Basic Tools, Increasing Functions, and Closure Operations in Generalized Ordered Sets

Árpád Száz

In Honor of Constantin Carathéodory

Abstract Having in mind Galois connections, we establish several consequences of the following definitions.

An ordered pair $X(\leq) = (X, \leq)$ consisting of a set X and a relation $\leq$ on X is called a goset (generalized ordered set).

For any $x \in X$ and $A \subseteq X$, we write $x \in \mathrm{ub}_X(A)$ if $a \leq x$ for all $a \in A$, and $x \in \mathrm{int}_X(A)$ if $\mathrm{ub}_X(x) \subseteq A$, where $\mathrm{ub}_X(x) = \mathrm{ub}_X\big(\{x\}\big)$.

Moreover, for any $A \subseteq X$, we also write $A \in \mathscr{U}_X$ if $A \subseteq \mathrm{ub}_X(A)$, and $A \in \mathscr{T}_X$ if $A \subseteq \mathrm{int}_X(A)$. And in particular, $A \in \mathscr{E}_X$ if $\mathrm{int}_X(A) \neq \emptyset$.

A function f of one goset X to another Y is called increasing if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$.

In particular, an increasing function φ of X to itself is called a closure operation if $x \leq \varphi(x)$ and $\varphi\big(\varphi(x)\big) \leq \varphi(x)$ for all $x \in X$.

The results obtained extend and supplement some former results on increasing functions and can be generalized to relator spaces.

1 Introduction

Ordered sets and *Galois connections* occur almost everywhere in mathematics [12]. They allow of transposing problems and results from one world of our imagination to another one.

Á. Száz (✉)
Institute of Mathematics, University of Debrecen, H-4002 Debrecen, Pf. 400, Hungary
e-mail: szaz@science.unideb.hu

P.M. Pardalos, T.M. Rassias (eds.), *Contributions in Mathematics and Engineering*,
DOI 10.1007/978-3-319-31317-7_28

In [48], having in mind a terminology of Birkhoff [2, p. 1], an ordered pair $X(\leq) = (X, \leq)$ consisting of a set X and a relation $\leq$ on X is called a *goset* (generalized ordered set).

In particular, a goset $X(\leq)$ is called a *proset* (preordered set) if the relation $\leq$ is reflexive and transitive. And, a proset $X(\leq)$ is called a *poset* (partially ordered set) if the relation $\leq$ is in addition antisymmetric.

In a goset X, we may define several algebraic and topological basic tools. For instance, for any $x \in X$ and $A \subseteq X$, we write $x \in \mathrm{ub}_X(A)$ if $a \leq x$ for all $a \in A$, and $x \in \mathrm{int}_X(A)$ if $\mathrm{ub}_X(x) \subseteq A$, where $\mathrm{ub}_X(x) = \mathrm{ub}_X(\{x\})$.

Moreover, we write $A \in \mathscr{U}_X$ if $A \subseteq \mathrm{ub}_X(A)$, $A \in \mathscr{T}_X$ if $A \subseteq \mathrm{int}_X(A)$, and $A \in \mathscr{E}_X$ if $\mathrm{int}_X(A) \neq \emptyset$. However, these families are in general much weaker tools than the relations ub_X and int_X which are actually equivalent tools.

In [58], in accordance with [11, Definition 7.23], an ordered pair (f, g) of functions f of one goset X to another Y and g of Y to X is called a *Galois connection* if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$.

In this case, by taking $\varphi = g \circ f$, we can at once see that $f(u) \leq f(v) \iff u \leq g\big(f(v)\big) \iff u \leq (g \circ f)(v) \iff u \leq \varphi(v)$ for all $u, v \in X$. Therefore, the ordered pair (f, φ) is a *Pataki connection* by a terminology of Száz [58].

A function f of one goset X to another Y is called *increasing* if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$. And, an increasing function φ of X to itself is called a *closure operation* on X if $x \leq \varphi(x)$ and $\varphi\big(\varphi(x)\big) \leq \varphi(x)$ for all $x \in X$.

In [53], we have proved that if (f, φ) is a Pataki connection between the prosets X and Y, then f is increasing and φ is a closure operation such that $f \leq f \circ \varphi$ and $f \circ \varphi \leq f$. Thus, $f = f \circ \varphi$ if in particular Y is a poset.

Moreover, we have also proved that a function φ of a proset X to itself is a closure operation if and only if (φ, φ) is a Pataki connection or equivalently (f, φ) is a Pataki connection for some function f of X to another proset Y.

Thus, increasing functions are, in a certain sense, natural generalizations of not only closure operations but also Pataki and Galois connections. Therefore, it seems plausible to extend some results on these connections to increasing functions.

For instance, having in mind a supremum property of Galois connections [51], we shall show that a function f of one goset X to another Y is increasing if and only if $f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$ for all $A \subseteq X$.

If X is *reflexive* in the sense that the inequality relation in it is reflexive, then we may write max instead of ub. While, if X and Y are sup-*complete* and *antisymmetric* and f is increasing, then we can state that $\sup_Y\big(f[A]\big) \leq f\big(\sup_X(A)\big)$.

Here, the relations $\max_X$ and $\sup_X$ are defined by $\max_X(A) = A \cap \mathrm{ub}_X(A)$ and $\sup_X(A) = \min_X(\mathrm{ub}_X(A)) = \mathrm{ub}_X(A) \cap \mathrm{lb}_X(\mathrm{ub}_X(A))$ for all $A \subseteq X$. Moreover, the goset X is called sup-complete if $\sup_X(A) \neq \emptyset$ for all $A \subseteq X$.

In particular, we shall show that if φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X, then $\varphi\big(\sup_X(A)\big) = \varphi\big(\sup_X(\varphi[A])\big)$ for all $A \subseteq X$. Moreover, if $Y = \varphi[X]$ and $A \subseteq Y$, then $\sup_Y(A) = \varphi\big(\sup_X(A)\big)$.

In addition to the above results, we shall also show that a function f of one goset X to another Y is increasing if and only if $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big)$ for all $A \subseteq X$, or equivalently $f^{-1}[B] \in \mathscr{T}_X$ for all $B \in \mathscr{T}_Y$ if in particular Y is a proset.

Finally, by writing R and S in place of the inequalities in the gosets X and Y, we shall show that a function f of one *simple relator space* $X(R)$ to another $Y(S)$ is increasing if and only if $f \circ R \subseteq S \circ f$, or equivalently $R \subseteq f^{-1} \circ S \circ f$.

The latter fact, together with some basic operations for relators [56], allows of several natural generalizations of the notion of increasingness of functions to pairs $(\mathscr{F}, \mathscr{G})$ of relators on one relator space $(X, Y)(\mathscr{R})$ to another $(Z, W)(\mathscr{S})$.

Here, a family $\mathscr{R}$ of relations on X to Y is called a *relator*, and the ordered pair $(X, Y)(\mathscr{R}) = \big((X, Y), \mathscr{R}\big)$ is called a *relator space*. Thus, relator spaces are substantial generalizations of not only *ordered sets* but also *uniform spaces*.

Moreover, analogously to Galois and Pataki connections [55, 60], increasing functions are also very particular cases of *upper, lower, and mildly semicontinuous pairs of relators*. Unfortunately, these were not considered in [35, 46, 56].

2 Binary Relations and Ordered Sets

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. If in particular $F \subseteq X^2$, with $X^2 = X \times X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation* on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images* of x and A under F, respectively. If $(x, y) \in F$, then we may also write $x\, F\, y$.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *total relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. Thus, it can be shown $F^c = X \times Y \setminus F$ and $F^c[A]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$. (See [57].)

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$. Thus, the operations c and -1 are compatible in the sense $(F^c)^{-1} = (F^{-1})^c$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product relation* $F \boxtimes G$ can be naturally defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$. (See [57].)

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for an arbitrary family of relations too.

If F is a relation on X to Y, then a subset Φ of F is called a *partial selection relation* of F. Thus, we also have $D_\Phi \subseteq D_F$. Therefore, a partial selection relation Φ of F may be called *total* if $D_\Phi = D_F$.

The total selection relations of a relation F will usually be simply called the selection relations of F. Thus, the axiom of choice can be briefly expressed by saying that every relation F has a selection function.

If F is a relation on X to Y and $U \subseteq D_F$, then the relation $F \,|\, U = F \cap (U \times Y)$ is called the *restriction* of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subseteq D_G$ and $F = G \,|\, D_F$, then G is called an *extension* of F.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^\diamond$ of X to $\mathscr{P}(Y)$ and $F^{\Diamond}$ of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$, such that $F^\diamond(x) = F(x)$ for all $x \in X$ and $F^{\Diamond}(A) = F[A]$ for all $A \subset X$.

Functions of X to $\mathscr{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ are more general objects than relations on X to Y. They were called *corelations* on X to Y in [59].

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For instance, for $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder relation on X. (See [24, 52].) While, for a *pseudo-metric* d on X and $r > 0$, the *surrounding* $B_r^d = \big\{(x, y) \in X^2 : \ d(x, y) < r\big\}$ is a tolerance relation on X.

Moreover, we may recall that if $\mathscr{A}$ is a *partition* of X, i. e., a family of pairwise disjoint, nonvoid subsets of X such that $X = \bigcup \mathscr{A}$, then $S_{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} A^2$ is an equivalence relation on X, which can, to some extent, be identified with $\mathscr{A}$.

According to algebra, for any relation R on X, we may naturally define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also naturally define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is the smallest preorder relation containing R [16].

Note that R is a preorder on X if and only if $R = R^\infty$. Moreover, $R^\infty = R^{\infty\infty}$ and $(R^\infty)^{-1} = (R^{-1})^\infty$. Therefore, R^{-1} is also a preorder on X if R is a preorder on X. Moreover, R^∞ is already an equivalence on X if R is symmetric.

According to [48], an ordered pair $X(\leq) = \big(X, \leq\big)$, consisting of a set X and a relation $\leq$ on X, will be called a *generalized ordered set* or an *ordered set without axioms*. And, we shall usually write X in place of $X(\leq)$.

In the sequel, a generalized ordered set $X(\leq)$ will, for instance, be called *reflexive* if the relation $\leq$ is reflexive on X. Moreover, it is called a preordered (partially ordered) set if $\leq$ is a preorder (partial order) on X.

Having in mind a widely used terminology of Birkhoff [2, p. 1], a generalized ordered set will be briefly called a *goset*. Moreover, a preordered (partially ordered) set will be call a *proset (poset)*.

Thus, every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the *universal relation* X^2. And, the *power set* $\mathscr{P}(X)$ of X is a poset with the ordinary set inclusion $\subseteq$.

In this respect, it is also worth mentioning that if in particular X a goset, then for any $A, B \subseteq X$ we may also naturally write $A \leq B$ if $a \leq b$ for all $a \in A$ and $b \in B$. Thus, $\mathscr{P}(X)$ is also a goset with this extended inequality.

Moreover, if $X(\leq)$ is a goset and $Y \subseteq X$, then by taking $\leq_Y = \leq \cap Y^2$, we can also get a goset $Y(\leq_Y)$. This *subgoset* inherits several properties of the original goset. Thus, for instance, every family of sets is a poset with set inclusion.

In the sequel, trusting to the reader's good sense to avoid confusions, for any goset $X(\leq)$ and operation $\star$ on relations on X, we shall use the notation $X^\star$ for the goset $X(\leq^\star)$. Thus, for instance, X^{-1} will be called the *dual* of the goset X.

Several definitions on posets can be naturally extended to gosets [48]. And, even to arbitrary *relator spaces* [47] which include *ordered sets* [11], *context spaces* [15], and *uniform spaces* [14] as the most important particular cases.

Moreover, most of the definitions can also be naturally extended to *corelator spaces* $(X, Y)(\mathscr{U}) = \big((X, Y), \mathscr{U}\big)$ consisting of two sets X and Y and a family $\mathscr{U}$ of corelations on X to Y. However, it is convenient to investigate first gosets.

3 Upper and Lower Bounds

According to [48], for instance, we may naturally introduce the following

Definition 1. For any subset A of a goset X, the elements of the sets

$$\mathrm{ub}_X(A) = \big\{x \in X : \quad A \leq \{x\}\big\} \qquad \text{and} \qquad \mathrm{lb}_X(A) = \big\{x \in X : \quad \{x\} \leq A\big\}$$

will be called the *upper and lower bounds* of the set A in X, respectively.

Remark 1. Thus, for any $x \in X$ and $A \subseteq X$, we have

(1) $x \in \mathrm{ub}_X(A)$ if and only if $a \leq x$ for all $a \in A$,
(2) $x \in \mathrm{lb}_X(A)$ if and only if $x \leq a$ for all $a \in A$.

Remark 2. Hence, by identifying singletons with their elements, we can see that

(1) $\mathrm{ub}_X(x) = \leq(x) = [\,x, +\infty\,[= \big\{y \in X : \quad x \leq y\big\}$,
(2) $\mathrm{lb}_X(x) = \geq(x) = \,]-\infty, x\,] = \big\{y \in X : \quad x \geq y\big\}$.

This shows that the relation ub_X is somewhat more natural tool than lb_X.

By using Remark 1, we can easily establish the following

Theorem 1. *For any subset A of a goset X, we have*

(1) $\mathrm{ub}_X(A) = \mathrm{lb}_{X^{-1}}(A)$,
(2) $\mathrm{lb}_X(A) = \mathrm{ub}_{X^{-1}}(A)$.

Proof. If $x \in \mathrm{ub}_X(A)$, then by Remark 1 we have $a \leq x$ for all $a \in A$. This implies that $x \leq^{-1} a$ for all $a \in A$. Hence, since $X^{-1} = X(\leq^{-1})$, we can already see that $x \in \mathrm{lb}_{X^{-1}}(A)$. Therefore, $\mathrm{ub}_X(A) \subseteq \mathrm{lb}_{X^{-1}}(A)$.

The converse inclusion can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by taking X^{-1} in place of X.

Remark 3. This theorem shows that the relations ub_X and lb_X are equivalent tools in the goset X.

By using Remark 1, we can also easily establish the following theorem.

Theorem 2. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\mathrm{ub}_Y(A) = \mathrm{ub}_X(A) \cap Y$,
(2) $\mathrm{lb}_Y(A) = \mathrm{lb}_X(A) \cap Y$.

Concerning the relations ub_X and lb_X, we can also easily prove the following theorem.

Theorem 3. *For any family $(A_i)_{i \in I}$ subsets of a goset X, we have*

(1) $\mathrm{ub}_X\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \mathrm{ub}_X\big(A_i\big)$,
(2) $\mathrm{lb}_X\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \mathrm{lb}_X\big(A_i\big)$.

Proof. If $x \in \mathrm{ub}_X\big(\bigcup_{i \in I} A_i\big)$, then by Remark 1 we have $a \leq x$ for all $a \in \bigcup_{i \in I} A_i$. Hence, it is clear that we also have $a \leq x$ for all $a \in A_i$ with $i \in I$. Therefore, $x \in \mathrm{ub}_X(A_i)$ for all $i \in I$, and thus $x \in \bigcap_{i \in I} \mathrm{ub}_X(A_i)$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 1.

From the above theorem, by identifying singletons with their elements, we can immediately derive the following corollary.

Corollary 1. *For any subset A of a goset X, we have*

(1) $\mathrm{ub}_X(A) = \bigcap_{a \in A} \mathrm{ub}_X(a)$,
(2) $\mathrm{lb}_X(A) = \bigcap_{a \in A} \mathrm{lb}_X(a)$.

Remark 4. Hence, by using Remark 2 and a basic fact on complement relations mentioned in Sect. 2, we can immediately derive that

(1) $\mathrm{ub}_X(A) = \leq^c [A]^c$. (2) $\mathrm{lb}_X(A) = \geq^c [A]^c$.

From Corollary 1, we can also immediately derive the first two assertions of

Theorem 4. *If X is a goset, then*

(1) $\mathrm{ub}_X(\emptyset) = X$ *and* $\mathrm{lb}_X(\emptyset) = X$,
(2) $\mathrm{ub}_X(B) \subseteq \mathrm{ub}_X(A)$ *and* $\mathrm{lb}_X(B) \subseteq \mathrm{lb}_X(A)$ *if* $A \subseteq B \subseteq X$,
(3) $\bigcup_{i \in I} \mathrm{ub}_X\big(A_i\big) \subseteq \mathrm{ub}_X\Big(\bigcap_{i \in I} A_i\Big)$ *and* $\bigcup_{i \in I} \mathrm{lb}_X\big(A_i\big) \subseteq \mathrm{lb}_X\Big(\bigcap_{i \in I} A_i\Big)$ *if* $A_i \subseteq X$ *for all* $i \in I$.

Proof. To prove the first part of (3), we can note that if $A_i \subseteq X$ for all $i \in I$, then $\bigcap_{i \in I} A_i \subseteq A_i$ for all $i \in I$. Hence, by using (2), we can already infer that $\mathrm{ub}_X(A_i) \subseteq \mathrm{ub}_X\big(\bigcap_{i \in I} A_i\big)$ for all $i \in I$, and thus the required inclusion is also true.

However, it is now more important to note that, as an immediate consequence of the corresponding definitions, we can also state the following theorem which actually implies most of the properties of the relations ub_X and lb_X.

Theorem 5. *For any two subsets A and B of a goset X, we have*

$$B \subseteq \mathrm{ub}_X(A) \quad \Longleftrightarrow \quad A \subseteq \mathrm{lb}_X(B)\,.$$

Proof. By Remark 1, it is clear that each of the above inclusions is equivalent to the property that $a \leq b$ for all $a \in A$ and $b \in B$.

Remark 5. This property can be briefly expressed by writing that $A \leq B$, or equivalently $A \times B \subseteq \leq$, that is, $B \in \mathrm{Ub}_X(A)$, or equivalently $A \in \mathrm{Lb}_X(B)$ by the notations of our former paper [47].

From Theorem 5, it is clear that in particular we have

Corollary 2. *For any subset A of a goset X, we have*

(1) $\mathrm{ub}_X(A) = \big\{x \in X : \; A \subseteq \mathrm{lb}_X(x)\big\}$,
(2) $\mathrm{lb}_X(A) = \big\{x \in X : \; A \subseteq \mathrm{ub}_X(x)\big\}$.

Remark 6. Moreover, from Theorem 5, we can see that, for any $A, B \subseteq X$, we have

$$\mathrm{lb}_X(A) \subseteq^{-1} B \quad \Longleftrightarrow \quad A \subseteq \mathrm{ub}_X(B)\,.$$

This shows that the set-valued functions lb_X and ub_X form a *Galois connection* between the poset $\mathscr{P}(X)$ and its dual in the sense of [11, Definition 7.23], suggested by Schmidt's reformulation [36, p. 209] of Ore's definition of Galois connexions [30].

Remark 7. Hence, by taking $\Phi_X = \mathrm{ub}_X \circ \mathrm{lb}_X$, for any $A, B \subseteq X$, we can infer that

$$\mathrm{lb}_X(A) \subseteq^{-1} \mathrm{lb}_X(B) \quad \Longleftrightarrow \quad A \subseteq \Phi_X(B)\,.$$

This shows that the set-valued functions lb_X and Φ_X form a *Pataki connection* between the poset $\mathscr{P}(X)$ and its dual in the sense of [51, Remark 3.8] suggested by a fundamental unifying work of Pataki [32] on the basic refinements of relators studied each separately by the present author in [42].

Remark 8. By [53, Theorem 4.7], this fact implies that $\mathrm{lb}_X = \mathrm{lb}_X \circ \Phi_X$, and Φ_X is a *closure operation* on the poset $\mathscr{P}(X)$ in the sense of [2, p. 111].

By an observation, attributed to Richard Dedekind by Erné [12, p. 50], this is equivalent to the requirement that the set function Φ_X with itself forms a Pataki connection between the poset $\mathscr{P}(X)$ and itself.

4 Interiors and Closures

Because of Remark 2, we may also naturally introduce the following

Definition 2. For any subset A of a goset X, the sets

$$\operatorname{int}_X(A) = \big\{ x \in X : \quad \operatorname{ub}_X(x) \subseteq A \big\} \quad \text{and} \quad \operatorname{cl}_X(A) = \big\{ x \in X : \quad \operatorname{ub}_X(x) \cap A \neq \emptyset \big\}$$

will be called the *interior and closure* of the set A in X, respectively.

Remark 9. Recall that, by Remark 2, we have $\operatorname{ub}_X(x) = \leq (x) = [\, x, \ +\infty \,[$ for all $x \in X$.

Therefore, the present one-sided interiors and closures, when applied to subsets of the real line $\mathbb{R}$, greatly differ from the usual ones.

The latter ones can only be derived from a *relator* (family of relations) which has to consist of at least countable many tolerance or preorder relations.

By using Definition 2, we can easily prove the following theorem.

Theorem 6. *For any subset* A *of a goset* X, *we have*

(1) $\operatorname{int}_X(A) = X \setminus \operatorname{cl}_X(X \setminus A)$,
(2) $\operatorname{cl}_X(A) = X \setminus \operatorname{int}_X(X \setminus A)$.

Proof. If $x \in \operatorname{int}_X(A)$, then by Definition 2 we have $\operatorname{ub}(x) \subseteq A$. Hence, we can infer that $\operatorname{ub}(x) \cap (X \setminus A) = \emptyset$. Therefore, by Definition 2, we have $x \notin \operatorname{cl}_X(X \setminus A)$, and thus $x \in X \setminus \operatorname{cl}_X(X \setminus A)$. This shows that $\operatorname{int}_X(A) \subseteq X \setminus \operatorname{cl}_X(X \setminus A)$.

The converse inclusion can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by writing $X \setminus A$ in place of A, and applying complementation.

Remark 10. This theorem shows that the relations int_X and cl_X are also equivalent tools in the goset X.

By using the complement operation $\mathscr{C}$, defined by $\mathscr{C}(A) = A^c = X \setminus A$ for all $A \subseteq X$, the above theorem can be reformulated in a more concise form.

Corollary 3. *For any goset* X, *we have*

(1) $\operatorname{int}_X = \big(\operatorname{cl}_X \circ \mathscr{C}\big)^c = \operatorname{cl}_X^c \circ \mathscr{C}$,
(2) $\operatorname{cl}_X = \big(\operatorname{int}_X \circ \mathscr{C}\big)^c = \operatorname{int}_X^c \circ \mathscr{C}$.

Proof. To prove the second part of (1), note that by the corresponding definitions, for any $A \subseteq X$, we have

$$\big(\operatorname{cl}_X \circ \mathscr{C}\big)^c(A) = \big(\operatorname{cl}_X \circ \mathscr{C}\big)(A)^c = \operatorname{cl}_X\big(\mathscr{C}(A)\big)^c = \operatorname{cl}_X^c\big(\mathscr{C}(A)\big) = \big(\operatorname{cl}_X^c \circ \mathscr{C}\big)(A).$$

Now, in contrast to Theorems 1 and 2, we can only state the following two theorems.

Theorem 7. *For any subset A of a goset X, we have*

(1) $\operatorname{int}_{X^{-1}}(A) = \{x \in X : \ \operatorname{lb}_X(x) \subseteq A\}$,
(2) $\operatorname{cl}_{X^{-1}}(A) = \{x \in X : \ \operatorname{lb}_X(x) \cap A \neq \emptyset\}$.

Theorem 8. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\operatorname{int}_X(A) \cap Y \subseteq \operatorname{int}_Y(A)$,
(2) $\operatorname{cl}_Y(A) \subseteq \operatorname{cl}_X(A) \cap Y$.

However, concerning the relations int_X and cl_X, we can also easily prove

Theorem 9. *For any family $(A_i)_{i \in I}$ subsets of a goset X, we have*

(1) $\operatorname{int}_X\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} \operatorname{int}_X(A_i)$,
(2) $\operatorname{cl}_X\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} \operatorname{cl}_X(A_i)$.

Proof. If $x \in \operatorname{int}_X\big(\bigcap_{i\in I} A_i\big)$, then by Definition 2 we have $\operatorname{ub}_X(x) \subseteq \bigcap_{i\in I} A_i$. Therefore, $\operatorname{ub}_X(x) \subseteq A_i$, and thus $x \in \operatorname{int}_X(A_i)$ for all $i \in I$. Therefore, $x \in \bigcap_{i\in I} \operatorname{int}_X(A_i)$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 6.

Remark 11. This theorem shows that, despite Remark 10, there are cases when the relation cl_X is a more convenient tool than int_X.

Namely, from assertion (2), by identifying singletons with their elements, we can immediately derive the following corollary.

