Taylor's Formula and Integral Inequalities for Conformable Fractional Derivatives

Douglas R. Anderson

In Honor of Constantin Carathéodory

Abstract We derive Taylor's theorem using a variation of constants formula for conformable fractional derivatives. This is then employed to extend some recent and classical integral inequalities to the conformable fractional calculus, including the inequalities of Steffensen, Chebyshev, Hermite–Hadamard, Ostrowski, and Grüss.

1 Taylor Theorem

We use the conformable α-fractional derivative, recently introduced in [6, 7, 9], which for $\alpha \in (0, 1]$ is given by:

$$D_\alpha f(t) := \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \quad D_\alpha f(0) = \lim_{t \to 0^+} D_\alpha f(t). \tag{1}$$

Note that if f is differentiable, then

$$D_\alpha f(t) = t^{1-\alpha} f'(t), \tag{2}$$

where $f'(t) = \lim_{\varepsilon \to 0}[f(t+\varepsilon) - f(t)]/\varepsilon$.

We will consider Taylor's Theorem in the context of iterated fractional differential equations. In this setting, the theorem will be proven using the variation of constants formula, where we use an approach similar to that used for integer-order derivatives found in [8], and different from that found in Williams [14], where the Riemann–Liouville fractional derivative is employed. With this in mind, we begin

D.R. Anderson (✉)
Department of Mathematics, Concordia College, Moorhead, MN 56562, USA
e-mail: andersod@cord.edu

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this note with a general higher-order equation. For $n \in \mathbb{N}_0$ and continuous functions $p_i : [0, \infty) \to \mathbb{R}$, $1 \leq i \leq n$, we consider the higher-order linear α-fractional differential equation:

$$Ly = 0, \quad \text{where} \quad Ly = D_\alpha^n y + \sum_{i=1}^{n} p_i D_\alpha^{n-i} y, \tag{3}$$

where $D_\alpha^n y = D_\alpha^{n-1}(D_\alpha y)$. A function $y : [0, \infty) \to \mathbb{R}$ is a solution of Eq. (3) on $[0, \infty)$ provided y is n times α-fractional differentiable on $[0, \infty)$ and satisfies $Ly(t) = 0$ for all $t \in [0, \infty)$. It follows that $D_\alpha^n y$ is a continuous function on $[0, \infty)$.

Now let $f : [0, \infty) \to \mathbb{R}$ be continuous and consider the nonhomogeneous equation:

$$D_\alpha^n(t) + \sum_{i=1}^{n} p_i(t) D_\alpha^{n-i} y(t) = f(t). \tag{4}$$

Definition 1. We define the Cauchy function $y : [0, \infty) \times [0, \infty) \to \mathbb{R}$ for the linear fractional equation (3) to be, for each fixed $s \in [0, \infty)$, the solution of the initial value problem:

$$Ly = 0, \quad D_\alpha^i y(s, s) = 0, \quad 0 \leq i \leq n-2, \quad D_\alpha^{n-1} y(s, s) = 1.$$

Remark 1. Note that

$$y(t, s) := \frac{1}{(n-1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{n-1}$$

is the Cauchy function for $D_\alpha^n = 0$, which can be easily verified using (2).

Definition 2. Let $\alpha \in (0, 1]$ and $0 \leq a < b$. A function $f : [a, b] \to \mathbb{R}$ is α-fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(t) d_\alpha t := \int_a^b f(t) t^{\alpha - 1} dt$$

exists and is finite.

Theorem 1 (Variation of Constants). *Let $\alpha \in (0, 1]$ and $s, t \in [0, \infty)$. If f is continuous, then the solution of the initial value problem:*

$$Ly = f(t), \quad D_\alpha^i y(s) = 0, \quad 0 \leq i \leq n-1$$

is given by

$$y(t) = \int_s^t y(t, \tau) f(\tau) d_\alpha \tau,$$

where $y(t, \tau)$ is the Cauchy function for (3).

Proof. With y defined as above and by the properties of the Cauchy function, we have

$$D_\alpha^i y(t) = \int_s^t D_\alpha^i y(t,\tau) f(\tau) d_\alpha \tau + D_\alpha^{i-1} y(t,t) f(t) = \int_s^t D_\alpha^i y(t,\tau) f(\tau) d_\alpha \tau$$

for $0 \leq i \leq n-1$, and

$$\begin{aligned} D_\alpha^n y(t) &= \int_s^t D_\alpha^n y(t,\tau) f(\tau) d_\alpha \tau + D_\alpha^{n-1} y(t,t) f(t) \\ &= \int_s^t D_\alpha^n y(t,\tau) f(\tau) d_\alpha \tau + f(t). \end{aligned}$$

