Applications of Quasiconvexity

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In Honor of Constantin Carathéodory

Abstract This survey deals with functions called γ-quasiconvex functions and their relations to convexity and superquadracity.

For γ-quasiconvex functions and for superquadratic functions, we get analogs of inequalities satisfied by convex functions and we get refinements for those convex functions which are also γ-quasiconvex as well as superquadratic.

We show in which cases the refinements by γ-quasiconvex functions are better than those obtained by superquadratic functions and convex functions. The power functions defined on $x \geq 0$ where the power is greater or equal to two are examples of convex, quasiconvex, and superquadratic functions.

1 Introduction

In this survey we present functions called γ-quasiconvex functions and their relations to convexity and superquadracity.

This survey may serve as introductory work to a book on quasiconvexity by S. Abramovich, L. E. Persson, J. A. Oguntoase, and S. Samko.

For γ-quasiconvex functions and for superquadratic functions, we get analogs of inequalities satisfied by convex functions and we get refinements for those convex functions which are also γ-quasiconvex as well as superquadratic.

We show in which cases the refinements by γ-quasiconvex functions are better than those obtained by superquadratic functions and convex functions. The power

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functions defined on $x \geq 0$, where the power is greater or equal to two, are important examples of convex, quasiconvex, and superquadratic functions.

We demonstrate the applications of γ-quasiconvexity and superquadracity by putting together some results related mainly to Jensen's inequality, Hardy's inequality, and Average Sums inequalities. We quote here the results obtained in [1–5, 8, 9, 16, 18, 19, 23, 24]. For more on the subjects of superquadracity, γ-superquadracity, and γ-quasiconvexity, we refer the reader to the reference list [1–24] and their references.

We start with some definitions, lemmas, and remarks we used in the proofs of the results stated in the sequel.

Definition 1 ([4, 5]). A function $\varphi : [0, b) \to \mathbb{R}$ is superquadratic provided that for all $0 \leq x < b$, there exists a constant $C_\varphi(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) \geq C_\varphi(x)(y - x) + \varphi(|y - x|) \tag{1}$$

for every y, $0 \leq y < b$.

Definition 2 ([1]). A function $K : [0, b) \to \mathbb{R}$ that satisfies $K(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}$, where φ is a superquadratic function, is called γ-quasisuperquadratic function.

Definition 3 ([8, 9]). A function $K : [0, b) \to \mathbb{R}$ that satisfies $K(x) = x^\gamma \varphi(x)$, when $\gamma \in \mathbb{R}$, and φ is a convex function is called γ-quasiconvex function.

Lemma 1 ([5]). *Let φ be a superquadratic function with $C_\varphi(x)$ as in Definition 1. Then:*

(i) $\varphi(0) \leq 0$,
(ii) if $\varphi(0) = \varphi'(0) = 0$, *then* $C_\varphi(x) = \varphi'(x)$ *whenever φ is differentiable at* $0 < x < b$.
(iii) if $\varphi \geq 0$, *then φ is convex and* $\varphi(0) = \varphi'(0) = 0$.

Lemma 2 ([5]). *Suppose that $\varphi : [0, b) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.*

Lemma 3 ([9]). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be differentiable function $x, y \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$. Then*

$$\begin{aligned}
&\varphi(x)(y^\gamma - x^\gamma) + \varphi'(x) y^\gamma (y - x) \\
&\quad - \left[(x\varphi(x))\left(y^{\gamma-1} - x^{\gamma-1}\right) + (x\varphi(x))' y^{\gamma-1}(y - x)\right] \\
&\quad = y^{\gamma-1}\varphi'(x)(y - x)^2 .
\end{aligned} \tag{2}$$

Using Lemma 2 we get:

Lemma 4 ([3]). *Let $\psi : [0, b) \to \mathbb{R}$ be 1-quasiconvex function, where $\psi(x) = x\varphi(x)$, φ is differentiable nonnegative increasing convex function on $x \geq 0$ satisfying $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$, then ψ is also superquadratic.*

Lemma 5 ([9]). *Let* $0 < a \leq x_i \leq 2a$, $i = 1, \ldots, n$, *or let* $\alpha_{i_0} \geq \frac{1}{2}$ *be such that* $x_{i_0} \geq x_i \geq 0$, $i = 1, \ldots, n$, $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$, *(for instance, when* $n = 2$, *let* $\alpha_1 = \alpha_2 = \frac{1}{2}$, $x_1, x_2 > 0$*). Then* $|x_i - \bar{x}| \leq \bar{x} \iff 0 \leq x_i \leq 2\bar{x}$, $i = 1, \ldots, n$ *when* $\bar{x} = \sum_{i=1}^{n} \alpha_i x_i$.

2 Jensen's Type Inequalities

Jensen's theorem states that $\int_{\Omega} \varphi(f(s))\, d\mu(s) \geq \varphi\left(\int_{\Omega} f(s)\, d\mu(s)\right)$ holds when $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, μ is a probability measure and f is a μ-integrable function (see, for instance, [23]). In this section we quote theorems that deal with generalizations and refinements of this very important theorem.

2.1 Jensen's Type Inequalities for Superquadratic Functions

From the definition of superquadracity, we easily get Jensen's type inequalities:

Lemma 6 ([5]). *The function* φ *is superquadratic on* $[0, b)$, *if and only if*

$$\frac{1}{A_n} \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\bar{x}) \geq \frac{1}{A_n} \sum_{i=1}^{n} a_i \varphi(|x_i - \bar{x}|), \tag{3}$$

holds, where $x_i \in [0, b)$, $i = 1, \ldots, n$ *and* $a_i \geq 0$, $i = 1, \ldots, n$, *are such that* $A_n = \sum_{i=1}^{n} a_i > 0$, *and* $\bar{x} = \frac{1}{A_n} \sum_{i=1}^{n} a_i x_i$.

The function φ *is superquadratic on* $[0, b)$, *if and only if*

$$\begin{aligned} &\int_{\Omega} \varphi(f(s))\, d\mu(s) - \varphi\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\ &\quad \geq \int_{\Omega} \varphi\left(\left|f(s) - \int_{\Omega} f(\sigma)\, d\mu(\sigma)\right|\right) d\mu(s) \end{aligned} \tag{4}$$

where f *is any nonnegative* μ*-integrable function on a probability measure space* (Ω, μ).

The power functions $\varphi(x) = x^p$, $x \geq 0$ are superquadratic when $p \geq 2$ and subquadratic, that is, $-\varphi$ is superquadratic when $1 \leq p \leq 2$. When $\varphi(x) = x^2$ inequality (1) reduces to equality and therefore the same holds for (3) and (4).

It is obvious that when the superquadratic function is nonnegative on $[0, b)$, then inequalities (3) and (4) are refinements of Jensen's inequalities for convex functions.

