, \ \forall n \geq n(\delta)]; \tag{21}$$

wherefrom the d-Cauchy property of $(x_n; n \geq 0)$ is clear. To do this, an induction argument upon $i \geq 1$ will be used. The case $i = 1$ is evident, by the choice of $n(\delta)$. Assume that our relation holds for all $i \in \{1,\ldots,p\}$, where $p \geq 1$; we must establish that it holds as well for $i = p+1$. So, let $n \geq n(\delta)$ be arbitrary fixed. From the inductive hypothesis,

$$d(x_n, x_{n+p}) < \varepsilon + \delta/2 < \varepsilon + \delta.$$

Combining with the (complete form of) Meir–Keeler property gives

$$d(x_{n+1}, x_{n+p+1}) = d(Tx_n, Tx_{n+p}) \leq \varepsilon.$$

This, along with the triangular inequality, yields

$$d(x_n, x_{n+p+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p+1}) < \varepsilon + \delta/2,$$

and the assertion is retainable.

Part 3. By the completeness assumption, $x_n \xrightarrow{d} z$ as $n \to \infty$, for some (uniquely determined) $z \in X$. As T is d-continuous (see above), $y_n := Tx_n \xrightarrow{d} Tz$. On the other hand, $(y_n = x_{n+1}; n \geq 0)$ is a subsequence of (x_n); whence $y_n \xrightarrow{d} z$; and this yields (as d = separated), $z = Tz$; i.e., $z \in \text{Fix}(T)$. The proof is complete.

Note that further extensions of this result are available; see, for instance, Jachymski [33] or Samet et al. [60]. But, for the objectives of our exposition, this will suffice.

2.3 *Quasi-Metric Structures*

Let X be a nonempty set. By a *quasi-metric* on X, we mean any map $b : X \times X \to R_+$ with the properties

(qm-1) $x = y$ iff $b(x, y) = 0$ (reflexive sufficient)
(qm-2) $b(x, z) \leq b(x, y) + b(y, z)$, $\forall x, y, z \in X$ (triangular);

in this case, (X, b) will be referred to as a *quasi-metric* space. Given such an object, $b(.,.)$, let $c : X \times X \to R_+$ stand for its *conjugate*

(c01) $c(x, y) = b(y, x)$, $x, y \in X$.

Clearly, $c(.,.)$ is again a quasi-metric on X. Moreover, the attached map

(c02) $d(x, y) = \max\{b(x, y), c(x, y)\}$, $x, y \in X$ (in short: $d = \max(b, c)$)

is a *(standard) metric* on X; we do not give details.

The concept of quasi-metric seems to have a long tradition in metrical spaces theory. For example, in his 2001 PhD Thesis, Hitzler [30, Chap. 1, Sect. 1.2] introduced such a notion as a useful tool for performing a topological study of logic semantics. Later, in a 2004 paper, Turinici [69] used the same concept—referred to as "reflexive triangular sufficient pseudometric"—with the aim of establishing a Caristi-Kirk fixed point theorem over such structures. From a "purely" fixed point perspective, the quasi-metric techniques have been used in 2012 by Jleli and Samet [36]. Further aspects of this theory—related to simulation method—were developed in a recent 2014 contribution due to Alsulami et al. [2].

In the following, some convergence and Cauchy structures are introduced over quasi-metric spaces. Let X be a nonempty set. Take a quasi-metric $b(.,.)$ on X, and let $c(.,.)$ stand for its conjugate map; remember that $c(.,.)$ is a quasi-metric too and $d = \max(b, c)$ is a (standard) metric on X.

Let (x_n) be a sequence in X and x be some point of X. Define $x_n \xrightarrow{b} x$ as $b(x_n, x) \to 0$ as $n \to \infty$, i.e.,

for each $\varepsilon > 0$, there exists $i(\varepsilon)$, such that $n \geq i(\varepsilon) \implies b(x_n, x) \leq \varepsilon$.

The set of all such x will be denoted as $b - \lim_n(x_n)$; when it is nonempty, we say that (x_n) is *b-convergent*. Further, a corresponding property is to be defined for the conjugated quasi-metric $c(.,.)$. Namely, for the same (x_n) and x, define $x_n \xrightarrow{c} x$ as $c(x_n, x) \to 0$ as $n \to \infty$, i.e.,

for each $\varepsilon > 0$, there exists $i(\varepsilon)$, such that $n \geq i(\varepsilon) \implies c(x_n, x) \leq \varepsilon$.

The set of all such x will be denoted as $c - \lim_n(x_n)$; when it is nonempty, we say that (x_n) is *c-convergent*. Note, at this moment, that

$$x_n \xrightarrow{d} x \text{ iff } [x_n \xrightarrow{b} x \text{ and } x_n \xrightarrow{c} x]. \tag{22}$$

The set of all these points will be denoted as $d - \lim_n(x_n)$; when it is nonempty, we say that (x_n) is d-convergent. Clearly, the generic property is valid

$$(\forall \text{ sequence}): d\text{-convergent} \implies [b\text{-convergent and } c\text{-convergent}]. \tag{23}$$

Let (x_n) be a sequence in X; we call it *b-Cauchy*, provided $b(x_m, x_n) \to \infty$, as $m, n \to \infty$, $m \leq n$, i.e.,

$\forall \varepsilon > 0$, there exists $j(\varepsilon)$, such that $j(\varepsilon) \leq m \leq n \implies b(x_m, x_n) \leq \varepsilon$.

Accordingly, we say that the sequence (x_n) is c-Cauchy, provided $c(x_m, x_n) \to \infty$, as $m, n \to \infty$, $m \leq n$, i.e.,

$\forall \varepsilon > 0$, there exists $j(\varepsilon)$, such that $j(\varepsilon) \leq m \leq n \implies c(x_m, x_n) \leq \varepsilon$.

By this very definition, the generic property is valid

$$(\forall \text{ sequence}): d\text{-Cauchy} \iff [b\text{-Cauchy and } c\text{-Cauchy}]. \tag{24}$$

Finally, remember that (the metric space) (X, d) is complete, when each d-Cauchy sequence is d-convergent.

A basic example of quasi-metric structure is to be given as below. Let X be a nonempty set. By a *Mustafa-Sims metric* (in short: *MS-metric*) on X, we mean any map $G : X \times X \times X \to R_+$, with

(ms-1) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$, $\forall x, y, z \in X$ (symmetric)
(ms-2) $(x = y = z) \implies D(x, y, z) = 0$ (reflexive)
(ms-3) $G(x, x, y) = 0$ implies $x = y$ (plane sufficient)
(ms-4) $G(x, x, y) \leq G(x, y, z)$, $\forall x, y, z \in$X, $y \neq z$ (MS-property)
(ms-5) $G(x, y, z) \leq G(x, u, u) + G(u, y, z)$, $\forall x, y, z, u \in X$ (MS-triangular);

in this case, the couple (X, G) will be referred to as a *Mustafa-Sims metric space* (in short: *MS-metric space*).

The introduction of such objects, performed by Mustafa and Sims [50], is related to the necessity of correcting some technical drawbacks of Dhage metrical structures [18]. However, as established in Jleli and Samet [36], a fixed point theory over Mustafa-Sims metric spaces is pseudometric in nature. This is essentially deductible from the construction and auxiliary statement below.

Let (X, G) be a Mustafa-Sims metric space. Define a quadruple of maps $b, c, d, e : X \times X \to R_+$ according to: for each $x, y \in X$,

(c03) $b(x, y) = G(x, y, y), c(x, y) = G(x, x, y) = b(y, x)$
(c04) $d(x, y) = \max\{b(x, y), c(x, y)\}, e(x, y) = b(x, y) + c(x, y)$.

Proposition 4. *Under the above notations,*

K-1) *The mappings $b(.,.)$ and $c(.,.)$ are triangular and reflexive sufficient; hence, these are quasi-metrics on X.*
K-2) *The mappings $d(.,.)$ and $e(.,.)$ are triangular, reflexive sufficient, and symmetric; hence, these are (standard) metrics on X.*

Proof. K-1): It will suffice establishing the assertions concerning the map $b(.,.)$. The reflexive sufficient property is a direct consequence of symmetric, reflexive, and plane sufficient properties of G. On the other hand, the triangular property is a direct consequence of the MS-triangular property of G. In fact, by this condition, we have (taking $z = y$)

$$G(x, y, y) \leq G(x, u, u) + G(u, y, y);$$

and, from this, we are done.
K-2): Evident, by the involved definition.

Note, finally, that by the second part of this statement, we may ask whether the fixed point theory over Mustafa-Sims metric spaces is metrical in nature. A (positive) partial answer to this may be found in An et al. [3]; see also Turinici [73].

2.4 Main Result

Let X be a nonempty set. Take a quasi-metric $b(.,.)$ on X, and let $c(.,.)$ stand for its conjugate map; remember that $c(.,.)$ is a quasi-metric too and $d = \max(b, c)$ is a (standard) metric on X.

Let $\Omega \subseteq R_+^0 \times R_+^0$ be a relation over R_+^0; as usually, we write $(t, s) \in \Omega$ as $t\Omega s$. Call this object, *Meir–Keeler normal* provided

(d01) $t, s \in R_+^0$ and $t\Omega s$ imply $t < s$
(referred to as Ω is *upper diagonal*)

(d02) there are no sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R^0_+ and no elements $\lambda \in R^0_+$, with $(t_n \Omega s_n$, for all $n \geq 0)$ and $(t_n \to \lambda+, s_n \to \lambda+)$ (referred to as Ω has the *asymptotic Meir–Keeler property*).

Here, given a sequence $(r_n; n \geq 0)$ in R and an element $r \in R$, define

$r_n \to r+$ provided $(r_n > r, \forall n)$, and $r_n \to r$.

Given the relation Ω over R^0_+, call the selfmap T of X, $(b; \Omega)$*-contractive*, if

(d03) $b(Tx, Ty)\Omega b(x, y)$, $\forall x, y \in X, x \neq y, Tx \neq Ty$.

(Clearly, $x \neq y$ follows from $Tx \neq Ty$; but, this is not important for us.) Note that, by simply interchanging x with y, this relation also writes

(d04) $c(Tx, Ty)\Omega c(x, y)$, $\forall x, y \in X, x \neq y, Tx \neq Ty$,

or, in other words, T is $(c; \Omega)$*-contractive* as well.

Our main result in this exposition is

Theorem 6. *Assume that T is $(b; \Omega)$-contractive, for some Meir–Keeler normal relation Ω over R^0_+. In addition, let (X, d) be complete. Then*

I) *T is Meir–Keeler d-contractive.*
II) *T is a globally strong Picard operator (modulo d).*

Proof. I): There are two steps to be passed.

Step 1. Let $x, y \in X$ be such that
$x \neq y$; hence, $b(x, y), c(x, y), d(x, y) > 0$.
If $Tx = Ty$, then $d(Tx, Ty) = 0 < d(x, y)$; so, we may assume that
$Tx \neq Ty$, whence $b(Tx, Ty), c(Tx, Ty), d(Tx, Ty) > 0$.
As Ω is upper diagonal, we get (by the contractive condition applied to b and c)

$$b(Tx, Ty) < b(x, y); c(Tx, Ty) < c(x, y), \text{ whence } d(Tx, Ty) < d(x, y).$$

This, by the arbitrariness of (x, y) (taken as before), assures us that T is strictly nonexpansive (modulo d).

Step 2. Assume by contradiction that T does not have the Meir–Keeler property (modulo d); i.e., for some $\varepsilon > 0$,
for each $\delta > 0$, there exists $(x_\delta, y_\delta) \in X \times X$, such that
$d(x_\delta, y_\delta) < \varepsilon + \delta$, $d(Tx_\delta, Ty_\delta) > \varepsilon$.

Taking a zero converging sequence $(\delta_n; n \geq 0)$ in R^0_+, we get a couple of sequences $(x_n; n \geq 0)$ and $(y_n; n \geq 0)$ in X, so as

$$(\forall n): \quad d(x_n, y_n) < \varepsilon + \delta_n, \ d(Tx_n, Ty_n) > \varepsilon; \tag{25}$$

or equivalently (by the symmetry of $d(.,.)$)

$$(\forall n): \quad d(y_n, x_n) < \varepsilon + \delta_n, \; d(Ty_n, Tx_n) > \varepsilon. \tag{26}$$

Denote, for simplicity,

$N_b = \{n \in N; d(Tx_n, Ty_n) = b(Tx_n, Ty_n)\}$,
$N_c = \{n \in N; d(Tx_n, Ty_n) = c(Tx_n, Ty_n)\}$.

As $N = N_b \cup N_c$, at least one of these subsets is infinite. Without loss, one may assume that N_b is infinite; otherwise, if N_c is infinite, we have

$$d(Ty_n, Tx_n) = b(Ty_n, Tx_n), \; \forall n \in N_c;$$

so, by simply passing to the pairs $((y_n, x_n); n \geq 0)$, we fall [via (26)] within the preceding alternative. By the imposed condition, $N_b = \{i(n); n \geq 0\}$, where $n \mapsto i(n)$ is strictly ascending. So, if we put

$u_n = x_{i(n)}, v_n = y_{i(n)}, n \geq 0,$

one gets, via (25) above, for all $n \geq 0$,

$$b(u_n, v_n) \leq d(u_n, v_n) < \varepsilon + \delta_{i(n)}, \; b(Tu_n, Tv_n) > \varepsilon. \tag{27}$$

Note that, as a consequence of the second relation, we must have, for all n,

$$u_n \neq v_n, Tu_n \neq Tv_n \; (hence, b(u_n, v_n) > 0, b(Tu_n, Tv_n) > 0).$$

By the contractive condition, we therefore get

$$b(Tu_n, Tv_n) \Omega b(u_n, v_n), \; \forall n; \tag{28}$$

so that, by the super-diagonal property of Ω and (27),

$$\varepsilon < b(Tu_n, Tv_n) < b(u_n, v_n) < \varepsilon + \delta_{i(n)}, \; \forall n \geq 0,$$

wherefrom (by a limit process)

$$b(Tu_n, Tv_n) \to \varepsilon+, b(u_n, v_n) \to \varepsilon+, \text{as } n \to \infty. \tag{29}$$

This, along with (28), contradicts the asymptotic Meir–Keeler property of Ω and proves that T has the Meir–Keeler property (modulo d).

II): By the preceding fact, the Meir–Keeler fixed point theorem is applicable to our data; and from this, we derive all desired conclusions.

2.5 Particular Aspects

In the following, a basic example of such contractions will be considered.

Given the sequence $(t_n; n \geq 0)$ in R, define

$\liminf_n(t_n) = \sup_k \inf\{t_k, t_{k+1}, \ldots\}$, $\limsup_n(t_n) = \inf_k \sup\{t_k, t_{k+1}, \ldots\}$.

By definition, we have (for any such sequence)

$$\liminf_n(t_n) \leq \limsup_n(t_n);$$

when equality occurs, the common value of these, t say, is denoted as $\lim_n(t_n)$, and we indicate this in the usual way: $t_n \to t$ (as $n \to \infty$). [Clearly, the case of $t \in \{-\infty, \infty\}$ cannot be avoided; but, if $t \notin \{-\infty, \infty\}$, we say that $(t_n; n \geq 0)$ is *convergent*.] A basic circumstance when this limit property holds (in a generalized sense) is characterized as

$(t_n; n \geq 0)$ is monotone (ascending or descending).

Then, by the definitions above,

$$\lim_n(t_n) = \sup_n(t_n), \text{ when } (t_n; n \geq 0) \text{ is ascending.}$$
$$\lim_n(t_n) = \inf_n(t_n), \text{ when } (t_n; n \geq 0) \text{ is descending.}$$

Since the verification is almost immediate, we do not give details.

Let the function $\varphi \in \mathscr{F}(R_+)$ be arbitrary for the moment. We say that $\xi : R_+^0 \times R_+^0 \to R$ is a φ-simulation function, provided

(e01) $\xi(.,.)$ is *strict φ-nonexpansive*:
$\xi(t,s) < \varphi(s) - \varphi(t)$, for all $t, s \in R_+^0$.

(e02) $\xi(.,.)$ is *separable*:
if (t_n) and (s_n) are sequences in R_+^0 and λ is an element in R_+^0 with $(t_n \to \lambda+, s_n \to \lambda+)$, then $\limsup_n \xi(t_n, s_n) < 0$.

The following auxiliary fact establishes the necessary connections with our previous developments.

Proposition 5. *Let $\xi : R_+^0 \times R_+^0 \to R$ be a φ-simulation function, for some $\varphi \in \mathscr{F}(R_+)$ endowed with the property*

(e03) φ is increasing on R_+^0.

Then, the relation Ω on R_+^0 introduced as

(e04) $t\Omega s$ iff $\xi(t,s) \geq 0$

is Meir–Keeler normal (see above).

Proof. There are two steps to be checked.

Step 1. Let $t, s \in R_+^0$ be such that $t\Omega s$. Combining with the strict φ-nonexpansive property of $\xi(.,.)$, we have

$$0 \le \xi(t,s) < \varphi(s) - \varphi(t).$$

This yields $\varphi(t) < \varphi(s)$, wherefrom (as φ is increasing), $t < s$. From the arbitrariness of couple (t,s), we then get that Ω is upper diagonal.

Step 2. Suppose that there exists a couple of sequences (t_n) and (s_n) in R_+^0 and an element λ in R_+^0, such that

$$(t_n \Omega s_n, \text{ for all } n); (t_n \to \lambda+ \text{ and } s_n \to \lambda+ \text{ as } n \to \infty).$$

By the very definition of our relation Ω, the former condition yields

$$0 \le \xi(t_n, s_n) < \varphi(s_n) - \varphi(t_n), \forall n.$$

Combining with the convergence-type conditions involving our sequences, one derives [via $\lim_n \varphi(t_n) = \lim_n \varphi(s_n) = \varphi(\lambda+)$]

$$0 \le \limsup_n \xi(t_n, s_n) \le \varphi(\lambda+) - \varphi(\lambda+) = 0, \text{ whence } \limsup_n \xi(t_n, s_n) = 0,$$

in contradiction with the separable property of $\xi(.,.)$. Hence, Ω has the asymptotic Meir–Keeler property, and the proof is complete.

Now, by simply combining this with our main result, we get the following practical statement. Let X be a nonempty set. Take a quasi-metric $b(.,.)$ on X, and let $c(.,.)$ stand for its conjugate map; remember that $c(.,.)$ is a quasi-metric too and $d = \max(b,c)$ is a (standard) metric on X. Further, let $T \in \mathscr{F}(X)$ be a map and $\xi : R_+^0 \times R_+^0 \to R$ be a function. We say that T is $(b;\xi)$*-contractive*, provided

(e05) $\xi(b(Tx,Ty), b(x,y)) \ge 0, \forall x,y \in X, x \ne y, Tx \ne Ty.$

Note that, by simply passing to (y,x), this relation also writes

(e06) $\xi(c(Tx,Ty), c(x,y)) \ge 0, \forall x,y \in X, x \ne y, Tx \ne Ty;$

or, in other words: T is $(c;\xi)$*-contractive*.

Theorem 7. *Suppose that T is $(b;\xi)$-contractive, for some φ-simulation function $\xi : R_+^0 \times R_+^0 \to R$, where $\varphi \in \mathscr{F}(R_+)$ is increasing on R_+^0. In addition, let (X,d) be complete. Then, T is a globally Picard operator (modulo d).*

In particular, when

$$\varphi(t) = t, t \in R_+ \text{ (i.e., } \varphi = \text{the identical function of } \mathscr{F}(R_+)),$$

this result is just the one in Alsulami et al. [2]. Further aspects may be found in Du and Khojasteh [22]; see also Roldan et al. [58].

Finally, it is worth noting that, in the metric structure (X,d), the program of reducing different classes of contractions to the Meir–Keeler ones includes

(i) the Matkowski-type d-contractions [44]

(M-con) $d(Tx, Ty) \leq \varphi(d(x, y)), \forall x, y \in X,$
where $\varphi \in \mathscr{F}(R_+)$ is an (increasing regressive) Matkowski function

(ii) the Dutta-Choudhury d-contractions [23]

(DC-con) $\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \forall x, y \in X,$
where (ψ, φ) is a couple of generalized altering functions in $\mathscr{F}(R_+)$

(iii) the Wardowski-type d-contractions [77]

(W-con) $\eta + F(d(Tx, Ty)) \leq F(d(x, y)), \forall x, y \in X, x \neq y, Tx \neq Ty,$
where $\eta > 0$ is a constant and $F : R^0_+ \to R$ is increasing

The first reducing question was clarified in Jachymski [33], and the second one is worked out in Turinici [74]. Concerning the last problem, some partial answers were obtained by Turinici [72]. However, it is not hard to see that, under the accepted conditions, the relation Ω over R^0_+ introduced as

(e07) $(t, s \in R^0_+)$: $t\Omega s$ iff $\eta + F(t) \leq F(s)$

is Meir–Keeler normal. As a consequence, any Wardowski-type d-contraction is Meir–Keeler d-contractive; so that the reducing program involving this class is clarified too. Further aspects will be delineated elsewhere.

3 Ultrametric Fixed Points in Reduced Axiomatic Systems

3.1 Introduction

Throughout this exposition, the axiomatic system in use is Zermelo–Fraenkel's (in short: ZF), as described by Cohen [15, Chap. 2]. The notations and basic facts about its axioms are more or less usual.

Remember that, an outstanding part of it is the *Axiom of Choice* (abbreviated: AC), which, in a convenient manner, may be written as

(AC) For each nonempty set X, there exists a (selective) function
$f : (2)^X \to X$ with $f(Y) \in Y$, for each $Y \in (2)^X$.

[Here, $(2)^X$ denotes the class of all nonempty parts in X.] There are many logical equivalents of (AC); see, for instance, Moore [49, Appendix 2]. A basic one is the *Zorn–Bourbaki Maximal Principle* (in short: ZB), expressed as

(ZB) Let the partially ordered set $(X, \leq)$ be *inductive* [any totally ordered part C of X is *bounded above*: $C \leq b$ (i.e., $x \leq b$, $\forall x \in C$), for some $b \in X$]. Then, for each (starting) $u \in X$, there exists a maximal element $v \in X$
(in the sense: $v \leq z \in X$ implies $v = z$), with $u \leq v$;

for a direct proof of this (avoiding transfinite induction), see Bourbaki [6].

Let X be a nonempty set. By a *sequence* in X, we mean any mapping $x : N \to X$, where $N := \{0, 1, \ldots\}$ is the set of *natural* numbers. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$ or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with

$(i(n); n \geq 0)$ is *divergent* [i.e.: $i(n) \to \infty$ as $n \to \infty$]

will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Call the subset Y of X, *almost singleton* (in short: *asingleton*) provided [$y_1, y_2 \in Y$ implies $y_1 = y_2$]; and *singleton* if, in addition, Y is nonempty; note that in this case, $Y = \{y\}$, for some $y \in X$. Further, let $d : X \times X \to R_+ := [0, \infty[$ be a *metric* over X; the couple (X, d) will be termed a *metric space*. Finally, let $T \in \mathscr{F}(X)$ be a selfmap of X. [Here, for each couple A, B of nonempty sets, $\mathscr{F}(A, B)$ stands for the class of all functions from A to B; when $A = B$, we write $\mathscr{F}(A)$ in place of $\mathscr{F}(A, A)$.] Denote $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. In the metrical fixed point theory, such points are to be determined according to the context below, comparable with the one described in Rus [59, Chap. 2, Sect 2.2]:

(pic-1) We say that T is a *Picard operator* (modulo d) if, for each $x \in X$, the iterative sequence $(T^n x; n \geq 0)$ is d-convergent.

(pic-2) We say that T is a *strong Picard operator* (modulo d) if, for each $x \in X$, $(T^n x; n \geq 0)$ is d-convergent with $\lim_n(T^n x) \in \mathrm{Fix}(T)$.

(pic-3) We say that T is *fix-asingleton* (resp., *fix-singleton*) if $\mathrm{Fix}(T)$ is asingleton (resp., singleton).

In this perspective, a basic answer to the posed question is the 1922 one due to Banach [4]. Given $\alpha \geq 0$, let us say that T is $(d; \alpha)$-*contractive*, provided

(a01) $d(Tx, Ty) \leq \alpha d(x, y)$, for all $x, y \in X$.

Theorem 8. *Suppose that T is $(d; \alpha)$-contractive, for some $\alpha \in [0, 1[$. In addition, let (X, d) be complete. Then, T is a strong Picard operator (modulo d) and fix-asingleton (hence, fix-singleton).*

This result—referred to as *Banach's contraction principle*—found a multitude of applications in operator equations theory; so, it was the subject of many extensions. A natural way of doing this is by considering "functional" contractive conditions

(a02) $d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$,
for all $x, y \in X$,

where $F : R_+^5 \to R_+$ is a function. Some important results in the area have been established by Boyd and Wong [8], Matkowski [44], and Leader [43]. For more details about other possible choices of F, we refer to the 1977 paper by Rhoades [57]; some extensions of these to quasi-ordered structures may be found in Turinici [66]. Further, a natural extension of the contractive condition above is

(a03) (T is *d-strictly nonexpansive*):
$d(Tx, Ty) < d(x, y)$, for all $x, y \in X, x \neq y$.

Note that a fixed point for such maps is to be reached when (X, d) is a *compact metric space*; cf. Edelstein [24]. Another circumstance when this conclusion holds is that of (X, d) being a *(transfinite) Cantor complete* ultrametric space; see, for instance, Petalas and Vidalis [54]. In this last case, a basic tool used in authors' proof is (ZB) (= the *Zorn–Bourbaki maximal principle*) or equivalently (see above) (AC) (= the *Axiom of Choice*); hence, this fixed point result is valid in the *complete* Zermelo–Fraenkel system (ZF). However, since all arguments used there are countable in nature, it is highly expectable that a denumerable version of (ZB) should suffice for the result's conclusion to hold. It is our aim in the present exposition to prove that this is indeed the case. Precisely, we show that the Zorn–Bourbaki maximal principle appearing there may be replaced with a countable version of it—namely, the Brezis–Browder ordering principle [9]—to solve the posed fixed point question; hence, the Petalas-Vidalis result is ultimately deductible in the *reduced* Zermelo–Fraenkel system (ZF-AC+DC), where (DC) is the *principle of dependent choices*. Note that the proposed reasoning is applicable as well to many other statements of this type, such as the ones due to Mishra and Pant [48]. Further aspects will be delineated in a separate paper.

3.2 Brezis–Browder Principles

Let M be a nonempty set. Take a *quasi-order* $(\leq)$ on M, i.e.,

$(\leq)$ is *reflexive* $(x \leq x, \forall x \in X)$
$(\leq)$ is *transitive* $(x \leq y, y \leq z \implies x \leq z)$;

the pair $(M, \leq)$ will be then referred to as a *quasi-ordered structure*. Let also $\varphi : M \to R_+$ be a function. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when $z \leq w \in M$ implies $\varphi(z) = \varphi(w)$. A basic result about such points is the 1976 Brezis–Browder ordering principle [9] (in short: BB).

Proposition 6. *Suppose that the quasi-ordered structure* $(M, \leq)$ *and the function* φ *(taken as before) fulfill*

(b01) $(M, \leq)$ *is sequentially inductive: each ascending sequence has an upper bound (modulo* $(\leq)$*)*
(b02) φ *is* $(\leq)$*-decreasing* $(x \leq y \implies \varphi(x) \geq \varphi(y))$.

Then, for each $u \in M$*, there exists a* $(\leq, \varphi)$*-maximal* $v \in M$ *with* $u \leq v$.

(A) In particular, assume that (in addition)

$(\leq)$ is *antisymmetric* ($x \leq y$ and $y \leq x$ imply $x = y$).

We then say that $(\leq)$ is a *(partial) order* on M and the pair $(M, \leq)$ will be called a *(partially) ordered structure*. In this case, by an appropriate choice of our structure (related to existence of functions $\varphi : M \to R_+$ fulfilling strict versions of (b02)), one gets a countable variant of the Zorn–Bourbaki maximal principle [6]. Some conventions are needed. Let $(<)$ stand for the associated relation

$x < y$ iff $x \leq y$ and $x \neq y$

Clearly,

$(<)$ is *irreflexive* ($x < x$ is false, $\forall x \in M$)
$(<)$ is *transitive* ($x < y$ and $y < z$ imply $x < z$);

as a consequence of this, $(<)$ will be referred to as the *strict order* attached to $(\leq)$. Call the point $z \in M$, $(\leq)$-*maximal*, provided

(b03) $w \in M, z \leq w \implies z = w$
or equivalently $M(z,<)(:= \{x \in M; z < x\})$ is empty.

The following (Zorn–Bourbaki) maximal version of (BB) (denoted, for simplicity, as (BB-Z)) is now available.

Proposition 7. *Suppose that the (partially) ordered structure* $(M, \leq)$ *is such that*

(b04) $(M,<)$ *is sequentially inductive:*
each $(<)$*-ascending sequence in M has an upper bound in M (modulo* $(<)$*).*
(b05) $(M,<)$ *is admissible:*
there exists at least one function $\varphi : M \to R_+$ *with the* $(<)$*-decreasing property (*$x < y \Longrightarrow \varphi(x) > \varphi(y)$*).*

Then, $(\leq)$ *is a Zorn order, in the sense: for each* $u \in M$*, there exists a* $(\leq)$*-maximal* $v \in M$ *with* $u \leq v$*.*

Proof. There are two steps to be passed.

Step 1. We claim that, under these conditions, $(M, \leq)$ is sequentially inductive. In fact, let $(x_n; n \geq 0)$ be a $(\leq)$-ascending sequence in M. If the alternative below is in force,

there exists $k \geq 0$, such that $x_k = x_n$, for all $n > k$,

we are done, because $y := x_k$ is an upper bound of $(x_n; n \geq 0)$. Suppose that the opposite alternative is true:

for each $k \geq 0$, there exists $h > k$ with $x_k < x_h$.

In this case, we get a $(<)$-ascending sequence of ranks $(i(n); n \geq 0)$, such that the subsequence $(y_n := x_{i(n)}; n \geq 0)$ is $(<)$-ascending. By the admitted hypothesis, there exists $y \in M$ such that $y_n < y$, for all n. This, along with the $(\leq)$-ascending property of $(x_n; n \geq 0)$, gives $x_n < y$, for each n, and the claim follows.

Step 2. As $(M,<)$ is admissible, there exists at least one function $\varphi : M \to R_+$ with the $(<)$-decreasing property: $x < y \Longrightarrow \varphi(x) > \varphi(y)$. Note that, by the very definition of our strict order $(<)$, we have the (converse) representation formula

$x \le y$ iff either $x < y$ or $x = y$.

As a direct consequence of this, one gets that

$$\varphi \text{ is } (\le)\text{-decreasing } (x \le y \Longrightarrow \varphi(x) \ge \varphi(y)).$$

Putting these together, (BB) is applicable to $(M, \le)$ and φ. From this principle, we are assured that, given $u \in M$, there exists a $(\le, \varphi)$-maximal $v \in M$ with $u \le v$. Suppose by contradiction that $v < w$, for some $w \in M$. As φ is $(<)$-decreasing, this gives $\varphi(v) > \varphi(w)$, in contradiction with the $(\le, \varphi)$-maximal property of v. Hence, v is $(\le)$-maximal; and we are done.

Note that, for the moment, (BB) $\Longrightarrow$ (BB-Z) in the *strongly reduced axiomatic system* (ZF-AC). On the other hand, this statement includes (see below) Ekeland's variational principle [25] (in short: EVP). As a consequence, many extensions of (BB) were proposed; see, for instance, Hyers et al. [32, Chap. 5]. For each (countable) variational principle (VP) of this type, one therefore has (VP) $\Longrightarrow$ (BB) $\Longrightarrow$ (EVP); so, we may ask whether these inclusions are effective. As we shall see, the answer to this is negative.

(B) Let M be a nonempty set and $\mathscr{R} \subseteq M \times M$ be a (nonempty) *relation* over M; for simplicity, we sometimes write $(x, y) \in \mathscr{R}$ as $x\mathscr{R}y$. Note that $\mathscr{R}$ may be viewed as a mapping between M and 2^M (= the class of all subsets in M). In fact, denote for each $x \in M$

$M(x, \mathscr{R}) = \{y \in M; x\mathscr{R}y\}$ (= the *section* of $\mathscr{R}$ through x);

then, the mapping representation of $\mathscr{R}$ is $(\mathscr{R}(x) = M(x, \mathscr{R}); x \in M)$.

Call the relation $\mathscr{R}$ over M, *proper* when

(b06) $M(c, \mathscr{R})$ is nonempty, for each $c \in M$.

Clearly, $\mathscr{R}$ may be then viewed as a mapping between M and $(2)^M$ (= the class of all nonempty subsets in M).

The following "principle of dependent choices" (in short: DC) is in effect for our future developments.

Proposition 8. *Suppose that $\mathscr{R}$ is a proper relation over M. Then, for each $a \in M$, there exists a sequence $(x_n; n \ge 0)$ in M with $x_0 = a$ and $x_n\mathscr{R}x_{n+1}$, for all n.*

This principle, due to Bernays [5] and Tarski [63], is deductible from AC (= the Axiom of Choice), but not conversely; cf. Wolk [78]. Moreover, the *reduced axiomatic system* (ZF-AC+DC) seems to be comprehensive enough for a large part of the "usual" mathematics; see Moore [49, Appendix 2, Table 4].

As an illustration of this assertion, we show that, ultimately, (BB) is contained in the underlying reduced system.

Proposition 9. *We have (DC) $\Longrightarrow$ (BB) in the strongly reduced system (ZF-AC); hence, (BB) is deductible in the reduced system (ZF-AC+DC).*

Proof. Let the premises of (BB) be admitted; i.e., the quasi-ordered structure $(M, \leq)$ is sequentially inductive and the function $\varphi : M \to R_+$ is $(\leq)$-decreasing. Define the function $\beta : M \to R_+$ as

$\beta(v) := \inf[\varphi(M(v, \leq))]$, $v \in M$.

Clearly, β is increasing and

$$\varphi(v) \geq \beta(v), \text{ for all } v \in M. \tag{30}$$

Moreover, ($\varphi =$ decreasing) yields a characterization of maximal elements like

$$v \text{ is } (\leq, \varphi)\text{-maximal iff } \varphi(v) = \beta(v). \tag{31}$$

Now, assume by contradiction that the conclusion in this statement is false, i.e., [in combination with (30)+(31)], there must be some $u \in M$ such that

(b07) for each $v \in M_u := M(u, \leq)$, one has $\varphi(v) > \beta(v)$.

Consequently (for all such v),

$$\varphi(v) > (1/2)(\varphi(v) + \beta(v)) > \beta(v);$$

hence

$$v \leq w \text{ and } (1/2)(\varphi(v) + \beta(v)) > \varphi(w), \tag{32}$$

for at least one w (belonging to M_u). The relation $\mathscr{R}$ over M_u introduced via (32) is proper on M_u, i.e.,

$M_u(v, \mathscr{R}) \neq \emptyset$, for all $v \in M_u$.

So, by (DC), there must be a sequence (u_n) in M_u with $u_0 = u$ and

$$u_n \leq u_{n+1}, (1/2)(\varphi(u_n) + \beta(u_n)) > \varphi(u_{n+1}), \text{ for all } n. \tag{33}$$

We have thus constructed an ascending sequence (u_n) in M_u for which the positive (real) sequence $(\varphi(u_n))$ is (via (b07)) strictly descending and bounded below; hence $\lambda := \lim_n \varphi(u_n)$ exists in R_+. As $(M, \leq)$ is sequentially inductive, (u_n) is bounded from above in M: there exists $v \in M$ such that $u_n \leq v$, for all n (whence, $v \in M_u$). Moreover, since ($\varphi =$ decreasing), we must have (by the properties of β)

j) $\varphi(u_n) \geq \varphi(v)$, $\forall n$; **jj)** $\varphi(v) \geq \beta(v) \geq \beta(u_n)$, $\forall n$.

The former of these relations gives $\lambda \geq \varphi(v)$ (passing to limit as $n \to \infty$). On the other hand, the latter of these relations yields (via (33))

$$(1/2)(\varphi(u_n) + \beta(v)) > \varphi(u_{n+1}), \text{ for all } n.$$

Passing to limit as $n \to \infty$ gives

$$(\varphi(v) \geq)\beta(v) \geq \lambda;$$

so, combining with the preceding one,

$$\varphi(v) = \beta(v)(= \lambda) \text{ contradiction.}$$

Hence, (b07) cannot be accepted; and the conclusion follows.

Note that a slightly different proof of this may be found in the 2007 monograph by Cârjă et al. [12, Chap. 2, Sect. 2.1]. Further metrical aspects of it may be found in Turinici [68].

(C) In the following, the relationships between (BB) and Ekeland's variational principle [25] (in short: EVP) are discussed.

Let (M, d) be a metric space and $\varphi : M \to R_+$ be a function. Assume that

(b08) (M, d) is *complete* (each d-Cauchy sequence in M is d-convergent)

(b09) φ is *d-lsc*: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \xrightarrow{d} x$;
or, equivalently: $\{x \in M; \varphi(x) \leq t\}$ is d-closed, for each $t \in R$.

Proposition 10. *Let these conditions hold. Then, for each (starting point) $u \in M$, there exists (another point) $v \in M$ with*

$$d(u, v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v)) \tag{34}$$

$$d(v, x) > \varphi(v) - \varphi(x), \text{ for each } x \in M \setminus \{v\}. \tag{35}$$

Proof. Let $(\preceq)$ stand for the relation (over M):

$x \preceq y$ iff $d(x, y) \leq \varphi(x) - \varphi(y)$.

Clearly, $(\preceq)$ acts as a (partial) order on M; note that, as a consequence of this, its associated relation

$x \prec y$ iff $0 < d(x, y) \leq \varphi(x) - \varphi(y)$

is a strict order on X. We claim that conditions of (BB-Z) are fulfilled on $(M, \preceq)$. In fact, by this very definition, φ is $(\prec)$-decreasing on M, so that $(M, \prec)$ is admissible. On the other hand, let (x_n) be a $(\prec)$-ascending sequence in M:

(b10) $0 < d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, if $n < m$.

The sequence $(\varphi(x_n))$ is strictly descending and bounded from below, hence a Cauchy one. This, along with our working hypothesis, tells us that (x_n) is a d-Cauchy sequence in M; wherefrom by completeness,

$x_n \xrightarrow{d} y$ as $n \to \infty$, for some $y \in M$.