It follows from these equations that

$$D_\alpha^i y(s) = 0, \quad 0 \leq i \leq n-1$$

and

$$Ly(t) = \int_s^t Ly(t,\tau) f(\tau) d_\alpha \tau + f(t) = f(t),$$

and the proof is complete. □

Theorem 2 (Taylor Formula). *Let $\alpha \in (0,1]$ and $n \in \mathbb{N}$. Suppose f is $(n+1)$ times α-fractional differentiable on $[0,\infty)$, and $s,t \in [0,\infty)$. Then we have*

$$f(t) = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau.$$

Proof. Let $g(t) := D_\alpha^{n+1} f(t)$. Then f solves the initial value problem:

$$D_\alpha^{n+1} x = g, \quad D_\alpha^k x(s) = D_\alpha^k f(s), \quad 0 \leq k \leq n.$$

Note that the Cauchy function for $D_\alpha^{n+1} y = 0$ is

$$y(t,s) = \frac{1}{n!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^n.$$

By the variation of constants formula,

$$f(t) = u(t) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n g(\tau) d_\alpha \tau,$$

where u solves the initial value problem:

$$D_\alpha^{n+1} u = 0, \quad D_\alpha^m u(s) = D_\alpha^m f(s), \quad 0 \le m \le n. \tag{5}$$

To validate the claim that $u(t) = \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k D_\alpha^k f(s)$, set

$$w(t) := \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k D_\alpha^k f(s).$$

Then $D_\alpha^{n+1} w = 0$, and we have that

$$D_\alpha^m w(t) = \sum_{k=m}^n \frac{1}{(k-m)!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{k-m} D_\alpha^k f(s).$$

It follows that

$$D_\alpha^m w(s) = \sum_{k=m}^n \frac{1}{(k-m)!} \left(\frac{s^\alpha - s^\alpha}{\alpha}\right)^{k-m} D_\alpha^k f(s) = D_\alpha^m f(s)$$

for $0 \le m \le n$. We consequently have that w also solves (5), and thus $u \equiv w$ by uniqueness. □

Corollary 1. *Let $\alpha \in (0, 1]$ and $s, r \in [0, \infty)$ be fixed. For any $t \in [0, \infty)$ and any positive integer n,*

$$\frac{1}{n!} \left(\frac{t^\alpha - r^\alpha}{\alpha}\right)^n = \sum_{k=0}^n \frac{1}{k!(n-k)!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k \left(\frac{s^\alpha - r^\alpha}{\alpha}\right)^{n-k}.$$

Proof. This follows immediately from the theorem if we take $f(t) = \frac{1}{n!} \left(\frac{t^\alpha - r^\alpha}{\alpha}\right)^n$ in Taylor's formula. It can also be shown directly. □

2 Steffensen Inequality

In this section we prove a new α-fractional version of Steffensen's inequality and of Hayashi's inequality. The results in this and subsequent sections differ from those in [10, 12, 13, 15].

Lemma 1. *Let $\alpha \in (0, 1]$ and $a, b \in \mathbb{R}$ with $0 \le a < b$. Let $A > 0$ and let $g : [a, b] \to [0, A]$ be an α-fractional integrable function on $[a, b]$. If*

$$\ell := \frac{\alpha(b-a)}{A(b^\alpha - a^\alpha)} \int_a^b g(t) d_\alpha t \in [0, b-a], \tag{6}$$

then

$$\int_{b-\ell}^{b} A\, d_\alpha t \le \int_a^b g(t)\, d_\alpha t \le \int_a^{a+\ell} A\, d_\alpha t. \tag{7}$$

Proof. Since $g(t) \in [0, A]$ for all $t \in [a, b]$, ℓ given in (6) satisfies

$$0 \le \ell = \frac{\alpha(b-a)}{A(b^\alpha - a^\alpha)} \int_a^b g(t) d_\alpha t \le \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b 1\, d_\alpha t = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \frac{b^\alpha - a^\alpha}{\alpha} = b - a.$$

As $\alpha \in (0, 1]$ we have that $t^{\alpha-1}$ is a decreasing function on $[a, b]$ or $(a, b]$ if $a = 0$. Thus using the fact that $d_\alpha t = t^{\alpha-1} dt$, we have the following inequalities, which are average values, namely,

$$\frac{1}{\ell} \int_{b-\ell}^{b} 1\, d_\alpha t \le \frac{1}{b-a} \int_a^b 1\, d_\alpha t \le \frac{1}{\ell} \int_a^{a+\ell} 1\, d_\alpha t.$$

This implies that

$$\int_{b-\ell}^{b} A\, d_\alpha t \le \frac{\ell}{b-a} \int_a^b A\, d_\alpha t \le \int_a^{a+\ell} A\, d_\alpha t,$$

which leads to (7) via (6). □

The next theorem is known as Steffensen's inequality if $A = 1$, and for general $A > 0$, it is known as Hayashi's inequality [1].

Theorem 3 (Fractional Hayashi–Steffensen Inequality). *Let* $\alpha \in (0, 1]$, $A > 0$, *and* $a, b \in \mathbb{R}$ *with* $0 \le a < b$. *Let* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to [0, A]$ *be* α*-fractional integrable functions on* $[a, b]$.

(i) If f *is nonnegative and nonincreasing, then*

$$A \int_{b-\ell}^{b} f(t) d_\alpha t \le \int_a^b f(t) g(t) d_\alpha t \le A \int_a^{a+\ell} f(t) d_\alpha t, \tag{8}$$

where ℓ *is given by* (6).