2.2 Jensen's Type Inequalities for γ-Quasisuperquadratic Functions

For γ-quasisuperquadratic functions defined in Definition 2 we get:

Lemma 7 ([8]). *Let $K(x) = x^{\gamma}\varphi(x)$, $\gamma \in \mathbb{R}_{+}$, where φ is superquadratic on $[0,b)$. Then, for this γ-quasisuperquadratic function K, the inequality*

$$K(y) - K(x) \geq \varphi(x)(y^{\gamma} - x^{\gamma}) + C_{\varphi}(x)\, y^{\gamma}(y-x) + y^{\gamma}\varphi(|y-x|) \tag{5}$$

holds for $x \in [0,b)$. $y \in [0,b)$.

Moreover,

$$\begin{aligned}
&\sum_{i=1}^{N}\alpha_i K(y_i) - K\left(\sum_{i=1}^{N}\alpha_i y_i\right) \\
&\geq \varphi\left(\sum_{j=1}^{N}\alpha_j y_j\right)\left(\sum_{i=1}^{N}\alpha_i y_i^{\gamma} - \left(\sum_{j=1}^{N}\alpha_j y_j\right)^{\gamma}\right) \\
&\quad + C_{\varphi}\left(\sum_{j=1}^{N}\alpha_j y_j\right)\sum_{i=1}^{N}\alpha_i y_i^{\gamma}\left(y_i - \sum_{j=1}^{N}\alpha_j y_j\right) \\
&\quad + \sum_{i=1}^{N}\alpha_i y_i^{\gamma}\varphi\left(\left|y_i - \sum_{j=1}^{N}\alpha_j y_j\right|\right)
\end{aligned} \tag{6}$$

holds for $x_i \in [0,b)$, $y_i \in [0,b)$, $0 \leq \alpha_i \leq 1$, $i = 1,\ldots,N$, and $\sum_{i=1}^{N}\alpha_i = 1$.

Also

$$\begin{aligned}
&\int_{\Omega} K(f(s))\, d\mu(s) - K\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\
&\geq \int_{\Omega}\left[\varphi(x)(f^{\gamma}(s) - x^{\gamma}) + C_{\varphi}(x) f^{\gamma}(s)(f(s)-x)\right. \\
&\quad \left. + f^{\gamma}(s)\varphi(|f(s)-x|)\right] d\mu(s).
\end{aligned} \tag{7}$$

holds, where f is any nonnegative μ-integrable function on the probability measure space (Ω,μ) and $x = \int_{\Omega} f(s)\, d\mu(s) > 0$.

If φ is subquadratic, then the reverse inequality of (5)–(7) hold, in particular

$$\begin{aligned}
&\int_{\Omega} K(f(s))\, d\mu(s) - K\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\
&\leq \int_{\Omega}\left[\varphi(x)(f^{\gamma}(s) - x^{\gamma}) + C_{\varphi}(x) f^{\gamma}(s)(f(s)-x)\right. \\
&\quad \left. + f^{\gamma}(s)\varphi(|f(s)-x|)\right] d\mu(s).
\end{aligned} \tag{8}$$

Inequalities (5)–(7) are satisfied by the power functions $K(x) = x^p$, $p \geq \gamma + 2$. *For* $\gamma < p \leq \gamma + 2$, *the reverse inequalities hold; in particular (8) holds. They reduce to equalities for* $p = \gamma + 2$.

The power functions are used to get from Jensen's type inequalities refined Hardy's type inequalities.

Equality (2) Lemmas 3 and 4 help in proving the following Theorem 1 about Jensen's type inequalities for γ-quasisuperquadratic functions. The results of this theorem refine Jensen's type inequalities stated in inequalities (3) and (4) for superquadratic functions. As nonnegative superquadratic functions (according to Lemma 1) are convex, Theorem 1 refines also Jensen's inequalities for these convex functions which are also superquadratic:

Theorem 1 ([8, Lemma 3.1]). *Let* $K(x) = x^{\gamma}\varphi(x) = x^{\gamma-1}\psi(x)$, $\gamma \geq 1$, *where* φ *is a differentiable nonnegative superquadratic function and* $\psi(x) = x\varphi(x)$. *Then the bound obtained for* $K(x) = x^{\gamma}\varphi(x)$ *is stronger than the bound obtained for* $K(x) = x^{\gamma-1}\psi(x)$, *which means that:*

$$K(y) - K(x) \geq \varphi(x)(y^{\gamma} - x^{\gamma}) + \varphi'(x)\, y^{\gamma}(y - x) + y^{\gamma}\varphi(|y - x|) \tag{9}$$

implies that

$$\begin{aligned} &K(y) - K(x) \\ &\quad\geq \psi(x)\left(y^{\gamma-1} - x^{\gamma-1}\right) + \psi'(x)\, y^{\gamma-1}(y - x) + y^{\gamma-1}\psi(|y - x|) \\ &\quad= x\varphi(x)(x)\left(y^{\gamma-1} - x^{\gamma-1}\right) + (x\varphi(x))'\, y^{\gamma-1}(y - x) + y^{\gamma-1}|y - x|\,\varphi(|y - x|)\,. \end{aligned} \tag{10}$$

Moreover, if $K(x) = x^n\varphi(x)$, $\psi_k(x) = x^k\varphi(x)$, *n is an integer,* $k = 1, 2, \ldots, n$, *and* $\varphi(x)$ *is nonnegative superquadratic, then the inequalities*

$$\begin{aligned} \int_{\Omega} K(f(s))\, d\mu(s) \; &- K\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\ &\geq \int_{\Omega} \left[\varphi(x)(f^n(s) - x^n) + C_{\varphi}(x) f^n(s)(f(s) - x)\right. \\ &\qquad \left. + f^n(s)\,\varphi(|f(s) - x|)\right] d\mu(s) \\ &\geq \int_{\Omega} \left[\psi_k(x)\left(f^{n-k}(s) - x^{n-k}\right) + C_{\psi_k}(x) f^{n-k}(s)(f(s) - x)\right. \\ &\qquad \left. + f^{n-k}(s)\,\psi_k(|f(s) - x|)\right] d\mu(s) \\ &\geq \int_{\Omega} \psi_n(|f(s) - x|)\, d\mu(s) = \int_{\Omega} K(|f(s) - x|)\, d\mu(s) \geq 0 \end{aligned} \tag{11}$$

hold for all probability measure spaces (Ω, μ) *of* μ*-integrable nonnegative functions* f, *where* $x = \int_{\Omega} f(s)\, d\mu(s) > 0$.