Passing to limit as $m \to \infty$ in the same working hypothesis, one derives

$$d(x_n, y) \le \varphi(x_n) - \varphi(y), \text{ (i.e., } x_n \preceq y\text{), for all } n.$$

This, combined with $(x_n; n \ge 0)$ being $(\prec)$-ascending, gives $x_n \prec y$, for all n and shows that $(M, \prec)$ is sequentially inductive. From (BB-Z), it then follows that, for the starting $u \in M$, there exists some $v \in M$ with

h) $u \preceq v$; **hh)** $v \preceq x \in M$ implies $v = x$.

The former of these is just (34) and the latter one gives at once (35).

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory, and global analysis; we refer to the quoted paper for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of (EVP) were proposed. For example, the dimensional way of extension refers to the ambient *positive halfline* R_+ of $\varphi(M)$ being substituted by a *convex cone* of a (topological or not) *vector space*. An account of the results in this area is to be found in the 2003 monograph by Goepfert et al. [28, Chap. 3]; see also Turinici [68]. On the other hand, the *(pseudo) metrical* one consists in the conditions imposed to the ambient metric over M being relaxed. Some basic results in this direction were obtained by Kang and Park [37]; see also Tataru [64].

(D) By the developments above, we therefore have the implications:

$$\text{(DC)} \implies \text{(BB)} \implies \text{(BB-Z)} \implies \text{(EVP)}.$$

So, we may ask whether these may be reversed. Clearly, the natural setting for solving this problem is (ZF-AC), referred to (see above) as the *strongly reduced* Zermelo–Fraenkel system.

Let X be a nonempty set and $(\le)$ be a (partial) order on it. We say that $(\le)$ has the *inf-lattice* property, provided:

$x \wedge y := \inf(x, y)$ exists, for all $x, y \in X$.

Remember that $z \in X$ is a $(\le)$-*maximal* element if $X(z, \le) = \{z\}$; the class of all these points will be denoted as $\max(X, \le)$. Call $(\le)$, a *Zorn order* when

$\max(X, \le)$ is nonempty and *cofinal* in X
(for each $u \in X$, there exists a $(\le)$-maximal $v \in X$ with $u \le v$).

Further aspects are to be described in a metric setting. Let $d : X \times X \to R_+$ be a metric over X and $\varphi : X \to R_+$ be some function. Then, the natural choice for $(\le)$ above is

$x \le_{(d,\varphi)} y$ iff $d(x, y) \le \varphi(x) - \varphi(y)$,

referred to as the *Brøndsted order* [10] attached to (d, φ). Denote

$X(x, \rho) = \{u \in X; d(x, u) < \rho\}, x \in X, \rho > 0$

[the *open* sphere with center x and radius ρ]. Call the ambient metric space (X, d), *discrete* when

for each $x \in X$, there exists $\rho = \rho(x) > 0$ such that $X(x, \rho) = \{x\}$.

Note that, under such an assumption, any function $\psi : X \to R$ is continuous over X. However, the *(global) d-Lipschitz property* of the same

$$|\psi(x) - \psi(y)| \leq Ld(x, y),\ x, y \in X, \text{ for some } L > 0$$

cannot be assured, in general.

Now, the statement below is a particular case of (EVP):

Proposition 11. *Let the metric space (X, d) and the function $\varphi : X \to R_+$ satisfy*

(b11) (X, d) *is discrete bounded and complete.*
(b12) $(\leq_{(d,\varphi)})$ *has the inf-lattice property.*
(b13) φ *is d-nonexpansive and $\varphi(X)$ is countable.*

Then, $(\leq_{(d,\varphi)})$ is a Zorn order.

We shall refer to it as the discrete Lipschitz countable version of EVP (in short: (EVP-dLc)). Clearly, (EVP) $\Longrightarrow$ (EVP-dLc). The remarkable fact to be added is that this last principle yields (DC); so, it completes the circle between all these.

Proposition 12. *The inclusion below is holding (in the strongly reduced Zermelo–Fraenkel system): (EVP-dLc) $\Longrightarrow$ (DC). So (by the above),*

i) *The maximal/variational principles (BB), (BB-Z), and (EVP) are all equivalent with (DC); hence, mutually equivalent.*
ii) *Each "intermediary" maximal/variational statement (VP) with (DC) $\Longrightarrow$ (VP) $\Longrightarrow$ (EVP) is equivalent with both (DC) and (EVP).*

For a complete proof, see Turinici [70]. In particular, when the discrete, bounded, inf-lattice, and nonexpansive properties are ignored in (EVP-dLc), the last result above reduces to the one in Brunner [11]. Note that, in the same particular setting, a different proof of (EVP) $\Longrightarrow$ (DC) was provided in Dodu and Morillon [20]. Further aspects may be found in Schechter [62, Chap. 19, Sect. 19.51].

3.3 Cantor Complete Ultrametrics

Let X be a nonempty set. By an *ultrametric* (or *non-Archimedean metric*) on X, we mean any mapping $d : X \times X \to R_+$ with the properties:

(c01) $x = y$ iff $d(x, y) = 0$ (reflexive sufficient)
(c02) $d(x, y = d(y, x)$, $\forall x, y \in X$ (symmetric)
(c03) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, $\forall x, y, z \in X$ (ultra-triangular);

in this case, the pair (X, d) will be referred to as an *ultrametric space*. Note that any ultrametric is a (standard) metric (on X), because

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \leq d(x,y) + d(y,z), \ \forall x,y,z \in X;$$

but, the converse is not in general valid. The class of these ultrametrics is nonempty. In fact, the *discrete* metric on X introduced as: for each $x, y \in X$

$d(x,y) = 1$ if $x \neq y$; $d(x,y) = 0$, if $x = y$,

is an ultrametric, as it can be directly seen. Further examples may be found in Rooij [75, Chap. 3].

Let in the following (X,d) be an ultrametric space. Note that the presence of ultra-triangular inequality induces a lot of dramatic changes with respect to the standard metrical case; some basic ones will be shown below. [These were stated without proof in Khamsi and Kirk [40, Chap. 5, Sect. 5.7]; see also Rooij [75, Chap. 2]; however, for completeness reasons, we shall provide a proof of them.]

Lemma 6. *Let $x,y,z \in X$ be such that $d(x,y) \neq d(y,z)$. Then, necessarily,*

$$d(x,z) = \max\{d(x,y), d(y,z)\};$$

hence, either $d(x,z) = d(x,y)$ or $d(x,z) = d(y,z)$. In other words: each triangle (x,y,z) in X is d-isosceles.

Proof. Suppose by contradiction that

$d(x,z) < \max\{d(x,y), d(y,z)\}$.

We have two alternatives to consider:

i) Suppose that $d(x,y) < d(y,z)$. By the working hypothesis, we then have $d(x,z) < d(y,z)$. In this case, the ultra-triangular inequality gives

$$d(y,z) \leq \max\{d(x,y), d(x,z)\} < d(y,z), \text{ contradiction.}$$

ii) Suppose that $d(y,z) < d(x,y)$. By the working hypothesis, we then have $d(x,z) < d(x,y)$; so, again from the ultra-triangular inequality,

$$d(x,y) \leq \max\{d(x,z), d(y,z)\} < d(x,y); \text{contradiction.}$$

Having discussed all possible alternatives, we are done.

By definition, any set of the form

(c04) $X[a,r] = \{x \in X; d(a,x) \leq r\}, a \in X, r \in R_+$,

will be referred to as a *d-closed sphere* with center $a \in X$ and radius $r \in R_+$; note that this is a nonempty subset of X, in view of $a \in X[a,r]$. In the following, some results involving the family of all d-closed spheres

$\mathscr{M} = \{X[a,r]; a \in X, r \in R_+\} \subseteq (2)^X$

will be discussed.

Lemma 7. *Let* $M_1 := X[a_1, r_1]$, $M_2 := X[a_2, r_2]$ *be a couple of non-disjoint d-closed spheres in X. Then,*

(i) $M_1 \subseteq M_2$*, whenever* $r_1 \leq r_2$
(ii) $M_1 = M_2$*, whenever* $r_1 = r_2$.

Proof. As $M_1 \cap M_2 \neq \emptyset$, there exists at least one element $b \in M_1 \cap M_2$.

i) Assume that $r_1 \leq r_2$ and let $x \in M_1$ be arbitrary fixed. From the ultra-triangular inequality, we have

$$d(x, b) \leq \max\{d(x, a_1), d(b, a_1)\} \leq r_1;$$

and this in turn yields (by the same procedure)

$$d(x, a_2) \leq \max\{d(x, b), d(b, a_2)\} \leq \max\{r_1, r_2\} = r_2; \text{ i.e., } x \in M_2.$$

As $x \in M_1$ was arbitrarily chosen, one derives $M_1 \subseteq M_2$.
ii) Evident, by the preceding step.

Lemma 8. *Let* $a, b \in X$ *and* $s \geq 0$ *be such that* $a \in X[b, s]$. *Then,*

$$X[a, r] \subseteq X[b, s], \textit{for each } r \in [0, s]. \tag{36}$$

Proof. Let $r \in [0, s]$ be arbitrary fixed. By the imposed hypothesis,

$$a \in X[a, r] \cap X[b, s], \text{ whence } X[a, r] \cap X[b, s] \neq \emptyset;$$

and then, from the previous result, we are done.

The next statement is, in a certain sense, a reciprocal of the previous one. Denote ($Y_1, Y_2 \in 2^X$): $Y_1 \subset Y_2$ iff $Y_1 \subseteq Y_2$ and $Y_1 \neq Y_2$.

Clearly, ($\subset$) is nothing else than the *strict order* (i.e., irreflexive and transitive relation) attached to the usual (partial) order ($\subseteq$) over 2^X.

Lemma 9. *Let* $M_1 := X[a_1, r_1]$, $M_2 := X[a_2, r_2]$ *be two d-closed balls in X. Then,*

$$M_1 \subset M_2 \Longrightarrow r_1 < r_2. \tag{37}$$

Proof. Suppose that $M_1 \subset M_2$; but (contrary to the conclusion) $r_2 \leq r_1$. As $M_1 \cap M_2 = M_1 \neq \emptyset$, one has by a preceding result (and the working hypothesis)

$$M_2 \subseteq M_1 \subset M_2, \text{ contradiction.}$$

This proves our assertion.

We are now introducing a basic notion. Call the ultrametric space (X,d), *Cantor strongly complete* (in short: *Cantor s-complete*), provided

(c05) each $(\supseteq)$-ascending sequence $(M_n := X[a_n, r_n]; n \geq 0)$ in $\mathscr{M}$ has a nonempty intersection.

A (formally) weaker variant of this definition is as follows. Call the ultrametric space (X,d), *Cantor complete*, provided

(c06) each $(\supset)$-ascending sequence $(M_n := X[a_n, r_n]; n \geq 0)$ in $\mathscr{M}$ has a nonempty intersection.

Clearly, we have

$$(\forall \text{ ultrametric structure}): \text{Cantor s-complete} \implies \text{Cantor complete}. \tag{38}$$

The reciprocal inclusion is also true, as results from

Lemma 10. *For each ultrametric structure* (X,d)*, we have*

$$\begin{aligned} &\textit{Cantor complete} \implies \textit{Cantor s-complete;} \\ &\textit{hence, Cantor complete} \iff \textit{Cantor s-complete.} \end{aligned} \tag{39}$$

Proof. Suppose that the ultrametric space (X,d) is Cantor complete; and let $(M_n := X[a_n, r_n]; n \geq 0)$ be a $(\supseteq)$-ascending sequence in $\mathscr{M}$. If one has that

$\exists (i \geq 0), \forall (j > i)$: $M_i = M_j$,

we are done, because $\cap\{M_n; n \geq 0\} = M_i$. Suppose now that the opposite alternative is holding:

$\forall (i \geq 0), \exists (j > i)$: $M_i \supset M_j$.

There exists then a strictly ascending sequence of ranks $(i(n); n \geq 0)$, such that the subsequence $(L_n := M_{i(n)}; n \geq 0)$ of $(M_n; n \geq 0)$ fulfills

$$(L_n) \text{ is } (\supset)\text{-ascending: } p < q \implies L_p \supset L_q.$$

By the imposed hypothesis, $L := \cap\{L_n; n \geq 0\}$ is nonempty. This, along with $L = \cap\{M_n; n \geq 0\}$, ends the argument.

Denote, for simplicity

$\Gamma = X \times R_+$; hence, $\Gamma = \{(a, \rho); a \in X, \rho \in R_+\}$.

A natural relation to be introduced here is the following:

$(a, \rho) \prec (b, \sigma)$ iff $X[a, \rho] \supset X[b, \sigma]$.

Clearly, $(\prec)$ is irreflexive and transitive, hence, a strict order on Γ. Let $(\preceq)$ stand for the associated (partial) order

$(a, \rho) \preceq (b, \sigma)$ iff either $(a, \rho) \prec (b, \sigma)$ or $(a, \rho) = (b, \sigma)$.

Having these precise, let us introduce the function

$(\varphi : \Gamma \to R_+)$: $\varphi(a, \rho) = \rho$, $(a, \rho) \in \Gamma$.

By a previous result, we have

φ is $(\prec)$-decreasing: $(a, \rho) \prec (b, \sigma) \Longrightarrow \varphi(a, \rho) > \varphi(b, \sigma)$.

This tells us that, necessarily,

$$(\Gamma, \prec) \text{ is admissible, hence, so is } (\Delta, \prec), \text{ where } \emptyset \neq \Delta \subseteq \Gamma. \tag{40}$$

As a consequence, the following (relative) maximal result is available.

Proposition 13. *Let the (nonempty) subset Δ of Γ be such that*

$(\Delta, \prec)$ *is sequentially inductive:*
each $(\prec)$-ascending sequence in Δ is bounded above in Δ (modulo $(\prec)$).

Then, $(\preceq)$ is a Zorn order on Δ; i.e., for each (starting element) $(a, \rho) \in \Delta$, there exists (another element) $(b, \sigma) \in \Delta$, with

i) $(a, \rho) \preceq (b, \sigma)$*, i.e., either* $(a, \rho) \prec (b, \sigma)$ *or* $(a, \rho) = (b, \sigma)$
ii) $(b, \sigma) \prec (c, \tau)$ *is impossible, for each* $(c, \tau) \in \Delta$.

Proof. By the admissible property for Γ, we have (see above)

(the strictly ordered structure) $(\Delta, \prec)$ is admissible.

Combining with the admitted hypothesis, it results that the sequential-type maximal result (BB-Z) is applicable to $(\Delta, \preceq)$; and, from this, we are done.

3.4 Application (Fixed Point Theorems)

In the following, an application of the above developments is given to the ultrametric fixed point theory.

Let (X, d) be an ultrametric space. We say that the selfmap $T \in \mathscr{F}(X)$ is *d-strictly nonexpansive*, provided

(d01) $d(Tx, Ty) < d(x, y)$, $\forall x, y \in X, x \neq y$.

Note that, in particular, T is *d-nonexpansive*:

(d02) $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in X$.

The following fixed point theorem over ultrametric spaces is available.

Theorem 9. *Suppose that T is d-strictly nonexpansive (see above). In addition, let (X, d) be Cantor complete. Then, T is fix-singleton; whence it has a unique fixed point in X.*

Proof. There are several steps to be followed.

Step 1. By the d-strict nonexpansive property, we have

$$\text{Fix}(T) \text{ is asingleton; i.e., } T \text{ is fix-asingleton.}$$

So, all we have to establish is that $\text{Fix}(T)$ appears as nonempty.

Step 2. Remember that, over $\Gamma := X \times R_+$, we introduced the strict ordering

$$(a, \rho) \prec (b, \sigma) \text{ iff } X[a, \rho] \supset X[b, \sigma]$$

as well as the associated ordering

$$(a, \rho) \preceq (b, \sigma) \text{ iff either } (a, \rho) \prec (b, \sigma) \text{ or } (a, \rho) = (b, \sigma).$$

Moreover, we have that

$$(\Gamma, \prec) \text{ is admissible, hence, so is } (\Delta, \prec), \text{ where } \emptyset \neq \Delta \subseteq \Gamma. \quad (41)$$

Step 3. Denote, for simplicity

$$\Delta = \{(a, d(a, Ta)); a \in X\};$$

this is a nonempty subset of Γ. By a previous relation, we have that

$$\text{(the strictly ordered structure) } (\Delta, \prec) \text{ is admissible.} \quad (42)$$

Moreover, we claim that the structure $(\Delta, \prec)$ is sequentially inductive. In fact, let $((a_n, d(a_n, Ta_n)); n \geq 0)$ be a $(\prec)$-ascending sequence in Δ; i.e.,

$$M_i \supset M_j, \text{ for } i < j; \text{ where } (M_n := X[a_n, d(a_n, Ta_n)]; n \geq 0).$$

As (X, d) is Cantor complete, it follows that

$$L := \cap\{M_n; n \geq 0\} \text{ is nonempty;}$$

let $b \in L$ be some point of it. By the very definition above (and the d-nonexpansive property of T)

$$d(Tb, Ta_n) \leq d(b, a_n) \leq d(a_n, Ta_n), \ \forall n \geq 0.$$

Combining with the ultra-triangular inequality, one gets (for the same ranks)

$$d(b, Tb) \leq \max\{d(b, a_n), d(a_n, Ta_n), d(Ta_n, Tb)\} \leq d(a_n, Ta_n);$$

and this, by a previous auxiliary fact, yields

$$X[a_n, d(a_n, Ta_n)] \supset X[b, d(b, Tb)], \text{ for all } n;$$

or equivalently (by definition)

$$(a_n, d(a_n, Ta_n)) \prec (b, d(b, Tb)) \in \Delta, \text{ for all } n,$$

which proves the desired fact.

Step 4. Putting these together, it follows that the previous maximal principle is applicable to $(\Delta, \preceq)$. So, for the starting element $(u, d(u, Tu))$ in Δ, there exists another element $(v, d(v, Tv))$ in Δ, with

i) $(u, d(u, Tu)) \preceq (v, d(v, Tv))$
ii) for each $w \in X$, $(v, d(v, Tv)) \prec (w, d(w, Tw))$ is impossible.

Suppose by contradiction that

$d(v, Tv) > 0$, hence, $d(Tv, T^2v) < d(v, Tv)$.

We claim that

$(v, d(v, Tv)) \prec (Tv, d(Tv, T^2v))$;

and this, by the previous maximal property of $(v, d(v, Tv))$, yields a contradiction. The desired relation may be written as

$X[v, d(v, Tv)] \supset X[Tv, d(Tv, T^2v)]$;

to establish it, we may proceed as follows.

I) Let $y \in X[Tv, d(Tv, T^2v)]$ be arbitrary fixed; hence,

$d(y, Tv) \leq d(Tv, T^2v)(< d(v, Tv))$.

By the ultra-triangular inequality,

$$d(y, v) \leq \max\{d(y, Tv), d(v, Tv)\} = d(v, Tv),$$

whence, $y \in X[v, d(v, Tv)]$; this, by the arbitrariness of y, gives

$$X[Tv, d(Tv, T^2v)] \subseteq X[v, d(v, Tv)].$$

II) From the working assumption about v, one must have

$$(v \in X[v, d(v, Tv)] \text{ and}) \ v \notin X[Tv, d(Tv, T^2v)];$$

hence, the above inclusion is strict. The proof is thereby complete.

By the argument above, this fixed point result is a consequence of the Brezis–Browder ordering principle [9]; hence, ultimately, it is deductible in the reduced Zermelo–Fraenkel system (ZF-AC+DC). Note that similar conclusions are to be derived for the related fixed point results over ultrametric spaces due to Gajić [26] and Pant [52]; see also Wang and Song [76]. Further aspects of this theory concerning fuzzy ultrametric spaces may be found in Sayed [61].

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Novel Tools to Determine Hyperbolic Triangle Centers

Abraham Albert Ungar

In Honor of Vladimir Arnold

Abstract Recently discovered tools to study analytic hyperbolic geometry in terms of analogies with analytic Euclidean geometry are presented and employed. Special attention is paid to the study of two novel hyperbolic triangle centers that we call hyperbolic Cabrera points of a hyperbolic triangle and to the way they descend to their novel Euclidean counterparts. The two novel hyperbolic Cabrera points are the (1) Cabrera gyrotriangle ingyrocircle gyropoint and the (2) Cabrera gyrotriangle exgyrocircle gyropoint. Accordingly, their Euclidean counterparts to which they descend are the two novel Euclidean Cabrera points, which are the (1) Cabrera triangle incircle point and the (2) Cabrera triangle excircle point.

1 Introduction

There are many excellent books on plane Euclidean geometry, exploring the subject at various levels. A recently published book by S.E. Louridas and M.Th. Rassias, *Problem-Solving and Selected Topics in Euclidean Geometry*, adds yet another facet to this colorful subject. This delightful book presents a collection of problems in plane Euclidean geometry in the spirit of mathematical Olympiads. One of the problems, attributed to Roberto B. Cabrera, is presented in the book without a solution as follows [30, p. 84]:

Let $A_1A_2A_3$ be a triangle with side midpoints M_{12}, M_{13}, M_{23}, and let T_1, T_2, T_3 be the points of tangency of the incircle of the medial triangle $M_{12}M_{13}M_{23}$ with sides $M_{12}M_{13}$, $M_{12}M_{23}$, $M_{13}M_{23}$, as shown in Fig. 15, p. 658. Cabrera proposed the problem of proving that the lines A_1T_1, A_2T_2, and A_3T_3 are concurrent.

Since Cabrera's problem is about a new triangle center, it naturally attracts the attention of triangle center hunters. Moreover, hyperbolic geometers may raise

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the question as to whether the resulting Cabrera point of a triangle possesses a counterpart in the hyperbolic geometry of Lobachevsky and Bolyai as well.

Accordingly, Cabrera's problem suggests the following formal definition.

Definition 1 (Cabrera Incircle Point). Let $A_1A_2A_3$ be a triangle in a Euclidean plane $\mathbb{R}^2$ with side midpoints M_{12}, M_{13}, M_{23}, and let F_1, F_2, F_3 be the points of tangency of the incircle of the medial triangle $M_{12}M_{13}M_{23}$ with its sides $M_{12}M_{13}$, $M_{12}M_{23}$, $M_{13}M_{23}$. The point of concurrency, F, of the lines A_1F_1, A_2F_2, A_3F_3 is the Cabrera incircle point of triangle $A_1A_2A_3$, shown in Fig. 16, p. 659.

Definition 1, in turn, naturally suggests the following definition of a second triangle Cabrera point in which the incircle of a medial triangle is replaced by the excircles of the medial triangle.

Definition 2 (Cabrera Excircle Point). Let $A_1A_2A_3$ be a triangle in a Euclidean plane $\mathbb{R}^2$ with side midpoints M_{12}, M_{13}, M_{23}, and let H_1, H_2, H_3 be the points of tangency of the excircles of the medial triangle $M_{12}M_{13}M_{23}$ with its sides $M_{12}M_{13}$, $M_{12}M_{23}$, $M_{13}M_{23}$. The point of concurrency, H, of the lines A_1H_1, A_2H_2, A_3H_3 is the Cabrera excircle point of triangle $A_1A_2A_3$, shown in Fig. 16, p. 659.

It is anticipated in each of Definitions 1–2 that given three lines are concurrent. We will see in this article that this is, indeed, the case.

The hunt for Euclidean triangle centers is an old tradition in Euclidean geometry, resulting in a repertoire of about six thousand triangle centers, each of which can be determined by its barycentric coordinate representation with respect to the vertices of its reference triangle [27].

The hunt for hyperbolic triangle centers is initiated in [53–56], where barycentric coordinates, commonly used as a tool in Euclidean geometry, are adapted for use as a tool in hyperbolic geometry as well.

The aim of this article is to present in detail the mathematical tools that enable the hyperbolic Cabrera points to be defined and determined. These mathematical tools prove useful for a general study of hyperbolic geometry in terms of novel analogies with Euclidean geometry, as demonstrated in [55, 56, 61]. However, we employ them in this article solely for the determination of the hyperbolic Cabrera points in the Beltrami–Klein model of hyperbolic geometry. Finally, in terms of Euclidean limits, we discover how the hyperbolic Cabrera points descend to their Euclidean counterparts.

Remarkably, our main tool in the study of hyperbolic geometry is the famous Einstein addition law of relativistically admissible velocities that Einstein introduced in his 1905 paper [15] that founded the special theory of relativity. Here Einstein addition proves useful due to the algebraic objects, *gyrogroups* and *gyrovector spaces*, to which it gives rise. It turns out that Einstein gyrovector spaces form the algebraic setting for the Beltrami–Klein model of hyperbolic geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

The Beltrami–Klein model of hyperbolic geometry is becoming known as the relativistic model of hyperbolic geometry [2, 5, 34, 37, 41, 45, 56]. Naturally, the story of the relativistic model of hyperbolic geometry begins with Einstein addition, the binary operation that expresses the Einstein composition law of relativistically admissible velocities that Einstein introduced in his 1905 paper [15, 16, 29].

2 Einstein Velocity Addition

Let c be any positive constant and let $(\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n \in \mathbb{N}$, equipped with the common vector addition, $+$, and inner product, $\cdot$. Furthermore, let

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\} \tag{1}$$

be the c-ball of all relativistically admissible velocities of material particles. It is the open ball of radius c, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c. $\mathbb{R}^n$ is said to be the ambient space of the ball $\mathbb{R}^n_c$.

Einstein velocity addition is a binary operation, $\oplus$, in the c-ball $\mathbb{R}^n_c$ of all relativistically admissible velocities, given by the equation [47], [35, Eq. 2.9.2], [31, p. 55], [24], v

$$\mathbf{u} \oplus \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\} \tag{2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, where $\gamma_{\mathbf{u}}$ is the gamma factor given by the equation

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{c^2}}}. \tag{3}$$

Here $\mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{v}^2$. A nonempty set with a binary operation is called a *groupoid* so that, accordingly, the pair $(\mathbb{R}^n_c, \oplus)$ is an *Einstein groupoid*.

In the Newtonian/Euclidean limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its ambient space $\mathbb{R}^n$, as we see from (1), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ descends to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (2) and (3).

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right), \tag{4}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. It can be written, equivalently, as

$$\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right), \tag{5}$$

by replacing $\mathbf{u}$ by $\ominus\mathbf{u} = -\mathbf{u}$.

A frequently used identity that follows immediately from (3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\|\mathbf{v}\|^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \tag{6}$$

and, similarly, a useful identity that follows immediately from (5) is

$$\frac{\mathbf{u}\cdot\mathbf{v}}{c^2} = 1 - \frac{\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}. \tag{7}$$

3 Einstein Addition Vs. Vector Addition

Vector addition, $+$, in $\mathbb{R}^n$ is both commutative and associative, that is,

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} && \text{Commutative Law} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && \text{Associative Law} \end{aligned} \tag{8}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}_c^n$ is neither commutative nor associative.

In order to measure the extent to which Einstein addition deviates from associativity, we introduce *gyrations*, which are maps that are *trivial* in the special cases when the application of $\oplus$ is associative. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$ the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is a map of the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$ onto itself. Gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in \mathrm{Aut}(\mathbb{R}_c^3, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v}) \oplus \{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{9}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}_c^3, \oplus)$.

We recall that an automorphism of a groupoid $(S, \oplus)$ is a one-to-one map f of S onto itself that respects the binary operation, that is, $f(a \oplus b) = f(a) \oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid $(S, \oplus)$ forms a group, denoted $\mathrm{Aut}(S, \oplus)$. To emphasize that the gyrations of an Einstein groupoid $(\mathbb{R}_c^3, \oplus)$ are automorphisms of the groupoid, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, is *trivial* if $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}_c^3$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0}, \mathbf{v}]$, $\mathrm{gyr}[\mathbf{v}, \mathbf{v}]$, and $\mathrm{gyr}[\mathbf{v}, \ominus\mathbf{v}]$ are trivial for all $\mathbf{v} \in \mathbb{R}_c^3$, as we see from (9).

Einstein gyrations possess their own rich structure. Moreover, they measure the extent to which Einstein addition deviates from commutativity and associativity as we see from the *gyrocommutative* and the *gyroassociative* laws of Einstein addition in the following identities [47, 48, 50]:

$$\begin{aligned} \mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law} \\ \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law} \\ (\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law} \end{aligned}$$

$$
\begin{aligned}
&\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Left Reduction Property}\\
&\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Right Reduction Property}\\
&\mathrm{gyr}[\ominus\mathbf{u},\ominus\mathbf{v}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Even Property}\\
&(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1} = \mathrm{gyr}[\mathbf{v},\mathbf{u}] && \text{Gyration Inversion Law}\\
&\mathbf{a}\cdot\mathbf{b} = \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} && \text{Inner Product Gyroinvariance}
\end{aligned}
\tag{10}
$$

for all $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{a},\mathbf{b}\in\mathbb{R}^n_c$.

The reduction properties in (10) present important gyration identities since they trigger remarkable reduction in complexity as, for instance, in the transition from Item (4) to Item (5) of (54), p. 578. These two gyration identities are, however, just the tip of a giant iceberg. Many other useful gyration identities are studied, for instance, in [47, 48, 50] and will be studied in the sequel.

The gyrocommutative-gyroassociative laws of Einstein addition were discovered in 1988 [42]. Euclidean geometry is very different from hyperbolic geometry, so that it was not clear before 1988 that lessons from Euclidean geometry would routinely translate into hyperbolic geometry. Recently, following the appearance of [47, 48, 50, 52, 55, 56, 61], along with the present article, the gamble has paid off owing to the gyrovector space structure that Einstein addition encodes. It is now clear that the Einstein gyrovector space approach to relativistic hyperbolic geometry is fully analogous to the standard vector space approach to Euclidean geometry. The resulting analogies allow, in particular, the adaptation of tools that are commonly used in Euclidean geometry for use in hyperbolic geometry as well.

In particular, barycentric coordinates, which are commonly used as a tool in Euclidean geometry, are adapted for use as a tool in hyperbolic geometry as well, where they are naturally called *gyrobarycentric coordinates* [55]. The latter will prove useful in the sequel.

In order to emphasize analogies with Euclidean geometry, the mathematical language that we use in the study of hyperbolic geometry, called *gyrolanguage*, involves the prefix *gyro*. In gyrolanguage we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. The prefix "gyro" stems from "gyration," which is the mathematical abstraction of the special relativistic effect known as "Thomas precession," studied in [61, Chap. 13].

The resulting group-like and vector-space-like structures that Einstein addition encodes are naturally called *gyrocommutative gyrogroups* and *gyrovector spaces*. Authors who apply the gyroformalism of gyrogroups and gyrovector spaces to deal with hyperbolic geometry may enrich gyrolanguage from time to time by contributing their own gyro-vocabulary. Thus, for instance, in a paper titled *Gyrolayout: A Hyperbolic Level-of-Detail Tree Layout* [62], Urribarri, Castro, and Martig employ the two-dimensional and three-dimensional Einstein gyrovector spaces to develop their computer hyperbolic visualization, thus introducing the new term "gyrolayout" into gyrolanguage. The history of gyrolanguage encompasses the history of gyrogroups and gyrovector spaces, unfolded in the seven books [47, 48, 50, 52, 55, 56, 61] on analytic hyperbolic geometry. A brief early history of gyrolanguage is presented in [50, Sect. 1.2] and [47, Remark 6.12, pp. 207–210].

In gyrolanguage, the hyperbolic counterparts of Definitions 1 and 2 of the two triangle Cabrera points give the following definitions of the two gyrotriangle Cabrera gyropoints.

Definition 3 (Cabrera Ingyrocircle Gyropoint). Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector plane $\mathbb{R}_s^2$ with gyroside gyromidpoints M_{12}, M_{13}, M_{23}, and let F_1, F_2, F_3 be the gyropoints of tangency of the ingyrocircle of the gyromedial gyrotriangle $M_{12}M_{13}M_{23}$ with its gyrosides $M_{12}M_{13}$, $M_{12}M_{23}$, $M_{13}M_{23}$. The gyropoint of concurrency, F, of the gyrolines A_1F_1, A_2F_2, A_3F_3 is the Cabrera ingyrocircle gyropoint of gyrotriangle $A_1A_2A_3$, shown in Fig. 14, p. 652.

Definition 4 (Cabrera Exgyrocircle Gyropoint). Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector plane $\mathbb{R}_s^2$ with gyroside gyromidpoints M_{12}, M_{13}, M_{23}, and let H_1, H_2, H_3 be the gyropoints of tangency of the exgyrocircles of the gyromedial gyrotriangle $M_{12}M_{13}M_{23}$ with its gyrosides $M_{12}M_{13}$, $M_{12}M_{23}$, $M_{13}M_{23}$. The gyropoint of concurrency, H, of the gyrolines A_1H_1, A_2H_2, A_3H_3 is the Cabrera exgyrocircle gyropoint of gyrotriangle $A_1A_2A_3$, shown in Fig. 14, p. 652.

It is anticipated in each of Definitions 3 and 4 that given three gyrolines are concurrent. We will see in this article that this is, indeed, the case.

4 Gyrations

The extension by abstraction of the special relativistic effect known as "Thomas precession" [61, Chap. 13] gives rise to gyrations which, in turn, result from the nonassociativity of Einstein addition.

Due to its nonassociativity, Einstein addition gives rise in (9) to gyrations

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}] : \mathbb{R}_c^n \to \mathbb{R}_c^n \tag{11}$$

for any $\mathbf{u},\mathbf{v} \in \mathbb{R}_c^n$ in an Einstein groupoid $(\mathbb{R}_c^n, \oplus)$. Gyrations, in turn, regulate Einstein addition, endowing it with the rich structure of a gyrocommutative gyrogroup, as we will see in Sect. 5, and a gyrovector space, as we will see in Sect. 13. Clearly, gyrations measure the extent to which Einstein addition is nonassociative, where associativity corresponds to trivial gyrations.

An explicit presentation of the gyrations of Einstein groupoids $(\mathbb{R}_c^n, \oplus)$ is, therefore, desirable. Indeed, the gyration equation (9) can be manipulated into the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w} + \frac{A\mathbf{u} + B\mathbf{v}}{D} \tag{12}$$

where

$$A = -\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{(\gamma_{\mathbf{u}}+1)}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w}) + \frac{1}{c^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})$$

$$+\frac{2}{c^4}\frac{\gamma_{\mathbf{u}}^2\gamma_{\mathbf{v}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) \tag{13}$$

$$B = -\frac{1}{c^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot\mathbf{w}) + (\gamma_{\mathbf{u}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\}$$

$$D = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}) + 1 = \gamma_{\mathbf{u}\oplus\mathbf{v}} + 1 > 1$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$. Allowing $\mathbf{w} \in \mathbb{R}^n \supset \mathbb{R}_c^n$ in (12)–(13), gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ are expendable from maps of $\mathbb{R}_c^n$ to linear maps of $\mathbb{R}^n$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

In each of the three special cases when (1) $\mathbf{u} = \mathbf{0}$, (2) $\mathbf{v} = \mathbf{0}$, or (3) $\mathbf{u}$ and $\mathbf{v}$ are parallel in $\mathbb{R}^n$, $\mathbf{u}\|\mathbf{v}$, we have $A\mathbf{u} + B\mathbf{v} = \mathbf{0}$ so that $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is trivial. Thus, we have

$$\begin{aligned}\mathrm{gyr}[\mathbf{0}, \mathbf{v}]\mathbf{w} &= \mathbf{w}\\ \mathrm{gyr}[\mathbf{u}, \mathbf{0}]\mathbf{w} &= \mathbf{w}\\ \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} &= \mathbf{w}, \qquad \mathbf{u}\|\mathbf{v}\,,\end{aligned} \tag{14}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, and all $\mathbf{w} \in \mathbb{R}^n$.

It follows from (12) that

$$\mathrm{gyr}[\mathbf{v}, \mathbf{u}](\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}) = \mathbf{w} \tag{15}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, $\mathbf{w} \in \mathbb{R}^n$, so that gyrations are invertible linear maps of $\mathbb{R}^n$, the inverse, $\mathrm{gyr}^{-1}[\mathbf{u}, \mathbf{v}]$, of $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ being $\mathrm{gyr}[\mathbf{v}, \mathbf{u}]$. We thus have the gyration inversion property

$$\mathrm{gyr}^{-1}[\mathbf{u}, \mathbf{v}] = \mathrm{gyr}[\mathbf{v}, \mathbf{u}] \tag{16}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

Gyrations keep the inner product of elements of the ball $\mathbb{R}_c^n$ invariant, that is,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{17}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$. Hence, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an *isometry* of $\mathbb{R}_c^n$, keeping the norm of elements of the ball $\mathbb{R}_c^n$ invariant,

$$\|\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}\| = \|\mathbf{w}\|\,. \tag{18}$$

Accordingly, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ represents a rotation of the ball $\mathbb{R}_c^n$ about its origin for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

The invertible self-map gyr[**u**, **v**] of $\mathbb{R}_c^n$ respects Einstein addition in $\mathbb{R}_c^n$,

$$\text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a} \oplus \mathbf{b}) = \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \oplus \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} \tag{19}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$, so that gyr[**u**, **v**] is an automorphism of the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$.

5 From Einstein Velocity Addition to Gyrogroups

Taking the key features of the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$ as axioms, and guided by analogies with groups, we are led to the formal gyrogroup definition in which gyrogroups turn out to form a most natural generalization of groups. Definitions related to groups and gyrogroups thus follow.

Definition 5 (Groups). A groupoid $(G, +)$ is a group if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

(G1) $0 + \text{a} = \text{a}$
for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $-a \in G$, called a left inverse of a, satisfying
(G2) $-a + a = 0$
Moreover, the binary operation obeys the associative law
(G3) $(a + b) + c = a + (b + c)$
for all $a, b, c \in G$.

Groups are classified into commutative and noncommutative groups.

Definition 6 (Commutative Groups). A group $(G, +)$ is commutative if its binary operation obeys the commutative law

(G6) $a + b = b + a$

for all $a, b \in G$.

Definition 7 (Subgroups). A subset H of a subgroup $(G, +)$ is a subgroup of G if it is nonempty, and H is closed under group compositions and inverses in G, that is, $x, y \in H$ implies $x + y \in H$ and $-x \in H$.

Theorem 1 (The Subgroup Criterion). *A subset H of a group G is a subgroup if and only if (1) H is nonempty, and (2) $x, y \in H$ implies $x - y \in H$.*

For a proof of the subgroup criterion, see any book on group theory.