(ii) If f *is nonpositive and nondecreasing, then the inequalities in* (8) *are reversed.*

Proof. For (i), assume f is nonnegative and nonincreasing; we will prove only the case in (8) for the left inequality; the proof for the right inequality is similar and relies on (7). By the definition of ℓ in (6) and the conditions on g, we know that (7) holds. After subtracting within the left inequality of (8), we see that

$$\begin{aligned} &\int_a^b f(t) g(t) d_\alpha t - A \int_{b-\ell}^{b} f(t) d_\alpha t \\ &\quad = \int_a^{b-\ell} f(t) g(t) d_\alpha t - \int_{b-\ell}^{b} f(t)(A - g(t)) d_\alpha t \end{aligned}$$

$$\geq \int_a^{b-\ell} f(t)g(t)d_\alpha t - f(b-\ell)\int_{b-\ell}^b (A-g(t))d_\alpha t$$

$$\overset{(7)}{\geq} \int_a^{b-\ell} f(t)g(t)d_\alpha t - f(b-\ell)\int_a^{b-\ell} g(t)d_\alpha t$$

$$= \int_a^{b-\ell} (f(t)-f(b-\ell))\, g(t)d_\alpha t \geq 0,$$

since f is nonincreasing, and f and g are nonnegative. Therefore, the left-hand side of (8) holds.

For (ii), assume f is nonpositive and nondecreasing; we will prove only the case in (8) for the reversed right inequality; the proof for the reversed left inequality is similar and also relies on (7). We see that we have

$$\int_a^b f(t)g(t)d_\alpha t - A\int_a^{a+\ell} f(t)d_\alpha t$$

$$= \int_{a+\ell}^b f(t)g(t)d_\alpha t + \int_a^{a+\ell} f(t)(g(t)-A)d_\alpha t$$

$$\geq \int_{a+\ell}^b f(t)g(t)d_\alpha t + f(a+\ell)\int_a^{a+\ell} (g(t)-A)d_\alpha t$$

$$\overset{(7)}{\geq} \int_{a+\ell}^b f(t)g(t)d_\alpha t - f(a+\ell)\int_{a+\ell}^b g(t)d_\alpha t$$

$$= \int_{a+\ell}^b (f(t)-f(a+\ell))\, g(t)d_\alpha t \geq 0,$$

since f is nondecreasing and nonpositive, and g is nonnegative. Therefore the right-hand side of the reversed (8) holds. □

Remark 2. The requirement in Steffensen's Theorem 3 that f be nonincreasing when f is nonnegative is essential. Let $a = 0$, $b = 1 = A$, $\alpha \in (0,1)$, $g(t) = t$, and $f(t) = t^{1-\alpha}$. Then $\ell = \frac{\alpha}{1+\alpha}$, and if (8) were to hold in this case, we would need

$$\int_{\frac{1}{1+\alpha}}^1 t^{1-\alpha}d_\alpha t \leq \int_0^1 t^{2-\alpha}d_\alpha t \leq \int_0^{\frac{\alpha}{1+\alpha}} t^{1-\alpha}d_\alpha t$$

to hold, that is to say

$$\frac{\alpha}{1+\alpha} = 1 - \frac{1}{1+\alpha} \leq \frac{1}{2} \leq \frac{\alpha}{1+\alpha}.$$

But this holds only if $\alpha = 1$, a contradiction even if we reverse the inequalities.

3 Taylor Remainder

Let $\alpha \in (0, 1]$ and suppose f is $n + 1$ times α-fractional differentiable on $[0, \infty)$. Using Taylor's Theorem, Theorem 2, we define the remainder function by

$$R_{-1,f}(\cdot, s) := f(s),$$

and for $n > -1$,

$$\begin{aligned} R_{n,f}(t, s) &:= f(s) - \sum_{k=0}^{n} \frac{D_\alpha^k f(t)}{k!} \left(\frac{s^\alpha - t^\alpha}{\alpha} \right)^k \\ &= \frac{1}{n!} \int_t^s \left(\frac{s^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau. \end{aligned} \tag{9}$$

Lemma 2. *Let $\alpha \in (0, 1]$. The following identity involving α-fractional Taylor's remainder holds:*

$$\int_a^b \frac{D_\alpha^{n+1} f(s)}{(n+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{n+1} d_\alpha s = \int_a^t R_{n,f}(a, s) d_\alpha s + \int_t^b R_{n,f}(b, s) d_\alpha s.$$

Proof. We proceed by mathematical induction on n. For $n = -1$,

$$\int_a^b D_\alpha^0 f(s) d_\alpha s = \int_a^b f(s) d_\alpha s = \int_a^t f(s) d_\alpha s + \int_t^b f(s) d_\alpha s.$$

Assume the result holds for $n = k - 1$:

$$\int_a^b \frac{D_\alpha^k f(s)}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k d_\alpha s = \int_a^t R_{k-1,f}(a, s) d_\alpha s + \int_t^b R_{k-1,f}(b, s) d_\alpha s.$$