Corollary 4. *For any subset A of a goset X, we have*

$$\operatorname{cl}_X(A) = \bigcup_{a\in A} \operatorname{cl}_X(a).$$

Remark 12. Note that, for any $x, y \in X$, we have

$$y \in \operatorname{cl}_X(x) \iff \operatorname{ub}_X(y) \cap \{x\} \neq \emptyset \iff x \in \operatorname{ub}_X(y) \iff y \in \operatorname{lb}_X(x),$$

and thus also $\operatorname{cl}_X(x) = \operatorname{lb}_X(x)$. Hence, by using Theorem 1, we can immediately infer that $\operatorname{cl}_X(x) = \operatorname{ub}_{X^{-1}}(x)$.

Therefore, as an immediate consequence of the above results, we can also state

Theorem 10. *For any subset A of a goset X, we have*

$$\operatorname{cl}_X(A) = \bigcup_{a\in A} \operatorname{lb}_X(a) = \bigcup_{a\in A} \operatorname{ub}_{X^{-1}}(a).$$

Remark 13. Hence, by using Remark 2 and Theorem 1, we can at once see that

$$\mathrm{cl}_X(A) = \bigcup_{a \in A} \geq (a) = \geq [A] \qquad \text{and} \qquad \mathrm{cl}_{X^{-1}}(A) = \bigcup_{a \in A} \leq (a) = \leq [A] .$$

And thus, by Theorem 6, also $\mathrm{int}_X(A) = \geq [A^c]^c$ and $\mathrm{int}_{X^{-1}}(A) = \leq [A^c]^c$.

Now, analogously to Theorem 4, we can also easily establish the following

Theorem 11. *If X is a goset, then*

(1) $\mathrm{cl}_X(\emptyset) = \emptyset$ *and* $\mathrm{int}_X(X) = X$,
(2) $\mathrm{cl}_X(A) \subseteq \mathrm{cl}_X(B)$ *and* $\mathrm{int}_X(A) \subseteq \mathrm{int}_X(B)$ *if* $A \subseteq B \subseteq X$,
(3) $\mathrm{cl}_X\Big(\bigcap_{i \in I} A_i\Big) \subseteq \bigcap_{i \in I} \mathrm{cl}_X\big(A_i\big)$ *and* $\bigcup_{i \in I} \mathrm{int}_X\big(A_i\big) \subseteq \mathrm{int}_X\Big(\bigcup_{i \in I} A_i\Big)$ *if* $A_i \subseteq X$ *for all* $i \in I$.

However, it is now more important to note that, analogously to Theorem 5, we also have the following theorem which actually implies most of the properties of the relations int_X and cl_X.

Theorem 12. *For any two subsets A and B of a goset X, we have*

$$B \subseteq \mathrm{int}_X(A) \quad \Longleftrightarrow \quad \mathrm{cl}_{X^{-1}}(B) \subseteq A .$$

Proof. If $B \subseteq \mathrm{int}_X(A)$, then by Definition 2, we have $\mathrm{ub}_X(b) \subseteq A$ for all $b \in B$. Hence, by Theorem 10, we can already see that $\mathrm{cl}_{X^{-1}}(B) = \bigcup_{b \in B} \mathrm{ub}_X(b) \subseteq A$.

The converse implication can be proved quite similarly by reversing the above argument.

Remark 14. Recall that, by Remark 13, we have $\mathrm{cl}_{X^{-1}}(B) = \leq [B]$. Therefore, by Theorem 12, the inclusion $B \subseteq \mathrm{int}_X(A)$ can also be reformulated by stating that $\leq [B] \subseteq A$, or equivalently $\leq [B] \cap A^c = \emptyset$. That is, $B \in \mathrm{Int}_X(A)$, or equivalently $B \notin \mathrm{Cl}_X(A^c)$ by the notations of Száz [47].

From Theorem 12, it is clear that in particular we have

Corollary 5. *For any subset A of a goset X, we have*

$$\mathrm{int}_X(A) = \big\{ x \in X : \quad \mathrm{cl}_{X^{-1}}(x) \subseteq A \big\} .$$

Remark 15. From Theorem 12, we can also see that, for any A, $B \subseteq X$, we have

$$\mathrm{cl}_{X^{-1}}(A) \subseteq B \quad \Longleftrightarrow \quad A \subseteq \mathrm{int}_X(B) .$$

This shows that the set-valued functions $\mathrm{cl}_{X^{-1}}$ and int_X form a Galois connection between the poset $\mathscr{P}(X)$ and itself.

Remark 16. Thus, by taking $\Phi_X = \text{int}_X \circ \text{cl}_{X^{-1}}$, for any $A, B \subseteq X$ we can state that

$$\text{cl}_{X^{-1}}(A) \subseteq \text{cl}_{X^{-1}}(B) \iff A \subseteq \Phi_X(B).$$

This shows that the set-valued functions $\text{cl}_{X^{-1}}$ and Φ_X form a Pataki connection between the poset $\mathscr{P}(X)$ and itself. Thus, $\text{cl}_{X^{-1}} = \text{cl}_{X^{-1}} \circ \Phi_X$, and Φ_X is closure operation on the poset $\mathscr{P}(X)$.

Remark 17. The upper- and lower-bound Galois connection, described in Remark 6, was first studied by Birkhoff [2, p. 122] under the name *polarities*.

While, the closure–interior Galois connection, described in Remark 15, has been only considered in [61] with reference to Davey and Priestly [11, Exercise 7.18].

5 Open and Closed Sets

Definition 3. For any goset X, the members of the families

$$\mathscr{T}_X = \{A \subseteq X : \quad A \subseteq \text{int}_X(A)\} \qquad \text{and} \qquad \mathscr{F}_X = \{A \subseteq X : \quad \text{cl}_X(A) \subseteq A\}$$

are called the *open and closed subsets* of X, respectively.

Remark 18. Thus, by Definition 2 and Theorem 10, for any $A \subseteq X$, we have

(1) $A \in \mathscr{T}_X$ if and only if $\text{ub}_X(a) \subseteq A$ for all $a \in A$.
(2) $A \in \mathscr{F}_X$ if and only if $\text{lb}_X(a) \subseteq A$ for all $a \in A$.

Namely, by Definition 2, for any $a \in A$ we have $a \in \text{int}_X(A)$ if and only if $\text{ub}_X(a) \subseteq A$. Moreover, by Theorem 10, we have $\text{cl}_X(A) = \bigcup_{a \in A} \text{lb}_X(a)$.

Remark 19. Because of Remarks 2 and 18, the members of the families $\mathscr{T}_X$ and $\mathscr{F}_X$ may also be called the *ascending and descending subsets* of X.

Namely, for instance, by the above mentioned remarks, for any $A \subseteq X$ we have $A \in \mathscr{T}_X$ if and only if for any $a \in A$ and $x \in X$, with $a \leq x$, we also have $x \in A$.

Remark 20. Moreover, from Remarks 2 and 18, we can also see that
(1) $\mathscr{T}_X = \{A \subseteq X : \ \leq [A] \subseteq A\}$. (2) $\mathscr{F}_X = \{A \subseteq X : \ \geq [A] \subseteq A\}$.
Namely, for instance, by a basic definition on relations and Remark 2, for any $A \subseteq X$ we have $\leq [A] = \bigcup_{a \in A} \leq (a) = \bigcup_{a \in A} \text{ub}_X(a)$.

By using Definition 3 and Theorem 6, we can also easily prove the following theorem.

Theorem 13. *For any goset X, we have*

(1) $\mathscr{T}_X = \{A \subseteq X : \quad A^c \in \mathscr{F}_X\}$,
(2) $\mathscr{F}_X = \{A \subseteq X : \quad A^c \in \mathscr{T}_X\}$.

Proof. If $A \in \mathscr{T}_X$, then by Definition 3 we have we have $A \subseteq \mathrm{int}_X(A)$. Hence, by using Theorem 6, we can infer that $\mathrm{cl}_X(A^c) = \mathrm{int}_X(A)^c \subseteq A^c$. Therefore, by Definition 3, the inclusion $A^c \in \mathscr{F}_X$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 6.

Remark 21. This theorem shows that the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are also equivalent tools in the goset X.

By using the element-wise complementation, defined by $\mathscr{A}^c = \{A^c : A \in \mathscr{A}\}$ for all $\mathscr{A} \subseteq \mathscr{P}(X)$, Theorem 13 can also be reformulated in a more concise form.

Corollary 6. *For any goset X, we have*

(1) $\mathscr{T}_X = \mathscr{F}_X{}^c$,
(2) $\mathscr{F}_X = \mathscr{T}_X{}^c$.

Now, as an immediate consequence of Remark 20, we can also state the following theorem which can also be easily proved with the help of Definition 3 and Theorem 12.

Theorem 14. *For any goset X, we have*

(1) $\mathscr{T}_X = \mathscr{F}_{X^{-1}}$,
(2) $\mathscr{F}_X = \mathscr{T}_{X^{-1}}$.

Proof. If $A \in \mathscr{T}_X$, then by Definition 3, we have $A \subseteq \mathrm{int}_X(A)$. Hence, by using Theorem 12, we can infer that $\mathrm{cl}_{X^{-1}}(A) \subseteq A$. Therefore, $A \in \mathscr{F}_{X^{-1}}$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by writing X^{-1} in place of X.

Remark 22. Moreover, because of Remark 14 and Theorem 13, for any $A \subseteq X$ we can also state that $A \in \mathscr{T}_X$ if and only if $A \in \mathrm{Int}_X(A)$, and $A \in \mathscr{F}_X$ if and only if $A^c \notin \mathrm{Cl}_X(A)$.

By using Definition 3 and Theorem 8, we can easily establish the following theorem.

Theorem 15. *For any subset Y of a goset X, we have*

(1) $\mathscr{T}_X \cap \mathscr{P}(Y) \subseteq \mathscr{T}_Y$,
(2) $\mathscr{F}_X \cap \mathscr{P}(Y) \subseteq \mathscr{F}_Y$.

Proof. Namely, if, for instance, $A \in \mathscr{T}_X \cap \mathscr{P}(Y)$, then $A \in \mathscr{T}_X$ and $A \in \mathscr{P}(Y)$. Therefore, $A \subseteq \mathrm{int}_X(A)$ and $A \subseteq Y$. Hence, by Theorem 8, we can already see that $A \subset \mathrm{int}_X(A) \cap Y \subseteq \mathrm{int}_Y(A)$, and thus $A \in \mathscr{T}_Y$ also holds.

Moreover, by using Definition 3 and Theorems 9 and 11, we can also easily prove the following.

Theorem 16. *For any goset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are ultratopologies [10] (complete rings [1]) in the sense that they are closed under arbitrary unions and intersections.*

Proof. Namely, if, for instance, $A_i \in \mathscr{T}_X$ for all $i \in I$, then $A_i \subseteq \text{int}_X(A_i)$ for all $i \in I$. Hence, by using Theorems 9 and 11, we can already infer that

$$\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \text{int}_X(A_i) = \text{int}_X\Big(\bigcap_{i \in I} A_i\Big) \quad \text{and} \quad \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} \text{int}_X(A_i) \subseteq \text{int}_X\Big(\bigcup_{i \in I} A_i\Big).$$

Therefore, the sets $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are also in $\mathscr{T}_X$.

Remark 23. From the above theorem, by taking the empty subfamily of $\mathscr{T}_X$ and $\mathscr{F}_X$, we can immediately infer that $\{\emptyset, X\} \subseteq \mathscr{T}_X \cap \mathscr{F}_X$.

Finally, we note that the following theorem is also true

Theorem 17. *For any subset A of a goset X, we have*

(1) $\bigcup \mathscr{T}_X \cap \mathscr{P}(A) \subseteq \text{int}_X(A)$,
(2) $\text{cl}_X(A) \subseteq \bigcap \mathscr{F}_X \cap \mathscr{P}^{-1}(A)$.

Proof. Define $B = \bigcup \mathscr{T}_X \cap \mathscr{P}(A)$. Then, we evidently have $B \subseteq A$. Moreover, by Theorem 16, we can see that $B \in \mathscr{T}_X$. Hence, by using Definition 3 and Theorem 11, we can already infer that $B \subseteq \text{int}_X(B) \subseteq \text{int}_X(A)$. Therefore, (1) is true.

Moreover, from (1), by using Theorem 12 and the fact that $U \in \mathscr{P}^{-1}(V)$ if and only if $V \subseteq U$, we can easily see that (2) is also true.

Example 1. If, for instance, $X = \mathbb{R}$ and $\preceq$ is a relation on X such that

$$\preceq(x) = \{x-1\} \cup [x, +\infty[$$

for all $x \in X$, then by using Remarks 2 and 18 we can easily see that $\mathscr{T}_X = \{\emptyset, X\}$, and thus by Corollary 6 also $\mathscr{F}_X = \{\emptyset, X\}$.

Namely, if $A \in \mathscr{T}_X$ such that $A \neq \emptyset$, then there exists $x \in X$ such that $x \in A$, and thus by the abovementioned remarks $\preceq(x) = \text{ub}_X(x) \subseteq A$. Therefore,

$$\{x-1\} \cup [x, +\infty[\subseteq A.$$

Hence, we can see that $x - 1 \in A$. Therefore, $\preceq(x-1) \subseteq A$, and thus

$$\{x-2\} \cup [x-1, +\infty[\subseteq A.$$

Hence, by induction, it is clear that for any $n \in \mathbb{N}$ we also have

$$\{x-n-1\} \cup [x-n, +\infty[\subseteq A.$$

Thus, by the Archimedean property of $\mathbb{N}$ in $\mathbb{R}$, we necessarily have $A = X$.

Now, by using that $\mathscr{F}_X = \{ \emptyset, X \}$, we can easily see that

$$\bigcup \mathscr{F}_X \cap \mathscr{P}^{-1}(A) = \emptyset \quad \text{if} \quad A = \emptyset \qquad \text{and} \qquad \bigcup \mathscr{F}_X \cap \mathscr{P}^{-1}(A) = X \quad \text{if} \quad A \neq \emptyset .$$

Moreover, we can also easily see that, for any $x, y \in X$,

$$y \in \mathrm{lb}_X(x) \iff x \in \mathrm{ub}_X(y) \iff x \in \preceq (y) \iff x \in \{y-1\} \cup [y, +\infty[$$
$$\iff x = y-1 \text{ or } y \leq x \iff y \leq x \text{ or } y = x+1 \iff y \in]-\infty, x] \cup \{x+1\} .$$

Therefore,

$$\mathrm{lb}_X(x) =]-\infty, x] \cup \{x+1\} .$$

Thus, by Theorem 10,

$$\mathrm{cl}_X(A) = \bigcup_{a \in A} \mathrm{lb}_X(a) = \bigcup_{a \in A} \big( ]-\infty, a] \cup \{a+1\} \big) .$$

for all $A \subseteq X$. Hence, it is clear that equality in the assertion (2) of Theorem 17 need not be true.

Remark 24. This shows that the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are, in general, much weaker tools in the goset X than the relations int_X and cl_X. However, later we see that this is not the case if X is in particular a proset.

6 Fat and Dense Sets

Note that a subset A of a goset X may be called *upper bounded* if $\mathrm{ub}_X(A) \neq \emptyset$. Therefore, in addition to Definition 3, we may also naturally introduce the following.

Definition 4. For any goset X, the members of the families

$$\mathscr{E}_X = \{ A \subseteq X : \ \mathrm{int}_X(A) \neq \emptyset \} \qquad \text{and} \qquad \mathscr{D}_X = \{ A \subseteq X : \ \mathrm{cl}_X(A) = X \}$$

are called the *fat and dense subsets* of X, respectively.

Remark 25. Thus, by Definition 2, for any $A \subseteq X$, we have

(1) $A \in \mathscr{E}_X$ if and only if $\mathrm{ub}_X(x) \subseteq A$ for some $x \in X$.
(2) $A \in \mathscr{D}_X$ if and only if $\mathrm{ub}_X(x) \cap A \neq \emptyset$ for all $x \in X$.

Remark 26. Moreover, by Remark 13 and Theorem 10, we can also see that

$$\mathscr{D}_X = \{ A \subseteq X : \quad X =_{\geq} [A] \} = \{ A \subseteq X : \quad X = \bigcup_{a \in A} \mathrm{lb}_X(a) \} .$$

Therefore, for any $A \subseteq X$, we have $A \in \mathscr{D}_X$ if and only if for any $x \in X$ there exists $a \in A$ such that $x \in \mathrm{lb}_X(a)$, i. e., $x \leq a$.

Remark 27. Because of the above two remarks, the members of the families $\mathscr{E}_X$ and $\mathscr{D}_X$ may also be called the *residual and cofinal subsets* of X.

Namely, for instance, by Remarks 2 and 25, for any $A \subseteq X$, we have $A \in \mathscr{E}_X$ if and only if there exists $x \in X$ such that for any $y \in X$, with $x \leq y$, we have $y \in A$.

By using Definition 4 and Theorem 6, we can easily prove the following.

Theorem 18. *For any goset X, we have*

(1) $\mathscr{E}_X = \{ A \subseteq X : \quad A^c \notin \mathscr{D}_X \}$,
(2) $\mathscr{D}_X = \{ A \subseteq X : \quad A^c \notin \mathscr{E}_X \}$.

Proof. If $A \in \mathscr{E}_X$, then by Definition 4 we have $\mathrm{int}_X(A) \neq \emptyset$. Hence, by Theorem 6, we can infer that $\mathrm{cl}_X(A^c) = X \setminus \mathrm{int}_X(A) \neq X$. Therefore, $A^c \notin \mathscr{D}_X$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 6.

Remark 28. This theorem shows that the families $\mathscr{E}_X$ and $\mathscr{D}_X$ are also equivalent tools in the goset X.

By using element-wise complementation, Theorem 18 can also be written in a more concise form.

Corollary 7. *For any goset X, we have*

(1) $\mathscr{E}_X = \big(\mathscr{P}(X) \setminus \mathscr{D}_X \big)^c$,
(2) $\mathscr{D}_X = \big(\mathscr{P}(X) \setminus \mathscr{E}_X \big)^c$.

Moreover, concerning the families $\mathscr{E}_X$ and $\mathscr{D}_X$, we can also prove the following.

Theorem 19. *For any goset X, we have*

(1) $\mathscr{E}_X = \{ E \subseteq X : \quad \forall\ D \in \mathscr{D}_X : \quad E \cap D \neq \emptyset \}$,
(2) $\mathscr{D}_X = \{ D \subseteq X : \quad \forall\ E \in \mathscr{E}_X : \quad E \cap D \neq \emptyset \}$.

Proof. If $E \in \mathscr{E}_X$, then by Remark 25, there exists $x \in X$ such that $\mathrm{ub}_X(x) \subseteq E$. Moreover, if $D \in \mathscr{D}_X$, then by Remark 25, we have $\mathrm{ub}_X(x) \cap D \neq \emptyset$. Therefore, $E \cap D \neq \emptyset$ also holds.

Conversely, if $E \subseteq X$ such that $E \cap D \neq \emptyset$ for all $D \in \mathscr{D}_X$, then we can also easily see that $E \in \mathscr{E}_X$. Namely, if $E \notin \mathscr{E}_X$, then by Theorem 18 we necessarily have $E^c \in \mathscr{D}_X$. Therefore, $E \cap E^c \neq \emptyset$ which is a contradiction.

Hence, it is clear that (1) is true. Assertion (2) can be proved quite similarly.

Now, a counterpart of Theorem 14 is not true. However, analogously to Theorems 15 and 16, we can also state the following two theorems.

Theorem 20. *For any subset Y of a goset X, we have*

(1) $\mathscr{E}_X \cap \mathscr{P}(Y) \subseteq \mathscr{E}_Y$,
(2) $\mathscr{D}_X \cap \mathscr{P}(Y) \subseteq \mathscr{D}_Y$.

Theorem 21. *For any goset X, the families $\mathscr{E}_X$ and $\mathscr{D}_X$ are ascending subfamilies of the poset $\mathscr{P}(X)$ such that*

(1) $\mathscr{T}_X \setminus \{\emptyset\} \subseteq \mathscr{E}_X$,
(2) $\mathscr{F}_X \cap \mathscr{D}_X \subseteq \{X\}$.

From this theorem, we can immediately derive the following

Corollary 8. *For any subset A of a goset X, the following assertions are true:*

(1) If $B \subseteq A$ for some $B \in \mathscr{T}_X \setminus \{\emptyset\}$, then $A \in \mathscr{E}_X$.
(2) If $A \in \mathscr{D}_X$, then $A \setminus B \neq \emptyset$ for all $B \in \mathscr{F}_X \setminus \{X\}$.

Proof. To check (2), note that if the conclusion of (2) does not hold, then there exists $B \in \mathscr{F}_X \setminus \{X\}$ such that $A \setminus B = \emptyset$, and thus $A \cap B^c = \emptyset$. Hence, by defining $C = B^c$ and using Theorem 13, we can already see that $C \in \mathscr{T}_X \setminus \{\emptyset\}$ such that $A \cap C = \emptyset$, and thus $C \subseteq A^c$. Therefore, by (1), $A^c \in \mathscr{E}_X$, and thus by Theorem 18, we have $A \notin \mathscr{D}_X$.

Remark 29. The converses of the above assertions need not be true. Namely, if X is as in Example 1, then $\mathscr{T}_X = \{\emptyset, X\}$, but $\mathscr{E}_X$ is quite a large subfamily of $\mathscr{P}(X)$.

This shows that there are cases when even the families $\mathscr{E}_X$ and $\mathscr{D}_X$ are better tools in a goset X than $\mathscr{T}_X$ and $\mathscr{F}_X$. However, later we shall see that this is not the case if X is in particular a proset.

The duality and several advantages of fat and dense sets in relator spaces, over the open and closed ones, were first revealed by the present author at a Prague Topological Symposium in 1991 [40]. However, nobody was willing to accept this.

Remark 30. An ascending subfamily $\mathscr{A}$ of the poset $\mathscr{P}(X)$ is usually called a *stack* in X. It is called proper if $\emptyset \notin \mathscr{A}$ or equivalently $\mathscr{A} \neq \mathscr{P}(X)$.

In particular, a stack $\mathscr{A}$ in X is called a *filter* if $A, B \in \mathscr{A}$ implies $A \cap B \in \mathscr{A}$. And, $\mathscr{A}$ is called a *grill* if $A \cup B \in \mathscr{A}$ implies $A \in \mathscr{A}$ or $B \in \mathscr{A}$. These are usually assumed to be nonempty and proper.

Several interesting historical facts on stacks, lters, grills and nets can be found in the works [62, 63] of Thron

Concerning the families $\mathscr{E}_X$ and $\mathscr{D}_X$, we can also easily establish the following two theorems.

Theorem 22. *For any poset X, the following assertions are equivalent :*

(1) $\mathscr{E}_X \neq \emptyset$,
(2) $X \in \mathscr{E}_X$,

(3) $\emptyset \notin \mathscr{D}_X$,
(4) $X \neq \emptyset$.

Proof. To prove the equivalence of (1) and (4), note that, by Theorem 21, assertions (1) and (2) are equivalent. Moreover, by Remark 25, assertion (2) holds if and only there exists $x \in X$ such that $\operatorname{ub}_X(x) \subseteq X$. That is, (4) holds.

Theorem 23. *For any poset X, the following assertions are equivalent:*

(1) $\emptyset \notin \mathscr{E}_X$,
(2) $\mathscr{D}_X \neq \emptyset$,
(3) $X \in \mathscr{D}_X$,
(4) $X = \geq [X]$.

Proof. To prove the equivalence of (1) and (4), note that by Remark 25 assertion (1) holds if and only if, for any $x \in X$, we have $\operatorname{ub}_X(x) \not\subseteq \emptyset$. That is, $\operatorname{ub}_X(x) \neq \emptyset$, or equivalently $\leq (x) \neq \emptyset$. That is, the relation $\leq$ is total in the sense that its domain is the whole X.

Remark 31. A subset $\mathscr{B}$ of a stack $\mathscr{A}$ in X is called a *base* of $\mathscr{A}$ if for each $A \in \mathscr{A}$ there exists $B \in \mathscr{B}$ such that $B \subseteq A$. That is, $\mathscr{B}$ is a cofinal subset of the poset $\mathscr{A}^{-1} = \mathscr{A}(\subseteq^{-1}) = \mathscr{A}(\supseteq)$.

Note that if $\mathscr{B} \subseteq \mathscr{P}(X)$, then the family

$$\mathscr{B}^* = \operatorname{cl}_{\mathscr{P}^{-1}}(\mathscr{B}) = \{A \subseteq X : \quad \exists\, B \in \mathscr{B} : \quad B \subseteq A\}$$

is already a stack in X such that $\mathscr{B}$ is a base of $\mathscr{B}^*$.