Definition 8 (Gyrogroups). A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying

(G1) $0 \oplus a = a$
for all $a \in G$. There is an element $0 \in G$ satisfying axiom ($G1$) such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a, satisfying
(G2) $\ominus a \oplus a = 0$.
Moreover, for any $a, b, c \in G$, there exists a unique element $\mathrm{gyr}[a,b]c \in G$ such that the binary operation obeys the left gyroassociative law
(G3) $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c$.
The map $\mathrm{gyr}[a,b] : G \to G$ given by $c \mapsto \mathrm{gyr}[a,b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,
(G4) $\mathrm{gyr}[a,b] \in \mathrm{Aut}(G, \oplus)$,
and the automorphism $\mathrm{gyr}[a,b]$ of G is called the gyroautomorphism, or the gyration, of G generated by $a, b \in G$. The operator $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called the gyrator of G. Finally, the gyroautomorphism $\mathrm{gyr}[a,b]$ generated by any $a, b \in G$ possesses the left reduction property
(G5) $\mathrm{gyr}[a,b] = \mathrm{gyr}[a \oplus b, b]$.

The gyrogroup axioms ($G1$)–($G5$) in Definition 8 are classified into three classes:

(1) The first pair of axioms, ($G1$) and ($G2$), is a reminiscent of the group axioms.
(2) The last pair of axioms, ($G4$) and ($G5$), presents the gyrator axioms.
(3) The middle axiom, ($G3$), is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and nongyrocommutative gyrogroups.

Definition 9 (Gyrocommutative Gyrogroups). A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law

(G6) $a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$
for all $a, b \in G$.

In order to capture analogies with groups, we introduce into the abstract gyrogroup $(G, \oplus)$ a second binary operation, $\boxplus$, called the gyrogroup *cooperation*, or coaddition.

Definition 10 (The Gyrogroup Cooperation (Coaddition)). Let $(G, \oplus)$ be a gyrogroup. The gyrogroup cooperation (or coaddition), $\boxplus$, is a second binary operation in G related to the gyrogroup operation (or addition), $\oplus$, by the equation

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \tag{20}$$

for all $a, b \in G$.

Naturally, we use the notation $a \boxminus b = a \boxplus (\ominus b)$ where $\ominus b = -b$, so that

$$a \boxminus b = a \ominus \mathrm{gyr}[a, b]b\,. \tag{21}$$

Being a natural extension of the algebra of groups, the algebra of gyrogroups has been explored and employed by several authors; see, for instance, [1–7, 13, 14, 17–22, 28, 33, 34, 36–40, 62], and [41, 43, 44, 46, 49, 58–60]. We realize that, as noted in [11, p. 523], the computation language that Einstein addition encodes plays a universal computational role, which extends far beyond the domain of special relativity.

It is clear how to define a right identity and a right inverse in a gyrogroup. However, as in group theory, the existence of such elements is not presumed. Indeed, the existence of a unique identity and a unique inverse, both left and right, is a consequence of the gyrogroup axioms, as the following theorem shows, along with other immediate results.

Theorem 2 (First Gyrogroup Properties). *Let* $(G, \oplus)$ *be a gyrogroup. For any elements* $a, b, c, x \in G$*, we have:*

1. *If* $a \oplus b = a \oplus c$*, then* $b = c$ *(general left cancellation law; see item (9) below).*
2. $\mathrm{gyr}[0, a] = I$ *for any left identity* 0 *in* G*.*
3. $\mathrm{gyr}[x, a] = I$ *for any left inverse* x *of* a *in* G*.*
4. $\mathrm{gyr}[a, a] = I$
5. *There is a left identity which is a right identity.*
6. *There is only one left identity.*
7. *Every left inverse is a right inverse.*
8. *There is only one left inverse,* $\ominus a$*, of* a*, and* $\ominus(\ominus a) = a$*.*
9. *The left cancellation law:*

$$\ominus a \oplus (a \oplus b) = b \tag{22}$$

10. *The gyrator identity:*

$$\mathrm{gyr}[a, b]x = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus x)\} \tag{23}$$

11. $\mathrm{gyr}[a, b]0 = 0\,.$
12. $\mathrm{gyr}[a, b](\ominus x) = \ominus \mathrm{gyr}[a, b]x\,.$
13. $\mathrm{gyr}[a, 0] = I\,.$

Proof. 1. Let x be a left inverse of a corresponding to a left identity, 0, in G. We have $x \oplus (a \oplus b) = x \oplus (a \oplus c)$, implying $(x \oplus a) \oplus \mathrm{gyr}[x, a]b = (x \oplus a) \oplus \mathrm{gyr}[x, a]c$ by left gyroassociativity. Since 0 is a left identity, $\mathrm{gyr}[x, a]b = \mathrm{gyr}[x, a]c$. Since automorphisms are bijective, $b = c$.

2. By left gyroassociativity we have for any left identity 0 of G, $a \oplus x = 0 \oplus (a \oplus x) = (0 \oplus a) \oplus \mathrm{gyr}[0, a]x = a \oplus \mathrm{gyr}[0, a]x$. Hence, by item 1 above we have $x = \mathrm{gyr}[0, a]x$ for all $x \in G$ so that $\mathrm{gyr}[0, a] = I$.

3. By the left reduction property and by item 2 above, we have $\text{gyr}[x, a] = \text{gyr}[x \oplus a, a] = \text{gyr}[0, a] = I$.
4. Follows from an application of the left reduction property and item 2 above.
5. Let x be a left inverse of a corresponding to a left identity, 0, of G. Then, by left gyroassociativity and item 3 above, $x \oplus (a \oplus 0) = (x \oplus a) \oplus \text{gyr}[x, a]0 = 0 \oplus 0 = 0 = x \oplus a$. Hence, by (1), $a \oplus 0 = a$ for all $a \in G$ so that 0 is a right identity.
6. Suppose 0 and 0^* are two left identities, one of which, say 0, is also a right identity. Then $0 = 0^* \oplus 0 = 0^*$.
7. Let x be a left inverse of a. Then $x \oplus (a \oplus x) = (x \oplus a) \oplus \text{gyr}[x, a]x = 0 \oplus x = x = x \oplus 0$, by left gyroassociativity, (G2) of Definition 8 and items 3, 5, 6 above. By item 1 we have $a \oplus x = 0$ so that x is a right inverse of a.
8. Suppose x and y are left inverses of a. By item 7 above, they are also right inverses, so $a \oplus x = 0 = a \oplus y$. By item 1, $x = y$. Let $\ominus a$ be the resulting unique inverse of a. Then $\ominus a \oplus a = 0$ so that the inverse $\ominus(\ominus a)$ of $\ominus a$ is a.
9. By left gyroassociativity and by 3, we have

$$\ominus a \oplus (a \oplus b) = (\ominus a \oplus a) \oplus \text{gyr}[\ominus a, a]b = b \tag{24}$$

10. By an application of the left cancellation law in item 9 to the left gyroassociative law (G3) in Definition 8, we obtain the result in item 10.
11. We obtain item 11 from item 10 with $x = 0$.
12. Since $\text{gyr}[a, b]$ is an automorphism of $(G, \oplus)$, we have from item 11

$$\text{gyr}[a, b](\ominus x) \oplus \text{gyr}[a, b]x = \text{gyr}[a, b](\ominus x \oplus x) = \text{gyr}[a, b]0 = 0 \tag{25}$$

and hence the result.
13. We obtain item 13 from item 10 with $b = 0$ and a left cancellation, item 9. □

6 Elements of Gyrogroup Theory

Einstein gyrogroups $(G, \oplus)$ possess the *gyroautomorphic inverse property*, according to which $\ominus(a \oplus b) = \ominus a \ominus b$ for all $a, b \in G$. In general, however, $\ominus(a \oplus b) \neq \ominus a \ominus b$ in other gyrogroups. Hence, the following theorem is important.

Theorem 3 (Gyrosum Inversion Law). *For any two elements a, b of a gyrogroup $(G, \oplus)$, we have the gyrosum inversion law*

$$\ominus (a \oplus b) = \text{gyr}[a, b](\ominus b \ominus a)\,. \tag{26}$$

Proof. By the gyrator identity in Theorem 2(10) and a left cancellation, Theorem 2(9), we have

$$\begin{aligned}\mathrm{gyr}[a,b](\ominus b \ominus a) &= \ominus(a \oplus b) \oplus (a \oplus (b \oplus (\ominus b \ominus a)))\\ &= \ominus(a \oplus b) \oplus (a \ominus a)\\ &= \ominus(a \oplus b)\,,\end{aligned} \tag{27}$$

as desired. □

Theorem 4. *For any two elements, a and b, of a gyrogroup* $(G, \oplus)$*, we have*

$$\begin{aligned}\mathrm{gyr}[a,b]b &= \ominus\{\ominus(a \oplus b) \oplus a\}\,,\\ \mathrm{gyr}[a,\ominus b]b &= \ominus(a \ominus b) \oplus a\end{aligned} \tag{28}$$

Proof. The first identity in (28) follows from Theorem 2(10) with $x = \ominus b$, and Theorem 2(12), and the second part of Theorem 2(8). The second identity in (28) follows from the first one by replacing b by $\ominus b$. □

A nested gyroautomorphism is a gyration generated by gyropoints that depend on another gyration. Thus, for instance, some gyrations in (29)–(31) below are nested.

Theorem 5. *Any three elements* a, b, c *of a gyrogroup* $(G, \oplus)$ *satisfy the nested gyroautomorphism identities*

$$\mathrm{gyr}[a, b \oplus c]\mathrm{gyr}[b,c] = \mathrm{gyr}[a \oplus b, \mathrm{gyr}[a,b]c]\mathrm{gyr}[a,b] \tag{29}$$

$$\mathrm{gyr}[a \oplus b, \ominus\mathrm{gyr}[a,b]b]\mathrm{gyr}[a,b] = I \tag{30}$$

$$\mathrm{gyr}[a, \ominus\mathrm{gyr}[a,b]b]\mathrm{gyr}[a,b] = I \tag{31}$$

and the gyroautomorphism product identities

$$\mathrm{gyr}[\ominus a, a \oplus b]\mathrm{gyr}[a,b] = I \tag{32}$$

$$\mathrm{gyr}[b, a \oplus b]\mathrm{gyr}[a,b] = I\,. \tag{33}$$

Proof. By two successive applications of the left gyroassociative law in two different ways, we obtain the following two chains of equations for all $a, b, c, x \in G$,

$$\begin{aligned}a \oplus (b \oplus (c \oplus x)) &= a \oplus ((b \oplus c) \oplus \mathrm{gyr}[b,c]x)\\ &= (a \oplus (b \oplus c)) \oplus \mathrm{gyr}[a, b \oplus c]\mathrm{gyr}[b,c]x\end{aligned} \tag{34}$$

and

$$\begin{aligned} a \oplus (b \oplus (c \oplus x)) &= (a \oplus b) \oplus \mathrm{gyr}[a, b](c \oplus x) \\ &= (a \oplus b) \oplus (\mathrm{gyr}[a, b]c \oplus \mathrm{gyr}[a, b]x) \\ &= ((a \oplus b) \oplus \mathrm{gyr}[a, b]c) \oplus \mathrm{gyr}[a \oplus b, \mathrm{gyr}[a, b]c]\mathrm{gyr}[a, b]x \\ &= (a \oplus (b \oplus c)) \oplus \mathrm{gyr}[a \oplus b, \mathrm{gyr}[a, b]c]\mathrm{gyr}[a, b]x\,. \end{aligned} \tag{35}$$

By comparing the extreme right sides of these two chains of equations, and by employing the left cancellation law, Theorem 2(1), we obtain the identity

$$\mathrm{gyr}[a, b \oplus c]\mathrm{gyr}[b, c]x = \mathrm{gyr}[a \oplus b, \mathrm{gyr}[a, b]c]\mathrm{gyr}[a, b]x \tag{36}$$

for all $x \in G$, thus verifying (29).

In the special case when $c = \ominus b$, (29) reduces to (30), noting that the left side of (29) becomes trivial owing to items (2) and (3) of Theorem 2.

Identity (31) results from the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned} I &\overbrace{=\!=}^{(1)} \mathrm{gyr}[a \oplus b,\ \ominus\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b] \\ &\overbrace{=\!=}^{(2)} \mathrm{gyr}[(a \oplus b) \ominus \mathrm{gyr}[a, b]b, \ominus\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b] \\ &\overbrace{=\!=}^{(3)} \mathrm{gyr}[a \oplus (b \ominus b), \ominus\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b] \\ &\overbrace{=\!=}^{(4)} \mathrm{gyr}[a, \ominus\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b]\,. \end{aligned} \tag{37}$$

Derivation of the numbered qualities in (37) follows:

(1) Follows from (30).
(2) Follows from Item (1) by the left reduction property.
(3) Follows from Item (2) by the left gyroassociative law. Indeed, an application of the left gyroassociative law to the first entry of the left gyration in Item (3) gives the first entry of the left gyration in Item (2), that is, $a \oplus (b \ominus b) = (a \oplus b) \ominus \mathrm{gyr}[a, b]b$.
(4) Follows from Item (3) immediately, since $b \ominus b = 0$.

To verify (32) we consider the special case of (29) when $b = \ominus a$, obtaining

$$\mathrm{gyr}[a, \ominus a \oplus c]\mathrm{gyr}[\ominus a, c] = \mathrm{gyr}[0, \mathrm{gyr}[a, \ominus a]c]\mathrm{gyr}[a, \ominus a] = I \tag{38}$$

where the second identity in (38) follows from items (2) and (3) of Theorem 2. Replacing a by $\ominus a$ and c by b in (38), we obtain (32).

Finally, (33) is derived from (32) by an application of the left reduction property to the first gyroautomorphism in (32) followed by a left cancellation, Theorem 2(9). Accordingly,

$$\begin{aligned} I &= \text{gyr}[\ominus a, a \oplus b]\text{gyr}[a, b] \\ &= \text{gyr}[\ominus a \oplus (a \oplus b), a \oplus b]\text{gyr}[a, b] \\ &= \text{gyr}[b, a \oplus b]\text{gyr}[a, b]\,, \end{aligned} \tag{39}$$

as desired. □

The nested gyroautomorphism identity (31) in Theorem 5 allows the equation that defines the coaddition $\boxplus$ to be dualized with its corresponding equation in which the roles of the binary operations $\boxplus$ and $\oplus$ are interchanged, as shown in the following theorem:

Theorem 6. *Let $(G, \oplus)$ be a gyrogroup with cooperation $\boxplus$ given in Definition 10, p. 571, by the equation*

$$a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b \tag{40}$$

Then

$$a \oplus b = a \boxplus \text{gyr}[a, b]b\,. \tag{41}$$

Proof. Let a and b be any two elements of G. By (40) and (31) we have

$$\begin{aligned} a \boxplus \text{gyr}[a, b]b &= a \oplus \text{gyr}[a, \ominus\text{gyr}[a, b]b]\text{gyr}[a, b]b \\ &= a \oplus b \end{aligned} \tag{42}$$

thus verifying (41). □

We naturally use the notation

$$a \boxminus b = a \boxplus (\ominus b) \tag{43}$$

in a gyrogroup $(G, \oplus)$, so that, by (43), (40) and Theorem 2(12),

$$\begin{aligned} a \boxminus b &= a \boxplus (\ominus b) \\ &= a \oplus \text{gyr}[a, b](\ominus b) \\ &= a \ominus \text{gyr}[a, b]b \end{aligned} \tag{44}$$

and, hence,

$$a \boxminus a = a \ominus a = 0 \tag{45}$$

as it should. Identity (45), in turn, implies the equality between the inverses of $a \in G$ with respect to $\oplus$ and $\boxplus$,

$$\boxminus a = \ominus a \tag{46}$$

for all $a \in G$.

Theorem 7. *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c \tag{47}$$

for all $a, b, c \in G$.

Proof. By the left gyroassociative law and the left cancellation law, and using the notation $d = \ominus b \oplus c$, we have,

$$\begin{aligned}(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) &= (\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b]d \\ &= \ominus a \oplus (b \oplus d) \\ &= \ominus a \oplus (b \oplus (\ominus b \oplus c)) \\ &= \ominus a \oplus c\,,\end{aligned} \tag{48}$$

as desired. □

Theorem 8 (The Gyrotranslation Theorem, I). *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$\ominus\, (\ominus a \oplus b) \oplus (\ominus a \oplus c) = \mathrm{gyr}[\ominus a, b](\ominus b \oplus c) \tag{49}$$

for all $a, b, c \in G$.

Proof. Identity (49) is a rearrangement of Identity (47) obtained by a left cancellation. □

The importance of Identity (49) lies in the analogy it shares with its group counterpart, $-(-a + b) + (-a + c) = -b + c$ in any group $(Group, +)$.

The identity of Theorem 7 can readily be generalized to any number of terms, for instance,

$$(\ominus a \oplus b) \oplus \mathrm{gyr}[\ominus a, b]\{(\ominus b \oplus c) \oplus \mathrm{gyr}[\ominus b, c](\ominus c \oplus d)\} = \ominus a \oplus d \tag{50}$$

which generalizes the obvious group identity $(-a + b) + (-b + c) + (-c + d) = -a + d$ in any group $(Group, +)$.

7 The Two Basic Equations of Gyrogroups

The two basic equations of gyrogroup theory are

$$a \oplus x = b \tag{51}$$

and

$$x \oplus a = b \tag{52}$$

$a, b, x \in G$, each for the unknown x in a gyrogroup $(G, \oplus)$.

Let x be a solution of the first basic equation, (51). Then we have by (51) and the left cancellation law, Theorem 2(9),

$$\ominus a \oplus b = \ominus a \oplus (a \oplus x) = x\,. \tag{53}$$

Hence, if a solution x of (51) exists, then it must be given by $x = \ominus a \oplus b$, as we see from (53).

Conversely, $x = \ominus a \oplus b$ is, indeed, a solution of (51) as we see by substituting $x = \ominus a \oplus b$ into (51) and applying the left cancellation law in Theorem 2(9). Hence, the gyrogroup equation (51) possesses the unique solution $x = \ominus a \oplus b$.

The solution of the second basic gyrogroup equation, (52), is quite different from that of the first, (51), owing to the noncommutativity of the gyrogroup operation. Let x be a solution of (52). Then we have the following chain of equations, which are numbered for subsequent derivation:

$$\begin{aligned} x &\overbrace{=\!=}^{(1)} x \oplus 0 \overbrace{=\!=}^{(2)} x \oplus (a \ominus a) \\ &\overbrace{=\!=}^{(3)} (x \oplus a) \oplus \mathrm{gyr}[x, a](\ominus a) \\ &\overbrace{=\!=}^{(4)} (x \oplus a) \ominus \mathrm{gyr}[x, a]a \\ &\overbrace{=\!=}^{(5)} (x \oplus a) \ominus \mathrm{gyr}[x \oplus a, a]a \\ &\overbrace{=\!=}^{(6)} b \ominus \mathrm{gyr}[b, a]a \\ &\overbrace{=\!=}^{(7)} b \boxminus a\,. \end{aligned} \tag{54}$$

Derivation of the numbered equalities in (54) follows:

(1) Follows from the existence of a unique identity element, 0, in the gyrogroup $(G, \oplus)$ by Theorem 2.
(2) Follows from the existence of a unique inverse element $\ominus a$ of a in the gyrogroup $(G, \oplus)$ by Theorem 2.
(3) Follows from Item (2) by the left gyroassociative law in Axiom (G3) of gyrogroups in Definition 8, p. 570.
(4) Follows from Item (3) by Theorem 2(12).
(5) Follows from Item (4) by the left reduction property (G5) of gyrogroups in Definition 8.
(6) Follows from Item (5) by the assumption that x is a solution of (52).
(7) Follows from Item (6) by (44).

Hence, if a solution x of (52) exists, then it must be given by $x = b \boxminus a$, as we see from (54).

Conversely, $x = b \boxminus a$ is, indeed, a solution of (52), as we see from the following chain of equations:

$$\begin{aligned}
x \oplus a &\overbrace{=\!=}^{(1)} (b \boxminus a) \oplus a \\
&\overbrace{=\!=}^{(2)} (b \ominus \mathrm{gyr}[b, a]a) \oplus a \\
&\overbrace{=\!=}^{(3)} (b \ominus \mathrm{gyr}[b, a]a) \oplus \mathrm{gyr}[b, \ominus\mathrm{gyr}[b, a]]\mathrm{gyr}[b, a]a \\
&\overbrace{=\!=}^{(4)} b \oplus (\ominus\mathrm{gyr}[b, a]a \oplus \mathrm{gyr}[b, a]a) \\
&\overbrace{=\!=}^{(5)} b \oplus 0 \\
&\overbrace{=\!=}^{(6)} b\,.
\end{aligned} \tag{55}$$

Derivation of the numbered equalities in (55) follows:

(1) Follows from the assumption that $x = b \boxminus a$.
(2) Follows from Item (1) by (44).
(3) Follows from Item (2) by Identity (31) of Theorem 5, according to which the gyration product applied to a in (3) is trivial.
(4) Follows from Item (3) by the left gyroassociative law. Indeed, an application of the left gyroassociative law to (4) results in (3).
(5) Follows from Item (4) since $\ominus\mathrm{gyr}[b, a]a$ is the unique inverse of $\mathrm{gyr}[b, a]a$.
(6) Follows from Item (5) since 0 is the unique identity element of the gyrogroup $(G, \oplus)$.

Formalizing the results of this section, we have the following theorem:

Theorem 9 (The Two Basic Gyrogroup Equations). *Let* $(G, \oplus)$ *be a gyrogroup, and let* $a, b \in G$*. The unique solution of the equation*

$$a \oplus x = b \tag{56}$$

in G *for the unknown* x *is*

$$x = \ominus a \oplus b \tag{57}$$

and the unique solution of the equation

$$x \oplus a = b \tag{58}$$

in G *for the unknown* x *is*

$$x = b \boxminus a\,. \tag{59}$$

Let $(G, \oplus)$ be a gyrogroup, and let $a \in G$. The maps λ_a and ρ_a of G, given by

$$\begin{aligned} \lambda_a : \ G &\to G, \qquad & \lambda_a : \ g &\mapsto a \oplus g\,, \\ \rho_a : \ G &\to G, \qquad & \rho_a : \ g &\mapsto g \oplus a\,, \end{aligned} \tag{60}$$

are called, respectively, a *left gyrotranslation* of G by a and a *right gyrotranslation* of G by a. Theorem 9 asserts that each of these transformations of G is bijective, that is, it maps G onto itself in a one-to-one manner.

8 The Basic Gyrogroup Cancellation Laws

The basic cancellation laws of gyrogroup theory are obtained in this section from the basic equations of gyrogroups solved in Sect. 7. Substituting the solution (57) into its Eq. (56), we obtain the left cancellation law

$$a \oplus (\ominus a \oplus b) = b \tag{61}$$

for all $a, b \in G$, already verified in Theorem 2(9).

Similarly, substituting the solution (59) into its Eq. (58), we obtain the first right cancellation law

$$(b \boxminus a) \oplus a = b \tag{62}$$

for all $a, b \in G$. The latter can be dualized, obtaining the second right cancellation law

$$(b \ominus a) \boxplus a = b \tag{63}$$

for all $a, b \in G$. Indeed, (63) results from the following chain of equations

$$\begin{aligned} b &= b \oplus 0 \\ &= b \oplus (\ominus a \oplus a) \\ &= (b \ominus a) \oplus \text{gyr}[b, \ominus a]a \\ &= (b \ominus a) \oplus \text{gyr}[b \ominus a, \ominus a]a \\ &= (b \ominus a) \boxplus a \end{aligned} \tag{64}$$

where we employ the left gyroassociative law, the left reduction property, and the definition of the gyrogroup cooperation. Identities (61)–(63) form the three basic cancellation laws of gyrogroup theory.

9 Commuting Automorphisms with Gyroautomorphisms

In this section we will find that automorphisms of a gyrogroup commute with its gyroautomorphisms in a special, interesting way.

Theorem 10. *For any two elements a, b of a gyrogroup $(G, \oplus)$ and any automorphism A of $(G, \oplus)$, $A \in \text{Aut}(G, \oplus)$,*

$$A\text{gyr}[a, b] = \text{gyr}[Aa, Ab]A\,. \tag{65}$$

Proof. For any three elements $a, b, x \in (G, \oplus)$ and any automorphism $A \in \text{Aut}(G, \oplus)$, we have by the left gyroassociative law

$$\begin{aligned} (Aa \oplus Ab) \oplus A\text{gyr}[a, b]x &= A((a \oplus b) \oplus \text{gyr}[a, b]x) \\ &= A(a \oplus (b \oplus x)) \\ &= Aa \oplus (Ab \oplus Ax) \\ &= (Aa \oplus Ab) \oplus \text{gyr}[Aa, Ab]Ax\,. \end{aligned} \tag{66}$$

Hence, by the left cancellation in Theorem 2(1),

$$A\text{gyr}[a, b]x = \text{gyr}[Aa, Ab]Ax$$

for all $x \in G$, implying (65). □

Theorem 11. *Let* a, b *be any two elements of a gyrogroup* $(G, \oplus)$ *and let* $A \in \mathrm{Aut}(G)$ *be an automorphism of* G*. Then*

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[Aa, Ab] \tag{67}$$

if and only if the automorphisms A *and* $\mathrm{gyr}[a, b]$ *are commutative.*

Proof. If $\mathrm{gyr}[Aa, Ab] = \mathrm{gyr}[a, b]$, then by Theorem 10 the automorphisms $\mathrm{gyr}[a, b]$ and A commute, $A\mathrm{gyr}[a, b] = \mathrm{gyr}[Aa, Ab]A = \mathrm{gyr}[a, b]A$. Conversely, if $\mathrm{gyr}[a, b]$ and A commute, then by Theorem 10 $\mathrm{gyr}[Aa, Ab] = A\mathrm{gyr}[a, b]A^{-1} = \mathrm{gyr}[a, b]AA^{-1} = \mathrm{gyr}[a, b]$, as desired. □

10 The Gyrosemidirect Product

Definition 11 (Gyroautomorphism Groups, Gyrosemidirect Product). Let $G = (G, \oplus)$ be a gyrogroup, and let $\mathrm{Aut}(G) = \mathrm{Aut}(G, \oplus)$ be the automorphism group of G. A gyroautomorphism group, $\mathrm{Aut}_0(G)$, of G is any subgroup of $\mathrm{Aut}(G)$ containing all the gyroautomorphisms $\mathrm{gyr}[a, b]$ of G, $a, b \in G$. The *gyrosemidirect product group*

$$G \times \mathrm{Aut}_0(G) \tag{68}$$

of a gyrogroup G and any gyroautomorphism group, $\mathrm{Aut}_0(G)$ of G, is a group of pairs (x, X), where $x \in G$ and $X \in \mathrm{Aut}_0(G)$, with operation given by the *gyrosemidirect product*

$$(x, X)(y, Y) = (x \oplus Xy, \ \mathrm{gyr}[x, Xy]XY) \, . \tag{69}$$

It is anticipated in Definition 11 that the gyrosemidirect product set (68) of a gyrogroup and any one of its gyroautomorphism groups is a set that forms a group with group operation given by the gyrosemidirect product (69). The following theorem shows that this is indeed the case.

Theorem 12. *Let* $(G, \oplus)$ *be a gyrogroup, and let* $\mathrm{Aut}_0(G, \oplus)$ *be a gyroautomorphism group of* G*. Then the gyrosemidirect product* $G \times \mathrm{Aut}_0(G)$ *is a group, with group operation given by the gyrosemidirect product* (69).

The proof of Theorem 12 is found in [56, Sect. 1.11].

The gyrosemidirect product group enables problems in gyrogroups to be converted to the group setting, thus gaining access to the powerful group theoretic techniques.

11 Basic Gyration Properties

The most important basic gyration properties that we establish in this section are the *gyration even property*

$$\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a, b] \tag{70}$$

and the *gyroautomorphism inversion law*

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a] \tag{71}$$

for any two elements a and b of a gyrogroup $(G, \oplus)$, where $\mathrm{gyr}^{-1}[a, b] = (\mathrm{gyr}[a, b])^{-1}$ is the inverse of the gyration $\mathrm{gyr}[a, b]$.

Theorem 13 (Gyrosum Inversion, Gyroautomorphism Inversion). *For any two elements a, b of a gyrogroup $(G, \oplus)$, we have the gyrosum inversion law*

$$\ominus (a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a) \tag{72}$$

and the gyroautomorphism inversion law

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[\ominus b, \ominus a] \,. \tag{73}$$

Proof. Let $\mathrm{Aut}_0(G)$ be any gyroautomorphism group of $(G, \oplus)$, and let $G \times \mathrm{Aut}_0(G)$ be the gyrosemidirect product of the gyrogroup G and the group $\mathrm{Aut}_0(G)$ according to Definition 11. Being a group, the product of two elements of the gyrosemidirect product group $G \times \mathrm{Aut}_0(G)$ has a unique inverse. This inverse can be calculated in two different ways.

On the one hand, the inverse of the left side of the gyrosemidirect product

$$(a, I)(b, I) = (a \oplus b, \mathrm{gyr}[a, b]) \tag{74}$$

in $G \times \mathrm{Aut}_0(G)$ is

$$\begin{aligned} (b, I)^{-1}(a, I)^{-1} &= (\ominus b, I)(\ominus a, I) \\ &= (\ominus b \ominus a, \mathrm{gyr}[\ominus b, \ominus a]) \,. \end{aligned} \tag{75}$$

On the other hand, noting the gyrosemidirect product inversion law

$$(a, A)^{-1} = (\ominus A^{-1} a, A^{-1}) \,, \tag{76}$$

the inverse of the right side of the product (74) is, by (76),

$$(\ominus \mathrm{gyr}^{-1}[a, b](a \oplus b),\ \mathrm{gyr}^{-1}[a, b]) \tag{77}$$

for all $a, b \in G$. Comparing corresponding entries in (75) and (77), we have

$$\ominus b \ominus a = \ominus\mathrm{gyr}^{-1}[a, b](a \oplus b) \tag{78}$$

and

$$\mathrm{gyr}[\ominus b, \ominus a] = \mathrm{gyr}^{-1}[a, b]\,. \tag{79}$$

Eliminating $\mathrm{gyr}^{-1}[a, b]$ between (78) and (79), we have

$$\ominus b \ominus a = \ominus\mathrm{gyr}[\ominus b, \ominus a](a \oplus b) \tag{80}$$

Replacing (a, b) by $(\ominus b, \ominus a)$, (80) becomes

$$a \oplus b = \ominus\mathrm{gyr}[a, b](\ominus b \ominus a)\,. \tag{81}$$

Identities (81) and (79) complete the proof. □

The gyrosum inversion law (72) is verified here as a by-product along with the gyroautomorphism inversion law (73) in Theorem 13 in terms of the gyrosemidirect product group. A direct proof of (72) is, however, simpler as we see in Theorem 3, p. 573.

Theorem 14. *Let* $(G, \oplus)$ *be a gyrogroup. Then for all* $a, b \in G$

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[a, \ominus\mathrm{gyr}[a, b]b] \tag{82}$$

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[\ominus a, a \oplus b] \tag{83}$$

$$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a \oplus b] \tag{84}$$

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[b, \ominus b \ominus a] \tag{85}$$

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[\ominus a, \ominus b \ominus a] \tag{86}$$

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[\ominus(a \oplus b), a]\,. \tag{87}$$

Proof. Identity (82) follows from (31).

Identity (83) follows from (32).

Identity (84) results from an application to (83) of the left reduction property followed by a left cancellation.

Identity (85) follows from the gyroautomorphism inversion law (73) and from (83),

$$\mathrm{gyr}[a, b] = \mathrm{gyr}^{-1}[\ominus b, \ominus a] = \mathrm{gyr}[b, \ominus b \ominus a]\,. \tag{88}$$

Identity (86) follows from an application, to the right side of (85), of the left reduction property followed by a left cancellation.

Identity (87) follows by inverting (83) by means of the gyroautomorphism inversion law (73). □

Theorem 15 (Gyration Inversion Law and Gyration Even Property). *The gyroautomorphisms of any gyrogroup* $(G, \oplus)$ *obey the gyration inversion law*

$$\mathrm{gyr}^{-1}[a,b] = \mathrm{gyr}[b,a] \tag{89}$$

and possess the gyration even property

$$\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a,b] \tag{90}$$

satisfying the four mutually equivalent nested gyroautomorphism identities

$$\begin{aligned} \mathrm{gyr}[b, \ominus\mathrm{gyr}[b,a]a] &= \mathrm{gyr}[a,b] \\ \mathrm{gyr}[b, \mathrm{gyr}[b,\ominus a]a] &= \mathrm{gyr}[a,\ominus b] \\ \mathrm{gyr}[\ominus\mathrm{gyr}[a,b]b, a] &= \mathrm{gyr}[a,b] \\ \mathrm{gyr}[\mathrm{gyr}[a,\ominus b]b, a] &= \mathrm{gyr}[a,\ominus b] \end{aligned} \tag{91}$$

for all $a, b \in G$.

Proof. By the left reduction property and (84), we have

$$\begin{aligned} \mathrm{gyr}^{-1}[a \oplus b, b] &= \mathrm{gyr}^{-1}[a,b] \\ &= \mathrm{gyr}[b, a \oplus b] \end{aligned} \tag{92}$$

for all $a, b \in G$. Let us substitute $a = c \boxminus b$ into (92), so that by a right cancellation $a \oplus b = c$, obtaining the identity

$$\mathrm{gyr}^{-1}[c,b] = \mathrm{gyr}[b,c] \tag{93}$$

for all $c, b \in G$. Renaming c in (93) as a, we obtain (89), as desired.

Identity (90) results from (73) and (89),

$$\begin{aligned} \mathrm{gyr}[\ominus a, \ominus b] &= \mathrm{gyr}^{-1}[b,a] \\ &= \mathrm{gyr}[a,b]\,. \end{aligned} \tag{94}$$

Finally, the first identity in (91) follows from (82) and (89).

By means of the gyroautomorphism inversion law (89), the third identity in (91) is equivalent to the first one.

The second (fourth) identity in (91) follows from the first (third) by replacing a by $\ominus a$ (or, alternatively, by replacing b by $\ominus b$), noting that gyrations are even by (90). □

The left gyroassociative law and the left reduction property of gyrogroups admit right counterparts, as we see from the following theorem.

Theorem 16. *For any three elements a, b, and c of a gyrogroup* $(G, \oplus)$*, we have*

(i) $(a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr}[b, a]c)$ *Right Gyroassociative Law,*

(ii) $\text{gyr}[a, b] = \text{gyr}[a, b \oplus a]$ *Right Reduction Property.*

Proof. The right gyroassociative law follows from the left gyroassociative law and the gyration inversion law (89) of gyroautomorphisms,

$$\begin{aligned} a \oplus (b \oplus \text{gyr}[b, a]c) &= (a \oplus b) \oplus \text{gyr}[a, b]\text{gyr}[b, a]c \\ &= (a \oplus b) \oplus c\,. \end{aligned} \tag{95}$$

The right reduction property results from (84) and the gyration inversion law (89),

$$\begin{aligned} \text{gyr}[b, a \oplus b] &= \text{gyr}^{-1}[a, b] \\ &= \text{gyr}[b, a]\,. \end{aligned} \tag{96}$$

□

A right and a left reduction give rise to the identities in the following theorem.

Theorem 17. *Let* $(G, \oplus)$ *be a gyrogroup. Then*

$$\begin{aligned} \text{gyr}[a \oplus b, \ominus a] &= \text{gyr}[a, b]\,, \\ \text{gyr}[\ominus a, a \oplus b] &= \text{gyr}[b, a]\,, \end{aligned} \tag{97}$$

for all $a, b \in G$.

Proof. By a right reduction, a left cancellation, and a left reduction, we have

$$\begin{aligned} \text{gyr}[a \oplus b, \ominus a] &= \text{gyr}[a \oplus b, \ominus a \oplus (a \oplus b)] \\ &= \text{gyr}[a \oplus b, b] \\ &= \text{gyr}[a, b] \end{aligned} \tag{98}$$

thus verifying the first identity in (97). The second identity in (97) follows from the first one by gyroautomorphism inversion, (89). □

12 Gyrocommutative Gyrogroups

Definition 12 (Gyroautomorphic Inverse Property). A gyrogroup $(G, \oplus)$ possesses the *gyroautomorphic inverse property* if for all $a, b \in G$,

$$\ominus(a \oplus b) = \ominus a \ominus b\,. \tag{99}$$

Theorem 18 (The Gyroautomorphic Inverse Property). *A gyrogroup is gyrocommutative if and only if it possesses the gyroautomorphic inverse property.*

Proof. Let $(G, \oplus)$ be a gyrogroup possessing the gyroautomorphic inverse property. Then the gyrosum inversion law (26), p. 573, specializes, by means of Theorem 2(12), p. 572, to the gyrocommutative law $(G6)$ in Definition 9, p. 571,

$$\begin{aligned} a \oplus b &= \ominus \mathrm{gyr}[a, b](\ominus b \ominus a) \\ &= \mathrm{gyr}[a, b]\{\ominus(\ominus b \ominus a)\} \\ &= \mathrm{gyr}[a, b](b \oplus a) \end{aligned} \tag{100}$$

for all $a, b \in G$.

Conversely, if the gyrocommutative law is valid, then by Theorem 2(12) and the gyrosum inversion law, (26), p. 573, we have

$$\mathrm{gyr}[a, b]\{\ominus(\ominus b \ominus a)\} = \ominus \mathrm{gyr}[a, b](\ominus b \ominus a) = a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \tag{101}$$

so that by eliminating the gyroautomorphism $\mathrm{gyr}[a, b]$ on both extreme sides of (101) and inverting the gyro-sign, we recover the gyroautomorphic inverse property,

$$\ominus(b \oplus a) = \ominus b \ominus a \tag{102}$$

for all $a, b \in G$. □

Theorem 19 (The Gyrotranslation Theorem, II). *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. For all $a, b, c \in G$,*

$$\begin{aligned} \ominus(a \oplus b) \oplus (a \oplus c) &= \mathrm{gyr}[a, b](\ominus b \oplus c) \\ (a \oplus b) \ominus (a \oplus c) &= \mathrm{gyr}[a, b](b \ominus c)\,. \end{aligned} \tag{103}$$

Proof. The first identity in (103) follows from the Gyrotranslation Theorem 8, p. 577, with a replaced by $\ominus a$. Hence, it is valid in nongyrocommutative gyrogroups as well. The second identity in (103) follows from the first by the gyroautomorphic inverse property of gyrocommutative gyrogroups, Theorem 18, p. 587. Hence, it is valid in gyrocommutative gyrogroups. □

13 Einstein Scalar Multiplication and Gyrovector Spaces

Definition 13 (Einstein Scalar Multiplication and Einstein Gyrovector Spaces). An Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is an Einstein gyrogroup $(\mathbb{R}_s^n, \oplus)$ with scalar multiplication $\otimes$ given by

$$\begin{aligned} r \otimes \mathbf{v} &= s \frac{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s \tanh(r \tanh^{-1} \frac{\|\mathbf{v}\|}{s}) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \qquad (104)$$

where r is any real number, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_s^n$, $\mathbf{v} \neq \mathbf{0}$, and $r \otimes \mathbf{0} = \mathbf{0}$ and with which we use the notation $\mathbf{v} \otimes r = r \otimes \mathbf{v}$.