Let $n = k$. Using integration by parts, we have

$$\begin{aligned} \int_a^b \frac{D_\alpha^{k+1} f(s)}{(k+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^{k+1} d_\alpha s = &\frac{D_\alpha^k f(b)}{(k+1)!} \left(\frac{t^\alpha - b^\alpha}{\alpha} \right)^{k+1} \\ &- \frac{D_\alpha^k f(a)}{(k+1)!} \left(\frac{t^\alpha - a^\alpha}{\alpha} \right)^{k+1} \\ &+ \int_a^b \frac{D_\alpha^k f(s)}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k d_\alpha s. \end{aligned}$$

By the induction assumption,

$$\begin{aligned}\int_a^b \frac{D_\alpha^{k+1} f(s)}{(k+1)!}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{k+1} d_\alpha s &= \int_a^t R_{k-1,f}(a,s) d_\alpha s + \int_t^b R_{k-1,f}(b,s) d_\alpha s \\ &\quad + \frac{D_\alpha^k f(b)}{(k+1)!}\left(\frac{t^\alpha - b^\alpha}{\alpha}\right)^{k+1} \\ &\quad - \frac{D_\alpha^k f(a)}{(k+1)!}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{k+1} \\ &= \int_a^t R_{k-1,f}(a,s) d_\alpha s + \int_t^b R_{k-1,f}(b,s) d_\alpha s \\ &\quad + \frac{D_\alpha^k f(b)}{k!}\int_b^t \left(\frac{s^\alpha - b^\alpha}{\alpha}\right)^{k} d_\alpha s \\ &\quad - \frac{D_\alpha^k f(a)}{k!}\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha}\right)^{k} d_\alpha s \\ &= \int_a^t \left[R_{k-1,f}(a,s) - \frac{D_\alpha^k f(a)}{k!}\left(\frac{s^\alpha - a^\alpha}{\alpha}\right)^{k}\right] d_\alpha s \\ &\quad + \int_t^b \left[R_{k-1,f}(b,s) - \frac{D_\alpha^k f(b)}{k!}\left(\frac{s^\alpha - b^\alpha}{\alpha}\right)^{k}\right] d_\alpha s \\ &= \int_a^t R_{k,f}(a,s) d_\alpha s + \int_t^b R_{k,f}(b,s) d_\alpha s.\end{aligned}$$

This completes the proof. □

Corollary 2. *Let* $\alpha \in (0,1]$. *For* $n \geq -1$,

$$\int_a^b \frac{D_\alpha^{n+1} f(s)}{(n+1)!}\left(\frac{a^\alpha - s^\alpha}{\alpha}\right)^{n+1} d_\alpha s = \int_a^b R_{n,f}(b,s) d_\alpha s,$$

$$\int_a^b \frac{D_\alpha^{n+1} f(s)}{(n+1)!}\left(\frac{b^\alpha - s^\alpha}{\alpha}\right)^{n+1} d_\alpha s = \int_a^b R_{n,f}(a,s) d_\alpha s.$$

4 Applications of the Steffensen Inequality

Let $\alpha \in (0,1]$. In the following we adapt to the α-fractional setting some results from [5] by applying the fractional Steffensen inequality, Theorem 3.

Theorem 4. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an $n+1$ times α-fractional differentiable function such that $D_\alpha^{n+1} f$ is increasing and $D_\alpha^n f$ is decreasing on $[a,b]$. If*

$$\ell := \frac{b-a}{n+2},$$

then

$$D_\alpha^n f(a+\ell) - D_\alpha^n f(a) \leq (n+1)! \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^{n+1} \int_a^b R_{n,f}(a,s) d_\alpha s$$
$$\leq D_\alpha^n f(b) - D_\alpha^n f(b-\ell).$$

If $D_\alpha^{n+1} f$ is decreasing and $D_\alpha^n f$ is increasing on $[a,b]$, then the above inequalities are reversed.

Proof. Assume $D_\alpha^{n+1} f$ is increasing and $D_\alpha^n f$ is decreasing on $[a,b]$, and let

$$F := -D_\alpha^{n+1} f.$$

Because $D_\alpha^n f$ is decreasing, $D_\alpha^{n+1} f \leq 0$, so that $F \geq 0$ and decreasing on $[a,b]$. Define

$$g(t) := \left(\frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha}\right)^{n+1} \in [0,1], \quad t \in [a,b], \quad n \geq -1.$$

Note that F, g satisfy the assumptions of Steffensen's inequality (i), Theorem 3, with $A = 1$; using (6),

$$\ell = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b g(t) d_\alpha t = \frac{b-a}{n+2},$$

and

$$-\int_{b-\ell}^b D_\alpha^{n+1} f(t) d_\alpha t \leq -\int_a^b D_\alpha^{n+1} f(t) \left(\frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha}\right)^{n+1} d_\alpha t \leq -\int_a^{a+\ell} D_\alpha^{n+1} f(t) d_\alpha t.$$

By Corollary 2 this simplifies to

$$D_\alpha^n f(t)\Big|_{t=a}^{a+\ell} \leq (n+1)! \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^{n+1} \int_a^b R_{n,f}(a,t) d_\alpha t \leq D_\alpha^n f(t)\Big|_{t=b-\ell}^{b}.$$

This completes the proof of the first part. If $D_\alpha^{n+1} f$ is decreasing and $D_\alpha^n f$ is increasing on $[a,b]$, then take $F := D_\alpha^{n+1} f$. □

The following corollary is the first Hermite–Hadamard inequality, derived from Theorem 4 with $n = 0$.

