Generalized Minkowski Functionals

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In Honor of Constantin Carathéodory

Abstract In the paper we present the generalized Minkowski functionals. We also establish some useful properties of the Minkowski functionals, criterium of the continuity of such functionals, and a generalization of a Kolmogorov result.

1 Introduction

We shall introduce basic ideas, which will be used in the paper.

Let X be a linear topological (Hausdorff) space over the set of real numbers $\mathbb{R}$. Denote $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^+ = [0, \infty]$. Also $0 \cdot \infty = \infty \cdot 0 = 0$. Let A be a subset of X. As usual for $\alpha \in \mathbb{R}$,

$$\alpha A := \{y \in X : y = \alpha x \text{ for } x \in A\}.$$

We shall call A a symmetric, provided that $A = -A$. Moreover, a set $A \subset X$ is said to be bounded (sequentially) (see [6]) iff for every sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to 0$ as $n \to \infty$ and every sequence $\{x_n\} \subset A$, the sequence $\{t_n \cdot x_n\} \subset X$ satisfies $t_n \cdot x_n \to 0$ as $n \to \infty$.

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We also recall the idea of generalized metric space (briefly gms) introduced by Luxemburg (see [5] and also [2]). Let X be a set. A function

$$d: X \times X \to [0, \infty]$$

is called a generalized metric on X, provided that for all $x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$,

A pair (X, d) is called a generalized metric space.

Clearly, every metric space is a generalized metric space.

Analogously, for a linear space X, we can define a generalized norm and a generalized normed space.

Let's note that any generalized metric d is a continuous function.

For if $x_n, x, y_n, y \in (X, d)$ for $n \in \mathbb{N}$ (the set of all natural numbers) and

$$x_n \to x, \quad y_n \to y \quad \text{as} \quad n \to \infty$$

i.e. $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$ as $n \to \infty$, then in the case $d(x, y) < \infty$, we can prove, in standard way, that

$$d(x_n, y_n) \to d(x, y) \quad \text{as} \quad n \to \infty.$$

But if $d(x, y) = \infty$, we have for $\varepsilon > 0$

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

and, for $n > n_0$, $n, n_0 \in \mathbb{N}$

$$\infty = d(x, y) \leq d(x_n, y_n) + \varepsilon,$$

i.e. $d(x_n, y_n) = \infty$ for $n > n_0$ and consequently

$$\infty = d(x_n, y_n) \to d(x, y) = \infty \quad \text{as} \quad n \to \infty,$$

as claimed.

2 Generalized Minkowski Functionals

Now we shall prove the following basic result.

Theorem 1. *Let X be a linear topological (Hausdorff) space over $\mathbb{R}$ and let a subset U of X satisfy the conditions:*

(i) U is a convex (nonempty) set,
(ii) U is a symmetric set.

Then the function $p\colon X \to \mathbb{R}^+$ defined by the formula

$$p(x) := \begin{cases} \inf\{t > 0\colon x \in tU\}, & x \in X, \text{ if } A_x \neq \emptyset, \\ \infty, & \text{if } A_x = \emptyset, \end{cases} \tag{1}$$

where

$$A_x := \{t > 0\colon x \in tU\}, \quad x \in X, \tag{2}$$

has the properties:

$$\text{if} \quad x = 0, \text{ then } p(x) = 0, \tag{3}$$

$$p(\alpha x) = |\alpha| p(x) \text{ for } x \in X \text{ and } \alpha \in \mathbb{R}, \tag{4}$$

$$p(x + y) \leq p(x) + p(y) \text{ for } x, y \in X. \tag{5}$$

Proof. Clearly, $p(x) \in [0, \infty]$ for $x \in X$. Since $0 \in U$, by the definition (1) we get (3). To prove (4), consider at first the case $\alpha > 0$ (if $\alpha = 0$, the property (4) is obvious). Assume that $p(x) < \infty$ for $x \in X$. Then we have

$$\begin{aligned} \alpha p(x) &= \alpha \inf\{s > 0\colon x \in sU\} = \alpha \inf\left\{\frac{t}{\alpha} > 0\colon x \in \frac{t}{\alpha} U\right\} \\ &= \inf\left\{\frac{t}{\alpha} \cdot \alpha > 0\colon x \in \frac{t}{\alpha} U\right\} = \inf\left\{t > 0\colon x \in \frac{t}{\alpha} U\right\} \\ &= \inf\{t > 0\colon \alpha x \in tU\} = p(\alpha x). \end{aligned}$$