Example 1 (The Einstein Half). In the special case when $r = 1/2$, (104) reduces to

$$\tfrac{1}{2} \otimes \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}} \mathbf{v} \qquad (105)$$

so that

$$\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}} \mathbf{v} \oplus \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}} \mathbf{v} = \mathbf{v} \,. \qquad (106)$$

Einstein gyrovector spaces are studied in [50, Sect. 6.18]. Einstein scalar multiplication does not distribute with Einstein addition, but it possesses other properties of vector spaces. For any positive integer k, and for all real numbers $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}_s^n$, we have

$$\begin{aligned} k \otimes \mathbf{v} &= \mathbf{v} \oplus \cdots \oplus \mathbf{v} && k \text{ terms} \\ (r_1 + r_2) \otimes \mathbf{v} &= r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v} && \text{Scalar Distributive Law} \\ (r_1 r_2) \otimes \mathbf{v} &= r_1 \otimes (r_2 \otimes \mathbf{v}) && \text{Scalar Associative Law} \end{aligned} \qquad (107)$$

in any Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$.

Additionally, Einstein gyrovector spaces possess the *scaling property*

$$\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \qquad (108)$$

$\mathbf{a} \in \mathbb{R}_s^n$, $\mathbf{a} \neq \mathbf{0}$, $r \in \mathbb{R}$, $r \neq 0$, the *gyroautomorphism property*

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \tag{109}$$

$\mathbf{a}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, $r \in \mathbb{R}$, and the identity gyroautomorphism

$$\mathrm{gyr}[r_1 \otimes \mathbf{v}, r_2 \otimes \mathbf{v}] = I \tag{110}$$

$r_1, r_2 \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_s^n$.

Any Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ inherits an inner product and a norm from its vector space $\mathbb{R}^n$. These turn out to be invariant under gyrations, that is,

$$\begin{aligned} \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{u} \cdot \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v} &= \mathbf{u} \cdot \mathbf{v} \\ \|\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v}\| &= \|\mathbf{v}\| \end{aligned} \tag{111}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$.

Unlike vector spaces, Einstein gyrovector spaces $(\mathbb{R}_s^n, \oplus, \otimes)$ do not possess the distributive law since, in general,

$$r \otimes (\mathbf{u} \oplus \mathbf{v}) \neq r \otimes \mathbf{u} \oplus r \otimes \mathbf{v} \tag{112}$$

for $r \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$. However, a weak form of the distributive law does exist, as we see from the following theorem:

Theorem 20 (The Monodistributive Law). *Let* $(\mathbb{R}_s^n, \oplus, \otimes)$ *be an Einstein gyrovector space. Then,*

$$r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) = r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}) \tag{113}$$

for all $r, r_1, r_2 \in \mathbb{R}$ *and* $\mathbf{v} \in \mathbb{R}_s^n$.

Proof. By the scalar distributive and associative laws, (107), we have

$$\begin{aligned} r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) &= r \otimes \{(r_1 + r_2) \otimes \mathbf{v}\} \\ &= (r(r_1 + r_2)) \otimes \mathbf{v} \\ &= (rr_1 + rr_2) \otimes \mathbf{v} \\ &= (rr_1) \otimes \mathbf{v} \oplus (rr_2) \otimes \mathbf{v} \\ &= r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}) \end{aligned} \tag{114}$$

as desired. □

Since scalar multiplication in Einstein gyrovector spaces does not distribute with Einstein addition, the following theorem is interesting.

Theorem 21 (The Two-Sum Identity). *Let* $\mathbf{u}, \mathbf{v}$ *be any two gyropoints of an Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$. *Then*

$$2 \otimes (\mathbf{u} \oplus \mathbf{v}) = \mathbf{u} \oplus (2 \otimes \mathbf{v} \oplus \mathbf{u}) \tag{115}$$

Proof. Employing the right gyroassociative law in (10), the identity $\text{gyr}[\mathbf{v}, \mathbf{v}] = I$, Theorem 2(4), the left gyroassociative law, and the gyrocommutative law in (10), we have the following chain of equations:

$$\begin{aligned}
\mathbf{u} \oplus (2 \otimes \mathbf{v} \oplus \mathbf{u}) &= \mathbf{u} \oplus ((\mathbf{v} \oplus \mathbf{v}) \oplus \mathbf{u}) \\
&= \mathbf{u} \oplus (\mathbf{v} \oplus (\mathbf{v} \oplus \text{gyr}[\mathbf{v}, \mathbf{v}]\mathbf{u})) \\
&= \mathbf{u} \oplus (\mathbf{v} \oplus (\mathbf{v} \oplus \mathbf{u})) \\
&= (\mathbf{u} \oplus \mathbf{v}) \oplus \text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v} \oplus \mathbf{u}) \\
&= (\mathbf{u} \oplus \mathbf{v}) \oplus (\mathbf{u} \oplus \mathbf{v}) \\
&= 2 \otimes (\mathbf{u} \oplus \mathbf{v})
\end{aligned} \tag{116}$$

which gives (115). □

14 The Euclidean Line

We introduce Cartesian coordinates into $\mathbb{R}^n$ in the usual way in order to specify uniquely each point P of the Euclidean n-space $\mathbb{R}^n$ by an n-tuple of real numbers, called the coordinates, or components, of P. Cartesian coordinates provide a method of indicating the position of points and rendering graphs on a two-dimensional Euclidean plane $\mathbb{R}^2$ and in a three-dimensional Euclidean space $\mathbb{R}^3$.

As an example, Fig. 1 presents a Euclidean plane $\mathbb{R}^2$ equipped with an unseen Cartesian coordinate system Σ. The position of points A and B and their midpoint m_{AB} with respect to Σ are shown. The missing Cartesian coordinates in Fig. 1 are shown in Fig. 3.

The set of all points

$$A + (-A + B)t \tag{117}$$

$t \in \mathbb{R}$, forms a Euclidean line. The segment of this line, corresponding to $1 \leq t \leq 1$, and a generic point P on the segment are shown in Fig. 1. Being collinear, the points A, P, and B obey the triangle equality $d(A, P) + d(P, B) = d(A, B)$, where $d(A, B) = \| -A + B\|$ is the Euclidean distance function in $\mathbb{R}^n$.

Figure 1 demonstrates the use of the standard Cartesian model of Euclidean geometry for graphical presentations. In a fully analogous way, Fig. 2 demonstrates the use of the Cartesian–Beltrami–Klein model of hyperbolic geometry.

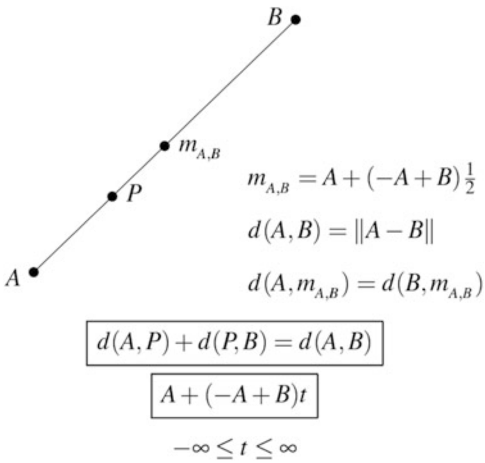


Fig. 1 The Euclidean line. The line $A + (-A + B)t$, $t \in \mathbb{R}$, in a Euclidean plane is shown. The points A and B correspond to $t = 0$ and $t = 1$, respectively. The point P is a generic point on the line through the points A and B lying between these points. The Euclidean sum, $+$, of the distance from A to P and from P to B equals the distance from A to B, so that distance along a line is additive. The point $m_{A,B}$ is the midpoint of the points A and B, corresponding to $t = 1/2$. This figure sets the stage for its hyperbolic counterpart, shown in Fig. 2

15 Gyrolines: The Hyperbolic Lines

Let $A, B \in \mathbb{R}_s^n$ be two distinct gyropoints of the Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, and let $t \in \mathbb{R}$ be a real parameter. Then, in full analogy with the Euclidean line (117), the graph of the set of all gyropoints

$$A \oplus (\ominus A \oplus B) \otimes t \tag{118}$$

$t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is a chord of the ball $\mathbb{R}_s^n$, shown in Fig. 2. As such, it is a geodesic line of the Cartesian–Beltrami–Klein ball model of hyperbolic geometry, shown in Fig. 2 for $n = 2$. The geodesic line (118) is the unique geodesic passing through the gyropoints A and B. It passes through the gyropoint A when $t = 0$ and, owing to the left cancellation law, (22), it passes through the gyropoint B when $t = 1$. Furthermore, it passes through the gyromidpoint $m_{A,B}$ of A and B when $t = 1/2$. Accordingly, the *gyrosegment* that joins the gyropoints A and B in Fig. 2 is obtained from gyroline (118) with $0 \le t \le 1$.

Each gyropoint of gyroline (118) with $0 < t < 1$ is said to lie *between* gyropoints A and B. Thus, for instance, the gyropoint P in Fig. 2 lies between the gyropoints A and B. As such, the gyropoints A, P, and B obey the *gyrotriangle equality* according to which $d(A, P) \oplus d(P.B) = d(A, B)$, in full analogy with Euclidean geometry.

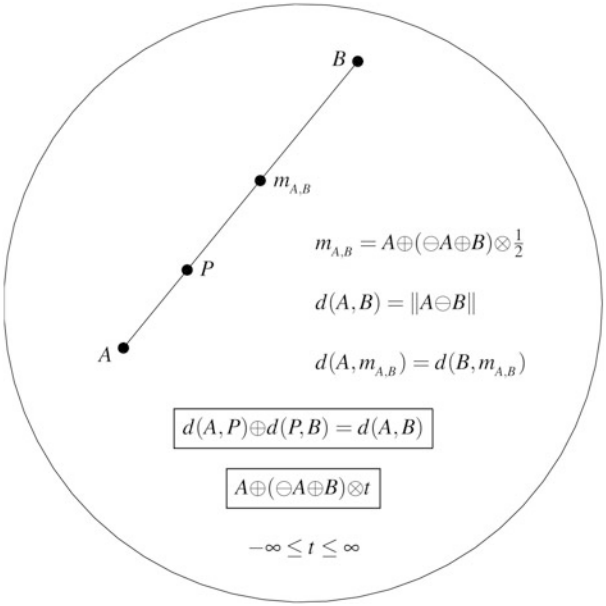
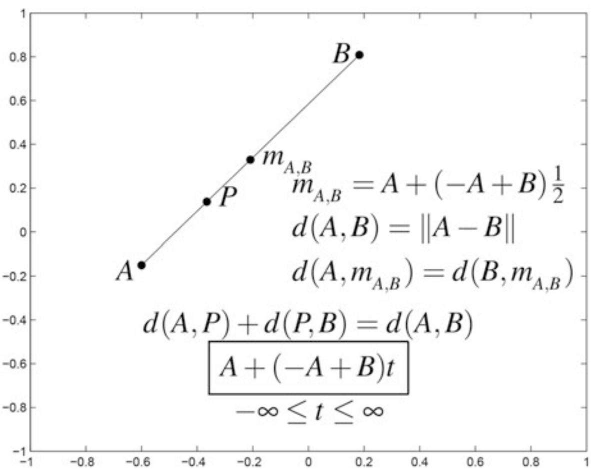


Fig. 2 Gyroline, the hyperbolic line. The gyroline $A \oplus (\ominus A \oplus B) \otimes t$, $t \in \mathbb{R}$, in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a geodesic line in the Beltrami–Klein ball model of hyperbolic geometry, fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in the Euclidean geometry of $\mathbb{R}^n$. The gyropoints A and B correspond to $t = 0$ and $t = 1$, respectively. The gyropoint P is a generic gyropoint on the gyroline through the gyropoints A and B lying between these gyropoints. The Einstein sum, $\oplus$, of the gyrodistance from A to P and from P to B equals the gyrodistance from A to B, so that gyrodistance along a gyroline is gyroadditive. The gyropoint $m_{A,B}$ is the gyromidpoint of the gyropoints A and B, corresponding to $t = 1/2$. The analogies between lines and gyrolines, as illustrated in Figs. 1 and 2, are obvious

Fig. 3 The Cartesian coordinates for the Euclidean plane $\mathbb{R}^2$, (x_1, x_2), $x_1^2 + x_2^2 < \infty$, unseen in Fig. 1, are shown here. The points A and B are given, with respect to these Cartesian coordinates by $A = (-0.60, -0.15)$ and $B = (0.18, 0.80)$



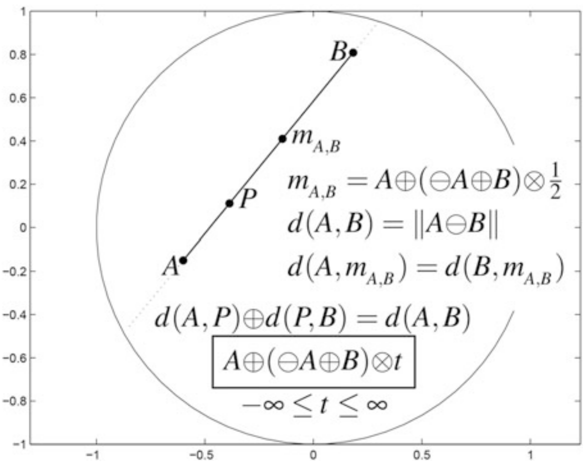


Fig. 4 The Cartesian coordinates for the unit disc in the Euclidean plane $\mathbb{R}^2$, (x_1, x_2), $x_1^2 + x_2^2 < 1$, unseen in Fig. 2, are shown here. The gyropoints A and B are given, with respect to these Cartesian coordinates by $A = (-0.60, -0.15)$ and $B = (0.18, 0.80)$

The gyropoints in Fig. 2 are drawn with respect to an unseen Cartesian coordinate system. The missing Cartesian coordinates for the hyperbolic disc in Fig. 2 are shown in Fig. 4.

The introduction of Cartesian coordinates $(x_1, x_2, \ldots, x_n)$, $x_1^2 + x_2^2 + \ldots + x_n^2 < \infty$ into the Euclidean n-space $\mathbb{R}^n$, along with the common vector addition in Cartesian coordinates, results in the Cartesian model of Euclidean geometry. The latter, in turn, enables Euclidean geometry to be studied analytically. In full analogy, the introduction of Cartesian coordinates $(x_1, x_2, \ldots, x_n)$, $x_1^2 + x_2^2 + \ldots + x_n^2 < s^2$ into the s-ball $\mathbb{R}_s^n$ of the Euclidean n-space $\mathbb{R}^n$, along with the common Einstein addition in Cartesian coordinates, results in the Cartesian model of hyperbolic geometry. The latter, in turn, enables hyperbolic geometry to be studied analytically. Indeed, Figs. 3 and 4 indicate the way we study analytic hyperbolic geometry, guided by analogies with analytic Euclidean geometry.

In Figs. 3 and 4, the Cartesian coordinates of A and B are equal. Yet, A and B in Fig. 3 are points of the Euclidean plane $\mathbb{R}^2$, while the same A and B in Fig. 4 are *gyropoints* of the hyperbolic *gyroplane* $\mathbb{R}_s^2$. Accordingly, for instance, in the Euclidean limit

$$\lim_{s\to\infty} (\ominus A \oplus B) = -A + B \tag{119}$$

A and B on the left side represent gyropoints of the ball $\mathbb{R}_s^n$, while A and B on the right side represent points of the ambient space $\mathbb{R}^n$.

Ambiguously, it is convenient to use the same notation for both $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ and $\mathbf{a}_{12} = -A_1 + A_2$, where it is always clear from the context whether $\mathbf{a}_{12}$ is the Euclidean difference between points or the hyperbolic gyrodifference between gyropoints. Hence, accordingly, we may write the Euclidean limit

$$\lim_{s\to\infty} \|\mathbf{a}_{12}\| = \|\mathbf{a}_{12}\| \tag{120}$$

where it is understood that $\|\mathbf{a}_{12}\| = \|\ominus A_1 \oplus A_2\|$ on the left side is the gyrodistance between the gyropoints A_1 and A_2 in the ball $\mathbb{R}^n_s$, while $\|\mathbf{a}_{12}\| = \|-A_1 + A_2\|$ on the right side is the distance between the points A_1 and A_2 in the ambient space $\mathbb{R}^n$ of the ball. This ambiguous notation will prove useful in the transition from hyperbolic geometry to Euclidean geometry in Sects. 41–42.

16 The Euclidean Group of Motions

The Euclidean group of motions of $\mathbb{R}^n$ consists of the commutative group of all translations of $\mathbb{R}^n$ and the group of all rotations of $\mathbb{R}^n$ about its origin.

For any $\mathbf{x} \in \mathbb{R}^n$, a translation of $\mathbb{R}^n$ by $\mathbf{x} \in \mathbb{R}^n$ is the map $L_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$L_{\mathbf{x}}\mathbf{v} = \mathbf{x} + \mathbf{v} \tag{121a}$$

for all $\mathbf{v} \in \mathbb{R}^n$.

A rotation R of $\mathbb{R}^n$ about its origin is an element of the group $SO(n)$ of all $n \times n$ orthogonal matrices with determinant 1. The rotation of $\mathbf{v} \in \mathbb{R}^n$ by $R \in SO(n)$ is given by $R\mathbf{v}$. The map $R \in SO(n)$ is a linear map of $\mathbb{R}^n$ that keeps the inner product invariant, that is

$$\begin{aligned} R(\mathbf{u} + \mathbf{v}) &= R\mathbf{u} + R\mathbf{v} \\ R\mathbf{u} \cdot R\mathbf{v} &= \mathbf{u} \cdot \mathbf{v} \end{aligned} \tag{121b}$$

for all $R \in \mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

The *Euclidean group of motions* is the *semidirect product group*

$$\mathbb{R}^n \times SO(n) \tag{122}$$

of the Euclidean commutative group $\mathbb{R}^n = (\mathbb{R}^n, +)$ and the rotation group $SO(n)$. It is a group of pairs $(\mathbf{x}, R)$, $\mathbf{x} \in (\mathbb{R}^n, +)$, $R \in SO(n)$, acting on elements $\mathbf{v} \in \mathbb{R}^n$ according to the equation

$$(\mathbf{x}, R)\mathbf{v} = \mathbf{x} + R\mathbf{v} \tag{123}$$

The group operation of the semidirect product group (122) is given by action composition. The latter, in turn, is determined by the following chain of equations, in which we employ the associative law:

$$\begin{aligned}(\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2)\mathbf{v} &= (\mathbf{x}_1, R_1)(\mathbf{x}_2 + R_2\mathbf{v}) \\ &= \mathbf{x}_1 + R_1(\mathbf{x}_2 + R_2\mathbf{v}) \\ &= \mathbf{x}_1 + (R_1\mathbf{x}_2 + R_1R_2\mathbf{v}) \\ &= (\mathbf{x}_1 + R_1\mathbf{x}_2) + R_1R_2\mathbf{v} \\ &= (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1R_2)\mathbf{v}\end{aligned} \tag{124}$$

for all $\mathbf{v} \in \mathbb{R}^n$.

Hence, by (124), the group operation of the semidirect product group (122) is given by the *semidirect product*

$$(\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2) = (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1R_2) \tag{125}$$

for any $(\mathbf{x}_1, R_1), (\mathbf{x}_2, R_2) \in \mathbb{R}^n \times SO(n)$.

Definition 14 (Covariance). An identity in $\mathbb{R}^n$ that remains invariant in form under the action of the Euclidean group of motions of $\mathbb{R}^n$ is said to be covariant.

Euclidean barycentric coordinate representations of points of $\mathbb{R}^n$ are covariant, as stated in Theorem 24, p. 605.

17 The Hyperbolic Group of Motions

The hyperbolic group of motions of $\mathbb{R}_s^n$ consists of the gyrocommutative gyrogroup of all left gyrotranslations of $\mathbb{R}_s^n$ and the group of all rotations of $\mathbb{R}_s^n$ about its center.

For any $\mathbf{x} \in \mathbb{R}_s^n$, a left gyrotranslation of $\mathbb{R}_s^n$ by $\mathbf{x} \in \mathbb{R}_s^n$ is the map $L_\mathbf{x} : \mathbb{R}_s^n \to \mathbb{R}_s^n$ given by

$$L_\mathbf{x}\mathbf{v} = \mathbf{x} \oplus \mathbf{v} \tag{126a}$$

for all $\mathbf{v} \in \mathbb{R}_s^n$.

The group of all rotations of the ball $\mathbb{R}_s^n$ about its center is $SO(n)$. Following (121b) we have

$$\begin{aligned}R(\mathbf{u} \oplus \mathbf{v}) &= R\mathbf{u} \oplus R\mathbf{v} \\ R\mathbf{u} \cdot R\mathbf{v} &= \mathbf{u} \cdot \mathbf{v}\end{aligned} \tag{126b}$$

for all $R \in SO(n)$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

The *hyperbolic group of motions* is the *gyrosemidirect product group*

$$\mathbb{R}_s^n \times SO(n) \tag{127}$$

of the Einsteinian gyrocommutative gyrogroup $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus)$ and the rotation group $SO(n)$. It is a group of pairs $(\mathbf{x}, R)$, $\mathbf{x} \in (\mathbb{R}_s^n, \oplus)$, $R \in SO(n)$, acting on elements $\mathbf{v} \in \mathbb{R}_s^n$ according to the equation

$$(\mathbf{x}, R)\mathbf{v} = \mathbf{x} \oplus R\mathbf{v} \,. \tag{128}$$

The group operation of the gyrosemidirect product group (127) is given by action composition. The latter, in turn, is determined by the following chain of equations, in which we employ the left gyroassociative law:

$$\begin{aligned}
(\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2)\mathbf{v} &= (\mathbf{x}_1, R_1)(\mathbf{x}_2 \oplus R_2\mathbf{v}) \\
&= \mathbf{x}_1 \oplus R_1(\mathbf{x}_2 \oplus R_2\mathbf{v}) \\
&= \mathbf{x}_1 \oplus (R_1\mathbf{x}_2 \oplus R_1R_2\mathbf{v}) \\
&= (\mathbf{x}_1 \oplus R_1\mathbf{x}_2) \oplus \mathrm{gyr}[\mathbf{x}, R_1\mathbf{x}_2]R_1R_2\mathbf{v} \\
&= (\mathbf{x}_1 \oplus R_1\mathbf{x}_2, \mathrm{gyr}[\mathbf{x}, R_1\mathbf{x}_2]R_1R_2)\mathbf{v}
\end{aligned} \tag{129}$$

for all $\mathbf{v} \in \mathbb{R}_s^n$.

Hence, by (129), the group operation of the gyrosemidirect product group (127) is given by the *gyrosemidirect product*

$$(\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2) = (\mathbf{x}_1 \oplus R_1\mathbf{x}_2, \mathrm{gyr}[\mathbf{x}, R_1\mathbf{x}_2]R_1R_2) \tag{130}$$

for any $(\mathbf{x}_1, R_1), (\mathbf{x}_2, R_2) \in \mathbb{R}_s^n \times SO(n)$. Indeed, the gyrosemidirect product is a group operation, as demonstrated in Sect. 10, p. 582.

Definition 15 (Gyrocovariance). An identity in $\mathbb{R}_s^n$ that remains invariant in form under the action of the hyperbolic group of motions of $\mathbb{R}_s^n$ is said to be gyrocovariant.

We will see that hyperbolic barycentric (i.e., gyrobarycentric) coordinate representations of gyropoints of $\mathbb{R}_s^n$ are gyrocovariant, by the Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 25, p. 609. A deeper study of Euclidean and hyperbolic motion groups in terms of analogies they share is presented in [61, Sects. 3.9–3.12].

18 Lorentz Transformation and Minkowski's Four-Velocity

Barycentric coordinates are commonly used as a tool in the study of Euclidean geometry. The notions of Lorentz transformation and relativistic velocity-dependent mass enable the adaptation of barycentric coordinates for use as a tool in hyperbolic geometry as well.

Einstein addition underlies the Lorentz transformation group of special relativity. A Lorentz transformation is a linear transformation of spacetime coordinates that fixes the spacetime origin. A Lorentz boost, $L(\mathbf{v})$, is a Lorentz transformation without rotation, parametrized by a velocity parameter $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_c$.

Being linear, the Lorentz boost has a matrix representation $L_m(\mathbf{v})$, which turns out to be [31],

$$L_m(\mathbf{v}) = \begin{pmatrix} \gamma_{\mathbf{v}} & c^{-2}\gamma_{\mathbf{v}} v_1 & c^{-2}\gamma_{\mathbf{v}} v_2 & c^{-2}\gamma_{\mathbf{v}} v_3 \\ \gamma_{\mathbf{v}} v_1 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_3 \\ \gamma_{\mathbf{v}} v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_2 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2 v_3 \\ \gamma_{\mathbf{v}} v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_1 v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_2 v_3 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} v_3^2 \end{pmatrix}. \tag{131}$$

Employing the matrix representation (131) of the Lorentz transformation boost, the Lorentz boost application to spacetime coordinates takes the form

$$L(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L_m(\mathbf{v}) \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} =: \begin{pmatrix} t' \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} \tag{132}$$

where $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}^3_c$, $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, $\mathbf{x}' = (x'_1, x'_2, x'_3)^t \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$, where exponent t denotes transposition.

In our approach to special relativity, analogies with classical results form the right tool. Hence, we emphasize that in the Newtonian/Euclidean limit of large vacuum speed of light c, $c \to \infty$, the Lorentz boost $L(\mathbf{v})$, (131)–(132), reduces to the Galilei boost $G(\mathbf{v})$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$\begin{aligned} G(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} &= \lim_{c\to\infty} L(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix} \end{aligned} \tag{133}$$

where $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

As we see from (132)–(133), our spacetime coordinates are $(t, \mathbf{x})^t$, and, as a result, the Lorentz boost matrix representation $L_m(\mathbf{v})$ in (131) is *nonsymmetric* for $c \neq 1$. In contrast, some authors present spacetime coordinates as $(ct, \mathbf{x})^t$, resulting in a *symmetric* Lorentz boost matrix representation found, for instance, in [25, Eq. (11.98), pp. 541].

Since in our approach to special relativity analogies with classical results from the right tool, the representation of spacetime coordinates as $(t, \mathbf{x})^t$ is more advantageous than its representation as $(ct, \mathbf{x})^t$. Indeed, unlike the latter representation, the former representation of spacetime coordinates allows one to recover the Galilei boost from the Lorentz boost by taking the Newtonian/Euclidean limit of large speed of light c, as shown in the transition from (132) to (133).

As a result of adopting $(t, \mathbf{x})^t$ rather than $(ct, \mathbf{x})^t$ as our four-vector that represents four-position, our four-velocity is given by $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}}\mathbf{v})$ rather than $(\gamma_{\mathbf{v}} c, \gamma_{\mathbf{v}}\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^3_c$. Similarly, our four-momentum is given by

$$\begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \frac{E}{c^2} \\ \mathbf{p} \end{pmatrix} = m \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix} \tag{134}$$

rather than the standard four-momentum, which is given by $(p_0, \mathbf{p})^t = (E/c, \mathbf{p})^t = (m\gamma_{\mathbf{v}} c, m\gamma_{\mathbf{v}}\mathbf{v})^t$, as found in most relativity physics books. According to (134) the relativistically invariant mass (i.e., rest mass) m of a particle is the ratio of the particle's four-momentum $(p_0, \mathbf{p})^t$ to its four-velocity $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}}\mathbf{v})^t$.

For the sake of simplicity, and without loss of generality, some authors normalize the vacuum speed of light to $c = 1$ as, for instance, in [23]. We, however, prefer to leave c as a free positive parameter, enabling related modern results to be reduced to classical ones under the limit of large c, $c \to \infty$ as, for instance, in the transition from a Lorentz boost into a corresponding Galilei boost in (131)–(133).

The Lorentz boost (131)–(132) can be written vectorially in the form

$$L(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}}(t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t + \mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix} . \tag{135}$$

Being written in a vector form, the Lorentz boost in (135) survives unimpaired in higher dimensions. Rewriting (135) in higher dimensional spaces, with $\mathbf{x} = \mathbf{v}t$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c \subset \mathbb{R}^n$, we have

$$\begin{aligned} L(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{v}t \end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}}(t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}t) \\ \gamma_{\mathbf{u}}\mathbf{u}t + \mathbf{v}t + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v}t)\mathbf{u} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}} t \\ \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}}(\mathbf{u}\oplus\mathbf{v})t \end{pmatrix} . \end{aligned} \tag{136}$$

Equation (136) reveals explicitly the way Einstein velocity addition underlies the Lorentz boost. The second equation in (136) follows from the first by (4), p. 565, and (2), p. 565.

The special case of $t = \gamma_{\mathbf{v}}$ in (136) proves useful, giving rise to the elegant identity

$$L(\mathbf{u})\begin{pmatrix}\gamma_{\mathbf{v}}\\ \gamma_{\mathbf{v}}\mathbf{v}\end{pmatrix} = \begin{pmatrix}\gamma_{\mathbf{u}\oplus\mathbf{v}}\\ \gamma_{\mathbf{u}\oplus\mathbf{v}}(\mathbf{u}\oplus\mathbf{v})\end{pmatrix} \tag{137}$$

of the Lorentz boost of four velocities, $\mathbf{u},\mathbf{v} \in \mathbb{R}_c^n$. Since in physical applications $n = 3$, in the context of n-dimensional special relativity, we call $\mathbf{v}$ a three-vector and $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}}\mathbf{v})^t$ a four-vector.

The four-vector $m(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}}\mathbf{v})^t$ is the four-momentum of a particle with invariant mass (or rest mass) m and velocity $\mathbf{v}$ relative to a given inertial rest frame $\Sigma_{\mathbf{0}}$. Let $\Sigma_{\ominus\mathbf{u}}$ be an inertial frame that moves with velocity $\ominus\mathbf{u} = -\mathbf{u}$ relative to the rest frame $\Sigma_{\mathbf{0}}$, $\mathbf{u},\mathbf{v} \in \mathbb{R}_c^n$. Then, the particle with velocity $\mathbf{v}$ relative to $\Sigma_{\mathbf{0}}$ has velocity $\mathbf{u}\oplus\mathbf{v}$ relative to the frame $\Sigma_{\ominus\mathbf{u}}$. In full agreement and, owing to the linearity of the Lorentz boost, it follows from (137) that the four-momentum of the particle relative to the frame $\Sigma_{\ominus\mathbf{u}}$ is

$$\begin{aligned} L(\mathbf{u})m\begin{pmatrix}\gamma_{\mathbf{v}}\\ \gamma_{\mathbf{v}}\mathbf{v}\end{pmatrix} &= mL(\mathbf{u})\begin{pmatrix}\gamma_{\mathbf{v}}\\ \gamma_{\mathbf{v}}\mathbf{v}\end{pmatrix} \\ &= m\begin{pmatrix}\gamma_{\mathbf{u}\oplus\mathbf{v}}\\ \gamma_{\mathbf{u}\oplus\mathbf{v}}(\mathbf{u}\oplus\mathbf{v})\end{pmatrix}. \end{aligned} \tag{138}$$

It follows from the linearity of the Lorentz boost and from (137) that

$$\begin{aligned} L(\mathbf{w})\sum_{k=1}^{N} m_k\begin{pmatrix}\gamma_{\mathbf{v}_k}\\ \gamma_{\mathbf{v}_k}\mathbf{v}_k\end{pmatrix} &= \sum_{k=1}^{N} m_k L(\mathbf{w})\begin{pmatrix}\gamma_{\mathbf{v}_k}\\ \gamma_{\mathbf{v}_k}\mathbf{v}_k\end{pmatrix} \\ &= \sum_{k=1}^{N} m_k\begin{pmatrix}\gamma_{\mathbf{w}\oplus\mathbf{v}_k}\\ \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)\end{pmatrix} \\ &= \begin{pmatrix}\sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}\\ \sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)\end{pmatrix}. \end{aligned} \tag{139}$$

The chain of Eqs. (139) reveals the interplay of Einstein addition, $\oplus$, in the ball $\mathbb{R}_c^n$ and vector addition, $+$, in the ambient space $\mathbb{R}^n$ that appears implicitly in the Σ-notation for scalar and vector addition. This harmonious interplay between $\oplus$

and $+$, which is crucially important in our mission to determine hyperbolic triangle centers, reveals itself in (139) where Einstein's three-vector formalism of special relativity meets Minkowski's four-vector formalism of special relativity.

The (Minkowski) norm of a four-vector is Lorentz transformation invariant. The norm of the four-position $(t, \mathbf{x})^t$ is

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \sqrt{t^2 - \frac{\|\mathbf{x}\|^2}{c^2}} \tag{140}$$

and, accordingly, the norm of the four-velocity $(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}}\mathbf{v})^t$ is

$$\left\| \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix} \right\| = \gamma_{\mathbf{v}} \left\| \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix} \right\| = \gamma_{\mathbf{v}} \sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}} = 1\,. \tag{141}$$

19 Invariant Mass of a System of Particles

In obtaining the result in (138), we exploit the linearity of the Lorentz boost. We will now further exploit that linearity, demonstrated in (139), to obtain the relativistically invariant mass of a system of particles. Being invariant, we refer the Newtonian, rest mass, m, to as the (relativistically) invariant mass, as opposed to the common relativistic mass, $m\gamma_{\mathbf{v}}$, which is velocity dependent.

Let

$$S = S(m_k, \mathbf{v}_k, \Sigma_{\mathbf{0}}, k = 1, \ldots, N) \tag{142}$$

be an isolated system of N noninteracting material particles the kth particle of which has invariant mass $m_k > 0$ and velocity $\mathbf{v}_k \in \mathbb{R}^n_c$ relative to an inertial frame $\Sigma_{\mathbf{0}}$, $k = 1, \ldots, N$.

Classically, the Newtonian mass m_{newton} of the system S equals the sum of the Newtonian masses of its constituent particles, that is

$$m_{\text{newton}} = \sum_{k=1}^{N} m_k\,, \tag{143}$$

and it forms the total mass of the system. Relativistically, however, this need not be the case since dark matter may emerge, as we will see in Theorem 22 of Sect. 20.

Accordingly, we wish to determine the relativistically invariant mass m_0 of the system S and the velocity $\mathbf{v}_0$ relative to $\Sigma_{\mathbf{0}}$ of a fictitious inertial frame, called the center of momentum frame, relative to which the three-momentum of S vanishes.

Assuming that the four-momentum is additive, the sum of the four-momenta of the N particles of the system S gives the four-momentum $(m_0\gamma_{\mathbf{v}_0}, m_0\gamma_{\mathbf{v}_0}\mathbf{v}_0)^t$ of S. Accordingly,

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} , \qquad (144)$$

where

1. the invariant masses $m_k > 0$ and the velocities $\mathbf{v}_k \in \mathbb{R}_c^n$, $k = 1, \ldots, N$, relative to $\Sigma_{\mathbf{0}}$ of the constituent particles of S are given
2. the invariant mass m_0 of S and the velocity $\mathbf{v}_0$ of the center of momentum frame of S relative to $\Sigma_{\mathbf{0}}$ are to be determined uniquely by the *Resultant Relativistically Invariant Mass Theorem*, which is Theorem 22 in Sect. 20.

If $m_0 > 0$ and $\mathbf{v}_0 \in \mathbb{R}_c^n$ that satisfy (144) exist, then, as anticipated, the three-momentum of the system S relative to its center of momentum frame vanishes since, by (138) and (144), the four-momentum of S relative to its center of momentum frame is given by

$$\begin{aligned} L(\ominus\mathbf{v}_0)\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} &= L(\ominus\mathbf{v}_0) m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \\ &= m_0 \begin{pmatrix} \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0} \\ \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0}(\ominus\mathbf{v}_0 \oplus \mathbf{v}_0) \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} , \end{aligned} \qquad (145)$$

noting that $\gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_0} = \gamma_{\mathbf{0}} = 1$.

20 The Resultant Relativistically Invariant Mass Theorem

Einstein velocity addition law (2), p. 565, admits the following theorem that involves expressions, in (148)–(150) below, which are covariant under left gyrotranslations (and, hence, are *gyrocovariant*).

Theorem 22 (Resultant Relativistically Invariant Mass Theorem). *Let* $(\mathbb{R}_c^n, \oplus)$ *be an Einstein gyrogroup, and let* $m_k \in \mathbb{R}$ *and* $\mathbf{v}_k \in \mathbb{R}_c^n$, $k = 1, 2, \ldots, N$, *be* N *real numbers and* N *elements of* $\mathbb{R}_c^n$ *satisfying*

$$\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \neq 0 \qquad (146)$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \qquad (147)$$

be an $(n + 1)$*-vector equation for the two unknowns* $m_0 \in \mathbb{R}$ *and* $\mathbf{v}_0 \in \mathbb{R}^n$.

Then (147) *possesses a unique solution* $(m_0, \mathbf{v}_0)$*,* $m_0 \neq 0$*,* $\mathbf{v}_0 \in \mathbb{R}_c^n$*, satisfying the following three identities for all* $\mathbf{w} \in \mathbb{R}_c^n$ *(including, in particular, the interesting special case of* $\mathbf{w} = \mathbf{0}$*):*

$$\mathbf{w} \oplus \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k} (\mathbf{w} \oplus \mathbf{v}_k)}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}} \tag{148}$$

$$\gamma_{\mathbf{w}\oplus\mathbf{v}_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}}{m_0} \tag{149}$$

$$\gamma_{\mathbf{w}\oplus\mathbf{v}_0} (\mathbf{w} \oplus \mathbf{v}_0) = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k} (\mathbf{w} \oplus \mathbf{v}_0)}{m_0} \tag{150}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus(\mathbf{w}\oplus\mathbf{v}_j)\oplus(\mathbf{w}\oplus\mathbf{v}_k)} - 1)}\,. \tag{151}$$

The proof of Theorem 22 is found in [56, Sect. 3.4].