Corollary 3 (Hermite–Hadamard Inequality I). *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable function such that $D_\alpha f$ is increasing and f is decreasing on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s \le f(b) + f(a) - f\left(\frac{a+b}{2}\right).$$

If $D_\alpha f$ is decreasing and f is increasing on $[a,b]$, then the above inequalities are reversed.

Theorem 5. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an $n+1$ times α-fractional differentiable function such that*

$$m \le D_\alpha^{n+1} f \le M$$

on $[a,b]$ for some real numbers $m < M$. Then

$$\frac{m}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} + \frac{M-m}{(n+2)!}\left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha}\right)^{n+2} \le \int_a^b R_{n,f}(a,t)d_\alpha t$$
$$\le \frac{M}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} + \frac{m-M}{(n+2)!}\left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha}\right)^{n+2}, \tag{10}$$

where ℓ is given by:

$$\ell = \frac{\alpha(b-a)}{(b^\alpha - a^\alpha)(M-m)}\left(D_\alpha^n f(b) - D_\alpha^n f(a) - m\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)\right)$$

Proof. Let

$$F(t) := \frac{1}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1},$$
$$k(t) := \frac{1}{M-m}\left(f(t) - \frac{m}{(n+1)!}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{n+1}\right),$$
$$G(t) := D_\alpha^{n+1}k(t) = \frac{1}{M-m}\left(D_\alpha^{n+1}f(t) - m\right) \in [0,1].$$

Observe that F is nonnegative and decreasing, and

$$\int_a^b G(t)d_\alpha t = \frac{1}{M-m}\left(D_\alpha^n f(b) - D_\alpha^n f(a) - m\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)\right).$$

Since F, G satisfy the hypotheses of Theorem 3(i), we compute the various integrals given in (8), after using (6) to set

$$\ell = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b G(t) d_\alpha t.$$

We have

$$\int_{b-\ell}^b F(t) d_\alpha t = \int_{b-\ell}^b \frac{1}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} d_\alpha t = \frac{1}{(n+2)!} \left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha} \right)^{n+2},$$

and

$$\int_a^{a+\ell} F(t) d_\alpha t = \frac{1}{(n+2)!} \left[\left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2} - \left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha} \right)^{n+2} \right].$$

Moreover, using Corollary 2, we have

$$\begin{aligned} \int_a^b F(t) G(t) d_\alpha t &= \frac{1}{(M-m)(n+1)!} \int_a^b \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} \left(D_\alpha^{n+1} f(t) - m \right) d_\alpha t \\ &= \frac{1}{M-m} \int_a^b R_{n,f}(a,t) d_\alpha t - \frac{m}{(M-m)(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}. \end{aligned}$$

Using Steffensen's inequality (8) and some rearranging, we obtain (10). □

Corollary 4. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable function such that*

$$m \le D_\alpha f \le M$$

on $[a,b]$ for some real numbers $m < M$. Then

$$\begin{aligned} &\frac{m}{2} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^2 + \frac{M-m}{2} \left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha} \right)^2 \\ &\le \int_a^b f(t) d_\alpha t - f(a) \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \\ &\le \frac{M}{2} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^2 + \frac{m-M}{2} \left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha} \right)^2, \end{aligned} \tag{11}$$

where ℓ is given by:

$$\ell = \frac{\alpha(b-a)}{(b^\alpha - a^\alpha)(M-m)} \left(f(b) - f(a) - m \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \right)$$

Proof. Use the previous theorem with $n = 0$ and Corollary 2. □

5 Applications of the Chebyshev Inequality

Let $\alpha \in (0, 1]$. We begin with Chebyshev's inequality for α-fractional integrals, then apply it to obtain a Hermite–Hadamard-type inequality.

Theorem 6 (Chebyshev Inequality). *Let f and g be both increasing or both decreasing in $[a, b]$, and let $\alpha \in (0, 1]$. Then*

$$\int_a^b f(t)g(t)d_\alpha t \geq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)d_\alpha t \int_a^b g(t)d_\alpha t.$$

If one of the functions is increasing and the other is decreasing, then the above inequality is reversed.

Proof. The proof is very similar to the classical case with $\alpha = 1$. □

The following is an application of Chebyshev's inequality, which extends a similar result in [5] for q-calculus to this α-fractional case.