If $p(x) = \infty$, then $\{t > 0\colon x \in tU\} = \emptyset$. Therefore,

$$\{t > 0\colon \alpha x \in tU\} = \alpha \left\{\frac{t}{\alpha} > 0\colon \alpha x \in tU\right\} = \alpha \left\{\frac{t}{\alpha} > 0\colon x \in \frac{t}{\alpha} U\right\} = \emptyset,$$

and consequently $p(\alpha x) = \infty$, i.e. (4) holds true.

Now consider the case $\alpha < 0$. Taking into account that (ii) implies that also tU for $t \in \mathbb{R}$ is a symmetric, one gets for $x \in X$ and $p(x) < \infty$

$$p(-x) = \inf\{t > 0\colon -x \in tU\} = \inf\{t > 0\colon x \in tU\} = p(x).$$

If $p(x) = \infty$, then

$$\emptyset = \{t > 0\colon x \in tU\} = \{t > 0\colon -x \in tU\},$$

which implies also $p(-x) = \infty$, and consequently

$$p(-x) = p(x) \quad \text{for any } x \in X. \tag{6}$$

Thus, for $\alpha < 0$, $x \in X$ and in view of the first part of the proof,

$$p(\alpha x) = p(-\alpha x) = -\alpha p(x) = |\alpha| p(x),$$

i.e. (4) has been verified.

Finally, if $p(x) = \infty$ or $p(y) = \infty$, then (5) is satisfied. So assume that $x, y \in X$ and

$$p(x) < \infty \quad \text{and} \quad p(y) < \infty.$$

Take an $\varepsilon > 0$. From the definition (1), there exist numbers $t_1 \geq p(x)$ and $t_2 \geq p(y)$, $t_1 \in A_x$, $t_2 \in A_y$ such that

$$0 < t_1 < p(x) + \frac{1}{2}\varepsilon, \qquad 0 < t_2 < p(y) + \frac{1}{2}\varepsilon.$$

The convexity of U implies that

$$\frac{x+y}{t_1+t_2} = \frac{t_1}{t_1+t_2} \cdot \frac{x}{t_1} + \frac{t_2}{t_1+t_2} \cdot \frac{y}{t_2} \in U$$

and consequently

$$x + y \in (t_1 + t_2)U,$$

which means that $t_1 + t_2 \in A_{x+y}$.

Hence

$$p(x+y) \leq t_1 + t_2 \leq p(x) + \frac{1}{2}\varepsilon + p(y) + \frac{1}{2}\varepsilon = p(x) + p(y) + \varepsilon$$

i.e.

$$p(x+y) \leq p(x) + p(y) + \varepsilon,$$

and since ε is arbitrarily chosen, this concludes the proof. □

Example 1. Consider $X = \mathbb{R} \times \mathbb{R}$, $U = (-1, 1)$.

Then

$$p(x) = \begin{cases} \inf\{t > 0 \colon (x, 0) \in tU\}, & \text{for } x = (x, 0), \\ \infty, & \text{for } x = (x_1, y_1),\ y_1 \neq 0, \end{cases}$$

$$p(x) = \begin{cases} |x|, \text{ for } x = (x, 0), \\ \infty, \text{ for } x = (x_1, y_1),\ y_1 \neq 0, \end{cases}$$

because $\{t > 0 \colon (x_1, y_1) \in tU\} = \emptyset$ for $y_1 \neq 0$.

We see that p is a generalized norm in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (p takes values in $[0, \infty]$).

