Panos M. Pardalos
Themistocles M. Rassias *Editors*

Contributions in Mathematics and Engineering

In Honor of Constantin Carathéodory

Foreword by
Terry Rockafellar

Springer

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Foreword

Constantin Carathéodory, 1873–1950, was one of the most important and influential mathematicians of his era and certainly the preeminent Greek on the scene. His work combined theory-building with practical applications in a way that is no longer so common. His career spanned a remarkable range of locations and historical upheavals, even by today's standards.

Carathéodory's accomplishments are too voluminous to describe in balanced detail, but from a personal perspective, my connection began with his famous theorem on convex hulls. That result, with major consequences in convex analysis and optimization technology, confirms that in n-dimensional space, the hull can be formed by taking convex combinations of just $n + 1$ elements at a time. His innovative developments in the calculus of variations, beginning with his doctoral dissertation and carried on in retirement with his editing of Euler's works in that subject, also attracted me early on because of their modern ties to optimization and control.

Theory-building was certainly the theme in those efforts in the calculus of variations, influenced by considerations in optics, but even more so in his ground-breaking axiomatic formulation of thermodynamics. These achievements under-score the back-and-forth between mathematics and physics that was the centerpiece of science a century ago but now is just one of many lines of inspiration and progress. They also reflect Carathéory's practical bent, having once worked for the British Colonial Service as an engineer in dam construction in Egypt.

We think it normal now that faculties of universities include multilingual professors of many nationalities who have lived, studied, and taught in many places, but that was true in Carathéodory's time as well. His own life provides an instructive example which is full of reminders of how world events can intervene. Born in Berlin of Greek parents in the diplomatic circuit, not for Greece but the Ottoman Empire (with Constantinople as their home), he grew up in Brussels, summering on the French Riviera. He went on to Germany as a university student and eventually to a prominent academic career with posts shifting from Göttingen to Bonn to Hannover to Breslau to Göttingen again and finally to Berlin. World War I brought him hardships and emptied the universities of colleagues and students of military age.

In 1920, after a formal invitation from the prime minister of Greece Eleftherios Venizelos, he moved to Greece in order to oversee the founding of a new university in Smyrna. However, that came quickly to an abrupt end when the Turks overran the region and forced the Greeks out. In 1922 he moved to Athens obtaining professorial positions at the University of Athens and subsequently at the National Technical University of Athens. In 1924, on the recommendation of A. Sommerfeld, he was appointed professor at the University of Munich succeeding F. Lindemann. A few years later, in 1930, Carathéodory was invited yet again by the Greek prime minister E. Venizelos in order to undertake administrative duties at the University of Athens as well as at the Aristotle University of Thessaloniki, which he accepted and then offered his valuable services for the next 2 years.

Carathéodory visited the United States twice and could have accepted an offer from Stanford, but did not. Then came World War II, which he rode out in Munich. That marked the demise of Germany's longtime dominance in mathematics and other sciences.

The dedication of this volume to Constantin Carathéodory is a fitting tribute to a great discoverer of facts and ideas which continue to enrich us all.

Seattle, WA, USA R. Tyrrell Rockafellar
November 2015

Preface

This volume consists of scientific articles dedicated to the work of Constantin Carathéodory. These articles deepen our understanding of some of the current research problems and theories which have their origin or have been influenced by Carathéodory.

The presentation of concepts and methods featured in this volume make it an invaluable reference for teachers and other professionals in Mathematics, who are interested in pure and applied research.

It is our pleasure to express our warmest thanks to all of the scientists who contributed to this volume and to Professor R. Tyrrell Rockafellar for writing the Foreword to this volume. We would also like to acknowledge the superb assistance that the staff of Springer has provided for this publication.

Gainesville, FL, USA — Panos M. Pardalos
Athens, Greece — Themistocles M. Rassias

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Applications of Quasiconvexity

S. Abramovich

In Honor of Constantin Carathéodory

Abstract This survey deals with functions called γ-quasiconvex functions and their relations to convexity and superquadracity.

For γ-quasiconvex functions and for superquadratic functions, we get analogs of inequalities satisfied by convex functions and we get refinements for those convex functions which are also γ-quasiconvex as well as superquadratic.

We show in which cases the refinements by γ-quasiconvex functions are better than those obtained by superquadratic functions and convex functions. The power functions defined on $x \geq 0$ where the power is greater or equal to two are examples of convex, quasiconvex, and superquadratic functions.

1 Introduction

In this survey we present functions called γ-quasiconvex functions and their relations to convexity and superquadracity.

This survey may serve as introductory work to a book on quasiconvexity by S. Abramovich, L. E. Persson, J. A. Oguntoase, and S. Samko.

For γ-quasiconvex functions and for superquadratic functions, we get analogs of inequalities satisfied by convex functions and we get refinements for those convex functions which are also γ-quasiconvex as well as superquadratic.

We show in which cases the refinements by γ-quasiconvex functions are better than those obtained by superquadratic functions and convex functions. The power

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functions defined on $x \geq 0$, where the power is greater or equal to two, are important examples of convex, quasiconvex, and superquadratic functions.

We demonstrate the applications of γ-quasiconvexity and superquadracity by putting together some results related mainly to Jensen's inequality, Hardy's inequality, and Average Sums inequalities. We quote here the results obtained in [1–5, 8, 9, 16, 18, 19, 23, 24]. For more on the subjects of superquadracity, γ-superquadracity, and γ-quasiconvexity, we refer the reader to the reference list [1–24] and their references.

We start with some definitions, lemmas, and remarks we used in the proofs of the results stated in the sequel.

Definition 1 ([4, 5]). A function $\varphi : [0, b) \to \mathbb{R}$ is superquadratic provided that for all $0 \leq x < b$, there exists a constant $C_\varphi(x) \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) \geq C_\varphi(x)(y - x) + \varphi(|y - x|) \tag{1}$$

for every y, $0 \leq y < b$.

Definition 2 ([1]). A function $K : [0, b) \to \mathbb{R}$ that satisfies $K(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}$, where φ is a superquadratic function, is called γ-quasisuperquadratic function.

Definition 3 ([8, 9]). A function $K : [0, b) \to \mathbb{R}$ that satisfies $K(x) = x^\gamma \varphi(x)$, when $\gamma \in \mathbb{R}$, and φ is a convex function is called γ-quasiconvex function.

Lemma 1 ([5]). *Let φ be a superquadratic function with $C_\varphi(x)$ as in Definition 1. Then:*

(i) $\varphi(0) \leq 0$,
(ii) if $\varphi(0) = \varphi'(0) = 0$, *then* $C_\varphi(x) = \varphi'(x)$ *whenever φ is differentiable at* $0 < x < b$.
(iii) if $\varphi \geq 0$, *then φ is convex and* $\varphi(0) = \varphi'(0) = 0$.

Lemma 2 ([5]). *Suppose that $\varphi : [0, b) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.*

Lemma 3 ([9]). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be differentiable function $x, y \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$. Then*

$$\begin{aligned} &\varphi(x)(y^\gamma - x^\gamma) + \varphi'(x) y^\gamma (y - x) \\ &\quad - \left[(x\varphi(x))\left(y^{\gamma-1} - x^{\gamma-1}\right) + (x\varphi(x))' y^{\gamma-1}(y - x)\right] \\ &\quad = y^{\gamma-1}\varphi'(x)(y - x)^2 . \end{aligned} \tag{2}$$

Using Lemma 2 we get:

Lemma 4 ([3]). *Let $\psi : [0, b) \to \mathbb{R}$ be 1-quasiconvex function, where $\psi(x) = x\varphi(x)$, φ is differentiable nonnegative increasing convex function on $x \geq 0$ satisfying $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$, then ψ is also superquadratic.*

Lemma 5 ([9]). *Let* $0 < a \leq x_i \leq 2a$*,* $i = 1, \ldots, n$*, or let* $\alpha_{i_0} \geq \frac{1}{2}$ *be such that* $x_{i_0} \geq x_i \geq 0$*,* $i = 1, \ldots, n$*,* $0 \leq \alpha_i \leq 1$*,* $i = 1, \ldots, n$*,* $\sum_{i=1}^{n} \alpha_i = 1$*, (for instance, when* $n = 2$*, let* $\alpha_1 = \alpha_2 = \frac{1}{2}$*,* $x_1, x_2 > 0$*). Then* $|x_i - \overline{x}| \leq \overline{x} \iff 0 \leq x_i \leq 2\overline{x}$*,* $i = 1, \ldots, n$ *when* $\overline{x} = \sum_{i=1}^{n} \alpha_i x_i$*.*

2 Jensen's Type Inequalities

Jensen's theorem states that $\int_\Omega \varphi(f(s))\, d\mu(s) \geq \varphi\left(\int_\Omega f(s)\, d\mu(s)\right)$ holds when $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, μ is a probability measure and f is a μ-integrable function (see, for instance, [23]). In this section we quote theorems that deal with generalizations and refinements of this very important theorem.

2.1 Jensen's Type Inequalities for Superquadratic Functions

From the definition of superquadracity, we easily get Jensen's type inequalities:

Lemma 6 ([5]). *The function* φ *is superquadratic on* $[0, b)$*, if and only if*

$$\frac{1}{A_n} \sum_{i=1}^{n} a_i \varphi(x_i) - \varphi(\overline{x}) \geq \frac{1}{A_n} \sum_{i=1}^{n} a_i \varphi(|x_i - \overline{x}|), \tag{3}$$

holds, where $x_i \in [0, b)$*,* $i = 1, \ldots, n$ *and* $a_i \geq 0$*,* $i = 1, \ldots, n$*, are such that* $A_n = \sum_{i=1}^{n} a_i > 0$*, and* $\overline{x} = \frac{1}{A_n} \sum_{i=1}^{n} a_i x_i$*.*

The function φ *is superquadratic on* $[0, b)$*, if and only if*

$$\int_\Omega \varphi(f(s))\, d\mu(s) - \varphi\left(\int_\Omega f(s)\, d\mu(s)\right) \geq \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(\sigma)\, d\mu(\sigma)\right|\right) d\mu(s) \tag{4}$$

where f *is any nonnegative* μ*-integrable function on a probability measure space* (Ω, μ)*.*

The power functions $\varphi(x) = x^p$, $x \geq 0$ are superquadratic when $p \geq 2$ and subquadratic, that is, $-\varphi$ is superquadratic when $1 \leq p \leq 2$. When $\varphi(x) = x^2$ inequality (1) reduces to equality and therefore the same holds for (3) and (4).

It is obvious that when the superquadratic function is nonnegative on $[0, b)$, then inequalities (3) and (4) are refinements of Jensen's inequalities for convex functions.

2.2 Jensen's Type Inequalities for γ-Quasisuperquadratic Functions

For γ-quasisuperquadratic functions defined in Definition 2 we get:

Lemma 7 ([8]). *Let $K(x) = x^{\gamma}\varphi(x)$, $\gamma \in \mathbb{R}_+$, where φ is superquadratic on $[0,b)$. Then, for this γ-quasisuperquadratic function K, the inequality*

$$K(y) - K(x) \geq \varphi(x)(y^{\gamma} - x^{\gamma}) + C_{\varphi}(x)\, y^{\gamma}(y - x) + y^{\gamma}\varphi(|y - x|) \tag{5}$$

holds for $x \in [0,b)$. $y \in [0,b)$.

Moreover,

$$\begin{aligned} &\sum_{i=1}^{N}\alpha_i K(y_i) - K\left(\sum_{i=1}^{N}\alpha_i y_i\right) \\ &\geq \varphi\left(\sum_{j=1}^{N}\alpha_j y_j\right)\left(\sum_{i=1}^{N}\alpha_i y_i^{\gamma} - \left(\sum_{j=1}^{N}\alpha_j y_j\right)^{\gamma}\right) \\ &\quad + C_{\varphi}\left(\sum_{j=1}^{N}\alpha_j y_j\right)\sum_{i=1}^{N}\alpha_i y_i^{\gamma}\left(y_i - \sum_{j=1}^{N}\alpha_j y_j\right) \\ &\quad + \sum_{i=1}^{N}\alpha_i y_i^{\gamma}\varphi\left(\left|y_i - \sum_{j=1}^{N}\alpha_j y_j\right|\right) \end{aligned} \tag{6}$$

holds for $x_i \in [0,b)$, $y_i \in [0,b)$, $0 \leq \alpha_i \leq 1$, $i = 1,\ldots,N$, and $\sum_{i=1}^{N}\alpha_i = 1$.

Also

$$\begin{aligned} &\int_{\Omega} K(f(s))\, d\mu(s) - K\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\ &\geq \int_{\Omega}\left[\varphi(x)(f^{\gamma}(s) - x^{\gamma}) + C_{\varphi}(x) f^{\gamma}(s)(f(s) - x)\right. \\ &\quad \left. + f^{\gamma}(s)\varphi(|f(s) - x|)\right] d\mu(s). \end{aligned} \tag{7}$$

holds, where f is any nonnegative μ-integrable function on the probability measure space (Ω,μ) and $x = \int_{\Omega} f(s)\, d\mu(s) > 0$.

If φ is subquadratic, then the reverse inequality of (5)–(7) hold, in particular

$$\begin{aligned} &\int_{\Omega} K(f(s))\, d\mu(s) - K\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\ &\leq \int_{\Omega}\left[\varphi(x)(f^{\gamma}(s) - x^{\gamma}) + C_{\varphi}(x) f^{\gamma}(s)(f(s) - x)\right. \\ &\quad \left. + f^{\gamma}(s)\varphi(|f(s) - x|)\right] d\mu(s). \end{aligned} \tag{8}$$

Inequalities (5)–(7) are satisfied by the power functions $K(x) = x^p$, $p \geq \gamma + 2$. *For* $\gamma < p \leq \gamma + 2$, *the reverse inequalities hold; in particular (8) holds. They reduce to equalities for* $p = \gamma + 2$.

The power functions are used to get from Jensen's type inequalities refined Hardy's type inequalities.

Equality (2) Lemmas 3 and 4 help in proving the following Theorem 1 about Jensen's type inequalities for γ-quasisuperquadratic functions. The results of this theorem refine Jensen's type inequalities stated in inequalities (3) and (4) for superquadratic functions. As nonnegative superquadratic functions (according to Lemma 1) are convex, Theorem 1 refines also Jensen's inequalities for these convex functions which are also superquadratic:

Theorem 1 ([8, Lemma 3.1]). *Let* $K(x) = x^\gamma \varphi(x) = x^{\gamma-1}\psi(x)$, $\gamma \geq 1$, *where* φ *is a differentiable nonnegative superquadratic function and* $\psi(x) = x\varphi(x)$. *Then the bound obtained for* $K(x) = x^\gamma \varphi(x)$ *is stronger than the bound obtained for* $K(x) = x^{\gamma-1}\psi(x)$, *which means that:*

$$K(y) - K(x) \geq \varphi(x)(y^\gamma - x^\gamma) + \varphi'(x)\, y^\gamma (y - x) + y^\gamma \varphi(|y - x|) \tag{9}$$

implies that

$$\begin{aligned} & K(y) - K(x) \\ & \geq \psi(x)\left(y^{\gamma-1} - x^{\gamma-1}\right) + \psi'(x)\, y^{\gamma-1}(y - x) + y^{\gamma-1}\psi(|y - x|) \\ & = x\varphi(x)(x)\left(y^{\gamma-1} - x^{\gamma-1}\right) + (x\varphi(x))'\, y^{\gamma-1}(y - x) + y^{\gamma-1}|y - x|\,\varphi(|y - x|) . \end{aligned} \tag{10}$$

Moreover, if $K(x) = x^n \varphi(x)$, $\psi_k(x) = x^k \varphi(x)$, n *is an integer,* $k = 1, 2, \ldots, n$, *and* $\varphi(x)$ *is nonnegative superquadratic, then the inequalities*

$$\begin{aligned} & \int_\Omega K(f(s))\, d\mu(s) \; - K\left(\int_\Omega f(s)\, d\mu(s)\right) \\ & \geq \int_\Omega \left[\varphi(x)(f^n(s) - x^n) + C_\varphi(x) f^n(s)(f(s) - x) \right. \\ & \qquad \left. + f^n(s)\,\varphi(|f(s) - x|)\right] d\mu(s) \\ & \geq \int_\Omega \left[\psi_k(x)\left(f^{n-k}(s) - x^{n-k}\right) + C_{\psi_k}(x) f^{n-k}(s)(f(s) - x) \right. \\ & \qquad \left. + f^{n-k}(s)\,\psi_k(|f(s) - x|)\right] d\mu(s) \\ & \geq \int_\Omega \psi_n(|f(s) - x|)\, d\mu(s) = \int_\Omega K(|f(s) - x|)\, d\mu(s) \geq 0 \end{aligned} \tag{11}$$

hold for all probability measure spaces (Ω, μ) *of* μ*-integrable nonnegative functions* f, *where* $x = \int_\Omega f(s)\, d\mu(s) > 0$.

Furthermore, if φ *is differentiable nonnegative increasing, convex subquadratic, and* $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$*, then according to Lemma 4,* $x\varphi(x)$ *is superquadratic and for* $k = 1, \ldots, n$

$$\begin{aligned}
&\int_{\Omega} \left[\varphi(x)(f^n(s) - x^n) + \varphi'(x) f^n(s)(f(s) - x) + f^n(s)\varphi(|f(s) - x|)\right] d\mu(s) \\
&\geq \int_{\Omega} K(f(s))\, d\mu(s) - K\left(\int_{\Omega} f(s)\, d\mu(s)\right) \\
&\geq \int_{\Omega} \left[\psi_1(x)\left(f^{n-1}(s) - x^{n-1}\right) + \psi_k'(x) f^{n-1}(s)(f(s) - x)\right. \\
&\quad \left. + f^{n-1}(s)\psi_1(|f(s) - x|)\right] d\mu(s) \\
&\geq \int_{\Omega} \left[\psi_k(x)\left(f^{n-k}(s) - x^{n-k}\right) + \psi_k'(x) f^{n-k}(s)(f(s) - x)\right. \\
&\quad \left. + f^{n-k}(s)\psi_k(|f(s) - x|)\right] d\mu(s) \\
&\geq \int_{\Omega} \psi_n(|f(s) - x|)\, d\mu(s) = \int_{\Omega} K(|f(s) - x|)\, d\mu(s) \geq 0. \qquad (12)
\end{aligned}$$

In particular, if $\varphi(x) = x^p$, $x \geq 0$, $p \geq 1$*, then (9)–(11) are satisfied when* $p \geq 2$ *and (12) is satisfied when* $1 \leq p \leq 2$*. When* $p = 2$ *equality holds in the first inequality of (11) and in the first inequality of (12).*

2.3 Jensen's Type Inequalities for γ-Quasiconvex Functions

In [1, 8, 9], Jensen's type inequalities for γ-quasiconvex functions (Definition 3) are derived and discussed.

A convex function φ on $[0, b)$, $0 < b \leq \infty$, is characterized by the following inequality:

$$\varphi(y) - \varphi(x) \geq C_{\varphi}(x)(y - x), \quad \forall x, y \in (0, b]. \qquad (13)$$

In [8] we proved for γ-quasiconvex functions $K : [0, b) \to \mathbb{R}$:

Lemma 8 ([8, Lemma 1]). *Let* $K(x) = x^{\gamma}\varphi(x)$, $\gamma \in \mathbb{R}$*, where* φ *is convex on* $[0, b)$*. Then*

$$K(y) - K(x) = y^{\gamma}\varphi(y) - x^{\gamma}\varphi(x) \geq \varphi(x)(y^{\gamma} - x^{\gamma}) + C_{\varphi}(x)\, y^{\gamma}(y - x) \qquad (14)$$

holds for $x \in [0, b)$, $y \in [0, b)$*, where* $C_{\varphi}(x)$ *is defined by (13). Moreover, the Jensen's type inequality*

$$\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right)$$
$$= \int_{\Omega} \varphi(f(s)) f^{\gamma}(s)\,d\mu(s) - \varphi(x)\,x^{\gamma}$$
$$\geq \int_{\Omega} \left[\varphi(x)(f^{\gamma}(s) - x^{\gamma}) + C_{\varphi}(x) f^{\gamma}(s)(f(s) - x)\right] d\mu(s) \qquad (15)$$

holds, where f is a nonnegative function, $x = \int_{\Omega} f(s)\,d\mu(s) > 0$, f and $K \circ f$ are μ-integrable functions on the probability measure space (Ω, μ).

In particular, for $\gamma = 1$, we get when $K = xf$ that

$$\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right)$$
$$\geq \int_{\Omega} \left[C_{\varphi}(x) f(s)(f(s) - x)\right] d\mu(s) = \int_{\Omega} C_{\varphi}(x)(f(s) - x)^2\,d\mu(s). \qquad (16)$$

If φ is concave, then the reverse inequalities of (13)–(16) hold. In particular

$$\int_{\Omega} K(f(s))\,d\mu(s) \; - K\left(\int_{\Omega} f(s)\,d\mu(s)\right)$$
$$\leq \int_{\Omega} \left[\varphi(x)(f^{\gamma}(s) - x^{\gamma}) + C_{\varphi}(x) f^{\gamma}(s)(f(s) - x)\right] d\mu(s)$$

holds.

Example 1. Inequalities (13)–(15) are satisfied by $K(x) = x^p$, $p \geq \gamma + 1$. For $\gamma < p \leq \gamma + 1$, the reverse inequalities hold. They reduce to equalities for $p = \gamma + 1$.

From Lemma 8, we get a refinement of Jensen's inequality:

Theorem 2 ([8, Theorem 1]). *Let $\gamma \in \mathbb{R}_+$ and f be nonnegative function. Let f and $\varphi \circ f$ be μ-integrable functions on the probability measure space (Ω, μ) and $x = \int_{\Omega} f(s)\,d\mu(s) > 0$. If φ is a differentiable, nonnegative, convex, increasing function on $[0, b)$, $0 < b \leq \infty$ and $\varphi(0) = \lim_{z \to 0^+} z\varphi'(z) = 0$, then the Jensen's type inequalities*

$$\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right)$$
$$= \int_{\Omega} \varphi(f(s)) f^{\gamma}(s)\,d\mu(s) - \varphi(x)\,x^{\gamma}$$
$$\geq \varphi(x) \int_{\Omega} (f^{\gamma}(s) - x^{\gamma})\,d\mu(s) + \varphi'(x) \int_{\Omega} f^{\gamma}(s)(f(s) - x)\,d\mu(s) \geq 0 \qquad (17)$$

hold.