Now, as a more important addition to Theorem 21, we can also easily prove

Theorem 24. *For any goset X, the stack $\mathscr{E}_X$ has a base $\mathscr{B}$ with* $\operatorname{card}(\mathscr{B}) \leq \operatorname{card}(X)$.

Proof. By Remarks 25 and 31, it is clear that the family $\mathscr{B}_X = \{\operatorname{ub}_X(x) : \quad x \in X\}$ is a base of $\mathscr{E}_X$.

Moreover, we can note that the function f, defined by $f(x) = \operatorname{ub}_X(x)$ for $x \in X$, is onto $\mathscr{B}_X$. Hence, by the axiom of choice, the cardinality condition follows.

Namely, now f^{-1} is a relation of $\mathscr{B}_X$ to X. Hence, by choosing a selection function φ of f^{-1}, we can see that φ is an injection of $\mathscr{B}$ to X.

Remark 32. Now, a corresponding property of the family $\mathscr{D}_X$ should, in principle, be derived from the above theorem by using either Theorem 18 or 19.

Remark 33. The importance of the study of the cardinalities of the bases of the stack of all fat sets in a relator space, concerning a problem of mine on paratopologically simple relators, was first recognized by J. Deák (1994) and G. Pataki (1998). (For the corresponding results, see Pataki [31].)

7 Maximum, Minimum, Supremum, and Infimum

According to [48], we may also naturally introduce the following.

Definition 5. For any subset A of a goset X, the members of the sets

$$\max_X(A) = A \cap \mathrm{ub}_X(A) \qquad \text{and} \qquad \min_X(A) = A \cap \mathrm{lb}_X(A)$$

are called the *maxima* and *minima* of the set A in X, respectively.

Remark 34. Thus, for any subset A of a goset X, we have

(1) $\mathrm{ub}_X(A) = \max_X(A)$ if and only if $\mathrm{ub}_X(A) \subseteq A$.
(2) $\mathrm{lb}_X(A) = \min_X(A)$ if and only if $\mathrm{lb}_X(A) \subseteq A$.

Moreover, from Definition 5, we can see that the properties of the relations $\max_X$ and $\min_X$ can be immediately derived from the results of Sect. 3.

For instance, from Theorems 1 and 2 and Corollaries 1 and 2, by using Definition 5, we can immediately derive the following four theorems.

Theorem 25. *For any subset A of a goset X, we have*

(1) $\max_X(A) = \min_{X^{-1}}(A)$,
(2) $\max_X(A) = \min_{X^{-1}}(A)$.

Remark 35. This theorem shows that the relations $\max_X$ and $\min_X$ are also equivalent tools in the goset X.

Theorem 26. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\max_Y(A) = \max_X(A)$,
(2) $\min_Y(A) = \min_X(A)$.

Theorem 27. *For any subset A of a goset X, we have*

(1) $\max_X(A) = \bigcap_{a \in A} A \cap \mathrm{ub}_X(a)$,
(2) $\min_X(A) = \bigcap_{a \in A} A \cap \mathrm{lb}_X(a)$.

Theorem 28. *For any subset A of a goset X, we have*

(1) $\max_X(A) = \{x \in A : A \subseteq \mathrm{lb}_X(x)\}$,
(2) $\min_X(A) = \{x \in A : A \subseteq \mathrm{ub}_X(x)\}$.

Remark 36. By Corollary 2, for instance, we may also naturally define

$$\mathrm{ub}^*_X(A) = \{x \in X : A \cap \mathrm{ub}_X(x) \subseteq \mathrm{lb}_X(x)\},$$

and also $\max^*_X(A) = A \cap \mathrm{ub}^*_X(A)$ for all $A \subseteq X$.

Thus, for any $x \in X$ and $A \subseteq X$, we have $x \in \mathrm{ub}^*_X(A)$ if and only if $x \leq a$ implies $a \leq x$ for all $a \in A$. Therefore, $\max^*_X(A)$ is just the family of all *maximal elements* of A.

The most important theorems on a poset X give some sufficient conditions in order that the set $\max^*(X)$ be nonempty. (See, for instance, [18, p. 33] and the references of [54] .)

Now, by using Definition 5, we may also naturally introduce

Definition 6. For any subset A of a goset X, the members of the sets

$$\sup_X(A) = \min_X\big(\mathrm{ub}_X(A)\big) \qquad \text{and} \qquad \inf_X(A) = \max_X\big(\mathrm{lb}_X(A)\big)$$

are called the *suprema* and *infima* of the set A in X, respectively.

Thus, by Definition 5, we evidently have the following

Theorem 29. *For any subset A of a goset X, we have*

(1) $\sup_X(A) = \mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$,
(2) $\inf_X(A) = \mathrm{lb}_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$.

Hence, by Theorem 1, it is clear that we also have the following.

Theorem 30. *For any subset A of a goset X, we have*

(1) $\sup_X(A) = \inf_{X^{-1}}(A)$,
(2) $\inf_X(A) = \sup_{X^{-1}}(A)$.

Remark 37. This theorem shows that the relations $\sup_X$ and $\inf_X$ are also equivalent tools in the goset X.

However, instead of an analogue of Theorem 2, we can only prove

Theorem 31. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\sup_X(A) \cap Y \subseteq \sup_Y(A)$,
(2) $\inf_X(A) \cap Y \subseteq \inf_Y(A)$.

Proof. To prove (1), by using Theorems 2, 4, and, 29 we can see that

$$\begin{aligned}
\sup_Y(A) &= \mathrm{ub}_Y(A) \cap \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big)\\
&= \mathrm{ub}_X(A) \cap Y \cap \ \mathrm{lb}_X\big(\mathrm{ub}_X(A) \cap Y\big) \cap \mathrm{ub}_X(A) \cap Y\\
&= \mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A) \cap Y\big) \cap Y \supseteq \mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big) \cap Y\\
&= \sup_X(A) \cap Y.
\end{aligned}$$

Remark 38. In connection with inclusion (2), Tamás Glavosits, my PhD student, showed that the corresponding equality need not be true even if X is a finite poset.

For this, he took $X = \{a, b, c, d\}$, $Y = X \setminus \{b\}$ and $A = Y \setminus \{a\}$, and considered the preorder $\leq$ on X generated by the relation $R = \{(a,b), (b,c), (b, d)\}$.

Thus, he could at once see that $\inf_Y(A) = \max_Y\big(\mathrm{lb}_Y(A)\big) = \max_Y\big(\{a\}\big) = \{a\}$, but $\inf_X(A) = \max_X\big(\mathrm{lb}_X(A)\big) = \max_X\big(\{a, b\}\big) = \{b\}$, and thus $\inf_X(A) \cap Y = \emptyset$.

Now, by using Theorem 29, we can also easily prove the following theorem which shows that the relations $\sup_X$ and $\inf_X$ are, in a certain sense, better tools in the goset X than $\max_X$ and $\min_X$.

Theorem 32. *For any subset A of a goset X, we have*

(1) $\max_X(A) = A \cap \sup_X(A)$,
(2) $\min_X(A) = A \cap \inf_X(A)$.

Proof. To prove (2), note that by Theorem 29 and Definition 5, we have

$$A \cap \inf_X(A) = A \cap \mathrm{lb}_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) = \min_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big).$$

Moreover, by Definition 5 and Remark 8, we have

$$\min_X(A) \subseteq A \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big), \qquad \text{and so} \qquad \min_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) = \min_X(A).$$

Remark 39. By the above theorem, for any subset A of a goset X, we have

(1) $\max_X(A) = \sup_X(A)$ if and only if $\sup_X(A) \subseteq A$.
(2) $\min_X(A) = \inf_X(A)$ if and only if $\inf_X(A) \subseteq A$.

Moreover, by using Theorem 29, we can also easily prove the following theorem which will make a basic theorem on supremum and infimum completeness properties to be completely obvious.

Theorem 33. *For any subset A of a goset X, we have*

(1) $\sup_X(A) = \inf_X\big(\mathrm{ub}_X(A)\big)$,
(2) $\inf_X(A) = \sup_X\big(\mathrm{lb}_X(A)\big)$.

Proof. To prove (2), note that by Theorem 29 and Remark 8, we have

$$\begin{aligned}\inf_X(A) &= \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \cap \mathrm{lb}_X(A) \\ &= \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \cap \mathrm{lb}_X\Big(\mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)\Big) = \sup_X\big(\mathrm{lb}_X(A)\big).\end{aligned}$$

Remark 40. Concerning our references to Remark 8 in the proofs of Theorems 32 and 33, note that the assertions

$$A \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \qquad \text{and} \qquad \mathrm{lb}_X(A) = \mathrm{lb}_X\Big(\mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)\Big)$$

can also be easily proved directly, by using Definition 1, without using the corresponding theorems on Pataki connections.

Definition 7. A goset X is called *inf-complete* (*sup-complete*) if $\inf_X(A) \neq \emptyset$ ($\sup_X(A) \neq \emptyset$) for all $A \subseteq X$.

Remark 41. Quite similarly, a goset X may, for instance, be also naturally called *min-complete* if $\min_X(A) \neq \emptyset$ for all nonvoid subset A of X.

Thus, the set $\mathbb{Z}$ of all integers is min-, but not inf-complete. While, the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is inf-, but not min-complete.

Now, as an immediate consequence of Theorem 33, we can state the following straightforward extension of [2, Theorem 3, p. 112].

Theorem 34. *For a goset X, the following assertions are equivalent:*

(1) X is inf-complete,
(2) X is sup-complete.

Remark 42. Similar equivalences of several modified inf- and sup-completeness properties of gosets have been established in [3, 4].

Finally, we note that, by Definition 5 and Theorem 27, we evidently have

Theorem 35. *For any subset A of a goset X, we have*

(1) $\inf_X(A) = \big\{ x \in \mathrm{lb}_X(A) : \quad \mathrm{lb}_X(A) \subseteq \mathrm{lb}_X(x) \big\}$,
(2) $\sup_X(A) = \big\{ x \in \mathrm{ub}_X(A) : \quad \mathrm{ub}_X(A) \subseteq \mathrm{ub}_X(x) \big\}$.

Moreover, by using this theorem, we can also easily prove the following.

Theorem 36. *For any subset A of a proset X, we have*

(1) $\inf_X(A) = \big\{ x \in X : \quad \mathrm{lb}_X(x) = \mathrm{lb}_X(A) \big\}$,
(2) $\sup_X(A) = \big\{ x \in X : \quad \mathrm{ub}_X(x) = \mathrm{ub}_X(A) \big\}$.

Proof. Define

$$\Phi(A) = \big\{ x \in X : \quad \mathrm{lb}_X(x) = \mathrm{lb}_X(A) \big\}.$$

Now, if $x \in \Phi(A)$, we can see that

(a) $\mathrm{lb}_X(x) \subseteq \mathrm{lb}_X(A)$,
(b) $\mathrm{lb}_X(A) \subseteq \mathrm{lb}_X(x)$.

From (a), since X is reflexive, and thus $x \leq x$, i.e., $x \in \mathrm{lb}_X(x)$, we can infer that $x \in \mathrm{lb}_X(A)$. Hence, by (b) and Theorem 35, we can already see that $x \in \inf_X(A)$. Therefore, $\Phi(A) \subseteq \inf_X(A)$ even if X is assumed to be only a reflexive goset.

Conversely, if $x \in \inf_X(A)$, then by Theorem 35 we also have
(c) $x \in \mathrm{lb}_X(A)$, (d) $\mathrm{lb}_X(A) \subseteq \mathrm{lb}_X(x)$.

From (c), we can infer that $x \leq a$ for all $a \in A$. Hence, by using the transitivity of X we can easily see that if $y \in \mathrm{lb}_X(x)$, and thus $y \leq x$, then $y \leq a$ also holds for all $a \in A$, and thus $y \in \mathrm{lb}_X(A)$. Therefore, $\mathrm{lb}_X(x) \subseteq \mathrm{lb}_X(A)$ even if X is assumed to be only a transitive goset. Hence, by using (d), we can already see that $\mathrm{lb}_X(x) = \mathrm{lb}_X(A)$, and thus $x \in \Phi(x)$. Therefore, $\inf_X(A) \subseteq \Phi(A)$ even if X is assumed to be only a transitive goset.

The above arguments show that (1) is true. Moreover, from (1) by using Theorems 1 and 30, we can at once see that (2) is also true.

8 Self-bounded Sets

Analogously to Definition 3, for instance, we may also naturally introduce

Definition 8. For any goset X, the members of the family

$$\mathcal{U}_X = \{ A \subseteq X : \quad A \subset \mathrm{ub}_X(A) \}$$

are called the *self-upper-bounded subsets* of X.

Remark 43. Thus, by the corresponding definitions, for any $A \subseteq X$, we have $A \in \mathcal{U}_X$ if and only if $x \leq y$ for all $x, y \in A$.

Therefore, $A \in \mathcal{U}_X$ if and only if $A \leq A$ or equivalently $A^2 \subseteq \leq$. That is, by the notations of Száz [47], we have $A \in \mathrm{Ub}_X(A)$ or equivalently $A \in \mathrm{Lb}_X(A)$.

Because of the above remark, we evidently have the following three theorems.

Theorem 37. *For any goset X, we have $\mathcal{U}_X = \mathcal{U}_{X^{-1}}$.*

Theorem 38. *For any subset Y of goset X, we have $\mathcal{U}_Y = \mathcal{U}_X \cap \mathcal{P}(Y)$.*

Theorem 39. *For any goset X, we have*

$$\mathcal{U}_X = \{ A \subseteq X : \quad \forall\ x, y \in A : \quad \{x, y\} \in \mathcal{U}_X \}.$$

Hence, it is clear that, in particular, we also have the following corollary.

Corollary 9. *For any goset X, the family $\mathcal{U}_X$ is a descending subset of the poset $\mathcal{P}(X)$ such that $\bigcup \mathcal{V} \in \mathcal{U}_X$ for any chain $\mathcal{V}$ in $\mathcal{U}_X$.*

However, it is now more important to note that, by using the corresponding definitions, we can also prove the following

Theorem 40. *For any subset A of a goset X, the following assertions are equivalent:*

(1) $A \in \mathcal{U}_X$,
(2) $A = \max_X(A)$,
(3) $A \subseteq \sup_X(A)$,
(4) $A \subseteq \mathrm{lb}_X(A)$,
(5) $A = \min_X(A)$,
(6) $A \subseteq \inf_X(A)$.

Proof. By Definitions 5 and 8, we evidently have

$$A \in \mathcal{U}_X \iff A \subseteq \mathrm{ub}_X(A) \iff A \subseteq A \cap \mathrm{ub}_X(A) \iff A \subseteq \max_X(A).$$

Hence, since $\max_X(A) \subseteq A$, it is clear that (1) and (2) are equivalent.

Moreover, by using Definition 8 and Theorem 5, we can at once see that (1) and (4) are also equivalent. Hence, by using the inclusion $A \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$ and Theorem 29, we can also easily see that

$$A \in \mathscr{U}_X \iff A \subseteq \mathrm{lb}_X(A) \iff A \subseteq \mathrm{lb}_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \iff A \subseteq \mathrm{inf}_X(A)\,.$$

Therefore, (1) and (6) are also equivalent. The proofs of the remaining implications are quite similar.

Remark 44. This theorem shows that, in a goset X, the family $\mathscr{U}_X$ is just the collection of all fixed elements of the set-valued functions $\max_X$ and $\min_X$.

Now, as some immediate consequences of Theorem 40 and Definition 6, we can also state

Corollary 10. *For any subset A of a goset X, the following assertions are equivalent:*

(1) $\mathrm{ub}_X(A) \in \mathscr{U}_X$;
(2) $\mathrm{ub}_X(A) = \sup_X(A)$;
(3) $\mathrm{ub}_X(A) \subseteq \mathrm{ub}_X\big(\mathrm{ub}_X(A)\big)$;
(4) $\mathrm{ub}_X(A) \subseteq \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$.

Corollary 11. *For any subset A of a goset X, the following assertions are equivalent:*

(1) $\mathrm{lb}_X(A) \in \mathscr{U}_X$;
(2) $\mathrm{lb}_X(A) = \inf_X(A)$;
(3) $\mathrm{lb}_X(A) \subseteq \mathrm{lb}_X\big(\mathrm{lb}_X(A)\big)$;
(4) $\mathrm{lb}_X(A) \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$.

However, it is now more important to note that, by using Theorem 40, we can also easily prove the following theorem.

Theorem 41. *For any goset X, we have*

(1) $\mathscr{U}_X = \big\{ \max_X(A) : \quad A \subseteq X \big\}$,
(2) $\mathscr{U}_X = \big\{ \min_X(A) : \quad A \subseteq X \big\}$.

Proof. If $V \in \mathscr{U}_X$, then by Theorem 40, we have $V = \max_X(V)$. Therefore, V is in the family $\mathscr{A} = \{ \max_X(A) : \ A \subseteq X \}$.

Conversely, if $V \in \mathscr{A}$, then there exists $A \in \mathscr{A}$ such that $V = \max_X(A)$. Hence, by Definition 5, it follows that $V \subseteq A$ and $V \subseteq \mathrm{ub}_X(A)$. Now, by Theorem 4, we can also see that $\mathrm{ub}_X(A) \subseteq \mathrm{ub}_X(V)$. Therefore, $V \subseteq \mathrm{ub}_X(V)$, and thus $V \in \mathscr{U}_X$ also holds.

This proves (1). Moreover, (2) can be derived from (1) by using Theorems 25 and 37.

Remark 45. This theorem shows that, in a goset X, the family $\mathscr{U}_X$ is just the range of the set-valued functions $\max_X$ and $\min_X$.

By using Remark 43, we can also easily prove the following three theorems.

Theorem 42. *For any goset X, the following assertions are equivalent:*

(1) *X is reflexive,*
(2) *$\{x\} \in \mathscr{U}_X$ for all $x \in X$.*

Theorem 43. *If X is an antisymmetric goset, then for any $A \in \mathscr{U}_X$ we have* $\operatorname{card}(A) \leq 1$.

Proof. If $A \in \mathscr{U}_X$ and $x, y \in A$, then by Remark 43 we have $x \leq y$ and $y \leq x$. Hence, by the assumed antisymmetry of $\leq$, it follows that $x = y$.

Theorem 44. *If X is reflexive goset such that* $\operatorname{card}(A) \leq 1$ *for all $A \in \mathscr{U}_X$, then X is antisymmetric.*

Proof. If $x, y \in X$ such that $x \leq y$ and $y \leq x$, then by taking $A = \{x, y\}$ we can see that $A \leq A$, and thus $A \in \mathscr{U}_X$. Hence, by the assumption, it follows that $\operatorname{card}(A) \leq 1$. Therefore, we necessarily have $x = y$.

From the latter two theorems, by using Theorem 41, Definition 6 and Theorem 32, we can immediately derive the following two theorems.

Theorem 45. *If X is an antisymmetric goset, then under the notation $\Phi = \max_X$, $\min_X$, $\sup_X$, or $\inf_X$, for any $A \subseteq X$ we have* $\operatorname{card}\big(\Phi(A)\big) \leq 1$.

Theorem 46. *If X is a reflexive goset such that, under the notation $\Phi = \max_X$, $\min_X$, $\sup_X$, or $\inf_X$, for any $A \subseteq X$ we have* $\operatorname{card}\big(\Phi(A)\big) \leq 1$, *then X is antisymmetric.*

Proof. Note that, if, for instance, $\operatorname{card}\big(\sup_X(A)\big) \leq 1$ for all $A \subseteq X$, then by Theorem 32, we also have $\operatorname{card}\big(\max_X(A)\big)$ for all $A \subseteq X$. Hence, by using Theorem 41, we can infer that $\operatorname{card}(A) \leq 1$ for all $A \in \mathscr{U}_X$. Therefore, by Theorem 44, we can state that X is antisymmetric.

Remark 46. In connection with the above results, it is worth noticing that the goset X considered in Example 1 is reflexive, but not antisymmetric.

Namely, concerning the relation $\preceq$, we can easily see that, for any $x, y \in X$, we have both $x \preceq y$ and $y \preceq x$ if and only if $x = y$ or $x = y - 1$ or $y = x - 1$.

Therefore, for any $A \subseteq X$, we have $A \in \mathscr{U}_X$ if and only if $A = \emptyset$ or $A = \{x\}$ or $A = \{x, x-1\}$ for some $x \in X$.

This fact, together with $\mathscr{T}_X = \{\emptyset, X\}$, shows that there are cases when even the family $\mathscr{U}_X$ is also a better tool than the family $\mathscr{T}_X$.

In the sequel, beside reflexivity and antisymmetry, we shall also need a further, similarly simple and important, property of gosets.

Definition 9. A goset X will be called *linear* if for any $x, y \in X$, with $x \neq y$, we have either $x \leq y$ or $y \leq x$.

Remark 47. If X is a goset, then for any $x, y \in X$,we may also write $x < y$ if both $x \leq y$ and $x \neq y$.

Therefore, if the goset X is linear, then for any $x, y \in X$, with $x \neq y$, we actually have either $x < y$ or $y < x$.

Moreover, as an immediate consequence of the corresponding definitions, we can also state the following.

Theorem 47. *For a goset X, the following assertions are equivalent :*

(1) *X is reflexive and linear,*
(2) *For any $x, y \in X$, we have either $x \leq y$ or $y \leq x$,*
(3) $\max_X(A) \neq \emptyset$ $(\min_X(A) \neq \emptyset)$ *for all* $A \subseteq X$ *with* $1 \leq \operatorname{card}(A) \leq 2$.

Proof. To check the implication (3) $\Longrightarrow$ (2), note that if $x, y \in X$, then $A = \{x, y\}$ is a subset of X such that $1 \leq \operatorname{card}(A) \leq 2$. Therefore, if (3) holds, then there exists $z \in X$ such that $z \in \max_X(A)$. Hence, by Definition 5, it follows that $z \in A$ and $z \in \operatorname{ub}_X(A)$. Therefore, we have either $z = x$ or $z = y$. Moreover, we have $x \leq z$ and $y \leq z$. Hence, if $z = x$, we can see that $y \leq x$. While, if $z = y$, we can see that $x \leq y$. Therefore, (2) also holds.

From this theorem, it is clear that in particular we have

Corollary 12. *If X is a min-complete (max-complete) goset, then X is reflexive and linear.*

The importance of reflexive, linear, and antisymmetric gosets is also apparent from the next two simple theorems.

Theorem 48. *If X is an antisymmetric goset, then $x < y$ implies $y \not\leq x$ for all $x, y \in X$.*

Theorem 49. *If X is a reflexive and linear goset, then $x \not\leq y$ implies $y < x$ for all $x, y \in X$.*

Proof. If $x, y \in X$ such that $x \not\leq y$, then by Theorem 47 we have $y \leq x$. Moreover, by the reflexivity of X, we also have $x \neq y$, and hence $y \neq x$. Therefore, $y < x$ also holds.

Remark 48. Therefore, if X is a reflexive, linear, and antisymmetric goset, then for any $x, y \in X$,we have

$$x \not\leq y \quad \Longleftrightarrow \quad x <^{-1} y .$$

Note that, analogously to the equivalences in Remarks 6 and 15, this is again a Galois connection property.

9 The Importance of Reflexivity and Transitivity

Several simple characterizations of reflexivity and transitivity of a goset X, in terms of the relations ub_X and lb_X, and their compositions considered in Sect. 7, have been given in [49].

Now, by using the techniques of the theory of relator spaces, we shall give some more delicate characterizations of these properties in terms of the relations int_X and cl_X and the families $\mathscr{T}_X$ and $\mathscr{F}_X$.

Theorem 50. *For any goset X, the following assertions are equivalent:*

(1) X is reflexive,
(2) $x \in \mathrm{ub}_X(x)$ for all $x \in X$,
(3) $\mathrm{int}_X(A) \subseteq A$ for all $A \subseteq X$,
(4) $\mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \subseteq \mathrm{ub}_X(x)$ for all $x \in X$.

Proof. By Remark 2, it is clear that (1) and (2) are equivalent. Moreover, if $A \subseteq X$ and $x \in \mathrm{int}_X(A)$, then by Definition 2 we have $\mathrm{ub}_X(x) \subseteq A$. Hence, if (2) holds, we can infer that $x \in A$, and thus (3) also holds.

Now, since (3) trivially implies (4), it remains to show only that (4) also implies (2). However, for this, it is enough to note only that, for any $x \in X$, we have $\mathrm{ub}_X(x) \subseteq \mathrm{ub}_X(x)$, and hence $x \in \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$ by Definition 2.