Remark 1. The function $p: X \to \mathbb{R}^+$ defined by (1) we shall call the generalized Minkowski functional of U (also a generalized seminorm).

Remark 2. Under some stronger assumptions (see e.g. [3]), the function p is called the Minkowski functional of U.

The next basic property of the functional p is given in

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied. If, moreover, U is bounded (sequentially), then*

$$p(x) = 0 \quad \Rightarrow \quad x = 0. \tag{7}$$

Proof. Assume that $p(x) = 0$ for $x \in X$. Suppose that $x \neq 0$. From the definition of $p(x)$ for every $\varepsilon_n = \frac{1}{n}$, there exists a $t_n > 0$ such that $x \in t_n U$, $n \in \mathbb{N}$ and $t_n < \frac{1}{n}$. Hence, $x = t_n x_n$, $x_n \in U$ for $n \in \mathbb{N}$ and by the boundedness of U, $x = t_n x_n \to 0$ as $n \to \infty$. But clearly $x \to x$, whence $x = 0$, which is a contradiction and completes the proof. □

Remark 3. Under the assumptions of Theorem 2, the generalized Minkowski functional is a generalized norm in X.

Let's note the following useful

Lemma 1. *Let $U \subset X$ be a convex set and $0 \in U$. Then*

$$\alpha U \subset U \tag{8}$$

for all $0 \leq \alpha \leq 1$.

The simple proof of this Lemma is omitted here.
Next we prove

Lemma 2. *Let U be as in Theorem 1. If, moreover, U does not contain half-lines, then*

$$p(x) = 0 \quad \Rightarrow \quad x = 0.$$

Proof. For the contrary, suppose that $x \neq 0$. By the definition of $p(x)$ for every $\varepsilon > 0$, there exists a $0 < t < \varepsilon$ such that $x \in tU$. Take $r > 0$ and $\varepsilon < \frac{1}{r}$. Clearly, $\frac{x}{t} \in U$. Furthermore,

$$rx = \frac{x}{t}(tr) = \alpha \frac{x}{t}, \qquad \text{where } \alpha = tr < 1.$$

By Lemma 1, $rx = \alpha\frac{x}{t} \in \alpha U \subset U$, which means that there exists an $x \neq 0$ such that for every $r > 0$, $rx \in U$ what contradicts the assumptions on U. This yields our statement. □

We have also

Lemma 3. *Let U be as in Theorem 1. Then*

$$[p(x) = 0 \;\Rightarrow\; x = 0] \quad \Rightarrow \quad U \text{ does not contain half-lines.} \tag{9}$$

Proof. For the contrary, suppose that there exists an $x \neq 0$ such that for every $r > 0$ we have $rx \in U$. Hence

$$x \in \frac{1}{r}U \quad \text{for} \quad r > 0$$

and therefore

$$\frac{1}{r} \in \{t > 0 : x \in tU\}$$

which implies that $p(x) = 0$. From (9) we get $x = 0$, which is a contradiction. Eventually, one gets the implication (9) and this ends the proof. □

Therefore, Lemmas 2 and 3, we can rewrite as the following

Proposition 1. *Let the assumptions of Theorem 1 be satisfied. Then the generalized Minkowski functional p for U is a generalized norm iff U does not contain half-lines.*

3 Properties of the Generalized Minkowski Functionals

In this part we start with the following

Theorem 3. *Let X be a linear topological (Hausdorff) space over $\mathbb{R}$ and let $f : X \to \mathbb{R}^+$ be any function with properties:*

$$f(\alpha x) = |\alpha| f(x) \text{ for all } x \in X \text{ and } \alpha \in \mathbb{R}, \tag{10}$$

$$f(x + y) \leq f(x) + f(y) \text{ for all } x, y \in X. \tag{11}$$

Define

$$U := \{x \in X : f(x) < 1\}. \tag{12}$$

Then

a) U is a symmetric set,
b) U is a convex (nonempty) set,
c) $f = p$, i.e. f is the generalized Minkowski functional of U.