It follows from (151) that the relativistically invariant mass m_0 of a particle system is greater than the sum $\sum_{k=1}^{N} m_k$ of the Newtonian Masses of its constituents. The excessive mass, $m_0 - \sum_{k=1}^{N} m_k$, is *dark* in the sense that (1) it is generated by internal relative velocities between the constituents of the particle system and that (2) it reveals its presence only gravitationally, since it emits no radiation and it involves no collisions [51, 57]. Interestingly, the relativistically invariant mass m_0 of a particle system in (151) is precisely what we need in order to adapt the Euclidean notion of barycentric coordinates for use in hyperbolic geometry without losing covariance.

To appreciate the power and elegance of Theorem 22 in relativistic mechanics in terms of novel analogies that it shares with familiar results in classical mechanics, we present below the classical counterpart, Theorem 23, of Theorem 22. The latter is obtained from the former by approaching the Newtonian/Euclidean limit when c tends to infinity. The resulting Theorem 23 is immediate, and its importance in classical mechanics is well known. Like Theorem 22, Theorem 23 involves an expression, in (154) below, which is covariant under translations.

Theorem 23 (Resultant Newtonian Invariant Mass Theorem). *Let* $(\mathbb{R}^n, +)$ *be a Euclidean n-space, and let* $m_k \in \mathbb{R}$ *and* $\mathbf{v}_k \in \mathbb{R}^n$*,* $k = 1, 2, \ldots, N$*, be N real numbers and N elements of* $\mathbb{R}^n$ *satisfying*

$$\sum_{k=1}^{N} m_k \neq 0 \tag{152}$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ \mathbf{v}_0 \end{pmatrix} \tag{153}$$

be an $(n+1)$*-vector equation for the two unknowns* $m_0 \in \mathbb{R}$ *and* $\mathbf{v}_0 \in \mathbb{R}^n$*.*

Then (153) *possesses a unique solution* $(m_0, \mathbf{v}_0)$*,* $m_0 \neq 0$*, satisfying the following equations for all* $\mathbf{w} \in \mathbb{R}^n$ *(including, in particular, the interesting special case of* $\mathbf{w} = \mathbf{0}$*):*

$$\mathbf{w} + \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k)}{\sum_{k=1}^{N} m_k} \tag{154}$$

and

$$m_0 = \sum_{k=1}^{N} m_k\,. \tag{155}$$

The proof of Theorem 23 is immediate.

Unlike Identity (154) of Theorem 23, which is immediate, its counterpart in Theorem 22, Identity (148), is not immediate. Yet, in full analogy with Theorem 23, the validity of Identity (148) in Theorem 22 for all $\mathbf{w} \in \mathbb{R}_c^n$ is geometrically important. This geometric importance of Identity (148) lies on its implication that the velocity $\mathbf{v}_0$ of the center of momentum frame of a particle system relative to a given inertial rest frame in relativistic mechanics is independent of the choice of the origin of the relativistic velocity space $\mathbb{R}_c^n$ with its underlying Cartesian–Beltrami–Klein ball model of hyperbolic geometry.

Not unexpectedly, the Newtonian mass m_0 in (155) of a particle system plays an important role in Theorem 24, p. 605, on the covariance of barycentric coordinates under the motions of Euclidean geometry, which are translations and rotations. Remarkably, the relativistic invariant mass m_0 in (151) of a particle system plays an analogous important role in Theorem 25, p. 609, on the gyrocovariance of gyrobarycentric coordinates under the gyromotions of hyperbolic geometry, which are left gyrotranslations and rotations. Left gyrotranslations, in turn, will play in the sequel an important role in the application of gyrobarycentric coordinates for determining the gyrotriangle Cabrera gyropoint.

21 Euclidean Barycentric Coordinates

The notion of barycentric coordinates dates back to Möbius. The use of barycentric coordinates in Euclidean geometry is described in [64], and the historical contribution of Möbius' barycentric coordinates to vector analysis is described in [12, pp. 48–50].

In this section we set the stage for the introduction in Sect. 22 of barycentric coordinates into hyperbolic geometry by illustrating the way Theorem 23, p. 602, suggests the introduction of barycentric coordinates into Euclidean geometry.

For any positive integer N, let $m_k \in \mathbb{R}$ be N given real numbers such that

$$\sum_{k=1}^{N} m_k \neq 0 \tag{156}$$

and let $A_k \in \mathbb{R}^n$ be N given points in the Euclidean n-space $\mathbb{R}^n$, $k = 1, \ldots, N$. Theorem 23, p. 602, states the trivial, but geometrically significant, result that the equation

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ A_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ P \end{pmatrix} \tag{157}$$

for the unknowns $m_0 \in \mathbb{R}$ and $P \in \mathbb{R}^n$ possesses the unique solution given by

$$m_0 = \sum_{k=1}^{N} m_k \tag{158}$$

and

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{159}$$

satisfying for all $X \in \mathbb{R}^n$,

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k}\,. \tag{160}$$

We view (159) as the representation of a point $P \in \mathbb{R}^n$ in terms of its *barycentric coordinates* m_k, $k = 1, \ldots, N$, with respect to the set of points $S = \{A_1, \ldots, A_N\}$. Identity (160), then, insures that the barycentric coordinate representation (159) of P with respect to the set S is *covariant* (or *invariant in form*) in the following sense. The point P and the points of the set S of its barycentric coordinate representation vary together under translations. Indeed, a translation $X + A_k$ of A_k by X, $k = 1, \ldots, N$, in (160) results in the translation $X + P$ of P by X.

In order to insure that barycentric coordinate representations with respect to a set S are unique, we require S to be pointwise independent.

Definition 16 (Euclidean Pointwise Independence). A set S of N points $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n$, $n \geq 2$, is *pointwise independent* if the $N - 1$ vectors $-A_1 + A_k$, $k = 2, \ldots, N$, are linearly independent.

Definition 17 (Barycentric Coordinates). Let

$$S = \{A_1, \dots, A_N\} \tag{161}$$

be a pointwise independent set of N points in $\mathbb{R}^n$. The real numbers $m_1, \dots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \neq 0 \tag{162}$$

are barycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \, . \tag{163}$$

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates $(m_1, \dots, m_N)$ of the point P in (163) are equivalent to the barycentric coordinates $(\lambda m_1, \dots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates $(m_1, \dots, m_N)$ are also written as $(m_1 : \dots : m_N)$.

Barycentric coordinates that are normalized by the condition

$$\sum_{k=1}^{N} m_k = 1 \tag{164}$$

are called *special barycentric coordinates*.

Equation (163) is said to be the (unique) barycentric coordinate representation of P with respect to the set S.

Theorem 24 (Covariance of Barycentric Coordinate Representations). *Let*

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{165}$$

be the barycentric coordinate representation of a point $P \in \mathbb{R}^n$ in a Euclidean n-space $\mathbb{R}^n$ with respect to a pointwise independent set $S = \{A_1, \dots, A_N\} \subset \mathbb{R}^n$. The barycentric coordinate representation (165) is covariant, that is,

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k} \tag{166}$$

for all $X \in \mathbb{R}^n$, *and*

$$RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k} \tag{167}$$

for all $R \in SO(n)$.

Proof. The proof is immediate, noting that rotations $R \in SO(n)$ of $\mathbb{R}^n$ about its origin are linear maps of $\mathbb{R}^n$. □

Following the vision of Felix Klein in his *Erlangen Program* [8, 32], it is owing to the covariance with respect to translations and rotations that barycentric coordinate representations possess geometric significance. Indeed, translations and rotations in Euclidean geometry form the *group of motions* of the geometry, studied in Sect. 16, and according to Felix Klein's Erlangen Program, a geometric property is a property that remains invariant in form under the motions of the geometry.

22 Gyrobarycentric Coordinates

Guided by analogies with Sect. 21, in this section we introduce barycentric coordinates into hyperbolic geometry [53], where they are called *gyrobarycentric coordinates*. Gyrobarycentric coordinates prove useful in the determination of various gyrotriangle gyrocenters, just as barycentric coordinates prove useful in the determination of various triangle centers.

For any positive integer N, let $m_k \in \mathbb{R}$ be N given real numbers, and let $A_k \in \mathbb{R}_s^n$ be N given gyropoints in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, $k = 1, \ldots, N$, satisfying,

$$\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \ > \ 0 \tag{168}$$

Theorem 22, p. 601 presents the result that the equation

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{A_k} \\ \gamma_{A_k} A_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_P \\ \gamma_P P \end{pmatrix} \tag{169}$$

for the unknowns $m_0 \in \mathbb{R}$ and $P \in \mathbb{R}_s^n$ possesses the unique solution given by

$$m_0 \ = \ \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)} \tag{170}$$

$m_0 > 0$, satisfying

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus(X\oplus A_j)\oplus(X\oplus A_k)} - 1)} \tag{171}$$

for all $X \in \mathbb{R}_s^n$, and

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{172}$$

satisfying

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}(X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}} \tag{173}$$

for all $X \in \mathbb{R}_s^n$.

Furthermore, Theorem 22, p. 601, also states that P and m_0 satisfy the two identities

$$\gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0} \tag{174}$$

and

$$\gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0} \tag{175}$$

and, more generally,

$$\gamma_{X\oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}}{m_0} \tag{176}$$

and

$$\gamma_{X\oplus P}(X \oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}(X \oplus A_k)}{m_0} \tag{177}$$

for all $X \in \mathbb{R}_s^n$.

We view (172) as the representation of a gyropoint $P \in \mathbb{R}_s^n$ in terms of its *hyperbolic barycentric coordinates* m_k, $k = 1, \ldots, N$, with respect to the set of gyropoints $S = \{A_1, \ldots, A_N\}$. Naturally in gyrolanguage, hyperbolic barycentric coordinates are called *gyrobarycentric coordinates*. Identity (173) insures that the *gyrobarycentric coordinate representation* (172) of P with respect to the set S is *gyrocovariant* in the sense of Definition 15, p. 596, as shown in Theorem 25 below. The gyropoint P and the gyropoints of the set S of its gyrobarycentric coordinate representation vary together under left gyrotranslations. Indeed, a left gyrotranslation $X \oplus A_k$ of A_k by X, $k = 1, \dots, N$ in (173) results in the left gyrotranslation $X \oplus P$ of P by X.

In order to insure that gyrobarycentric coordinate representations with respect to a set S are unique, we require S to be hyperbolically pointwise independent.

Definition 18 (Hyperbolic Pointwise Independence). A set S of N gyropoints $S = \{A_1, \dots, A_N\}$ in $\mathbb{R}_s^n$, $n \geq 2$, is *gyropointwise independent* if the $N - 1$ gyrovectors in $\mathbb{R}_s^n$, $\ominus A_1 \oplus A_k$, $k = 2, \dots, N$, considered as vectors in $\mathbb{R}^n$, are linearly independent.

We are now in the position to present the formal definition of gyrobarycentric coordinates, that is, hyperbolic barycentric coordinates, as motivated by mass and center of momentum velocity of Einsteinian particle systems.

Definition 19 (Gyrobarycentric Coordinates). Let

$$S = \{A_1, \dots, A_N\} \tag{178}$$

be a gyropointwise independent set of N gyropoints in $\mathbb{R}_s^n$. The real numbers $m_1, \dots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \gamma_{A_k} > 0 \tag{179}$$

are gyrobarycentric coordinates of a gyropoint $P \in \mathbb{R}_s^n$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} . \tag{180}$$

Gyrobarycentric coordinates are homogeneous in the sense that the gyrobarycentric coordinates $(m_1, \dots, m_N)$ of the gyropoint P in (180) are equivalent to the gyrobarycentric coordinates $(\lambda m_1, \dots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates $(m_1, \dots, m_N)$ are also written as $(m_1 : \dots : m_N)$.

Gyrobarycentric coordinates that are normalized by the condition

$$\sum_{k=1}^{N} m_k = 1 \tag{181}$$

are called *special gyrobarycentric coordinates*.

Equation (180) is said to be the gyrobarycentric coordinate representation of P with respect to the set S.

Finally, the constant of the gyrobarycentric coordinate representation of P in (180) is $m_0 > 0$, given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)}\ . \tag{182}$$

Theorem 25 (Gyrocovariance of Gyrobarycentric Coordinate Representations). *Let*

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{183a}$$

be a gyrobarycentric coordinate representation of a gyropoint $P \in \mathbb{R}_s^n$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with respect to a gyropointwise independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n$.

Then

$$\gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0} \tag{183b}$$

and

$$\gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0} \tag{183c}$$

where m_0, given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)}\ , \tag{183d}$$

$m_0 > 0$, is the constant of the gyrobarycentric coordinate representation (183a).

Furthermore, the gyrobarycentric coordinate representation (183a) and its associated identities in (183b)–(183d) are gyrocovariant, that is,

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{184a}$$

$$\gamma_{X\oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}}{m_0} \tag{184b}$$

$$\gamma_{X\oplus P}(X \oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}(X \oplus A_k)}{m_0} \tag{184c}$$

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus(X\oplus A_j)\oplus(X\oplus A_k)} - 1)} \tag{184d}$$

for all $X \in \mathbb{R}_s^n$, *and*

$$RP = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} RA_k}{\sum_{k=1}^{N} m_k \gamma_{RA_k}} \tag{185a}$$

$$\gamma_{RP} = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}}{m_0} \tag{185b}$$

$$\gamma_{RP}(RP) = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}(RA_k)}{m_0} \tag{185c}$$

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus(RA_j)\oplus(RA_k)} - 1)} \tag{185d}$$

for all $R \in SO(n)$.

The proof of Theorem 25 is found in [56, Sect. 4.2].

Following the vision of Felix Klein in his *Erlangen Program* [8, 32], it is owing to the gyrocovariance, that is, covariance with respect to left gyrotranslations and rotations, that gyrobarycentric coordinate representations are geometrically significant. Indeed, left gyrotranslations and rotations in hyperbolic geometry form the group of motions of the geometry, studied in Sect. 17, and according to Felix Klein's Erlangen Program, a geometric property is a property that remains invariant in form under the motions of the geometry.

The following two corollaries of Theorem 25 prove useful.

Corollary 1. *Let* $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n$ *be a gyropointwise independent set of* N *gyropoints in* $\mathbb{R}_s^n$, *and let*

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{186}$$

be a gyrobarycentric coordinate representation of a gyropoint $P \in \mathbb{R}^n$ with respect to the set S. Furthermore, let m_0 be the representation constant, given by

$$m_0^2 = \left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus A_j \oplus A_k} - 1)\,. \tag{187}$$

Then, the point P lies in the ball $\mathbb{R}_s^n$, $P \in \mathbb{R}_s^n$, if and only if $m_0^2 > 0$ (in other words, the point P is a gyropoint if and only if $m_0^2 > 0$).

The proof of Corollary 1 is found in [56, Corollary 4.9].

Corollary 2. *Let $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n$ be a gyropointwise independent set of N gyropoints in $\mathbb{R}_s^n$, and let*

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{188}$$

be a gyrobarycentric coordinate representation of a point $P \in \mathbb{R}^n$ with respect to the set S, with positive gyrobarycentric coordinates $m_k > 0$, $k = 1, \ldots, N$. Then, $P \in \mathbb{R}_s^n$. Moreover, P lies on the convex span of S if and only if $m_k > 0$, $k = 1, \ldots, N$.

Proof. The gyrobarycentric coordinate representation (188) possesses the constant m_0 in (187). This representation constant is positive since $m_k > 0$, $k = 1, \ldots, N$ and since the gamma factors in (187) are greater than 1. Hence, by Corollary 1, $P \in \mathbb{R}_s^n$.

The gyrobarycentric combination (188) is positive if all the coefficients m_k, $k = 1, \ldots, N$ are positive. The set of all positive gyrobarycentric combinations of the points of the set S is the convex span of S. By convexity considerations, it is a subset of $\mathbb{R}_s^n$. Hence, P lies on the convex span of S if and only if $m_k > 0$, $k = 1, \ldots, N$. Owing to the homogeneity of gyrobarycentric coordinates, it is agreed that if all the gyrobarycentric coordinates of a point have equal signs, then the signs are selected to be positive. □

23 Uniqueness of Gyrobarycentric Coordinate Representations

Theorem 26 (Uniqueness of Gyrobarycentric Coordinate Representations). *A gyrobarycentric coordinate representation of a gyropoint in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with respect to a gyropointwise independent set $S = \{A_1, \ldots, A_N\}$ is unique.*

The proof of Theorem 26 is found in [56, Sect. 4.3].

Remark 1 (Gyrobarycentric Coordinates and Quantum Field Theory). The outstanding book [9] studies the geometry of classical quantum states by means of barycentric coordinates of *probability simplices*, where barycentric coordinates are interpreted as probabilities [9, pp. 3–5, 235]. Suggestively, relativistic quantum states may be studied by means of gyrobarycentric coordinates of *gyroprobability gyrosimplices*, where gyrobarycentric coordinates are interpreted as gyroprobabilities [59]. Gyrosimplices, the hyperbolic counterparts of simplices, are studied in [61]. Accordingly, the study of relativistic quantum mechanics, known as *Quantum Field Theory*, can be based on gyrobarycentric coordinates of gyroprobability gyrosimplices, just as the study of quantum mechanics in [9] is based on barycentric coordinates of probability simplices. The suitability of Bayesian probabilities for use in quantum mechanics is known [10]. Following Einstein addition, Bayesian probabilities form the appropriate interpretation for gyrobarycentric coordinates of probability simplices in quantum field theory.

24 Triangle Centroid

The triangle centroid is located at the intersection of the triangle medians. In this section we demonstrate the use of barycentric coordinates by determining the triangle centroid in $\mathbb{R}^n$.

Let $A_1A_2A_3$ be a triangle with vertices A_1, A_2, and A_3 in a Euclidean n-space $\mathbb{R}^n$, and let G be the triangle centroid, as shown in Fig. 5 for $n = 2$. Then, G is given by its barycentric coordinate representation (163), p. 605, with respect to the set $\{A_1, A_2, A_3\}$,

$$G = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} \tag{189}$$

where the barycentric coordinates m_1, m_2, and m_3 of G in (189) are to be determined in (195).

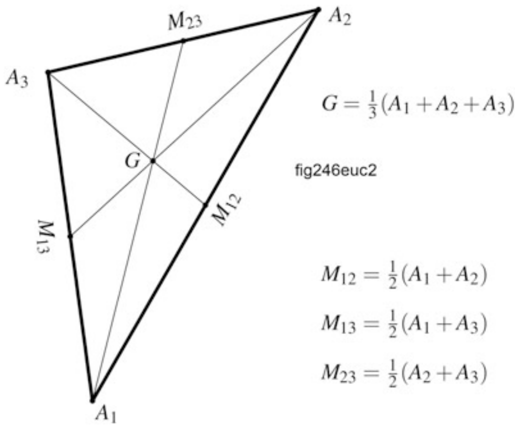


Fig. 5 The triangle medians and centroid in a Euclidean plane $\mathbb{R}^2$. The centroid G is the point of concurrency of the triangle medians. The midpoints M_{12}, M_{13}, and M_{23} of the three sides, A_1A_2, A_1A_3, and A_2A_3 of triangle $A_1A_2A_3$ in a Euclidean n-space $\mathbb{R}^n$ are shown here for $n = 2$, along with its medians $A_1M_{23}, A_2M_{13}, A_3M_{12}$, and its centroid G. This figure sets the stage for its hyperbolic counterpart, shown in Fig. 7, p. 617

The midpoint M_{12} of side A_1A_2, shown in Fig. 5, is given by

$$M_{12} = \frac{A_1 + A_2}{2} \tag{190}$$

so that an equation of the line L_{123} through the points M_{12} and A_3 is given by

$$L_{123}(t_1) = A_3 + (-A_3 + \frac{A_1 + A_2}{2})t_1 \tag{191}$$

with the parameter $t_1 \in \mathbb{R}$.

The line $L_{123}(t_1)$ contains one of the three medians of the sides of triangle $A_1A_2A_3$. Equations of the lines L_{123}, L_{231}, and L_{312} that contain, respectively, the three triangle medians are therefore obtained from (191) by index cyclic permutations,

$$\begin{aligned}
L_{123}(t_1) &= \frac{t_1}{2}A_1 + \frac{t_1}{2}A_2 + (1 - t_1)A_3 \\
L_{231}(t_2) &= \frac{t_2}{2}A_2 + \frac{t_2}{2}A_3 + (1 - t_2)A_1 \\
L_{312}(t_3) &= \frac{t_3}{2}A_3 + \frac{t_3}{2}A_1 + (1 - t_3)A_2
\end{aligned} \tag{192}$$

$t_1, t_2, t_3 \in \mathbb{R}$.

The triangle centroid G, shown in Fig. 5, is the point of concurrency of the three lines in (192). It is found by solving the equation $L_{123}(t_1) = L_{231}(t_2) = L_{312}(t_3)$ for the unknowns $t_1, t_2, t_3 \in \mathbb{R}$, obtaining $t_1 = t_2 = t_3 = 2/3$. Hence, G is given by the equation

$$G = \frac{A_1 + A_2 + A_3}{3} \tag{193}$$

as we see by substituting $t_1 = t_2 = t_3 = 2/3$ into (192).

Comparing (193) with (189) we find that the special barycentric coordinates (m_1, m_2, m_3) of G with respect to the set $\{A_1, A_2, A_3\}$ are given by

$$m_1 = m_2 = m_3 = \frac{1}{3}\,. \tag{194}$$

Accordingly, convenient barycentric coordinates $(m_1 : m_2 : m_3)$ of G are

$$(m_1 : m_2 : m_3) = (1 : 1 : 1) \tag{195}$$

as it is well known in the literature; see, for instance, [26, 27].

25 Gyromidpoint

In this section we demonstrate the use of gyrobarycentric coordinates by determining the gyromidpoints of gyrosegments in Einstein gyrovector spaces. Gyromidpoints, in turn, play an important role in the definitions of gyromedial gyrotriangles and Cabrera gyropoints, shown in Fig. 16, p. 659.

Let A_1A_2 be a gyrosegment in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq 1$, formed by two distinct gyropoints $A_1, A_2 \in \mathbb{R}^n_s$. The gyromidpoint M_{12} of gyrosegment A_1A_2 in Fig. 6 is the gyropoint of the gyrosegment that is equigyrodistant from A_1 and A_2, that is,

$$\| \ominus A_1 \oplus M_{12}\| = \| \ominus A_2 \oplus M_{12}\| \,. \tag{196}$$

In order to determine the gyromidpoint M_{12} of gyrosegment A_1A_2, let M_{12} be given by its gyrobarycentric coordinate representation (180) with respect to the set $S = \{A_1, A_2\}$,

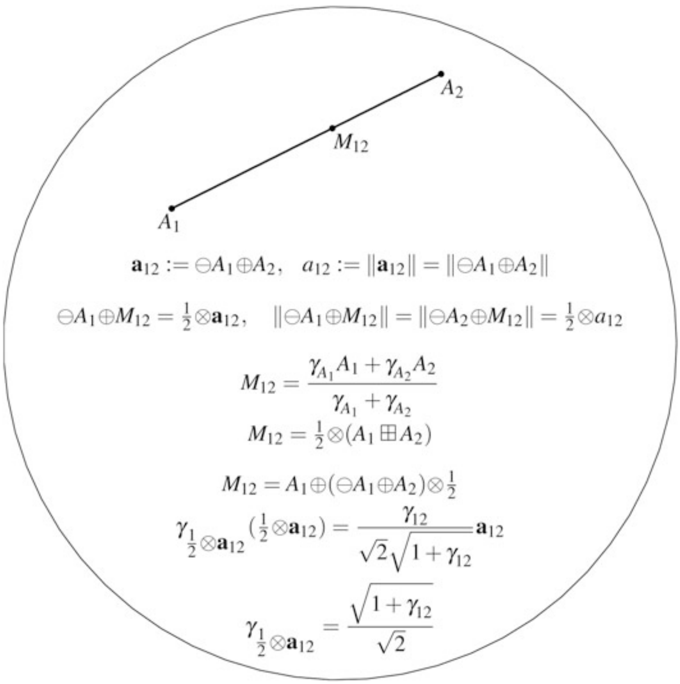


Fig. 6 The Einstein Gyromidpoint. The Einstein gyromidpoint M_{12} of a gyrosegment A_1A_2 in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is shown for $n = 2$, along with several useful identities that the gyromidpoint possesses

$$M_{12} = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2}{m_1\gamma_{A_1} + m_2\gamma_{A_2}}\,, \tag{197}$$

where the gyrobarycentric coordinates m_1 and m_2 are to be determined in (205) below.

The constant m_0 of the gyrobarycentric coordinate representation (197) of M_{12} turns out to be

$$m_0 \;=\; \sqrt{(m_1+m_2)^2 + 2m_1m_2(\gamma_{12}-1)} \tag{198}$$

according to (182).

Following the gyrocovariance of gyrobarycentric coordinate representations, Theorem 25, we have from (184a) with $X = \ominus A_1$ and $X = \ominus A_2$, respectively,

$$\begin{aligned}
\ominus A_1 \oplus M_{12} &= \frac{m_1\gamma_{\ominus A_1\oplus A_1}(\ominus A_1 \oplus A_1) + m_2\gamma_{\ominus A_1\oplus A_2}(\ominus A_1 \oplus A_2)}{m_1\gamma_{\ominus A_1\oplus A_1} + m_2\gamma_{\ominus A_1\oplus A_2}} = \frac{m_2\gamma_{12}\mathbf{a}_{12}}{m_1 + m_2\gamma_{12}} \\
\ominus A_2 \oplus M_{12} &= \frac{m_1\gamma_{\ominus A_2\oplus A_1}(\ominus A_2 \oplus A_1) + m_2\gamma_{\ominus A_2\oplus A_2}(\ominus A_2 \oplus A_2)}{m_1\gamma_{\ominus A_2\oplus A_1} + m_2\gamma_{\ominus A_2\oplus A_2}} = \frac{m_1\gamma_{12}\mathbf{a}_{21}}{m_1\gamma_{21} + m_2}
\end{aligned} \tag{199}$$

where, as shown in Fig. 6, we use the convenient index notation

$$\begin{aligned}
\mathbf{a}_{12} &= \ominus A_1 \oplus A_2, & a_{12} &= \|\mathbf{a}_{12}\|, & \gamma_{12} &= \gamma_{a_{12}}\,, \\
\mathbf{a}_{21} &= \ominus A_2 \oplus A_1, & a_{21} &= \|\mathbf{a}_{21}\|, & \gamma_{21} &= \gamma_{a_{21}}\,.
\end{aligned} \tag{200}$$

We note that $a_{12} = a_{21}$, $\gamma_{12} = \gamma_{21}$, and $\gamma_{\mathbf{0}} = 0$, while, in general, $\mathbf{a}_{21} \neq \mathbf{a}_{12}$ since, by the gyrocommutative law, $\mathbf{a}_{21} = \mathrm{gyr}[\ominus A_2, A_1]\mathbf{a}_{12}$.

Taking magnitudes of the extreme sides of each of the two equations in (199), we have

$$\begin{aligned}
\|\ominus A_1 \oplus M_{12}\| &= \frac{m_2}{m_1 + m_2\gamma_{12}}\gamma_{12}a_{12} \\
\|\ominus A_2 \oplus M_{12}\| &= \frac{m_1}{m_1\gamma_{12} + m_2}\gamma_{12}a_{12}
\end{aligned} \tag{201}$$

so that by (201) and (196) we have

$$\frac{m_1}{m_1\gamma_{12} + m_2} = \frac{m_2}{m_1 + m_2\gamma_{12}} \tag{202}$$

implying $m_1 = \pm m_2$.

For $m_1 = m_2 =: m$, the constant $m_{M_{12}}$ of the gyrobarycentric representation (197) of M_{12} is given by

$$m_{M_{12}}^2 = (m_1 + m_2)^2 + 2m_1m_2(\gamma_{12} - 1) = 2m^2(\gamma_{12} + 1) \; > \; 0 \tag{203}$$

so that, being positive, m_0^2 is acceptable.

In contrast, for $m_1 = -m_2 =: m$, the constant m_0 of the gyrobarycentric coordinate representation (197) of M_{12} is given by

$$m_{M_{12}}^2 = (m_1 + m_2)^2 + 2m_1m_2(\gamma_{12} - 1) = -2m^2(\gamma_{12} - 1) \; < \; 0 \tag{204}$$

so that, being negative, m_0^2 is rejected. Indeed, if $m_0^2 < 0$, then m_0 is purely imaginary so that, by (169), p. 606, also γ_P is purely imaginary, implying that while M_{12} lies in $\mathbb{R}^n$, it does not lie in the ball, $M_{12} \notin \mathbb{R}_s^n$. Hence, the solution $m_1 = -m_2$ of (202) is rejected, allowing the unique solution $m_1 = m_2$.

The unique solution for the gyrobarycentric coordinates of the gyromidpoint M_{12} (modulo a multiplicative normalization constant) is, therefore, $(m_1 : m_2) = (m : m)$ or, equivalently,

$$(m_1 : m_2) = (1 : 1). \tag{205}$$

Substituting the gyrobarycentric coordinates (205) into (197), we, finally, express the gyromidpoint M_{12} in terms of its vertices A_1 and A_2 by the equation

$$M_{12} = \frac{\gamma_{A_1} A_1 + \gamma_{A_2} A_2}{\gamma_{A_1} + \gamma_{A_2}}. \tag{206}$$

Following (203) and (205)–(206), the constant m_0 of the gyrobarycentric coordinate representation of the gyromidpoint M_{12} in (206) is

$$m_{M_{12}} = \sqrt{2(\gamma_{12} + 1)}. \tag{207}$$

Hence, by the Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 25, p. 609, the gyromidpoint M_{12} possesses the following three identities:

$$X \oplus M_{12} = \frac{\gamma_{X\oplus A_1}(X \oplus A_1) + \gamma_{X\oplus A_2}(X \oplus A_2)}{\gamma_{X\oplus A_1} + \gamma_{X\oplus A_2}} \tag{208a}$$

$$\gamma_{X\oplus M_{12}} = \frac{\gamma_{X\oplus A_1} + \gamma_{X\oplus A_2}}{\sqrt{2}\sqrt{\gamma_{12} + 1}} \tag{208b}$$

$$\gamma_{X\oplus M_{12}}(X \oplus M_{12}) = \frac{\gamma_{X\oplus A_1}(X \oplus A_1) + \gamma_{X\oplus A_2}(X \oplus A_2)}{\sqrt{2}\sqrt{\gamma_{12} + 1}} \tag{208c}$$

for all $X \in \mathbb{R}_s^n$.

Following (208a) with $X = \ominus A_1$, by Einstein half (105), p. 588, we have

$$\ominus A_1 \oplus M_{12} = \frac{\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2)}{1 + \gamma_{\ominus A_1 \oplus A_2}} = \frac{\gamma_{12}}{1 + \gamma_{12}}\mathbf{a}_{12} = \tfrac{1}{2} \otimes \mathbf{a}_{12} \tag{209}$$

so that by the scaling property (108), p. 588,

$$\| \ominus A_1 \oplus M_{12}\| = \|\tfrac{1}{2} \otimes \mathbf{a}_{12}\| = \tfrac{1}{2} \otimes \|\mathbf{a}_{12}\| = \tfrac{1}{2} \otimes a_{12}\,. \tag{210}$$

Similarly, following (208b)–(208c) with $X = \ominus A_1$, we have

$$\gamma_{\ominus A_1 \oplus M_{12}} = \frac{1 + \gamma_{\ominus A_1 \oplus A_2}}{\sqrt{2}\sqrt{1 + \gamma_{12}}} = \frac{1 + \gamma_{12}}{\sqrt{2}\sqrt{1 + \gamma_{12}}}a = \frac{\sqrt{1 + \gamma_{12}}}{\sqrt{2}} \tag{211}$$

and

$$\gamma_{\ominus A_1 \oplus M_{12}}(\ominus A_1 \oplus M_{12}) = \frac{\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2)}{\sqrt{2}\sqrt{1 + \gamma_{12}}} = \frac{\gamma_{12}\mathbf{a}_{12}}{\sqrt{2}\sqrt{1 + \gamma_{12}}}. \tag{212}$$

Hence, by (210) and (211),

$$\gamma_{\frac{1}{2} \otimes \mathbf{a}_{12}} = \frac{\sqrt{1 + \gamma_{12}}}{\sqrt{2}} \tag{213}$$

and, by (209)–(210) and (212),

$$\gamma_{\frac{1}{2} \otimes \mathbf{a}_{12}}(\tfrac{1}{2} \otimes \mathbf{a}_{12}) = \frac{\gamma_{12}\mathbf{a}_{12}}{\sqrt{2}\sqrt{1 + \gamma_{12}}}\,. \tag{214}$$

As shown in Fig. 2, p. 592, the gyromidpoint M_{12} in (206) can be written as

$$M_{12} = A_1 \oplus (\ominus A_1 \oplus A_2) \otimes \tfrac{1}{2}\,. \tag{215}$$

26 Gyrotriangle Gyrocentroid

The hyperbolic triangle centroid is called, in gyrolanguage, the gyrotriangle gyrocentroid.

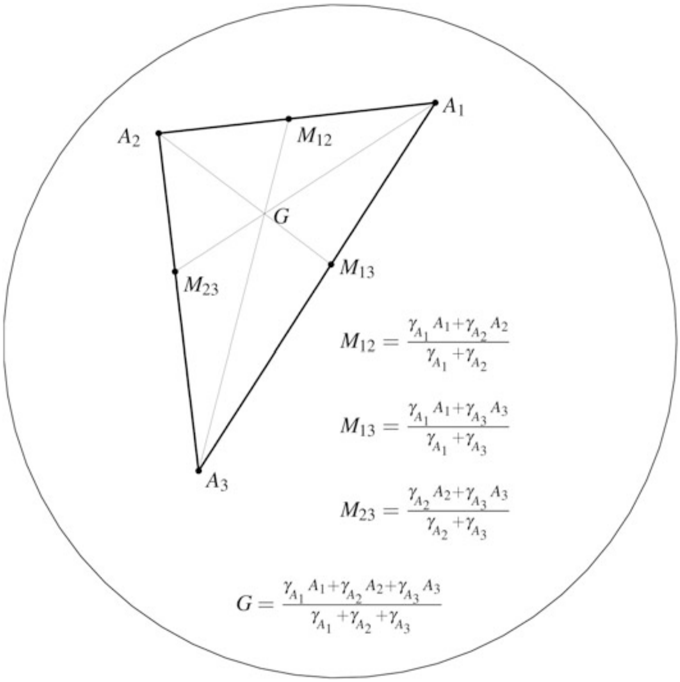


Fig. 7 The gyrotriangle gyromedians and gyrocentroid in an Einstein gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. The gyrocentroid G is the gyropoint of concurrency of the gyrotriangle gyromedians. The gyromidpoints M_{12}, M_{13}, and M_{23} of the three gyrosides, A_1A_2, A_1A_3, and A_2A_3, of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ are shown here for $n = 2$, along with its gyromedians $A_1M_{23}, A_2M_{13}, A_3M_{12}$, and its gyrocentroid G. The Euclidean counterpart of this figure is shown in Fig. 5, p. 612

In this section we demonstrate the use of gyrobarycentric coordinates by determining the gyrotriangle gyrocentroid in Einstein gyrovector spaces.

Definition 20 (Gyromedians, Gyrotriangle Gyrocentroids). A gyromedian of a gyrotriangle in an Einstein gyrovector space is the gyrosegment joining a vertex of the gyrotriangle with the gyromidpoint of the opposing gyroside, shown in Fig. 7. The gyrocentroid, G, of a gyrotriangle is the gyropoint of concurrency of the gyrotriangle gyromedians, shown in Fig. 7.

Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, and let the gyromidpoints of its gyrosides be M_{12}, M_{13} and M_{23}, as shown in Fig. 7. Hence, by (206), M_{12} and, in a similar way, M_{13} and M_{23}, are given by the equations

$$M_{12} = \frac{\gamma_{A_1} A_1 + \gamma_{A_2} A_2}{\gamma_{A_1} + \gamma_{A_2}}$$
$$M_{13} = \frac{\gamma_{A_1} A_1 + \gamma_{A_3} A_3}{\gamma_{A_1} + \gamma_{A_3}} \qquad (216)$$
$$M_{23} = \frac{\gamma_{A_2} A_2 + \gamma_{A_3} A_3}{\gamma_{A_2} + \gamma_{A_3}} .$$

The three gyromedians of gyrotriangle $A_1A_2A_3$ in Fig. 7 are the gyrosegments A_1M_{23}, A_2M_{13}, and A_3M_{12}. Since gyrosegments in Einstein gyrovector spaces coincide with Euclidean segments, one can employ methods of linear algebra to determine the gyropoint of concurrency, that is, the gyrocentroid, of the three gyromedians of gyrotriangle $A_1A_2A_3$ in Fig. 7.

The details of the use of methods of linear algebra for the determination of the gyrobarycentric coordinates of the gyrotriangle gyrocentroid in Einstein gyrovector spaces are presented below.

In order to determine the gyrobarycentric coordinates of the gyrotriangle gyrocentroid in Einstein gyrovector spaces, we begin with some gyroalgebraic manipulations that reduce the task we face to a problem in linear algebra.

Let the gyrocentroid G of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, shown in Fig. 7, be given by its gyrobarycentric coordinate representation, (180), p. 608, with respect to the set $S = \{A_1, A_2, A_3\}$ of the gyrotriangle vertices,

$$G = \frac{m_1\gamma_{A_1} A_1 + m_2\gamma_{A_2} A_2 + m_3\gamma_{A_3} A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}} , \qquad (217)$$

where the gyrobarycentric coordinates (m_1, m_2, m_3) of G in (217) are to be determined in (240) below.

Left gyrotranslating gyrotriangle $A_1A_2A_3$ by $\ominus A_1$, the gyrotriangle becomes gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$, where $O = \ominus A_1 \oplus A_1$ is the arbitrarily selected origin of the Einstein gyrovector space $\mathbb{R}_s^n$, so that

$$O = \mathbf{0} = (0, \ldots, 0) \qquad (218)$$

with respect to the Cartesian coordinates of $\mathbb{R}_s^n$.

Following the left gyrotranslation by $\ominus A_1$, the gyrotriangle gyroside gyromidpoints M_{12}, M_{13}, and M_{23} become, respectively, $\ominus A_1 \oplus M_{12}$, $\ominus A_1 \oplus M_{13}$, and $\ominus A_1 \oplus M_{23}$. These are calculated in (219) below by employing the gyrocovariance of gyrobarycentric coordinate representations, Theorem 25, p. 609.