From this theorem, by using Theorem 6, we can immediately derive

Corollary 13. *For any goset X, the following assertions are equivalent:*

(1) X is reflexive,
(3) $A \subseteq \mathrm{cl}_X(A)$ for all $A \subseteq X$.

Proof. For instance, if (1) holds, then by Theorem 50, for any $A \subseteq X$, we have $\mathrm{int}_X(A^c) \subseteq A^c$. Hence, by using Theorem 6, we can already infer that $A \subseteq \mathrm{int}_X(A^c)^c = \mathrm{cl}_X(A)$. Therefore, (2) also holds.

From the above results, by Definition 3, it is clear that we also have

Theorem 51. *If X is a reflexive goset, then*

(1) $\mathscr{T}_X = \big\{A \subseteq X : \ A = \mathrm{int}_X(A)\big\}$,
(2) $\mathscr{F}_X = \big\{A \subseteq X : \ A = \mathrm{cl}_X(A)\big\}$.

Remark 49. This theorem shows that, in a reflexive goset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are just the collections of all fixed elements of the set-valued functions int_X and cl_X, respectively.

However, it is now more important to note that, in addition to Theorem 50, we can also prove the following.

Theorem 52. *For any goset X, the following assertions are equivalent :*

(1) X *is transitive,*
(2) $\mathrm{ub}_X(x) \in \mathscr{T}_X$ *for all* $x \in X$,
(3) $\mathrm{int}_X(A) \in \mathscr{T}_X$ *for all* $A \subseteq X$,
(4) $\mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \in \mathscr{T}_X$ *for all* $x \in X$,
(5) $x \in \mathrm{int}_X\Big(\mathrm{int}_X\big(\mathrm{ub}_X(x)\big)\Big)$ *for all* $x \in X$.

Proof. If (1) holds, then the inequality relation $\leq$ in X is transitive. Therefore, if $x \in X$ and $y \in \mathrm{ub}_X(x)$, then by Remark 2, for any $z \in \mathrm{ub}_X(y)$ we also have $z \in \mathrm{ub}_X(x)$. Hence, we can see that $\mathrm{ub}_X(y) \subseteq \mathrm{ub}_X(x)$, and thus by Definition 2 we have $y \in \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. This shows that $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$, and thus by Definition 3 we have $\mathrm{ub}_X(x) \in \mathscr{T}_X$. Therefore, (2) also holds.

Conversely, if (2) holds, then by Definition 3, for any $x \in X$, we have $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. Therefore, by Definition 2, for any $y \in \mathrm{ub}_X(x)$ we have $\mathrm{ub}_X(y) \subseteq \mathrm{ub}_X(x)$. Therefore, $z \in \mathrm{ub}_X(y)$ implies $z \in \mathrm{ub}_X(x)$. Hence, by Remark 2, it is clear that the inequality relation $\leq$ in X is transitive, and (1) also holds.

Next, we show that (2) also implies (3). For this, note that if $A \subseteq X$ and $x \in \mathrm{int}_X(A)$, then by Definition 2 we have $\mathrm{ub}_X(x) \subseteq A$. Hence, by using Theorem 11, we can infer that $\mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \subseteq \mathrm{int}_X(A)$. Moreover, if (2) holds, then by Definition 3 we also have $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. Thus, $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X(A)$ is also true. Hence, by Definition 2, it follows that $x \in \mathrm{int}_X\big(\mathrm{int}_X(A)\big)$. This shows that $\mathrm{int}_X(A) \subseteq \mathrm{int}_X\big(\mathrm{int}_X(A)\big)$, and thus by Definition 3 we also have $\mathrm{int}_X(A) \in \mathscr{T}_X$. Therefore, (3) also holds.

Now, since (3) trivially implies (4), it remains only to show only that (4) implies (5), and (5) implies (2). For this, note that if (4) holds, then by Definitions 2 and 3, for any $x \in X$, we have $x \in \mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \subseteq \mathrm{int}_X\Big(\mathrm{int}_X\big(\mathrm{ub}_X(x)\big)\Big)$. Therefore, (5) also holds. Moreover, if (5) holds, then by Definition 2, for any $x \in X$, we have $\mathrm{ub}_X(x) \subset \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. Therefore, $\mathrm{ub}_X(x) \in \mathscr{T}_X$, and thus (2) also holds.

From this theorem, by using Theorems 6 and 13, we can immediately derive

Corollary 14. *For any goset X, the following assertions are equivalent :*

(1) X *is transitive,*
(2) $\mathrm{cl}_X(A) \in \mathscr{F}_X$ *for all* $A \subseteq X$.

Now, as an immediate consequence of the above results, we can also state

Theorem 53. *For a proset X, we have*

(1) $\mathscr{T}_X = \big\{ \mathrm{int}_X(A) : \quad A \subseteq X \big\}$,
(2) $\mathscr{F}_X = \big\{ \mathrm{cl}_X(A) : \quad A \subseteq X \big\}$.

Remark 50. This theorem shows that in a proset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are just the ranges of the set-valued functions int_X and cl_X, respectively.

However, it is now more important to note that, by using Theorems 50 and 52, we can also easily prove the following.

Theorem 54. *For any goset X, the following assertions are equivalent:*

(1) X is reflexive and transitive,
(2) $\operatorname{int}_X(A) = \bigcup \mathscr{T}_X \cap \mathscr{P}(A)$ for all $A \subseteq X$,
(3) $\operatorname{cl}_X(A) = \bigcap \mathscr{F}_X \cap \mathscr{P}^{-1}(A)$ for all $A \subseteq X$.

Proof. Suppose that (1) holds and $A \subseteq X$. Define

$$B = \operatorname{int}_X(A) \qquad \text{and} \qquad C = \bigcup \mathscr{T}_X \cap \mathscr{P}(A)\,.$$

Then, by Theorems 50 and 52, we can see that $B \subseteq A$ and $B \in \mathscr{T}_X$, and thus $B \in \mathscr{T}_X \cap \mathscr{P}(A)$. Therefore, $B \subseteq \bigcup \mathscr{T}_X \cap \mathscr{P}(A) = C$. Moreover, from Theorem 17, we can see that $C \subseteq B$ is always true. Therefore, (2) also holds.

Conversely, if (2) holds, then for any $A \subseteq X$ we evidently have $\operatorname{int}_X(A) \subseteq A$. Thus, by Theorem 50, X is reflexive. Moreover, by Theorem 16, we can see that $\operatorname{int}_X(A) \in \mathscr{T}_X$. Therefore, by Theorem 52, X is also transitive. Thus, (1) also holds.

Now, to complete the proof, it remains to note only that the equivalence of (2) and (3) is an immediate consequence of Theorems 6 and 13.

Remark 51. This theorem shows that in a proset X the relation int_X or cl_X and the family $\mathscr{T}_X$ or $\mathscr{F}_X$ are also equivalent tools.

Now, by using Theorems 50 and 52, we can also easily prove the following.

Theorem 55. *For any subset A of a proset X, we have*

(1) $A \in \mathscr{E}_X$ if and only if $B \subseteq A$ for some $B \in \mathscr{T}_X \setminus \{\emptyset\}$,
(2) $A \in \mathscr{D}_X$ if and only if $A \setminus B \neq \emptyset$ for all $B \in \mathscr{F}_X \setminus \{X\}$.

Proof. According to Remark 31, define $\mathscr{B} = \mathscr{T}_X \setminus \{\emptyset\}$ and $\mathscr{A} = \mathscr{B}^*$. Then, for any $A \subseteq X$, we have $A \in \mathscr{A}$ if and only if $B \subseteq A$ for some $B \in \mathscr{B}$.

Now, if $A \in \mathscr{E}_X$, then by Remark 25, there exists $x \in X$ such that $\operatorname{ub}_X(x) \subseteq A$. Moreover, by Theorems 50 and 52, we have $x \in \operatorname{ub}_X(x)$ and $\operatorname{ub}_X(x) \in \mathscr{T}_X$, and hence $\operatorname{ub}_X(x) \in \mathscr{B}$. Therefore, $A \in \mathscr{A}$ also holds. This shows that $\mathscr{E}_X \subseteq \mathscr{A}$.

Moreover, from Corollary 8, we can see that $\mathscr{A} \subseteq \mathscr{E}_X$ is always true. Therefore, (1) also holds. Now, (2) can be easily derived from (1) by using Theorems 13 and 18.

Remark 52. By Remark 31, assertion (1) means only that, in a proset X, the family $\mathscr{T}_X \setminus \{\emptyset\}$ is also a base for the stack $\mathscr{E}_X$.

Beside Remark 51, this also shows that, in a proset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are better tools than the families $\mathscr{E}_X$ and $\mathscr{D}_X$.

10 An Interior Operation and the Preorder Closure

Because of Theorems 50 and 52, in addition to the operations c, -1, and ∞ mentioned in Sect. 2, we may also naturally introduce some further unary operations on relations and thus also on gosets.

For instance, in accordance with [44, Definition 3.1], we may naturally introduce

Definition 10. For any goset X, we define a relation $\leq^\circ$ on X such that

$$\leq^\circ(x) = \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$$

for all $x \in X$. Moreover, according to a notation of Sect. 2, we write $X^\circ = X(\leq^\circ)$.

Remark 53. Thus, by the corresponding definitions, for any $x, y \in X$, we have

$$x \leq^\circ y \iff y \in\, \leq^\circ(x) \iff y \in \mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \iff \mathrm{ub}_X(y) \subseteq \mathrm{ub}_X(x)\,.$$

Therefore, $\leq^\circ$ is already a preorder relation on X, and thus X° is a proset.

Moreover, as an immediate consequence of Theorems 50 and 52, we can state

Theorem 56. *For any goset X, we have*

(1) $\leq^\circ \subseteq \leq$ *if X is reflexive,*
(2) $\leq \subseteq \leq^\circ$ *if and only if X is transitive.*

Proof. To derive (2) from Theorem 52, note that for any $x \in X$ we have

$$\leq(x) \subseteq \leq^\circ(x) \iff \mathrm{ub}_X(x) \subseteq \mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \iff x \in \mathrm{int}_X\Big(\mathrm{int}_X\big(\mathrm{ub}_X(x)\big)\Big)\,.$$

From this theorem, by Remark 53, it is clear that in particular we also have

Corollary 15. *For any goset X, the following assertions are equivalent:*

(1) $\leq = \leq^\circ$,
(2) *X is a proset,*
(3) $y \in \mathrm{ub}_X(x) \iff \mathrm{ub}_X(y) \subseteq \mathrm{ub}_X(x)$ *for all $x, y \in X$.*

Remark 54. Note that, analogously to the statements of Remarks 7 and 16, assertion (3) is again a Pataki connection property.

Concerning assertion (3), it is also worth mentioning that $\leq$ is an equivalence relation on X if and only if it is total and, under the notation $X = X(\leq)$, for any $x, y \in X$ we have $y \in \mathrm{ub}_X(x)$ if and only if $\mathrm{ub}_X(x) \cap \mathrm{ub}_X(y) \neq \emptyset$.

Moreover, from Theorem 56, by using Remark 53 and a basic property of the relation $\leq^\infty$, we can also immediately derive the following.

Theorem 57. *For any goset X, we have*

(1) $\leq^\circ \subseteq \leq^\infty$ *if X is reflexive,*
(2) $\leq^\infty \subseteq \leq^\circ$ *if and only if X is transitive.*

Hence, it is clear that in particular we also have the following.

Corollary 16. *For a reflexive goset X, the following assertions are equivalent :*

(1) $\leq^{\circ} = \leq^{\infty}$,

(2) X *is transitive.*

Remark 55. Now, analogously to Definition 10, for any goset X, we may also naturally define a relation $\leq^{-}$ on X such that

$$\leq^{-}(x) = \mathrm{cl}_X\big(\mathrm{ub}_X(x)\big)$$

for all $x \in X$. Moreover, now we may also naturally write $X^{-} = X(\leq^{-})$.

Thus, in addition to the inclusions $\leq \subseteq \leq^{-}$ and $\leq^{-} \subseteq \leq$, we may also naturally investigate the inclusions $\leq^{\circ} \subseteq \leq^{-}$ and $\leq^{-} \subseteq \leq^{\circ}$. (See [44].)

However, it now is more important to note that the generated preorder relations can always be expressed in terms of the Pervin relations of the open sets defined by the original relations [26, 27] .

Theorem 58. *If X is a goset, then for any $x \in X$, we have*

$$\leq^{\infty}(x) = \bigcap_{A \in \mathscr{T}_X} R_A = \bigcap \big\{ A \in \mathscr{T}_X : \quad x \in A \big\}.$$

Proof. Recall that, for any $A \subseteq X$, we have $R_A = A^2 \cup A^c \times X$. Therefore,

$$R_A(x) = A \quad \text{if} \quad x \in A \qquad \text{and} \qquad R_A(x) = X \quad \text{if} \quad x \in A^c .$$

Hence, we can easily see that $x \in R_A(x)$ and

$$\big(R_A \circ R_A\big)(x) = R_A\,[R_A(x)\,] = \bigcup_{x \in A} R_A(x) \subseteq R_A(x)$$

for all $x \in X$. Therefore, $\Delta_X \subseteq R_A$ and $R_A \circ R_A \subseteq R_A$, and thus R_A is a preorder relation on X.

Hence, by a basic theorem on preorder relations, it is clear that $S = \bigcap_{A \in \mathscr{T}_X} R_A$ is also a preorder relation on X. Moreover, we can note that, for any $x \in X$, we have

$$S(x) = \Big(\bigcap_{A \in \mathscr{T}_X} R_A\Big)(x) = \bigcap_{A \in \mathscr{T}_X} R_A(x) = \bigcap \big\{ A \in \mathscr{T}_X : \quad x \in A \big\}.$$

Furthermore, if $x \in X$ and $y \in \leq^{\infty}(x)$, then by using the inclusion $\leq \subseteq \leq^{\infty}$ and the transitivity of $\leq^{\infty}$, we can also easily see that

$$\mathrm{ub}_X(y) = \leq_X(y) \subseteq \leq [\leq^{\infty}(x)] \subseteq \leq^{\infty}[\leq^{\infty}(x)] = \big(\leq^{\infty} \circ \leq^{\infty}\big)(x) \subseteq \leq^{\infty}(x).$$

Therefore, $y \in \operatorname{int}_X\big(\leq^\infty (x)\big)$. This shows that $\leq^\infty (x) \subseteq \operatorname{int}_X\big(\leq^\infty (x)\big)$ and thus $\leq^\infty (x) \in \mathscr{T}_X$. Hence, since $x \in \leq^\infty (x)$ also holds, we can already infer that $S(x) \subseteq \leq^\infty (x)$. Therefore, $S \subseteq \leq^\infty$ is also true.

On the other hand, if $A \in \mathscr{T}_X$, then by Remark 18, for any $x \in A$, we have $\leq (x) = \operatorname{ub}_X(x) \subseteq A = R_A(x)$. Therefore, $\leq \subseteq R_A$. Hence, since R_A is a preorder relation on X, we can already infer that $\leq^\infty \subseteq R_A^\infty = R_A$. Therefore, $\leq^\infty \subseteq S$, and thus the required assertion is also true.

Remark 56. Note that if X is a goset, then by using Theorem 16 from the above theorem, we can also see that $\leq^\infty (x) \in \mathscr{T}_X$ for all $x \in X$.

From Theorem 58, we can also immediately derive the following

Corollary 17. *For any goset X, the following assertions are equivalent :*

(1) X is proset,

(2) $\leq = \bigcap_{A \in \mathscr{T}_X} R_A$.

Now, according to the definitions of [21, 33], we may also have

Definition 11. A goset X is called *well-chained* if the inequality relation $\leq$ in it is well-chained in the sense that $\leq^\infty = X^2$.

Remark 57. By using the definition of $\leq^\infty$, the above property can be reformulated in a detailed form that for any $x, y \in X$, with $x \neq y$, there exists a finite sequence $(x_i)_{i=0}^n$ in X, with $x_0 = x$ and $x_n = y$, such that $x_{i-1} \leq x_i$ for all $i = 1, 2, \dots, n$.

Remark 58. During the long evolution of the concept of "connected", the denition of "chain connectedness", and also that of "archwise connectedness", has been replaced by the present "modern denition of connectedness". (See Thron [62, p. 29] and Wilder [66].)

However, in the theory relator spaces, it has turned out that the latter, celebrated connectedness is a particular case of well-chainedness, and well-chainedness is a particular case of *simplicity*. Unfortunately, our fundamental works [20, 21, 31, 33] on on these subjects were also strongly rejected by the leading topologists working in the editorial boards of various mathematical journals.

In this respect, it is also worth mentioning that Császár [9] also observed that "the concept of a connected set belongs rather to the theory of generalized topological spaces instead of topology in the strict sense." However, he has not quoted our former paper [33], despite that he knew that each increasing operation γ on $\mathscr{P}(X)$, with $\gamma(X) = X$, can be written in the form $\gamma = \operatorname{int}_{\mathscr{R}}$ with some nonvoid relator $\mathscr{R}$ on X. (For the proof of this and some more general results, see [41] and the references therein.)

By using Definition 11, from Theorem 58, we can easily derive the following.

Theorem 59. *For a goset X, the following assertions are equivalent:*

(1) X *is well-chained,*
(2) $\mathscr{T}_X = \{\emptyset, X\}$,
(2) $\mathscr{F}_X = \{\emptyset, X\}$.

Proof. To see that (1) implies (2), note that, by Theorem 58, for any $x \in X$, we have

$$\leq^{\infty}(x) = \bigcap \{A \in \mathscr{T}_X : \quad x \in A\}.$$

Therefore, for any $A \in \mathscr{T}_X$ and $x \in A$, we have $\leq^{\infty}(x) \subseteq A$. Moreover, if (1) holds, then $\leq^{\infty} = X^2$, and thus $\leq^{\infty}(x) = X$ for all $x \in X$. Therefore, if $A \neq \emptyset$, then $A = X$, and thus (2) also holds.

Remark 59. This theorem shows that, analogously to Example 1, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ in a well-chained goset X are also quite useless tools.

Now, in addition to Theorem 59, we can also easily prove the following.

Theorem 60. *For a proset X, the following assertions are equivalent:*

(1) X *is well-chained,*
(2) $\mathscr{E}_X = \{X\}$,
(3) $\mathscr{D}_X = \mathscr{P}(X) \setminus \{\emptyset\}$.

Proof. If (1) holds, then by Theorems 55 and 59, it is clear that (2) also holds. (Note that this implication can also be easily proved by using the corresponding definitions.)

On the other hand, if (2) holds, then by Remark 25, for any $x \in X$, we necessarily have $\mathrm{ub}_X(x) = X$, and thus $\leq(x) = X$. Therefore, $\leq = X^2$, and thus (1) also holds.

This shows that (1) and (2) are equivalent. Moreover, by Theorem 19, it is clear that (2) and (3) are always equivalent.

Remark 60. In [33], as a consequence of some other results, we have proved that if $X = X(\mathscr{R})$ is a relator space with $\mathscr{R} \neq \emptyset$ and $\mathrm{card}(X) > 1$, then X is paratopologically well-chained if and only if $\mathscr{E}_X = \{X\}$.

Moreover, X is paratopologically connected if and only if $\mathscr{E}_X \subseteq \mathscr{D}_X$. Therefore, the "hyperconnectedness," introduced by Levine [22] and studied by several further authors, is a particular case of our paratopological connectedness.

11 Comparisons of Inequalities

Because of the inclusion $\leq \subseteq \leq^{\infty}$, it is also of some interest to prove the following.

Theorem 61. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$, *the following assertions are equivalent :*

(1) $\leq_1 \subseteq \leq_2$,
(2) $\mathrm{ub}_{X_1} \subseteq \mathrm{ub}_{X_2}$,
(3) $\mathrm{lb}_{X_1} \subseteq \mathrm{lb}_{X_2}$.

Proof. If (1) holds, then by Remark 2, we have $\mathrm{ub}_{X_1}(x) = \leq_1 (x) \subseteq \leq_2 (x) = \mathrm{ub}_{X_2}(x)$ for all $x \in X$. Hence, by using Corollary 1, we can already infer that

$$\mathrm{ub}_{X_1}(A) = \bigcap_{a \in A} \mathrm{ub}_{X_1}(a) \subseteq \bigcap_{a \in A} \mathrm{ub}_{X_2}(a) = \mathrm{ub}_{X_2}(A)$$

for all $A \subseteq X$. Therefore, (2) also holds.

Conversely, if (2) holds, then in particular, we have

$$\mathrm{ub}_{X_1}(x) = \mathrm{ub}_{X_1}\big(\{x\}\big) \subseteq \mathrm{ub}_{X_2}\big(\{x\}\big) = \mathrm{ub}_{X_2}(x),$$

and hence $\leq_1 (x) \subseteq \leq_2 (x)$ for all $x \in X$. Therefore, (1) also holds.

This shows that (1) and (2) are equivalent. Hence, by using Theorem 1, we can easily see that (1) and (3) are also equivalent.

From this theorem, by Definition 8, it is clear that in particular we also have

Corollary 18. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$, *with* $\leq_1 \subseteq \leq_2$, *we have* $\mathscr{U}_{X_1} \subseteq \mathscr{U}_{X_2}$.

Proof. Namely, if $A \in \mathscr{U}_{X_1}$, then by Definition 8 we have $A \subseteq \mathrm{ub}_{X_1}(A)$. Moreover, by Theorem 61, now we also have $\mathrm{ub}_{X_1}(A) \subseteq \mathrm{ub}_{X_2}(A)$. Therefore, $A \subseteq \mathrm{ub}_{X_2}(A)$, and thus $A \in \mathscr{U}_{X_2}$ also holds.

Remark 61. Note if X is a reflexive and antisymmetric goset, then by Theorems 42 and 43 we have $\mathscr{U}_X = \{\{\emptyset\}\} \cup \{\{x\}\}_{x \in X}$.

Therefore, the converse of the above corollary need not be true even if in particular $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$ are posets.

However, by using Theorem 61, we can also easily prove the following.

Theorem 62. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$, *the following assertions are equivalent :*

(1) $\leq_1 \subseteq \leq_2$,
(2) $\mathrm{int}_{X_2} \subseteq \mathrm{int}_{X_1}$,
(3) $\mathrm{cl}_{X_1} \subseteq \mathrm{cl}_{X_2}$.

Proof. If $A \subseteq X$ and $x \in \operatorname{int}_{X_2}(A)$, then by Definition 2 we have $\operatorname{ub}_{X_2}(x) \subseteq A$. Moreover, if (1) holds, then by Theorem 61 we also have $\operatorname{ub}_{X_1}(x) \subseteq \operatorname{ub}_{X_2}(x)$. Therefore, $\operatorname{ub}_{X_1}(x) \subseteq A$, and thus $x \in \operatorname{int}_{X_1}(A)$ is also true. This, shows that $\operatorname{int}_{X_2}(A) \subseteq \operatorname{int}_{X_1}(A)$ for all $A \subseteq X$. Therefore, (2) also holds.

Moreover, if (2) holds, then by using Theorem 6 we can easily see that (3) also holds. Therefore, we need only show that (3) also implies (1). For this, note that if (3) holds, then in particular by Remark 12 we have

$$\operatorname{lb}_{X_1}(x) = \operatorname{cl}_{X_1}\big(\{x\}\big) \subseteq \operatorname{cl}_{X_2}\big(\{x\}\big) = \operatorname{lb}_{X_2}(x)$$

for all $x \in X$. Hence, by using Corollary 1, we can see that $\operatorname{lb}_{X_1}(A) \subseteq \operatorname{lb}_{X_2}(A)$ for all $A \subseteq X$. Therefore, $\operatorname{lb}_{X_1} \subseteq \operatorname{lb}_{X_2}$, and thus by Theorem 61 assertion (1) also holds.

From this theorem, by Definitions 3 and 4, it is clear that we also have

Corollary 19. *For any two gosets $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$, with $\leq_1 \subseteq \leq_2$, we have*

(1) $\mathscr{T}_{X_2} \subseteq \mathscr{T}_{X_1}$,
(2) $\mathscr{F}_{X_2} \subseteq \mathscr{F}_{X_1}$,
(3) $\mathscr{E}_{X_2} \subseteq \mathscr{E}_{X_1}$,
(4) $\mathscr{D}_{X_1} \subseteq \mathscr{D}_{X_2}$.

Proof. For instance, if $A \in \mathscr{D}_{X_1}$, then by Definition 4 we have $X = \operatorname{cl}_{X_1}(A)$. Moreover, by Theorem 62, now we also have $\operatorname{cl}_{X_1}(A) \subseteq \operatorname{cl}_{X_2}(A)$. Therefore, $X = \operatorname{cl}_{X_2}(A)$, and thus $A \in \mathscr{D}_{X_2}$ also holds. Therefore, (4) is true.