Proof. The conditions a) and b) follow directly from the definition (12) and properties (10) and (11), respectively. To prove c), assume that $x \in U$, thus $f(x) < \infty$. Therefore, for $t > 0$

$$x \in tU = tf^{-1}([0,1)) \Leftrightarrow \frac{x}{t} \in f^{-1}([0,1))$$
$$\Leftrightarrow f\left(\frac{x}{t}\right) \in [0,1) \Leftrightarrow \frac{1}{t}f(x) \in [0,1) \Leftrightarrow f(x) \in [0,t),$$

i.e. $x \in tU \Leftrightarrow f(x) \in [0,t)$ for $t > 0$.

Thus

$$A_x = \{t > 0: x \in tU\} = \{t > 0: f(x) \in [0,t)\},$$

whence

$$p(x) = \inf A_x = \inf\{t > 0: f(x) \in [0,t)\} = f(x).$$

Now let $f(x) = \infty$. For the contrary, assume that $p(x) < \infty$. Then by the definition of p,

$$\{t > 0: x \in tU\} \neq \emptyset,$$

which implies that there exists $t > 0$ such that $x \in tU$, thus also

$$\frac{x}{t} \in U = f^{-1}([0,1)),$$
$$\frac{1}{t}f(x) \in [0,1)$$

and finally $f(x) \in [0,t)$ which is impossible. This completes the proof. □

The next result reads as follows.

Theorem 4. *Let X, f, U be as in the Theorem 3. If U is sequentially bounded, then $f = p$ is a generalized norm.*

Proof. Assume that $f(x) = p(x) = 0$. One has

$$p(x) = \inf\{t > 0: x \in tU\} = 0,$$

therefore, for every $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$, there exists $0 < t_n < \frac{1}{n}$ such that $x \in t_n U$, i.e. $x = t_n u_n$, where $u_n \in U$ for $n \in \mathbb{N}$. Since U is bounded

$$x = t_n u_n \to 0 \quad \text{as} \quad n \to \infty,$$

thus $x = 0$, as claimed. □

4 Continuity of the Generalized Minkowski Functionals

Let's note the following

Theorem 5. *Let the assumptions of Theorem 1 be satisfied. Then*

$$p \text{ is continuous at zero} \quad \Rightarrow \quad 0 \in \text{int } U. \tag{13}$$

Proof. From the assumption, for $0 < \varepsilon < 1$, there exists a neighbourhood V of zero such that

$$p(u) < \varepsilon \quad \text{for} \quad u \in V.$$

But $p(u) < 1$ for $u \in V$, whence by Lemma 1 $u \in U$, and therefore, $V \subset U$, which proves the implication (13). □

We have also

Theorem 6. *Let the assumptions of Theorem 1 be satisfied. Then*

$$0 \in \text{int } U \quad \Rightarrow \quad p \text{ is continuous at zero.} \tag{14}$$

Proof. Let U_0 be a neighbourhood of zero such that $U_0 \subset U$. For the contrary, suppose that there exists an $\varepsilon_0 > 0$ such that for every neighbourhood V of zero there exists an $x \in V$ with $p(x) \geq \varepsilon_0$. Take $V = V_n = \frac{1}{n} U_0$, $n \in \mathbb{N}$ (clearly V_n is a neighbourhood of zero). Then there exists an $x_n \in \frac{1}{n} U_0 \subset \frac{1}{n} U$, such that

$$p(x_n) \geq \varepsilon_0 \quad \text{for} \quad n \in \mathbb{N}. \tag{15}$$

Take n such that $\frac{1}{n} < \varepsilon_0$. Thus, one has

$$p(x_n) = \inf\{t > 0 \colon x_n \in tU\} \leq \frac{1}{n},$$

i.e.

$$p(x_n) \le \frac{1}{n} < \varepsilon_0$$

which contradicts the inequality (15) and completes the proof. □

We have even more.