Accordingly, we obtain from (184a), p. 609, with $X = \ominus A_1$ the following left gyrotranslation by $\ominus A_1$ of the gyromidpoints M_{12}, M_{13}, and M_{23} in (216):

$$\begin{aligned}
\ominus A_1 \oplus M_{12} &= \ominus A_1 \oplus \frac{\gamma_{A_1} A_1 + \gamma_{A_2} A_2}{\gamma_{A_1} + \gamma_{A_2}} = \frac{\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2)}{1 + \gamma_{\ominus A_1 \oplus A_2}} = \frac{\gamma_{12}\mathbf{a}_{12}}{\gamma_{12} + 1} \\
\ominus A_1 \oplus M_{13} &= \ominus A_1 \oplus \frac{\gamma_{A_1} A_1 + \gamma_{A_3} A_3}{\gamma_{A_1} + \gamma_{A_3}} = \frac{\gamma_{\ominus A_1 \oplus A_3}(\ominus A_1 \oplus A_3)}{1 + \gamma_{\ominus A_1 \oplus A_3}} = \frac{\gamma_{13}\mathbf{a}_{13}}{\gamma_{13} + 1} \\
\ominus A_1 \oplus M_{23} &= \ominus A_1 \oplus \frac{\gamma_{A_2} A_2 + \gamma_{A_3} A_3}{\gamma_{A_2} + \gamma_{A_3}} = \frac{\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2) + \gamma_{\ominus A_1 \oplus A_3}(\ominus A_1 \oplus A_3)}{\gamma_{\ominus A_1 \oplus A_2} + \gamma_{\ominus A_1 \oplus A_3}} \\
&= \frac{\gamma_{12}\mathbf{a}_{12} + \gamma_{13}\mathbf{a}_{13}}{\gamma_{12} + \gamma_{13}}.
\end{aligned} \tag{219}$$

As in (200), in (219) we use the convenient notation

$$\mathbf{a}_{ij} = \ominus A_i \oplus A_j, \qquad a_{ij} = \|\mathbf{a}_{ij}\|, \qquad \gamma_{ij} = \gamma_{a_{ij}}, \tag{220}$$

$i, j = 1, 2, 3$, noting that $a_{ij} = a_{ji}$, $\gamma_{ij} = \gamma_{ji}$.

Note that, by Definition 19, p. 608, the set of gyropoints $S = \{A_1, A_2, A_3\}$ is gyropointwise independent in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$. Hence, the two gyrovectors $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ and $\mathbf{a}_{13} = \ominus A_1 \oplus A_3$ in $\mathbb{R}_s^n \subset \mathbb{R}^n$ in (219), considered as vectors in $\mathbb{R}^n$, are linearly independent in $\mathbb{R}^n$.

Similarly to the gyroalgebra in (219), under a left gyrotranslation by $\ominus A_1$, the gyrocentroid G in (217) becomes

$$\ominus A_1 \oplus G = \frac{m_2\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2) + m_3\gamma_{\ominus A_1 \oplus A_3}(\ominus A_1 \oplus A_3)}{m_1 + m_2\gamma_{\ominus A_1 \oplus A_2} + m_3\gamma_{\ominus A_1 \oplus A_3}} = \frac{m_2\gamma_{12}\mathbf{a}_{12} + m_3\gamma_{13}\mathbf{a}_{13}}{m_1 + m_2\gamma_{12} + m_3\gamma_{13}}. \tag{221}$$

The gyromedian of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$ that joins the vertex

$$\ominus A_1 \oplus A_1 = O = \mathbf{0} \tag{222}$$

with the gyromidpoint of its opposing gyroside, as calculated in (219),

$$\ominus A_1 \oplus M_{23} = \frac{\gamma_{12}\mathbf{a}_{12} + \gamma_{13}\mathbf{a}_{13}}{\gamma_{12} + \gamma_{13}} \tag{223}$$

is contained in the Euclidean line

$$L_1 = O + (-O + \{\ominus A_1 \oplus M_{23}\})t_1 = \frac{\gamma_{12}\mathbf{a}_{12} + \gamma_{13}\mathbf{a}_{13}}{\gamma_{12} + \gamma_{13}} t_1 \tag{224}$$

where $t_1 \in \mathbb{R}$ is the line parameter. This line passes through the point $O = \mathbf{0} \in \mathbb{R}_s^n \subset \mathbb{R}^n$ when $t_1 = 0$, and it passes through the point

$$\ominus A_1 \oplus M_{23} = \frac{\gamma_{12}\mathbf{a}_{12} + \gamma_{13}\mathbf{a}_{13}}{\gamma_{12} + \gamma_{13}} \in \mathbb{R}_s^n \subset \mathbb{R}^n \tag{225}$$

when $t_1 = 1$.

Similarly to (222)–(224), the gyromedian of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$ that joins the vertex

$$\ominus A_1 \oplus A_2 = \mathbf{a}_{12} \tag{226}$$

with the gyromidpoint of its opposing gyroside, as calculated in (219),

$$\ominus A_1 \oplus M_{13} = \frac{\gamma_{13}\mathbf{a}_{13}}{\gamma_{13} + 1} \tag{227}$$

is contained in the Euclidean line

$$L_2 = \mathbf{a}_{12} + (-\mathbf{a}_{12} + \{\ominus A_1 \oplus M_{13}\})t_2 = \mathbf{a}_{12} + (-\mathbf{a}_{12} + \frac{\gamma_{13}\mathbf{a}_{13}}{\gamma_{13} + 1})t_2 \tag{228}$$

where $t_2 \in \mathbb{R}$ is the line parameter. This line passes through the point $\mathbf{a}_{12} \in \mathbb{R}_s^n \subset \mathbb{R}^n$ when $t_2 = 0$, and it passes through the point $\ominus A_1 \oplus M_{13} = \gamma_{13}\mathbf{a}_{13}/(\gamma_{13}+1) \in \mathbb{R}_s^n \subset \mathbb{R}^n$ when $t_2 = 1$.

Similarly to (222)–(224), and similarly to (226)–(228), the gyromedian of the left gyrotranslated gyrotriangle $O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3)$ that joins the vertex

$$\ominus A_1 \oplus A_3 = \mathbf{a}_{13} \tag{229}$$

with the gyromidpoint of its opposing gyroside, as calculated in (219),

$$\ominus A_1 \oplus M_{12} = \frac{\gamma_{12}\mathbf{a}_{12}}{\gamma_{12} + 1} \tag{230}$$

is contained in the Euclidean line

$$L_3 = \mathbf{a}_{13} + (-\mathbf{a}_{13} + \{\ominus A_1 \oplus M_{12}\})t_3 = \mathbf{a}_{13} + (-\mathbf{a}_{13} + \frac{\gamma_{12}\mathbf{a}_{12}}{\gamma_{12} + 1})t_3 \tag{231}$$

where $t_3 \in \mathbb{R}$ is the line parameter. This line passes through the point $\mathbf{a}_{13} \in \mathbb{R}_s^n \subset \mathbb{R}^n$ when $t_3 = 0$, and it passes through the point $\ominus A_1 \oplus M_{12} = \gamma_{12}\mathbf{a}_{12}/(\gamma_{12}+1) \in \mathbb{R}_s^n \subset \mathbb{R}^n$ when $t_3 = 1$.

Hence, if the gyrocentroid G exists, its left gyrotranslated gyrocentroid, $\ominus A_1 \oplus G$, given by (221), is contained in each of the three Euclidean lines L_k, $k = 1, 2, 3$, in (224), (228), and (231). Formalizing, if G exists, then the point P in (221),

$$P = \ominus A_1 \oplus G = \frac{m_2\gamma_{12}\mathbf{a}_{12} + m_3\gamma_{13}\mathbf{a}_{13}}{m_1 + m_2\gamma_{12} + m_3\gamma_{13}}\,, \tag{232}$$

lies on each of the lines L_k, $k = 1, 2, 3$. Imposing the normalization condition $m_1 + m_2 + m_3 = 1$ of special gyrobarycentric coordinates, (232) can be simplified by means of the resulting equation $m_1 = 1 - m_2 - m_3$, obtaining

$$P = \ominus A_1 \oplus G = \frac{m_2\gamma_{12}\mathbf{a}_{12} + m_3\gamma_{13}\mathbf{a}_{13}}{1 + m_2(\gamma_{12} - 1) + m_3(\gamma_{13} - 1)}\,. \tag{233}$$

Since the point P lies on each of the three lines L_k, $k = 1, 2, 3$, there exist values $t_{k,0}$ of the line parameters t_k, $k = 1, 2, 3$, respectively, such that

$$\begin{aligned} P - \frac{\gamma_{12}\mathbf{a}_{12} + \gamma_{13}\mathbf{a}_{13}}{\gamma_{12} + \gamma_{13}} t_{1,0} \qquad\qquad &= 0 \\ P - \mathbf{a}_{12} - (-\mathbf{a}_{12} + \frac{\gamma_{13}\mathbf{a}_{13}}{\gamma_{13} + 1}) t_{2,0} &= 0 \\ P - \mathbf{a}_{13} - (-\mathbf{a}_{13} + \frac{\gamma_{12}\mathbf{a}_{12}}{\gamma_{12} + 1}) t_{3,0} &= 0\,. \end{aligned} \tag{234}$$

The kth equation in (234), $k = 1, 2, 3$, is equivalent to the condition that point P lies on line L_k.

The system of Eqs. (234) was obtained by methods of gyroalgebra and will be solved below by a common method of linear algebra.

Substituting P from (233) into (234), and rewriting each equation in (234) as a linear combination of $\mathbf{a}_{12}$ and $\mathbf{a}_{13}$ equals zero, one obtains the following linear homogeneous system of three gyrovector equations

$$\begin{aligned} c_{11}\mathbf{a}_{12} + c_{12}\mathbf{a}_{13} &= \mathbf{0} \\ c_{21}\mathbf{a}_{12} + c_{22}\mathbf{a}_{13} &= \mathbf{0} \\ c_{31}\mathbf{a}_{12} + c_{32}\mathbf{a}_{13} &= \mathbf{0} \end{aligned} \tag{235}$$

where each coefficient c_{ij}, $i = 1, 2, 3, j = 1, 2$, is a function of γ_{12}, γ_{13}, γ_{23}, and the five unknowns m_2, m_3, and $t_{k,0}$, $k = 1, 2, 3$.

Since the set $S = \{A_1, A_2, A_3\}$ is gyropointwise independent, the two gyrovectors $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ and $\mathbf{a}_{13} = \ominus A_1 \oplus A_3$ in $\mathbb{R}^n_s$, considered as vectors in $\mathbb{R}^n$, are linearly independent. Hence, each coefficient c_{ij} in (235) equals zero. Accordingly, the three gyrovector equations in (235) are equivalent to the following six scalar equations,

$$c_{11} = c_{12} = c_{21} = c_{22} = c_{31} = c_{32} = 0 \tag{236}$$

for the five unknowns m_2, m_3, and $t_{k,0}$, $k = 1, 2, 3$.

Explicitly, the six scalar equations in (236) are equivalent to the following six equations:

$$\begin{aligned}
&[1 + m_2(\gamma_{12} - 1) + m_3(\gamma_{13} - 1)]t_{1,0} - m_2(\gamma_{12} + \gamma_{13}) &&= 0\\
&[1 + m_2(\gamma_{12} - 1) + m_3(\gamma_{13} - 1)]t_{1,0} - m_3(\gamma_{12} + \gamma_{13}) &&= 0\\
&[1 + m_2(\gamma_{12} - 1) + m_3(\gamma_{13} - 1)]t_{2,0} - m_3(\gamma_{13} + 1) &&= 0\\
&[1 + m_2(\gamma_{12} - 1) + m_3(\gamma_{13} - 1)]t_{2,0} - m_3(\gamma_{13} - 1) + m_2 - 1 &&= 0\\
&[1 + m_2(\gamma_{12} - 1) + m_3(\gamma_{13} - 1)]t_{3,0} - m_2(\gamma_{12} + 1) &&= 0\\
&[1 + m_2(\gamma_{12} - 1) + m_3(\gamma_{13} - 1)]t_{3,0} - m_2(\gamma_{12} - 1) + m_3 - 1 &&= 0.
\end{aligned} \tag{237}$$

The unique solution of (237) is given by

$$t_{1,0} = \frac{\gamma_{12} + \gamma_{13}}{\gamma_{12} + \gamma_{13} + 1}, \qquad t_{2,0} = \frac{\gamma_{13} + 1}{\gamma_{12} + \gamma_{13} + 1}, \qquad t_{3,0} = \frac{\gamma_{12} + 1}{\gamma_{12} + \gamma_{13} + 1}, \tag{238}$$

and

$$m_2 = m_3 = \frac{1}{3} \tag{239}$$

so that by the normalization condition $m_1 + m_2 + m_3 = 1$, also $m_1 = 1/3$.

Hence, the special gyrobarycentric coordinates of the gyrocentroid of a gyrotriangle $A_1A_2A_3$ with respect to the gyropointwise independent set $\{A_1, A_2, A_3\}$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Fig. 7, are given by $(m_1, m_2, m_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, so that convenient gyrobarycentric coordinates of the gyrotriangle gyrocentroid are

$$(m_1 : m_2 : m_3) = (1, 1, 1)\,. \tag{240}$$

Finally, following (240) and (217), the gyrocentroid of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Fig. 7, is given by the equation

$$G = \frac{\gamma_{A_1} A_1 + \gamma_{A_2} A_2 + \gamma_{A_3} A_3}{\gamma_{A_1} + \gamma_{A_2} + \gamma_{A_3}} \tag{241}$$

The similarity between the gyrotriangle gyrocentroid G of gyrotriangle $A_1A_2A_3$ in (241) and the gyromidpoint M_{12} of gyrosegment A_1A_2 in (216) is remarkable. The extension to higher dimensions is now obvious. Indeed, the gyrocentroid G_t of an $(N-1)$-gyrosimplex $S_N := A_1 \ldots A_N$, $N \ge 2$, in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \ge N-1$, is given by its gyrobarycentric representation [50, Eq. 6.338], [61, Eq. 10.31],

$$G_t = \frac{\sum_{i=1}^{N} \gamma_{A_i} A_i}{\sum_{i=1}^{N} \gamma_{A_i}}\,. \tag{242}$$

27 Gyrodistance Between Gyropoints

Let P and P' be two gyropoints in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with gyrobarycentric coordinate representations

$$
\begin{aligned}
P &= \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \\
P' &= \frac{\sum_{k=1}^{N} m'_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m'_k \gamma_{A_k}}
\end{aligned}
\tag{243}
$$

with respect to the set of gyropoints $S = \{A_1, \ldots, A_N\}$.

By the gyrocovariance in (184b), p. 610, of γ_P, with $X = \ominus P'$, and following (243), we obtain the two gamma factors in (244)–(245):

$$
\begin{aligned}
\gamma_{\ominus P' \oplus P} &= \frac{\sum_{j=1}^{N} m_j \gamma_{\ominus P' \oplus A_j}}{m_0} \\
&= \frac{\sum_{j=1}^{N} m_j \gamma_{\ominus A_j \oplus P'}}{m_0}
\end{aligned}
\tag{244a}
$$

where, following (170), p. 606, $m_0 > 0$ is given by

$$
m_0^2 = \left(\sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1) \tag{244b}
$$

and

$$
\gamma_{\ominus A_j \oplus P'} = \frac{\sum_{k=1}^{N} m'_k \gamma_{\ominus A_j \oplus A_k}}{m'_0} \tag{245a}
$$

where, as in (244b), $m'_0 > 0$ is given by

$$
(m'_0)^2 = \left(\sum_{k=1}^{N} m'_k \right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m'_j m'_k (\gamma_{\ominus A_j \oplus A_k} - 1) \, . \tag{245b}
$$

Note that while, in general, $\ominus P' \oplus A_j \neq \ominus A_j \oplus P'$, their norms are equal, $\| \ominus P' \oplus A_j \| = \| \ominus A_j \oplus P' \|$, implying the equality of their gamma factors, thus justifying the second equation in (244a).

Substituting (245a) into the extreme right side of (244a), we obtain the gamma factor of $\ominus P' \oplus P$ in terms of the parameters of the gyrobarycentric coordinate representations (243), according to the following theorem:

Theorem 27 (Gyrodistance Between Gyropoints). *Let P and P' be two gyropoints in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with gyrobarycentric coordinate representations (243)–(244) with respect to the set of gyropoints $S = \{A_1, \ldots, A_N\}$.*

Then, the squared gyrodistance between the gyropoints P and P' is given by the equation

$$\| \ominus P' \oplus P\|^2 = s^2 \frac{\gamma^2_{\ominus P' \oplus P} - 1}{\gamma^2_{\ominus P' \oplus P}} \tag{246}$$

where

$$\begin{aligned}
\gamma_{\ominus P' \oplus P} &= \frac{1}{m_0 m_0'} \sum_{j=1}^{N} \sum_{k=1}^{N} m_j m_k' \gamma_{\ominus A_j \oplus A_k} \\
&= \frac{1}{m_0 m_0'} \left\{ \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k' \gamma_{\ominus A_j \oplus A_k} + \sum_{\substack{j,k=1 \\ j>k}}^{N} m_j m_k' \gamma_{\ominus A_j \oplus A_k} + \sum_{\substack{j,k=1 \\ j=k}}^{N} m_j m_k' \gamma_{\ominus A_j \oplus A_k} \right\} \\
&= \frac{1}{m_0 m_0'} \left\{ \sum_{\substack{j,k=1 \\ j<k}}^{N} (m_j m_k' + m_j' m_k) \gamma_{\ominus A_j \oplus A_k} + \sum_{i=1}^{N} m_i m_i' \right\} .
\end{aligned} \tag{247}$$

Proof. The gamma factor $\gamma_{\ominus P' \oplus P}$ is obtained by substituting (245a) into the extreme right side of (244a). The resulting gyrodistance $\| \ominus P' \oplus P\|$, in (246), between the gyropoints P and P' is obtained from the gamma factor $\gamma_{\ominus P' \oplus P}$ in (247) by means of Identity (6), p. 566. □

28 The Law of Gyrocosines

Let $\ominus A \oplus B$ and $\ominus A \oplus C$ be two gyrovectors that form two sides of gyrotriangle ABC and include the gyrotriangle gyroangle α in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, as shown in Fig. 8 for $n = 2$.

By the Gyrotranslation Theorem 8, p. 577,

$$\ominus (\ominus A \oplus B) \oplus (\ominus A \oplus C) = \mathrm{gyr}[\ominus A, B](\ominus B \oplus C) . \tag{248}$$

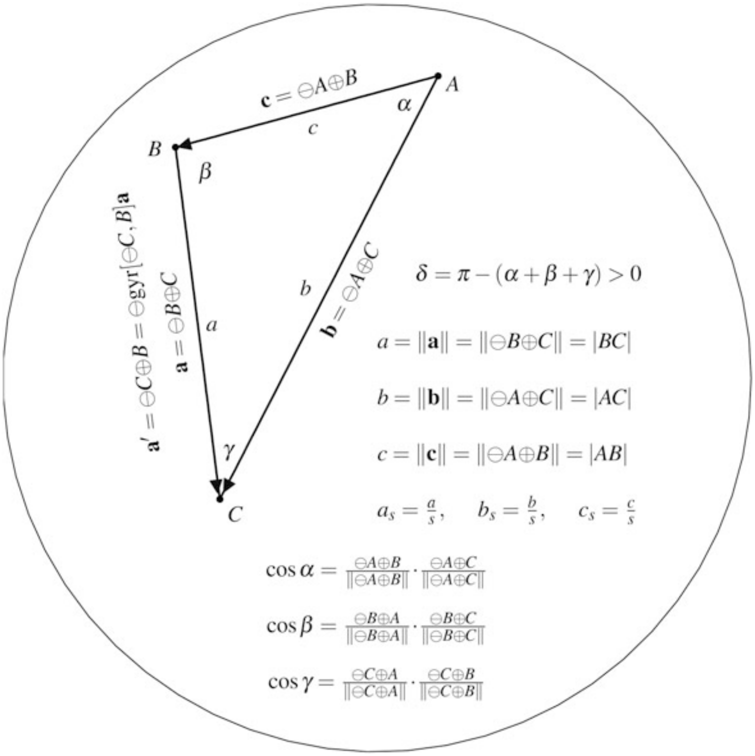


Fig. 8 Gyrotriangle *ABC*, along with its standard notation, in an Einstein gyrovector space. The notation that we use with a gyrotriangle *ABC*, its gyrovector sides, and its gyroangles in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is shown here for the Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$

Since gyrations preserve the norm, (18), p. 569,

$$\| \ominus (\ominus A \oplus B) \oplus (\ominus A \oplus C)\| = \|\text{gyr}[\ominus A, B](\ominus B \oplus C)\| = \| \ominus B \oplus C\| \,. \tag{249}$$

In the notation of Fig. 8 for gyrotriangle *ABC*, (249) is written as

$$\| \ominus \mathbf{c} \oplus \mathbf{b}\| = \|\mathbf{a}\| \tag{250}$$

implying

$$\gamma_{\ominus\mathbf{c}\oplus\mathbf{b}} = \gamma_{\mathbf{a}} = \gamma_a \,. \tag{251}$$

By (5), p. 565, we have

$$\gamma_{\ominus\mathbf{c}\oplus\mathbf{b}} = \gamma_b\gamma_c(1 - \frac{\mathbf{b}\cdot\mathbf{c}}{s^2}) \,. \tag{252}$$

In full analogy with Euclidean geometry, the gyrocosine of gyroangle $\alpha = \angle BAC$ of gyrotriangle ABC in Fig. 8 is given by

$$\cos\alpha = \frac{\ominus A \oplus B}{\| \ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\| \ominus A \oplus C\|} = \frac{\mathbf{c}}{c} \cdot \frac{\mathbf{b}}{b} \,. \tag{253}$$

Hence,

$$\frac{\mathbf{b}\cdot\mathbf{c}}{s^2} = \frac{bc}{s^2}\cos\alpha = b_s c_s \cos\alpha \tag{254}$$

where $b_s = b/s$, etc.

Following (251)–(254) we have

$$\gamma_a = \gamma_b\gamma_c(1 - b_s c_s \cos\alpha) \tag{255}$$

where a, b, c are the gyroside-gyrolengths of gyrotriangle ABC in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, as shown in Fig. 8 for $n = 2$.

Identity (255) is the *law of gyrocosines* in the gyrotrigonometry of Einstein gyrovector spaces. As in trigonometry, it is useful for calculating one side, a, of a gyrotriangle ABC, shown in Fig. 8, when the gyroangle α opposite to gyroside a and the other two gyrosides (i.e., their gyrolengths), b and c, are known.

Remarkably, in the Euclidean limit of large s, $s \to \infty$, gamma factors tend to 1 and the law of gyrocosines (255) reduces to the trivial identity $1 = 1$. Hence, (255) has no immediate Euclidean counterpart, thus presenting a disanalogy between hyperbolic and Euclidean geometry. As a result, each of Theorems 28 and 29 has no Euclidean counterpart as well.

29 The *SSS* to *AAA* Conversion Law

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \otimes, \oplus)$ with its standard notation in Fig. 8. According to (255) the gyrotriangle ABC possesses the following three identities, each of which represents its law of gyrocosines,

$$\begin{aligned} \gamma_a &= \gamma_b\gamma_c(1 - b_s c_s \cos\alpha) \\ \gamma_b &= \gamma_a\gamma_c(1 - a_s c_s \cos\beta) \\ \gamma_c &= \gamma_a\gamma_b(1 - a_s b_s \cos\gamma)\,. \end{aligned} \tag{256}$$

Like Euclidean triangles, the gyroangles of a gyrotriangle are uniquely determined by its gyrosides. Solving the system (256) of three identities for the three unknowns, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, and employing (6), p. 566, we obtain the following theorem:

Theorem 28 (The Law of Gyrocosines: The *SSS* to *AAA* Conversion Law). *Let ABC be a gyrotriangle in an Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$. *Then, in the gyrotriangle notation in Fig. 8,*

$$\begin{aligned}
\cos\alpha &= \frac{-\gamma_a + \gamma_b\gamma_c}{\gamma_b\gamma_c b_s c_s} = \frac{-\gamma_a + \gamma_b\gamma_c}{\sqrt{\gamma_b^2 - 1}\sqrt{\gamma_c^2 - 1}} \\
\cos\beta &= \frac{-\gamma_b + \gamma_a\gamma_c}{\gamma_a\gamma_c a_s c_s} = \frac{-\gamma_b + \gamma_a\gamma_c}{\sqrt{\gamma_a^2 - 1}\sqrt{\gamma_c^2 - 1}} \\
\cos\gamma &= \frac{-\gamma_c + \gamma_a\gamma_b}{\gamma_a\gamma_b a_s b_s} = \frac{-\gamma_c + \gamma_a\gamma_b}{\sqrt{\gamma_a^2 - 1}\sqrt{\gamma_b^2 - 1}} \,.
\end{aligned} \tag{257}$$

The identities in (257) form the *SSS* (Side-Side-Side) to *AAA* (gyroAngle-gyroAngle-gyroAngle) conversion law in Einstein gyrovector spaces. This law is useful for calculating the gyroangles of a gyrotriangle in an Einstein gyrovector space when its gyrosides (i.e., its gyroside-gyrolengths) are known.

In full analogy with the trigonometry of triangles, the *gyrosine* of a gyrotriangle gyroangle α is nonnegative, given by the equation

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} \geq 0\,. \tag{258}$$

Hence, it follows from Theorem 28 that the gyrosines of the gyrotriangle gyroangles in that Theorem are given by

$$\begin{aligned}
\sin\alpha &= \frac{\sqrt{1 + 2\gamma_a\gamma_b\gamma_c - \gamma_a^2 - \gamma_b^2 - \gamma_c^2}}{\sqrt{\gamma_b^2 - 1}\sqrt{\gamma_c^2 - 1}} \\
\sin\beta &= \frac{\sqrt{1 + 2\gamma_a\gamma_b\gamma_c - \gamma_a^2 - \gamma_b^2 - \gamma_c^2}}{\sqrt{\gamma_a^2 - 1}\sqrt{\gamma_c^2 - 1}} \\
\sin\gamma &= \frac{\sqrt{1 + 2\gamma_a\gamma_b\gamma_c - \gamma_a^2 - \gamma_b^2 - \gamma_c^2}}{\sqrt{\gamma_a^2 - 1}\sqrt{\gamma_b^2 - 1}} \,.
\end{aligned} \tag{259}$$

Any gyrotriangle gyroangle α satisfies $0 < \alpha < \pi$, so that $\sin\alpha > 0$ for any gyrotriangle gyroangle. Following (259) we have the inequality

$$1 + 2\gamma_a\gamma_b\gamma_c - \gamma_a^2 - \gamma_b^2 - \gamma_c^2 > 0 \tag{260}$$

for any gyrotriangle in an Einstein gyrovector space, in the notation of Theorem 28 and Fig. 8.

Identities (259) immediately give rise to the law of gyrosines

$$\frac{\sin\alpha}{\sqrt{\gamma_a^2-1}} = \frac{\sin\beta}{\sqrt{\gamma_b^2-1}} = \frac{\sin\gamma}{\sqrt{\gamma_c^2-1}}\,. \qquad (261)$$

30 The *AAA* to *SSS* Conversion Law

Unlike Euclidean triangles, the gyroside-gyrolengths of a gyrotriangle are uniquely determined by its gyroangles, as the following theorem demonstrates.

Theorem 29 (The ***AAA*** **to** ***SSS*** **Conversion Law I).** *Let ABC be a gyrotriangle in an Einstein gyrovector space* $(\mathbb{R}_s^n, \oplus, \otimes)$. *Then, in the gyrotriangle notation in Fig. 8,*

$$\begin{aligned}\gamma_a &= \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\\ \gamma_b &= \frac{\cos\beta + \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\\ \gamma_c &= \frac{\cos\gamma + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\end{aligned} \qquad (262)$$

where, following (258), the gyrosine of the gyrotriangle gyroangle α, $\sin\alpha$, *is given by the nonnegative value of* $\sqrt{1-\cos^2\alpha}$, *and similarly for* β *and* γ.

Proof. Let *ABC* be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \otimes, \oplus)$ with its standard notation in Fig. 8. It follows straightforwardly from the *SSS* to *AAA* conversion law (257) that

$$\left(\frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)^2 = \frac{(\cos\alpha + \cos\beta\cos\gamma)^2}{(1-\cos^2\beta)(1-\cos^2\gamma)} = \gamma_a^2 \qquad (263)$$

implying the first identity in (262). The remaining two identities in (262) are obtained from (257) in a similar way by vertex cyclic permutations. □

The identities in (262) form the *AAA* to *SSS* conversion law. This law is useful for calculating the gyrosides (i.e., the gyroside-gyrolengths) of a gyrotriangle in an Einstein gyrovector space when its gyroangles are known. Thus, for instance, γ_a is obtained from the first identity in (262), and a is obtained from γ_a by Identity (6), p. 566.

Solving the third identity in (262) for $\cos\gamma$, we have

$$\begin{aligned}\cos\gamma &= -\cos\alpha\cos\beta + \gamma_c \sin\alpha\sin\beta \\ &= -\cos(\alpha+\beta) + (\gamma_c - 1)\sin\alpha\sin\beta\end{aligned} \tag{264}$$

implying

$$\cos\gamma = \cos(\pi - \alpha - \beta) + (\gamma_c - 1)\sin\alpha\sin\beta\,. \tag{265}$$

In the Euclidean limit of large s, $s \to \infty$, γ_c tends to 1, so that the gyrotrigonometric identity (265) in hyperbolic geometry reduces to the trigonometric identity

$$\cos\gamma = \cos(\pi - \alpha - \beta) \qquad (\textit{Euclidean Geometry}) \tag{266}$$

in Euclidean geometry. The latter, in turn, is equivalent to the familiar result,

$$\alpha + \beta + \gamma = \pi \qquad (\textit{Euclidean Geometry}) \tag{267}$$

of Euclidean geometry, according to which the triangle angle sum is π.

31 Right Gyrotriangles

Theorem 30 (The Einstein–Pythagoras Theorem). *Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$. It is right gyroangled with legs* $\mathbf{a}$ *and* $\mathbf{b}$ *and hypotenuse* $\mathbf{c}$*, shown in Fig. 9, if and only if*

$$\gamma_a\,\gamma_b = \gamma_c\ . \tag{268}$$

Proof. Let ABC be a right gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with the right gyroangle $\gamma = \pi/2$, as shown in Fig. 9 for $n = 2$. It follows from (262) with $\gamma = \pi/2$ that the gyrosides a, b, and c of gyrotriangle ABC in Fig. 9 are related to the acute gyroangles α and β of the gyrotriangle by the equations

$$\gamma_a = \frac{\cos\alpha}{\sin\beta}, \qquad \gamma_b = \frac{\cos\beta}{\sin\alpha}, \qquad \gamma_c = \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}\,. \tag{269}$$

The identities in (269) imply the Einstein–Pythagoras identity (268) for a right gyrotriangle ABC with hypotenuse c and legs a and b in an Einstein gyrovector space, shown in Fig. 9. The converse statement is obvious owing to the one-to-one correspondence between gyrotriangle gyroangles and gyroside-gyrolengths. □

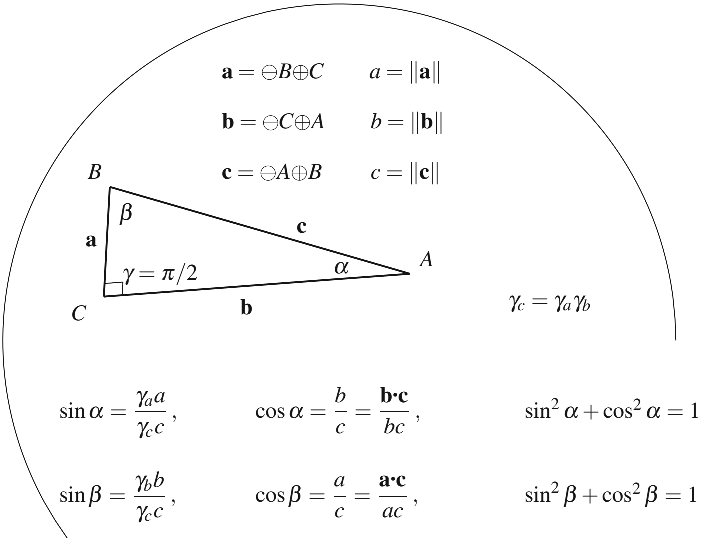


Fig. 9 Gyrotrigonometry in an Einstein gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. Here $\sin\alpha$ and $\cos\alpha$ are two elementary gyrotrigonometric functions of a gyrotriangle α, called *gyrosine* and *gyrocosine*. The use of the same notation for both elementary trigonometric functions and elementary gyrotrigonometric functions emphasizes the obvious analogies that gyrotrigonometry shares with trigonometry

It follows from (268) that $\gamma_a^2\gamma_b^2 = \gamma_c^2 = (1 - c_s^2)^{-1}$, implying that the gyrolength of the hypotenuse is given by the equation

$$c_s = \frac{\sqrt{\gamma_a^2\gamma_b^2 - 1}}{\gamma_a\gamma_b}. \tag{270}$$

32 Gyrotrigonometry

Right-angled triangles in Euclidean geometry along with the Pythagorean identity that each right triangle obeys are useful for the presentation of the elementary trigonometric functions. In full analogy, right-gyroangled gyrotriangles in the hyperbolic geometry of Einstein gyrovector spaces along with the two Einsteinian–Pythagorean identities that each right gyrotriangle obeys are useful for the presentation of the elementary gyrotrigonometric functions, as shown below.

Let a, b, and c be the respective gyrolengths of the two legs $\mathbf{a}$ *and* $\mathbf{b}$ and the hypotenuse $\mathbf{c}$ of a right gyrotriangle ABC in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, shown in Fig. 9. By (6), p. 566, and (269) we have

$$\begin{aligned}\left(\frac{a}{c}\right)^2 &= \frac{(\gamma_a^2-1)/\gamma_a^2}{(\gamma_c^2-1)/\gamma_c^2} = \cos^2\beta \\ \left(\frac{b}{c}\right)^2 &= \frac{(\gamma_b^2-1)/\gamma_b^2}{(\gamma_c^2-1)/\gamma_c^2} = \cos^2\alpha\end{aligned} \tag{271}$$

where γ_a, γ_b, and γ_c are related by (268).

Similarly, by (6), p. 566, and (269), we also have

$$\begin{aligned}\left(\frac{\gamma_a a}{\gamma_c c}\right)^2 &= \frac{\gamma_a^2-1}{\gamma_c^2-1} = \sin^2\alpha \\ \left(\frac{\gamma_b b}{\gamma_c c}\right)^2 &= \frac{\gamma_b^2-1}{\gamma_c^2-1} = \sin^2\beta\,.\end{aligned} \tag{272}$$

Identities (271) and (272) imply

$$\begin{aligned}\left(\frac{a}{c}\right)^2 + \left(\frac{\gamma_b b}{\gamma_c c}\right)^2 &= 1 \\ \left(\frac{\gamma_a a}{\gamma_c c}\right)^2 + \left(\frac{b}{c}\right)^2 &= 1\end{aligned} \tag{273}$$

so that, as shown in Fig. 9,

$$\begin{aligned}\cos\alpha &= \frac{b}{c} \\ \cos\beta &= \frac{a}{c}\end{aligned} \tag{274}$$

and

$$\begin{aligned}\sin\alpha &= \frac{\gamma_a a}{\gamma_c c} \\ \sin\beta &= \frac{\gamma_b b}{\gamma_c c}\,.\end{aligned} \tag{275}$$

Interestingly, we see from (274)–(275) that the gyrocosine function of an acute gyroangle of a right gyrotriangle in an Einstein gyrovector space has the same form as its Euclidean counterpart, the cosine function. In contrast, it is only modulo gamma factors that the gyrosine function has the same form as its Euclidean counterpart.

Identities (273) give rise to the following two distinct Einsteinian-Pythagorean identities,

$$a^2 + \left(\frac{\gamma_b}{\gamma_c}\right)^2 b^2 = c^2, \qquad \left(\frac{\gamma_a}{\gamma_c}\right)^2 a^2 + b^2 = c^2, \tag{276}$$

for a right gyrotriangle with hypotenuse c and legs a and b in an Einstein gyrovector space. The two distinct Einsteinian-Pythagorean identities in (276) that each Einsteinian right gyrotriangle obeys converge in the Newtonian/Euclidean limit of large s, $s \to \infty$, to the single Pythagorean identity

$$a^2 + b^2 = c^2 \qquad \text{(Euclidean Geometry)} \tag{277}$$

that each Euclidean right-angled triangle obeys.

Some explorers believe that "in the hyperbolic model the Pythagorean theorem is not valid" [63, p. 363], so that "the Pythagorean theorem is strictly Euclidean." It is therefore interesting to realize that while Euclidean geometry possesses a single Pythagorean identity, (277), hyperbolic geometry possesses two Pythagorean identities, (276), that capture the missing analogy.

33 In-Exgyrocircle Tangency Gyropoints

Theorem 31 (Ingyrocircle Tangency Gyropoints). *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $\mathbb{R}^n_s$ and let T_k, $k = 1, 2, 3$, be the gyropoint in which the ingyrocircle of the gyrotriangle meets the opposite gyroside of A_k, shown in Fig. 10. A gyrotrigonometric gyrobarycentric coordinate representation of each gyropoint T_k is given by*

$$\begin{aligned} T_1 &= \frac{\tan\frac{\alpha_2}{2}\gamma_{A_2}A_2 + \tan\frac{\alpha_3}{2}\gamma_{A_3}A_3}{\tan\frac{\alpha_2}{2}\gamma_{A_2} + \tan\frac{\alpha_3}{2}\gamma_{A_3}} \\ T_2 &= \frac{\tan\frac{\alpha_1}{2}\gamma_{A_1}A_1 + \tan\frac{\alpha_3}{2}\gamma_{A_3}A_3}{\tan\frac{\alpha_1}{2}\gamma_{A_1} + \tan\frac{\alpha_3}{2}\gamma_{A_3}} \\ T_3 &= \frac{\tan\frac{\alpha_1}{2}\gamma_{A_1}A_1 + \tan\frac{\alpha_2}{2}\gamma_{A_2}A_2}{\tan\frac{\alpha_1}{2}\gamma_{A_1} + \tan\frac{\alpha_2}{2}\gamma_{A_2}}\,. \end{aligned} \tag{278}$$

Theorem 31 and its proof are presented in [56, Sect. 7.14, pp. 191–194] in an equivalent form (noting that gyrobarycentric coordinates are homogeneous). This theorem proves useful in the determination of the tangency gyropoints T_k, $k = 1, 2, 3$, shown in Fig. 10, and, consequently, in the determination of the Cabrera gyropoint C of gyrotriangle $A_1A_2A_3$, shown in Fig. 12, p. 648.