Now, by using the above results and Theorems 17 and 54, we can also prove

Theorem 63. *For any goset $X_1 = X(\leq_1)$ and proset $X_2 = X(\leq_2)$, the following assertions are equivalent:*

(1) $\leq_1 \subseteq \leq_2$,
(2) $\mathscr{T}_{X_2} \subseteq \mathscr{T}_{X_1}$,
(3) $\mathscr{F}_{X_2} \subseteq \mathscr{F}_{X_1}$.

Proof. If (1) holds, then by Corollary 19 assertion (2) also holds. Conversely, if (2) holds, then by Theorems 17 and 54 we have

$$\operatorname{int}_{X_2}(A) = \bigcup \mathscr{T}_{X_2} \cap \mathscr{P}(A) \subseteq \bigcup \mathscr{T}_{X_1} \cap \mathscr{P}(A) \subseteq \operatorname{int}_{X_1}(A)$$

for all $A \subseteq X$. Therefore, $\operatorname{int}_{X_2} \subseteq \operatorname{int}_{X_2}$, and thus by Theorem 62 assertion (1) also holds.

This shows that (1) and (2) are equivalent. Moreover, by Theorem 13, it is clear that (2) and (3) are always equivalent.

However, concerning fat and dense sets, we can only prove the following.

Theorem 64. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$*, the following assertions are equivalent:*

(1) $\mathscr{E}_{X_1} \subseteq \mathscr{E}_{X_2}$,

(2) $\mathscr{D}_{X_2} \subseteq \mathscr{D}_{X_1}$,

(3) There exists a function φ *of* X *to itself such that* $\leq_2 \circ \varphi \subseteq \leq_1$,

(4) There exists a relation R *of* X *to itself such that* $\leq_2 \circ R \subseteq \leq_1$.

Proof. By Remarks 2 and 25, for any $x \in X$, we have $\leq_1 (x) \in \mathscr{E}_{X_1}$. Therefore, if (1) holds, then we also have $\leq_1 (x) \in \mathscr{E}_{X_2}$. Hence, by using Remarks 2 and 25, we can infer that there exists $y \in X$ such that $\leq_2 (y) \subseteq \leq_1 (x)$.

Hence, by the axiom of choice, it is clear that there exists a function φ of X to itself such that $\leq_2 \big(\varphi(x)\big) \subseteq \leq_1 (x)$, and thus $(\leq_2 \circ \varphi)(x) \subseteq \leq_1 (x)$ for all $x \in X$. Therefore, (3) also holds.

On the other hand, if (3) holds, then by Remark 2 for any $x \in X$, we have $\mathrm{ub}_{X_2}\big(\varphi(x)\big) = \leq_2 \big(\varphi(x)\big) \subseteq \leq_1 (x) = \mathrm{ub}_{X_1}(x)$. Hence, by Remark 25, it is clear that (1) also holds.

Now, since (3) trivially implies (4), and (3) follows from (4) by choosing a selection function φ of R, it remains only to note that, by Theorem 18, assertions (1) and (2) are also equivalent.

Finally, we note that, by using the above theorem, we can also easily prove the following theorem whose converse seems not to be true.

Theorem 65. *If* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$ *are gosets, with* $\leq_1 \subseteq \leq_2$*, such that either* $\mathscr{E}_{X_1} \subseteq \mathscr{E}_{X_2}$ *or* $\mathscr{D}_{X_2} \subseteq \mathscr{D}_{X_1}$*, then there exists a function* φ *of* X *to itself such that* $\leq_1 = \leq_1 \circ \varphi^{\infty}$.

Proof. Now, by Theorem 64, there exists a function φ of X to itself such that $\leq_2 \circ \varphi \subseteq \leq_1$. Hence, by using that $\leq_1 \subseteq \leq_2$, we can already infer that

$$\leq_1 \circ \varphi \subseteq \leq_2 \circ \varphi \subseteq \leq_1 \subseteq \leq_2, \qquad \text{and thus} \qquad \leq_1 \circ \varphi^2 \subseteq \leq_2 \circ \varphi \subseteq \leq_1 .$$

Hence, by induction, it is clear that we actually have $\leq_1 \circ \varphi^n \subseteq \leq_1$ for all $n \in \mathbb{N}$. Moreover, we can also note that $\leq_1 \circ \varphi^0 = \leq_1 \circ \Delta_X = \leq_1$.

Hence, by using a basic theorem on relations, we can infer that

$$\leq_1 \circ \varphi^{\infty} = \leq_1 \circ \bigcup_{n=0}^{\infty} \varphi^n = \bigcup_{n=0}^{\infty} \leq_1 \circ \varphi^n \subseteq \bigcup_{n=0}^{\infty} \leq_1 = \leq_1 .$$

Thus, since $\leq_1 = \leq_1 \circ \varphi^0 \subseteq \leq_1 \circ \varphi^{\infty}$, the required equality is also true.

12 The Importance of the Preorder Closure and Complementation

From the inclusion $\leq \subseteq \leq^{\infty}$, by using Theorems 61 and 62 and the notation $X^{\infty} = X(\leq^{\infty})$, we can immediately derive the following.

Theorem 66. *For any goset X, we have*

(1) $\mathrm{ub}_X \subseteq \mathrm{ub}_{X^{\infty}}$,
(2) $\mathrm{lb}_X \subseteq \mathrm{lb}_{X^{\infty}}$,
(3) $\mathrm{int}_{X^{\infty}} \subseteq \mathrm{int}_X$,
(4) $\mathrm{cl}_X \subseteq \mathrm{cl}_{X^{\infty}}$.

Moreover, by using Corollary 19, Remark 56, and Theorem 13, we can also prove the following.

Theorem 67. *For any goset X, we have*

(1) $\mathscr{T}_X = \mathscr{T}_{X^{\infty}}$,
(2) $\mathscr{F}_X = \mathscr{F}_{X^{\infty}}$,
(3) $\mathscr{E}_{X^{\infty}} \subseteq \mathscr{E}_X$,
(4) $\mathscr{D}_X \subset \mathscr{D}_{X^{\infty}}$.

Proof. From Corollary 19, we can at once see that the inclusions (3), (4), and $\mathscr{T}_{X^{\infty}} \subseteq \mathscr{T}_X$ are true.

On the other hand, if $A \in \mathscr{T}_X$, then by Theorem 58 we have $\leq^{\infty}(x) \subseteq A$ for all $x \in A$. Hence, by Remark 18, we can see that $A \in \mathscr{T}_{X^{\infty}}$. Therefore, $\mathscr{T}_{X^{\infty}} \subseteq \mathscr{T}_X$, and thus (1) is also true. Hence, by Theorem 13, it is clear that (2) is also true.

Remark 62. Note that if X is as in Example 1, then $\mathscr{T}_X = \{\emptyset, X\}$, and thus by Theorems 59 and 60, we have $\mathscr{E}_{X^{\infty}} = \{X\}$. However, because of Remark 25, $\mathscr{E}_X$ is quite a large subfamily of $\mathscr{P}(X)$. Therefore, the equalities in (3) and (4) need not be true.

Now, by using Theorems 63 and 67 and Corollary 18, we can also prove

Theorem 68. *For any two gosets $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$, the following assertions are equivalent:*

(1) $\mathscr{T}_{X_2} \subseteq \mathscr{T}_{X_1}$,
(2) $\mathscr{F}_{X_2} \subseteq \mathscr{F}_{X_1}$,
(3) $\leq_1 \subseteq \leq_2^{\infty}$,
(4) $\leq_1^{\infty} \subseteq \leq_2^{\infty}$.

Proof. If (1) holds, then by Theorem 67 we can see that $\mathscr{T}_{X_2^{\infty}} \subseteq \mathscr{T}_{X_1}$ also holds. Hence, by using Theorem 63, we can already infer that (3) also holds.

Moreover, if (3) holds, then by using the corresponding properties of the operation ∞, we can also easily see that $\leq_1^{\infty} \subseteq \leq_2^{\infty\,\infty} = \leq_2^{\infty}$, and thus (4) also holds.

On the other hand, if (4) holds, then because of $\leq_1 \subseteq \leq_1^\infty$ it is clear that (3) also holds. Moreover, if (3) holds, then by using Theorem 67 and Corollary 19, we can see that $\mathscr{T}_{X_2} = \mathscr{T}_{X_2^\infty} \subseteq \mathscr{T}_{X_1}$, and thus (1) also holds. Now, to complete the proof, it remains only to note that, by Theorem 13, assertions (1) and (2) are also equivalent.

Remark 63. From this theorem, we can at once see that, for any two gosets $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$, we have

$$\mathscr{T}_{X_1} \subseteq^{-1} \mathscr{T}_{X_2} \iff X_1 \leq X_2^\infty,$$

in the sense that $\leq_1 \subseteq \leq_2^\infty$.

This shows that, analogously to Remarks 7 and 16, the set-valued functions $\mathscr{T}$ and ∞ also form a Pataki connection.

Thus, the counterparts of the corresponding parts of Remarks 8 and 16 can also be stated. However, it would be more interesting to look for a generating Galois connection.

Now, by Theorems 64 and 65, we can also state the following two theorems.

Theorem 69. *For any goset X, the following assertions are equivalent :*

(1) $\mathscr{E}_X \subseteq \mathscr{E}_{X^\infty}$,
(2) $\mathscr{D}_{X^\infty} \subseteq \mathscr{D}_X$,
(3) there exists a function φ of X to itself such that $\leq^\infty \circ \varphi \subseteq \leq$,
(4) there exists a relation R of X to itself such that $\leq^\infty \circ R \subseteq \leq$.

Remark 64. Note that, by Theorem 67, we may write equality in the assertions (1) and (2) of the above theorem and also in the conditions of the following.

Theorem 70. *If X is a goset, such that $\mathscr{E}_X \subseteq \mathscr{E}_{X^\infty}$, or equivalently $\mathscr{D}_{X^\infty} \subseteq \mathscr{D}_X$, then there exists function φ of X to itself such that $\leq = \leq \circ \varphi^\infty$.*

Finally, we note that, by using the notation $X^c = X(\leq^c)$, we can also prove the following particular case of [47, Theorem 4.11], which in addition to the results of [17, 57] also shows the importance of complement relations.

Theorem 71. *For any goset X, we have*

(1) $\text{lb}_X = \big(\text{cl}_{X^c}\big)^c$,
(2) $\text{cl}_X = \big(\text{lb}_{X^c}\big)^c$.

Proof. By using Remarks 4 and 13, instead of Corollary 1 and Theorem 10, we can at once see that

$$\text{lb}_X^c(A) = \text{lb}_X(A)^c = \geq^c [A] = \text{cl}_{X^c}(A)$$

for all $A \subseteq X$. Therefore, $\text{lb}_X^c = \text{cl}_{X^c}$, and thus (1) is also true.

Now, (2) can be immediately derived from (1) by writing X^c in place of X and applying complementation.

Remark 65. This theorem shows that the relations lb_X and cl_X are also equivalent tools in the goset X.

Hence, by Remarks 3 and 10, it is clear that the relations ub_X and int_X are also equivalent tools in the goset X.

Remark 66. By using Theorem 1 and Corollary 3, and the corresponding properties of inversion and complementations, the assertions (1) and (2) of Theorem 71 can be reformulated in several different forms.

For instance, as an immediate consequence of Theorem 71 and Corollary 3, we can at once state the following.

Corollary 20. *For any goset X, we have*

(1) $\mathrm{lb}_X = \mathrm{int}_{X^c} \circ \mathscr{C}$,
(2) $\mathrm{int}_X = \mathrm{lb}_{X^c} \circ \mathscr{C}$.

Remark 67. Analogously to Theorem 10, the above results also show that, despite Remark 2, there are cases when the relation lb_X is a more convenient tool in the goset X than ub_X.

13 Some Further Results on the Basic Tools

As some converses to Theorems 3, 9, 16, and 24, we can also easily prove the following theorems.

Theorem 72. *If Φ is a relation on $\mathscr{P}(X)$ to X, for some set X, such that*

$$\Phi\Big(\bigcup_{i\in I} A_i\Big) = \bigcap_{i\in I} \Phi\big(A_i\big)$$

for any family $(A_i)_{i\in I}$ subsets of X, then there exists a relation $\leq$ on X such that, under the notation $X = X(\leq)$, we have $\Phi = \mathrm{ub}_X$ $(\Phi = \mathrm{lb}_X)$.

Proof. For any $x, y \in X$, define $x \leq y$ if $y \in \Phi(x)$, where $\Phi(x) = \Phi\big(\{x\}\big)$. Then, by Remark 2, we have $\Phi(x) = \mathrm{ub}_X(x)$ for all $x \in X$. Hence, by using the assumed union-reversingness of Φ and Corollary 1, we can already see that

$$\Phi(A) = \bigcap_{a\in A} \Phi(a) = \bigcap_{a\in A} \mathrm{ub}_X(a) = \mathrm{ub}_X(A)$$

for all $A \subseteq X$. Therefore, $\Phi = \mathrm{ub}_X$ is also true.

This proves the first statement of the theorem. The second statement can be derived from the first one by using Theorem 1.

Theorem 73. *If Ψ is a relation on $\mathscr{P}(X)$ to X, for some set X, such that*

$$\Psi\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} \Psi\big(A_i\big)$$

for any family $(A_i)_{i\in I}$ *subsets of* X*, then there exists a relation* $\leq$ *on* X *such that, under the notation* $X = X(\leq)$*, we have* $\Psi = \mathrm{cl}_X$.

Proof. For any $x, y \in X$, define $x \leq y$ if $x \in \Psi(y)$, where $\Psi(y) = \Psi(\{y\})$. Then, by Remark 2, we have $\mathrm{lb}_X(y) = \Psi(y)$ for all $y \in X$. Hence, by using the assumed union preservingness of Ψ and Theorem 10, we can already see that

$$\Psi(A) = \bigcup_{a\in A} \Psi(a) = \bigcup_{a\in A} \mathrm{lb}_X(a) = \mathrm{cl}_X(A)$$

for all $A \subseteq X$. Therefore, the required equality is also true.

From this theorem, by using Corollary 3, we can easily derive the following.

Corollary 21. *If* Φ *is a relation on* $\mathscr{P}(X)$ *to* X*, for some set* X*, such that*

$$\Phi\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} \Phi\big(A_i\big)$$

for any family $\big(A_i\big)_{i\in I}$ *of subsets of* X*, then there exists a relation* $\leq$ *on* X *such that, under the notation* $X = X(\leq)$*, we have* $\Phi = \mathrm{int}_X$.

Proof. Define $\Psi = (\Phi \circ \mathscr{C})^c$. Then, by using the assumed intersection-preservingness of Φ and De Morgan's law, we can see that Ψ is an union-preserving relation on $\mathscr{P}(X)$ to X. Therefore, by Theorem 73, there exists a relation $\leq$ on X such that in the goset $X = X(\leq)$ we have $\Psi = \mathrm{cl}_X$. Hence, by using the definition of Ψ and Corollary 3, we can see that $\Phi = \big(\Psi \circ \mathscr{C}\big)^c = \big(\mathrm{cl}_X \circ \mathscr{C}\big)^c = \mathrm{int}_X$ also holds.

Theorem 74. *If* $\mathscr{A}$ *is a family of subsets of a set* X *such that* $\mathscr{A}$ *is closed under arbitrary unions and intersections, then there exists a preorder relation* $\leq$ *on* X *such that, under the notation* $X = X(\leq)$*, we have* $\mathscr{A} = \mathscr{T}_X$ $(\mathscr{A} = \mathscr{F}_X)$.

Proof. Define

$$\leq\, = \bigcap_{A\in A} R_A \qquad \text{where} \qquad R_A = A^2 \cup A^c \times X\,.$$

Then, from the proof of Theorem 58, we know that $\leq$ is a preorder relation on X such that, under the notation $X = X(\leq)$, for any $x \in X$ we have

$$\mathrm{ub}_X(x) = \leq(x) = \bigcap \big\{A \in \mathscr{A} : \quad x \in A\big\}.$$

Hence, since $\mathscr{A}$ is closed under arbitrary intersections, it is clear that $\mathrm{ub}_X(x) \in \mathscr{A}$ for all $x \in X$. Moreover, we can also note that $x \in \mathrm{ub}_X(x)$ for all $x \in X$.

Therefore, if $V \in \mathscr{T}_X$, that is, by Remark 18 we have $\mathrm{ub}_X(x) \subseteq V$ for all $x \in V$, then we necessarily have $V = \bigcup_{x\in V} \mathrm{ub}_X(x)$. Hence, since $\mathscr{A}$ is also closed under arbitrary unions, it is clear that $V \in \mathscr{A}$. Therefore, $\mathscr{T}_X \subseteq \mathscr{A}$.

Conversely, if $V \in \mathscr{A}$, then for any $x \in V$ we have

$$\text{ub}_X(x) = \bigcap \{ A \in \mathscr{A} : \quad x \in A \} \subseteq V .$$

Therefore, by Remark 18, we have $V \in \mathscr{T}_X$. Thus, $\mathscr{A} \subseteq \mathscr{T}_X$ also holds.

This proves that $\mathscr{A} = \mathscr{T}_X$, and thus the first statement of the theorem is true. The second statement of the theorem can be derived from the first one by using Theorem 14.

Remark 68. In principle, the first statement of the above theorem can also be proved with the help of Corollary 21 . However, this proof requires an intimate connection between interior operations and families of sets.

For this, one can note that if Φ is a relation on $\mathscr{P}(X)$ to X such that

$$\Phi(B) = \bigcup \big(\mathscr{A} \cap \mathscr{P}(B) \big)$$

for all $B \subseteq X$, then by this definition and the assumed union property of $\mathscr{A}$, we have

(a) $\mathscr{A} = \{ B \subseteq X : \quad B = \Phi(B) \}$, (b) $\Phi(B) \in \mathscr{A} \cap \mathscr{P}(B)$ for all $B \subseteq X$.

Moreover, by using (b), the assumed intersection property of $\mathscr{A}$ and the definition of Φ, we can see that Φ is union preserving.

However, it is now more important to note that, analogously to Theorem 74, we also have the following.

Theorem 75. *If $\mathscr{A}$ is a nonvoid stack in X, for some set X, having a base $\mathscr{B}$ with $\text{card}(\mathscr{B}) \leq \text{card}(X)$, then there exists a relation $\leq$ on X such that, under the notation $X = X(\leq)$, we have $\mathscr{A} = \mathscr{E}_X$.*

Proof. Since $\text{card}(\mathscr{B}) \leq \text{card}(X)$, there exists an injective function φ of $\mathscr{B}$ onto a subset Y of X. Choose $B \in \mathscr{B}$ and define a relation $\leq$ on X such that

$$\leq (x) = \varphi^{-1}(x) \quad \text{if} \quad x \in Y \qquad \text{and} \qquad \leq (x) = B \quad \text{if} \quad x \in Y^c .$$

Then, under the notation $X = X(\leq)$, we evidently have

$$\mathscr{B} = \{ \text{ub}_X(x) : \quad x \in X \} .$$

Hence, since $\mathscr{B}$ is a base of $\mathscr{A}$, we can already infer that

$$\mathscr{A} = \{ A \subseteq X : \quad \exists\, x \in X : \quad \text{ub}_X(x) \subseteq A \} = \mathscr{E}_X .$$

Remark 69. Now, a corresponding theorem for the family $\mathscr{D}_X$ should, in principle, be derived from the above theorem by using either Theorem 18 or 19 .

However, it would now be even more interesting to prove a counterpart of Theorems 74 and 75 for the family $\mathscr{U}_X$.

14 Increasing Functions

Increasing functions are usually called isotone, monotone, or order-preserving in algebra. Moreover, in [11, p. 186] even the extensive maps are called increasing. However, we prefer to use the following terminology of analysis [38, p. 128].

Definition 12. If f is a function of one goset X to another Y, then we say that:

(1) f is *increasing* if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$.
(2) f is *strictly increasing* if $u < v$ implies $f(u) < f(v)$ for all $u, v \in X$.

Remark 70. Quite similarly, the function f may, for instance, be called *decreasing* if $u \leq v$ implies $f(v) \leq f(u)$ for all $u, v \in X$.

Thus, we can note that f is a decreasing function of X to Y if and only if it is an increasing function of X to the dual Y^{-1} of Y.

Therefore, the study of decreasing functions can be traced back to that of the increasing ones. The following two obvious theorems show that almost the same is true in connection with the strictly increasing ones.

Theorem 76. *If f is an injective, increasing function of one goset X to another Y, then f is strictly increasing.*

Remark 71. Conversely, we can at once see that if f is a strictly increasing function of an arbitrary goset X to a reflexive one Y, then f is increasing.

Moreover, we can also easily prove the following

Theorem 77. *If f is a strictly increasing function of a linear goset X to an arbitrary one Y, then f is injective.*

Proof. If $u, v \in X$ such that $u \neq v$, then by Remark 47 we have either $u < v$ or $v < u$. Hence, by using the strict increasingness of f, we can already infer that either $f(u) < f(v)$ or $f(v) < f(u)$, and thus $f(u) \neq f(v)$.

Now, as an immediate consequence of the above results, we can also state

Corollary 22. *For a function f of a linear goset X to a reflexive one Y, the following assertions are equivalent:*

(1) f is strictly increasing,
(2) f is injective and increasing.

In this respect, the following is also worth proving.

Theorem 78. *If f is a strictly increasing function of a linear goset X onto an antisymmetric one Y, then f^{-1} is a strictly increasing function of Y onto X.*

Proof. From Theorem 77, we know that f is injective. Hence, since $f[X] = Y$, we can see that f^{-1} is a function of Y onto X. Therefore, we need only show that f^{-1} is also strictly increasing.

For this, suppose that $z, w \in Y$ such that $z < w$. Define $u = f^{-1}(z)$ and $v = f^{-1}(w)$. Then, $u, v \in X$ such that $z = f(u)$ and $w = f(v)$. Hence, since $z \neq w$, we can also see that $u \neq v$. Moreover, by Remark 47, we have either $u < v$ or $v < u$. However, if $v < u$, then by the strict increasingness of f we also have $f(v) < f(u)$, and thus $w < z$. Hence, by using the inequality $z < w$ and the antisymmetry of Y, we can already infer that $z = w$. This contradiction proves that $u < v$, and thus $f^{-1}(z) < f^{-1}(w)$.

Hence, by using Theorem 76 and Remark 71, we can immediately derive

Corollary 23. *If f is an injective, increasing function of a reflexive, linear goset X onto an antisymmetric one Y, then f^{-1} is an injective, increasing function of Y onto X.*

Analogously to [58], we shall now also use the following.

Definition 13. If φ is an unary operation on a goset X, then we say that:

(1) φ is *extensive (intensive)* if $\Delta_X \leq \varphi$ $\big(\varphi \leq \Delta_X\big)$.
(2) φ is *upper (lower) semi-idempotent* if $\varphi \leq \varphi^2$ $\big(\varphi^2 \leq \varphi\big)$.

Remark 72. Moreover, φ may be naturally called *upper (lower) semi-involutive* if φ^2 is extensive (intensive). That is, $\Delta_X \leq \varphi^2$ $(\varphi^2 \leq \Delta_X)$.

Remark 73. In this respect, it is also worth noticing that φ is upper (lower) semi-idempotent if and only if its restriction to its range is extensive (intensive). Therefore, if φ is extensive (intensive), then φ is upper (lower) semi-idempotent.

The importance of extensive operations is also apparent from the following.

Theorem 79. *If φ is a strictly increasing operation on a min-complete, antisymmetric goset X, then φ is extensive.*

Proof. If φ is not extensive, then the set $A = \{x \in X : \ x \not\leq \varphi(x)\}$ is not void. Thus, by the min-completeness of X, there exists $a \in \min_X(A)$. Hence, by the definition of $\min_X$, we can see that $a \in A$ and $a \in \operatorname{lb}_X(A)$. Thus, in particular, by the definition of A, we have $a \not\leq \varphi(a)$. Hence, by using Corollary 12 and Theorem 49, we can infer that $\varphi(a) < a$. Thus, since φ is strictly increasing, we also have $\varphi\big(\varphi(a)\big) < \varphi(a)$. Hence, by using Theorem 48, we can infer that $\varphi(a) \not\leq \varphi\big(\varphi(a)\big)$. Thus, by the definition of A, we also have $\varphi(a) \in A$. Hence, by using that $a \in \operatorname{lb}_X(A)$, we can infer that $a \leq \varphi(a)$. This contradiction shows that φ is extensive.

Remark 74. To feel the importance of extensive operations, it is also worth noticing that if φ is an extensive operation on an antisymmetric goset, then each maximal element x of X is already a fixed point of φ in the sense that $\varphi(x) = x$.