Furthermore, for an integer n, we get:

$$\begin{aligned}
&\int_{\Omega} \varphi\left(f\left(s\right)\right) f^{n}\left(s\right) d\mu\left(s\right) - \varphi\left(x\right) x^{n} \\
&\geq \varphi\left(x\right) \int_{\Omega} \left(f^{n}\left(s\right) - x^{n}\right) d\mu\left(s\right) + \varphi^{'}\left(x\right) \int_{\Omega} f^{n}\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&\geq x\varphi\left(x\right) \int_{\Omega} \left(f^{n-1}\left(s\right) - x^{n-1}\right) d\mu\left(s\right) \\
&\quad + \left(x\varphi\left(x\right)\right)^{'} \int_{\Omega} f^{n-1}\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&\geq x^{k}\varphi\left(x\right) \int_{\Omega} \left(f^{n-k}\left(s\right) - x^{n-k}\right) d\mu\left(s\right) \\
&\quad + \left(x^{k}\varphi\left(x\right)\right)^{'} \int_{\Omega} f^{n-k}\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&\geq \left(x^{n-1}\varphi\left(x\right)\right)^{'} \int_{\Omega} f\left(s\right)\left(f\left(s\right) - x\right) d\mu\left(s\right) \\
&= \left(x^{n-1}\varphi\left(x\right)\right)^{'} \int_{\Omega} \left(f\left(s\right) - x\right)^{2} d\mu\left(s\right) \geq 0, \qquad k = 1, \ldots, n-1. \qquad (18)
\end{aligned}$$

Remark 1. Note that when $n = 0$ (17) and the first inequality in (18) coincide with Jensen's inequality.

Theorem 2 is used to prove the theorems in Sect. 3 related to Hardy's inequality.

Corollary 1. *By applying (15) with $\mu(s) = \sum_{i=1}^{N} a_i \delta_i$ with $\sum_{i=1}^{N} a_i = 1$ and δ_i unit masses at $x = x_i$, $y_i = f(x_i)$, $i = 1, \ldots, N$, $N \in \mathbb{Z}_+$, we obtain that the following special case of (15) yields the inequality*

$$\begin{aligned}
&\sum_{i=1}^{N} \alpha_i K\left(y_i\right) - K\left(\sum_{i=1}^{N} \alpha_i y_i\right) \\
&\geq \varphi\left(\sum_{j=1}^{N} \alpha_j y_j\right)\left(\sum_{i=1}^{N} \alpha_i y_i^{\gamma} - \left(\sum_{j=1}^{N} \alpha_j y_j\right)^{\gamma}\right) \\
&\quad + C_{\varphi}\left(\sum_{j=1}^{N} \alpha_j y_j\right) \sum_{i=1}^{N} \alpha_i y_i^{\gamma}\left(y_i - \sum_{j=1}^{N} \alpha_j y_j\right), \qquad (19)
\end{aligned}$$

which holds for $x_i \in [0, b)$, $y_i \in [0, b)$, $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, N$, and $\sum_{i=1}^{N} \alpha_i = 1$. Moreover, under the conditions on φ in Theorem 2, as φ is differentiable so that $C_{\varphi} = \varphi'$, then the right-hand side of (19) is nonnegative and therefore we get that (19) is a genuine scale of refined discrete Jensen's type inequalities.

The definition of γ- quasiconvex function K, $K(x) = x^{\gamma}\varphi(x)$ can be meaningful even if $\gamma < 0$. We quote for example the following complement of Theorem 2:

Theorem 3 ([9]). *Let $-1 \le \gamma \le 0$, and let f be nonnegative μ-integrable function on the probability measure space (Ω, μ) $x = \int_{\Omega} f(s)\, d\mu(s) > 0$. If φ is a differentiable, nonnegative, convex increasing function that satisfies $\varphi(0) = 0 = \lim_{z\to 0+} z\varphi'(z)$, then*

$$\int_{\Omega} \varphi(f(s))(f(s))^{\gamma}\, d\mu(s) - \varphi(x)\, x^{\gamma}$$
$$\ge \varphi(x) \int_{\Omega} ((f(s))^{\gamma} - x^{\gamma})\, d\mu(s) + \varphi'(x) \int_{\Omega} (f(s))^{\gamma}(f(s) - x)\, d\mu(s) \quad (20)$$

holds and the right-hand side expression of (20) is nonpositive.

Remark 2. From the case $\gamma = -1$, it follows that when φ is convex and $\varphi(0) = 0 = \lim_{z\to 0+} (z\varphi'(z))$ and $\frac{\varphi(x)}{x}$ is concave, we get a negative lower bound to our Jensen's type difference. This important fact is further stated in the next subsection

2.4 Some Two-Sided Reversed Jensen's Type Inequalities

In [9] we deal with γ-quasiconvex functions when $-1 \le \gamma \le 0$, from which we derive some two-sided Jensen's type inequalities.

The results in this subsection are quoted mainly from that paper. First we state the following consequence of Theorem 3:

Theorem 4. *Let the conditions in Theorem 3 be satisfied and assume in addition that $\frac{\varphi(x)}{x}$ is concave. Then the following two-sided Jensen's type inequality holds:*

$$\varphi(x) \int_{\Omega} \left((f(s))^{-1} - x^{-1}\right) d\mu(s) + \varphi'(x) \int_{\Omega} (f(s))^{-1}(f(s) - x)\, d\mu(s)$$
$$\le \int_{\Omega} \frac{\varphi(f(s))}{f(s)} d\mu(s) - \frac{\varphi(x)}{x} \le 0.$$

Corollary 2. *Let $0 < p \le 1$, and let f be a μ-measurable and positive function on the probability measure space (μ, Ω) and $x = \int_{\Omega} f(s)\, d\mu(s) > 0$. Then*

$$-I_1 + \left(\int_{\Omega} f(s)\, d\mu(s)\right)^p \le \int_{\Omega} (f(s))^p\, d\mu(s) \le \left(\int_{\Omega} f(s)\, d\mu(s)\right)^p,$$

where

$$I_1 = p\left(\int_{\Omega} f(s)\, d\mu(s)\right)^p \left(1 - \int_{\Omega} f(s)\, d\mu(s) \int_{\Omega} (f(s))^{-1}\, d\mu(s)\right) > 0.$$

Using (2) in Lemma 3, our next two-sided reversed Jensen's type inequality quoted from [9] reads:

Theorem 5. *Let f be a nonnegative μ-measurable function on the probability measure space (μ, Ω) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Assume that φ is a differentiable nonnegative, convex function, $\varphi(0) = \lim_{z\to 0+} z\varphi'(z) = 0$. Moreover, assume that $\frac{\varphi(x)}{x}$ is concave. Then the following two-sided Jensen's type inequality holds:*

$$-\left(\frac{\varphi(x)}{x}\right)' \int_\Omega \frac{(f(s)-x)^2}{f(s)} d\mu(s) \le \int_\Omega \frac{\varphi(f(s))}{f(s)} d\mu(s) - \frac{\varphi(x)}{x} \le 0.$$

By applying Theorem 5 with $\varphi(x) = x^{1+p}$, $0 < p \le 1$, we get the following result:

Corollary 3. *Let $0 < p \le 1$, let f be a nonnegative μ-measurable function on the probability measure space (Ω, μ) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Then*

$$-I_2 + \left(\int_\Omega f(s)\, d\mu(s)\right)^p \le \int_\Omega (f(s))^p\, d\mu(s) \le \left(\int_\Omega f(s)\, d\mu(s)\right)^p,$$

where

$$I_2 = p\left(\int_\Omega f(s)\, d\mu(s)\right)^{p-1} \int_\Omega \frac{(f(s)-x)^2}{f(s)} d\mu(s).$$

Next we state additional two-sided inequalities:

Theorem 6. *Let f be a nonnegative μ-measurable function on the probability measure space (Ω, μ) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Assume that φ is a differentiable nonnegative function such that $\varphi(0) = \lim_{z\to 0_+} z\varphi'(z) = 0$, $\frac{\varphi(x)}{x^2}$ is convex, and $\frac{\varphi(x)}{x}$ is concave. Then the following two-sided Jensen's type inequality holds:*

$$\left(\frac{\varphi(x)}{x^2}\right)' \int_\Omega (f(s)-x)^2\, d\mu(s) \le \int_\Omega \frac{\varphi(f(s))}{f(s)} d\mu(s) - \frac{\varphi(x)}{x} \le 0.$$

By applying Theorem 6 with $\varphi(x) = x^{1+p}$, $0 < p \le 1$, we obtain the following Corollary 4:

Corollary 4. *Let $0 < p \le 1$, and let f be a μ-measurable function on the probability measure space (Ω, μ) and $x = \int_\Omega f(s)\, d\mu(s) > 0$. Then*

$$-I_3 + \left(\int_\Omega f(s)\, d\mu(s)\right)^p \le \int_\Omega (f(s))^p\, d\mu(s) \le \left(\int_\Omega f(s)\, d\mu(s)\right)^p,$$

where

$$I_3 = (1-p)\left(\int_\Omega f(s)\,d\mu(s)\right)^{p-2}\int_\Omega (f(s)-x)^2\,d\mu(s)\,.$$

2.5 Comparing Jensen's Type Inequalities

Using Lemmas 4 and 5, we compare Jensen's type inequality obtained by using the 1-quasiconvexity of ψ, where $\psi(x) = x\varphi(x)$, φ is differentiable nonnegative increasing convex function on $x \geq 0$ satisfying $\varphi(0) = \lim_{x\to 0+} x\varphi'(x) = 0$ and the superquadracity of the function ψ. The comparison shows that when $\gamma = 1$, (17) is sharper than (4) for the same ψ, that is:

Theorem 7 ([9]). *Let $\psi(x) = x\varphi(x)$, where φ is nonnegative, convex, increasing, and differentiable function on $[0,b)$, and $\varphi(0) = 0 = \lim_{x\to 0+} x\varphi'(x)$. Then the inequalities*

$$\sum_{j=1}^m \alpha_j\psi(x_j) - \psi(\bar{x}) \geq \sum_{j=1}^m \alpha_j\varphi'(\bar{x})(x_j-\bar{x})^2 \geq \sum_{j=1}^m \alpha_j\psi(|x_j-\bar{x}|) \geq 0$$

hold for

$$x_j \leq 2\bar{x}, \quad \bar{x} = \sum_{j=1}^m \alpha_j x_j, \quad 0 \leq \alpha_j \leq 1, \quad \sum_{j=1}^m \alpha_j = 1, \quad j = 1,\ldots,m.$$

In particular, the theorem holds under the conditions stated in Lemma 5.

Similarly we also get that:

Theorem 8. *Under the conditions of Theorem 2 for $\psi(x) = x\varphi(x)$, the inequalities*

$$\begin{aligned}&\int_\Omega \psi(f(s))\,d\mu(s) - \psi\left(\int_\Omega f(s)\,d\mu(s)\right)\\ &\geq \int_\Omega \varphi'\left(\int_\Omega f(\sigma)\,d\mu(\sigma)\right)\left(f(s)-\int_\Omega f(\sigma)\,d\mu(\sigma)\right)^2 d\mu(s)\\ &\geq \int_\Omega \psi\left(\left|f(s)-\int_\Omega f(\sigma)\,d\mu(\sigma)\right|\right)d\mu(s) \geq 0,\end{aligned}$$

hold when $0 < a \leq f(s) \leq 2a$, $s \in \Omega$.

Theorem 7 for $m = 2$ leads to the proofs of Theorems 16 and 19 which deal with the behavior of averages of $A_n(f)$ and $B_n(f)$ discussed in Sect. 4.

3 Hardy's Type Inequalities Related to Quasiconvexity and Superquadracity

In 1928 Hardy [16] obtained and proved the inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(y)\, dy \right)^p x^\alpha dx \le \left(\frac{p}{p-\alpha-1} \right)^p \int_0^\infty f^p(x)\, x^\alpha dx \tag{21}$$

which holds for all measurable and nonnegative functions f on $(0, \infty)$ whenever $\alpha < p-1$, $p \ge 1$. In [24] sufficient conditions for a variant

$$\int_0^b \left(\frac{1}{x} \int_0^x f(y)\, dy \right)^p x^{-1} dx \le \int_0^b f^p(x)\, x^{-1} \left(1 - \left(\frac{x}{b} \right)^{\frac{p-\alpha-1}{p}} \right) dx \tag{22}$$

of (21) to hold are given for $p \ge 1$. In particular it is shown there that inequality (22) is equivalent to the following variant of (21):

$$\int_0^b \left(\frac{1}{x} \int_0^x f(y)\, dy \right)^p x^\alpha dx$$
$$\le \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b f^p(x)\, x^\alpha \left(1 - \left(\frac{x}{b} \right)^{\frac{p-\alpha-1}{p}} \right) dx \tag{23}$$

for $p \ge 1$, $\alpha < p-1$ or $p < 0$, $\alpha > p-1$ and $0 \le b \le \infty$.

In 2008 Oguntuase and Persson proved the following refined Hardy's inequality with "breaking point" $p = 2$ (see [18] and also [19]):

Theorem 9. *Let $p \ge 1$, $\alpha < p-1$ and $0 < b \le \infty$. If $p \ge 2$, and the function f is nonnegative and locally integrable on $(0, b)$ and $\int_0^b x^\alpha f^p(x)\, dx < \infty$, then*

$$\int_0^b \left(\frac{1}{x} \int_0^x f(y)\, dy \right)^p x^\alpha dx$$
$$+ \frac{p-1-\alpha}{p} \int_0^b \int_t^b \left| \frac{p}{p-\alpha-1} \left(\frac{t}{x} \right)^{1-\frac{p-\alpha-1}{p}} f(t) \right.$$
$$\left. - \frac{1}{x} \int_0^x f(\tau)\, d\tau \right|^p x^{\alpha - \frac{p-\alpha-1}{p}} dx \cdot t^{\frac{p-\alpha-1}{p}-1} dt$$
$$\le \left(\frac{p}{p-\alpha-1} \right)^p \int_0^b f^p(x)\, x^\alpha \left(1 - \left(\frac{x}{b} \right)^{\frac{p-\alpha-1}{p}} \right) dx. \tag{24}$$

If $1 < p \le 2$, then (24) holds in the reversed direction. In particular, for $p = 2$, we have equality in (24).

In [2] another theorem about refined Hardy's inequality with "breaking point" at $p = 2$ is proved by using the quasiconvexity of the power functions for $p \geq 2$:

Theorem 10. *Let* $p \geq 2$, $k > 1$, $0 < b \leq \infty$, *and let the function* f *be nonnegative and locally integrable on* $(0, b)$. *Then*

$$\left(\frac{p}{k-1}\right)^p \int_0^b \left(1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right) x^{p-k} f^p(x)\, dx - \int_0^b x^{-k} \left(\int_0^x f(t)\, dt\right)^p dx$$
$$\geq \frac{(p-1)}{p} (k-1) \int_0^b \int_t^b \left(f(t) \frac{p}{k-1} \left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} - \frac{1}{x} \int_0^x f(\sigma)\, d\sigma\right)^2$$
$$\times \left(\frac{1}{x} \int_0^x f(\sigma)\, d\sigma\right)^{p-2} x^{\left(1-\frac{k-1}{p}\right)(p+1)} t^{\frac{k-1}{p}-1} \frac{dx}{x^2} dt. \tag{25}$$

Moreover, the double integral of the right-hand side of (25) is nonnegative. If $1 < p \leq 2$, *then the inequality (25) holds in reverse direction. Equality holds when* $p = 2$.

There, in [2] an additional theorem is proved about a "breaking point" at $p = 3$ for Hardy's type inequality by using the quasisuperquadracity of the power functions for $p \geq 3$:

Theorem 11. *Let* $p \geq 3$, $k > 1$, $0 < b \leq \infty$, *and let the function* f *be nonnegative and locally integrable on* $(0, b)$. *Then*

$$\left(\frac{p}{k-1}\right)^p \int_0^b \left(1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right) x^{p-k} f^p(x)\, dx - \int_0^b x^{-k} \left(\int_0^x f(t)\, dt\right)^p dx$$
$$\geq \frac{p-1}{p} (k-1) \int_0^b \int_t^b \left(f(t) \frac{p}{k-1} \left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} - \frac{1}{x} \int_0^x f(\sigma)\, d\sigma\right)^2$$
$$\times \left(\frac{1}{x} \int_0^x f(\sigma)\, d\sigma\right)^{p-2} x^{\left(1-\frac{k-1}{p}\right)(p+1)} t^{\frac{k-1}{p}-1} \frac{dx}{x^2} dt$$
$$+ \int_0^b \int_t^b \left(f(t)\, t^{1-\frac{k-1}{p}}\right) \left(\left| f(t) \frac{p}{k-1} \left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} - \frac{1}{x} \int_0^x f(\sigma)\, d\sigma \right|\right)^{p-1}$$
$$\times x^{\left(1-\frac{k-1}{p}\right)p} t^{\frac{k-1}{p}-1} \frac{dx}{x^2} dt. \tag{26}$$

Moreover, each double integral of the right-hand side of (26) is nonnegative.

If $1 < p \leq 3$, *then the inequality (26) holds in the reverse direction. Equality holds when* $p = 3$.

Using the γ-quasiconvexity of the power function when the power is $p+\gamma$, $p \geq 1$, $\gamma \geq 0$ we get in [8] and Hardy's type inequality:

Theorem 12. *Let $p \geq 1$, $k > 1$, $0 < b \leq \infty$, and $\gamma \in \mathbb{R}_+$, and let the function f be nonnegative and locally integrable on $[0, b)$. Then*

$$\left(\frac{p+\gamma}{k-1}\right)^{p+\gamma} \int_0^b \left[\left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p+\gamma}}\right) x^{p+\gamma} f^{p+\gamma}(x) - \left(\int_0^x f(t)\,dt\right)^{p+\gamma}\right] \frac{dx}{x^k}$$
$$\geq \left(\frac{k-1}{p+\gamma}\right) \int_0^b \int_t^b \left(\left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} - \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{\gamma}\right)$$
$$\times \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma-1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2}dt$$
$$+p\left(\frac{k-1}{p+\gamma}\right)^{1-\gamma} \int_0^b \int_t^b \left(f(t)\,t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} \left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right.$$
$$\left.-\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-1} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2}dt \geq 0. \tag{27}$$

holds.

Moreover, when $\gamma = 0$ (27) coincides with (23) and therefore also with (21).

By using the γ-superquadracity, we get for the power function with the power greater than $p+\gamma$, $p \geq 2$, $\gamma \geq 0$ that the following Hardy's type inequality holds.

Theorem 13 ([1]). *Let $p \geq 2$, $k > 1$, $0 < b \leq \infty$, and $\gamma \in \mathbb{R}_+$, and let the function f be nonnegative and locally integrable on $(0, b)$. Then*

$$\left(\frac{p+\gamma}{k-1}\right)^{p+\gamma} \int_0^b \left[\left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p+\gamma}}\right) x^{p+\gamma} f^{p+\gamma}(x) - \left(\int_0^x f(t)\,dt\right)^{p+\gamma}\right] \frac{dx}{x^k}$$
$$\geq \left(\frac{k-1}{p+\gamma}\right) \int_0^b \int_t^b \left(\left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} - \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{\gamma}\right)$$
$$\times \left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma-1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2}dt$$
$$+\left(\frac{k-1}{p+\gamma}\right)^{1-\gamma} \int_0^b \int_t^b \left(f(t)\,t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma} \left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right.$$
$$\left.-\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)\cdot p\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-1} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2}dt$$
$$+\left(\frac{k-1}{p+\gamma}\right)^{1-\gamma} \int_0^b \int_t^b \left(f(t)\,t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma}$$
$$\times \left(\left|f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}} - \frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right|\right)^{p} x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)} t^{\frac{k-1}{p+\gamma}-1} \frac{dx}{x^2}dt. \tag{28}$$

Moreover, if γ is a nonnegative integer, then the right-hand side of (28) is nonnegative. If $1 < p \leq 2$, then inequality (28) is reversed. Equality holds when $p = 2$. When $\gamma = 0$, inequality (28) coincide with (24).

4 Averages

In this section we deal with the lower bounds of differences of averages where the functions f involved with are quasiconvex.

For a function f and a sequence a_n, $n = 0, 1, \ldots$, we define

$$A_n(f) = \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right), \quad n \geq 2,$$

and

$$B_n(f) = \frac{1}{a_{n+1}} \sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \quad n \geq 1.$$

In [4, Theorems 3.1 and 5.3], the following results concerning averages for superquadratic functions are proved:

Theorem 14. *Let a_i, $i = 0, 1 \ldots$, be an increasing sequence with $a_0 = 0$, and $a_1 > 0$, and let $a_{i+1} - a_i$ be decreasing. Suppose that f is superquadratic and nonnegative on $[0, b)$. Then, for $n \geq 2$*

$$\begin{aligned} A_{n+1}(f) - A_n(f) &= \frac{1}{a_n} \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_n a_{n-1}} \sum_{i=1}^{n-1} a_i f\left(\left|\frac{a_{i+1}}{a_{n+1}} - \frac{a_i}{a_n}\right|\right) \\ &\quad + \frac{1}{a_n a_{n-1}} \sum_{i=1}^{n-1} (a_n - a_i) f\left(\left|\frac{a_i}{a_n} - \frac{a_i}{a_{n+1}}\right|\right). \end{aligned} \tag{29}$$

In the special case where $a_i = i$, $i = 0, 1 \ldots$, we get for

$$A_n(f) = \frac{1}{n-1} \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right), \quad n \geq 2$$

that if f is superquadratic on $[0,1]$, then for $n \geq 2$

$$A_{n+1}(f) - A_n(f) \geq \sum_{r=1}^{n-1} \frac{2r}{n(n-1)} f\left(\frac{n-r}{n(n+1)}\right).$$

holds.