Furthermore, if φ is differentiable nonnegative increasing, convex subquadratic, and $\varphi(0) = 0 = \lim_{x\to 0+} x\varphi'(x)$, then according to Lemma 4, $x\varphi(x)$ is superquadratic and for $k = 1, \dots, n$

$$
\begin{aligned}
&\int_\Omega \left[\varphi(x)\left(f^n(s) - x^n\right) + \varphi'(x) f^n(s)\left(f(s) - x\right) + f^n(s)\varphi\left(\left|f(s) - x\right|\right)\right] d\mu(s) \\
&\geq \int_\Omega K\left(f(s)\right) d\mu(s) - K\left(\int_\Omega f(s)\, d\mu(s)\right) \\
&\geq \int_\Omega \Big[\psi_1(x)\left(f^{n-1}(s) - x^{n-1}\right) + \psi_k'(x) f^{n-1}(s)\left(f(s) - x\right) \\
&\quad + f^{n-1}(s)\,\psi_1\left(\left|f(s) - x\right|\right)\Big] d\mu(s) \\
&\geq \int_\Omega \Big[\psi_k(x)\left(f^{n-k}(s) - x^{n-k}\right) + \psi_k'(x) f^{n-k}(s)\left(f(s) - x\right) \\
&\quad + f^{n-k}(s)\,\psi_k\left(\left|f(s) - x\right|\right)\Big] d\mu(s) \\
&\geq \int_\Omega \psi_n\left(\left|f(s) - x\right|\right) d\mu(s) = \int_\Omega K\left(\left|f(s) - x\right|\right) d\mu(s) \geq 0.
\end{aligned}
\tag{12}
$$

In particular, if $\varphi(x) = x^p$, $x \geq 0$, $p \geq 1$, then (9)–(11) are satisfied when $p \geq 2$ and (12) is satisfied when $1 \leq p \leq 2$. When $p = 2$ equality holds in the first inequality of (11) and in the first inequality of (12).

2.3 Jensen's Type Inequalities for γ-Quasiconvex Functions

In [1, 8, 9], Jensen's type inequalities for γ-quasiconvex functions (Definition 3) are derived and discussed.

A convex function φ on $[0, b)$, $0 < b \leq \infty$, is characterized by the following inequality:

$$\varphi(y) - \varphi(x) \geq C_\varphi(x)(y - x), \quad \forall x, y \in (0, b]. \tag{13}$$

In [8] we proved for γ-quasiconvex functions $K : [0, b) \to \mathbb{R}$:

Lemma 8 ([8, Lemma 1]). *Let $K(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}$, where φ is convex on $[0, b)$. Then*

$$K(y) - K(x) = y^\gamma \varphi(y) - x^\gamma \varphi(x) \geq \varphi(x)\left(y^\gamma - x^\gamma\right) + C_\varphi(x)\, y^\gamma (y - x) \tag{14}$$

holds for $x \in [0, b)$, $y \in [0, b)$, where $C_\varphi(x)$ is defined by (13). Moreover, the Jensen's type inequality

$$\begin{aligned}
&\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right) \\
&\quad = \int_{\Omega} \varphi(f(s)) f^{\gamma}(s)\,d\mu(s) - \varphi(x)\,x^{\gamma} \\
&\quad \geq \int_{\Omega} \left[\varphi(x)\left(f^{\gamma}(s) - x^{\gamma}\right) + C_{\varphi}(x) f^{\gamma}(s)\left(f(s) - x\right)\right] d\mu(s) \qquad (15)
\end{aligned}$$

holds, where f is a nonnegative function, $x = \int_{\Omega} f(s)\,d\mu(s) > 0$, f and $K \circ f$ are μ-integrable functions on the probability measure space (Ω, μ).

In particular, for $\gamma = 1$, we get when $K = xf$ that

$$\begin{aligned}
&\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right) \\
&\quad \geq \int_{\Omega} \left[C_{\varphi}(x) f(s)\left(f(s) - x\right)\right] d\mu(s) = \int_{\Omega} C_{\varphi}(x)\left(f(s) - x\right)^{2} d\mu(s)\,. \qquad (16)
\end{aligned}$$

If φ is concave, then the reverse inequalities of (13)–(16) hold. In particular

$$\begin{aligned}
&\int_{\Omega} K(f(s))\,d\mu(s) \; - K\left(\int_{\Omega} f(s)\,d\mu(s)\right) \\
&\quad \leq \int_{\Omega} \left[\varphi(x)\left(f^{\gamma}(s) - x^{\gamma}\right) + C_{\varphi}(x) f^{\gamma}(s)\left(f(s) - x\right)\right] d\mu(s)
\end{aligned}$$

holds.

Example 1. Inequalities (13)–(15) are satisfied by $K(x) = x^{p}$, $p \geq \gamma + 1$. For $\gamma < p \leq \gamma + 1$, the reverse inequalities hold. They reduce to equalities for $p = \gamma + 1$.

From Lemma 8, we get a refinement of Jensen's inequality:

Theorem 2 ([8, Theorem 1]). *Let $\gamma \in \mathbb{R}_{+}$ and f be nonnegative function. Let f and $\varphi \circ f$ be μ-integrable functions on the probability measure space (Ω, μ) and $x = \int_{\Omega} f(s)\,d\mu(s) > 0$. If φ is a differentiable, nonnegative, convex, increasing function on $[0, b)$, $0 < b \leq \infty$ and $\varphi(0) = \lim_{z \to 0^{+}} z\varphi'(z) = 0$, then the Jensen's type inequalities*

$$\begin{aligned}
&\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right) \\
&\quad = \int_{\Omega} \varphi(f(s)) f^{\gamma}(s)\,d\mu(s) - \varphi(x)\,x^{\gamma} \\
&\quad \geq \varphi(x) \int_{\Omega} \left(f^{\gamma}(s) - x^{\gamma}\right) d\mu(s) + \varphi'(x) \int_{\Omega} f^{\gamma}(s)\left(f(s) - x\right) d\mu(s) \geq 0 \qquad (17)
\end{aligned}$$

hold.

Furthermore, for an integer n, we get:

$$
\begin{aligned}
&\int_{\Omega} \varphi\left(f\left(s\right)\right) f^{n}\left(s\right) d\mu\left(s\right) - \varphi\left(x\right) x^{n} \\
&\geq \varphi\left(x\right) \int_{\Omega} \left(f^{n}\left(s\right) - x^{n}\right) d\mu\left(s\right) + \varphi^{'}\left(x\right) \int_{\Omega} f^{n}\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&\geq x\varphi\left(x\right) \int_{\Omega} \left(f^{n-1}\left(s\right) - x^{n-1}\right) d\mu\left(s\right) \\
&\quad + \left(x\varphi\left(x\right)\right)^{'} \int_{\Omega} f^{n-1}\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&\geq x^{k}\varphi\left(x\right) \int_{\Omega} \left(f^{n-k}\left(s\right) - x^{n-k}\right) d\mu\left(s\right) \\
&\quad + \left(x^{k}\varphi\left(x\right)\right)^{'} \int_{\Omega} f^{n-k}\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&\geq \left(x^{n-1}\varphi\left(x\right)\right)^{'} \int_{\Omega} f\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&= \left(x^{n-1}\varphi\left(x\right)\right)^{'} \int_{\Omega} \left(f\left(s\right) - x\right)^{2} d\mu\left(s\right) \geq 0, \qquad k = 1, \ldots, n-1. \qquad (18)
\end{aligned}
$$

Remark 1. Note that when $n = 0$ (17) and the first inequality in (18) coincide with Jensen's inequality.

Theorem 2 is used to prove the theorems in Sect. 3 related to Hardy's inequality.