This fact has also been strongly emphasized by Brøndsted [6]. Moreover, fixed point theorems for extensive maps (which are sometimes called expansive, progressive, increasing, or inflationary) were also proved in [19], [11, p. 188], and [29].

The following theorem shows that, in contrast to the injective, increasing functions, the inverse of an injective, extensive operation need not be extensive.

Theorem 80. *If φ is an injective, extensive operation on an antisymmetric goset X such that $X = \varphi[X]$ and φ^{-1} is also extensive, then $\varphi = \Delta_X$.*

Proof. By the extensivity of φ and φ^{-1}, for every $x \in X$, we have $x \leq \varphi(x)$ and $\varphi(x) \leq \varphi^{-1}(\varphi(x))$. Hence, by noticing that $\varphi^{-1}(\varphi(x)) = x$ and using the antisymmetry of X, we can already infer that $\varphi(x) = x$, and thus $\varphi(x) = \Delta_X(x)$. Therefore, the required equality is also true.

From this theorem, by using Theorems 78 and 79, we can immediately derive

Corollary 24. *If φ is a strictly increasing operation on a min-complete, antisymmetric goset X such that $X = \varphi[X]$, then $\varphi = \Delta_X$.*

Proof. Now, from Corollary 12 and Theorem 78, we can see that φ^{-1} is also strictly increasing. Thus, by Theorem 79, both φ and φ^{-1} are extensive. Therefore, by Theorem 80, the required equality is also true.

In general, the idempotent operations are quite different from both upper and lower semi-idempotent ones. However, we may still naturally have the following.

Definition 14. An increasing, extensive (intensive) operation is called a *preclosure (preinterior) operation*. And, a lower semi-idempotent (upper semi-idempotent) preclosure (preinterior) operation is called a *closure (interior) operation*.

Moreover, an extensive (intensive) lower semi-idempotent (upper semi-idempotent) operation is called a *semiclosure (semi-interior) operation*. While, an increasing and upper (lower) semi-idempotent operation is called an *upper (lower) semimodification operation*.

Remark 75. Thus, φ is, for instance, an interior operation on a goset X if and only if it is a closure operation on the dual X^{-1} of X.

15 Algebraic Properties of Increasing Functions

Concerning increasing functions, we can also prove the following.

Theorem 81. *For a function f of one goset X to another Y, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $f[\mathrm{ub}_X(x)] \subseteq \mathrm{ub}_Y(f(x))$ *for all* $x \in X$,
(3) $f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y(f[A])$ *for all* $A \subseteq X$.

Proof. If $A \subseteq X$ and $y \in f[\mathrm{ub}_X(A)]$, then there exists $x \in \mathrm{ub}_X(A)$ such that $y = f(x)$. Thus, for any $a \in A$, we have $a \leq x$. Hence, if (1) holds, we can infer that $f(a) \leq f(x)$, and thus $f(a) \leq y$. Therefore, $y \in \mathrm{ub}_Y(f[A])$, and thus (3) also holds.

The remaining implications $(3) \Longrightarrow (2) \Longrightarrow (1)$ are even more obvious.

From this theorem, by using Definition 8, we can immediately derive

Corollary 25. *If f is an increasing function of one goset X to another Y, then for any $A \in \mathcal{U}_X$ we have $f[A] \in \mathcal{U}_Y$.*

Proof. Namely, if $A \in \mathcal{U}_X$, then by Definition 8, we have $A \subseteq \mathrm{ub}_X(A)$. Hence, by using Theorem 81, we can infer that $f[A] \subseteq f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$. Thus, by Definition 8, we also have $f[A] \in \mathcal{U}_Y$.

Moreover, by using Theorem 81, we can also prove the following.

Theorem 82. *If f is an increasing function of one goset X onto another Y, then for any $B \subseteq Y$ we have*

$$\mathrm{ub}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{ub}_Y(B)].$$

Proof. Now, by Theorem 81 and a basic theorem on relations, we have

$$f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] \subseteq \mathrm{ub}_Y\left(f\left[f^{-1}[B]\right]\right) = \mathrm{ub}_Y\left(\left(f \circ f^{-1}\right)[B]\right).$$

Moreover, by using that Y is the range of f, we can easily see that $\Delta_Y \subseteq f \circ f^{-1}$. Hence, we can immediately infer that $B \subseteq \left(f \circ f^{-1}\right)[B]$, and thus also

$$\mathrm{ub}_Y\left(\left(f \circ f^{-1}\right)[B]\right) \subseteq \mathrm{ub}_Y(B).$$

Therefore, we actually have $f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] \subseteq \mathrm{ub}_Y(B)$, and thus also

$$\left(f^{-1} \circ f\right)\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] = f^{-1}\left[f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right]\right] \subseteq f^{-1}[\mathrm{ub}_Y(B)].$$

Moreover, since X is the domain of f, we can note that $\Delta_X \subseteq f^{-1} \circ f$, and thus

$$\mathrm{ub}_X\big(f^{-1}[B]\big) \subseteq \left(f^{-1} \circ f\right)\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right].$$

Therefore, the required inclusion is also true.

Now, as a partial converse to this theorem, we can also prove the following.

Theorem 83. *If f is an injective function of one goset X to another Y such that*

$$\mathrm{ub}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{ub}_Y(B)]$$

for all $B \subseteq X$, then f is increasing.

Proof. Now, by some basic theorems on relations, for any $B \subseteq Y$, we also have

$$f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] \subseteq f\left[f^{-1}[\mathrm{ub}_Y(B)]\right] = \left(f \circ f^{-1}\right)[\mathrm{ub}_Y(B)].$$

Moreover, since f is a function, we also have $f \circ f^{-1} \subseteq \Delta_X$, and thus also $\big(f \circ f^{-1}\big)\,[\,\mathrm{ub}_Y(B)\,] \subseteq \mathrm{ub}_Y(B)$. Therefore, we actually have

$$f\big[\,\mathrm{ub}_X\big(f^{-1}[\,B\,]\big)\,\big] \subseteq \mathrm{ub}_Y(B)\,.$$

Hence, it is clear that, for any $A \subseteq X$, we have

$$f\big[\,\mathrm{ub}_X\,\big(\big(f^{-1}\circ f\,\big)\,[\,A\,]\big)\,\big] = f\big[\,\mathrm{ub}_X\,\big(f^{-1}\big[\,f\,[\,A\,]\,\big]\big)\,\big] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)\,.$$

Moreover, by using that f is injective, we can note that $f^{-1} \circ f \subseteq \Delta_X$, and thus also $\big(f^{-1}\circ f\,\big)\,[\,A\,] \subseteq A$. Hence, we can infer that $\mathrm{ub}_x(A) \subseteq \mathrm{ub}_X\,\big(\big(f^{-1}\circ f\,\big)\,[\,A\,]\big)$, and thus also

$$f\,[\,\mathrm{ub}_X(A)\,] \subseteq f\big[\,\mathrm{ub}_X\,\big(\big(f^{-1}\circ f\,\big)\,[\,A\,]\big)\,\big].$$

Therefore, we actually have

$$f\,[\,\mathrm{ub}_X(A)\,] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)\,.$$

Hence, by Theorem 81, we can already see that f is increasing.

Remark 76. Note that f is an increasing function of X to Y if and only if it is an increasing function of X^{-1} to Y^{-1}.

Therefore, in the above theorems, we may write lb in place of ub. However, because of Theorems 29 and 4, we cannot write sup instead of ub.

Despite this, by using Theorem 81, we can also prove the following.

Theorem 84. *For a function f of a reflexive goset X to an arbitrary one Y, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $f\,[\,\max_X(A)\,] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$,
(3) $f\,[\,\max_X(A)\,] \subseteq \max_Y\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$,
(4) $f\,[\,\max_X(A)\,] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$ *with* $\mathrm{card}(A) \leq 2$.

Proof. If (1) holds, then by Theorem 81 and a basic theorem on relations, for any $A \subseteq X$, we have

$$\begin{aligned} f\,[\,\max_X(A)\,] &= f\,[\,A \cap \mathrm{ub}_X(A)\,] \subseteq f\,[\,A\,] \cap f\,[\,\mathrm{ub}_X(A)\,] \\ &\subseteq f\,[\,A\,] \cap \mathrm{ub}_Y\big(f\,[\,A\,]\big) = \max_Y\big(f\,[\,A\,]\big). \end{aligned}$$

Therefore, (3) also holds even if X is not assumed to be reflexive.

Thus, since the implication $(3) \Longrightarrow (2) \Longrightarrow (4)$ trivially hold, we need only show that (4) also implies (1). For this, note that if $u,\ v \in X$ such that $u \leq v$, then by taking $A = \{u, v\}$ and using the reflexivity of X we can see that $v \in \mathrm{ub}_X(A)$, and thus

$$v \in A \cap \mathrm{ub}_X(A) = \max_X(A).$$

Hence, if (4) holds, we can infer that

$$f(v) \in f[\max_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big) = \mathrm{ub}_Y\big(\{f(u), f(v)\}\big).$$

Thus, in particular $f(u) \leq f(v)$, and thus (1) also holds.

Now, as a useful consequence of this theorem, we can also easily prove

Corollary 26. *If f is a function on a reflexive goset X to an arbitrary one Y such that*

$$f[\sup_X(A)] \subseteq \sup_Y\big(f[A]\big)$$

for all $A \subseteq X$ with $\mathrm{card}(A) \leq 2$, then f is already increasing.

Proof. If A is as above, then by Theorems 29 and 32 we have

$$f[\max_X(A)] \subseteq f[\sup_X(A)] \subseteq \sup_Y\big(f[A]\big) \subseteq \mathrm{ub}_Y\big(f[A]\big).$$

Therefore, by Theorem 84, the required assertion is also true.

Because of Theorems 29 and 4, a converse of this corollary is certainly not true. However, by using Theorem 81, we can also prove the following two theorems.

Theorem 85. *If f is an increasing function of one goset X to another Y, then for any $A \subseteq X$ we have*

$$\mathrm{lb}_Y\Big(\mathrm{ub}_Y\big(f[A]\big)\Big) \subseteq \mathrm{lb}_Y\big(f[\mathrm{ub}_X(A)]\big).$$

Proof. Now, by Theorem 81, we have $f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$. Hence, by using Theorem 4, we can immediately derive the required inclusion.

Theorem 86. *If f is an increasing function of one sup-complete, antisymmetric goset X to another Y, then for any $A \subseteq X$ we have*

$$\sup_Y\big(f[A]\big) \leq f\big(\sup_X(A)\big).$$

Proof. If $\alpha = \sup_X(A)$, then by Theorems 29 and 45 and, and the usual identification of singletons with their elements, we also have $\alpha \in \mathrm{ub}_X(A)$, and thus $f(\alpha) \in f[\mathrm{ub}_X(A)]$. Hence, by using Theorem 81, we can already infer that $f(\alpha) \in \mathrm{ub}_Y\big(f[A]\big)$.

While, if $\beta = \sup_Y\big(f[A]\big)$, then by Theorems 29 and 45, and the usual identification of singletons with their elements, we also have $\beta \in \mathrm{lb}_Y\big(\mathrm{ub}_Y\big(f[A]\big)\big)$. Hence, by using that $f(\alpha) \in \mathrm{ub}_Y\big(f[A]\big)$, we can already infer that $\beta \leq f(\alpha)$, and thus the required equality is also true.

By using the dual of Theorem 81 mentioned in Remark 76, we can quite similarly prove the following theorem which can also be derived from Theorem 86 by dualization.

Theorem 87. *If f is an increasing function of one inf-complete, antisymmetric goset X to another Y, then for any $A \subseteq X$ we have*

$$f\big(\inf_X(A)\big) \leq \inf_Y\big(f[A]\big).$$

Remark 77. Note that, by Theorem 34, in the latter theorem we may also write sup-complete instead of inf-complete.

Therefore, as an immediate consequence of Theorems 86 and 87, we can state

Corollary 27. *If f is an increasing function of a sup-complete, antisymmetric goset X to a sup-complete, transitive and antisymmetric goset Y, and A is a nonvoid subset of X such that $f\big(\inf_X(A)\big) = f\big(\sup_X(A)\big)$, then*

$$\inf_Y\big(f[A]\big) = f\big(\inf_X(A)\big) \qquad \textit{and} \qquad \sup_Y\big(f[A]\big) = f\big(\sup_X(A)\big).$$

16 Topological Properties of Increasing Functions

In principle, the following theorem can be derived from the dual Theorem 81 by using Theorem 71. However, it is now more convenient to give a direct proof.

Theorem 88. *For a function f of one goset X to another Y, the following assertions are equivalent:*

(1) f is increasing,
(2) $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big)$ for all $A \subseteq X$,
(3) $\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B)]$ for all $B \subseteq B \subseteq Y$,
(4) $f^{-1}[\mathrm{int}_Y(B)] \subseteq \mathrm{int}_X\big(f^{-1}[B]\big)$ for all $B \subseteq Y$.

Proof. If $A \subseteq X$ and $y \in f[\mathrm{cl}_X(A)]$, then there exists $x \in \mathrm{cl}_X(A)$ such that $y = f(x)$. Thus, by Definition 2, we have $\mathrm{ub}_X(x) \cap A \neq \emptyset$. Therefore, there exists $a \in A$ such that $a \in \mathrm{ub}_X(x)$, and thus $x \leq a$. Hence, if (1) holds, we can infer that $f(x) \leq f(a)$, and thus $f(a) \in \mathrm{ub}_Y\big(f(x)\big) = \mathrm{ub}_Y(y)$. Now, since $f(a) \in f[A]$ also holds, we can already see that $f(a) \in \mathrm{ub}_Y(y) \cap f[A]$, and thus $\mathrm{ub}_Y(y) \cap f[A] \neq \emptyset$. Therefore, by Definition 2, we also have $y \in \mathrm{cl}_Y\big(f[A]\big)$. This shows that $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big)$, and thus (2) also holds.

While, if $B \subseteq Y$, then $f^{-1}[B] \subseteq X$. Therefore, if (2) holds, then we have

$$f\big[\mathrm{cl}_X(f^{-1}[B])\big] \subseteq \mathrm{cl}_Y\big(f\big[f^{-1}[B]\big]\big) = \mathrm{cl}_Y\big(\big(f \circ f^{-1}\big)[B]\big).$$

Moreover, since f is a function, we can easily see that $f \circ f^{-1} \subseteq \Delta_Y$, and thus $\big(f \circ f^{-1}\big)[B] \subseteq B$. Hence, by using Theorem 11, we can infer that

$$\mathrm{cl}_Y\big(\big(f \circ f^{-1}\big)[B]\big) \subseteq \mathrm{cl}_Y(B).$$

Therefore, we actually have $f\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big] \subseteq \mathrm{cl}_Y(B)$, and thus also

$$\big(f^{-1} \circ f\big)\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big] = f^{-1}\big[f\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big]\big] \subseteq f^{-1}\big[\mathrm{cl}_Y(B)\big].$$

Moreover, since X is the domain of f, we can note that $\Delta_X \subseteq f^{-1} \circ f$, and thus

$$\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq \big(f^{-1} \circ f\big)\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big].$$

Therefore, we actually have $\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B)]$, and thus (3) also holds.

On the other hand, if $B \subseteq Y$, then by using Theorem 6 and a basic fact on inverse images, we can also see that

$$f^{-1}[\mathrm{int}_Y(B)] = f^{-1}\big[\mathrm{cl}_Y(B^c)^c\big] = f^{-1}\big[\mathrm{cl}_Y(B^c)\big]^c.$$

Moreover, if (3) holds, then we can also see that $\mathrm{cl}_X\big(f^{-1}[B^c]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B^c)]$, and thus

$$f^{-1}[\mathrm{cl}_Y(B^c)]^c \subseteq \mathrm{cl}_X\big(f^{-1}[B^c]\big)^c = \mathrm{cl}_X\big(f^{-1}[B]^c\big)^c = \mathrm{int}_X\big(f^{-1}[B]\big).$$

This shows that $f^{-1}[\mathrm{int}_Y(B)] \subseteq \mathrm{int}_X\big(f^{-1}[B]\big)$, and thus (4) also holds.

Now, it remains to show that (4) also implies (1). For this, note that, by Definition 2, for any $x \in X$ we have $f(x) \in \mathrm{int}_Y\big(\mathrm{ub}_Y\big(f(x)\big)\big)$, and thus

$$x \in f^{-1}\big(f(x)\big) \subseteq f^{-1}\big[\mathrm{int}_Y\big(\mathrm{ub}_Y\big(f(x)\big)\big)\big].$$

Moreover, if (4) holds, then we also have

$$f^{-1}\big[\mathrm{int}_Y\big(\mathrm{ub}_Y\big(f(x)\big)\big)\big] \subseteq \mathrm{int}_X\big(f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]\big).$$

This shows that $x \in \mathrm{int}_X\big(f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]\big)$, and thus by Definition 2 we have $\mathrm{ub}_X(x) \subseteq f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]$. Hence, we can already infer that

$$f[\mathrm{ub}_X(x)] \subseteq f\big[f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]\big] = \big(f \circ f^{-1}\big)\big[\mathrm{ub}_Y\big(f(x)\big)\big] \subseteq \mathrm{ub}_Y\big(f(x)\big).$$

Therefore, by Theorem 81, assertion (1) also holds.

From this theorem, by using Definition 3, we can immetiately derive

Corollary 28. *If f is an increasing function of one goset X to another Y, then*

(1) $B \in \mathscr{T}_Y$ *implies* $f^{-1}[B] \in \mathscr{T}_X$,
(2) $B \in \mathscr{F}_Y$ *implies* $f^{-1}[B] \in \mathscr{F}_X$.

Proof. If $B \in \mathscr{T}_Y$, then by Definition 3 we have $B \subseteq \operatorname{int}_Y(B)$. Hence, by using Theorem 88 and the increasingness of f, we can already infer that

$$f^{-1}[B] \subseteq f^{-1}[\operatorname{int}_Y(B)] \subseteq \operatorname{int}_X\big(f^{-1}[B]\big).$$

Therefore, by Definition 3, we also have $f^{-1}[B] \in \mathscr{T}_X$.

This shows that (1) is true. Moreover, by using Theorem 13, we can easily see that (1) and (2) are equivalent even if f is not assumed to be increasing.

For instance, if $B \in \mathscr{F}_Y$, then by Theorem 13, we have $B^c \in \mathscr{T}_X$. Hence, if (1) holds, we can infer $f^{-1}[B^c] \in \mathscr{T}_X$. Now, by using that $f^{-1}[B^c] = f^{-1}[B]^c$, we can already see that $f^{-1}[B]^c \in \mathscr{T}_X$, and thus by Theorem 13 we also have $f^{-1}[B] \in \mathscr{F}_X$. Therefore, (2) also holds.

Remark 78. Moreover, if f is as in the above corollary, then by using the assertion (2) of Theorem 88 we can immediately see that if $A \subseteq X$ such that $f[A] \in \mathscr{F}_Y$, then $f[\operatorname{cl}_X(A)] \subseteq f[A]$. Note that this fact can also be derived from Corollary 28.

However, it is now more important to note that, in addition to the Corollary 28, we can also prove the following.

Theorem 89. *For a function f of a goset X to a proset Y, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $B \in \mathscr{T}_Y$ *implies* $f^{-1}[B] \in \mathscr{T}_X$,
(3) $B \in \mathscr{F}_Y$ *implies* $f^{-1}[B] \in \mathscr{F}_X$.

Proof. Now, by Corollary 28 and its proof, we need actually show only that (3) also implies (1). For this, note that if $B \subseteq Y$, then by Corollary 14 we have $\operatorname{cl}_Y(B) \in \mathscr{F}_Y$. Hence, if (3) holds, we can infer that $f^{-1}[\operatorname{cl}_Y(B)] \in \mathscr{F}_X$. Therefore, by Definition 3, we have

$$\operatorname{cl}_X\big(f^{-1}[\operatorname{cl}_Y(B)]\big) \subseteq f^{-1}[\operatorname{cl}_Y(B)].$$

Moreover, by Corollary 13, now we also have $B \subseteq \operatorname{cl}_Y(B)$, and thus also $f^{-1}[B] \subseteq f^{-1}[\operatorname{cl}_Y(B)]$. Hence, by using Theorem 11, we can infer that

$$\operatorname{cl}_X\big(f^{-1}[B]\big) \subseteq \operatorname{cl}_X\big(f^{-1}[\operatorname{cl}_Y(B)]\big).$$

This shows that

$$\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\,\mathrm{cl}_Y(B)\,].$$

Therefore, by Theorem 88, assertion (1) also holds.

Remark 79. Note that the assertion (2) of Theorem 88, and the assertions (3) of Theorems 81 and 84, are more natural than the assertions (3) and (4) of Theorem 88 and the assertions (2) and (3) of Theorem 89.

Namely, the assertion (2) of Theorem 88, in a detailed form, means only that, for any $A \subseteq X$, the inclusion $x \in \mathrm{cl}_X((A)$ implies $f(x) \in \mathrm{cl}_Y(f[A]\big)$. That is, if x is "near" to A in X, then $f(x)$ is also "near" to $f[A]$ in Y.

Actually, the nearness of one set to another is an even more natural concept than that of a point to a set. Note that, according to a general definition of Száz [47], for any two subsets A and B of a goset X, we have $B \in \mathrm{Cl}_X(A)$ if and only if $\mathrm{cl}_X(A) \cap B \neq \emptyset$.

Now, by using Theorem 88, we can also prove the following.

Theorem 90. *If f is an increasing function of one goset X onto another Y, then*

(1) $A \in \mathscr{D}_X$ *implies* $f[A] \in \mathscr{D}_Y$*,*
(2) $B \in \mathscr{E}_Y$ *implies* $f^{-1}[B] \in \mathscr{E}_X$*.*

Proof. If $A \in \mathscr{D}_X$, then by Definition 4 we have $X = \mathrm{cl}_X(A)$. Hence, by using Theorem 88 and our assumptions on f, we can already infer that

$$Y = f[X] = f[\,\mathrm{cl}_X(A)\,] \subseteq \mathrm{cl}_Y\big(f[A]\big),$$

and thus $Y = \mathrm{cl}_Y\big(f[A]\big)$. Therefore, by Definition 4, we also have $f[A] \in \mathscr{D}_Y$.

This shows that (1) is true. Moreover, by using Theorem 19, we can easily see that (1) and (2) are equivalent even if f is not assumed to be increasing and onto Y.

For instance, if $A \in \mathscr{D}_X$ and (1) holds, then $f[A] \in \mathscr{D}_Y$. Therefore, if $B \in \mathscr{E}_Y$, then by Theorem 19 we have $f[A] \cap B \neq \emptyset$. Hence, it follows that $A \cap f^{-1}[B] \neq \emptyset$. Therefore, by Theorem 19, we have $f^{-1}[B] \in \mathscr{E}_X$, and thus (2) also holds.

Remark 80. Moreover, if f is as in the above theorem, then by using the assertion (3) of Theorem 88 we can also easily see that if $B \subseteq Y$ such that $f^{-1}[B] \in \mathscr{D}_X$, then $B \in \mathscr{D}_Y$. However, this fact can be more easily derived from Theorem 90.

17 Algebraic Properties of Closure Operations

Theorem 91. *If φ is a closure operation on an inf-complete, antisymmetric goset X, then for any $A \subseteq X$ we have*

$$\inf_X\big(\varphi[A]\big) = \varphi\left(\inf_X\big(\varphi[A]\big)\right).$$

Proof. Now, by Theorem 87, we have $\varphi\big(\inf_X(A)\big) \leq \inf_X\big(\varphi[A]\big)$. Hence, by writing $\varphi[A]$ in place of A, we can see that

$$\varphi\,\big(\inf_X\big(\varphi[A]\big)\big) \leq \inf_X\left(\varphi[\varphi[A]]\right).$$

Moreover, by using the antisymmetry of X, we can see that φ is now idempotent. Therefore, $\varphi[\varphi[A]] = (\varphi\circ\varphi)[A] = \varphi^2[A] = \varphi[A]$. Thus, we actually have

$$\varphi\,\big(\inf_X\big(\varphi[A]\big)\big) \leq \inf_X\left(\varphi[A]\right).$$

Moreover, by extensivity of φ, the converse inequality is also true. Hence, by using the antisymmetry of X, we can see that the required equality is also true.

Remark 81. It can be easily seen that an operation φ on a set X is idempotent if and only if $\varphi[X]$ is the family of all fixed points of φ.