Further,

$$A_{n+1}(f) - A_n(f) \geq f\left(\tfrac{1}{3n}\right) + \sum_{r=1}^{n-1} \lambda_r f(y_r),$$

where $\lambda_r = \frac{2r}{n(n-1)}$, $y_r = \frac{|2n-1-3r|}{3n(n+1)}$, $r = 1, \ldots, n-1$.

Moreover if f is superquadratic and nonnegative, then for $n \geq 3$

$$A_{n+1}(f) - A_n(f) \geq f\left(\tfrac{1}{3n}\right) + f\left(\tfrac{16}{81(n+3)}\right).$$

Theorem 15 ([4, Theorems 3.2 and 5.6]). *Let $a_i > 0$, and $a_i - a_{i-1}$, $i = 1, \ldots$, be increasing sequences and let $a_0 = 0$. Suppose that f is superquadratic and nonnegative on $[0,b)$.*

Then,

$$\begin{aligned} B_{n-1}(f) - B_n(f) &= \frac{1}{a_n} \sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}} \sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_n a_{n+1}} \sum_{i=1}^{n-1} a_i f\left(\left|\frac{a_{i-1}}{a_{n-1}} - \frac{a_i}{a_n}\right|\right) \\ &\quad + \frac{1}{a_n a_{n+1}} \sum_{i=0}^{n-1} (a_n - a_i) f\left(\left|\frac{a_i}{a_n} - \frac{a_i}{a_{n-1}}\right|\right), \end{aligned} \tag{30}$$

and in the special case where $a_i = i$, $i = 0, 1, \ldots$, we get for

$$B_n(f) = \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{r}{n}\right), \quad n \geq 1$$

that if f is superquadratic on [0,1], then for $n \geq 2$

$$B_{n-1}(f) - B_n(f) \geq \sum_{r=1}^{n} \frac{2r}{n(n+1)} f\left(\frac{n-r}{n(n-1)}\right).$$

holds.

Further,

$$B_{n-1}(f) - B_n(f) \geq f\left(\tfrac{1}{3n}\right) + \sum_{r=1}^{n} \lambda_r f(y_r),$$

where $\lambda_r = \frac{2r}{n(n+1)}$, $y_r = \frac{|2n+1-3r|}{3n(n-1)}$, $r = 1, \ldots, n$.

Moreover, if f *is also nonnegative, then for* $n \geq 2$

$$B_{n-1}(f) - B_n(f) \geq f\left(\tfrac{1}{3n}\right) + f\left(\tfrac{16}{81n}\right).$$

Using Theorem 7 we get for functions which are simultaneously quasiconvex and superquadratic functions a better lower bound for the difference $A_{n+1}(f) - A_n(f)$ when we use the quasiconvexity of the function f than when we use its superquadracity.

To show it we first state results analog to those in Theorem 14, but now instead of superquadratic functions we deal with quasiconvex functions.

The results quoted below are mainly from [3].

Theorem 16. *Let* $\varphi : [0, b) \to \mathbb{R}_+$, $0 < b \leq \infty$ *be differentiable convex increasing function satisfying* $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$, *and let* $f = x\varphi$. *Let the sequence* $a_i > 0$, $i = 1, \ldots$ *be such that* a_i *is increasing and* $a_{i+1} - a_i$ *is decreasing and let* $a_0 = 0$. *Then, for* $n \geq 2$, *we get from the quasiconvexity of* f *that the inequalities*

$$\begin{aligned} A_{n+1}(f) - A_n(f) &= \frac{1}{a_n} \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i(a_n + a_{i+1} - a_i)}{a_n a_{n+1}}\right)\left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2}\right) \geq 0 \end{aligned} \tag{31}$$

hold, and as φ' *is increasing the inequalities*

$$\begin{aligned} A_{n+1}(f) - A_n(f) &= \frac{1}{a_n} \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) - \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i(a_n + a_{i+1} - a_i)}{a_n a_{n+1}}\right)\left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2}\right) \\ &\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i}{a_n}\right)\left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2}\right) \geq 0 \end{aligned}$$

hold.

From the superquadracity of $f = x\varphi$, we get that

$$\begin{aligned}
&A_{n+1}(f) - A_n(f) \\
&\geq \frac{1}{a_n a_{n-1}} \sum_{i=1}^{n-1} \left(a_i f\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right. \\
&\quad \left. + (a_n - a_i) f\left(\frac{a_i (a_{i+1} - a_i)}{a_n (a_{n+1})} \right) \right) \\
&= \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i) a_i (a_n - a_i)}{a_n^2 a_{n+1}} \left(\varphi\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right. \\
&\quad \left. + \varphi\left(\frac{a_i (a_{i+1} - a_i)}{a_n (a_{n+1})} \right) \right) \geq 0. \qquad (32)
\end{aligned}$$

As φ is convex we get that

$$\begin{aligned}
&A_{n+1}(f) - A_n(f) \\
&\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i) a_i (a_n - a_i)}{a_n^2 a_{n+1}} \left(\varphi\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right. \\
&\quad \left. + \varphi\left(\frac{a_i (a_{i+1} - a_i)}{a_n (a_{n+1})} \right) \right) \\
&\geq \frac{2}{a_{n-1} a_n^2 a_{n+1}} \sum_{i=1}^{n-1} \varphi\left(\frac{a_{i+1} - a_i}{2a_{n+1}} \right) (a_{i+1} - a_i) a_i (a_n - a_i) \geq 0.
\end{aligned}$$

Further, if φ' is also convex, then

$$\begin{aligned}
&A_{n+1}(f) - A_n(f) \\
&\geq \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_{n-1} a_n^2 a_{n+1}^2} \varphi'\left(\frac{\sum_{i=1}^{n-1} a_i^2 (a_n - a_i)(a_{i+1} - a_i)^2}{a_n \sum_{i=1}^{n-1} a_i (a_n - a_i)(a_{i+1} - a_i)^2} \right) \geq 0.
\end{aligned}$$

Finally, the bound obtained in (31) by the quasiconvexity of f is better than the bound obtained by its superquadracity in (32) and in (29), that is:

$$\begin{aligned}
&A_{n+1}(f) - A_n(f) \\
&\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \varphi'\left(\frac{a_i (a_n + a_{i+1} - a_i)}{a_n a_{n+1}} \right) \left(\frac{(a_{i+1} - a_i)^2 a_i (a_n - a_i)}{a_n^2 a_{n+1}^2} \right) \\
&\geq \frac{1}{a_{n-1}} \sum_{i=1}^{n-1} \frac{(a_{i+1} - a_i) a_i (a_n - a_i)}{a_n^2 a_{n+1}} \left(\varphi\left(\frac{(a_n - a_i)(a_{i+1} - a_i)}{a_n a_{n+1}} \right) \right.
\end{aligned}$$

$$+\varphi\left(\frac{a_i\left(a_{i+1}-a_i\right)}{a_n\left(a_{n+1}\right)}\right)\Bigg)$$

$$\geq \frac{2}{a_{n-1}a_n^2a_{n+1}}\sum_{i=1}^{n-1}\varphi\left(\frac{a_{i+1}-a_i}{2a_{n+1}}\right)\left(a_{i+1}-a_i\right)a_i\left(a_n-a_i\right)\geq 0.$$

Example 2. Let $f = x\varphi$, be quasiconvex function where φ is nonnegative convex increasing and differentiable function on $[0, b)$, and $\varphi(0) = 0 = \lim_{x\to 0^+} x\varphi'(x)$. Let $a_i = i$, $i = 0, \ldots, n$. Then by Theorem 16

$$A_{n+1}(f) - A_n(f) \geq \frac{1}{(n-1)}\sum_{r=1}^{n-1}\varphi'\left(\frac{r}{n}\right)\frac{r(n-r)}{n^2(n+1)^2}\geq 0,$$

and the lower bound obtained by the quasiconvexity of f is better than the lower bound obtained by its quasiconvexity, that is:

$$A_{n+1}(f) - A_n(f) \geq \frac{1}{(n-1)}\sum_{r=1}^{n-1}\varphi'\left(\frac{r}{n}\right)\frac{r(n-r)}{n^2(n+1)^2}$$

$$\geq \frac{1}{(n-1)}\sum_{r=1}^{n-1}\frac{2r}{n}f\left(\frac{n-r}{n(n+1)}\right)\geq 0.$$

If φ' is also convex, we get also:

$$A_{n+1}(f) - A_n(f) \geq \frac{1}{6n(n+1)}\varphi'\left(\frac{1}{2}\right)\geq \frac{1}{3n}\varphi\left(\frac{1}{2(n+1)}\right)\geq 0.$$

Now we present results related to the behavior of $B_n(f)$ when n changes.

First we state a theorem about $B_{n-1}(f) - B_n(f)$ when the function f is quasiconvex and $f = x\varphi$:

Theorem 17. *Let $\varphi : [0, b) \to \mathbb{R}_+$, $0 < b \leq \infty$ be differentiable convex increasing function and let $f = x\varphi$. Let the sequence $a_i > 0$, $i = 1, \ldots$, be such that $a_i - a_{i-1}$, $i = 1, \ldots$ is increasing and let $a_0 = 0$. Then, for $n \geq 2$*

$$\begin{aligned}
&B_{n-1}(f) - B_n(f)\\
&= \frac{1}{a_n}\sum_{i=0}^{n-1}f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n}f\left(\frac{a_i}{a_n}\right)\\
&\geq \frac{1}{a_{n+1}a_n^2a_{n-1}^2}\sum_{i=1}^{n-1}\left(a_i-a_{i-1}\right)^2 a_i\left(a_n-a_i\right)\varphi'\left(\frac{a_i\left(a_n+a_{i-1}-a_i\right)}{a_n}\right)\geq 0.
\end{aligned} \tag{33}$$

If in addition φ' *is convex on* $[0,\infty)$*, then*

$$\begin{aligned}
&\frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\
&\geq \sum_{i=1}^{n-1} \frac{(a_i-a_{i-1})^2 a_i (a_n-a_i)\varphi'}{a_{n+1}a_n^2 a_{n-1}^2}\left(\frac{\sum_{i=1}^{n-1} a_i^2 (a_n-a_i)(a_i-a_{i-1})^2}{a_n \sum_{i=1}^{n-1} a_i (a_n-a_i)(a_i-a_{i-1})^2}\right) \\
&\geq 0.
\end{aligned}$$

We state now a theorem about the superquadratic function f where $f = x\varphi$.

Theorem 18. *Let* $\varphi : [0,b) \to \mathbb{R}_+$ $0 < b \leq \infty$ *be differentiable convex increasing function satisfying* $\varphi(0) = 0 = \lim_{x\to 0^+} x\varphi'(x)$ *and let* $f = x\varphi$*. Let* $a_0 = 0$ *and* $a_i > 0$*,* $i = 1, \ldots,$ *be a sequence for which* $a_i - a_{i-1}$ *is increasing for* $i = 1, \ldots,$*. Then*

$$\begin{aligned}
B_{n-1}(f) - B_n(f) &= \frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\
&\geq \frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\left(\frac{a_i}{a_n} f\left(\frac{(a_n-a_i)(a_i-a_{i-1})}{a_{n-1}a_n}\right)\right. \\
&\quad \left. + \frac{a_n-a_i}{a_n} f\left(\frac{a_i(a_i-a_{i-1})}{a_{n-1}a_n}\right)\right) \\
&= \sum_{i=1}^{n-1}\frac{(a_i-a_{i-1})a_i(a_n-a_i)}{a_{n+1}a_n^2 a_{n-1}}\left(\varphi\left(\frac{a_i(a_i-a_{i-1})}{a_{n-1}a_n}\right)\right. \\
&\quad \left. + \varphi\left(\frac{(a_n-a_i)(a_i-a_{i-1})}{a_{n-1}a_n}\right)\right),
\end{aligned} \tag{34}$$

and as φ *is also convex we get that*

$$\begin{aligned}
B_{n-1}(f) - B_n(f) &= \frac{1}{a_n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) - \frac{1}{a_{n+1}}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \\
&\geq \frac{1}{a_{n+1}}\sum_{i=1}^{n-1}\left(\frac{a_i}{a_n} f\left(\frac{(a_n-a_i)(a_i-a_{i-1})}{a_{n-1}a_n}\right)\right. \\
&\quad \left. + \frac{a_n-a_i}{a_n} f\left(\frac{a_i(a_i-a_{i-1})}{a_{n-1}a_n}\right)\right)
\end{aligned}$$

$$= \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})\, a_i\, (a_n - a_i)}{a_{n+1} a_n^2 a_{n-1}} \left(\varphi\left(\frac{a_i\,(a_i - a_{i-1})}{a_{n-1} a_n} \right) + \varphi\left(\frac{(a_n - a_i)(a_i - a_{i-1})}{a_{n-1} a_n} \right) \right)$$

$$\geq \sum_{i=1}^{n-1} \frac{2a_i\,(a_n - a_i)(a_i - a_{i-1})}{a_{n+1} a_n^2 a_{n-1}} \varphi\left(\frac{(a_i - a_{i-1})}{2a_{n-1}} \right) \geq 0.$$

The proof of (35) in Theorem 19 uses Theorem 7 to show that the bound obtained in (33) is better than the bound obtained in (30) and in (34).

Theorem 19. *Let $\varphi : [0,b) \to \mathbb{R}_+$ $0 < b \leq \infty$ be differentiable convex increasing function satisfying $\varphi(0) = 0 = \lim_{x \to 0^+} x\varphi'(x)$ and let $f = x\varphi$. Let $a_0 = 0$ and $a_i > 0$, $i = 1, \ldots$ be a sequence for which $a_i - a_{i-1}$ is increasing for $i = 1, \ldots,$. Then the inequalities*

$$B_{n-1}(f) - B_n(f) = \frac{1}{a_n} \sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}} \right) - \frac{1}{a_{n+1}} \sum_{i=0}^{n} f\left(\frac{a_i}{a_n} \right)$$

$$\geq \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})(a_n - a_i)\, a_i}{a_{n-1}^2 a_n^2 a_{n+1}} \varphi'\left(\frac{a_i\,(a_n + a_{i-1} - a_i)}{a_n a_{n-1}} \right)$$

$$\geq \sum_{i=1}^{n-1} \frac{(a_i - a_{i-1})(a_n - a_i)\, a_i}{a_{n-1} a_n^2 a_{n+1}} \left(\varphi\left(\frac{a_i\,(a_i - a_{i-1})}{a_n a_{n-1}} \right) + \varphi\left(\frac{(a_n - a_i)(a_i - a_{i-1})}{a_n a_{n-1}} \right) \right) \geq 0 \tag{35}$$

hold.

Example 3. Let $f = x\varphi$, then under the conditions of Theorem 19 on φ, when $a_i = i$, $i = 0, \ldots, n$ we get that

$$B_{n-1}(f) - B_n(f) = \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n-1} \right) - \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{r}{n} \right)$$

$$\geq \frac{1}{(n+1)(n-1)^2 n^2} \sum_{r=0}^{n} \varphi'\left(\frac{r}{n} \right) r\,(n-r)$$

$$\geq 2 \sum_{r=0}^{n} \frac{r}{n} f\left(\frac{n-r}{n\,(n-1)} \right) \geq 0$$

holds.

Further, if φ' is also convex on $[0, \infty)$, we get that

$$B_{n-1}(f) - B_n(f) \geq \frac{1}{6n(n-1)}\varphi'\left(\frac{1}{2}\right) \geq \frac{1}{3n}\varphi\left(\frac{1}{2(n-1)}\right).$$

Similarly, the bounds of the differences

$$\frac{1}{2n-1}\sum_{i=1}^{n} f\left(\frac{2i+1}{2n+1}\right) - \frac{1}{2n-3}\sum_{i=1}^{n-1} f\left(\frac{2i+1}{2n-1}\right)$$

$$\frac{1}{2n-1}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n-3}\right) - \frac{1}{2n+1}\sum_{i=1}^{n-1} f\left(\frac{2i-1}{2n-1}\right)$$

obtained by using the quasiconvexity of f are better than the bound obtained by using superquadracity.

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Taylor's Formula and Integral Inequalities for Conformable Fractional Derivatives

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In Honor of Constantin Carathéodory

Abstract We derive Taylor's theorem using a variation of constants formula for conformable fractional derivatives. This is then employed to extend some recent and classical integral inequalities to the conformable fractional calculus, including the inequalities of Steffensen, Chebyshev, Hermite–Hadamard, Ostrowski, and Grüss.

1 Taylor Theorem

We use the conformable α-fractional derivative, recently introduced in [6, 7, 9], which for $\alpha \in (0, 1]$ is given by:

$$D_\alpha f(t) := \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \quad D_\alpha f(0) = \lim_{t \to 0^+} D_\alpha f(t). \tag{1}$$

Note that if f is differentiable, then

$$D_\alpha f(t) = t^{1-\alpha} f'(t), \tag{2}$$

where $f'(t) = \lim_{\varepsilon \to 0}[f(t+\varepsilon) - f(t)]/\varepsilon$.

We will consider Taylor's Theorem in the context of iterated fractional differential equations. In this setting, the theorem will be proven using the variation of constants formula, where we use an approach similar to that used for integer-order derivatives found in [8], and different from that found in Williams [14], where the Riemann–Liouville fractional derivative is employed. With this in mind, we begin

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this note with a general higher-order equation. For $n \in \mathbb{N}_0$ and continuous functions $p_i : [0,\infty) \to \mathbb{R}$, $1 \le i \le n$, we consider the higher-order linear α-fractional differential equation:

$$Ly = 0, \quad \text{where} \quad Ly = D_\alpha^n y + \sum_{i=1}^{n} p_i D_\alpha^{n-i} y, \tag{3}$$

where $D_\alpha^n y = D_\alpha^{n-1}(D_\alpha y)$. A function $y : [0,\infty) \to \mathbb{R}$ is a solution of Eq. (3) on $[0,\infty)$ provided y is n times α-fractional differentiable on $[0,\infty)$ and satisfies $Ly(t) = 0$ for all $t \in [0,\infty)$. It follows that $D_\alpha^n y$ is a continuous function on $[0,\infty)$.

Now let $f : [0,\infty) \to \mathbb{R}$ be continuous and consider the nonhomogeneous equation:

$$D_\alpha^n(t) + \sum_{i=1}^{n} p_i(t) D_\alpha^{n-i} y(t) = f(t). \tag{4}$$

Definition 1. We define the Cauchy function $y : [0,\infty) \times [0,\infty) \to \mathbb{R}$ for the linear fractional equation (3) to be, for each fixed $s \in [0,\infty)$, the solution of the initial value problem:

$$Ly = 0, \quad D_\alpha^i y(s,s) = 0, \quad 0 \le i \le n-2, \quad D_\alpha^{n-1} y(s,s) = 1.$$

Remark 1. Note that

$$y(t,s) := \frac{1}{(n-1)!}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{n-1}$$

is the Cauchy function for $D_\alpha^n = 0$, which can be easily verified using (2).

Definition 2. Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$\int_a^b f(t) d_\alpha t := \int_a^b f(t) t^{\alpha-1} dt$$

exists and is finite.

Theorem 1 (Variation of Constants). *Let $\alpha \in (0,1]$ and $s,t \in [0,\infty)$. If f is continuous, then the solution of the initial value problem:*

$$Ly = f(t), \quad D_\alpha^i y(s) = 0, \quad 0 \le i \le n-1$$

is given by

$$y(t) = \int_s^t y(t,\tau) f(\tau) d_\alpha \tau,$$

where $y(t,\tau)$ is the Cauchy function for (3).

Proof. With y defined as above and by the properties of the Cauchy function, we have

$$D_\alpha^i y(t) = \int_s^t D_\alpha^i y(t,\tau) f(\tau) d_\alpha \tau + D_\alpha^{i-1} y(t,t) f(t) = \int_s^t D_\alpha^i y(t,\tau) f(\tau) d_\alpha \tau$$

for $0 \le i \le n-1$, and

$$\begin{aligned} D_\alpha^n y(t) &= \int_s^t D_\alpha^n y(t,\tau) f(\tau) d_\alpha \tau + D_\alpha^{n-1} y(t,t) f(t) \\ &= \int_s^t D_\alpha^n y(t,\tau) f(\tau) d_\alpha \tau + f(t). \end{aligned}$$

It follows from these equations that

$$D_\alpha^i y(s) = 0, \quad 0 \le i \le n-1$$

and

$$Ly(t) = \int_s^t Ly(t,\tau) f(\tau) d_\alpha \tau + f(t) = f(t),$$

and the proof is complete. □

Theorem 2 (Taylor Formula). *Let* $\alpha \in (0,1]$ *and* $n \in \mathbb{N}$. *Suppose* f *is* $(n+1)$ *times* α*-fractional differentiable on* $[0,\infty)$*, and* $s,t \in [0,\infty)$. *Then we have*

$$f(t) = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau.$$

Proof. Let $g(t) := D_\alpha^{n+1} f(t)$. Then f solves the initial value problem:

$$D_\alpha^{n+1} x = g, \quad D_\alpha^k x(s) = D_\alpha^k f(s), \quad 0 \le k \le n.$$

Note that the Cauchy function for $D_\alpha^{n+1} y = 0$ is

$$y(t,s) = \frac{1}{n!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^n.$$

By the variation of constants formula,

$$f(t) = u(t) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n g(\tau) d_\alpha \tau,$$

where u solves the initial value problem:

$$D_\alpha^{n+1} u = 0, \quad D_\alpha^m u(s) = D_\alpha^m f(s), \quad 0 \le m \le n. \tag{5}$$

To validate the claim that $u(t) = \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k D_\alpha^k f(s)$, set

$$w(t) := \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k D_\alpha^k f(s).$$

Then $D_\alpha^{n+1} w = 0$, and we have that

$$D_\alpha^m w(t) = \sum_{k=m}^n \frac{1}{(k-m)!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{k-m} D_\alpha^k f(s).$$

It follows that

$$D_\alpha^m w(s) = \sum_{k=m}^n \frac{1}{(k-m)!} \left(\frac{s^\alpha - s^\alpha}{\alpha}\right)^{k-m} D_\alpha^k f(s) = D_\alpha^m f(s)$$

for $0 \le m \le n$. We consequently have that w also solves (5), and thus $u \equiv w$ by uniqueness. □

Corollary 1. *Let $\alpha \in (0,1]$ and $s, r \in [0,\infty)$ be fixed. For any $t \in [0,\infty)$ and any positive integer n,*

$$\frac{1}{n!} \left(\frac{t^\alpha - r^\alpha}{\alpha}\right)^n = \sum_{k=0}^n \frac{1}{k!(n-k)!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k \left(\frac{s^\alpha - r^\alpha}{\alpha}\right)^{n-k}.$$

Proof. This follows immediately from the theorem if we take $f(t) = \frac{1}{n!} \left(\frac{t^\alpha - r^\alpha}{\alpha}\right)^n$ in Taylor's formula. It can also be shown directly. □

2 Steffensen Inequality

In this section we prove a new α-fractional version of Steffensen's inequality and of Hayashi's inequality. The results in this and subsequent sections differ from those in [10, 12, 13, 15].