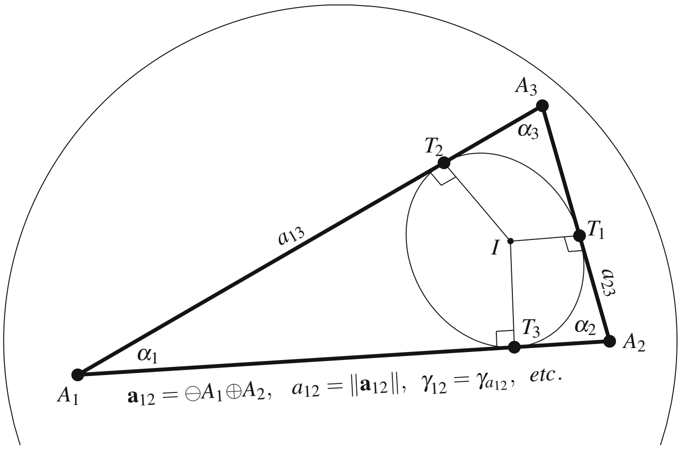


Fig. 10 The ingyrocircle of gyrotriangle $A_1A_2A_3$ is shown along with its gyrocenter I and its gyrotangency gyropoints T_k, $k = 1, 2, 3$, in an Einstein gyrovector plane $\mathbb{R}_s^2$. The gyropoint T_k is the gyropoint in which the ingyrocircle of the gyrotriangle meets the gyrotriangle gyroside opposite to gyrovertex A_k. The gyrotriangle gyroangle at gyrovertex A_k is α_k

Theorem 32 (Exgyrocircle Tangency Gyropoints). *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $\mathbb{R}_s^n$, and let H_k, $k = 1, 2, 3$, be the gyropoint in which an exgyrocircle of the gyrotriangle meets the opposite gyroside of A_k, as shown in Fig. 14, p. 652. A gyrotrigonometric gyrobarycentric coordinate representation of each gyropoint H_k is given by*

$$
\begin{aligned}
H_1 &= \frac{\cot\frac{\alpha_2}{2}\gamma_{A_2}A_2 + \cot\frac{\alpha_3}{2}\gamma_{A_3}A_3}{\cot\frac{\alpha_2}{2}\gamma_{A_2} + \cot\frac{\alpha_3}{2}\gamma_{A_3}} \\
H_2 &= \frac{\cot\frac{\alpha_1}{2}\gamma_{A_1}A_1 + \cot\frac{\alpha_3}{2}\gamma_{A_3}A_3}{\cot\frac{\alpha_1}{2}\gamma_{A_1} + \cot\frac{\alpha_3}{2}\gamma_{A_3}} \\
H_3 &= \frac{\cot\frac{\alpha_1}{2}\gamma_{A_1}A_1 + \cot\frac{\alpha_2}{2}\gamma_{A_2}A_2}{\cot\frac{\alpha_1}{2}\gamma_{A_1} + \cot\frac{\alpha_2}{2}\gamma_{A_2}} \,.
\end{aligned}
\tag{279}
$$

Theorem 32 and its proof are presented in [56, Sect. 8.13, pp. 246–248] in an equivalent form (noting that gyrobarycentric coordinates are homogeneous). This theorem proves useful in the determination of the tangency gyropoints H_k, $k = 1, 2, 3$, shown in Fig. 14, and, consequently, in the determination of the Cabrera exgyrocircle gyropoint H of gyrotriangle $A_1A_2A_3$, shown in Fig. 14, p. 652.

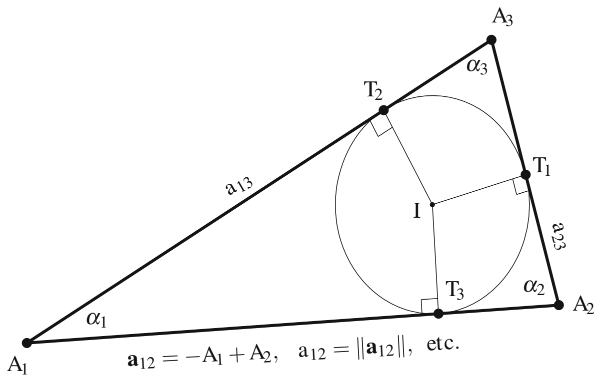


Fig. 11 The incircle of triangle $A_1A_2A_3$ is shown along with its center I and its tangency points T_k, $k = 1, 2, 3$, in a Euclidean plane $\mathbb{R}2$. The point T_k is the point in which the incircle of the triangle meets the triangle side opposite to vertex A_k. The triangle angles are α_k

34 Incircle Points of Tangency

It should be noted that, owing to our notation that emphasizes analogies, in the Euclidean limit

$$\lim_{s\to\infty} \frac{\tan\frac{\alpha_1}{2}\gamma_{A_1}A_1 + \tan\frac{\alpha_2}{2}\gamma_{A_2}A_2}{\tan\frac{\alpha_1}{2}\gamma_{A_1} + \tan\frac{\alpha_2}{2}\gamma_{A_2}} = \frac{\tan\frac{\alpha_1}{2}A_1 + \tan\frac{\alpha_2}{2}A_2}{\tan\frac{\alpha_1}{2} + \tan\frac{\alpha_2}{2}}, \tag{280}$$

A_k, $k = 1, 2$, are gyropoints on the left side, and the same A_k are points on the right side. Similarly, $\tan(\phi_k/2)$ on the left side is the gyrotangent of a gyroangle $\phi_k/2)$, while the same $\tan(\phi_k/2)$ on the right side is the tangent of an angle $\phi_k/2)$, as shown in Figs. 10 and 11.

Figure 11 is the Euclidean counterpart of Fig. 10 and, correspondingly, Theorem 33 below is the Euclidean counterpart of Theorem 31.

Theorem 33. *Let $A_1A_2A_3$ be a triangle in a Euclidean vector space $\mathbb{R}^n$, and let T_k, $k = 1, 2, 3$, be the point in which the incircle of the triangle meets the opposite side of A_k, shown in Fig. 11. A trigonometric barycentric representation of each point T_k is given by*

$$\begin{aligned} T_1 &= \frac{\tan\frac{\alpha_2}{2}A_2 + \tan\frac{\alpha_3}{2}A_3}{\tan\frac{\alpha_2}{2} + \tan\frac{\alpha_3}{2}} \\ T_2 &= \frac{\tan\frac{\alpha_1}{2}A_1 + \tan\frac{\alpha_3}{2}A_3}{\tan\frac{\alpha_1}{2} + \tan\frac{\alpha_3}{2}} \\ T_3 &= \frac{\tan\frac{\alpha_1}{2}A_1 + \tan\frac{\alpha_2}{2}A_2}{\tan\frac{\alpha_1}{2} + \tan\frac{\alpha_2}{2}}. \end{aligned} \tag{281}$$

The results of Theorem 33 are the Euclidean limit of corresponding results of Theorem 31, noting that in the Euclidean limit, $s \to \infty$, gamma factors tend to 1 and gyrotriangle gyroangles α_i tend to corresponding triangle angles, also denoted by α_i, $i = 1, 2, 3$. Theorem 33 proves useful in the determination of the tangency points T_k, $k = 1, 2, 3$, shown in Fig. 11, and, consequently, in the determination of the Cabrera point C, shown in Fig. 15, p. 658.

35 Gyromedial Gyrotriangle

Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, and let M_{12}, M_{13} and M_{23} be the gyromidpoints of gyrosides A_1A_2, A_1A_3, and A_2A_3 of the gyrotriangle, respectively, as shown in Fig. 12.

Following (206)–(207), p. 616, M_{12} possesses the gyrobarycentric representation

$$M_{12} = \frac{\gamma_{A_1}A_1 + \gamma_{A_2}A_2}{\gamma_{A_1} + \gamma_{A_2}} \tag{282}$$

with respect to the set $\{A_1, A_2\}$, and the constant of the gyrobarycentric representation (282) is

$$m_{M_{12}} = \sqrt{2(\gamma_{12} + 1)}. \tag{283}$$

Furthermore, the gamma factor of M_{12} is, by (208b),

$$\gamma_{M_{12}} = \frac{\gamma_{A_1} + \gamma_{A_2}}{m_{M_{12}}}\,. \tag{284}$$

Thus, by cyclic gyrovertex permutations of gyrotriangle $A_1A_2A_3$, we have

$$\begin{aligned} M_{ij} &= \frac{\gamma_{A_i}A_i + \gamma_{A_j}A_j}{\gamma_{A_i} + \gamma_{A_j}} \\ m_{M_{ij}} &= \sqrt{2(1 + \gamma_{ij})} \\ \gamma_{M_{ij}} &= \frac{\gamma_{A_i} + \gamma_{A_j}}{m_{M_{ij}}} = \frac{\gamma_{A_i} + \gamma_{A_j}}{\sqrt{2(1 + \gamma_{ij})}}, \end{aligned} \tag{285}$$

$i, j = 1, 2, 3,\ i < j$.

In order to determine the gyrodistance between the gyromidpoints M_{12} and M_{23}, we represent these two gyropoints gyrobarycentrically with respect to the set

$\{A_1, A_2, A_3\}$, obtaining by means of (282),

$$\begin{aligned} m_{12} &= \frac{m_1\gamma_{_{A_1}}A_1 + m_2\gamma_{_{A_2}}A_2 + m_3\gamma_{_{A_3}}A_3}{m_1\gamma_{_{A_1}} + m_2\gamma_{_{A_2}} + m_3\gamma_{_{A_3}}} \\ m_{23} &= \frac{m'_1\gamma_{_{A_1}}A_1 + m'_2\gamma_{_{A_2}}A_2 + m'_3\gamma_{_{A_3}}A_3}{m'_1\gamma_{_{A_1}} + m'_2\gamma_{_{A_2}} + m'_3\gamma_{_{A_3}}} \end{aligned} \tag{286}$$

where

$$\begin{aligned} m_1 &= 1, \quad m_2 = 1, \quad m_3 = 0 \\ m'_1 &= 0, \quad m'_2 = 1, \quad m'_3 = 1\,. \end{aligned} \tag{287}$$

Then, by Theorem 27, p. 625, the gamma factor $\gamma_{\ominus M12 \oplus M_{23}}$ of the gyrodistance $\|\ominus M_{12} \oplus M_{23}\|$ between the gyromidpoints M_{12} and M_{23} is given by

$$\begin{aligned} \gamma_{\ominus M12 \oplus M_{23}} &= \frac{1}{m_{M_{12}}m_{M_{23}}}\{(m'_1 m_2 + m_1 m'_2)\gamma_{12} + (m'_1 m_3 + m_1 m'_3)\gamma_{13} \\ &\quad + (m'_2 m_3 + m_2 m'_3)\gamma_{23} + (m'_1 m_1 + m'_2 m_2 + m'_3 m_3)\} \\ &= \frac{1}{m_{M_{12}}m_{M_{23}}}(1 + \gamma_{12} + \gamma_{13} + \gamma_{23}). \end{aligned} \tag{288}$$

Hence, by (288) and (283), and by cyclic gyrovertex permutations of gyrotriangle $A_1A_2A_3$, we have

$$\begin{aligned} \sigma_{12} &:= \gamma_{\ominus M_{13} \oplus M_{23}} = \frac{1 + \gamma_{12} + \gamma_{13} + \gamma_{23}}{2\sqrt{(1 + \gamma_{13})(1 + \gamma_{23})}} \\ \sigma_{13} &:= \gamma_{\ominus M_{12} \oplus M_{23}} = \frac{1 + \gamma_{12} + \gamma_{13} + \gamma_{23}}{2\sqrt{(1 + \gamma_{12})(1 + \gamma_{23})}} \\ \sigma_{23} &:= \gamma_{\ominus M_{12} \oplus M_{13}} = \frac{1 + \gamma_{12} + \gamma_{13} + \gamma_{23}}{2\sqrt{(1 + \gamma_{12})(1 + \gamma_{13})}}\,. \end{aligned} \tag{289}$$

Here, σ_{12}, σ_{13}, and σ_{23} are, respectively, the gamma factors of gyrosides $M_{13}M_{23}$, $M_{12}M_{23}$, and $M_{12}M_{13}$ of the gyromedial gyrotriangle $M_{12}M_{13}M_{23}$ of gyrotriangle $A_1A_2A_3$, shown in Fig. 12.

Remark 2. If the gyromedial gyrotriangle $M_{12}M_{13}M_{23}$ of gyrotriangle $A_1A_2A_3$, shown in Fig. 12, is right gyroangled, say $\beta_3 = \pi/2$, then, by the Einstein–Pythagoras identity (268), p. 630, we have

$$\sigma_{13}\sigma_{23} = \sigma_{12} \tag{290}$$

or equivalently, by (289),

$$-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1 = 0\,. \tag{291}$$

Indeed, (291) insures that $\cos\beta_3 = 0$, as we see from (294), p. 639.

36 Gyromedial Gyrotriangle Gyroangles

Let β_1, β_2, and β_3 be the gyroangles of the gyromedial gyrotriangle $M_{12}M_{13}M_{23}$ of gyrotriangle $A_1A_2A_3$, shown in Fig. 12. By the *SSS* to *AAA* conversion law (257)–(259), p. 628, we have

$$\begin{aligned}
\cos\beta_1 &= \frac{-\sigma_{23} + \sigma_{12}\sigma_{13}}{\sqrt{\sigma_{12}^2 - 1}\sqrt{\sigma_{13}^2 - 1}} \\
\cos\beta_2 &= \frac{-\sigma_{13} + \sigma_{12}\sigma_{23}}{\sqrt{\sigma_{12}^2 - 1}\sqrt{\sigma_{23}^2 - 1}} \\
\cos\beta_3 &= \frac{-\sigma_{12} + \sigma_{13}\sigma_{23}}{\sqrt{\sigma_{13}^2 - 1}\sqrt{\sigma_{23}^2 - 1}}
\end{aligned} \tag{292}$$

and

$$\begin{aligned}
\sin\beta_1 &= \frac{\sqrt{1 + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2}}{\sqrt{\sigma_{12}^2 - 1}\sqrt{\sigma_{13}^2 - 1}} \\
\sin\beta_2 &= \frac{\sqrt{1 + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2}}{\sqrt{\sigma_{12}^2 - 1}\sqrt{\sigma_{23}^2 - 1}} \\
\sin\beta_3 &= \frac{\sqrt{1 + 2\sigma_{12}\sigma_{13}\sigma_{23} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2}}{\sqrt{\sigma_{13}^2 - 1}\sqrt{\sigma_{23}^2 - 1}}\,.
\end{aligned} \tag{293}$$

Substitutions from (289) into (292)–(293) yield the equations

$$\cos^2\beta_1 = \frac{(1+\gamma_{12}+\gamma_{13}+\gamma_{23})^2(\gamma_{12}+\gamma_{13}-\gamma_{23}-1)^2}{D_1^2}$$

$$\cos^2\beta_2 = \frac{(1+\gamma_{12}+\gamma_{13}+\gamma_{23})^2(\gamma_{12}-\gamma_{13}+\gamma_{23}-1)^2}{D_2^2} \tag{294}$$

$$\cos^2\beta_3 = \frac{(1+\gamma_{12}+\gamma_{13}+\gamma_{23})^2(-\gamma_{12}+\gamma_{13}+\gamma_{23}-1)^2}{D_3^2}$$

and

$$\sin^2\beta_1 = \frac{8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)}{D_1^2}(1+\gamma_{23})$$

$$\sin^2\beta_2 = \frac{8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)}{D_2^2}(1+\gamma_{13}) \tag{295}$$

$$\sin^2\beta_3 = \frac{8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)}{D_3^2}(1+\gamma_{12})\,,$$

where

$$\begin{aligned}
D_1^2 &= \{(\gamma_{12}^2-1)+(\gamma_{13}^2-1)+(\gamma_{23}^2-1)+2(\gamma_{12}\gamma_{13}-\gamma_{23})\}^2 \\
&\quad - \{2(\gamma_{12}\gamma_{23}-\gamma_{13})-2(\gamma_{13}\gamma_{23}-\gamma_{12})\}^2 \\
D_2^2 &= \{(\gamma_{12}^2-1)+(\gamma_{13}^2-1)+(\gamma_{23}^2-1)+2(\gamma_{12}\gamma_{23}-\gamma_{13})\}^2 \\
&\quad - \{2(\gamma_{12}\gamma_{13}-\gamma_{23})-2(\gamma_{13}\gamma_{23}-\gamma_{12})\}^2 \\
D_3^2 &= \{(\gamma_{12}^2-1)+(\gamma_{13}^2-1)+(\gamma_{23}^2-1)+2(\gamma_{13}\gamma_{23}-\gamma_{12})\}^2 \\
&\quad - \{2(\gamma_{12}\gamma_{13}-\gamma_{23})-2(\gamma_{12}\gamma_{23}-\gamma_{13})\}^2\,.
\end{aligned} \tag{296}$$

In order to see that $D_k^2 > 0$, $k = 1, 2, 3$, it is useful to rewrite D_k^2 equivalently in the following form:

$$\begin{aligned} D_1^2 &= (1+\gamma_{12}+\gamma_{13}+\gamma_{23})^2(\gamma_{12}+\gamma_{13}-\gamma_{23}-1)^2 \\ &\quad + 8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)(1+\gamma_{23}) \\ D_2^2 &= (1+\gamma_{12}+\gamma_{13}+\gamma_{23})^2(\gamma_{12}-\gamma_{13}+\gamma_{23}-1)^2 \\ &\quad + 8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)(1+\gamma_{13}) \\ D_3^2 &= (1+\gamma_{12}+\gamma_{13}+\gamma_{23})^2(-\gamma_{12}+\gamma_{13}+\gamma_{23}-1)^2 \\ &\quad + 8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)(1+\gamma_{12})\,. \end{aligned} \tag{297}$$

Indeed, it follows from (297) and Inequality (260), p. 628, that

$$D_k^2 > 0\,, \tag{298}$$

$k = 1, 2, 3$, as expected from (294)–(295).

Owing to (298), $D_k = \sqrt{D_k^2} > 0$. Hence, by Inequality (260), (294)–(295) can be written conveniently as

$$\begin{aligned} \cos\beta_1 &= \frac{1+\gamma_{12}+\gamma_{13}+\gamma_{23}}{D_1}(\gamma_{12}+\gamma_{13}-\gamma_{23}-1) \\ \cos\beta_2 &= \frac{1+\gamma_{12}+\gamma_{13}+\gamma_{23}}{D_2}(\gamma_{12}-\gamma_{13}+\gamma_{23}-1) \\ \cos\beta_3 &= \frac{1+\gamma_{12}+\gamma_{13}+\gamma_{23}}{D_3}(-\gamma_{12}+\gamma_{13}+\gamma_{23}-1) \end{aligned} \tag{299}$$

$-1 < \cos\beta_k < 1$, $k = 1, 2, 3$, and

$$\begin{aligned} \sin\beta_1 &= \frac{\sqrt{8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)}}{D_1}\sqrt{1+\gamma_{23}} \;>\; 0 \\ \sin\beta_2 &= \frac{\sqrt{8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)}}{D_2}\sqrt{1+\gamma_{13}} \;>\; 0 \\ \sin\beta_3 &= \frac{\sqrt{8(1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2)}}{D_3}\sqrt{1+\gamma_{12}} \;>\; 0 \end{aligned} \tag{300}$$

where $D_k > 0$, $k = 1, 2, 3$, are given by (297) or, equivalently, by (296).

If we use the notation

$$\begin{aligned} A &= 1 + \gamma_{12} + \gamma_{13} + \gamma_{23} \\ B &= \sqrt{8(1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2)} \end{aligned} \tag{301}$$

then

$$\begin{aligned} \cot\beta_1 &= \frac{A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)}{B\sqrt{1 + \gamma_{23}}} \\ \frac{1}{\sin\beta_1} &= \frac{D_1}{B\sqrt{1 + \gamma_{23}}} \,. \end{aligned} \tag{302}$$

We now employ the trigonometric/gyrotrigonometric identity

$$\cot\frac{\beta_1}{2} = \cot\beta_1 + \frac{1}{\sin\beta_1} \,, \tag{303}$$

obtaining the equation

$$\cot\frac{\beta_1}{2} = \frac{D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)}{B\sqrt{1 + \gamma_{23}}} \,. \tag{304}$$

Finally, it follows from (297), (301), and (304) along with cyclic gyrovertex permutations of gyrotriangle $A_1A_2A_3$ that

$$\begin{aligned} \cot\frac{\beta_1}{2} &= \frac{D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)}{B\sqrt{1 + \gamma_{23}}} \\ \cot\frac{\beta_2}{2} &= \frac{D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)}{B\sqrt{1 + \gamma_{13}}} \\ \cot\frac{\beta_3}{2} &= \frac{D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)}{B\sqrt{1 + \gamma_{12}}} \end{aligned} \tag{305}$$

where

$$A = 1 + \gamma_{12} + \gamma_{13} + \gamma_{23}$$
$$B^2 = 8(1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2) \tag{306}$$

and

$$D_1^2 = A^2(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)^2 + B^2(1 + \gamma_{23})$$
$$D_2^2 = A^2(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)^2 + B^2(1 + \gamma_{13}) \tag{307}$$
$$D_3^2 = A^2(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 + B^2(1 + \gamma_{12}),$$

$B > 0, D_k > 0, k = 1, 2, 3.$

37 Gyromedial Gyrotriangle Ingyrocircle Tangency Gyropoints

By Theorem 31, p. 633, the tangency gyropoint T_3 in Fig. 12, p. 648, is given by

$$T_3 = \frac{m_1\gamma_{M_{23}}M_{23} + m_2\gamma_{M_{13}}M_{13}}{m_1\gamma_{M_{23}} + m_2\gamma_{M_{13}}} \tag{308}$$

where

$$m_1 = \tan\frac{\beta_1}{2}$$
$$m_2 = \tan\frac{\beta_2}{2} \tag{309}$$

where, by (305),

$$\tan\frac{\beta_1}{2} = \frac{B\sqrt{1+\gamma_{23}}}{D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)}$$
$$\tan\frac{\beta_2}{2} = \frac{B\sqrt{1+\gamma_{13}}}{D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)} \tag{310}$$
$$\tan\frac{\beta_3}{2} = \frac{B\sqrt{1+\gamma_{12}}}{D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)}$$

and where, by (285), p. 636,

$$M_{ij} = \frac{\gamma_{A_i} A_i + \gamma_{A_j} A_j}{\gamma_{A_i} + \gamma_{A_j}}$$
$$\gamma_{M_{ij}} = \frac{\gamma_{A_i} + \gamma_{A_j}}{\sqrt{2(1 + \gamma_{ij})}}, \tag{311}$$

$i, j = 1, 2, 3, i < j$.

Substituting (309)–(311) into (308), we obtain T_3 in (314a).

The remaining tangency gyropoints T_1 and T_2 that are shown in Fig. 12, p. 648, are derived from T_3 in (314a) by cyclic gyrovertex permutations of gyrotriangle $A_1A_2A_3$, obtaining

$$T_1 = \frac{m_{11}\gamma_{A_1} A_1 + m_{21}\gamma_{A_2} A_2 + m_{31}\gamma_{A_3} A_3}{m_{11}\gamma_{A_1} + m_{21}\gamma_{A_2} + m_{31}\gamma_{A_3}} \tag{312a}$$

$$m_{11} = D_2 + D_3 + 2A(\gamma_{23} - 1) = m_{21} + m_{31}$$
$$m_{21} = D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1) \tag{312b}$$
$$m_{31} = D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)$$

$$T_2 = \frac{m_{12}\gamma_{A_1} A_1 + m_{22}\gamma_{A_2} A_2 + m_{32}\gamma_{A_3} A_3}{m_{12}\gamma_{A_1} + m_{22}\gamma_{A_2} + m_{32}\gamma_{A_3}} \tag{313a}$$

$$m_{12} = D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)$$
$$m_{22} = D_1 + D_3 + 2A(\gamma_{13} - 1) = m_{12} + m_{32} \tag{313b}$$
$$m_{32} = D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)$$

$$T_3 = \frac{m_{13}\gamma_{A_1} A_1 + m_{23}\gamma_{A_2} A_2 + m_{33}\gamma_{A_3} A_3}{m_{13}\gamma_{A_1} + m_{23}\gamma_{A_2} + m_{33}\gamma_{A_3}} \tag{314a}$$

$$m_{13} = D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)$$
$$m_{23} = D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1) \tag{314b}$$
$$m_{33} = D_1 + D_2 + 2A(\gamma_{12} - 1) = m_{13} + m_{23}\,.$$

A left gyrotranslation of gyrotriangle $A_1A_2A_3$ by $\ominus A_1$ results in the $\ominus A_1$-*left gyrotranslated gyrotriangle*

$$(\ominus A_1 \oplus A_1)(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3) = O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3) = \mathbf{0}\mathbf{a}_{12}\mathbf{a}_{13}, \tag{315}$$

where $O = \ominus A_1 \oplus A_1$ is the arbitrarily selected origin of the Einstein gyrovector space $\mathbb{R}^n_s$, that is,

$$O = \mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n \tag{316}$$

with respect to the Cartesian coordinates of $\mathbb{R}^n_s$, and where we use the convenient *index notation*

$$\begin{aligned}
\mathbf{a}_{12} &= \ominus A_1 \oplus A_2, & a_{12} &= \|\mathbf{a}_{12}\|, & \gamma_{21} &= \gamma_{12} = \gamma_{a_{12}} = \gamma_{\mathbf{a}_{12}}\,, \\
\mathbf{a}_{13} &= \ominus A_1 \oplus A_3, & a_{13} &= \|\mathbf{a}_{13}\|, & \gamma_{31} &= \gamma_{13} = \gamma_{a_{13}} = \gamma_{\mathbf{a}_{13}}\,, \\
\mathbf{a}_{23} &= \ominus A_2 \oplus A_3, & a_{23} &= \|\mathbf{a}_{23}\|, & \gamma_{32} &= \gamma_{23} = \gamma_{a_{23}} = \gamma_{\mathbf{a}_{23}}\,.
\end{aligned} \tag{317}$$

Following the left gyrotranslation by $\ominus A_1$ of gyrotriangle $A_1A_2A_3$, the tangency gyropoints T_k in (312)–(314), $k = 1, 2, 3$, become $\ominus A_1 \oplus T_k$. Employing the Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 25, p. 609, we have from (312)–(314),

$$\begin{aligned}
&\ominus A_1 \oplus T_1 \\
&= \frac{m_{11}\gamma_{\ominus A_1 \oplus A_1}(\ominus A_1 \oplus A_1) + m_{21}\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2) + m_{31}\gamma_{\ominus A_1 \oplus A_3}(\ominus A_1 \oplus A_3)}{m_{11}\gamma_{\ominus A_1 \oplus A_1} + m_{21}\gamma_{\ominus A_1 \oplus A_2} + m_{31}\gamma_{\ominus A_1 \oplus A_3}} \\
&= \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}}\,,
\end{aligned} \tag{318}$$

noting the trivial equations $\ominus A_1 \oplus A_1 = \mathbf{0} = (0, \ldots, 0)$ and $\gamma_{\mathbf{0}} = 1$.

Similarly, $\ominus A_1 \oplus T_k$, $k = 1, 2, 3$, are given by

$$\begin{aligned}
\ominus A_1 \oplus T_1 &= \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}} \\
\ominus A_1 \oplus T_2 &= \frac{m_{22}\gamma_{12}\mathbf{a}_{12} + m_{32}\gamma_{13}\mathbf{a}_{13}}{m_{12} + m_{22}\gamma_{12} + m_{32}\gamma_{13}} \\
\ominus A_1 \oplus T_3 &= \frac{m_{23}\gamma_{12}\mathbf{a}_{12} + m_{33}\gamma_{13}\mathbf{a}_{13}}{m_{13} + m_{23}\gamma_{12} + m_{33}\gamma_{13}}\,.
\end{aligned} \tag{319}$$

The gyroline L_1 that passes through the gyropoints $\ominus A_1 \oplus A_1 = \mathbf{0}$ and $\ominus A_1 \oplus T_1$ is contained in the Euclidean line

$$L_1: \quad \mathbf{0} + (-\mathbf{0} + [\ominus A_1 \oplus T_1])t_1 = \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}} t_1 \tag{320}$$

where $t_1 \in \mathbb{R}$ is the line parameter.

The gyroline L_2 that passes through the gyropoints $\ominus A_1 \oplus A_2 = \mathbf{a}_{12}$ and $\ominus A_1 \oplus T_2$ is contained in the Euclidean line

$$L_2: \quad \mathbf{a}_{12} + (-\mathbf{a}_{12} + [\ominus A_1 \oplus T_2])t_2 = \mathbf{a}_{12}(1 - t_2) + \frac{m_{22}\gamma_{12}\mathbf{a}_{12} + m_{32}\gamma_{13}\mathbf{a}_{13}}{m_{12} + m_{22}\gamma_{12} + m_{32}\gamma_{13}} t_2 \tag{321}$$

where $t_2 \in \mathbb{R}$ is the line parameter.

Similarly, the gyroline L_3 that passes through the gyropoints $\ominus A_1 \oplus A_3 = \mathbf{a}_{13}$ and $\ominus A_1 \oplus T_3$ is contained in the Euclidean line

$$L_3: \quad \mathbf{a}_{13} + (-\mathbf{a}_{13} + [\ominus A_1 \oplus T_3])t_3 = \mathbf{a}_{13}(1 - t_3) + \frac{m_{23}\gamma_{12}\mathbf{a}_{12} + m_{33}\gamma_{13}\mathbf{a}_{13}}{m_{13} + m_{23}\gamma_{12} + m_{33}\gamma_{13}} t_3 \tag{322}$$

where $t_3 \in \mathbb{R}$ is the line parameter.

The three lines L_k, $k = 1, 2, 3$, will prove useful in the study of the Cabrera ingyrocircle gyropoint.

38 Cabrera Ingyrocircle Gyropoint

Let the gyropoint of concurrency, C, of gyrolines A_1T_1, A_2T_2, and A_3T_3, as shown in Figs. 12 and 13, be given by its gyrobarycentric representation with respect to the set $\{A_1, A_2, A_3\}$,

$$C = \frac{m_{1c}\gamma_{A_1}A_1 + m_{2c}\gamma_{A_2}A_2 + m_{3c}\gamma_{A_3}A_3}{m_{1c}\gamma_{A_1} + m_{2c}\gamma_{A_2} + m_{3c}\gamma_{A_3}}, \tag{323}$$

where the gyrobarycentric coordinates m_{ic}, $i = 1, 2, 3$, of C are to be determined.

Employing the Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 25, p. 609, we find that the $\ominus A_1$ left gyrotranslated concurrency gyropoint C is given gyrobarycentrically by

$$
\begin{aligned}
P :&= \ominus A_1 \oplus C \\
&= \frac{m_{1c}\gamma_{\ominus A_1 \oplus A_1}(\ominus A_1 \oplus A_1) + m_{2c}\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2) + m_{3c}\gamma_{\ominus A_1 \oplus A_3}(\ominus A_1 \oplus A_3)}{m_{1c}\gamma_{\ominus A_1 \oplus A_1} + m_{2c}\gamma_{\ominus A_1 \oplus A_2} + m_{3c}\gamma_{\ominus A_1 \oplus A_3}} \\
&= \frac{m_{2c}\gamma_{12}\mathbf{a}_{12} + m_{3c}\gamma_{13}\mathbf{a}_{13}}{m_{1c} + m_{2c}\gamma_{12} + m_{3c}\gamma_{13}}\,,
\end{aligned}
\tag{324}
$$

noting the trivial equations $\ominus A_1 \oplus A_1 = \mathbf{0} = (0, \ldots, 0)$ and $\gamma_{\mathbf{0}} = 1$.

We assume that gyropoint C lies on each of the three gyrolines A_1T_1, A_2T_2, and A_3T_3, as shown in Fig. 12. This assumption implies that the gyropoint P lies on each of the three lines L_k, $k = 1, 2, 3$ in (320)–(322). Hence, there exist values t_{k0} of the line parameters t_k, $k = 1, 2, 3$, respectively, such that

$$
\begin{aligned}
P - \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}} t_{10} &= 0 \\
P - \mathbf{a}_{12}(1 - t_{20}) - \frac{m_{22}\gamma_{12}\mathbf{a}_{12} + m_{32}\gamma_{13}\mathbf{a}_{13}}{m_{12} + m_{22}\gamma_{12} + m_{32}\gamma_{13}} t_{20} &= 0 \\
P - \mathbf{a}_{13}(1 - t_{30}) - \frac{m_{23}\gamma_{12}\mathbf{a}_{12} + m_{33}\gamma_{13}\mathbf{a}_{13}}{m_{13} + m_{23}\gamma_{12} + m_{33}\gamma_{13}} t_{30} &= 0,
\end{aligned}
\tag{325}
$$

where the gyrobarycentric coordinates m_{ij} in (325) are given in (312)–(314).

The kth equation in (325) expresses the condition that point P lies on line L_k, $k = 1, 2, 3$.

The system of Eqs. (325) was obtained by methods of gyroalgebra and will be solved below by a common method of linear algebra.

Substituting P from (324) into (325), and rewriting each equation in (325) as a linear combination of $\mathbf{a}_{12}$ and $\mathbf{a}_{13}$ equals zero, one obtains the following linear homogeneous system of three gyrovector equations

$$
\begin{aligned}
c_{11}\mathbf{a}_{12} + c_{12}\mathbf{a}_{13} &= \mathbf{0} \\
c_{21}\mathbf{a}_{12} + c_{22}\mathbf{a}_{13} &= \mathbf{0} \\
c_{31}\mathbf{a}_{12} + c_{32}\mathbf{a}_{13} &= \mathbf{0},
\end{aligned}
\tag{326}
$$

where each coefficient c_{ij}, $i = 1, 2, 3, j = 1, 2$, is a function of γ_{12}, γ_{13}, γ_{23}, and the six unknowns m_{kc} and t_{k0}, $k = 1, 2, 3$.

Since the set $S = \{A_1, A_2, A_3\}$ is gyrobarycentrically independent, the two gyrovectors $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ and $\mathbf{a}_{13} = \ominus A_1 \oplus A_3$ in $\mathbb{R}^n_s$, considered as vectors in the ambient space $\mathbb{R}^n$, are linearly independent. Hence, each coefficient c_{ij} in (326) equals zero. Accordingly, the three gyrovector equations in (326) are equivalent to the following six scalar equations,

$$
c_{11} = c_{12} = c_{21} = c_{22} = c_{31} = c_{32} = 0 \tag{327}
$$

for the six unknowns m_{kc} and t_{k0}, $k = 1, 2, 3$.

The six scalar equations in (327) are not independent. For convenience one may initially select $m_{1c} = 1$ and obtain a unique solution for the remaining unknowns. Finally, owing to the homogeneity of the gyrobarycentric coordinates m_{kc}, one can multiply each of them by a convenient nonzero common factor, obtaining

$$\begin{aligned} m_{1c} &= D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1) \\ m_{2c} &= D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1) \\ m_{3c} &= D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)\,. \end{aligned} \tag{328}$$

It is clear from the definition of D_k, $k = 1, 2, 3$, in (307), p. 642, that these terms satisfy the inequalities

$$\begin{aligned} D_1 &> A|\gamma_{12} + \gamma_{13} - \gamma_{23} - 1| \\ D_2 &> A|\gamma_{12} - \gamma_{13} + \gamma_{23} - 1| \\ D_3 &> A| - \gamma_{12} + \gamma_{13} + \gamma_{23} - 1|\,. \end{aligned} \tag{329}$$

Hence, all the gyrobarycentric coordinates m_{kc}, $k = 1, 2, 3$, in (328) of Cabrera ingyrocircle gyropoint C in (323) are positive so that, by Corollary 2, p. 611, Cabrera ingyrocircle gyropoint always lies on the interior of its reference gyrotriangle $A_1A_2A_3$. However, it need not lie on the interior of its gyromedial gyrotriangle, as we see from Figs. 12 and 13.

Formalizing the main result of this section, we have the following theorem.

Theorem 34 (Cabrera Ingyrocircle Gyropoint). *The Cabrera ingyrocircle gyropoint C of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Figs. 12 and 13, lies on the interior of its reference gyrotriangle $A_1A_2A_3$. It is given by its gyrobarycentric representation with respect to the set $\{A_1, A_2, A_3\}$,*

$$C = \frac{m_{1c}\gamma_{A_1} A_1 + m_{2c}\gamma_{A_2} A_2 + m_{3c}\gamma_{A_3} A_3}{m_{1c}\gamma_{A_1} + m_{2c}\gamma_{A_2} + m_{3c}\gamma_{A_3}} \tag{330}$$

where

$$\begin{aligned} m_{1c} &= D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1) \\ m_{2c} &= D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1) \\ m_{3c} &= D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)\,. \end{aligned} \tag{331}$$

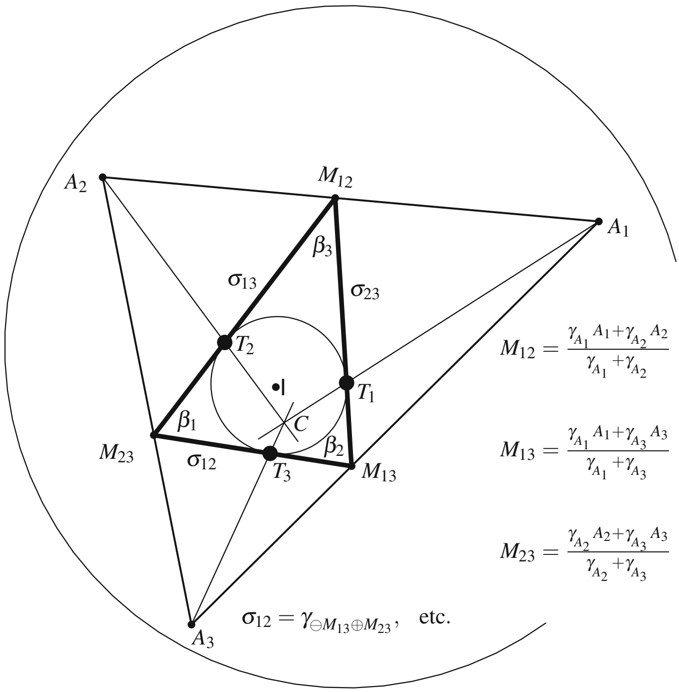


Fig. 12 Cabrera gyropoint. The gyromidpoints of the gyrosides of gyrotriangle $A_1A_2A_3$ are M_{12}, M_{13}, and M_{23}. The gyromedial gyrotriangle of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector plane $(\mathbb{R}^2, \oplus, \otimes)$ is $M_{12}M_{13}M_{23}$. The gyroangles of the gyromedial gyrotriangle are β_1, β_2, and β_3, its ingyrocenter is I, and its ingyrocircle gyrotangency gyropoints are T_1, T_2, and T_3. The gamma factors of the gyromedial gyrotriangle gyrosides are σ_{12}, σ_{13}, and σ_{23}. The gyrolines A_1T_1, A_2T_2, and A_3T_3 are concurrent. The concurrency gyropoint, C, is the Cabrera gyropoint of gyrotriangle $A_1A_2A_3$, determined gyrobarycentrically in Theorem 34, p. 647

where

$$
\begin{aligned}
A &= 1 + \gamma_{12} + \gamma_{13} + \gamma_{23} \\
B^2 &= 8(1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2)
\end{aligned}
\tag{332}
$$

and

$$
\begin{aligned}
D_1^2 &= A^2(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)^2 + B^2(1 + \gamma_{23}) \\
D_2^2 &= A^2(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)^2 + B^2(1 + \gamma_{13}) \\
D_3^2 &= A^2(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 + B^2(1 + \gamma_{12}) \,,
\end{aligned}
\tag{333}
$$

$B > 0$, $D_k > 0$, $k = 1, 2, 3$.