Corollary 1. *By applying (15) with* $\mu(s) = \sum_{i=1}^{N} a_i \delta_i$ *with* $\sum_{i=1}^{N} a_i = 1$ *and* δ_i *unit masses at* $x = x_i$, $y_i = f(x_i)$, $i = 1, \ldots, N$, $N \in \mathbb{Z}_{+}$, *we obtain that the following special case of (15) yields the inequality*

$$
\begin{aligned}
&\sum_{i=1}^{N} \alpha_i K\left(y_i\right) - K\left(\sum_{i=1}^{N} \alpha_i y_i\right) \\
&\quad \geq \varphi\left(\sum_{j=1}^{N} \alpha_j y_j\right)\left(\sum_{i=1}^{N} \alpha_i y_i^{\gamma} - \left(\sum_{j=1}^{N} \alpha_j y_j\right)^{\gamma}\right) \\
&\quad\quad + C_{\varphi}\left(\sum_{j=1}^{N} \alpha_j y_j\right) \sum_{i=1}^{N} \alpha_i y_i^{\gamma}\left(y_i - \sum_{j=1}^{N} \alpha_j y_j\right), \qquad (19)
\end{aligned}
$$

which holds for $x_i \in [0, b)$, $y_i \in [0, b)$, $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, N$, *and* $\sum_{i=1}^{N} \alpha_i = 1$. *Moreover, under the conditions on* φ *in Theorem 2, as* φ *is differentiable so that* $C_{\varphi} = \varphi'$, *then the right-hand side of (19) is nonnegative and therefore we get that (19) is a genuine scale of refined discrete Jensen's type inequalities.*

The definition of γ- quasiconvex function K, $K(x) = x^{\gamma}\varphi(x)$ can be meaningful even if $\gamma < 0$. We quote for example the following complement of Theorem 2:

Theorem 3 ([9]). *Let $-1 \le \gamma \le 0$, and let f be nonnegative μ-integrable function on the probability measure space (Ω, μ) $x = \int_{\Omega} f(s)\, d\mu(s) > 0$. If φ is a differentiable, nonnegative, convex increasing function that satisfies $\varphi(0) = 0 = \lim_{z\to 0+} z\varphi'(z)$, then*

$$\int_{\Omega} \varphi(f(s))(f(s))^{\gamma}\, d\mu(s) - \varphi(x)x^{\gamma} \\ \ge \varphi(x)\int_{\Omega}((f(s))^{\gamma} - x^{\gamma})\, d\mu(s) + \varphi'(x)\int_{\Omega}(f(s))^{\gamma}(f(s) - x)\, d\mu(s) \quad (20)$$

holds and the right-hand side expression of (20) is nonpositive.

Remark 2. From the case $\gamma = -1$, it follows that when φ is convex and $\varphi(0) = 0 = \lim_{z\to 0+}(z\varphi'(z))$ and $\frac{\varphi(x)}{x}$ is concave, we get a negative lower bound to our Jensen's type difference. This important fact is further stated in the next subsection

2.4 Some Two-Sided Reversed Jensen's Type Inequalities

In [9] we deal with γ-quasiconvex functions when $-1 \le \gamma \le 0$, from which we derive some two-sided Jensen's type inequalities.

The results in this subsection are quoted mainly from that paper. First we state the following consequence of Theorem 3:

Theorem 4. *Let the conditions in Theorem 3 be satisfied and assume in addition that $\frac{\varphi(x)}{x}$ is concave. Then the following two-sided Jensen's type inequality holds:*

$$\varphi(x)\int_{\Omega}\left((f(s))^{-1} - x^{-1}\right) d\mu(s) + \varphi'(x)\int_{\Omega}(f(s))^{-1}(f(s) - x)\, d\mu(s) \\ \le \int_{\Omega}\frac{\varphi(f(s))}{f(s)}\, d\mu(s) - \frac{\varphi(x)}{x} \le 0.$$

Corollary 2. *Let $0 < p \le 1$, and let f be a μ-measurable and positive function on the probability measure space (μ, Ω) and $x = \int_{\Omega} f(s)\, d\mu(s) > 0$. Then*

$$-I_1 + \left(\int_{\Omega} f(s)\, d\mu(s)\right)^p \le \int_{\Omega}(f(s))^p\, d\mu(s) \le \left(\int_{\Omega} f(s)\, d\mu(s)\right)^p,$$

where

$$I_1 = p\left(\int_{\Omega} f(s)\, d\mu(s)\right)^p\left(1 - \int_{\Omega} f(s)\, d\mu(s)\int_{\Omega}(f(s))^{-1}\, d\mu(s)\right) > 0.$$

Using (2) in Lemma 3, our next two-sided reversed Jensen's type inequality quoted from [9] reads:

Theorem 5. *Let f be a nonnegative μ-measurable function on the probability measure space (μ, Ω) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Assume that φ is a differentiable nonnegative, convex function, $\varphi(0) = \lim_{z\to 0+} z\varphi'(z) = 0$. Moreover, assume that $\frac{\varphi(x)}{x}$ is concave. Then the following two-sided Jensen's type inequality holds:*

$$-\left(\frac{\varphi(x)}{x}\right)' \int_\Omega \frac{(f(s)-x)^2}{f(s)} d\mu(s) \leq \int_\Omega \frac{\varphi(f(s))}{f(s)} d\mu(s) - \frac{\varphi(x)}{x} \leq 0.$$

By applying Theorem 5 with $\varphi(x) = x^{1+p}$, $0 < p \leq 1$, we get the following result:

Corollary 3. *Let $0 < p \leq 1$, let f be a nonnegative μ-measurable function on the probability measure space (Ω, μ) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Then*

$$-I_2 + \left(\int_\Omega f(s)\, d\mu(s)\right)^p \leq \int_\Omega (f(s))^p\, d\mu(s) \leq \left(\int_\Omega f(s)\, d\mu(s)\right)^p,$$

where

$$I_2 = p\left(\int_\Omega f(s)\, d\mu(s)\right)^{p-1} \int_\Omega \frac{(f(s)-x)^2}{f(s)} d\mu(s).$$

Next we state additional two-sided inequalities:

Theorem 6. *Let f be a nonnegative μ-measurable function on the probability measure space (Ω, μ) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Assume that φ is a differentiable nonnegative function such that $\varphi(0) = \lim_{z\to 0_+} z\varphi'(z) = 0$, $\frac{\varphi(x)}{x^2}$ is convex, and $\frac{\varphi(x)}{x}$ is concave. Then the following two-sided Jensen's type inequality holds:*

$$\left(\frac{\varphi(x)}{x^2}\right)' \int_\Omega (f(s)-x)^2\, d\mu(s) \leq \int_\Omega \frac{\varphi(f(s))}{f(s)} d\mu(s) - \frac{\varphi(x)}{x} \leq 0.$$

By applying Theorem 6 with $\varphi(x) = x^{1+p}$, $0 < p \leq 1$, we obtain the following Corollary 4:

Corollary 4. *Let $0 < p \leq 1$, and let f be a μ-measurable function on the probability measure space (Ω, μ) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Then*

$$-I_3 + \left(\int_\Omega f(s)\, d\mu(s)\right)^p \leq \int_\Omega (f(s))^p\, d\mu(s) \leq \left(\int_\Omega f(s)\, d\mu(s)\right)^p,$$

where

$$I_3 = (1-p)\left(\int_{\Omega} f(s)\,d\mu(s)\right)^{p-2} \int_{\Omega} (f(s)-x)^2\,d\mu(s).$$

2.5 Comparing Jensen's Type Inequalities

Using Lemmas 4 and 5, we compare Jensen's type inequality obtained by using the 1-quasiconvexity of ψ, where $\psi(x) = x\varphi(x)$, φ is differentiable nonnegative increasing convex function on $x \geq 0$ satisfying $\varphi(0) = \lim_{x\to 0+} x\varphi'(x) = 0$ and the superquadracity of the function ψ. The comparison shows that when $\gamma = 1$, (17) is sharper than (4) for the same ψ, that is:

Theorem 7 ([9]). *Let $\psi(x) = x\varphi(x)$, where φ is nonnegative, convex, increasing, and differentiable function on $[0, b)$, and $\varphi(0) = 0 = \lim_{x\to 0+} x\varphi'(x)$. Then the inequalities*

$$\sum_{j=1}^{m} \alpha_j \psi(x_j) - \psi(\bar{x}) \geq \sum_{j=1}^{m} \alpha_j \varphi'(\bar{x})(x_j - \bar{x})^2 \geq \sum_{j=1}^{m} \alpha_j \psi(|x_j - \bar{x}|) \geq 0$$

hold for

$$x_j \leq 2\bar{x}, \quad \bar{x} = \sum_{j=1}^{m} \alpha_j x_j, \quad 0 \leq \alpha_j \leq 1, \quad \sum_{j=1}^{m} \alpha_j = 1, \quad j = 1, \ldots, m.$$

In particular, the theorem holds under the conditions stated in Lemma 5.

Similarly we also get that:

Theorem 8. *Under the conditions of Theorem 2 for $\psi(x) = x\varphi(x)$, the inequalities*

$$\begin{aligned}
&\int_{\Omega} \psi(f(s))\,d\mu(s) - \psi\left(\int_{\Omega} f(s)\,d\mu(s)\right) \\
&\geq \int_{\Omega} \varphi'\left(\int_{\Omega} f(\sigma)\,d\mu(\sigma)\right)\left(f(s) - \int_{\Omega} f(\sigma)\,d\mu(\sigma)\right)^2 d\mu(s) \\
&\geq \int_{\Omega} \psi\left(\left|f(s) - \int_{\Omega} f(\sigma)\,d\mu(\sigma)\right|\right) d\mu(s) \geq 0,
\end{aligned}$$

hold when $0 < a \leq f(s) \leq 2a$, $s \in \Omega$.

Theorem 7 for $m = 2$ leads to the proofs of Theorems 16 and 19 which deal with the behavior of averages of $A_n(f)$ and $B_n(f)$ discussed in Sect. 4.

3 Hardy's Type Inequalities Related to Quasiconvexity and Superquadracity

In 1928 Hardy [16] obtained and proved the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty f^p(x)\,x^\alpha dx \tag{21}$$

which holds for all measurable and nonnegative functions f on $(0,\infty)$ whenever $\alpha < p-1$, $p \ge 1$. In [24] sufficient conditions for a variant

$$\int_0^b \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^{-1} dx \le \int_0^b f^p(x)\,x^{-1}\left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right) dx \tag{22}$$

of (21) to hold are given for $p \ge 1$. In particular it is shown there that inequality (22) is equivalent to the following variant of (21):

$$\int_0^b \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b f^p(x)\,x^\alpha \left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right) dx \tag{23}$$

for $p \ge 1$, $\alpha < p-1$ or $p<0$, $\alpha > p-1$ and $0 \le b \le \infty$.

In 2008 Oguntuase and Persson proved the following refined Hardy's inequality with "breaking point" $p=2$ (see [18] and also [19]):

Theorem 9. *Let $p \ge 1$, $\alpha < p-1$ and $0 < b \le \infty$. If $p \ge 2$, and the function f is nonnegative and locally integrable on $(0,b)$ and $\int_0^b x^\alpha f^p(x)\,dx < \infty$, then*

$$\begin{aligned}&\int_0^b \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \\ &+\frac{p-1-\alpha}{p}\int_0^b\int_t^b \left|\frac{p}{p-\alpha-1}\left(\frac{t}{x}\right)^{1-\frac{p-\alpha-1}{p}} f(t)\right.\\ &\left.-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|^p x^{\alpha-\frac{p-\alpha-1}{p}} dx\cdot t^{\frac{p-\alpha-1}{p}-1} dt \\ &\le \left(\frac{p}{p-\alpha-1}\right)^p \int_0^b f^p(x)\,x^\alpha\left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right) dx.\end{aligned} \tag{24}$$

If $1 < p \le 2$, then (24) holds in the reversed direction. In particular, for $p=2$, we have equality in (24).

In [2] another theorem about refined Hardy's inequality with "breaking point" at $p = 2$ is proved by using the quasiconvexity of the power functions for $p \geq 2$:

Theorem 10. *Let $p \geq 2$, $k > 1$, $0 < b \leq \infty$, and let the function f be nonnegative and locally integrable on $(0, b)$. Then*

$$\left(\frac{p}{k-1}\right)^p \int_0^b \left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right) x^{p-k} f^p(x)\, dx - \int_0^b x^{-k}\left(\int_0^x f(t)\, dt\right)^p dx$$
$$\geq \frac{(p-1)}{p}(k-1)\int_0^b \int_t^b \left(f(t)\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} - \frac{1}{x}\int_0^x f(\sigma)\, d\sigma\right)^2$$
$$\times \left(\frac{1}{x}\int_0^x f(\sigma)\, d\sigma\right)^{p-2} x^{\left(1-\frac{k-1}{p}\right)(p+1)} t^{\frac{k-1}{p}-1}\frac{dx}{x^2}dt. \tag{25}$$

Moreover, the double integral of the right-hand side of (25) is nonnegative. If $1 < p \leq 2$, then the inequality (25) holds in reverse direction. Equality holds when $p = 2$.

There, in [2] an additional theorem is proved about a "breaking point" at $p = 3$ for Hardy's type inequality by using the quasisuperquadracity of the power functions for $p \geq 3$:

Theorem 11. *Let $p \geq 3$, $k > 1$, $0 < b \leq \infty$, and let the function f be nonnegative and locally integrable on $(0, b)$. Then*

$$\left(\frac{p}{k-1}\right)^p \int_0^b \left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right) x^{p-k} f^p(x)\, dx - \int_0^b x^{-k}\left(\int_0^x f(t)\, dt\right)^p dx$$
$$\geq \frac{p-1}{p}(k-1)\int_0^b \int_t^b \left(f(t)\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} - \frac{1}{x}\int_0^x f(\sigma)\, d\sigma\right)^2$$
$$\times \left(\frac{1}{x}\int_0^x f(\sigma)\, d\sigma\right)^{p-2} x^{\left(1-\frac{k-1}{p}\right)(p+1)} t^{\frac{k-1}{p}-1}\frac{dx}{x^2}dt$$
$$+ \int_0^b \int_t^b \left(f(t)\, t^{1-\frac{k-1}{p}}\right)\left(\left|f(t)\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} - \frac{1}{x}\int_0^x f(\sigma)\, d\sigma\right|\right)^{p-1}$$
$$\times x^{\left(1-\frac{k-1}{p}\right)p} t^{\frac{k-1}{p}-1}\frac{dx}{x^2}dt. \tag{26}$$

Moreover, each double integral of the right-hand side of (26) is nonnegative.