Lemma 1. *Let $\alpha \in (0,1]$ and $a, b \in \mathbb{R}$ with $0 \le a < b$. Let $A > 0$ and let $g : [a,b] \to [0,A]$ be an α-fractional integrable function on $[a,b]$. If*

$$\ell := \frac{\alpha(b-a)}{A(b^\alpha - a^\alpha)} \int_a^b g(t) d_\alpha t \in [0, b-a], \tag{6}$$

then

$$\int_{b-\ell}^{b} A\, d_\alpha t \le \int_a^b g(t)\, d_\alpha t \le \int_a^{a+\ell} A\, d_\alpha t. \tag{7}$$

Proof. Since $g(t) \in [0, A]$ for all $t \in [a, b]$, ℓ given in (6) satisfies

$$0 \le \ell = \frac{\alpha(b-a)}{A(b^\alpha - a^\alpha)} \int_a^b g(t) d_\alpha t \le \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b 1\, d_\alpha t = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \frac{b^\alpha - a^\alpha}{\alpha} = b - a.$$

As $\alpha \in (0, 1]$ we have that $t^{\alpha-1}$ is a decreasing function on $[a, b]$ or $(a, b]$ if $a = 0$. Thus using the fact that $d_\alpha t = t^{\alpha-1} dt$, we have the following inequalities, which are average values, namely,

$$\frac{1}{\ell} \int_{b-\ell}^{b} 1\, d_\alpha t \le \frac{1}{b-a} \int_a^b 1\, d_\alpha t \le \frac{1}{\ell} \int_a^{a+\ell} 1\, d_\alpha t.$$

This implies that

$$\int_{b-\ell}^{b} A\, d_\alpha t \le \frac{\ell}{b-a} \int_a^b A\, d_\alpha t \le \int_a^{a+\ell} A\, d_\alpha t,$$

which leads to (7) via (6). □

The next theorem is known as Steffensen's inequality if $A = 1$, and for general $A > 0$, it is known as Hayashi's inequality [1].

Theorem 3 (Fractional Hayashi–Steffensen Inequality). *Let* $\alpha \in (0, 1]$, $A > 0$, *and* $a, b \in \mathbb{R}$ *with* $0 \le a < b$. *Let* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to [0, A]$ *be* α*-fractional integrable functions on* $[a, b]$.

(i) If f *is nonnegative and nonincreasing, then*

$$A \int_{b-\ell}^{b} f(t) d_\alpha t \le \int_a^b f(t) g(t) d_\alpha t \le A \int_a^{a+\ell} f(t) d_\alpha t, \tag{8}$$

where ℓ *is given by* (6).

(ii) If f *is nonpositive and nondecreasing, then the inequalities in* (8) *are reversed.*

Proof. For (i), assume f is nonnegative and nonincreasing; we will prove only the case in (8) for the left inequality; the proof for the right inequality is similar and relies on (7). By the definition of ℓ in (6) and the conditions on g, we know that (7) holds. After subtracting within the left inequality of (8), we see that

$$\begin{aligned} &\int_a^b f(t) g(t) d_\alpha t - A \int_{b-\ell}^{b} f(t) d_\alpha t \\ &= \int_a^{b-\ell} f(t) g(t) d_\alpha t - \int_{b-\ell}^{b} f(t)(A - g(t)) d_\alpha t \end{aligned}$$

$$\geq \int_a^{b-\ell} f(t)g(t)d_\alpha t - f(b-\ell)\int_{b-\ell}^b (A-g(t))d_\alpha t$$

$$\overset{(7)}{\geq} \int_a^{b-\ell} f(t)g(t)d_\alpha t - f(b-\ell)\int_a^{b-\ell} g(t)d_\alpha t$$

$$= \int_a^{b-\ell} (f(t)-f(b-\ell))\, g(t)d_\alpha t \geq 0,$$

since f is nonincreasing, and f and g are nonnegative. Therefore, the left-hand side of (8) holds.

For (ii), assume f is nonpositive and nondecreasing; we will prove only the case in (8) for the reversed right inequality; the proof for the reversed left inequality is similar and also relies on (7). We see that we have

$$\int_a^b f(t)g(t)d_\alpha t - A\int_a^{a+\ell} f(t)d_\alpha t$$

$$= \int_{a+\ell}^b f(t)g(t)d_\alpha t + \int_a^{a+\ell} f(t)(g(t)-A)d_\alpha t$$

$$\geq \int_{a+\ell}^b f(t)g(t)d_\alpha t + f(a+\ell)\int_a^{a+\ell} (g(t)-A)d_\alpha t$$

$$\overset{(7)}{\geq} \int_{a+\ell}^b f(t)g(t)d_\alpha t - f(a+\ell)\int_{a+\ell}^b g(t)d_\alpha t$$

$$= \int_{a+\ell}^b (f(t)-f(a+\ell))\, g(t)d_\alpha t \geq 0,$$

since f is nondecreasing and nonpositive, and g is nonnegative. Therefore the right-hand side of the reversed (8) holds. □

Remark 2. The requirement in Steffensen's Theorem 3 that f be nonincreasing when f is nonnegative is essential. Let $a = 0$, $b = 1 = A$, $\alpha \in (0,1)$, $g(t) = t$, and $f(t) = t^{1-\alpha}$. Then $\ell = \frac{\alpha}{1+\alpha}$, and if (8) were to hold in this case, we would need

$$\int_{\frac{1}{1+\alpha}}^1 t^{1-\alpha}d_\alpha t \leq \int_0^1 t^{2-\alpha}d_\alpha t \leq \int_0^{\frac{\alpha}{1+\alpha}} t^{1-\alpha}d_\alpha t$$

to hold, that is to say

$$\frac{\alpha}{1+\alpha} = 1 - \frac{1}{1+\alpha} \leq \frac{1}{2} \leq \frac{\alpha}{1+\alpha}.$$

But this holds only if $\alpha = 1$, a contradiction even if we reverse the inequalities.

3 Taylor Remainder

Let $\alpha \in (0, 1]$ and suppose f is $n + 1$ times α-fractional differentiable on $[0, \infty)$. Using Taylor's Theorem, Theorem 2, we define the remainder function by

$$R_{-1,f}(\cdot, s) := f(s),$$

and for $n > -1$,

$$\begin{aligned} R_{n,f}(t, s) &:= f(s) - \sum_{k=0}^{n} \frac{D_\alpha^k f(t)}{k!} \left(\frac{s^\alpha - t^\alpha}{\alpha}\right)^k \\ &= \frac{1}{n!} \int_t^s \left(\frac{s^\alpha - \tau^\alpha}{\alpha}\right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau. \end{aligned} \tag{9}$$

Lemma 2. *Let $\alpha \in (0, 1]$. The following identity involving α-fractional Taylor's remainder holds:*

$$\int_a^b \frac{D_\alpha^{n+1} f(s)}{(n+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{n+1} d_\alpha s = \int_a^t R_{n,f}(a, s) d_\alpha s + \int_t^b R_{n,f}(b, s) d_\alpha s.$$

Proof. We proceed by mathematical induction on n. For $n = -1$,

$$\int_a^b D_\alpha^0 f(s) d_\alpha s = \int_a^b f(s) d_\alpha s = \int_a^t f(s) d_\alpha s + \int_t^b f(s) d_\alpha s.$$

Assume the result holds for $n = k - 1$:

$$\int_a^b \frac{D_\alpha^k f(s)}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k d_\alpha s = \int_a^t R_{k-1,f}(a, s) d_\alpha s + \int_t^b R_{k-1,f}(b, s) d_\alpha s.$$

Let $n = k$. Using integration by parts, we have

$$\begin{aligned} \int_a^b \frac{D_\alpha^{k+1} f(s)}{(k+1)!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{k+1} d_\alpha s = &\frac{D_\alpha^k f(b)}{(k+1)!} \left(\frac{t^\alpha - b^\alpha}{\alpha}\right)^{k+1} \\ &- \frac{D_\alpha^k f(a)}{(k+1)!} \left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{k+1} \\ &+ \int_a^b \frac{D_\alpha^k f(s)}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^k d_\alpha s. \end{aligned}$$

By the induction assumption,

$$\begin{aligned}
\int_a^b \frac{D_\alpha^{k+1}f(s)}{(k+1)!}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)^{k+1} d_\alpha s &= \int_a^t R_{k-1,f}(a,s)d_\alpha s + \int_t^b R_{k-1,f}(b,s)d_\alpha s \\
&\quad + \frac{D_\alpha^k f(b)}{(k+1)!}\left(\frac{t^\alpha - b^\alpha}{\alpha}\right)^{k+1} \\
&\quad - \frac{D_\alpha^k f(a)}{(k+1)!}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{k+1} \\
&= \int_a^t R_{k-1,f}(a,s)d_\alpha s + \int_t^b R_{k-1,f}(b,s)d_\alpha s \\
&\quad + \frac{D_\alpha^k f(b)}{k!}\int_b^t \left(\frac{s^\alpha - b^\alpha}{\alpha}\right)^k d_\alpha s \\
&\quad - \frac{D_\alpha^k f(a)}{k!}\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha}\right)^k d_\alpha s \\
&= \int_a^t \left[R_{k-1,f}(a,s) - \frac{D_\alpha^k f(a)}{k!}\left(\frac{s^\alpha - a^\alpha}{\alpha}\right)^k\right] d_\alpha s \\
&\quad + \int_t^b \left[R_{k-1,f}(b,s) - \frac{D_\alpha^k f(b)}{k!}\left(\frac{s^\alpha - b^\alpha}{\alpha}\right)^k\right] d_\alpha s \\
&= \int_a^t R_{k,f}(a,s)d_\alpha s + \int_t^b R_{k,f}(b,s)d_\alpha s.
\end{aligned}$$

This completes the proof. □

Corollary 2. *Let $\alpha \in (0,1]$. For $n \geq -1$,*

$$\int_a^b \frac{D_\alpha^{n+1}f(s)}{(n+1)!}\left(\frac{a^\alpha - s^\alpha}{\alpha}\right)^{n+1} d_\alpha s = \int_a^b R_{n,f}(b,s)d_\alpha s,$$

$$\int_a^b \frac{D_\alpha^{n+1}f(s)}{(n+1)!}\left(\frac{b^\alpha - s^\alpha}{\alpha}\right)^{n+1} d_\alpha s = \int_a^b R_{n,f}(a,s)d_\alpha s.$$

4 Applications of the Steffensen Inequality

Let $\alpha \in (0,1]$. In the following we adapt to the α-fractional setting some results from [5] by applying the fractional Steffensen inequality, Theorem 3.

Theorem 4. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an $n+1$ times α-fractional differentiable function such that $D_\alpha^{n+1} f$ is increasing and $D_\alpha^n f$ is decreasing on $[a,b]$. If*

$$\ell := \frac{b-a}{n+2},$$

then

$$D_\alpha^n f(a+\ell) - D_\alpha^n f(a) \le (n+1)! \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^{n+1} \int_a^b R_{n,f}(a,s) d_\alpha s$$
$$\le D_\alpha^n f(b) - D_\alpha^n f(b-\ell).$$

If $D_\alpha^{n+1} f$ is decreasing and $D_\alpha^n f$ is increasing on $[a,b]$, then the above inequalities are reversed.

Proof. Assume $D_\alpha^{n+1} f$ is increasing and $D_\alpha^n f$ is decreasing on $[a,b]$, and let

$$F := -D_\alpha^{n+1} f.$$

Because $D_\alpha^n f$ is decreasing, $D_\alpha^{n+1} f \le 0$, so that $F \ge 0$ and decreasing on $[a,b]$. Define

$$g(t) := \left(\frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha}\right)^{n+1} \in [0,1], \quad t \in [a,b], \quad n \ge -1.$$

Note that F, g satisfy the assumptions of Steffensen's inequality (i), Theorem 3, with $A = 1$; using (6),

$$\ell = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b g(t) d_\alpha t = \frac{b-a}{n+2},$$

and

$$-\int_{b-\ell}^b D_\alpha^{n+1} f(t) d_\alpha t \le -\int_a^b D_\alpha^{n+1} f(t) \left(\frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha}\right)^{n+1} d_\alpha t \le -\int_a^{a+\ell} D_\alpha^{n+1} f(t) d_\alpha t.$$

By Corollary 2 this simplifies to

$$D_\alpha^n f(t)\big|_{t=a}^{a+\ell} \le (n+1)! \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^{n+1} \int_a^b R_{n,f}(a,t) d_\alpha t \le D_\alpha^n f(t)\big|_{t=b-\ell}^{b}.$$

This completes the proof of the first part. If $D_\alpha^{n+1} f$ is decreasing and $D_\alpha^n f$ is increasing on $[a,b]$, then take $F := D_\alpha^{n+1} f$. $\square$

The following corollary is the first Hermite–Hadamard inequality, derived from Theorem 4 with $n = 0$.

Corollary 3 (Hermite–Hadamard Inequality I). *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable function such that $D_\alpha f$ is increasing and f is decreasing on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \le f(b) + f(a) - f\left(\frac{a+b}{2}\right).$$

If $D_\alpha f$ is decreasing and f is increasing on $[a,b]$, then the above inequalities are reversed.

Theorem 5. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an $n+1$ times α-fractional differentiable function such that*

$$m \le D_\alpha^{n+1} f \le M$$

on $[a,b]$ for some real numbers $m < M$. Then

$$\begin{aligned} &\frac{m}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} + \frac{M-m}{(n+2)!}\left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha}\right)^{n+2} \le \int_a^b R_{n,f}(a,t) d_\alpha t \\ &\le \frac{M}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} + \frac{m-M}{(n+2)!}\left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha}\right)^{n+2}, \end{aligned} \tag{10}$$

where ℓ is given by:

$$\ell = \frac{\alpha(b-a)}{(b^\alpha - a^\alpha)(M-m)}\left(D_\alpha^n f(b) - D_\alpha^n f(a) - m\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)\right)$$

Proof. Let

$$\begin{aligned} F(t) &:= \frac{1}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1}, \\ k(t) &:= \frac{1}{M-m}\left(f(t) - \frac{m}{(n+1)!}\left(\frac{t^\alpha - a^\alpha}{\alpha}\right)^{n+1}\right), \\ G(t) &:= D_\alpha^{n+1} k(t) = \frac{1}{M-m}\left(D_\alpha^{n+1} f(t) - m\right) \in [0,1]. \end{aligned}$$

Observe that F is nonnegative and decreasing, and

$$\int_a^b G(t) d_\alpha t = \frac{1}{M-m}\left(D_\alpha^n f(b) - D_\alpha^n f(a) - m\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)\right).$$

Since F, G satisfy the hypotheses of Theorem 3(i), we compute the various integrals given in (8), after using (6) to set

$$\ell = \frac{\alpha(b-a)}{b^\alpha - a^\alpha} \int_a^b G(t) d_\alpha t.$$

We have

$$\int_{b-\ell}^{b} F(t) d_\alpha t = \int_{b-\ell}^{b} \frac{1}{(n+1)!} \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} d_\alpha t = \frac{1}{(n+2)!} \left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha} \right)^{n+2},$$

and

$$\int_a^{a+\ell} F(t) d_\alpha t = \frac{1}{(n+2)!} \left[\left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2} - \left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha} \right)^{n+2} \right].$$

Moreover, using Corollary 2, we have

$$\begin{aligned} \int_a^b F(t) G(t) d_\alpha t &= \frac{1}{(M-m)(n+1)!} \int_a^b \left(\frac{b^\alpha - t^\alpha}{\alpha} \right)^{n+1} \left(D_\alpha^{n+1} f(t) - m \right) d_\alpha t \\ &= \frac{1}{M-m} \int_a^b R_{n,f}(a,t) d_\alpha t - \frac{m}{(M-m)(n+2)!} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^{n+2}. \end{aligned}$$

Using Steffensen's inequality (8) and some rearranging, we obtain (10). □

Corollary 4. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable function such that*

$$m \le D_\alpha f \le M$$

on $[a,b]$ for some real numbers $m < M$. Then

$$\begin{aligned} &\frac{m}{2} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^2 + \frac{M-m}{2} \left(\frac{b^\alpha - (b-\ell)^\alpha}{\alpha} \right)^2 \\ &\le \int_a^b f(t) d_\alpha t - f(a) \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \\ &\le \frac{M}{2} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right)^2 + \frac{m-M}{2} \left(\frac{b^\alpha - (a+\ell)^\alpha}{\alpha} \right)^2, \end{aligned} \tag{11}$$

where ℓ is given by:

$$\ell = \frac{\alpha(b-a)}{(b^\alpha - a^\alpha)(M-m)} \left(f(b) - f(a) - m \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \right)$$

Proof. Use the previous theorem with $n = 0$ and Corollary 2. □

5 Applications of the Chebyshev Inequality

Let $\alpha \in (0,1]$. We begin with Chebyshev's inequality for α-fractional integrals, then apply it to obtain a Hermite–Hadamard-type inequality.

Theorem 6 (Chebyshev Inequality). *Let f and g be both increasing or both decreasing in $[a,b]$, and let $\alpha \in (0,1]$. Then*

$$\int_a^b f(t)g(t)d_\alpha t \geq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t \int_a^b g(t)d_\alpha t.$$

If one of the functions is increasing and the other is decreasing, then the above inequality is reversed.

Proof. The proof is very similar to the classical case with $\alpha = 1$. □

The following is an application of Chebyshev's inequality, which extends a similar result in [5] for q-calculus to this α-fractional case.

Theorem 7. *Let $\alpha \in (0,1]$. Assume that $D_\alpha^{n+1}f$ is monotonic on $[a,b]$. If $D_\alpha^{n+1}f$ is increasing, then*

$$\begin{aligned} 0 &\geq \int_a^b R_{n,f}(a,t)d_\alpha t - \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+1} \\ &\geq \left(\frac{D_\alpha^{n+1}f(a) - D_\alpha^{n+1}f(b)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}. \end{aligned}$$

If $D_\alpha^{n+1}f$ is decreasing, then the inequalities are reversed.

Proof. The situation where $D_\alpha^{n+1}f$ is decreasing is analogous to that of $D_\alpha^{n+1}f$ increasing. Thus, assume $D_\alpha^{n+1}f$ is increasing and set

$$F(t) := D_\alpha^{n+1}f(t), \quad G(t) := \frac{1}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1}.$$

Then F is increasing by assumption, and G is decreasing, so that by Chebyshev's inequality:

$$\int_a^b F(t)G(t)d_\alpha t \leq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b F(t)d_\alpha t \int_a^b G(t)d_\alpha t.$$

By Corollary 2,

$$\int_a^b F(t)G(t)d_\alpha t = \int_a^b \frac{D_\alpha^{n+1}f(t)}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1} d_\alpha t = \int_a^b R_{n,f}(a,t)d_\alpha t.$$

We also have

$$\int_a^b F(t)d_\alpha t = D_\alpha^n f(b) - D_\alpha^n f(a), \quad \int_a^b G(t)d_\alpha t = \frac{1}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}.$$

Thus Chebyshev's inequality implies

$$\int_a^b R_{n,f}(a,t)d_\alpha t \le \frac{\alpha}{b^\alpha - a^\alpha}\left(D_\alpha^n f(b) - D_\alpha^n f(a)\right)\frac{1}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2},$$

which subtracts to the left side of the inequality. Since $D_\alpha^{n+1} f$ is increasing on $[a,b]$,

$$\begin{aligned}\frac{D_\alpha^{n+1} f(a)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} &\le \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+1} \\ &\le \frac{D_\alpha^{n+1} f(b)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2},\end{aligned}$$

and we have

$$\begin{aligned}&\int_a^b R_{n,f}(a,t)d_\alpha t - \left(\frac{D_\alpha^n f(b) - D_\alpha^n f(a)}{(n+2)!}\right)\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+1} \\ &\ge \int_a^b R_{n,f}(a,t)d_\alpha t - \frac{D_\alpha^{n+1} f(b)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}.\end{aligned}$$

Now Corollary 2 and $D_\alpha^{n+1} f$ is increasing imply that

$$\begin{aligned}\int_a^b \frac{D_\alpha^{n+1} f(b)}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1} d_\alpha t &\ge \int_a^b R_{n,f}(a,t)d_\alpha t \\ &\ge \int_a^b \frac{D_\alpha^{n+1} f(a)}{(n+1)!}\left(\frac{b^\alpha - t^\alpha}{\alpha}\right)^{n+1} d_\alpha t,\end{aligned}$$

which simplifies to

$$\frac{D_\alpha^{n+1} f(b)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2} \ge \int_a^b R_{n,f}(a,t)d_\alpha t \ge \frac{D_\alpha^{n+1} f(a)}{(n+2)!}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)^{n+2}.$$

This, together with the earlier lines, gives the right side of the inequality. □

Corollary 5 (Hermite–Hadamard Inequality II). *Let* $\alpha \in (0,1]$. *If* $D_\alpha f$ *is increasing on* $[a,b]$, *then*

$$\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)d_\alpha t \le \frac{f(b) + f(a)}{2}. \tag{12}$$

If $D_\alpha f$ *is decreasing on* $[a,b]$, *then the inequality is reversed.*

Remark 3. Combining Corollary 3 with Corollary 5, we can state the following. If $\alpha \in (0, 1]$ and $f : [a, b] \to \mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha f$ is increasing and f is decreasing on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \le \frac{f(b) + f(a)}{2}.$$

If $\alpha = 1$ this is the Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(t) dt \le \frac{f(b) + f(a)}{2},$$

which holds for all convex functions $f : [a, b] \to \mathbb{R}$. In the α-fractional case, however, the assumption that f is decreasing on $[a, b]$ seems to be crucial. Let $[a, b] = [0, 1]$, $\alpha = 1/2$, and $f(t) = \frac{2}{3}t^{3/2}$. Then f is increasing and convex on $[0, 1]$, but

$$\begin{aligned} f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\right) = \frac{2}{3}\left(\frac{1}{2}\right)^{3/2} &> \frac{1}{6} \\ &= \frac{1}{2}\int_0^1 \frac{2}{3} t^{3/2} t^{1/2-1} dt = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t. \end{aligned}$$

6 Ostrowski Inequality

In this section we prove Ostrowski's α-fractional inequality using a Montgomery identity. For more on Ostrowski's inequalities, see [3] and the references therein.