Namely, $\varphi^2 = \varphi$ if and only if $\varphi^2(x) = \varphi(x)$, i.e., $\varphi\big(\varphi(x)\big) = \varphi(x)$ for all $x \in X$. That is, $\varphi(x) \in \operatorname{Fix}(\varphi)$ for all $x \in X$, or equivalently $\varphi[X] \subseteq \operatorname{Fix}(\varphi)$. Thus, since the converse inclusion always holds, the required assertion is also true.

Therefore, by using Theorem 91, we can also prove the following.

Corollary 29. *Under the conditions of Theorem 91, for any $A \subseteq \varphi[X]$, we have*

$$\inf_X\big(A\big) = \varphi\big(\inf_X(A)\big).$$

Proof. Now, because of the antisymmetry of X, the operation φ is idempotent. Thus, by Remark 81, we have $\varphi(y) = y$ for all $y \in \varphi[X]$. Hence, by using the assumption $A \subseteq \varphi[X]$, we can see that $\varphi[A] = A$. Thus, Theorem 91 gives the required equality.

Remark 82. Note that if φ is an extensive, idempotent operation on a reflexive, antisymmetric goset X, then $\varphi[X]$ is also the family of all elements x of X which are φ-closed in the sense that $\varphi(x) \leq x$.

Therefore, if in addition to the conditions of Theorem 91, X is reflexive, then the assertion of Corollary 29 can also be expressed by stating that the infimum of any family of φ-closed elements of X is also φ-closed.

Now, instead of an analogue of Theorem 91 for supremum, we can only prove

Theorem 92. *If φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X, then for any $A \subseteq X$ we have*

$$\varphi\big(\sup_X(A)\big) = \varphi\,\big(\sup_X\big(\varphi[A]\big)\big).$$

Proof. Define $\alpha = \sup_X(A)$ and $\beta = \sup_X\big(\varphi[A]\big)$. Then, by Theorem 86, we have $\beta \leq \varphi(\alpha)$. Hence, since φ is increasing, we can infer that $\varphi(\beta) \leq \varphi\big(\varphi(\alpha)\big)$. Moreover, since φ is now idempotent, we also have $\varphi\big(\varphi(\alpha)\big) = \varphi(\alpha)$. Therefore, $\varphi(\beta) \leq \varphi(\alpha)$.

On the other hand, since φ is extensive, for any $x \in A$ we have $x \leq \varphi(x)$. Moreover, since $\beta \in \operatorname{ub}_X\big(\varphi[A]\big)$, we also have $\varphi(x) \leq \beta$. Hence, by using the transitivity of X, we can infer that $x \leq \beta$. Therefore, $\beta \in \mathrm{ub}_X(A)$. Now, by using that $\alpha \in \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$, we can see that $\alpha \leq \beta$. Hence, by using the increasingness of φ, we can infer that $\varphi(\alpha) \leq \varphi(\beta)$. Therefore, by the antisymmetry of X, we actually have $\varphi(\alpha) = \varphi(\beta)$, and thus the required equality is also true.

From this theorem, we only get the following counterpart of Theorem 91.

Corollary 30. *Under the conditions of Theorem 92, for any $A \subseteq X$, the following assertions are equivalent:*

(1) $\sup_X\big(\varphi[A]\big) = \varphi\big(\sup_X(A)\big)$,
(2) $\sup_X\big(\varphi[A]\big) = \varphi\big(\sup_X\big(\varphi[A]\big)\big)$.

Now, in addition to Theorems 26 and 31, we can also prove

Theorem 93. *If φ is a closure operation on an inf-complete, antisymmetric goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\inf_Y(A) = \inf_X(A).$$

Proof. If $\alpha = \inf_X(A)$, then by Corollary 29 we have $\alpha = \varphi(\alpha)$, and hence $\alpha \in Y$. Therefore, under the usual identification of singletons with their elements, $\alpha = \inf_X(A) \cap Y$ also holds.

On the other hand, by Theorem 31, we always have $\inf_X(A) \cap Y \subseteq \inf_Y(A)$. Therefore, $\alpha \in \inf_Y(A)$ also holds. Hence, by using Theorem 45, we can already see that $\alpha = \inf_Y(A)$ is also true.

From this theorem, it is clear that in particular we also have

Corollary 31. *Under the conditions of Theorem 93, the subgoset Y is also inf-complete.*

Remark 83. Hence, by Theorem 34, we can see that the subgoset Y is also sup-complete.

Now, instead of establishing an analogue of Theorem 93 for supremum, it is convenient to prove first some more general theorems.

Theorem 94. *If φ is an idempotent operation on a goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\mathrm{ub}_Y(A) \subseteq \varphi[\mathrm{ub}_X(A)].$$

Proof. If $\beta \in \mathrm{ub}_Y(A)$, then by Theorem 2 we have $\beta \in Y$ and $\beta \in \mathrm{ub}_X(A)$. Hence, by Remark 81, we can see that $\beta = \varphi(\beta)$, and thus $\beta \in \varphi[\mathrm{ub}_X(A)]$. Therefore, the required inclusion is also true.

Remark 84. By dualization, it is clear that in the above theorem we may also write lb in place of ub.

However, it is now more important to note that we also have the following.

Theorem 95. *If* φ *is an extensive operation on a transitive goset* X *and* $Y = \varphi[X]$*, then for any* $A \subseteq Y$ *we have*

$$\varphi[\,\mathrm{ub}_X(A)\,] \subseteq \mathrm{ub}_Y(A)\,.$$

Proof. If $\beta \in \mathrm{ub}_X(A)$, then because of $\beta \leq \varphi(\beta)$ and the transitivity of X, we also have $\varphi(\beta) \in \mathrm{ub}_X(A)$. Hence, since $\varphi(\beta) \in Y$, we can already see that $\varphi(\beta) \in \mathrm{ub}_X(A) \cap Y = \mathrm{ub}_Y(A)$, and thus the required inclusion is also true.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 32. *If* φ *is a semiclosure operation on a transitive, antisymmetric goset* X *and* $Y = \varphi[X]$*, then for any* $A \subseteq Y$ *we have*

$$\mathrm{ub}_Y(A) = \varphi[\,\mathrm{ub}_X(A)\,]\,.$$

However, it is now more important to note that, in addition to Theorem 95, we can also prove the following.

Theorem 96. *If* φ *is a lower semimodification operation on a transitive goset* X *and* $Y = \varphi[X]$*, then for any* $A \subseteq Y$ *we have*

$$\varphi\big[\,\mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)\,\big] \subseteq \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big)\,.$$

Proof. Suppose that $\beta \in \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$. If $v \in \mathrm{ub}_Y(A)$, then by Theorem 2 we have $v \in Y$ and $v \in \mathrm{ub}_X(A)$. Hence, by using the assumed property of β, we can infer that $\beta \leq v$. Now, since φ is increasing, we can also state that $\varphi(\beta) \leq \varphi(v)$.

Moreover, since $v \in Y$, we can see that there exists $u \in X$ such that $v = \varphi(u)$. Hence, by using that φ is lower semi-idempotent, we can infer that

$$\varphi(v) = \varphi\big(\varphi(u)\big) = \varphi^2(u) \leq \varphi(u) = v\,.$$

Now, by using the transitivity of X, we can also see that $\varphi(\beta) \leq v$. Therefore, $\varphi(\beta) \in \mathrm{lb}_X\big(\mathrm{ub}_Y(A)\big)$. Hence, since $\varphi(\beta) \in Y$ also holds, we can already infer that $\varphi(\beta) \in \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big)$. Therefore, the required inclusion is also true.

Now, by using Theorems 95 and 96, we can also prove the following.

Theorem 97. *If* φ *is a closure operation on a transitive goset* X *and* $Y = \varphi[A]$*, then for any* $A \subseteq Y$ *we have*

$$\varphi[\,\sup_X(A)\,] \subseteq \sup_Y(A)\,.$$

Proof. By Theorems 29, 95, and 96, and a basic fact on relations, we have

$$\varphi\,[\,\sup_X(A)\,] = \varphi\,[\,\mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)\,]$$
$$\subseteq \varphi\,[\,\mathrm{ub}_X(A)\,] \,\cap\, \varphi\Big[\,\mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)\,\Big] \subseteq \mathrm{ub}_Y(A) \cap \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big) = \sup_Y(A)\,.$$

Hence, it is clear that, analogously to Corollary 31, we can also state

Corollary 33. *If in addition to the conditions of Theorem 97, the goset X sup-complete, then the subgoset Y is also sup-complete.*

From Theorem 97, by using Theorem 45, we can also immediately derive the following counterpart of Theorem 93 and Corollary 29.

Theorem 98. *If φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X and $Y = \varphi\,[A\,]$, then for any $A \subseteq Y$ we have*

$$\sup_Y(A) = \varphi\big(\sup_X(A)\big)\,.$$

18 Generalizations of Increasingness to Relator Spaces

A family $\mathscr{R}$ of relations on one set X to another Y is called a *relator* on X to Y. And, the ordered pair $(X, Y)(\mathscr{R}) = \big((X, Y), \mathscr{R}\big)$ is called a *relator space*. (For the origins, see [65], [28], [14], [39], and the references therein.)

If in particular $\mathscr{R}$ is a relator on X to itself, then we may simply say that $\mathscr{R}$ is a relator on X. And, by identifying singletons with their elements, we may naturally write $X(\mathscr{R})$ in place of $(X,X)(\mathscr{R})$, since $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [11] and *uniform spaces* [14]. However, they are insufficient for several important purposes. (See, for instance, [15, 46].)

A relator $\mathscr{R}$ on X to Y, or a relator space $(X, Y)(\mathscr{R})$ is called *simple* if there exists a relation R on X to Y such that $\mathscr{R} = \{R\}$. In this case, by identifying singletons with their elements, we may write $(X, Y)(R)$ in place of $(X, Y)(\{R\})$.

According to our former definition, a simple relator space $X(R)$ may be called a *goset* (generalized ordered set). Moreover, by Ganter and Wille [15, p. 17], a simple relator space $\big(X, Y\big)(R)$ may be called called a *formal context* or *context space*.

A relator $\mathscr{R}$ on X, or a relator space $X(\mathscr{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathscr{R}$ is a reflexive relation on X. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators*.

For any family $\mathscr{A}$ of subsets of X, the family $\mathscr{R}_{\mathscr{A}} = \{R_A : A \in \mathscr{A}\}$ is a preorder relator on X. While, for any family $\mathscr{D}$ of *pseudo-metrics* on X, the family $\mathscr{R}_{\mathscr{D}} = \{B_r^d : r > 0,\ d \in \mathscr{D}\}$ is a tolerance relator on X.

Moreover, if $\mathfrak{S}$ is a family of partitions of X, then $\mathscr{R}_{\mathfrak{S}} = \{ S_{\mathscr{A}} : \mathscr{A} \in \mathfrak{S} \}$ is an equivalence relator on X. Uniformities generated by such practically important relators seem to have been investigated only by Levine [23].

Now, according to Definition 12, a function f of one simple relator space $X(R)$ to another $Y(S)$ may be naturally called *increasing* if for any $u, v \in X$

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v)\,.$$

Hence, by noticing that

$$u\,R\,v \quad \Longleftrightarrow \quad v \in R(u) \quad \Longleftrightarrow \quad (u, v) \in R,$$

and

$$f(u)\,S\,f(v) \quad \Longleftrightarrow \quad f(v) \in S\big(f(u)\big) \quad \Longleftrightarrow \quad \big(f(u), f(v)\big) \in S,$$

that is,

$$f(u)Sf(v) \quad \Longleftrightarrow \quad f(v) \in (S \circ f)(u) \quad \Longleftrightarrow \quad (f \boxtimes f)(u, v) \in S,$$

we can easily establish the following.

Theorem 99. *For a function f of one simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent :*

(1) f *is increasing,*
(2) $f \circ R \subseteq S \circ f$,
(3) $(f \boxtimes f)[R] \subseteq S$,
(4) $f \circ R \circ f^{-1} \subseteq S$,
(5) $R \subseteq (f \boxtimes f)^{-1}[S]$,
(6) $R \subseteq f^{-1} \circ S \circ f$.

Proof. By the above argument and the corresponding definitions, it is clear that

$$(1) \quad \Longleftrightarrow \quad \forall\ (u, v) \in R : \quad (f \boxtimes f)(u, v) \in S \quad \Longleftrightarrow \quad (3)$$

and

$$\begin{aligned} (1) \quad &\Longleftrightarrow \quad \forall\ u \in X : \ \ \forall\ v \in R(u) : \ \ f(v) \in (S \circ f)(u) \\ &\Longleftrightarrow \quad \forall\ u \in X : \ \ f[R(u)] \subseteq (S \circ f)(u) \\ &\Longleftrightarrow \quad \forall\ u \in X : \ \ \big(f \circ R\big)(u) \subseteq (S \circ f)(u) \quad \Longleftrightarrow \quad (2)\,. \end{aligned}$$

Moreover, if (2) holds, then by using that $f \circ f^{-1} \subseteq \Delta_Y$ we can see that

$$f \circ R \circ f^{-1} \subseteq S \circ f \circ f^{-1} \subseteq S \circ \Delta_Y = S,$$

and thus (4) also holds.

Conversely, if (4) holds, then by using that $\Delta_X \subseteq f^{-1} \circ f$ we can similarly see that

$$f \circ R = f \circ R \circ \Delta_X \subseteq f \circ R \circ f^{-1} \circ f \subseteq S \circ f,$$

and thus (2) also holds. Therefore, (2) and (4) are also equivalent.

Now, it is enough to prove only that (3) and (2) are also equivalent to (5) and (6), respectively.

For this, it is convenient to note that if φ is a function of one set U to another V, then because of the inclusions $\Delta_U \subseteq \varphi^{-1} \circ \varphi$ and $\varphi \circ \varphi^{-1} \subseteq \Delta_V$, for any $A \subseteq U$ and $B \subseteq Y$, we have

$$\varphi[A] \subseteq B \iff A \subseteq \varphi^{-1}[B].$$

That is, the set functions φ and φ^{-1} also form a Galois connection.

Namely, if, for instance, (2) holds, then for any $x \in X$ we have

$$f[R(x)] = (f \circ R)(x) \subseteq (S \circ f)(x).$$

Hence, by using the abovementioned fact, we can already infer that

$$R(x) \subseteq f^{-1}[(S \circ f)(x)] = \big(f^{-1} \circ S \circ f\big)(x).$$

Therefore, (6) also holds. While, if (6) holds, then by using a reverse argument, we can quite similarly see that (2) also holds.

From Theorem 99, by using the uniform closure operation $*$ defined by

$$\mathscr{R}^* = \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathscr{R} : \quad R \subseteq S \big\}$$

for any relator $\mathscr{R}$ on X to Y, we can immediately derive the following.

Corollary 34. *For a function f of one simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $S \circ f \in \big\{ f \circ R \big\}^*$,
(3) $S \in \big\{ (f \boxtimes f)[R] \big\}^*$,
(4) $S \in \big\{ f \circ R \circ f^{-1} \big\}^*$,
(5) $(f \boxtimes f)^{-1}[S] \in \{R\}^*$,
(6) $f^{-1} \circ S \circ f \in \{R\}^*$.

Remark 85. Now, by using the notations $\mathscr{F} = \{f\}$, $\mathscr{R} = \{R\}$ and $\mathscr{S} = \{S\}$, instead of (2) we may also write the more instructive inclusions

$$\mathscr{S} \circ \mathscr{F} \subseteq (\mathscr{F} \circ \mathscr{R})^*, \quad (\mathscr{S}^* \circ \mathscr{F})^* \subseteq (\mathscr{F} \circ \mathscr{R}^*)^*, \quad (\mathscr{S}^* \circ \mathscr{F}^*)^* \subseteq (\mathscr{F}^* \circ \mathscr{R}^*)^*.$$

The second one, whenever we think arbitrary relators in place of $\mathscr{R}$ and $\mathscr{S}$, already shows the $*$-invariance of the increasingness of $\mathscr{F}$ with respect to those relators.

From Corollary 34, by using the following obvious extensions of the operations -1 and $\circ$ from relations to relators, defined by

$$\mathscr{R}^{-1} = \{R^{-1} : \quad R \in \mathscr{R}\} \qquad \text{and} \qquad \mathscr{S} \circ \mathscr{R} = \{S \circ R : \quad R \in \mathscr{R}, \ S \in \mathscr{S}\}$$

for any relator $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, we can easily derive the following generalization of [46, Definition 4.1], which is also closely related to [60, Definition 15.1].

Definition 15. Let $(X, Y)(\mathscr{R})$ and $(Z, W)(\mathscr{S})$ be relator spaces, and suppose that $\square$ is a direct unary operation for relators. Then, for any two relators $\mathscr{F}$ on X to Z and $\mathscr{G}$ on Y to W, we say that the pair

(1) $(\mathscr{F}, \mathscr{G})$ is *mildly $\square$-increasing* if

$$\left(\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square}\right)^{\square} \subseteq \mathscr{R}^{\square}.$$

(2) $(\mathscr{F}, \mathscr{G})$ is *upper $\square$-semi-increasing* if

$$\left(\mathscr{S}^{\square} \circ \mathscr{F}^{\square}\right)^{\square} \subseteq \left(\mathscr{G}^{\square} \circ \mathscr{R}^{\square}\right)^{\square}.$$

(3) $(\mathscr{F}, \mathscr{G})$ is *lower $\square$-semi-increasing* if

$$\left(\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square}\right)^{\square} \subseteq \left(\mathscr{R}^{\square} \circ \left(\mathscr{F}^{\square}\right)^{-1}\right)^{\square}.$$

Remark 86. A function $\square$ of the class of all relator spaces to that of all relators is called a *direct unary operation for relators* if, for any relator space $(X, Y)(\mathscr{R})$, the value $\square\big((X, Y)(\mathscr{R})\big)$ is a relator on X to Y.

In this case, trusting to the reader's good sense to avoid confusions, we shall simply write $\mathscr{R}^{\square}$ instead of $\mathscr{R}^{\square_{XY}} = \square\big((X, Y)(\mathscr{R})\big)$. Thus, $*$ is a direct, while -1 is a non-direct unary operation for relators.

19 Some Useful Simplifications of Definition 15

The rather difficult increasingness properties given in Definition 15 can be greatly simplified whenever the operation $\square$ has some useful additional properties.

For instance, by using an analogue of Definition 14, we can easily establish

Theorem 100. *If in addition to the assumptions of Definition 15, $\square$ is a closure operation for relators, then*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly $\square$-increasing if and only if*

$$\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square} \subseteq \mathscr{R}^{\square}.$$

(2) $(\mathscr{F}, \mathscr{G})$ *is upper $\square$-semi-increasing if and only if*

$$\mathscr{S}^{\square} \circ \mathscr{F}^{\square} \subseteq \left(\mathscr{G}^{\square} \circ \mathscr{R}^{\square}\right)^{\square}.$$

(3) $(\mathscr{F}, \mathscr{G})$ *is lower $\square$-semi-increasing if and only if*

$$\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \subseteq \left(\mathscr{R}^{\square} \circ \left(\mathscr{F}^{\square}\right)^{-1}\right)^{\square}.$$

Remark 87. To check this, note that an operation $\square$ for relators is a closure operation if and only if, for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y, we have

$$\mathscr{U}^{\square} \subseteq \mathscr{V}^{\square} \iff \mathscr{U} \subseteq \mathscr{V}^{\square}.$$

That is, the set functions $\square$ and $\square$ form a Pataki connection.

Now, by calling an operation $\square$ for relators to be *inversion and composition compatible* if

$$\left(\mathscr{R}^{\square}\right)^{-1} = \left(\mathscr{R}^{-1}\right)^{\square} \qquad \text{and} \qquad \left(\mathscr{S} \circ \mathscr{R}\right)^{\square} = \left(\mathscr{S}^{\square} \circ \mathscr{R}\right)^{\square} = \left(\mathscr{S} \circ \mathscr{R}^{\square}\right)^{\square}$$

for any relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, we can also easily establish

Theorem 101. *If in addition to the assumptions of Definition 15, $\square$ is an inversion and composition compatible operation for relators, then*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly $\square$-increasing if and only if*

$$\left(\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F}\right)^{\square} \subseteq \mathscr{R}^{\square}.$$

(2) $\big(\mathscr{F}, \mathscr{G}\big)$ *is upper* $\Box$*-semi-increasing if and only if*

$$\big(\mathscr{S} \circ \mathscr{F}\big)^{\Box} \subseteq \big(\mathscr{G} \circ \mathscr{R}\big)^{\Box}.$$

(3) $\big(\mathscr{F}, \mathscr{G}\big)$ *is lower* $\Box$*-semi-increasing if and only if*

$$\big(\mathscr{G}^{-1} \circ \mathscr{S}\big)^{\Box} \subseteq \big(\mathscr{R} \circ \mathscr{F}^{-1}\big)^{\Box}.$$

Remark 88. To check this, note that if $\Box$ is a composition compatible operation for relators, then for any three relators $\mathscr{R}$ on X to Y, $\mathscr{S}$ on Y to Z, and $\mathscr{T}$ on Z to W, we have

$$(\mathscr{S} \circ \mathscr{R})^{\Box} = (\mathscr{S}^{\Box} \circ \mathscr{R}^{\Box})^{\Box} \qquad \text{and} \qquad (\mathscr{T} \circ \mathscr{S} \circ \mathscr{R})^{\Box} = (\mathscr{T}^{\Box} \circ \mathscr{S}^{\Box} \circ \mathscr{R}^{\Box})^{\Box}.$$

From the above theorem, it is clear that in particular we also have

Corollary 35. *If in addition to the assumptions of Definition 15,* $\Box$ *is an inversion and composition compatible closure operation for relators, then*

(1) $\big(\mathscr{F}, \mathscr{G}\big)$ *is mildly* $\Box$*-increasing if and only if*

$$\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F} \subseteq \mathscr{R}^{\Box}.$$

(2) $\big(\mathscr{F}, \mathscr{G}\big)$ *is upper* $\Box$*-semi-increasing if and only if*

$$\mathscr{S} \circ \mathscr{F} \subseteq \big(\mathscr{G} \circ \mathscr{R}\big)^{\Box}.$$

(3) $\big(\mathscr{F}, \mathscr{G}\big)$ *is lower* $\Box$*-semi-increasing if and only if*

$$\mathscr{G}^{-1} \circ \mathscr{S} \subseteq \big(\mathscr{R} \circ \mathscr{F}^{-1}\big)^{\Box}.$$

Concerning inversion compatible operations, we can also prove the following.

Theorem 102. *If in addition to the assumptions of Definition 15,* $\Box$ *is an inversion compatible operation for relators, then*

(1) $\big(\mathscr{F}, \mathscr{G}\big)$ *is mildly* $\Box$*-increasing with respect to the relators* $\mathscr{R}$ *and* $\mathscr{S}$ *if and only if* $\big(\mathscr{G}, \mathscr{F}\big)$ *is mildly* $\Box$*-increasing with respect to the relators* $\mathscr{R}^{-1}$ *and* $\mathscr{S}^{-1}$*.*

(2) $\big(\mathscr{F}, \mathscr{G}\big)$ *is upper* $\Box$*-semi-increasing with respect to the relators* $\mathscr{R}$ *and* $\mathscr{S}$ *if and only if* $\big(\mathscr{G}, \mathscr{F}\big)$ *is lower* $\Box$*-semi-increasing with respect to the relators* $\mathscr{R}^{-1}$ *and* $\mathscr{S}^{-1}$*.*

Proof. To prove the "only if part" of (2), note that by the assumed inversion compatibility of $\square$ and a basic inversion property of the element-wise composition of relators, we have

$$\Big(\big(\mathscr{S}^{\square}\circ\mathscr{F}^{\square}\big)^{\square}\Big)^{-1}=\Big(\big(\mathscr{S}^{\square}\circ\mathscr{F}^{\square}\big)^{-1}\Big)^{\square}$$
$$=\Big(\big(\mathscr{F}^{\square}\big)^{-1}\circ\big(\mathscr{S}^{\square}\big)^{-1}\Big)^{\square}$$
$$=\Big(\big(\mathscr{F}^{\square}\big)^{-1}\circ\big(\mathscr{S}^{-1}\big)^{\square}\Big)^{\square},$$

and quite similarly

$$\Big(\big(\mathscr{G}^{\square}\circ\mathscr{R}^{\square}\big)^{\square}\Big)^{-1}=\Big(\big(\mathscr{R}^{-1}\big)^{\square}\circ\big(\mathscr{G}^{\square}\big)^{-1}\Big)^{\square}.$$

Therefore, if $\big(\mathscr{S}^{\square}\circ\mathscr{F}^{\square}\big)^{\square}\subseteq\big(\mathscr{G}^{\square}\circ\mathscr{R}^{\square}\big)$ holds, then we also have

$$\Big(\big(\mathscr{F}^{\square}\big)^{-1}\circ\big(\mathscr{S}^{-1}\big)^{\square}\Big)^{\square}\subseteq\Big(\big(\mathscr{R}^{-1}\big)^{\square}\circ\big(\mathscr{G}^{\square}\big)^{-1}\Big)^{\square}.$$

Remark 89. Such types of arguments indicate that we actually have to keep in mind only the definition of upper $\square$-semi-increasingness, since the other two ones can be easily derived from this one under some simplifying assumptions.