If $1 < p \leq 3$, then the inequality (26) holds in the reverse direction. Equality holds when $p = 3$.

Using the γ-quasiconvexity of the power function when the power is $p+\gamma$, $p \geq 1$, $\gamma \geq 0$ we get in [8] and Hardy's type inequality:

Theorem 12. *Let $p \geq 1$, $k > 1$, $0 < b \leq \infty$, and $\gamma \in \mathbb{R}_{+}$, and let the function f be nonnegative and locally integrable on $[0, b)$. Then*

$$\left(\frac{p+\gamma}{k-1}\right)^{p+\gamma} \int_0^b \left[\left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p+\gamma}}\right) x^{p+\gamma} f^{p+\gamma}(x) - \left(\int_0^x f(t)\,dt\right)^{p+\gamma}\right] \frac{dx}{x^k}$$
$$\geq \left(\frac{k-1}{p+\gamma}\right) \int_0^b \int_t^b \left(\left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} - \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{\gamma}\right)$$
$$\times \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma-1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2} dt$$
$$+ p\left(\frac{k-1}{p+\gamma}\right)^{1-\gamma} \int_0^b \int_t^b \left(f(t)\, t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} \left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right.$$
$$\left. - \frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right) \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-1} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2} dt \geq 0. \tag{27}$$

holds.

Moreover, when $\gamma = 0$ (27) coincides with (23) and therefore also with (21).

By using the γ-superquadracity, we get for the power function with the power greater than $p+\gamma$, $p \geq 2$, $\gamma \geq 0$ that the following Hardy's type inequality holds.

Theorem 13 ([1]). *Let $p \geq 2$, $k > 1$, $0 < b \leq \infty$, and $\gamma \in \mathbb{R}_{+}$, and let the function f be nonnegative and locally integrable on $(0, b)$. Then*

$$\left(\frac{p+\gamma}{k-1}\right)^{p+\gamma} \int_0^b \left[\left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p+\gamma}}\right) x^{p+\gamma} f^{p+\gamma}(x) - \left(\int_0^x f(t)\,dt\right)^{p+\gamma}\right] \frac{dx}{x^k}$$
$$\geq \left(\frac{k-1}{p+\gamma}\right) \int_0^b \int_t^b \left(\left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} - \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{\gamma}\right)$$
$$\times \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma-1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2} dt$$
$$+ \left(\frac{k-1}{p+\gamma}\right)^{1-\gamma} \int_0^b \int_t^b \left(f(t)\, t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} \left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right.$$
$$\left. - \frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right) \cdot p\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-1} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2} dt$$
$$+ \left(\frac{k-1}{p+\gamma}\right)^{1-\gamma} \int_0^b \int_t^b \left(f(t)\, t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma}$$
$$\times \left(\left|f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}} - \frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right|\right)^{p} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2} dt. \tag{28}$$

Moreover, if γ *is a nonnegative integer, then the right-hand side of (28) is nonnegative. If* $1 < p \leq 2$*, then inequality (28) is reversed. Equality holds when* $p = 2$*. When* $\gamma = 0$*, inequality (28) coincide with (24).*

4 Averages

In this section we deal with the lower bounds of differences of averages where the functions f involved with are quasiconvex.

For a function f and a sequence a_n, $n = 0, 1, \ldots$, we define

$$A_n(f) = \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right), \quad n \geq 2,$$

and

$$B_n(f) = \frac{1}{a_{n+1}} \sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \quad n \geq 1.$$

In [4, Theorems 3.1 and 5.3], the following results concerning averages for superquadratic functions are proved:

Theorem 14. *Let* a_i*,* $i = 0, 1 \ldots$*, be an increasing sequence with* $a_0 = 0$*, and* $a_1 > 0$*, and let* $a_{i+1} - a_i$ *be decreasing. Suppose that* f *is superquadratic and nonnegative on* $[0, b)$*. Then, for* $n \geq 2$

$$\begin{aligned} A_{n+1}(f) - A_n(f) &= \frac{1}{a_n} \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_n a_{n-1}} \sum_{i=1}^{n-1} a_i f\left(\left|\frac{a_{i+1}}{a_{n+1}} - \frac{a_i}{a_n}\right|\right) \\ &+ \frac{1}{a_n a_{n-1}} \sum_{i=1}^{n-1} (a_n - a_i) f\left(\left|\frac{a_i}{a_n} - \frac{a_i}{a_{n+1}}\right|\right). \end{aligned} \tag{29}$$

In the special case where $a_i = i$*,* $i = 0, 1 \ldots$*, we get for*

$$A_n(f) = \frac{1}{n-1} \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right), \quad n \geq 2$$

that if f is superquadratic on $[0,1]$, then for $n \geq 2$

$$A_{n+1}(f) - A_n(f) \geq \sum_{r=1}^{n-1} \frac{2r}{n(n-1)} f\left(\frac{n-r}{n(n+1)}\right).$$

holds.

Further,

$$A_{n+1}(f) - A_n(f) \geq f\left(\tfrac{1}{3n}\right) + \sum_{r=1}^{n-1} \lambda_r f(y_r),$$

where $\lambda_r = \frac{2r}{n(n-1)}$, $y_r = \frac{|2n-1-3r|}{3n(n+1)}$, $r = 1, \ldots, n-1$.

Moreover if f is superquadratic and nonnegative, then for $n \geq 3$

$$A_{n+1}(f) - A_n(f) \geq f\left(\tfrac{1}{3n}\right) + f\left(\tfrac{16}{81(n+3)}\right).$$

Theorem 15 ([4, Theorems 3.2 and 5.6]). *Let $a_i > 0$, and $a_i - a_{i-1}$, $i = 1, \ldots,$ be increasing sequences and let $a_0 = 0$. Suppose that f is superquadratic and nonnegative on $[0, b)$.*

Then,

$$\begin{aligned} B_{n-1}(f) - B_n(f) &= \frac{1}{a_n} \sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}} \sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_n a_{n+1}} \sum_{i=1}^{n-1} a_i f\left(\left|\frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_n}\right|\right) \\ &\quad + \frac{1}{a_n a_{n+1}} \sum_{i=0}^{n-1} (a_n - a_i) f\left(\left|\frac{a_i}{a_n} - \frac{a_i}{a_{n-1}}\right|\right), \end{aligned} \tag{30}$$

and in the special case where $a_i = i$, $i = 0, 1, \ldots,$ we get for

$$B_n(f) = \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{r}{n}\right), \quad n \geq 1$$

that if f is superquadratic on [0,1], then for $n \geq 2$

$$B_{n-1}(f) - B_n(f) \geq \sum_{r=1}^{n} \frac{2r}{n(n+1)} f\left(\frac{n-r}{n(n-1)}\right).$$

holds.