Theorem 7. *Let $\alpha \in (0, 1]$. Assume that $D_\alpha^{n+1} f$ is monotonic on $[a, b]$. If $D_\alpha^{n+1} f$ is increasing, then*

$$\begin{aligned} 0 &\geq \int_a^b R_{n,f}(a,t)d_\alpha t - \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+1} \\ &\geq \left(\frac{D_\alpha^{n+1} f(a) - D_\alpha^{n+1} f(b)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}. \end{aligned}$$

If $D_\alpha^{n+1} f$ is decreasing, then the inequalities are reversed.

Proof. The situation where $D_\alpha^{n+1} f$ is decreasing is analogous to that of $D_\alpha^{n+1} f$ increasing. Thus, assume $D_\alpha^{n+1} f$ is increasing and set

$$F(t) := D_\alpha^{n+1} f(t), \quad G(t) := \frac{1}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1}.$$

Then F is increasing by assumption, and G is decreasing, so that by Chebyshev's inequality:

$$\int_a^b F(t)G(t)d_\alpha t \leq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b F(t)d_\alpha t \int_a^b G(t)d_\alpha t.$$

By Corollary 2,

$$\int_a^b F(t)G(t)d_\alpha t = \int_a^b \frac{D_\alpha^{n+1} f(t)}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1} d_\alpha t = \int_a^b R_{n,f}(a,t)d_\alpha t.$$

We also have

$$\int_a^b F(t)d_\alpha t = D_\alpha^n f(b) - D_\alpha^n f(a), \quad \int_a^b G(t)d_\alpha t = \frac{1}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}.$$

Thus Chebyshev's inequality implies

$$\int_a^b R_{n,f}(a,t)d_\alpha t \leq \frac{\alpha}{b^\alpha - a^\alpha}\left(D_\alpha^n f(b) - D_\alpha^n f(a)\right)\frac{1}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2},$$

which subtracts to the left side of the inequality. Since $D_\alpha^{n+1}f$ is increasing on $[a,b]$,

$$\begin{aligned}\frac{D_\alpha^{n+1}f(a)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} &\leq \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+1} \\ &\leq \frac{D_\alpha^{n+1}f(b)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2},\end{aligned}$$

and we have

$$\begin{aligned}&\int_a^b R_{n,f}(a,t)d_\alpha t - \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+1} \\ &\geq \int_a^b R_{n,f}(a,t)d_\alpha t - \frac{D_\alpha^{n+1}f(b)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}.\end{aligned}$$

Now Corollary 2 and $D_\alpha^{n+1}f$ is increasing imply that

$$\begin{aligned}\int_a^b \frac{D_\alpha^{n+1}f(b)}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1} d_\alpha t &\geq \int_a^b R_{n,f}(a,t)d_\alpha t \\ &\geq \int_a^b \frac{D_\alpha^{n+1}f(a)}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1} d_\alpha t,\end{aligned}$$

which simplifies to

$$\frac{D_\alpha^{n+1}f(b)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} \geq \int_a^b R_{n,f}(a,t)d_\alpha t \geq \frac{D_\alpha^{n+1}f(a)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}.$$

This, together with the earlier lines, gives the right side of the inequality. □

Corollary 5 (Hermite–Hadamard Inequality II). *Let* $\alpha \in (0,1]$. *If* $D_\alpha f$ *is increasing on* $[a,b]$, *then*

$$\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t \leq \frac{f(b) + f(a)}{2}. \tag{12}$$

If $D_\alpha f$ *is decreasing on* $[a,b]$, *then the inequality is reversed.*

Remark 3. Combining Corollary 3 with Corollary 5, we can state the following. If $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha f$ is increasing and f is decreasing on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t \le \frac{f(b)+f(a)}{2}.$$

If $\alpha = 1$ this is the Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(b)+f(a)}{2},$$

which holds for all convex functions $f : [a,b] \to \mathbb{R}$. In the α-fractional case, however, the assumption that f is decreasing on $[a,b]$ seems to be crucial. Let $[a,b] = [0,1]$, $\alpha = 1/2$, and $f(t) = \frac{2}{3}t^{3/2}$. Then f is increasing and convex on $[0,1]$, but

$$\begin{aligned} f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\right) = \frac{2}{3}\left(\frac{1}{2}\right)^{3/2} &> \frac{1}{6} \\ &= \frac{1}{2}\int_0^1 \frac{2}{3}t^{3/2}t^{1/2-1}dt = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t. \end{aligned}$$

6 Ostrowski Inequality

In this section we prove Ostrowski's α-fractional inequality using a Montgomery identity. For more on Ostrowski's inequalities, see [3] and the references therein.