Theorem 7. *Let the assumptions of Theorem 1 be satisfied. Then*

$$0 \in int\ U \quad \Rightarrow \quad p \text{ is continuous.} \tag{16}$$

Proof. First of all, observe that since there exists a neighbourhood V of zero, contained in U, then for $x \in X$

$$\frac{1}{n}x \in V \quad \text{for} \quad n > n_0,$$

and hence

$$A_x = \{t > 0 \colon x \in tU\} \neq \emptyset \quad \text{for all } x \in X.$$

Therefore, we have $p(x) < \infty$ for any $x \in X$. Since p is also convex and, by Theorem 6, p is continuous at zero, then by the famous theorem of Bernstein–Doetsch (see e.g. [1]), p is continuous in X, which ends the proof. □

Remark 4. To see that the condition $0 \in int\ U$ is essential in Theorems 5 and 6, the reader is referred to Example 1.

Eventually, taking into account Theorems 5 and 7, we can state the following useful result about the continuity of the generalized Minkowski functionals.

Proposition 2. *Under the assumptions of Theorem 1, the equivalence*

$$p \text{ is continuous} \quad \Leftrightarrow \quad 0 \in int\ U \tag{17}$$

holds true.

5 Kolmogorov Type Result

Let X be a linear space (over $\mathbb{R}$ or $\mathbb{C}$—the set of all complex numbers) and a generalized metric space. We say that X is a generalized linear-metric space, if the operations of addition and multiplication by constant are continuous, i.e. if $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$ and $tx_n \to tx$ (with respect to a generalized metric in X).

For example, if generalized metric is introduced by a generalized norm, then we get a generalized linear-metric space.

We shall prove the following.

Theorem 8. *Let* (X, ϱ) *be a generalized linear-metric space over* $\mathbb{R}$. *Suppose that* $U \subset X$ *is an open, convex and sequentially bounded set. Then there exists a generalized norm* $\|\cdot\|$ *such that the generalized metric induced by this norm is equivalent to a generalized metric* ϱ.

Proof. Take a point $x_0 \in U$, then

$$V := (U - x_0) \cap (x_0 - U)$$

is an open, convex, symmetric and sequentially bounded subset of X (the details we omit here).

Define

$$\|x\| := \begin{cases} \inf\{t > 0 : x \in tV\}, & x \in X, \text{ if } A_x \neq \emptyset, \\ \infty, & \text{if } A_x = \emptyset. \end{cases} \tag{18}$$

By Theorem 2 we see that this function is a generalized norm.

At first we shall show the implication:

$$\varrho(x_n, 0) \to 0 \quad \Rightarrow \quad \|x_n\| \to 0. \tag{19}$$

To this end, take $\varepsilon > 0$. Then the set εV is also open: for it, because $f(x) = \frac{1}{\varepsilon}x$, $x \in X$, is a continuous function and

$$f^{-1}(V) = \varepsilon V,$$

we see that also εV is open. Therefore, $x_n \in \varepsilon V$ for $n > n_0$ and consequently

$$\|x_n\| < \varepsilon \quad \text{for} \quad n > n_0$$

i.e. (19) is satisfied.

Conversely, assume that $\|x_n\| \to 0$ as $n \to \infty$. By the definition (18) for every $\varepsilon_n = \frac{1}{n}$, $n > n_0$, there exists $t_n > 0$ such that

$$\|x_n\| \leq t_n < \|x_n\| + \varepsilon_n \quad \text{and} \quad x_n \in t_n V.$$

Let $\varepsilon_n \to 0$, then $t_n \to 0$ as $n \to \infty$. Also $\frac{x_n}{t_n} \in V$, but since V is bounded, then

$$t_n \left(\frac{x_n}{t_n}\right) = x_n \to 0 \quad \text{as} \quad n \to \infty$$

in the generalized metric ϱ thus $\varrho(x_n, 0) \to 0$, which ends the proof. □

Remark 5. If ϱ is a metric, from Theorem 7, we get the Kolmogorov result (see [4]).

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