Further,

$$B_{n-1}(f) - B_n(f) \geq f\left(\tfrac{1}{3n}\right) + \sum_{r=1}^{n} \lambda_r f(y_r),$$

where $\lambda_r = \frac{2r}{n(n+1)}$, $y_r = \frac{|2n+1-3r|}{3n(n-1)}$, $r = 1, \ldots, n$.

Moreover, if f *is also nonnegative, then for* $n \geq 2$

$$B_{n-1}(f) - B_n(f) \geq f\left(\tfrac{1}{3n}\right) + f\left(\tfrac{16}{81n}\right).$$

Using Theorem 7 we get for functions which are simultaneously quasiconvex and superquadratic functions a better lower bound for the difference $A_{n+1}(f) - A_n(f)$ when we use the quasiconvexity of the function f than when we use its superquadracity.

To show it we first state results analog to those in Theorem 14, but now instead of superquadratic functions we deal with quasiconvex functions.

The results quoted below are mainly from [3].

Theorem 16. *Let* $\varphi : [0, b) \to \mathbb{R}_+$, $0 < b \leq \infty$ *be differentiable convex increasing function satisfying* $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$, *and let* $f = x\varphi$. *Let the sequence* $a_i > 0$, $i = 1, \ldots$ *be such that* a_i *is increasing and* $a_{i+1} - a_i$ *is decreasing and let* $a_0 = 0$. *Then, for* $n \geq 2$, *we get from the quasiconvexity of* f *that the inequalities*

$$\begin{aligned} A_{n+1}(f) - A_n(f) &= \frac{1}{a_n} \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i(a_n + a_{i+1} - a_i)}{a_n a_{n+1}}\right)\left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2}\right) \geq 0 \end{aligned} \tag{31}$$

hold, and as φ' *is increasing the inequalities*

$$\begin{aligned} A_{n+1}(f) - A_n(f) &= \frac{1}{a_n} \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i(a_n + a_{i+1} - a_i)}{a_n a_{n+1}}\right)\left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2}\right) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i}{a_n}\right)\left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2}\right) \geq 0 \end{aligned}$$

hold.

From the superquadracity of $f = x\varphi$, *we get that*

$$\begin{aligned} &A_{n+1}(f) - A_n(f) \\ &\geq \frac{1}{a_n a_{n-1}} \sum_{i=1}^{n-1} \left(a_i f\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right. \\ &\quad \left. + (a_n - a_i) f\left(\frac{a_i (a_{i+1} - a_i)}{a_n (a_{n+1})} \right) \right) \\ &= \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i) a_i (a_n - a_i)}{a_n^2 a_{n+1}} \left(\varphi\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right. \\ &\quad \left. + \varphi\left(\frac{a_i (a_{i+1} - a_i)}{a_n (a_{n+1})} \right) \right) \geq 0. \end{aligned} \tag{32}$$

As φ *is convex we get that*

$$\begin{aligned} &A_{n+1}(f) - A_n(f) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i) a_i (a_n - a_i)}{a_n^2 a_{n+1}} \left(\varphi\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right. \\ &\quad \left. + \varphi\left(\frac{a_i (a_{i+1} - a_i)}{a_n (a_{n+1})} \right) \right) \\ &\geq \frac{2}{a_{n-1} a_n^2 a_{n+1}} \sum_{i=1}^{n-1} \varphi\left(\frac{a_{i+1} - a_i}{2a_{n+1}} \right) (a_{i+1} - a_i) a_i (a_n - a_i) \geq 0. \end{aligned}$$

Further, if φ' *is also convex, then*

$$\begin{aligned} &A_{n+1}(f) - A_n(f) \\ &\geq \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_{n-1} a_n^2 a_{n+1}^2} \varphi'\left(\frac{\sum_{i=1}^{n-1} a_i^2 (a_n - a_i)(a_{i+1} - a_i)^2}{a_n \sum_{i=1}^{n-1} a_i (a_n - a_i)(a_{i+1} - a_i)^2} \right) \geq 0. \end{aligned}$$

Finally, the bound obtained in (31) by the quasiconvexity of f *is better than the bound obtained by its superquadracity in (32) and in (29), that is:*

$$\begin{aligned} &A_{n+1}(f) - A_n(f) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i (a_n + a_{i+1} - a_i)}{a_n a_{n+1}} \right) \left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2} \right) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i) a_i (a_n - a_i)}{a_n^2 a_{n+1}} \left(\varphi\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right. \end{aligned}$$

$$+\varphi\left(\frac{a_i\left(a_{i+1}-a_i\right)}{a_n\left(a_{n+1}\right)}\right)\Bigg)$$

$$\geq \frac{2}{a_{n-1}a_n^2a_{n+1}}\sum_{i=1}^{n-1}\varphi\left(\frac{a_{i+1}-a_i}{2a_{n+1}}\right)\left(a_{i+1}-a_i\right)a_i\left(a_n-a_i\right)\geq 0.$$

Example 2. Let $f = x\varphi$, be quasiconvex function where φ is nonnegative convex increasing and differentiable function on $[0, b)$, and $\varphi(0) = 0 = \lim_{x\to 0^+} x\varphi'(x)$. Let $a_i = i$, $i = 0, \ldots, n$. Then by Theorem 16

$$A_{n+1}(f) - A_n(f) \geq \frac{1}{(n-1)}\sum_{r=1}^{n-1}\varphi'\left(\frac{r}{n}\right)\frac{r(n-r)}{n^2(n+1)^2}\geq 0,$$

and the lower bound obtained by the quasiconvexity of f is better than the lower bound obtained by its quasiconvexity, that is:

$$A_{n+1}(f) - A_n(f) \geq \frac{1}{(n-1)}\sum_{r=1}^{n-1}\varphi'\left(\frac{r}{n}\right)\frac{r(n-r)}{n^2(n+1)^2}$$

$$\geq \frac{1}{(n-1)}\sum_{r=1}^{n-1}\frac{2r}{n}f\left(\frac{n-r}{n(n+1)}\right)\geq 0.$$

If φ' is also convex, we get also:

$$A_{n+1}(f) - A_n(f) \geq \frac{1}{6n(n+1)}\varphi'\left(\frac{1}{2}\right)\geq \frac{1}{3n}\varphi\left(\frac{1}{2(n+1)}\right)\geq 0.$$

Now we present results related to the behavior of $B_n(f)$ when n changes.