Lemma 3 (Montgomery Identity). *Let $a, b, s, t \in \mathbb{R}$ with $0 \le a < b$, and let $f : [a, b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0, 1]$. Then*

$$f(t) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s + \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(t, s) D_\alpha f(s) d_\alpha s \tag{13}$$

where

$$p(t, s) := \begin{cases} \frac{s^\alpha - a^\alpha}{\alpha} & : a \le s < t, \\ \frac{s^\alpha - b^\alpha}{\alpha} & : t \le s \le b. \end{cases} \tag{14}$$

Proof. Integrating by parts, we have

$$\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(s) d_\alpha s = \frac{t^\alpha - a^\alpha}{\alpha} f(t) - \int_a^t f(s) d_\alpha s$$

and

$$\int_t^b \left(\frac{s^\alpha - b^\alpha}{\alpha}\right) D_\alpha f(s) d_\alpha s = \frac{b^\alpha - t^\alpha}{\alpha} f(t) - \int_t^b f(s) d_\alpha s.$$

Adding and solving for f yields the result. □

Theorem 8 (Ostrowski Inequality). *Let $a, b, s, t \in \mathbb{R}$ with $0 \le a < b$, and let $f : [a, b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0, 1]$. Then*

$$\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \le \frac{M}{2\alpha \left(b^\alpha - a^\alpha\right)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2 \right], \tag{15}$$

where

$$M := \sup_{t \in (a,b)} |D_\alpha f(t)| \,.$$

This inequality is sharp in the sense that the right-hand side of (15) *cannot be replaced by a smaller one.*

Proof. Using Lemma 3 with $p(t, s)$ defined in (14), we see that

$$\begin{aligned}
\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \right| &= \left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(t, s) D_\alpha f(s) d_\alpha s \right| \\
&\le \frac{M\alpha}{b^\alpha - a^\alpha} \left(\int_a^t \left| \frac{s^\alpha - a^\alpha}{\alpha} \right| d_\alpha s + \int_t^b \left| \frac{s^\alpha - b^\alpha}{\alpha} \right| d_\alpha s \right) \\
&= \frac{M\alpha}{b^\alpha - a^\alpha} \left(\int_a^t \left(\frac{s^\alpha - a^\alpha}{\alpha} \right) d_\alpha s + \int_t^b \left(\frac{b^\alpha - s^\alpha}{\alpha} \right) d_\alpha s \right) \\
&= \frac{M\alpha}{b^\alpha - a^\alpha} \left(\frac{1}{2} \left(\frac{s^\alpha - a^\alpha}{\alpha} \right)^2 \Big|_a^t - \frac{1}{2} \left(\frac{b^\alpha - s^\alpha}{\alpha} \right)^2 \Big|_t^b \right) \\
&= \frac{M}{2\alpha \left(b^\alpha - a^\alpha\right)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2 \right].
\end{aligned}$$

Now $p(t, a) = 0$, so the smallest value attaining the supremum in M is greater than a. To prove the sharpness of this inequality, let $f(t) = t^\alpha/\alpha$, $a = t_1$, $b = t_2 = t$. It follows that $D_\alpha f(t) = 1$ and $M = 1$. Examining the right-hand side of (15), we get

$$\frac{M}{2\alpha \left(b^\alpha - a^\alpha\right)} \left[(t^\alpha - a^\alpha)^2 + (b^\alpha - t^\alpha)^2 \right] = \frac{\left(t_2^\alpha - t_1^\alpha\right)^2}{2\alpha \left(t_2^\alpha - t_1^\alpha\right)} = \frac{t_2^\alpha - t_1^\alpha}{2\alpha}.$$

Starting with the left-hand side of (15), we have

$$\begin{aligned}
\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| &= \left| \frac{t^\alpha}{\alpha} - \frac{\alpha}{t_2^\alpha - t_1^\alpha} \int_{t_1}^{t_2} \frac{t^\alpha}{\alpha} d_\alpha t \right| \\
&= \left| \frac{t^\alpha}{\alpha} - \left(\frac{\alpha}{t_2^\alpha - t_1^\alpha} \right) \left(\frac{t^{2\alpha}}{2\alpha^2} \right) \Big|_{t_1}^{t_2} \right|
\end{aligned}$$

$$= \left| \frac{t^\alpha}{\alpha} - \left(\frac{1}{t_2^\alpha - t_1^\alpha} \right) \left(\frac{t_2^{2\alpha} - t_1^{2\alpha}}{2\alpha} \right) \right|$$
$$= \left| \frac{t^\alpha}{\alpha} - \left(\frac{t_2^\alpha + t_1^\alpha}{2\alpha} \right) \right|$$
$$= \frac{t_2^\alpha - t_1^\alpha}{2\alpha}.$$

Therefore, by the squeeze theorem, the sharpness of Ostrowski's inequality is shown. □

7 Grüss Inequality

In this section we prove the Grüss inequality, which relies on Jensen's inequality. Our approach is similar to that taken by Bohner and Matthews [2].

Theorem 9 (Jensen Inequality). *Let $\alpha \in (0,1]$ and $a,b,x,y \in [0,\infty)$. If $w:\mathbb{R}\to\mathbb{R}$ and $g : \mathbb{R} \to (x,y)$ are nonnegative, continuous functions with $\int_a^b w(t)d_\alpha t > 0$, and $F:(x,y)\to\mathbb{R}$ is continuous and convex, then*

$$F\left(\frac{\int_a^b w(t)g(t)d_\alpha t}{\int_a^b w(t)d_\alpha t} \right) \leq \frac{\int_a^b w(t)F(g(t))d_\alpha t}{\int_a^b w(t)d_\alpha t}.$$

Proof. The proof is the same as those found in Bohner and Peterson [4, Theorem 6.17] and Rudin [11, Theorem 3.3] and thus is omitted. □

Theorem 10 (Grüss Inequality). *Let $a,b,s \in [0,\infty)$, and let $f,g:[a,b]\to\mathbb{R}$ be continuous functions. Then for $\alpha \in (0,1]$ and*

$$m_1 \leq f(t) \leq M_1, \qquad m_2 \leq g(t) \leq M_2, \tag{16}$$

we have

$$\left| \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)g(t)d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^2 \int_a^b f(t)d_\alpha t \int_a^b g(t)d_\alpha t \right|$$
$$\leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

Proof. Initially we consider an easier case, namely, where $f = g$ and

$$\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t)d_\alpha t = 0.$$

If we define

$$v(t) := \frac{f(t) - m_1}{M_1 - m_1} \in [0, 1],$$

then $f(t) = m_1 + (M_1 - m_1)v(t)$. Since

$$\int_a^b v^2(t) d_\alpha t \le \int_a^b v(t) d_\alpha t = \frac{-m_1(b^\alpha - a^\alpha)}{\alpha(M_1 - m_1)},$$

we have

$$\begin{aligned} I(f,f) &:= \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f^2(t) d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right)^2 \\ &= \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b \left[m_1 + (M_1 - m_1)v(t)\right]^2 (t) d_\alpha t \\ &\le -m_1 M_1 = \frac{1}{4}\left[(M_1 - m_1)^2 - (M_1 + m_1)^2\right] \\ &\le \frac{1}{4}(M_1 - m_1)^2. \end{aligned}$$

Now consider the case:

$$r := \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \ne 0,$$

where $r \in \mathbb{R}$. If we take $h(t) := f(t) - r$, then $h(t) \in [m_1 - r, M_1 - r]$ and

$$\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b h(t) d_\alpha t = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (f(t) - r) d_\alpha t = r - \frac{r\alpha}{b^\alpha - a^\alpha} \int_a^b d_\alpha t = 0.$$

Consequently h satisfies the earlier assumptions and so

$$I(h, h) \le \frac{1}{4}\left[M_1 - r - (m_1 - r)\right]^2 = \frac{1}{4}(M_1 - m_1)^2.$$

Additionally we have

$$I(h, h) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (f(t) - r)^2 d_\alpha t = -r^2 + \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f^2(t) d_\alpha t = I(f,f).$$

As a result,

$$I(f,f) = I(h, h) \le \frac{1}{4}(M_1 - m_1)^2.$$

Let us now turn to the case involving general functions f and g under assumptions (16). Using

$$I(f,g) := \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t)g(t)d_\alpha t - \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2 \int_a^b f(t)d_\alpha t \int_a^b g(t)d_\alpha t$$

and the earlier cases, one can easily finish the proof as in the case with $\alpha = 1$. See [2] for complete details to mimic. □

Corollary 6. *Let $\alpha \in (0,1]$, $a,b,s,t \in [0,\infty)$, and $f : [a,b] \to \mathbb{R}$ be α-fractional differentiable. If $D_\alpha f$ is continuous and*

$$m \le D_\alpha f(t) \le M, \quad t \in [a,b],$$

then

$$\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s - \left[\frac{2t^\alpha - a^\alpha - b^\alpha}{2(b^\alpha - a^\alpha)}\right][f(b) - f(a)]\right| \le \frac{1}{4}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)(M-m). \tag{17}$$

for all $t \in [a,b]$.

Proof. Using Lemma 3 Montgomery's identity, we have

$$f(t) - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b p(t,s)D_\alpha f(s)d_\alpha s \tag{18}$$

for all $t \in [a,b]$, where $p(t,s)$ is given in (14). Now for all $t,s \in [a,b]$, we see that

$$\frac{t^\alpha - b^\alpha}{\alpha} \le p(t,s) \le \frac{t^\alpha - a^\alpha}{\alpha}.$$

Applying Theorem 10 Grüss' inequality to the mappings $p(t,\cdot)$ and $D_\alpha f$, we obtain

$$\left|\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b p(t,s)D_\alpha f(s)d_\alpha s - \left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2 \int_a^b p(t,s)d_\alpha s\int_a^b D_\alpha f(s)d_\alpha s\right| \le \frac{1}{4}\left(\frac{t^\alpha - a^\alpha}{\alpha} - \frac{t^\alpha - b^\alpha}{\alpha}\right)(M-m) = \frac{1}{4}\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)(M-m). \tag{19}$$

Computing the integrals involved, we obtain

$$\left(\frac{\alpha}{b^\alpha - a^\alpha}\right)^2 \int_a^b p(t,s)d_\alpha s = \frac{2t^\alpha - a^\alpha - b^\alpha}{2(b^\alpha - a^\alpha)}$$

and

$$\int_a^b D_\alpha f(s) d_\alpha s = f(b) - f(a),$$

so that (17) holds, after using (18) and (19). □

Compare the following corollary with Corollaries 3 and 5.

Corollary 7 (Hermite–Hadamard III). *Let* $\alpha \in (0,1]$*,* $a, b, s, t \in [0, \infty)$*, and* $f : [a,b] \to \mathbb{R}$ *be* α*-fractional differentiable. If* $D_\alpha f$ *is continuous and*

$$m \leq D_\alpha f(t) \leq M, \quad t \in [a,b],$$

then

$$\left| \frac{f(b) + f(a)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(s) d_\alpha s \right| \leq \frac{1}{4} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) (M - m).$$

for all $t \in [a,b]$.

Proof. Take $t = b$ in the previous corollary. □

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Sobolev-Type Inequalities on Manifolds in the Presence of Symmetries and Applications

Athanase Cotsiolis and Nikos Labropoulos

In Honor of Constantin Carathéodory

Abstract In this article, we present some Sobolev-type inequalities on compact Riemannian manifolds with boundary, the data and the functions being invariant under the action of a compact subgroup of the isometry group. We investigate the best constants for the Sobolev, trace Sobolev, Nash, and trace Nash inequalities. By developing particular geometric properties of the manifold as well as of the solid torus, we can calculate the precise values of the best constants in the presented Sobolev-type inequalities. We apply these results to solve nonlinear elliptic, type Dirichlet and Neumann, PDEs of upper critical Sobolev exponent.

1 Introduction

In this article, we present the most interesting aspects of some Sobolev-type inequalities, i.e., Sobolev inequalities and Nash inequalities on compact Riemannian manifolds, from the geometrical point of view. By developing particular geometrical properties of the manifold, we can calculate the precise values of the best constants in the presented Sobolev inequalities. The result of this analysis represents an improvement over the classic analysis and allows us to prove the existence of solutions for elliptic differential equations of scalar curvature of the generalized type with supercritical exponents. The Sobolev inequalities first appeared in 1938 by Sobolev [40] in the case of the whole the Euclidean space. The Nash inequality was introduced by Nash [39] and was used to prove the Hölder regularity of solutions of divergence form uniformly elliptic equations.

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The model manifolds studied in determining the best constants in the abovementioned inequalities are the general case of compact Riemannian manifolds (M, g) with boundary invariant under the action of a compact subgroup G of the isometry group $I(M, g)$ and the case of the solid torus.

We would like at this point to give an explanation as to why we study the solid torus even giving a special emphasis. In recent years, significant progress has been made on the analysis of a number of important features of nonlinear partial differential equations of elliptic and parabolic type. The study of these equations has received considerable attention, because of their special mathematical interest and because of practical applications of the torus in scientific research today. For example, in astronomy, investigators study the torus which is a significant topological feature surrounding many stars and black holes [22]. In physics, the torus is being explored at the National Spherical Torus Experiment (NSTX) at Princeton Plasma Physics Laboratory to test the fusion physics principles for the spherical torus concept at the MA level [36]. In biology, some investigators interested in circular DNA molecules detected a large number of viruses, bacteria, and higher organisms. In this topologically very interesting type of molecule, superhelical turns are formed as the Watson–Crick double helix winds in a torus formation [21].

This paper is organized as follows: In Sect. 2, a short survey on known results about the best constants is presented. Furthermore, a short review of the history and the development of Sobolev inequalities, presenting the most interesting examples and the critical role of the geometry in their studying as well as some applications, are discussed. Section 2.1 is devoted to the presentation of new results concerning the best constants in Sobolev inequalities on manifolds with boundary in the presence of symmetries, where the exponents are in the critical of the supercritical case and to the investigation of the behavior and the existence of positive and nonradially symmetric solutions to some of the most interesting nonlinear elliptic problems. In Sect. 2.2, we determine the best constants in Sobolev inequalities on the solid torus, and consequently we investigate the behavior and the existence of positive and nonradially symmetric solutions to some problems with Dirichlet and Neumann conditions as well as to the nonlinear exponential elliptic model problems. In Sect. 3, a short survey on known results about the best constants of Nash inequalities, presenting the most interesting results about the best constants, is presented. The case of a manifold with boundary in the presence of symmetries is discussed in Sect. 3.1. Section 3.2 is devoted to the case of the solid torus.

2 Sobolev Inequalities and Applications

Let (M, g) be a compact n-dimensional Riemannian manifold, $n \geq 3$, with boundary. If $1 \leq p < n$, $p^* = \frac{np}{n-p}$ and $\tilde{p}^* = \frac{(n-1)p}{n-p}$, according to Sobolev's theorem (see, for instance, [4]) the embeddings $H_1^p(M) \hookrightarrow L^q(M)$ and $H_1^p(M) \hookrightarrow L^{\tilde{q}}(\partial M)$

are compact for any $q \in [1, p^*)$ and $\tilde{q} \in [1, \tilde{p}^*)$, respectively, but the embeddings $H_1^p(M) \hookrightarrow L^{p^*}(M)$ and $H_1^p(M) \hookrightarrow L^{\tilde{p}^*}(\partial M)$ are only continuous. So, there exist constants A, B and $\tilde{A}, \tilde{B}$ such that for all $u \in H_1^p(M)$ the following inequalities hold

$$\left(\int_M |u|^{p^*}\, d\upsilon_g\right)^{\frac{1}{p^*}} \leq A \left(\int_M |\nabla u|^p\, d\upsilon_g\right)^{\frac{1}{p}} + B \left(\int_M |u|^p\, d\upsilon_g\right)^{\frac{1}{p}} \tag{1}$$

and

$$\left(\int_{\partial M} |u|^{\tilde{p}^*}\, ds_g\right)^{\frac{1}{\tilde{p}^*}} \leq \tilde{A} \left(\int_M |\nabla u|^p\, d\upsilon_g\right)^{\frac{1}{p}} + \tilde{B} \left(\int_{\partial M} |u|^p\, ds_g\right)^{\frac{1}{p}}. \tag{2}$$

When the compact manifold is without boundary, the best constant in front of the gradient term in inequality (1) is the same as the best constant for the Sobolev embedding for $M = R^n$ under the Euclidean metric [2], i.e.,

$$\frac{1}{K(n,p)} = \inf_{\substack{u \in L^{p^*}(R^n)\setminus\{0\} \\ \nabla u \in L^p(R^n)}} \frac{\int_{R^n} |\nabla u|^p dx}{\left(\int_{R^n} |u|^{p^*} dx\right)^{\frac{p}{p^*}}}$$

The value of $K(n,p)$ was explicitly computed independently by Aubin [1] and Talenti [41]:

$$K(n,1) = \frac{1}{n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}}}$$

$$K(n,p) = \frac{p-1}{n-p} \left(\frac{n-p}{n(p-1)}\right)^{\frac{1}{p}} \left[\frac{\Gamma(n+1)}{\omega_{n-1} \Gamma(\frac{n}{p}) \Gamma(n+1-\frac{n}{p})}\right]^{\frac{1}{n}},$$

where ω_{n-1} is the area of the unit sphere in R^n and Γ is the gamma function.

If (M, g) is a compact Riemannian manifold with boundary, then we denote $H_1^p(M)$ the completion of $C^\infty(M)$ under the norm

$$\|u\|_{H_1^p(M)} = \left(\int_M |\nabla u|^p\, d\upsilon_g + \int_M |u|^p\, d\upsilon_g\right)^{\frac{1}{p}}$$

and $H_1^p(M) \neq \overset{\circ}{H}{}_1^p(M)$. In this case, the critical Sobolev embedding is valid and therefore concerning the best constants of (1) the same questions are also raised, except that now we have to consider two distinct Sobolev spaces. When we consider

(1) on $\overset{\circ}{H}{}^{p}_{1}(M)$, the same results for best constants on compact manifolds without boundary described above remain true. On the other hand, if we consider (1) on $H_1^p(M)$, Cherrier [12] has shown that the first best constant is $2^{\frac{1}{n}}K(n,p)$.

Lions [37] proved that the best constant in front of the gradient term in inequality (2) in the Sobolev trace embedding for the Euclidean half-space $R_+^n = \{(x',t) : t \geq 0\}$, i.e.,

$$\frac{1}{\tilde{K}\,(n,p)} = \inf_{u\in L^{\tilde{p}*}(\partial R_+^n)\backslash\{o\},\ \nabla u\in L^p(R_+^n)} \frac{\int\limits_{R_+^n} |\nabla u|^p dx}{\left(\int\limits_{\partial R_+^n} |u|^{\tilde{p}*} dx'\right)^{\frac{p}{\tilde{p}*}}}.$$

Biezuner [9] showed that Lions's conclusion still remains valid for any smooth, compact n-dimensional Riemannian manifold, $n \geq 3$, with boundary and $1 < p < n$. The explicit value of $\tilde{K}(n,p)$ was computed independently by Escobar [25] and Beckner [8], only in the case $p = 2$:

$$\tilde{K}(n,2) = \frac{2}{n-2}\,\omega_{n-1}^{-\frac{1}{n-1}}.$$

For $p \neq 2$, the problem remains still open.

It is well known that Sobolev embeddings can be improved in the presence of symmetries (see [5, 6, 14–17, 23, 26, 27, 30, 32, 37] and the references therein).

Let G be a compact group of the isometries without finite subgroup and $p \geq 1$. Denote by $H_{1,G}^p(M)$ the subspace of $H_1^p(M)$ of all G-invariant functions. If k denotes the minimum orbit dimension of G, it's known (see [31]) that for a G-invariant manifold (M,g) without boundary the embeddings $H_{1,G}^p(M) \hookrightarrow L^q(M)$ are continuous for any $p \in [1, n-k)$ and $q \in \left[1, \frac{(n-k)p}{n-k-p}\right]$ and compact if $q \in \left[1, \frac{(n-k)p}{n-k-p}\right)$.

Also, we know [17] that for a G-invariant manifold (M,g) with boundary, the embeddings $H_{1,G}^p(M) \hookrightarrow L^q(\partial M)$ are continuous for any $p \in [1, n-k)$ and $q \in \left[1, \frac{(n-k-1)p}{n-k-p}\right]$ and compact if $q \in \left[1, \frac{(n-k-1)p}{n-k-p}\right)$.

The following two examples are representative of cases of manifolds which present symmetries.

Example 1. Let $\overline{T}$ be the three-dimensional solid torus

$$\overline{T} = \left\{(x,y,z) \in R^3 : \left(\sqrt{x^2+y^2} - l\right)^2 + z^2 \leq r^2,\ l > r > 0\right\},$$

with the metric induced by the R^3 metric. Let $G = O(2) \times I$ be the group of rotations around axis z. The G-orbits of T are circles. If P is a point of T, let O_P be its G-orbit. All G-orbits are of dimension 1, and the orbit of minimum volume is the circle of radius $l-r$ and of volume $2\pi(l-r)$. Then $\overline{T}$ is a compact three-dimensional manifold with boundary, invariant under the action of the subgroup G of the isometry group $O(3)$.