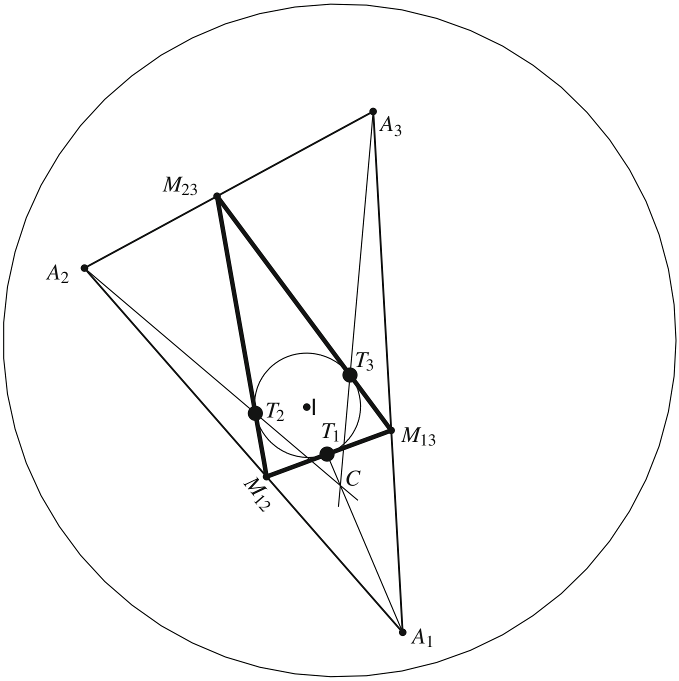


Fig. 13 Contrasting Fig. 12, the Cabrera ingyrocircle gyropoint, C, of a gyrotriangle $A_1A_2A_3$, which does not lie on the interior of the gyromedial gyrotriangle of the reference gyrotriangle $A_1A_2A_3$, is shown

39 Gyromedial Gyrotriangle Exgyrocircles Tangency Gyropoints

This section is similar to Sect. 37. By Theorem 32, p. 634, the tangency gyropoint H_3 is given by

$$H_3 = \frac{m_1\gamma_{M_{23}}M_{23} + m_2\gamma_{M_{13}}M_{13}}{m_1\gamma_{M_{23}} + m_2\gamma_{M_{13}}} \tag{334}$$

where

$$\begin{aligned} m_1 &= \cot\frac{\beta_1}{2} \\ m_2 &= \cot\frac{\beta_2}{2} \end{aligned} \tag{335}$$

where $\cot(\beta_k/2)$, $k = 1, 2, 3$, are given by (305)–(307) and where M_{ij} and $\gamma_{M_{ij}}$, $i, j = 1, 2, 3$, $i < j$, are given by (311).

Expressing H_3 in terms of D_1, D_2, A, B, and γ_{ij} by means of (335), (305)–(307), and (311), we obtain H_3 in (338a).

The remaining tangency gyropoints H_1 and H_2 that are shown in Fig. 12, p. 648, are derived from H_3 by cyclic gyrovertex permutations of gyrotriangle $A_1A_2A_3$, obtaining

$$H_1 = \frac{m_{11}\gamma_{A_1}A_1 + m_{21}\gamma_{A_2}A_2 + m_{31}\gamma_{A_3}A_3}{m_{11}\gamma_{A_1} + m_{21}\gamma_{A_2} + m_{31}\gamma_{A_3}} \tag{336a}$$

$$\begin{aligned} m_{11} &= m_{21} + m_{31} \\ m_{21} &= (D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1))(1 + \gamma_{13}) \\ m_{31} &= (D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1))(1 + \gamma_{12}) \end{aligned} \tag{336b}$$

$$H_2 = \frac{m_{12}\gamma_{A_1}A_1 + m_{22}\gamma_{A_2}A_2 + m_{32}\gamma_{A_3}A_3}{m_{12}\gamma_{A_1} + m_{22}\gamma_{A_2} + m_{32}\gamma_{A_3}} \tag{337a}$$

$$\begin{aligned} m_{12} &= (D_3 + A(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1))(1 + \gamma_{23}) \\ m_{22} &= m_{12} + m_{32} \\ m_{32} &= (D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1))(1 + \gamma_{12}) \end{aligned} \tag{337b}$$

$$H_3 = \frac{m_{13}\gamma_{A_1}A_1 + m_{23}\gamma_{A_2}A_2 + m_{33}\gamma_{A_3}A_3}{m_{13}\gamma_{A_1} + m_{23}\gamma_{A_2} + m_{33}\gamma_{A_3}} \tag{338a}$$

$$\begin{aligned} m_{13} &= \{D_2 + A(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)\}(1 + \gamma_{23}) \\ m_{23} &= \{D_1 + A(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)\}(1 + \gamma_{13}) \\ m_{33} &= m_{13} + m_{23}\,. \end{aligned} \tag{338b}$$

A left gyrotranslation of gyrotriangle $A_1A_2A_3$ by $\ominus A_1$ results in the $\ominus A_1$-*left gyrotranslated gyrotriangle*

$$(\ominus A_1 \oplus A_1)(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3) = O(\ominus A_1 \oplus A_2)(\ominus A_1 \oplus A_3) = \mathbf{0}\mathbf{a}_{12}\mathbf{a}_{13}, \tag{339}$$

where $O = \ominus A_1 \oplus A_1$ is the arbitrarily selected origin of the Einstein gyrovector space $\mathbb{R}^n_s$.

Following the left gyrotranslation by $\ominus A_1$ of gyrotriangle $A_1A_2A_3$, the tangency gyropoints H_k in (336)–(338), $k = 1, 2, 3$, become $\ominus A_1 \oplus H_k$. Employing the

Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 25, p. 609, we have from (336)–(338),

$$\begin{aligned}\ominus A_1 \oplus H_1 &= \frac{m_{11}\gamma_{\ominus A_1\oplus A_1}(\ominus A_1 \oplus A_1) + m_{21}\gamma_{\ominus A_1\oplus A_2}(\ominus A_1 \oplus A_2) + m_{31}\gamma_{\ominus A_1\oplus A_3}(\ominus A_1 \oplus A_3)}{m_{11}\gamma_{\ominus A_1\oplus A_1} + m_{21}\gamma_{\ominus A_1\oplus A_2} + m_{31}\gamma_{\ominus A_1\oplus A_3}} \\ &= \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}},\end{aligned} \tag{340}$$

noting the trivial equations $\ominus A_1 \oplus A_1 = \mathbf{0} = (0, \ldots, 0)$ and $\gamma_{\mathbf{0}} = 1$.

Similarly, $\ominus A_1 \oplus H_k$, $k = 1, 2, 3$, are given by

$$\begin{aligned}\ominus A_1 \oplus H_1 &= \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}} \\ \ominus A_1 \oplus H_2 &= \frac{m_{22}\gamma_{12}\mathbf{a}_{12} + m_{32}\gamma_{13}\mathbf{a}_{13}}{m_{12} + m_{22}\gamma_{12} + m_{32}\gamma_{13}} \\ \ominus A_1 \oplus H_3 &= \frac{m_{23}\gamma_{12}\mathbf{a}_{12} + m_{33}\gamma_{13}\mathbf{a}_{13}}{m_{13} + m_{23}\gamma_{12} + m_{33}\gamma_{13}}\,.\end{aligned} \tag{341}$$

The gyroline L_1 that passes through the gyropoints $\ominus A_1 \oplus A_1 = \mathbf{0}$ and $\ominus A_1 \oplus H_1$ is contained in the Euclidean line

$$L_1: \quad \mathbf{0} + (-\mathbf{0} + [\ominus A_1 \oplus H_1])t_1 = \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}}t_1 \tag{342}$$

where $t_1 \in \mathbb{R}$ is the line parameter.

The gyroline L_2 that passes through the gyropoints $\ominus A_1 \oplus A_2 = \mathbf{a}_{12}$ and $\ominus A_1 \oplus H_2$ is contained in the Euclidean line

$$L_2: \quad \mathbf{a}_{12} + (-\mathbf{a}_{12} + [\ominus A_1 \oplus H_2])t_2 = \mathbf{a}_{12}(1 - t_2) + \frac{m_{22}\gamma_{12}\mathbf{a}_{12} + m_{32}\gamma_{13}\mathbf{a}_{13}}{m_{12} + m_{22}\gamma_{12} + m_{32}\gamma_{13}}t_2 \tag{343}$$

where $t_2 \in \mathbb{R}$ is the line parameter.

Similarly, the gyroline L_3 that passes through the gyropoints $\ominus A_1 \oplus A_3 = \mathbf{a}_{13}$ and $\ominus A_1 \oplus H_3$ is contained in the Euclidean line

$$L_3: \quad \mathbf{a}_{13} + (-\mathbf{a}_{13} + [\ominus A_1 \oplus H_3])t_3 = \mathbf{a}_{13}(1 - t_3) + \frac{m_{23}\gamma_{12}\mathbf{a}_{12} + m_{33}\gamma_{13}\mathbf{a}_{13}}{m_{13} + m_{23}\gamma_{12} + m_{33}\gamma_{13}}t_3 \tag{344}$$

where $t_3 \in \mathbb{R}$ is the line parameter.

The three lines L_k, $k = 1, 2, 3$, will prove useful in the study of the Cabrera exgyrocircle gyropoint.

40 Cabrera Exgyrocircle Gyropoint

This section is similar to Sect. 38. Let the gyropoint of concurrency, H, of gyrolines A_1H_1, A_2H_2, and A_3H_3, as shown in Fig. 14, be given by its gyrobarycentric representation with respect to the set $\{A_1, A_2, A_3\}$,

$$H = \frac{m_{1h}\gamma_{A_1} A_1 + m_{2h}\gamma_{A_2} A_2 + m_{3h}\gamma_{A_3} A_3}{m_{1h}\gamma_{A_1} + m_{2h}\gamma_{A_2} + m_{3h}\gamma_{A_3}}, \tag{345}$$

where the gyrobarycentric coordinates m_{ih}, $i = 1, 2, 3$, of H are to be determined.

Employing the Gyrobarycentric Coordinate Representation Gyrocovariance Theorem 25, p. 609, we find that the $\ominus A_1$ left gyrotranslated concurrency gyropoint H is given gyrobarycentrically by

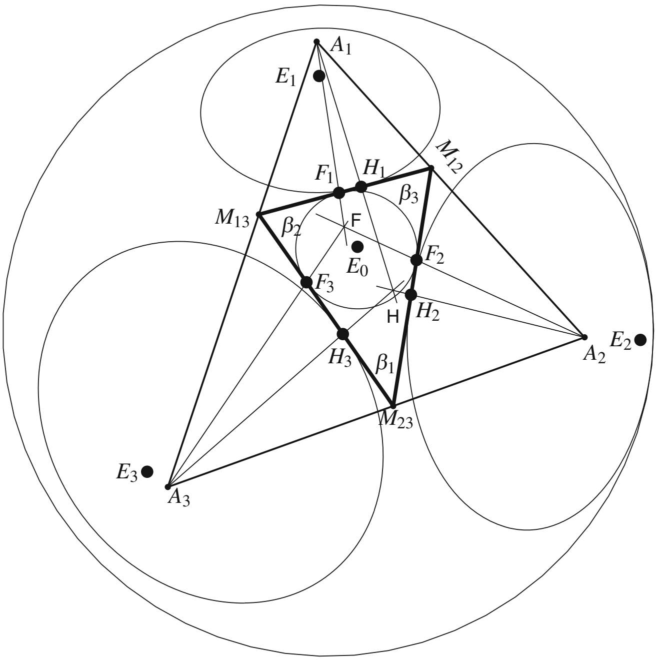


Fig. 14 Cabrera in-exgyrocircle gyropoints. The gyromidpoints, M_{12}, M_{13}, of the gyrosides of gyrotriangle $A_1A_2A_3$ form the gyrovertices of the gyromedial gyrotriangle of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. The gyroangles of the gyromedial gyrotriangle are β_1, β_2, and β_3. The ingyrocenter of gyrotriangle $A_1A_2A_3$ is E_0 and its exgyrocenters are E_1, E_2, and E_3. The tangency gyropoints where the ingyrocircle meets the gyrosides of gyrotriangle $A_1A_2A_3$ are T_k and the tangency gyropoints where the exgyrocircles meet the gyrosides of the gyrotriangle are H_k, $k = 1, 2, 3$. The gyrolines A_1F_1, A_2F_2, and A_3F_3 are concurrent, the concurrency gyropoint being the Cabrera ingyrocircle gyropoint, F, of gyrotriangle $A_1A_2A_3$. Similarly, the gyrolines A_1H_1, A_2H_2, and A_3H_3 are concurrent, the concurrency gyropoint being the Cabrera exgyrocircle gyropoint, H, of gyrotriangle $A_1A_2A_3$. The Cabrera ingyrocircle gyropoint is determined gyrobarycentrically in Theorem 34, p. 647. Similarly, the Cabrera exgyrocircle gyropoint is determined gyrobarycentrically in Theorem 35, p. 654

$$
\begin{aligned}
P &:= \ominus A_1 \oplus H \\
&= \frac{m_{1h}\gamma_{\ominus A_1 \oplus A_1}(\ominus A_1 \oplus A_1) + m_{2h}\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2) + m_{3h}\gamma_{\ominus A_1 \oplus A_3}(\ominus A_1 \oplus A_3)}{m_{1h}\gamma_{\ominus A_1 \oplus A_1} + m_{2h}\gamma_{\ominus A_1 \oplus A_2} + m_{3h}\gamma_{\ominus A_1 \oplus A_3}} \\
&= \frac{m_{2h}\gamma_{12}\mathbf{a}_{12} + m_{3h}\gamma_{13}\mathbf{a}_{13}}{m_{1h} + m_{2h}\gamma_{12} + m_{3h}\gamma_{13}}\,,
\end{aligned}
\tag{346}
$$

noting the trivial equations $\ominus A_1 \oplus A_1 = \mathbf{0} = (0, \ldots, 0)$ and $\gamma_{\mathbf{0}} = 1$.

We assume that gyropoint H lies on each of the three gyrolines A_1H_1, A_2H_2, and A_3H_3, as shown in Fig. 14. This assumption implies that the gyropoint P lies on each of the three lines L_k, $k = 1, 2, 3$ in (342)–(344). Hence, there exist values t_{k0} of the line parameters t_k, $k = 1, 2, 3$, respectively, such that

$$
\begin{aligned}
P - \frac{m_{21}\gamma_{12}\mathbf{a}_{12} + m_{31}\gamma_{13}\mathbf{a}_{13}}{m_{11} + m_{21}\gamma_{12} + m_{31}\gamma_{13}} t_{10} &= 0 \\
P - \mathbf{a}_{12}(1 - t_{20}) - \frac{m_{22}\gamma_{12}\mathbf{a}_{12} + m_{32}\gamma_{13}\mathbf{a}_{13}}{m_{12} + m_{22}\gamma_{12} + m_{32}\gamma_{13}} t_{20} &= 0 \\
P - \mathbf{a}_{13}(1 - t_{30}) - \frac{m_{23}\gamma_{12}\mathbf{a}_{12} + m_{33}\gamma_{13}\mathbf{a}_{13}}{m_{13} + m_{23}\gamma_{12} + m_{33}\gamma_{13}} t_{30} &= 0\,,
\end{aligned}
\tag{347}
$$

where the gyrobarycentric coordinates m_{ij} in (347) are given in (336)–(338).

The kth equation in (347) expresses the condition that point P lies on line L_k, $k = 1, 2, 3$.

The system of Eqs. (347) was obtained by methods of gyroalgebra and will be solved below by a common method of linear algebra.

Substituting P from (346) into (347), and rewriting each equation in (347) as a linear combination of $\mathbf{a}_{12}$ and $\mathbf{a}_{13}$ equals zero, one obtains the following linear homogeneous system of three gyrovector equations

$$
\begin{aligned}
c_{11}\mathbf{a}_{12} + c_{12}\mathbf{a}_{13} &= \mathbf{0} \\
c_{21}\mathbf{a}_{12} + c_{22}\mathbf{a}_{13} &= \mathbf{0} \\
c_{31}\mathbf{a}_{12} + c_{32}\mathbf{a}_{13} &= \mathbf{0}\,,
\end{aligned}
\tag{348}
$$

where each coefficient c_{ij}, $i = 1, 2, 3, j = 1, 2$, is a function of γ_{12}, γ_{13}, γ_{23}, and the six unknowns m_{kh} and t_{k0}, $k = 1, 2, 3$.

Since the set $S = \{A_1, A_2, A_3\}$ is gyrobarycentrically independent, the two gyrovectors $\mathbf{a}_{12} = \ominus A_1 \oplus A_2$ and $\mathbf{a}_{13} = \ominus A_1 \oplus A_3$ in $\mathbb{R}^n_s$, considered as vectors in the ambient space $\mathbb{R}^n$, are linearly independent. Hence, each coefficient c_{ij} in (348) equals zero. Accordingly, the three gyrovector equations in (348) are equivalent to the following six scalar equations,

$$c_{11} = c_{12} = c_{21} = c_{22} = c_{31} = c_{32} = 0 \tag{349}$$

for the six unknowns m_{kh} and t_{k0}, $k = 1, 2, 3$.

The six scalar equations in (349) are not independent. For convenience one may initially select $m_{1h} = 1$ and obtain a unique solution for the remaining unknowns. Finally, owing to the homogeneity of the gyrobarycentric coordinates m_{kh}, one can multiply each of them by a convenient nonzero common factor, obtaining the following gyrobarycentric coordinates:

$$\begin{aligned} m_{1h} &= \frac{1+\gamma_{23}}{D_1 + A(\gamma_{12}+\gamma_{13}-\gamma_{23}-1)} \\ m_{2h} &= \frac{1+\gamma_{13}}{D_2 + A(\gamma_{12}-\gamma_{13}+\gamma_{23}-1)} \\ m_{3h} &= \frac{1+\gamma_{12}}{D_3 + A(-\gamma_{12}+\gamma_{13}+\gamma_{23}-1)}. \end{aligned} \tag{350}$$

As in Sect. 38, it is clear that all the gyrobarycentric coordinates m_{kh}, $k = 1, 2, 3$, in (350) of Cabrera exgyrocircle gyropoint H in (345) are positive so that, by Corollary 2, p. 611, Cabrera exgyrocircle gyropoint always lies on the interior of its reference gyrotriangle $A_1A_2A_3$.

Formalizing the main result of this section, we have the following theorem.

Theorem 35 (Cabrera Exgyrocircle Gyropoint). *The Cabrera exgyrocircle gyropoint H of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Fig. 14, lies on the interior of its reference gyrotriangle $A_1A_2A_3$. It is given by its gyrobarycentric representation with respect to the set $\{A_1, A_2, A_3\}$,*

$$H = \frac{m_{1h}\gamma_{A_1}A_1 + m_{2h}\gamma_{A_2}A_2 + m_{3h}\gamma_{A_3}A_3}{m_{1h}\gamma_{A_1} + m_{2h}\gamma_{A_2} + m_{3h}\gamma_{A_3}}, \tag{351}$$

where

$$\begin{aligned} m_{1h} &= \frac{1+\gamma_{23}}{D_1 + A(\gamma_{12}+\gamma_{13}-\gamma_{23}-1)} = \frac{1+\gamma_{23}}{m_{1c}} \\ m_{2h} &= \frac{1+\gamma_{13}}{D_2 + A(\gamma_{12}-\gamma_{13}+\gamma_{23}-1)} = \frac{1+\gamma_{13}}{m_{2c}} \\ m_{3h} &= \frac{1+\gamma_{12}}{D_3 + A(-\gamma_{12}+\gamma_{13}+\gamma_{23}-1)} = \frac{1+\gamma_{12}}{m_{3c}}. \end{aligned} \tag{352}$$

where, as in (332)–(333),

$$\begin{aligned} A &= 1 + \gamma_{12} + \gamma_{13} + \gamma_{23} \\ B^2 &= 8(1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2) \end{aligned} \tag{353}$$

and

$$\begin{aligned} D_1^2 &= A^2(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)^2 + B^2(1 + \gamma_{23}) \\ D_2^2 &= A^2(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)^2 + B^2(1 + \gamma_{13}) \\ D_3^2 &= A^2(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 + B^2(1 + \gamma_{12}) \,, \end{aligned} \tag{354}$$

$B > 0$, $D_k > 0$, $m_{kh} > 0$, $k = 1, 2, 3$.

Interestingly, $B^2/8$ in (353) can be written as the determinant of an elegant 3×3 gamma matrix Γ_3,

$$\Gamma_3 = \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{pmatrix} . \tag{355}$$

Indeed,

$$\text{Det}\,\Gamma_3 = \begin{vmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{vmatrix} = 1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 > 0 \,. \tag{356}$$

The extension of the gamma matrix Γ_3 of order 3×3 to gamma matrices Γ_N of any order $N \times N$, $N \in \mathbb{N}$, is obvious. These matrices, in turn, along with their determinants and cofactors, play an important role in analytic hyperbolic geometry in N dimensions, as demonstrated in [61, Chap.10].

41 Useful Euclidean Limits

In order to extract the Cabrera point of Euclidean geometry from the Cabrera gyropoint of hyperbolic geometry, in (330)–(333), we need several Euclidean limits, where $s \to \infty$. Since gamma factors tend to 1 as s approaches infinity, we have the following trivial Euclidean limits of expressions that appear in (331)–(333),

$$\lim_{s\to\infty} (1 + \gamma_{ij}) = 2 \,, \tag{357}$$

$i, j = 1, 2, 3$, $i < j$, and by (332),

$$\lim_{s\to\infty} A = \lim_{s\to\infty}(1 + \gamma_{12} + \gamma_{13} + \gamma_{23}) = 4\,. \tag{358}$$

Additionally, we need the nontrivial Euclidean limits that are presented in (369).

Following [61, Eq. (7.148)], we have the Euclidean limit

$$\lim_{s\to\infty} s^2(\gamma_{ij} - 1) = \tfrac{1}{2}a_{ij}^2\,, \tag{359}$$

where $a_{ij} = \|-A_i + A_j\|$ are the side lengths of triangle $A_1A_2A_3$ in a Euclidean plane, as shown in Fig. 15.

Hence, by (359) we have

$$\begin{aligned}\lim_{s\to\infty} s^2(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1) &= \lim_{s\to\infty} s^2\{(\gamma_{12} - 1) + (\gamma_{13} - 1) - (\gamma_{23} - 1)\}\\ &= \tfrac{1}{2}(a_{12}^2 + a_{13}^2 - a_{23}^2)\,,\end{aligned} \tag{360}$$

so that, similarly,

$$\begin{aligned}\lim_{s\to\infty} 4s^4(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)^2 &= (-a_{12}^2 + a_{13}^2 + a_{23}^2)^2\\ \lim_{s\to\infty} 4s^4(\gamma_{12} - \gamma_{13} + \gamma_{23} - 1)^2 &= (a_{12}^2 - a_{13}^2 + a_{23}^2)^2\\ \lim_{s\to\infty} 4s^4(\gamma_{12} + \gamma_{13} - \gamma_{23} - 1)^2 &= (a_{12}^2 + a_{13}^2 - a_{23}^2)^2\,.\end{aligned} \tag{361}$$

Following [61, Eq. (7.156)], we have the Euclidean limit

$$\lim_{s\to\infty} 4s^4(1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2) = E^2\,, \tag{362}$$

where

$$E^2 := (a_{12}+a_{13}+a_{23})(-a_{12}+a_{13}+a_{23})(a_{12}-a_{13}+a_{23})(a_{12}+a_{13}-a_{23}). \tag{363}$$

By Heron's formula

$$E = 4|A_1A_2A_3|\,, \tag{364}$$

where $|A_1A_2A_3|$ is the area of triangle $A_1A_2A_3$.

Hence, by (332), (357), and (362),

$$\begin{aligned}\lim_{s\to\infty} s^4B^2(1 + \gamma_{ij}) &= 2\lim_{s\to\infty} 4s^4(1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2)\lim_{s\to\infty}(1 + \gamma_{ij})\\ &= 4E^2\,.\end{aligned} \tag{365}$$

Hence, by (333), (357)–(358), (361) and (365),

$$\begin{aligned}\lim_{s\to\infty} s^4 D_1^2 &= \lim_{s\to\infty} \frac{A^2}{4} \lim_{s\to\infty} 4s^4(\gamma_{12}+\gamma_{13}-\gamma_{23}-1)^2 + \lim_{s\to\infty} s^4 B^2 \lim_{s\to\infty}(1+\gamma_{23}) \\ &= 4(a_{12}^2+a_{13}^2-a_{23}^2)^2 + 4E^2 \,.\end{aligned} \tag{366}$$

Hence, similarly,

$$\begin{aligned}\lim_{s\to\infty} s^2 D_1 &= 2\sqrt{(a_{12}^2+a_{13}^2-a_{23}^2)^2+E^2} \\ \lim_{s\to\infty} s^2 D_2 &= 2\sqrt{(a_{12}^2-a_{13}^2+a_{23}^2)^2+E^2} \\ \lim_{s\to\infty} s^2 D_3 &= 2\sqrt{(-a_{12}^2+a_{13}^2+a_{23}^2)^2+E^2} \,.\end{aligned} \tag{367}$$

Hence, by means of (331), (367), (332) and (360), we have

$$\begin{aligned}\lim_{s\to\infty} s^2 m_{1c} &= \lim_{s\to\infty} s^2 D_1 + \lim_{s\to\infty} A \lim_{s\to\infty} s^2(\gamma_{12}+\gamma_{13}-\gamma_{23}-1) \\ &= 2\sqrt{(a_{12}^2+a_{13}^2-a_{23}^2)^2+E^2} + 2(a_{12}^2+a_{13}^2-a_{23}^2) \,.\end{aligned} \tag{368}$$

so that, similarly,

$$\begin{aligned}\lim_{s\to\infty} \frac{s^2}{2} m_{1c} &= \sqrt{(a_{12}^2+a_{13}^2-a_{23}^2)^2+E^2} + a_{12}^2+a_{13}^2-a_{23}^2 \\ \lim_{s\to\infty} \frac{s^2}{2} m_{2c} &= \sqrt{(a_{12}^2-a_{13}^2+a_{23}^2)^2+E^2} + a_{12}^2-a_{13}^2+a_{23}^2 \\ \lim_{s\to\infty} \frac{s^2}{2} m_{3c} &= \sqrt{(-a_{12}^2+a_{13}^2+a_{23}^2)^2+E^2} - a_{12}^2+a_{13}^2+a_{23}^2 .\end{aligned} \tag{369}$$

Furthermore, in a similar way, by means of (352), we have the Euclidean limits

$$\begin{aligned}\lim_{s\to\infty} \frac{1}{s^2} m_{1h} &= \frac{1}{\sqrt{(a_{12}^2+a_{13}^2-a_{23}^2)^2+E^2} + a_{12}^2+a_{13}^2-a_{23}^2} \\ \lim_{s\to\infty} \frac{1}{s^2} m_{2h} &= \frac{1}{\sqrt{(a_{12}^2-a_{13}^2+a_{23}^2)^2+E^2} + a_{12}^2-a_{13}^2+a_{23}^2} \\ \lim_{s\to\infty} \frac{1}{s^2} m_{3h} &= \frac{1}{\sqrt{(-a_{12}^2+a_{13}^2+a_{23}^2)^2+E^2} - a_{12}^2+a_{13}^2+a_{23}^2} \,.\end{aligned} \tag{370}$$

42 Cabrera In-Excircle Points

The Cabrera incircle point C' is shown in Fig. 15, where it is denoted by C, and in Fig. 16, where it is denoted by F. the Cabrera excircle point H' is shown in Fig. 16, where it is denoted by H.

Due to their homogeneity, the barycentric coordinates $(m_{1c} : m_{2c} : m_{3c})$ of the Cabrera ingyrocircle gyropoint C in Theorem 34 are equivalent to the barycentric coordinates $((s^2/2)m_{1c} : (s^2/2)m_{2c} : (s^2/2)m_{3c})$. The latter, in turn, have the advantage of possessing the Euclidean limits in (369).

The Cabrera incircle point, C' (C in Fig. 15) is the Euclidean limit of its hyperbolic counterpart in Theorem 34. Hence, following the Euclidean limits in (369), C' is given by its barycentric representation with respect to the set $\{A_1, A_2, A_3\}$ of the vertices of the reference triangle $A_1A_2A_3$,

$$C' = \frac{m'_{1c}A_1 + m'_{2c}A_2 + m'_{3c}A_3}{m'_{1c} + m'_{2c} + m'_{3c}} \tag{371}$$

where its barycentric coordinates are given by

$$\begin{aligned} m'_{1c} &= \lim_{s\to\infty} \frac{s^2}{2} m_{1c} = \sqrt{(a_{12}^2 + a_{13}^2 - a_{23}^2)^2 + E^2} + a_{12}^2 + a_{13}^2 - a_{23}^2 \\ m'_{2c} &= \lim_{s\to\infty} \frac{s^2}{2} m_{2c} = \sqrt{(a_{12}^2 - a_{13}^2 + a_{23}^2)^2 + E^2} + a_{12}^2 - a_{13}^2 + a_{23}^2 \\ m'_{3c} &= \lim_{s\to\infty} \frac{s^2}{2} m_{3c} = \sqrt{(-a_{12}^2 + a_{13}^2 + a_{23}^2)^2 + E^2} - a_{12}^2 + a_{13}^2 + a_{23}^2, \end{aligned} \tag{372}$$

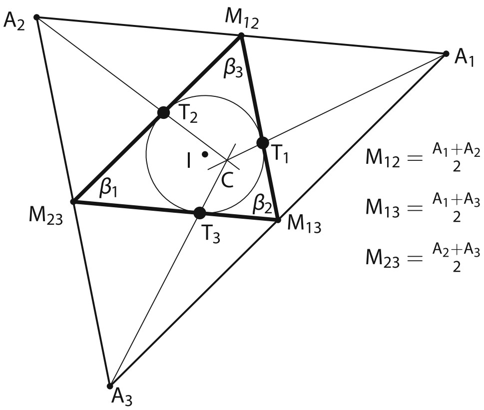


Fig. 15 Cabrera incircle point. The midpoints of the sides of triangle $A_1A_2A_3$ are M_{12}, M_{13}, and M_{23}. The medial triangle of triangle $A_1A_2A_3$ in a Euclidean plane $\mathbb{R}^2$ is $M_{12}M_{13}M_{23}$. The angles of the medial triangle are β_1, β_2, and β_3, its incenter is I, and its incircle tangency points are T_1, T_2 and T_3. The lines A_1T_1, A_2T_2 and A_3T_3 are concurrent. The concurrency point, C, is the Cabrera incircle point of triangle $A_1A_2A_3$, determined barycentrically in Theorem 36, p. 660

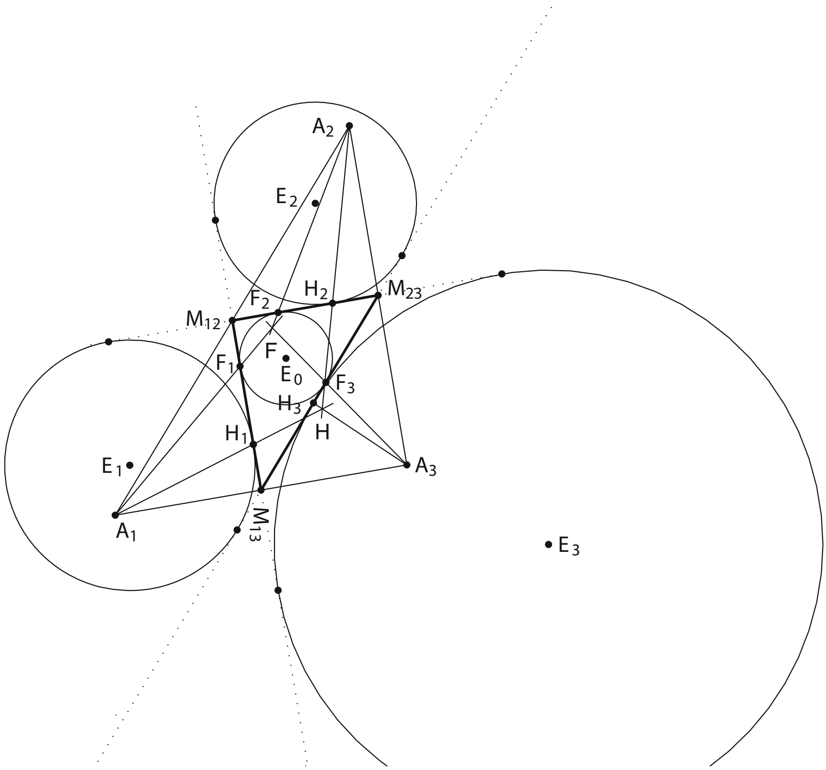


Fig. 16 Cabrera in-excircle points. The midpoints, M_{12}, M_{13}, of the sides of triangle $A_1A_2A_3$ form the vertices of the medial triangle of triangle $A_1A_2A_3$ in a Euclidean plane $\mathbb{R}^2$. The incenter of triangle $A_1A_2A_3$ is E_0 and its excenters are E_1, E_2, and E_3. The tangency points where the incircle meets the sides of triangle $A_1A_2A_3$ are T_k and the tangency points where the excircles meet the sides of the triangle are H_k, $k = 1, 2, 3$. The lines A_1F_1, A_2F_2, and A_3F_3 are concurrent, the concurrency point being the Cabrera incircle point, F, of triangle $A_1A_2A_3$. Similarly, the lines A_1H_1, A_2H_2, and A_3H_3 are concurrent, the concurrency point being the Cabrera excircle point, H, of triangle $A_1A_2A_3$. The Cabrera incircle point F and the Cabrera excircle point H are determined barycentrically in Theorem 36, p. 660

where a_{ij} are the side lengths of the reference triangle $A_1A_2A_3$ and where E^2 is given by (363).

In a similar manner, following the Euclidean limits in (370), the Cabrera excircle point H' (H in Fig. 16) is given by its barycentric representation with respect to the set $\{A_1, A_2, A_3\}$ of the vertices of the reference triangle $A_1A_2A_3$,

$$H' = \frac{m'_{1h}A_1 + m'_{2h}A_2 + m'_{3h}A_3}{m'_{1h} + m'_{2h} + m'_{3h}} \tag{373}$$

where its barycentric coordinates are given by

$$
\begin{aligned}
m'_{1h} &= \frac{1}{\sqrt{(a_{12}^2 + a_{13}^2 - a_{23}^2)^2 + E^2} + a_{12}^2 + a_{13}^2 - a_{23}^2} \\
m'_{2h} &= \frac{1}{\sqrt{(a_{12}^2 - a_{13}^2 + a_{23}^2)^2 + E^2} + a_{12}^2 - a_{13}^2 + a_{23}^2} \\
m'_{3h} &= \frac{1}{\sqrt{(-a_{12}^2 + a_{13}^2 + a_{23}^2)^2 + E^2} - a_{12}^2 + a_{13}^2 + a_{23}^2} .
\end{aligned}
\tag{374}
$$

Formalizing the two main results of this section, we obtain the following theorem.

Theorem 36 (Cabrera In-Excircle Points). *The Cabrera incircle point F and the Cabrera excircle point H of a triangle $A_1A_2A_3$ in a Euclidean plane lie on the interior of their reference triangle $A_1A_2A_3$.*

1. *The Cabrera incircle point F, shown in Fig. 16, is given by its barycentric representation with respect to the set $\{A_1, A_2, A_3\}$,*

$$
F = \frac{m_1A_1 + m_2A_2 + m_3A_3}{m_1 + m_2 + m_3} , \tag{375}
$$

2. *and the Cabrera excircle point H, shown in Fig. 16, is given by its barycentric representation with respect to the set $\{A_1, A_2, A_3\}$,*

$$
H = \frac{m_1^{-1}A_1 + m_2^{-1}A_2 + m_3^{-1}A_3}{m_1^{-1} + m_2^{-1} + m_3^{-1}} , \tag{376}
$$

where

$$
\begin{aligned}
m_1 &= \sqrt{(a_{12}^2 + a_{13}^2 - a_{23}^2)^2 + E^2} + a_{12}^2 + a_{13}^2 - a_{23}^2 \\
m_2 &= \sqrt{(a_{12}^2 - a_{13}^2 + a_{23}^2)^2 + E^2} + a_{12}^2 - a_{13}^2 + a_{23}^2 \\
m_3 &= \sqrt{(-a_{12}^2 + a_{13}^2 + a_{23}^2)^2 + E^2} - a_{12}^2 + a_{13}^2 + a_{23}^2 .
\end{aligned}
\tag{377}
$$

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