First we state a theorem about $B_{n-1}(f) - B_n(f)$ when the function f is quasiconvex and $f = x\varphi$:

Theorem 17. *Let $\varphi : [0, b) \to \mathbb{R}_+$, $0 < b \leq \infty$ be differentiable convex increasing function and let $f = x\varphi$. Let the sequence $a_i > 0$, $i = 1, \ldots$, be such that $a_i - a_{i-1}$, $i = 1, \ldots$ is increasing and let $a_0 = 0$. Then, for $n \geq 2$*

$$\begin{aligned}
&B_{n-1}(f) - B_n(f)\\
&= \frac{1}{a_n}\sum_{i=0}^{n-1}f\left(\frac{a_i}{a_{n-1}}\right)-\frac{1}{a_{n+1}}\sum_{i=0}^{n}f\left(\frac{a_i}{a_n}\right)\\
&\geq \frac{1}{a_{n+1}a_n^2a_{n-1}^2}\sum_{i=1}^{n-1}\left(a_i-a_{i-1}\right)^2a_i\left(a_n-a_i\right)\varphi'\left(\frac{a_i\left(a_n+a_{i-1}-a_i\right)}{a_n}\right)\geq 0.
\end{aligned} \tag{33}$$

If in addition φ' is convex on $[0,\infty)$, then

$$\begin{aligned}
&\frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\
&\quad \geq \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})^2 a_i (a_n - a_i)\varphi'}{a_{n+1} a_n^2 a_{n-1}^2} \left(\frac{\sum_{i=1}^{n-1} a_i^2 (a_n - a_i)(a_i - a_{i-1})^2}{a_n \sum_{i=1}^{n-1} a_i (a_n - a_i)(a_i - a_{i-1})^2}\right) \\
&\quad \geq 0.
\end{aligned}$$

We state now a theorem about the superquadratic function f where $f = x\varphi$.

Theorem 18. *Let $\varphi : [0,b) \to \mathbb{R}_+$ $0 < b \leq \infty$ be differentiable convex increasing function satisfying $\varphi(0) = 0 = \lim_{x\to 0^+} x\varphi'(x)$ and let $f = x\varphi$. Let $a_0 = 0$ and $a_i > 0$, $i = 1,\ldots,$ be a sequence for which $a_i - a_{i-1}$ is increasing for $i = 1,\ldots,$. Then*

$$\begin{aligned}
B_{n-1}(f) - B_n(f) &= \frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\
&\geq \frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\left(\frac{a_i}{a_n} f\left(\frac{(a_n - a_i)(a_i - a_{i-1})}{a_{n-1}a_n}\right)\right. \\
&\quad \left. + \frac{a_n - a_i}{a_n} f\left(\frac{a_i (a_i - a_{i-1})}{a_{n-1}a_n}\right)\right) \\
&= \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1}) a_i (a_n - a_i)}{a_{n+1} a_n^2 a_{n-1}} \left(\varphi\left(\frac{a_i (a_i - a_{i-1})}{a_{n-1}a_n}\right)\right. \\
&\quad \left. + \varphi\left(\frac{(a_n - a_i)(a_i - a_{i-1})}{a_{n-1}a_n}\right)\right), \qquad (34)
\end{aligned}$$

and as φ is also convex we get that

$$\begin{aligned}
B_{n-1}(f) - B_n(f) &= \frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\
&\geq \frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\left(\frac{a_i}{a_n} f\left(\frac{(a_n - a_i)(a_i - a_{i-1})}{a_{n-1}a_n}\right)\right. \\
&\quad \left. + \frac{a_n - a_i}{a_n} f\left(\frac{a_i (a_i - a_{i-1})}{a_{n-1}a_n}\right)\right)
\end{aligned}$$

$$= \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})\, a_i\, (a_n - a_i)}{a_{n+1} a_n^2 a_{n-1}} \left(\varphi \left(\frac{a_i\,(a_i - a_{i-1})}{a_{n-1} a_n} \right) + \varphi \left(\frac{(a_n - a_i)\,(a_i - a_{i-1})}{a_{n-1} a_n} \right) \right)$$

$$\geq \sum_{i=1}^{n-1} \frac{2a_i\,(a_n - a_i)\,(a_i - a_{i-1})}{a_{n+1} a_n^2 a_{n-1}} \varphi \left(\frac{(a_i - a_{i-1})}{2a_{n-1}} \right) \geq 0.$$

The proof of (35) in Theorem 19 uses Theorem 7 to show that the bound obtained in (33) is better than the bound obtained in (30) and in (34).

Theorem 19. *Let* $\varphi : [0, b) \to \mathbb{R}_+$ $0 < b \leq \infty$ *be differentiable convex increasing function satisfying* $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$ *and let* $f = x\varphi$. *Let* $a_0 = 0$ *and* $a_i > 0$, $i = 1, \ldots$ *be a sequence for which* $a_i - a_{i-1}$ *is increasing for* $i = 1, \ldots,$. *Then the inequalities*

$$B_{n-1}(f) - B_n(f) = \frac{1}{a_n} \sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}} \sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right)$$

$$\geq \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})\,(a_n - a_i)\, a_i}{a_{n-1}^2 a_n^2 a_{n+1}} \varphi' \left(\frac{a_i\,(a_n + a_{i-1} - a_i)}{a_n a_{n-1}} \right)$$

$$\geq \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})\,(a_n - a_i)\, a_i}{a_{n-1} a_n^2 a_{n+1}} \left(\varphi \left(\frac{a_i\,(a_i - a_{i-1})}{a_n a_{n-1}} \right) + \varphi \left(\frac{(a_n - a_i)\,(a_i - a_{i-1})}{a_n a_{n-1}} \right) \right) \geq 0 \tag{35}$$

hold.

Example 3. Let $f = x\varphi$, then under the conditions of Theorem 19 on φ, when $a_i = i$, $i = 0, \ldots, n$ we get that

$$B_{n-1}(f) - B_n(f) = \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n-1}\right) - \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{r}{n}\right)$$

$$\geq \frac{1}{(n+1)\,(n-1)^2\, n^2} \sum_{r=0}^{n} \varphi'\left(\frac{r}{n}\right) r\,(n-r)$$

$$\geq 2 \sum_{r=0}^{n} \frac{r}{n} f\left(\frac{n-r}{n\,(n-1)}\right) \geq 0$$

holds.

Further, if φ' is also convex on $[0, \infty)$, we get that

$$B_{n-1}(f) - B_n(f) \geq \frac{1}{6n(n-1)}\varphi'\left(\frac{1}{2}\right) \geq \frac{1}{3n}\varphi\left(\frac{1}{2(n-1)}\right).$$

Similarly, the bounds of the differences

$$\frac{1}{2n-1}\sum_{i=1}^{n} f\left(\frac{2i+1}{2n+1}\right) - \frac{1}{2n-3}\sum_{i=1}^{n-1} f\left(\frac{2i+1}{2n-1}\right)$$

$$\frac{1}{2n-1}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n-3}\right) - \frac{1}{2n+1}\sum_{i=1}^{n-1} f\left(\frac{2i-1}{2n-1}\right)$$

obtained by using the quasiconvexity of f are better than the bound obtained by using superquadracity.

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