Example 2. Let $R^n = R^k \times R^m$, $k \geq 2$, $m \geq 1$ and $\overline{\Omega} \subset \left(R^k \backslash \{0\}\right) \times R^m$. Denote by $G_{k,m} = O(k) \times Id_m$, the subgroup of the isometry group $O(n)$ of the type $(x_1, x_2) \to (\sigma(x_1), x_2)$, $\sigma \in O(k)$, $x_1 \in R^k$, $x_2 \in R^m$, and suppose that $\overline{\Omega}$ is invariant under the action of $G_{k,m}$ $\left(\tau(\overline{\Omega}) = \overline{\Omega}, \forall \tau \in G_{k,m}\right)$. Then $\overline{\Omega}$ is a compact n-dimensional manifold with boundary, invariant under the action of the subgroup $G_{k,m}$ of the isometry group $O\,(n)$.

The "best constant problem" consists in finding the smallest A, B and $\tilde{A}$, $\tilde{B}$, respectively, such that the inequalities (1) and (2) remain true for all $u \in H_1^p(M)$. Priority is given to the constants A and $\tilde{A}$ because of its importance. It is well known that the "best constant problem" is strongly connected with variational type problems. Knowing the precise value of the smallest A and $\tilde{A}$ called "first best constants," such that the above inequalities remain true for all $u \in H_1^p(M)$ allow us to solve nonlinear elliptic boundary value problems, of the following type:

(P_1)

$$\Delta_p u + a(x)u^{p-1} = f(x)u^{q-1},\ u > 0 \ \text{ on } M,\ u|_{\partial M} = 0,$$

$$\tfrac{2n(n-k)}{n(n-k)+2k} < p < n-k,\ q = \tfrac{(n-k)p}{n-k-p}$$

and

(P_2)

$$\begin{aligned} \Delta_p u + a(x)u^{p-1} &= f(x)u^{q-1},\ u > 0 \ \text{ on } M, \\ |\nabla u|^{p-2}\tfrac{\partial u}{\partial n} + b(x)u^{p-1} &= h(x)u^{\tilde{q}-1} \ \text{ on } \ \partial M, \\ \tfrac{2n(n-k)}{n(n-k)+2k} < p < n-k,\ q &= \tfrac{(n-k)p}{n-k-p},\ \tilde{q} = \tfrac{(n-k-1)p}{n-k-p}, \end{aligned}$$

where $\Delta_p u = -\mathrm{div}_g\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian and $p = 2$, $\Delta_2 = \Delta_g$ is the Laplace–Beltrami operator.

When $p = n$ it is known [4, 13] that $H_1^n(M) \not\subset L^\infty(M)$ and $H_1^n(M) \not\subset L^\infty(\partial M)$. However, when $\upsilon \in H_1^n(M)$ we have $e^\upsilon \in L^1(M)$ and $e^\upsilon \in L^1(\partial M)$. Then, we have another Sobolev inequalities to this case. More precisely, for any $\varepsilon > 0$, there exist three constants $\mu, \nu, C > 0$ and $\tilde{\mu}, \tilde{\nu}, \tilde{C} > 0$, respectively, such that for any $\upsilon \in H_1^n(M)$ the following inequalities hold:

$$\int_M e^\upsilon dV \leq C\exp\left((\mu+\varepsilon)\|\nabla\upsilon\|_n^n + \nu\int_M \upsilon dV\right) \tag{3}$$

and

$$\int_{\partial M} e^{\upsilon} dS \leq \tilde{C} \exp\left((\tilde{\mu} + \varepsilon) \|\nabla \upsilon\|_n^n + \tilde{\nu} \int_{\partial M} \upsilon dS \right). \tag{4}$$

The "best constants" are the smallest μ and $\tilde{\mu}$ such that inequalities (3) and (4), respectively, are true for any $\upsilon \in H_1^n(M)$.

Cherrier in [11] proved that $\mu_n = (n-1)^{n-1} n^{1-2n} \omega_{n-1}^{-1}$ is the smallest possible μ such that (3) remains true for any $\upsilon \in H_1^n(M)$. Cherrier, also, proved that μ_n is attained for the sphere S_n and μ_2 is attained for compact Riemannian manifolds of dimension 2.

Aubin [3] proved that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that for any $\upsilon \in H_1^n(M)$ with $\int_M \upsilon dV = 0$

$$\int_M e^{\upsilon} dV \leq C_\varepsilon \exp\left((\mu_n + \varepsilon) \|\nabla \upsilon\|_n^n \right). \tag{5}$$

This last result is a better result in the sense that (5) implies (3).

Faget in [28] proved that it is possible to take $\varepsilon = 0$ in (5), hence in (3), when M is locally conformally flat and in [29] she gave the complete answer in this problem.

Cherrier, also, in [11] proved that $\tilde{\mu}$ must be strictly greater than $2\mu_n$ such that (4) remains true for any $\upsilon \in H_1^n(M)$.

When $p = n - k$, $H_1^{n-k}(M) \not\hookrightarrow L^\infty(M)$. However, when $\upsilon \in H_1^{n-k}(M)$ we have that $e^{\upsilon} \in L^1(M)$ and $e^{\upsilon} \in L^1(\partial M)$.

The precise value of the best constants μ and $\tilde{\mu}$, appears in results of the existence of nontrivial solutions of exponential elliptic boundary value problems, of the following type:

(P_3)

$$\Delta_{n-k} \upsilon + \gamma = f(x) e^{\upsilon}, \ \upsilon > 0 \quad \text{on} \quad M, \quad \upsilon|_{\partial M} = 0,$$

and

(P_4)

$$\Delta_{n-k} \upsilon + a + f e^{\upsilon} = 0, \ \upsilon > 0 \quad \text{on} \quad M,$$

$$|\nabla \upsilon|^{n-k-2} \frac{\partial \upsilon}{\partial n} + b + g e^{\upsilon} = 0 \quad \text{on} \quad \partial M,$$

where $\Delta_{n-k} \upsilon = -\text{div}_g (|\nabla \upsilon|^{n-k-2} \nabla \upsilon)$ is the n-Laplacian operator and $\gamma, \alpha, \beta \in R$ and f, g are smooth functions. Especially, when $n = 2$ because of linearity of the Laplace–Beltrami operator Δ, the constants γ, α, and β can be replaced by smooth functions $\gamma(x), \alpha(x)$ and $\beta(x)$, respectively.

2.1 Sharp Sobolev Inequalities on Manifolds in the Presence of Symmetries

In the following, we assume the notations and background material. Given $(\widetilde{M}, g)$ a Riemannian manifold (complete or not, but connected), we denote by $I(\widetilde{M}, g)$ its group of isometries. Let (M, g) be a compact n-dimensional, $n \geq 3$, Riemannian manifold with boundary G-invariant under the action of a subgroup G of the isometry group $I(M, g)$. We assume that (M, g) is a smooth-bounded open subset of a slightly larger Riemannian manifold $(\widetilde{M}, g)$, invariant under the action of a subgroup G of the isometry group of $(\widetilde{M}, g)$.

Consider the spaces of all G-invariant functions under the action of the group G:

$$C_G^\infty(M) = \{u \in C^\infty(M) : u \circ \tau = u\,,\ \forall\ \tau \in G\}$$
$$C_{0,G}^\infty(M) = \{u \in C_0^\infty(M) : u \circ \tau = u\,,\ \forall\ \tau \in G\}\,.$$

Denote $H_1^p(M)$ the completion of $C^\infty(M)$ with respect to the norm:

$$\|u\|_{H_1^p(M)} = \left(\|\nabla u\|^p_{L^p(M)} + \|u\|^p_{L^p(M)}\right)^{\frac{1}{p}},$$

and $H_{1,G}^p(M)$ the space of all G-invariant functions of $H_1^p(M)$.

For reasons of completeness, we cite some background material and results from [17]:

Let $P \in M$ and $O_P = \{\tau(P), \tau \in G\}$ be its orbit of dimension k, $0 \leq k < n$. According to ([31, § 9] or [26]), the map $\Phi : G \to O_P$, defined by $\Phi\,(\tau) = \tau\,(P)$, is of rank k, and there exists a submanifold H of G of dimension k with $Id \in H$, such that Φ restricted to H is a diffeomorphism from H onto its image denoted $\mathscr{V}_P$. Let N be a submanifold of M of dimension $(n-k)$, such that $T_P\Phi\,(H) \oplus T_PN = T_PM$. Using the exponential map at P, we build a $(n-k)$-dimensional submanifold $\mathscr{W}_P$ of N, orthogonal to O_P at P and such that for any $Q \in \mathscr{W}_P$, the minimizing geodesics of (M, g) joining P and Q are all contained in $\mathscr{W}_P$.

Let $\Psi : H \times \mathscr{W}_P \to M$ be the map defined by $\Psi\,(\tau, Q) = \tau\,(Q)$. According to the local inverse theorem, there exists a neighborhood $\mathscr{V}_{(Id,P)} \subset H \times \mathscr{W}_P$ of (Id, P) and a neighborhood $\mathscr{M}_P \subset M$ such that $\Psi^{-1} = (\Psi_1 \times \Psi_2)$, from $\mathscr{M}_P$ onto $\mathscr{V}_{(Id,P)}$ is a diffeomorphism. Up to restricting $\mathscr{V}_P$, we choose a normal chart $(\mathscr{V}_P, \varphi_1)$ around P for the metric $\tilde{g}$ induced on O_P, with $\varphi_1\,(\mathscr{V}_P) = U \subset R^k$. In the same way, we choose a geodesic normal chart $(\mathscr{W}_P, \varphi_2)$ around P for the metric $\tilde{\tilde{g}}$ induced on $\mathscr{W}_P$, with $\varphi_2\,(\mathscr{W}_P) = W \subset R^{n-k}$.

By setting $\xi_1 = \varphi_1 \circ \Phi \circ \Psi_1$, $\xi_2 = \varphi_2 \circ \Psi_2$, $\xi = (\xi_1, \xi_2)$ and $\Omega = \mathscr{M}_P$, from the above and from the Lemmas 1 and 2 (see in [32]), the following lemma arises:

Lemma 1 ([17]). *Let (M, g) be a compact Riemannian n-manifold with boundary, G a compact subgroup of $I\,(M, g)$, $P \in M$ with orbit of dimension k, $0 \leq k < n$.*

Then, there exists a chart (Ω, ξ) *around* P *such that the following properties are valid:*

1) $\xi(\Omega) = U \times W$, *where* $U \subset R^k$ *and* $W \subset R^{n-k}$.
2) U, W *are bounded, and* W *has smooth boundary.*
3) (Ω, ξ) *is a normal chart of* M *around of* P, $(\mathscr{V}_P, \varphi_1)$ *is a normal chart around of* P *of submanifold* O_P *and* $(\mathscr{W}_P, \varphi_2)$ *is a normal geodesic chart around of* P *of submanifold* $\mathscr{W}_P$.
4) For any $\varepsilon > 0$, (Ω, ξ) *can be chosen such that:*

$$1 - \varepsilon \leq \sqrt{\det\left(g_{ij}\right)} \leq 1 + \varepsilon \text{ on } \Omega, \text{ for } 1 \leq i,j \leq n$$

$$1 - \varepsilon \leq \sqrt{\det\left(\tilde{g}_{ij}\right)} \leq 1 + \varepsilon \text{ on } \mathscr{V}_P, \text{ for } 1 \leq i,j \leq k.$$

For any $u \in C^\infty_G(M)$, $u \circ \xi^{-1}$ *depends only on* W *variables.*

We say that we choose a neighborhood of O_P when we choose $\delta > 0$ and we consider $O_{P,\,\delta} = \{Q \in \widetilde{M} : d(Q, O_P) < \delta\}$. Such a neighborhood of O_P is called a *tubular* neighborhood.

Let $P \in M$ and O_P be its orbit of dimension k. Since the manifold M is included in $\widetilde{M}$, we can choose a normal chart (Ω_P, ξ_P) around P such that Lemma 1 holds for some $\varepsilon_0 > 0$. For any $Q = \tau(P) \in O_P$, where $\tau \in G$, we build a chart around Q, denoted by $(\tau(\Omega_P), \xi_P \circ \tau^{-1})$ and "isometric" to (Ω_P, ξ_P). O_P is then covered by such charts. We denote by $(\Omega_{P,m})_{m=1,\ldots,M}$ a finite extract covering. Then, we can choose $\delta > 0$ small enough, depending on P and ε_0 such that the tubular neighborhood $O_{P,\,\delta}$, (where $d(\cdot, O_P)$ is the distance to the orbit) has the following properties: $\overline{O_{P,\,\delta}}$ is a submanifold of $\widetilde{M}$ with boundary, $d^2(\cdot, O_P)$, is a C^∞ function on $O_{P,\,\delta}$ and $O_{P,\,\delta}$ is covered by $(\Omega_m)_{m=1,\ldots,M}$. Clearly, M is covered by $\cup_{P\in M} O_{P,\,\delta}$. We denote by $(O_{j,\,\delta})_{j=1,\ldots,J}$ a finite extract covering of M, where all $O_{j,\,\delta}$'s are covered by $(\Omega_{jm})_{m=1,\ldots,M_j}$. Then, we will have:

$$M \subset \bigcup_{j=1}^{J} \bigcup_{m=1}^{M_j} \Omega_{jm} = \bigcup_{i=1}^{\sum_{j=1}^{J} M_j} \Omega_i.$$

So, we obtain a finite covering of M consisting of Ω_i's, $i = 1, \ldots, \sum_{j=1}^{J} M_j$. We choose such a covering in the following way:

(i) If P lies in the interior of M, then there exist j, $1 \leq j \leq J$ and m, $1 \leq m \leq M_j$ such that the tubular neighborhood $O_{j,\,\delta}$ and Ω_{jm}, with $P \in \Omega_{jm}$, lie entirely in M's interior, (that is, if $P \in M \backslash \partial M$, then $O_{j,\delta} \subset M \backslash \partial M$ and $\Omega_{jm} \subset M \backslash \partial M$).
(ii) If P lies on the boundary ∂M of M, then a j, $1 \leq j \leq J$ exists, such that the tubular neighborhood $O_{j,\,\delta}$ intersects the boundary ∂M and an m, $1 \leq m \leq M_j$ exists, such that Ω_{jm}, with $P \in \Omega_{jm}$, cuts a part of the boundary ∂M. Then, the Ω_{jm} covers a patch of the boundary of M, and the whole of the boundary is covered by charts around $P \in \partial M$.

We denote N the projection of the image of M, through the charts (Ω_{jm}, ξ_{jm}), $j = 1, \ldots, J, m = 1, \ldots, M_j$, on R^{n-k}. Then $(N, \bar{g})$ is a $(n-k)$-dimensional compact submanifold of R^{n-k} with boundary and N is covered by (W_i), $i = 1, \ldots, \sum_{j=1}^{J} M_j$, where W_i is the component of $\xi_i(\Omega_i)$ on R^{n-k} for all $i = 1, \ldots, \sum_{j=1}^{J} M_j$. Let p be the projection of $\xi_i(P), P \in M$ on R^{n-k}. Thus one of the following holds:

(i) If $p \in N \backslash \partial N$, then $W_i \subset N \backslash \partial N$ and W_i is a normal geodesic neighborhood with normal geodesic coordinates $(y_1, \ldots, y_{n-k})$.

(ii) If $p \in \partial N$, then W_i is a Fermi neighborhood with Fermi coordinates $(y_1, \ldots, y_{n-k-1}, t)$.

In these neighborhoods the following inequality holds:

$$1 - \varepsilon_0 \le \sqrt{\det\left(\bar{g}_{ij}\right)} \le 1 + \varepsilon_0 \text{ on } N, \text{ for } 1 \le i, j \le n-k,$$

where ε_0 can be as small as we want, depending on the chosen covering.

For convenience in the following we set $O_j = O_{j,\delta} = \{Q \in \widetilde{M} : d(Q, O_{P_j}) < \delta\}$ We still need the following lemma:

Lemma 2 ([17]).

a) For any $\upsilon \in H_{1,G}^{p}\left(O_j \cap M\right)$, $\upsilon \ge 0$ *the following properties are valid:*

1) $(1 - c\varepsilon_0) V_j \int_N \upsilon_2^p d\upsilon_{\bar{g}} \le \int_M \upsilon^p dV_g \le (1 + c\varepsilon_0) V_j \int_N \upsilon_2^p d\upsilon_{\bar{g}}$,

2) $(1 - c\varepsilon_0) V_j \int_N \left|\nabla_{\bar{g}} \upsilon_2\right|^p d\upsilon_{\bar{g}} \le \int_M \left|\nabla_g \upsilon\right|^p dV_g \le (1 + c\varepsilon_0) V_j \int_N \left|\nabla_{\bar{g}} \upsilon_2\right|^p d\upsilon_{\bar{g}}$

b) For any $\upsilon \in H_{1,G}^{p}\left(O_j \cap \partial M\right)$, $\upsilon \ge 0$ *the following property is valid:*

$$(1 - c\varepsilon_0) V_j \int_{\partial N} \upsilon_2 ds_{\bar{g}} \le \int_{\partial M} \upsilon dS_g \le (1 + c\varepsilon_0) V_j \int_{\partial N} \upsilon_2 ds_{\bar{g}},$$

where $V_j = Vol\left(O_j\right)$, $\upsilon_2 = \upsilon \circ \xi^{-1}$ *and c is a positive constant.*

Theorem 1 ([17]). *Let* (M, g) *be a smooth, compact n-dimensional Riemannian manifold,* $n \ge 3$, *with boundary, G-invariant under the action of a subgroup G of the isometry group* $I(M, g)$. *Let k denotes the minimum orbit dimension of G and V denotes the minimum of the volume of the k-dimensional orbits. Let* $p \in (1, n-k)$ *and* $\tilde{q} = \frac{(n-k-1)p}{n-k-p}$. *Then for any* $\varepsilon > 0$, *there exist positive constants* B_ε *and* $\tilde{B}_\varepsilon$ *depending on p, G and the geometry of* (M, g), *such that for all* $u \in H_{1,G}^{p}(M)$ *the following inequalities hold:*

$$\left(\int_M |u|^q dV\right)^{\frac{p}{q}} \le \left(2^{\frac{p}{n-k}} K_G^p + \varepsilon\right) \int_M |\nabla u|^p dV + B_\varepsilon \int_M |u|^p dV \qquad (6)$$

and

$$\left(\int_{\partial M} |u|^{\tilde{q}} dS\right)^{\frac{p}{q}} \leq \left(\tilde{K}_G^p + \varepsilon\right) \int_M |\nabla u|^p dV + \tilde{B}_\varepsilon \int_{\partial M} |u|^p dS, \tag{7}$$

where $K_G = \frac{K(n-k,p)}{V^{1/(n-k)}}$ *and* $\tilde{K}_G = \frac{\tilde{K}(n-k,p)}{V^{(p-1)/(n-k-1)p}}$.

Moreover, $2^{\frac{1}{n-k}} K_G$ *and* $\tilde{K}_G$ *are the best constants for these inequalities.*

We will use Faget's inequality (6) and inequality (7) to solve the following problems (P_1) and (P_2).

Consider the problem

$$(P_1) \qquad \Delta_p u + a(x)u^{p-1} = f(x)u^{q-1},\ u > 0 \ \text{ on } M,\ u|_{\partial M} = 0$$
$$\frac{2n(n-k)}{n(n-k)+2k} < p < n-k,\ q = \frac{(n-k)p}{n-k-p},$$

where a, f are G-invariant smooth functions.

Define the functional $J(u) = \int_M \left(|\nabla u|^p + a(x)|u|^p\right) dV$ and suppose that the operator $L_p(u) = \Delta_p u + a(x)u^{p-1}$ is coercive. That is, there exists a real number $\lambda > 0$, such that, for all $u \in H^p_{1,G}(M)$: $J(u) \geq \lambda \int_M |u|^p dV$.

If we denote

$$\mathscr{H} = \left\{u \in H^p_{1,G}(M)\ ,\ u > 0\ :\ \int_M f(x)u^q dV = 1\right\} \quad \text{and} \quad \mu = \inf J(u),$$

concerning the problem (P_1), for all $u \in \mathscr{H}$ the following theorem holds:

Theorem 2 ([17]). *Let a and f be two smooth functions and G-invariant and p, q be two real numbers defined as in* (P1). *Suppose that* $\sup_{x\in M} f(x) > 0$ *and the operator* L_p *is coercive. The problem* (P_1) *has a positive solution (in* $H^p_{1,G}(M)$*) that belongs to* $C^{1,\alpha}(M)$ *for some* $\alpha \in (0,1)$*, if* $\mu < K_G^{-p} (\sup f)^{\frac{-p}{q}}$.

For the problem

$$(P_2) \qquad \Delta_p u + a(x)u^{p-1} = f(x)u^{q-1},\ u > 0 \ \text{ on } M,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} + b(x)u^{p-1} = h(x)u^{\tilde{q}-1} \ \text{ on } \ \partial M,$$
$$\frac{2n(n-k)}{n(n-k)+2k} < p < n-k,\ q = \frac{(n-k)p}{n-k-p},\ \tilde{q} = \frac{(n-k-1)p}{n-k-p},$$

where a, f, b and h are four smooth functions G-invariant, we set:

$$\Lambda = \left\{c = (\alpha, \beta) \in R^2 : \alpha - \beta \geq \delta, p \leq \alpha \leq q, p \leq \beta \leq \tilde{q}\right\}$$

with $\delta \in (0, q - \tilde{q}) = \left(0, \frac{p}{n-k-p}\right)$, and for all $u \in H^p_{1,G}(M)$ and for any $c \in \Lambda$ we define the functionals:

$$I(u) = \int_M (|\nabla u|^p + a\,|u|^p)\,dV + \int_{\partial M} b\,|u|^p\,dS$$

and

$$I_c(u) = \int_M f\,|u|^\alpha\,dV + \frac{\alpha}{\beta}\int_{\partial M} h\,|u|^\beta\,dS,$$

and suppose that the operator $L_p(u) = \Delta_p u + a(x)u^{p-1}$ is coercive. The functionals $I(u)$ and $I_c(u)$ are well defined because the embeddings of $H^p_{1,G}(M)$ onto $L^q(M)$ and $L^{\tilde{q}}(M)$ are continuous according to the Sobolev theorem.