Lemma 3 (Montgomery Identity). *Let $a, b, s, t \in \mathbb{R}$ with $0 \le a < b$, and let $f : [a,b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0,1]$. Then*

$$f(t) = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s + \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b p(t,s)D_\alpha f(s)d_\alpha s \tag{13}$$

where

$$p(t,s) := \begin{cases} \frac{s^\alpha - a^\alpha}{\alpha} & : a \le s < t, \\ \frac{s^\alpha - b^\alpha}{\alpha} & : t \le s \le b. \end{cases} \tag{14}$$

Proof. Integrating by parts, we have

$$\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(s)d_\alpha s = \frac{t^\alpha - a^\alpha}{\alpha}f(t) - \int_a^t f(s)d_\alpha s$$

and

$$\int_t^b \left(\frac{s^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(s) d_\alpha s = \frac{b^\alpha - t^\alpha}{\alpha} f(t) - \int_t^b f(s) d_\alpha s.$$

Adding and solving for f yields the result. ◻

Theorem 8 (Ostrowski Inequality). *Let $a, b, s, t \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a,b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0,1]$. Then*

$$\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \frac{M}{2\alpha\,(b^\alpha - a^\alpha)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2 \right], \tag{15}$$

where

$$M := \sup_{t \in (a,b)} |D_\alpha f(t)|\,.$$

This inequality is sharp in the sense that the right-hand side of (15) cannot be replaced by a smaller one.

Proof. Using Lemma 3 with $p(t,s)$ defined in (14), we see that

$$\begin{aligned} \left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \right| &= \left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(t,s) D_\alpha f(s) d_\alpha s \right| \\ &\leq \frac{M\alpha}{b^\alpha - a^\alpha} \left(\int_a^t \left| \frac{s^\alpha - a^\alpha}{\alpha} \right| d_\alpha s + \int_t^b \left| \frac{s^\alpha - b^\alpha}{\alpha} \right| d_\alpha s \right) \\ &= \frac{M\alpha}{b^\alpha - a^\alpha} \left(\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha} \right) d_\alpha s + \int_t^b \left(\frac{b^\alpha - s^\alpha}{\alpha} \right) d_\alpha s \right) \\ &= \frac{M\alpha}{b^\alpha - a^\alpha} \left(\frac{1}{2} \left(\frac{s^\alpha - a^\alpha}{\alpha} \right)^2 \Big|_a^t - \frac{1}{2} \left(\frac{b^\alpha - s^\alpha}{\alpha} \right)^2 \Big|_t^b \right) \\ &= \frac{M}{2\alpha\,(b^\alpha - a^\alpha)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2 \right]. \end{aligned}$$

Now $p(t,a) = 0$, so the smallest value attaining the supremum in M is greater than a. To prove the sharpness of this inequality, let $f(t) = t^\alpha/\alpha$, $a = t_1$, $b = t_2 = t$. It follows that $D_\alpha f(t) = 1$ and $M = 1$. Examining the right-hand side of (15), we get

$$\frac{M}{2\alpha\,(b^\alpha - a^\alpha)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2 \right] = \frac{\left(t_2^\alpha - t_1^\alpha\right)^2}{2\alpha \left(t_2^\alpha - t_1^\alpha\right)} = \frac{t_2^\alpha - t_1^\alpha}{2\alpha}.$$

Starting with the left-hand side of (15), we have

$$\begin{aligned} \left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| &= \left| \frac{t^\alpha}{\alpha} - \frac{\alpha}{t_2^\alpha - t_1^\alpha} \int_{t_1}^{t_2} \frac{t^\alpha}{\alpha} d_\alpha t \right| \\ &= \left| \frac{t^\alpha}{\alpha} - \left(\frac{\alpha}{t_2^\alpha - t_1^\alpha} \right) \left(\frac{t^{2\alpha}}{2\alpha^2} \right) \Big|_{t_1}^{t_2} \right| \end{aligned}$$

$$= \left| \frac{t^\alpha}{\alpha} - \left(\frac{1}{t_2^\alpha - t_1^\alpha} \right) \left(\frac{t_2^{2\alpha} - t_1^{2\alpha}}{2\alpha} \right) \right|$$
$$= \left| \frac{t^\alpha}{\alpha} - \left(\frac{t_2^\alpha + t_1^\alpha}{2\alpha} \right) \right|$$
$$= \frac{t_2^\alpha - t_1^\alpha}{2\alpha}.$$

Therefore, by the squeeze theorem, the sharpness of Ostrowski's inequality is shown. □

7 Grüss Inequality

In this section we prove the Grüss inequality, which relies on Jensen's inequality. Our approach is similar to that taken by Bohner and Matthews [2].

Theorem 9 (Jensen Inequality). *Let $\alpha \in (0,1]$ and $a,b,x,y \in [0,\infty)$. If $w:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to(x,y)$ are nonnegative, continuous functions with $\int_a^b w(t)d_\alpha t > 0$, and $F:(x,y)\to\mathbb{R}$ is continuous and convex, then*

$$F\left(\frac{\int_a^b w(t)g(t)d_\alpha t}{\int_a^b w(t)d_\alpha t} \right) \le \frac{\int_a^b w(t)F(g(t))d_\alpha t}{\int_a^b w(t)d_\alpha t}.$$

Proof. The proof is the same as those found in Bohner and Peterson [4, Theorem 6.17] and Rudin [11, Theorem 3.3] and thus is omitted. □

Theorem 10 (Grüss Inequality). *Let $a,b,s \in [0,\infty)$, and let $f,g:[a,b]\to\mathbb{R}$ be continuous functions. Then for $\alpha \in (0,1]$ and*

$$m_1 \le f(t) \le M_1, \qquad m_2 \le g(t) \le M_2, \tag{16}$$

we have

$$\left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)g(t)d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^2 \int_a^b f(t)d_\alpha t \int_a^b g(t)d_\alpha t \right|$$
$$\le \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