Remark 90. Unfortunately, Theorems 102 and 101 have only a limited range of applicability since several important closure operations on relators are not inversion or composition compatible.

Remark 91. However, it can be easily seen that a union-preserving operation $\square$ for relators is inversion compatible if and only if $\{R^{-1}\}^{\square}\subseteq\big(\{R\}^{\square}\big)^{-1}$ for any relation R on X to Y.

Moreover, a closure operation $\square$ for relators is composition compatible if and only if

$$\mathscr{S}\circ\mathscr{R}^{\square}\subseteq\big(\mathscr{S}\circ\mathscr{R}\big)^{\square}\qquad\text{and}\qquad\mathscr{S}^{\square}\circ\mathscr{R}\subseteq\big(\mathscr{S}\circ\mathscr{R}\big)^{\square}$$

for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z.

Remark 92. By using the latter facts, one can more easily see that, for instance, the uniform closure operation $*$ is inversion and composition compatible.

20 Some Further Important Unary Operations for Relators

In addition to the operation $*$, the functions #, $\wedge$, and Δ, defined by
$$\mathscr{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \ \exists\ R \in \mathscr{R} : \quad R[A] \subseteq S[A] \big\},$$
$$\mathscr{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \ \exists\ R \in \mathscr{R} : \quad R(x) \subseteq S(x) \big\},$$
and
$$\mathscr{R}^{\Delta} = \big\{ S \subset X \times Y : \quad \forall\ x \in X : \ \exists\ u \in X : \ \exists\ R \in \mathscr{R} : \quad R(u) \subseteq S(x) \big\}$$
for any relator $\mathscr{R}$ on X to Y, are also important closure operations for relators.

Thus, we evidently have $\mathscr{R} \subseteq \mathscr{R}^{*} \subseteq \mathscr{R}^{\#} \subseteq \mathscr{R}^{\wedge} \subseteq \mathscr{R}^{\Delta}$ for any relator $\mathscr{R}$ on X to Y. Moreover, if in particular $X = Y$, then in addition to the above inclusions we can also easily prove that $\mathscr{R}^{\infty} \subseteq \mathscr{R}^{*\infty} \subseteq \mathscr{R}^{\infty *} \subseteq \mathscr{R}^{*}$, where
$$\mathscr{R}^{\infty} = \big\{ R^{\infty} : \quad R \in \mathscr{R} \big\}.$$

In addition to ∞, it is also worth considering the operation ∂, defined by
$$\mathscr{R}^{\partial} = \big\{ S \subseteq X^{2} : \quad S^{\infty} \in \mathscr{R} \big\}$$
for any relator $\mathscr{R}$ on X. Namely, for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X, we have
$$\mathscr{R}^{\infty} \subseteq \mathscr{S} \quad \Longleftrightarrow \quad \mathscr{R} \subseteq \mathscr{S}^{\partial}.$$
This shows that the set functions ∞ and ∂ also form a Galois connection. Therefore, $\infty = \infty\, \partial\, \infty$, and $\infty\, \partial$ is also closure operation for relators.

Moreover, for any relator $\mathscr{R}$ on X to Y, we may also naturally define
$$\mathscr{R}^{c} = \big\{ R^{c} : \quad R \in \mathscr{R} \big\},$$
where $R^{c} = X \times Y \setminus R$. Thus, for instance, we may also naturally consider the operation $\circledast = c * c$ which seems to play the same role in order theory as the operation $*$ does in topology.

Unfortunately, the operations $\wedge$ and Δ are not inversion Compatible; therefore, in addition to these operations we have also to consider the operations $\vee = \wedge - 1$ and $\nabla = \Delta - 1$, which already have very curious properties.

For instance, the operations $\vee\vee$ and $\nabla\nabla$ coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined by
$$\mathscr{R}^{\bullet} = \big\{ \delta_{\mathscr{R}} \big\}^{*}, \qquad \text{where} \qquad \delta_{\mathscr{R}} = \bigcap \mathscr{R},$$
and
$$\mathscr{R}^{\blacklozenge} = \mathscr{R} \quad \text{if} \quad \mathscr{R} = \big\{ X \times Y \big\} \qquad \text{and} \qquad \mathscr{R}^{\blacklozenge} = \mathscr{P}(X \times Y) \quad \text{if} \quad \mathscr{R} \neq \big\{ X \times Y \big\}.$$

Because of the above important operations for relators, Definition 15 offers an abundance of natural increasingness properties for relations. Moreover, from the results of Sects. 15 and 16, one can also immediately derive several reasonable definitions for the increasingness of relations.

However, in [58], a relation F on a goset X to a set Y has been called increasing if the induced set-valued function $F^{\diamond}$ is increasing. That is, $u \leq v$ implies $F(u) \subseteq F(v)$ for all $u, v \in X$. Thus, it can be easily seen that F is increasing if and only if F^{-1} is *ascending valued* in the sense that $F^{-1}(y)$ is an ascending subset of X for all $y \in Y$.

If $\mathscr{R}$ is a relator on X to Y, then by extending the corresponding parts of Definitions 1 and 2 ,we may also naturally define

$$\mathrm{Lb}_{\mathscr{R}}(B) = \big\{ A \subseteq X : \quad \exists\, R \in \mathscr{R} : \ A \times B \subseteq R \big\} \quad \text{and} \quad \mathrm{lb}_{\mathscr{R}}(B) = X \cap \mathrm{Lb}_{\mathscr{R}}(B),$$

and

$$\mathrm{Int}_{\mathscr{R}}(B) = \big\{ A \subseteq X : \quad \exists\, R \in \mathscr{R} : \ R[A] \subseteq B \big\} \quad \text{and} \quad \mathrm{int}_{\mathscr{R}}(B) = X \cap \mathrm{Int}_{\mathscr{R}}(B)$$

for all $B \subseteq Y$. However, these relations are again not independent of each other.

Namely, by the corresponding definitions, it is clear that

$$\begin{aligned} A \times B \subseteq R \quad &\Longleftrightarrow \quad \forall\, a \in A : \ B \subseteq R(a) \quad \Longleftrightarrow \quad \forall\, a \in A : \ R(a)^c \subseteq B^c \\ &\Longleftrightarrow \quad \forall\, a \in A : \ R^c(a) \subseteq B^c \quad \Longleftrightarrow \quad R^c[A] \subseteq B^c . \end{aligned}$$

Therefore, we have

$$A \in \mathrm{Lb}_{\mathscr{R}}(B) \quad \Longleftrightarrow \quad A \in \mathrm{Int}_{\mathscr{R}^c}(B^c) \quad \Longleftrightarrow \quad A \in \big(\mathrm{Int}_{\mathscr{R}^c} \circ \mathscr{C}\big)(B) .$$

Hence, we can already see that

$$\mathrm{Lb}_{\mathscr{R}} = \mathrm{Int}_{\mathscr{R}^c} \circ \mathscr{C}, \qquad \text{and thus also} \qquad \mathrm{lb}_{\mathscr{R}} = \mathrm{int}_{\mathscr{R}^c} \circ \mathscr{C} .$$

These formulas, proved first in [47], establish at least as important relationship between order and topological theories as the famous Euler formulas do between exponential and trigonometric functions [38, p. 227] .

To see the importance of the operations $\#$ and $\circledast = c \# c$, by using Pataki connections on power sets [50], it can be shown that, for any relator $\mathscr{R}$ on X to Y, $\mathscr{S} = \mathscr{R}^{\#}$ $\big(\mathscr{S} = \mathscr{R}^{\circledast}\big)$ is the largest relator on X to Y such that $\mathrm{Int}_{\mathscr{S}} = \mathrm{Int}_{\mathscr{R}}$ $\big(\mathrm{Lb}_{\mathscr{S}} = \mathrm{Lb}_{\mathscr{R}}\big)$.

Concerning the operations $\wedge$ and $\circled{\wedge} = c \wedge c$, we can quite similarly see that $\mathscr{S} = \mathscr{R}^{\wedge}$ $\big(\mathscr{S} = \mathscr{R}^{\circled{\wedge}}\big)$ is the largest relator on X to Y such that $\mathrm{int}_{\mathscr{S}} = \mathrm{int}_{\mathscr{R}}$ $\big(\mathrm{lb}_{\mathscr{S}} = \mathrm{lb}_{\mathscr{R}}\big)$. Moreover, if in particular $\mathscr{R}$ is a relator on X, then some similar assertions holds for the families

$$\tau_{\mathscr{R}} = \big\{ A \subseteq X : \quad A \in \mathrm{Int}_{\mathscr{R}}(A) \big\} \qquad \text{and} \qquad \ell_{\mathscr{R}} = \big\{ A \subseteq X : \quad A \in \mathrm{Lb}_{\mathscr{R}}(A) \big\} .$$

However, if $\mathscr{R}$ is a relator on X, then for the families

$$\mathscr{T}_{\mathscr{R}} = \{ A \subseteq X : \ A \subseteq \operatorname{int}_{\mathscr{R}}(A) \} \qquad \text{and} \qquad \mathscr{L}_{\mathscr{R}} = \{ A \subseteq X : \ A \subseteq \operatorname{lb}_{\mathscr{R}}(A) \}$$

there does not exist a largest relator $\mathscr{S}$ on X such that $\mathscr{T}_{\mathscr{S}} = \mathscr{T}_{\mathscr{R}}$ $\big(\mathscr{L}_{\mathscr{S}} = \mathscr{L}_{\mathscr{R}} \big)$.

In the light of this and some other disadvantages of the family $\mathscr{T}_{\mathscr{R}}$, it is rather curious that most of the works in topology and analysis have been based on open sets suggested by Tietze [64] and standardized by Bourbaki [5] and Kelley [18].

Moreover, it also a striking fact that, despite the results of Pervin [34], Fletcher and Lindgren [14], and the present author [52], topologies and their generalizations are still intensively investigated, without generalized uniformities, by a great number of mathematicians.

The study of the various generalized topologies is mainly motivated by some recent papers of Á. Császár. For instance, the authors of [7, 25] write that: "The theory of generalized topological spaces, which was founded by Á. Császár, is one of the most important developments of general topology in recent years."

For any relator $\mathscr{R}$ on X to Y, we may also naturally define

$$\mathscr{E}_{\mathscr{R}} = \{ B \subseteq Y : \ \operatorname{int}_{\mathscr{R}}(B) \neq \emptyset \} \qquad \text{and} \qquad \mathfrak{E}_{\mathscr{R}} = \{ B \subseteq Y : \ \operatorname{lb}_{\mathscr{R}}(B) \neq \emptyset \}.$$

In a relator space $X(\mathscr{R})$, the family $\mathscr{E}_{\mathscr{R}}$ of all *fat sets* is frequently a more important tool than the family $\mathscr{T}_{\mathscr{R}}$ of all *topologically open sets*. Namely, if $\mathscr{R}$ is a relator on X to Y, then it can be shown that $\mathscr{S} = \mathscr{R}^{\Delta}$ is the largest relator on X to Y such that $\mathscr{E}_{\mathscr{S}} = \mathscr{E}_{\mathscr{R}}$.

Moreover, if $\mathscr{R}$ is a relator on X to Y, then for any goset Γ, and *nets* $x \in X^{\Gamma}$ and $y \in Y^{\Gamma}$,we may naturally define $x \in \operatorname{Lim}_{\mathscr{R}}(y)$ if the net (x, y) is eventually in each $R \in \mathscr{R}$ in the sense that $(x, y)^{-1}[R] \in \mathscr{E}_{\Gamma}$. Now, for any $a \in X$, we may also naturally write $a \in \lim_{\mathscr{R}}(y)$ if $(a) \in \operatorname{Lim}_{\mathscr{R}}(y)$, where (a) is an abbreviation for the constant net $(a)_{\gamma \in \Gamma} = \Gamma \times \{a\}$.

In a relator space $(X, Y)(\mathscr{R})$, the *convergence relation* $\operatorname{Lim}_{\mathscr{R}}$, suggested by Efremović and Šwarc [13], is a much stronger tool than the *proximal interior relation* $\operatorname{Int}_{\mathscr{R}}$ suggested by Smirnov [37]. If $\mathscr{R}$ is a relator on X to Y, then it can be shown that $\mathscr{S} = \mathscr{R}^{*}$ is the largest relator on X to Y such that $\operatorname{Lim}_{\mathscr{S}} = \operatorname{Lim}_{\mathscr{R}}$.

Now, following the ideas of Császár [8], for any relator $\mathscr{R}$ on X to Y, we may also naturally consider the *hyperrelators*

$$\mathfrak{H}_{\mathscr{R}} = \{ \operatorname{Int}_R : \ R \in \mathscr{R} \} \qquad \text{and} \qquad \mathfrak{K}_{\mathscr{R}} = \{ \operatorname{Lim}_R : \ R \in \mathscr{R} \}.$$

By the corresponding definitions, it is clear that

$$\operatorname{Int}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \operatorname{Int}_R \qquad \text{and} \qquad \operatorname{Lim}_{\mathscr{R}} = \bigcap_{R \in \mathscr{R}} \operatorname{Lim}_R .$$

Therefore, the above hyperrelators are much stronger tools in the relator space $(X, Y)(\mathscr{R})$ than the relations $\operatorname{Int}_{\mathscr{R}}$ and $\operatorname{Lim}_{\mathscr{R}}$.

For instance, a net $y \in Y^{\Gamma}$ may be naturally called *convergence Cauchy* with respect to the relator $\mathscr{R}$ if $\lim_R(y) \neq \emptyset$ for all $R \in \mathscr{R}$. Hence, since

$$\lim_{\mathscr{R}}(y) = \bigcap_{R \in \mathscr{R}} \lim_R(y),$$

we can at once see that a convergent net is convergence Cauchy, but the converse statement need not be true.

However, it can be shown that the net y is convergent with respect to the relator $\mathscr{R}$ if and only if it convergence Cauchy with respect to the *topological closure* $\mathscr{R}^{\wedge}$ of $\mathscr{R}$. (See [43].) Therefore, the two notions are in a certain sense equivalent.

The same is true in connection with the notions *adherent* and *adherence Cauchy*, which are defined by using $\mathscr{D}_{\Gamma}$ instead of $\mathscr{E}_{\Gamma}$. Moreover, it is also noteworthy that a similar situation holds in connection with the concepts *compact* and *precompact*. (See [45].)

Now, according to the ideas of Száz [59], we may also naturally consider corelator spaces, mentioned in Sect. 2, instead of relator spaces. However, the increasingness properties (1) and (3) considered in Definition 15 cannot be immediately generalized to such spaces. Namely, in contrast to relations, the ordinary inverse of a correlation is usually not a correlation.

Finally, we note that, in addition to the results of Sect.17, it would also be desirable to to establish some topological properties of closure operations by supplementing the results of Sect. 16. Moreover, it would be desirable to extend the notion of closure operations to arbitrary relator spaces.

However, in this direction, we could only observe that a unary operation φ on a simple relator space $X(R)$ is extensive if and only if $\varphi \subseteq R$. Moreover, φ is lower semi-idempotent if and only if $\varphi \,|\, \varphi[X] \subseteq R^{-1}$.

Acknowledgements The work of the author was supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651.

References

1. Birkhoff, G.: Rings of sets. Duke Math. J. **3**, 443–454 (1937)
2. Birkhoff, G.: Lattice Theory, vol. 25. Amer. Math. Soc. Colloq. Publ., Providence, RI (1967)
3. Boros, Z., Száz, Á.: Finite and conditional completeness properties of generalized ordered sets. Rostock. Math. Kolloq. **59**, 75–86 (2005)
4. Boros, Z., Száz, Á.: Infimum and supremum completeness properties of ordered sets without axioms. An. St. Univ. Ovid. Constanta **16**, 1–7 (2008)
5. Bourbaki, N.: General Topology, Chaps. 1–4. Springer, Berlin (1989)
6. Brøndsted, A.: Fixed points and partial orders. Proc. Am. Math. Soc. **60**, 365–366 (1976)
7. Cao, C., Wang, B., Wang, W.: Generalized topologies, generalized neighborhood systems, and generalized interior operators. Acta Math. Hungar. **132**, 310–315 (2011)
8. Császár, Á.: Foundations of General Topology. Pergamon Press, London (1963)
9. Császár, Á.: γ-Connected sets. Acta Math. Hungar. **101**, 273–279 (2003)

10. Császár, Á.: Ultratopologies generated by generalized topologies. Acta Math. Hungar. **110**, 153–157 (2006)
11. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order. Cambridge University Press, Cambridge (2002)
12. Denecke, K., Erné, M., Wismath, S.L. (eds.): Galois Connections and Applications. Kluwer Academic Publisher, Dordrecht (2004)
13. Efremović, V.A., Švarc, A.S.: A new definition of uniform spaces. Metrization of proximity spaces. Dokl. Acad. Nauk. SSSR **89**, 393–396 (1953) [Russian]
14. Fletcher, P., Lindgren, W.F.: Quasi-Uniform Spaces. Marcel Dekker, New York (1982)
15. Ganter, B., Wille, R.: Formal Concept Analysis. Springer, Berlin (1999)
16. Glavosits, T.: Generated preorders and equivalences. Acta Acad. Paed. Agriensis Sect. Math. **29**, 95–103 (2002)
17. Graham, R.L., Knuth, D.E., Motzkin, T.S.: Complements and transitive closures. Discrete Math. **2**, 17–29 (1972)
18. Kelley, J.L.: General Topology. Van Nostrand Reinhold Company, New York (1955)
19. Kneser, H.: Eine direct Ableitung des Zornschen lemmas aus dem Auswahlaxiom. Math. Z. **53**, 110–113 (1950)
20. Kurdics, J., Száz, Á.: Connected relator spaces. Publ. Math. Debrecen **40**, 155–164 (1992)
21. Kurdics, J., Száz, Á.: Well-chained relator spaces. Kyungpook Math. J. **32**, 263–271 (1992)
22. Levine, N.: Dense topologies. Amer. Math. Monthly **75**, 847–852 (1968)
23. Levine, N.: On uniformities generated by equivalence relations. Rend. Circ. Mat. Palermo **18**, 62–70 (1969)
24. Levine, N.: On Pervin's quasi uniformity. Math. J. Okayama Univ. **14**, 97–102 (1970)
25. Li, Z., Zhu, W.: Contra continuity on generalized topological spaces. Acta Math. Hungar. **138**, 34–43 (2013)
26. Mala, J.: Relators generating the same generalized topology. Acta Math. Hungar. **60**, 291–297 (1992)
27. Mala, J., Száz, Á.: Modifications of relators. Acta Math. Hungar. **77**, 69–81 (1997)
28. Nakano, H., Nakano, K.: Connector theory. Pacific J. Math. **56**, 195–213 (1975)
29. Niederle, J.: A useful fixpoint theorem. Rend. Circ. Mat. Palermo II. Ser. **47**, 463–464 (1998)
30. Ore, O.: Galois connexions. Trans. Am. Math. Soc. **55**, 493–513 (1944)
31. Pataki, G.: Supplementary notes to the theory of simple relators. Radovi Mat. **9**, 101–118 (1999)
32. Pataki, G.: On the extensions, refinements and modifications of relators. Math. Balk. **15**, 155–186 (2001)
33. Pataki, G., Száz, Á.: A unified treatment of well-chainedness and connectedness properties. Acta Math. Acad. Paedagog. Nyházi. (N.S.) **19**, 101–165 (2003)
34. Pervin, W.J.: Quasi-uniformization of topological spaces. Math. Ann. **147**, 316–317 (1962)
35. Rakaczki, Cs., Száz, Á.: Semicontinuity and closedness properties of relations in relator spaces. Mathematica (Cluj) **45**, 73–92 (2003)
36. Schmidt, J.: Bieträge zur Filtertheorie II. Math. Nachr. **10**, 197–232 (1953)
37. Smirnov, Y.M.: On proximity spaces. Math. Sb. **31**, 543–574 (1952) [Russian]
38. Stromberg, K.R.: An Introduction to Classical Real Analysis. Wadsworth, Belmont, CA (1981)
39. Száz, Á.: Basic tools and mild continuities in relator spaces. Acta Math. Hungar. **50**, 177–201 (1987)
40. Száz, Á.: The fat and dense sets are more important than the open and closed ones. Abstracts of the Seventh Prague Topological Symposium, Institute of Mathematics Czechoslovak Academy Science, p. 106 (1991)
41. Száz, Á.: Structures derivable from relators. Singularité **3**, 14–30 (1992)
42. Száz, Á.: Refinements of relators. Tech. Rep., Inst. Math., Univ. Debrecen **76**, 19 pp (1993)
43. Száz, Á.: Cauchy nets and completeness in relator spaces. Colloq. Math. Soc. János Bolyai **55**, 479–489 (1993)
44. Száz, Á.: Topological characterizations of relational properties. Grazer Math. Ber. **327**, 37–52 (1996)

45. Száz, Á.: Uniformly, proximally and topologically compact relators. Math. Pannon. **8**, 103–116 (1997)
46. Száz, Á.: Somewhat continuity in a unified framework for continuities of relations. Tatra Mt. Math. Publ. **24**, 41–56 (2002)
47. Száz, Á.: Upper and lower bounds in relator spaces. Serdica Math. J. **29**, 239–270 (2003)
48. Száz, Á.: Lower and upper bounds in ordered sets without axioms. Tech. Rep., Inst. Math., Univ. Debrecen, 11 pp. (2004/1)
49. Száz, Á.: The importance of reflexivity, transitivity, antisymmetry and totality in generalized ordered sets. Tech. Rep., Inst. Math., Univ. Debrecen, 15 pp. (2004/2)
50. Száz, Á.: Galois-type connections on power sets and their applications to relators. Tech. Rep., Inst. Math., Univ. Debrecen, 328 pp. (2005/2)
51. Száz, Á.: Supremum properties of Galois-type connections. Comment. Math. Univ. Carol. **47**, 569–583 (2006)
52. Száz, Á.: Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities. Filomat **21**, 87–97 (2007)
53. Száz, Á.: Galois type connections and closure operations on preordered sets. Acta Math. Univ. Comen. **78**, 1–21 (2009)
54. Száz, Á.: Altman type generalizations of ordering and maximality principles of Brézis, Browder and Brønsted. Adv. Stud. Contemp. Math. (Kyungshang) **20**, 595–620 (2010)
55. Száz, Á.: Galois-type connections and continuities of pairs of relations. J. Int. Math. Virt. Inst. **2**, 39–66 (2012)
56. Száz, Á.: Lower semicontinuity properties of relations in relator spaces. Adv. Stud. Contemp. Math. (Kyungshang) **23**, 107–158 (2013)
57. Száz, Á.: Inclusions for compositions and box products of relations. J. Int. Math. Virt. Inst. **3** 97–125 (2013)
58. Száz, Á.: Galois and Pataki connections revisited. Tech. Rep., Inst. Math., Univ. Debrecen, 20 pp. (2013/3)
59. Száz, Á.: A particular Galois connection between relations and set functions. Acta Univ. Sapientiae Math. **6**, 73–91 (2014)
60. Száz, Á.: Generalizations of Galois and Pataki connections to relator spaces. J. Int. Math. Virt. Inst. **4**, 43–75 (2014)
61. Száz, Á.: Remarks and problems at the Conference on Inequalities and Applications, Hajdúszoboszló, Hungary, 2014. Tech. Rep., Inst. Math., Univ. Debrecen, 12 pp. (2014/5)
62. Thron, W.J.: Topological Structures. Holt, Rinehart and Winston, New York (1966)
63. Thron, W.J.: Proximity structures and grills. Math. Ann. **206**, 35–62 (1973)
64. Tietze, H.: Beiträge zur allgemeinen Topologie I. Axiome für verschiedene Fassungen des Umgebungsbegriffs. Math. Ann. **88**, 290–312 (1923)
65. Weil, A.: Sur les espaces á structure uniforme et sur la topologie générale. Actualites Scientifiques et Industriielles, vol. 551. Herman and Cie (Paris) (1937)
66. Wilder, R.L.: Evolution of the topological concept of “connected”. Am. Math. Mon. **85**, 720–726 (1978)