If we define $\Sigma_c = \left\{u \in H^p_{1,G}(M) : I_c(u) = 1\right\}$, $\mu_c = \inf\{I(u) : u \in \Sigma_c\}$, $c_0 = (q, \tilde{q})$ and $t^+ = \sup(t, 0)$, $t \in R$, we have the following theorem:

Theorem 3 ([17]). *Let a, f, b, and h be four smooth functions, G-invariant and p, $\tilde{p}$, q, and $\tilde{q}$ be four real numbers defined as in* (P$_2$). *Suppose that the function f has constant sign (e.g. $f \geq 0$). The problem* (P$_2$) *has a positive solution $u \in C^\infty_G(\overline{M})$ if the following inequality holds:*

$$\left(\sup_{\overline{M}} f\right)\left(2^{\frac{p}{n-k}} K^p_G \mu^+_{c_0}\right)^{\frac{q}{2}} + \frac{q}{\tilde{q}}\left(\sup_{\partial M} h\right)^+ \left(\tilde{K}^p_G \mu^+_{c_0}\right)^{\frac{\tilde{q}}{2}} < 1 \tag{8}$$

and if:

1) $f > 0$ everywhere and h arbitrary, or
2) $f \geq 0$, $h > 0$ everywhere and $(-\inf_M a)^+ K < 1$, where

$$K = \inf\left\{A > 0\ : \exists\, B > 0\ s.t.\ \|\psi\|^p_{L^p(M)} \leq A\,\|\nabla\psi\|^p_{L^p(M)} + B\,\|\psi\|^p_{L^p(\partial M)}\right\}.$$

2.2 Sharp Sobolev Inequalities on the Solid Torus

In this part of the paper, we present a thorough study devoted on the best constants in Sovolev-type inequalities in the solid torus, and in the following, we solve nonlinear elliptic boundary value problems with supercritical exponent (critical of supercritical). The possibility to solve problems of this type provided to us from the fact that the solid torus is invariant under the action of the subgroup $G = O(2) \times I$ of rotations of the isometry group $O(3)$ (see Example 1). The solutions we find are not radial but invariant under the same group, and this reflects the geometry of the solid torus. Additionally, we refer that these solutions accurately play the same role for the torus with those that play the radial solutions in case of the sphere.

Since the solid torus is invariant under the action of the subgroup $G = O(2) \times I$ of the isometry group $O(3)$, for any $q \in [1,2)$ real, the embedding $H^q_{1,G}(T) \hookrightarrow L^p_G(T)$ is compact for $1 \le p < \frac{2q}{2-q}$, while $H^q_{1,G}(T) \hookrightarrow L^{\frac{2q}{2-q}}_{1,G}(T)$, is only continuous. In addition, for any $q \in [1,2)$ real, the embedding $H^q_{1,G}(T) \hookrightarrow L^p_G(\partial T)$ is compact for $1 \le p < \frac{q}{2-q}$, while $H^q_{1,G}(T) \hookrightarrow L^{\frac{q}{2-q}}_G(\partial T)$, is only continuous.

We need now some notations and some background material.

Let the solid torus be represented by the equation:

$$\overline{T} = \left\{(x,y,z) \in R^3 : (\sqrt{x^2+y^2} - l)^2 + z^2 \le r^2,\ l > r > 0\right\},$$

and the subgroup $G = O(2) \times I$ of $O(3)$. Note that the solid torus $\overline{T} \subset R^3$ is invariant under the group G.

Let, also, $\mathscr{A} = \{(\Omega_i, \xi_i) : i = 1,2\}$ be an atlas on T defined by

$$\Omega_1 = \{(x,y,z) \in T : (x,y,z) \notin H^+_{XZ}\}, \quad \Omega_2 = \{(x,y,z) \in T : (x,y,z) \notin H^-_{XZ}\}$$

where

$$H^+_{XZ} = \{(x,y,z) \in R^3 : x > 0\,,\ y = 0\},\ \ H^-_{XZ} = \{(x,y,z) \in R^3 : x < 0\,,\ y = 0,\}$$

$$\xi_i : \Omega_i \to I_i \times D,\ i = 1,2, \ \text{ with } I_1 = (0, 2\pi),\ \ I_2 = (-\pi, \pi),$$

$$D = \{(t,s) \in R^2 : t^2 + s^2 < 1\}, \quad \partial D = \{(t,s) \in R^2 : t^2 + s^2 = 1\},$$

$$\xi_i(x,y,z) = (\omega_i, t, s),\ \ i = 1,2 \ \text{ with } \ \cos\omega_i = \frac{x}{\sqrt{x^2+y^2}},\ \sin\omega_i = \frac{y}{\sqrt{x^2+y^2}},$$

$$\omega_1 = \begin{cases} \arctan\frac{y}{x}, & x \ne 0 \\ \frac{\pi}{2},\ x = 0, & y > 0 \\ \frac{3\pi}{2},\ x = 0, & y < 0 \end{cases}, \quad \omega_2 = \begin{cases} \arctan\frac{y}{x}, & x \ne 0 \\ \frac{\pi}{2},\ x = 0, & y > 0 \\ -\frac{\pi}{2},\ x = 0, & y < 0 \end{cases}$$

and

$$t = \frac{\sqrt{x^2+y^2} - l}{r}, \quad s = \frac{z}{r}, \quad 0 \le t,\ s \le 1.$$

The Euclidean metric g on $(\Omega, \xi) \in \mathscr{A}$ can be expressed as

$$\left(\sqrt{g} \circ \xi^{-1}\right)(\omega, t, s) = r^2(l + rt).$$

Consider now the spaces $C^\infty_{0,G}(T)$ and $H^q_{1,G}(T)$ of all G-invariant functions under the action of the group $G = O(2) \times I \subset O(3)$ and denote $\overset{\circ}{H}{}^q_{1,G}(T)$ the completion of $C^\infty_{0,G}(T)$ with respect to the norm $\|u\|_{H^q_1(T)}$.

If for any G-invariant υ we define the functions $\phi(t,s) = (u \circ \xi^{-1})(\omega, t, s)$ then the following formulas hold:

$$||u||^p_{L^p(T)} = 2\pi r^2 \int_D |\phi(t,s)|^p (l + rt)\, dt\, ds \tag{9}$$

$$||\nabla u||^p_{L^p(T)} = 2\pi r^{2-p} \int_D |\nabla \phi(t,s)|^p (l + rt)\, dt\, ds \tag{10}$$

$$||u||^p_{L^p(\partial T)} = 2\pi r \int_{\partial D} |\phi(t,0)|^p (l + rt)\, d\sigma, \tag{11}$$

$$\int_T e^u dV = 2\pi r^2 \int_D e^{\phi(t,s)} (l + rt)\, dt\, ds \tag{12}$$

$$||\nabla u||^2_{L^2(T)} = 2\pi \int_D |\nabla \phi(t,s)|^2 (l + rt)\, dt\, ds \tag{13}$$

and

$$\int_{\partial T} e^u dS = 2\pi r \int_{\partial D} e^{\phi(t,0)} (l + rt)\, dt, \tag{14}$$

where by ϕ we denote the extension of ϕ on ∂D.

Let $K(2,q)$ be the best constant [4] of the Sobolev inequality

$$\left(\int_{R^2} |\varphi|^p \, dx \right)^{\frac{1}{p}} \le K(2,q) \left(\int_{R^2} |\nabla \varphi|^q \, dx \right)^{\frac{1}{q}},$$

for the Euclidean space R^2, where $1 \le q < 2$, $p = \frac{2q}{2-q}$ and $\tilde{K}(2,q)$ be the best constant [37] in the Sobolev trace embedding

$$\left(\int_{\partial R^2} |\varphi|^{\tilde{p}} \, dx' dt \right)^{\frac{1}{\tilde{p}}} \le \tilde{K}(2,q) \left(\int_{R^2} |\nabla \varphi|^q \, dx \right)^{\frac{1}{q}}$$

for the Euclidean half-space R^2_+, where $1 \le q < 2$, $\tilde{p} = \frac{q}{2-q}$.

In the following theorem, we determine the best constants of the classical Sobolev inequality where the exponent $p = \frac{2q}{2-q}$ is the largest possible exponent for this inequality and concerns the critical of supercritical and $q \in (1,2)$.

Theorem 4 ([15]). *Let $\overline{T}$ be the solid torus and p, q be two positive real numbers such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{2}$ with $1 \leq q < 2$. Then for all $\varepsilon > 0$, there exists a constant $B = B(\varepsilon, q)$ such that:*

1) For all $u \in \overset{\circ}{H}{}^q_{1,G}(T)$ the following inequality holds:

$$\left(\int_T |u|^p \, dV\right)^{\frac{q}{p}} \leq \left(\left(\frac{K(2,q)}{\sqrt{2\pi\,(l-r)}}\right)^q + \varepsilon\right) \int_T |\nabla u|^q \, dV + B \int_T |u|^q \, dV \tag{15}$$

2) For all $u \in H^q_{1,G}(T)$ the following inequality holds:

$$\left(\int_T |u|^p \, dV\right)^{\frac{q}{p}} \leq \left(\left(\frac{K(2,q)}{\sqrt{\pi\,(l-r)}}\right)^q + \varepsilon\right) \int_T |\nabla u|^q \, dV + B \int_T |u|^q \, dV. \tag{16}$$

Moreover, $\frac{K(2,q)}{\sqrt{2\pi(l-r)}}$ and $\frac{K(2,q)}{\sqrt{\pi(l-r)}}$ are the best constants for which the inequalities (15) and (16) hold for all $u \in \overset{\circ}{H}{}^q_{1,G}(T)$ and $u \in H^q_{1,G}(T)$ respectively.

We need now the definition of the concentration orbit.

Definition 1 (Concentration Orbit [26]). Set O_P a G-orbit of T. O_P is an orbit of concentration of the sequence (u_α) if for any $\delta > 0$, the following holds: $\lim_{\alpha \to \infty} \sup \int_{O_{P,\delta}} u^p_\alpha dv(g) > 0$, where $O_{P,\delta} = \{Q \in T : d(Q, O_P) < \delta\}$.

Because of the concentration phenomenon on the orbit of a sequence of solutions of nonlinear deferential equations, the following theorem holds:

Theorem 5 ([15]). *Let $\overline{T}$ be the solid torus and p, q be two positive real numbers such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{2}$ with $1 < q < 2$. Then, there exists $B = B(q) > 0$ such that:*

1) For all $u \in \overset{\circ}{H}{}^q_{1,G}(T)$

$$\left(\int_T |u|^p \, dV\right)^{\frac{q}{p}} \leq \left(\frac{K(2,q)}{\sqrt{2\pi\,(l-r)}}\right)^q \int_T |\nabla u|^q \, dV + B \int_T |u|^q \, dV \tag{17}$$

2) For all $u \in H^q_{1,G}(T)$

$$\left(\int_T |u|^p \, dV\right)^{\frac{q}{p}} \leq \left(\frac{K(2,q)}{\sqrt{\pi\,(l-r)}}\right)^q \int_T |\nabla u|^q \, dV + B \int_T |u|^q \, dV. \tag{18}$$

We give now an application resolving the problem

$$(\mathrm{P}_1') \qquad \Delta_q u + a(x)u^{q-1} = f(x)u^{p-1}, \; u > 0 \;\text{ on } T, \; u|_{\partial M} = 0,$$
$$\frac{3}{2} < q < 2, \; p = \frac{2q}{2-q}$$

Consider the functional $I(u) = \int_T (|\nabla u|^q + a(x)|u|^q)\, dV$ and suppose that the operator $L_q(u) = \Delta_q u + a(x)u^{q-1}$ is coercive.

For $\frac{3}{2} < q < 2$, $\frac{3+2}{3-2}+1 = 6 < p = \frac{2q}{2-q}$ and for all $u \in \mathscr{H}_p$ we set $\mu = \inf I(u)$, where

$$\mathscr{H}_p = \Big\{ u \in H^q_{1,G}(T), \; u > 0 \; : \; \int_T f(x)u^p dV = 1 \Big\}.$$

Consequently, for the problem (P_1') we have the following theorem:

Theorem 6 ([15]). *Let $\overline{T}$ be a solid torus , α and f be two smooth functions and G-invariant and p, q be two real numbers defined as in* (P_1'). *Suppose that* $\sup_{x\in T} f(x) > 0$ *and the operator* $L_q u = \Delta_q u + \alpha u^{q-1}$ *is coercive. The problem* (P_1') *accepts a positive solution, that belongs to* $C^{1,\alpha}(T)$ *for some* $\alpha \in (0,1)$*, if* $\mu < \left(\frac{K(2,q)}{\sqrt{2\pi(l-r)}}\right)^{-q} (\sup f)^{\frac{-q}{p}}$.

Corollary 1 ([15]). *Let $\overline{T}$ be a solid torus, and α, f be two smooth functions, G-invariant. Then the problem*

$$\Delta u + a(x)u = f(x)u^{p-1}, \; u > 0 \;\text{ on } T, \; u|_{\partial T} = 0, \;\; p > 1$$

accepts a positive solution that belongs to $H^2_{1,G}(T)$.

In this part, we determine the best constants of the Sobolev trace inequality, where the exponent $\tilde{p} = \frac{q}{2-q}$ is the critical of supercritical of this case and $q \in (1,2)$.

Theorem 7 ([16]). *Let $\overline{T}$ be the solid torus and $\tilde{p}$, q be two positive real numbers such that $\tilde{p} = \frac{q}{2-q}$ with $1 < q < 2$. Then for all $\varepsilon > 0$ there exists a real number B_ε such that for all $u \in H^q_{1,G}(T)$ the following inequality holds:*

$$\left(\int_{\partial T} |u|^{\tilde{p}} dS \right)^{\frac{q}{\tilde{p}}} \leq \left(\frac{\tilde{K}^q\,(2,q)}{[2\pi\,(l-r)]^{q-1}} + \varepsilon \right) \int_T |\nabla u|^q dV + B \int_{\partial T} |u|^q dS \qquad (19)$$

In addition, $\frac{\tilde{K}^q(2,q)}{[2\pi(l-r)]^{q-1}}$ is the best constant for the above inequality.

We use the best constant found in the above theorem to solve the following nonlinear elliptic problem with Neumann type boundary conditions:

$$(\mathrm{P}_2') \ \Delta_q u + a(x)u^{q-1} = \lambda f(x)u^{p-1}, \ u > 0 \quad \text{on} \quad T,$$
$$|\nabla u|^{q-2}\frac{\partial u}{\partial \nu} + b(x)u^{q-1} = \lambda g(x)u^{\tilde{p}-1} \quad \text{on} \quad \partial T,$$
$$p = \frac{2q}{2-q} > 6, \quad \tilde{p} = \frac{q}{2-q} > 4, \quad \frac{3}{2} < q < 2,$$

For the problem (P_2') we have the following theorem:

Theorem 8 ([16]). *Let a, f, b, and g be four smooth functions and G-invariant and q, p, $\tilde{p}$ be three real numbers defined as in* (P_2'). *Suppose that the function f has constant sign (e.g. $f \geq 0$). The problem* (P_2') *has a positive solution $u \in H^q_{1,G}(T)$ if the following holds:*

$$(\sup_T f)\left[\frac{K^q(2,q)\mu^+_{c_0}}{[\pi(l-r)]^{q/2}}\right]^{\frac{p}{2}} + \frac{p}{\tilde{p}}(\sup_{\partial T} g)^+\left[\frac{\widetilde{K}^q(2,q)\mu^+_{c_0}}{[2\pi(l-r)]^{q-1}}\right]^{\frac{\tilde{p}}{2}} < 1 \tag{20}$$

and if

(i) $f > 0$ everywhere and g is arbitrary, or
(ii) $f \geq 0$, $g > 0$ everywhere and $(-\inf_T a)^+\kappa < 1$, where

$$\kappa = \inf\{A > 0 : \exists B > 0 \ \text{ s.t. } \ \|\psi\|^q_{L^q(T)} \leq A\|\nabla\psi\|^q_{L^q(T)} + B\|\psi\|^q_{L^q(\partial T)}\} \tag{21}$$

We have proved that for any $p \in [1,2)$ real, the embedding $H^p_{1,G}(T) \hookrightarrow L^q_G(T)$ is compact for $1 \leq q < \frac{2p}{2-p}$, while the embedding $H^p_{1,G}(T) \hookrightarrow L^{\frac{2p}{2-p}}_G(T)$ is only continuous.

Also, we have proved that for any $p \in [1,2)$ real, the embedding $H^p_{1,G}(T) \hookrightarrow L^q_G(\partial T)$ is compact for $1 \leq q < \frac{p}{2-p}$, while the trace embedding $H^p_{1,G}(T) \hookrightarrow L^{\frac{p}{2-p}}_G(\partial T)$ is only continuous.

Additionally, we observe that if $\frac{3}{2} < p < 2$ then $q = \frac{2p}{2-p} > 6 = \frac{2\cdot 3}{3-2}$ and $\tilde{q} > \frac{p}{2-p} > 4 = \frac{2(3-1)}{3-2}$, that is the exponents q and $\tilde{q}$ are supercritical.

In this part, we study the exceptional case when $p = n - k = 3 - 1 = 2$. In this case, $H^2_{1,G}(T) \not\hookrightarrow L^\infty_G(T)$, however, when $\upsilon \in H^2_{1,G}(T)$ we have $e^\upsilon \in L^1_G(T)$, $e^\upsilon \in L^1_G(\partial T)$ and the exponent $p = 2$ is the critical of supercritical.

In the following theorem, we determine the best constants μ and $\tilde{\mu}$ in the exponential Sobolev inequalities (3) and (4) in the case of the solid torus:

Theorem 9 ([19]). *Let $\overline{T}$ be the solid torus, $2\pi^2 r^2 l$ be the volume of T and $4\pi^2 rl$ be the volume of ∂T, then there exists a constant C such that:*

1) For all functions $\upsilon \in \mathscr{H}_G(T)$, the following inequality holds:

$$\int_T e^{\upsilon} dV \le C\exp\left(\mu\|\nabla\upsilon\|_2^2 + \frac{1}{2\pi^2 r^2 l}\int_T \upsilon dV\right) \tag{22}$$

2) For all functions $\upsilon \in \mathscr{H}_G(T)$, the following inequality holds:

$$\int_{\partial T} e^{\upsilon} dS \le C\exp\left(\tilde{\mu}\|\nabla\upsilon\|_2^2 + \frac{1}{4\pi^2 rl}\int_{\partial T} \upsilon dS\right), \tag{23}$$

where, for the first inequality, $\mu = \frac{1}{32\pi^2(l-r)}$ if $\mathscr{H}_G(T) = \overset{\circ}{H}{}^2_{1,G}(T)$ and $\mu = \frac{1}{16\pi^2(l-r)}$ if $\mathscr{H}_G(T) = H^2_{1,G}(T)$. For the second inequality $\tilde{\mu} > \frac{1}{8\pi^2(l-r)}$ for all $\upsilon \in H^2_{1,G}(T)$.

Moreover, μ and $\tilde{\mu}$ are the best constants for the above inequalities.

Remark 1. In [29] Faget proved that for a compact three-dimensional manifold without boundary the first best constant for inequality (22) is $\mu_3 = \frac{2}{81\pi}$ and the map: $H_1^3 \ni \upsilon \to e^{\upsilon} \in L^1$ is compact. Clearly, the best constant μ_3 depends only on the dimension 3 of the manifold. For the solid torus, we prove that the first best constant for the same inequality is $\mu = \frac{1}{32\pi^2(l-r)}$ and the map: $H^2_{1,G} \ni \upsilon \to e^{\upsilon} \in L^1_G$ is compact. In this case, it is proved that the value of the best constant μ not only depends on the dimension but also on the geometry of torus.

Corollary 2 ([19]). *For all $\upsilon \in \overset{\circ}{H}{}^2_{1,G}(T)$ such that $\|\nabla\upsilon\|_2^2 \le 2\pi(l+r)$ and for all $\alpha \le 4\pi$ the following holds:*

$$\int_T e^{\alpha\upsilon^2} dV \le C\, 2\pi^2 r^2 l, \tag{24}$$

where the constant C is independent of $\upsilon \in \overset{\circ}{H}{}^2_{1,G}(T)$.

In addition, the constant $\alpha \le 4\pi$ is the best one, in the sense that, if $\alpha > 4\pi$ the integral in the inequality is finite, but it can be made arbitrarily large by an appropriate choice of υ.

Remark 2. Corollary 2 is a special case of the result of Moser [38].

For the problem

(P_3') $\Delta\upsilon + \gamma = f(x)e^{\upsilon},\ \upsilon > 0 \quad \text{on} \quad T, \quad \upsilon|_{\partial T} = 0.$

we have the theorem:

Theorem 10 ([19]). *Consider a solid torus $\overline{T}$ and the function f continuous and G-invariant.*

Then the problem (P_3') *accepts a solution that belongs to* C_G^∞*, if one of the following holds:*

(a) $\sup_T f < 0$ *if* $\gamma < 0$.
(b) $\int_T f dV < 0$ *and* $\sup_T f > 0$ *if* $\gamma = 0$.
(c) $\sup_T f > 0$ *if* $0 < \gamma < \frac{8(l-r)}{lr^2}$.

For the problem

$$(\mathrm{P}_4') \qquad \Delta \upsilon + a + f e^{\upsilon} = 0,\ \upsilon > 0 \quad \text{on} \quad T,$$

$$\frac{\partial \upsilon}{\partial n} + b + g e^{\upsilon} = 0 \quad \text{on} \quad \partial T,$$

we have the next theorem:

Theorem 11 ([19]). *Consider a solid torus* $\overline{T}$ *and the smooth functions* f, g *G-invariant and not both identical* 0*. If* $a, b \in R$ *and* $R = 2\pi^2 r^2 la + 4\pi^2 rlb$*, the problem* (P_4) *accepts a solution that belongs to* C_G^∞ *in each one of the following cases:*

(i) *If* $a = b = 0$ *the necessary and sufficient condition is* f *and* g *not both* ≥ 0 *and that* $\int_T f dV + \int_{\partial T} g dS > 0$.
(ii) *If* $a \geq 0$ *and* $b \geq 0$, f , g *not both* ≥ 0 *everywhere and* $0 < R < 4\pi^2(l-r)$*. Particularly, if* $g = 0$ *we can substitute the last condition with* $0 < R < 8\pi^2(l-r)$.
(iii) *If* $R > 0$ *(respectively* $R < 0$*) it is necessary that* f , g *not both* ≥ 0 *everywhere (respectively* ≤ 0*). Then, there exists a solution of the problem in each one of the following cases:*

(a) $a < 0, b > 0, f < 0, g \leq 0$ *and* $b < \frac{l-r}{lr}$ *if* $g \not\equiv 0$ *or* $b < \frac{2(l-r)}{lr}$ *if* $g \equiv 0$.
(b) $a > 0, b < 0, f \leq 0, g < 0$ *and* $a < \frac{2(l-r)}{lr^2}$.
(c) $a > 0, b < 0, f \geq 0, g > 0$ *and* $a < \frac{2(l-r)}{lr^2}$ *if* $g \not\equiv 0$ *or* $a < \frac{4(l-r)}{lr^2}$ *if* $g \equiv 0$.
(d) $a < 0, b > 0, f > 0, g \geq 0$ *and* $b < \frac{l-r}{lr}$.