Proof. Initially we consider an easier case, namely, where $f = g$ and

$$\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)d_\alpha t = 0.$$

If we define

$$v(t) := \frac{f(t) - m_1}{M_1 - m_1} \in [0, 1],$$

then $f(t) = m_1 + (M_1 - m_1)v(t)$. Since

$$\int_a^b v^2(t) d_\alpha t \leq \int_a^b v(t) d_\alpha t = \frac{-m_1(b^\alpha - a^\alpha)}{\alpha(M_1 - m_1)},$$

we have

$$\begin{aligned} I(f,f) &:= \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f^2(t) d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right)^2 \\ &= \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b \left[m_1 + (M_1 - m_1)v(t)\right]^2 (t) d_\alpha t \\ &\leq -m_1 M_1 = \frac{1}{4} \left[(M_1 - m_1)^2 - (M_1 + m_1)^2 \right] \\ &\leq \frac{1}{4} (M_1 - m_1)^2. \end{aligned}$$

Now consider the case:

$$r := \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \neq 0,$$

where $r \in \mathbb{R}$. If we take $h(t) := f(t) - r$, then $h(t) \in [m_1 - r, M_1 - r]$ and

$$\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b h(t) d_\alpha t = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (f(t) - r) d_\alpha t = r - \frac{r\alpha}{b^\alpha - a^\alpha} \int_a^b d_\alpha t = 0.$$

Consequently h satisfies the earlier assumptions and so

$$I(h,h) \leq \frac{1}{4} \left[M_1 - r - (m_1 - r)\right]^2 = \frac{1}{4} (M_1 - m_1)^2.$$

Additionally we have

$$I(h,h) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (f(t) - r)^2 d_\alpha t = -r^2 + \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f^2(t) d_\alpha t = I(f,f).$$

As a result,

$$I(f,f) = I(h,h) \leq \frac{1}{4} (M_1 - m_1)^2.$$

Let us now turn to the case involving general functions f and g under assumptions (16). Using

$$I(f,g) := \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)g(t)d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2 \int_a^b f(t)d_\alpha t \int_a^b g(t)d_\alpha t$$

and the earlier cases, one can easily finish the proof as in the case with $\alpha = 1$. See [2] for complete details to mimic. □

Corollary 6. *Let $\alpha \in (0,1]$, $a,b,s,t \in [0,\infty)$, and $f : [a,b] \to \mathbb{R}$ be α-fractional differentiable. If $D_\alpha f$ is continuous and*

$$m \le D_\alpha f(t) \le M, \quad t \in [a,b],$$

then

$$\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s - \left[\frac{2t^\alpha - a^\alpha - b^\alpha}{2(b^\alpha - a^\alpha)}\right][f(b) - f(a)]\right| \le \frac{1}{4}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)(M-m). \tag{17}$$

for all $t \in [a,b]$.

Proof. Using Lemma 3 Montgomery's identity, we have

$$f(t) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b p(t,s)D_\alpha f(s)d_\alpha s \tag{18}$$

for all $t \in [a,b]$, where $p(t,s)$ is given in (14). Now for all $t,s \in [a,b]$, we see that

$$\frac{t^\alpha - b^\alpha}{\alpha} \le p(t,s) \le \frac{t^\alpha - a^\alpha}{\alpha}.$$

Applying Theorem 10 Grüss' inequality to the mappings $p(t,\cdot)$ and $D_\alpha f$, we obtain

$$\left|\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b p(t,s)D_\alpha f(s)d_\alpha s - \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2\int_a^b p(t,s)d_\alpha s\int_a^b D_\alpha f(s)d_\alpha s\right| \le \frac{1}{4}\left(\frac{t^\alpha - a^\alpha}{\alpha} - \frac{t^\alpha - b^\alpha}{\alpha}\right)(M-m) = \frac{1}{4}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)(M-m). \tag{19}$$

Computing the integrals involved, we obtain

$$\left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2\int_a^b p(t,s)d_\alpha s = \frac{2t^\alpha - a^\alpha - b^\alpha}{2(b^\alpha - a^\alpha)}$$

and

$$\int_a^b D_\alpha f(s) d_\alpha s = f(b) - f(a),$$

so that (17) holds, after using (18) and (19). □

Compare the following corollary with Corollaries 3 and 5.

Corollary 7 (Hermite–Hadamard III). *Let* $\alpha \in (0,1]$, $a,b,s,t \in [0,\infty)$, *and* $f : [a,b] \to \mathbb{R}$ *be* α*-fractional differentiable. If* $D_\alpha f$ *is continuous and*

$$m \le D_\alpha f(t) \le M, \quad t \in [a,b],$$

then

$$\left| \frac{f(b)+f(a)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \right| \le \frac{1}{4} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) (M-m).$$

for all $t \in [a,b]$.

Proof. Take $t = b$ in the previous corollary. □

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