(iv) *If* $a \leq 0, b \leq 0$*, not both* $= 0$*, it is necessary* $\int_T f dV + \int_{\partial T} g dS > 0$*. Then there exists a nonempty subset* $S_{f,\,g}$ *of* $R_-^2 = \{(a,b) \neq (0,0) : a \leq 0,\ b \leq 0\}$ *with the property that if* $(c,d) \in S_{f,\,g}$ *then* $(c',d') \in S_{f,\,g}$ *for any* $c' \geq c$, $d' \geq d$ *and such that the problem* (P_4') *has a solution if and only if* $(a,b) \in S_{f,\,g}$. $S_{f,\,g} = R_-^2$ *if and only if the functions* f, g *are* $\not\equiv 0$ *and* ≥ 0*. For all* $(a,b) \in R_-^2$ *there exist functions* f *and* g *such that* $\int_T f dV + \int_{\partial T} g dS > 0$ *and* $(a,b) \notin S_{f,\,g}$.

3 Nash Inequalities

We say that the Nash inequality (25) is valid if there exists a constant $A > 0$ such that for all $u \in C_0^\infty(R^n)$, $n \geq 2$:

$$\left(\int_{R^n} u^2 dx\right)^{1+\frac{2}{n}} \leq A \int_{R^n} |\nabla u|^2 dx \left(\int_{R^n} |u|\, dx\right)^{\frac{4}{n}}. \tag{25}$$

Such an inequality first appeared in the celebrated paper of Nash [39], where he discussed the Hölder regularity of solutions of divergence form in uniformly elliptic equations. It is a particular case of the Gagliardo–Nirenberg type inequalities

$$||u||_r \leq C||\nabla u||_q^a \, ||u||_s^{1-a}$$

and it is well known that the Nash inequality (25) and the Euclidian-type Sobolev inequality are equivalent in the sense that if one of them is valid, the other one is also valid (i.e., see [7]). It is, also, well known that with this procedure of passing from the one type of inequalities to the other is impossible to compare the best constants, since the inequalities under use are not optimal.

As far as the optimal version of Nash inequality (25) is concerned, the best constant $A_0(n)$, that is

$$A_0\left(n\right)^{-1} = \inf\left\{ \frac{\int_{R^n} |\nabla u|^2 \, dx \left(\int_{R^n} |u| \, dx\right)^{\frac{4}{n}}}{\left(\int_{R^n} u^2 dx\right)^{1+\frac{2}{n}}} \middle| u \in C_0^\infty\left(R^n\right), u \not\equiv 0 \right\},$$

has been computed by Carlen and Loss in [10], together with the characterization of the extremals for the corresponding optimal inequality, as:

$$A_0\left(n\right) = \frac{\left(n+2\right)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n \lambda_1^N \left|\mathscr{B}^n\right|^{\frac{2}{n}}},$$

where $|\mathscr{B}^n|$ denotes the euclidian volume of the unit ball $\mathscr{B}^n$ in R^n and λ_1^N is the first Neumann eigenvalue for the Laplacian for radial functions in the unit ball $\mathscr{B}^n$.

For an example of application of the Nash inequality with the best constant, we refer to Kato [35] and for a geometric proof with an asymptotically sharp constant, we refer to Beckner [8].

For compact Riemannian manifolds, the Nash inequality still holds with an additional L^1-term and that is why we will refer this as the L^1-Nash inequality.

Given (M, g) a smooth compact Riemannian n-manifold, $n \geq 2$, we get here the existence of real constants A and B such that for any $u \in C^\infty(M)$:

$$\left(\int_M u^2 dV_g\right)^{1+\frac{2}{n}} \leq A \int_M |\nabla u|_g^2 dV_g \left(\int_M |u| \, dV_g\right)^{\frac{4}{n}} + B \left(\int_M |u| \, dV_g\right)^{2+\frac{4}{n}}. \quad (26)$$

The best constant for this inequality is defined as:

$$A_{\text{opt}}^1(M) = \inf\left\{A > 0 : \exists B > 0 \;\text{ s.t. } (26) \;\text{ is true} \;\forall\, u \in C^\infty\left(M\right)\right\}.$$

This inequality has been studied completely by Druet, Hebey, and Vaugon in [24]. They proved that $A_{\text{opt}}^1(M) = A_0(n)$, and (26) with its optimal constant $A = A_0(n)$ is sometimes valid and sometimes not, depending on the geometry of M.

Humbert in [33] studied the following L^2-Nash inequality:

$$\left(\int_M u^2 dV_g\right)^{1+\frac{2}{n}} \leq \left(A\int_M |\nabla u|_g^2 dV_g + B\int_M u^2 dV_g\right)\left(\int_M |u|\, dV_g\right)^{\frac{4}{n}}, \tag{27}$$

for all $u \in C^\infty(M)$, of which the best constant is defined as:

$$A_{\text{opt}}^2(M) = \inf\{A > 0 : \exists B > 0 \ \text{ s.t. } (27) \ \text{ is true} \ \forall\, u \in C^\infty(M)\}.$$

Contrary to the sharp L^1-Nash inequality, in this case, he proved that B always exists and $A_{\text{opt}}^2(M) = A_0(n)$.

We denote $R_+^n = R^{n-1} \times [0, +\infty)$ and $\partial R_+^n = R^{n-1} \times \{0\}$. The trace Nash inequality states that a constant $\tilde{A} > 0$ exists such that for all $u \in C_0^\infty(R_+^n)$, $n \geq 2$ with $\nabla u \in L^2(R^n)$ and $u|_{\partial R_+^n} \in L^1(\partial R_+^n) \cap L^2(\partial R_+^n)$:

$$\left(\int_{\partial R_+^n} u^2 ds\right)^{\frac{n}{n-1}} \leq \widetilde{A}\int_{R_+^n} |\nabla u|^2 dx \left(\int_{\partial R_+^n} |u|\, ds\right)^{\frac{2}{n-1}}, \tag{28}$$

where ds is the standard volume element on R^{n-1} and the trace of u on ∂R_+^n is also denoted by u.

Let $\widetilde{A}_0(n)$ be the best constant in Nash inequality (28). That is:

$$\widetilde{A}_0(n)^{-1} = \inf\left\{ \frac{\int_{R_+^n} |\nabla u|^2 dx \left(\int_{\partial R_+^n} |u|\, ds\right)^{\frac{2}{n-1}}}{\left(\int_{\partial R_+^n} u^2 ds\right)^{\frac{n}{n-1}}} \,\middle|\, u \in C_0^\infty\left(R_+^n\right), u \not\equiv 0 \right\}.$$

The computation problem of the exact value of $\widetilde{A}_0(n)$ still remains open.

For compact Riemannian manifolds with boundary, Humbert, also, studied in [34] the trace Nash inequality.

On smooth compact n-dimensional, $n \geq 2$, Riemannian manifolds with boundary, for all $u \in C^\infty(M)$, consider the following trace Nash inequality:

$$\left(\int_{\partial M} u^2 dS_g\right)^{\frac{n}{n-1}} \leq \left(\tilde{A}\int_M |\nabla u|_g^2 dV_g + \tilde{B}\int_{\partial M} u^2 dS_g\right)\left(\int_{\partial M} |u|\, dS_g\right)^{\frac{2}{n-1}}. \tag{29}$$

The best constant for the above inequality is defined as

$$\widetilde{A}_{\text{opt}}(M) = \inf\left\{\tilde{A} > 0 : \exists \tilde{B} > 0 \ \text{ s.t. } (29) \ \text{ is true} \ \forall\, u \in C^\infty(M)\right\}.$$

It was proved in [34] that $\widetilde{A}_{\text{opt}}(M) = \widetilde{A}_0(n)$, and (29) with its optimal constant $\widetilde{A} = \widetilde{A}_0(n)$ is always valid.

In this part of the paper, we prove that, when the functions are invariant under an isometry group, all orbits of which are of infinite cardinal, the Nash inequalities can be improved, in the sense that we can get a higher critical exponent.

More precisely we establish:

(i) The best constant for the *Nash inequality* on compact Riemannian manifolds with boundary, invariant under the action of an arbitrary compact subgroup G of the isometry group $Is(M, g)$, where all the orbits have infinite cardinal, and
(ii) The best constant for the *Trace Nash inequality* on compact Riemannian manifolds with boundary, invariant under the action of an arbitrary compact subgroup G of the isometry group $Is(M, g)$, where all the orbits have infinite cardinal.

These best constants are improvements over the classical cases due to the symmetries which arise and reflect the geometry of the manifold.

3.1 Sharp Nash Inequalities on Manifolds in the Presence of Symmetries

Theorem 12 ([18]). *Let (M, g) be a smooth, compact n-dimensional Riemannian manifold, $n \geq 3$, with boundary, G-invariant under the action of a subgroup G of the isometry group $Is(M, g)$. Let k denote the minimum orbit dimension of G and V denote the minimum of the volume of the k-dimension orbits. Then for any $\varepsilon > 0$, there exists a constant B_ε such that and for all $u \in H^2_{1,G}(M)$ the following inequality holds:*

$$\left(\int_M u^2 dV_g\right)^{\frac{n-k+2}{n-k}} \leq \left((A_G + \varepsilon)^{\frac{n-k}{n-k+2}} \int_M |\nabla u|_g^2 dV_g + B_\varepsilon \int_M u^2 dV_g\right) \times \left(\int_M |u|\, dV_g\right)^{\frac{4}{n-k}}, \tag{30}$$

where $A_G = \frac{A_0(n-k)}{V^{\frac{2}{n-k}}}$.

Moreover, the constants A_G is the best constant for this inequality.

Theorem 13 ([18]). *Let (M, g) be a smooth, compact n-dimensional Riemannian manifold, $n \geq 3$, with boundary, G-invariant under the action of a subgroup G of the isometry group $Is(M, g)$. Let k denote the minimum orbit dimension of G and V denote the minimum of the volume of the k-dimension orbits. Then for any $\varepsilon > 0$, there exists a constant $\tilde{B}_\varepsilon$ such that and for all $u \in H^2_{1,G}(M)$ the following inequality holds:*

$$\left(\int_{\partial M} u^2 dS_g\right)^{\frac{n-k}{n-k-1}} \leq \left((\tilde{A}_G + \varepsilon)^{\frac{n-k-1}{n-k}} \int_M |\nabla u|^2 dV_g + \tilde{B}_\varepsilon \int_{\partial M} u^2 dS_g\right) \times \left(\int_{\partial M} |u| dS_g\right)^{\frac{2}{n-k-1}}, \tag{31}$$

where $\tilde{A}_G = \frac{\tilde{A}_0(n-k)}{V^{\frac{1}{n-k-1}}}$.

Moreover, the constants $\tilde{A}_G$ is the best constant for this inequality.

Corollary 3 ([18]). *For any $\varepsilon > 0$, there exists a constant C_ε such that and for all $u \in H^2_{1,G}(T)$ the following inequality holds:*

$$\left(\int_{\partial T} u^2 dS\right)^2 \leq \left(\frac{\tilde{A}_0\,(2) + \varepsilon}{2\pi\,(l-r)} \int_T |\nabla u|^2\, dV + C_\varepsilon \int_{\partial T} u^2 dS\right)\left(\int_{\partial T} |u| dS\right)^2 \quad (32)$$

Moreover, the constant $\widetilde{A}_{\text{opt}}(\overline{T}) = \frac{\tilde{A}_0(2)}{2\pi(l-r)}$ is the best constant for this inequality and verifies:

$$\frac{3\sqrt{3}}{4\pi^2(l-r)} \leq \widetilde{A}_{\text{opt}}(\overline{T}) \leq \frac{2}{\pi^2(l-r)} \quad (33)$$

3.2 Sharp Nash Inequalities on the Solid Torus

In this part, we establish the best constant $\widetilde{A}_{\text{opt}}(\overline{T})$ for the trace Nash inequality on a three-dimensional solid torus $\overline{T}$, which is an improvement over the classical case due to the symmetries which arise and reflect the geometry of torus.

Theorem 14 ([20]). *For any $\varepsilon > 0$ and for all $u \in H^2_{1,G}(T)$, the following inequality holds:*

$$\left(\int_{\partial T} u^2 dS\right)^2 \leq \left(\frac{\tilde{A}_0\,(2) + \varepsilon}{2\pi\,(l-r)} \int_T |\nabla u|^2\, dV + C_\varepsilon \int_{\partial T} u^2 dS\right)\left(\int_{\partial T} |u| dS\right)^2 \quad (34)$$

Moreover, the constant $\frac{\tilde{A}_0(2)}{2\pi(l-r)}$ is the best constant for this inequality.

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Generalized Minkowski Functionals

Stefan Czerwik and Krzysztof Król

In Honor of Constantin Carathéodory

Abstract In the paper we present the generalized Minkowski functionals. We also establish some useful properties of the Minkowski functionals, criterium of the continuity of such functionals, and a generalization of a Kolmogorov result.

1 Introduction

We shall introduce basic ideas, which will be used in the paper.

Let X be a linear topological (Hausdorff) space over the set of real numbers $\mathbb{R}$. Denote $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}^+ = [0,\infty]$. Also $0\cdot\infty = \infty\cdot 0 = 0$. Let A be a subset of X. As usual for $\alpha \in \mathbb{R}$,

$$\alpha A := \{y \in X\colon y = \alpha x \text{ for } x \in A\}.$$

We shall call A a symmetric, provided that $A = -A$. Moreover, a set $A \subset X$ is said to be bounded (sequentially) (see [6]) iff for every sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to 0$ as $n \to \infty$ and every sequence $\{x_n\} \subset A$, the sequence $\{t_n \cdot x_n\} \subset X$ satisfies $t_n \cdot x_n \to 0$ as $n \to \infty$.

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We also recall the idea of generalized metric space (briefly gms) introduced by Luxemburg (see [5] and also [2]). Let X be a set. A function

$$d: X \times X \to [0, \infty]$$

is called a generalized metric on X, provided that for all $x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \le d(x, z) + d(z, y)$,

A pair (X, d) is called a generalized metric space.

Clearly, every metric space is a generalized metric space.

Analogously, for a linear space X, we can define a generalized norm and a generalized normed space.

Let's note that any generalized metric d is a continuous function.

For if $x_n, x, y_n, y \in (X, d)$ for $n \in \mathbb{N}$ (the set of all natural numbers) and

$$x_n \to x, \quad y_n \to y \quad \text{as} \quad n \to \infty$$

i.e. $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$ as $n \to \infty$, then in the case $d(x, y) < \infty$, we can prove, in standard way, that

$$d(x_n, y_n) \to d(x, y) \quad \text{as} \quad n \to \infty.$$

But if $d(x, y) = \infty$, we have for $\varepsilon > 0$

$$d(x, y) \le d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

and, for $n > n_0$, $n, n_0 \in \mathbb{N}$

$$\infty = d(x, y) \le d(x_n, y_n) + \varepsilon,$$

i.e. $d(x_n, y_n) = \infty$ for $n > n_0$ and consequently

$$\infty = d(x_n, y_n) \to d(x, y) = \infty \quad \text{as} \quad n \to \infty,$$

as claimed.

2 Generalized Minkowski Functionals

Now we shall prove the following basic result.

Theorem 1. *Let X be a linear topological (Hausdorff) space over $\mathbb{R}$ and let a subset U of X satisfy the conditions:*

(i) U is a convex (nonempty) set,
(ii) U is a symmetric set.

Then the function $p: X \to \mathbb{R}^+$ *defined by the formula*

$$p(x) := \begin{cases} \inf\{t > 0 : x \in tU\}, & x \in X, \text{ if } A_x \neq \emptyset, \\ \infty, & \text{if } A_x = \emptyset, \end{cases} \tag{1}$$

where

$$A_x := \{t > 0 : x \in tU\}, \quad x \in X, \tag{2}$$

has the properties:

$$\text{if} \quad x = 0, \text{then } p(x) = 0, \tag{3}$$

$$p(\alpha x) = |\alpha| p(x) \text{ for } x \in X \text{ and } \alpha \in \mathbb{R}, \tag{4}$$

$$p(x + y) \leq p(x) + p(y) \text{ for } x, y \in X. \tag{5}$$

Proof. Clearly, $p(x) \in [0, \infty]$ for $x \in X$. Since $0 \in U$, by the definition (1) we get (3). To prove (4), consider at first the case $\alpha > 0$ (if $\alpha = 0$, the property (4) is obvious). Assume that $p(x) < \infty$ for $x \in X$. Then we have

$$\begin{aligned} \alpha p(x) &= \alpha \inf\{s > 0 : x \in sU\} = \alpha \inf\left\{\frac{t}{\alpha} > 0 : x \in \frac{t}{\alpha} U\right\} \\ &= \inf\left\{\frac{t}{\alpha} \cdot \alpha > 0 : x \in \frac{t}{\alpha} U\right\} = \inf\left\{t > 0 : x \in \frac{t}{\alpha} U\right\} \\ &= \inf\{t > 0 : \alpha x \in tU\} = p(\alpha x). \end{aligned}$$

If $p(x) = \infty$, then $\{t > 0 : x \in tU\} = \emptyset$. Therefore,

$$\{t > 0 : \alpha x \in tU\} = \alpha \left\{\frac{t}{\alpha} > 0 : \alpha x \in tU\right\} = \alpha \left\{\frac{t}{\alpha} > 0 : x \in \frac{t}{\alpha} U\right\} = \emptyset,$$

and consequently $p(\alpha x) = \infty$, i.e. (4) holds true.

Now consider the case $\alpha < 0$. Taking into account that (ii) implies that also tU for $t \in \mathbb{R}$ is a symmetric, one gets for $x \in X$ and $p(x) < \infty$

$$p(-x) = \inf\{t > 0 : -x \in tU\} = \inf\{t > 0 : x \in tU\} = p(x).$$

If $p(x) = \infty$, then

$$\emptyset = \{t > 0 : x \in tU\} = \{t > 0 : -x \in tU\},$$

which implies also $p(-x) = \infty$, and consequently

$$p(-x) = p(x) \quad \text{for any } x \in X. \tag{6}$$

Thus, for $\alpha < 0$, $x \in X$ and in view of the first part of the proof,

$$p(\alpha x) = p(-\alpha x) = -\alpha p(x) = |\alpha| p(x),$$

i.e. (4) has been verified.

Finally, if $p(x) = \infty$ or $p(y) = \infty$, then (5) is satisfied. So assume that $x, y \in X$ and

$$p(x) < \infty \quad \text{and} \quad p(y) < \infty.$$

Take an $\varepsilon > 0$. From the definition (1), there exist numbers $t_1 \geq p(x)$ and $t_2 \geq p(y)$, $t_1 \in A_x$, $t_2 \in A_y$ such that

$$0 < t_1 < p(x) + \frac{1}{2}\varepsilon, \qquad 0 < t_2 < p(y) + \frac{1}{2}\varepsilon.$$

The convexity of U implies that

$$\frac{x+y}{t_1+t_2} = \frac{t_1}{t_1+t_2} \cdot \frac{x}{t_1} + \frac{t_2}{t_1+t_2} \cdot \frac{y}{t_2} \in U$$

and consequently

$$x + y \in (t_1 + t_2)U,$$

which means that $t_1 + t_2 \in A_{x+y}$.

Hence

$$p(x+y) \leq t_1 + t_2 \leq p(x) + \frac{1}{2}\varepsilon + p(y) + \frac{1}{2}\varepsilon = p(x) + p(y) + \varepsilon$$

i.e.

$$p(x+y) \leq p(x) + p(y) + \varepsilon,$$

and since ε is arbitrarily chosen, this concludes the proof. □

Example 1. Consider $X = \mathbb{R} \times \mathbb{R}$, $U = (-1, 1)$.

Then

$$p(x) = \begin{cases} \inf\{t > 0\colon (x, 0) \in tU\}, & \text{for } x = (x, 0), \\ \infty, & \text{for } x = (x_1, y_1),\ y_1 \neq 0, \end{cases}$$

$$p(x) = \begin{cases} |x|, \text{ for } x = (x, 0), \\ \infty, \text{ for } x = (x_1, y_1),\ y_1 \neq 0, \end{cases}$$

because $\{t > 0\colon (x_1, y_1) \in tU\} = \emptyset$ for $y_1 \neq 0$.

We see that p is a generalized norm in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (p takes values in $[0, \infty]$).

Remark 1. The function $p: X \to \mathbb{R}^+$ defined by (1) we shall call the generalized Minkowski functional of U (also a generalized seminorm).

Remark 2. Under some stronger assumptions (see e.g. [3]), the function p is called the Minkowski functional of U.

The next basic property of the functional p is given in

Theorem 2. *Suppose that the assumptions of Theorem 1 are satisfied. If, moreover, U is bounded (sequentially), then*

$$p(x) = 0 \quad \Rightarrow \quad x = 0. \tag{7}$$

Proof. Assume that $p(x) = 0$ for $x \in X$. Suppose that $x \neq 0$. From the definition of $p(x)$ for every $\varepsilon_n = \frac{1}{n}$, there exists a $t_n > 0$ such that $x \in t_n U$, $n \in \mathbb{N}$ and $t_n < \frac{1}{n}$. Hence, $x = t_n x_n$, $x_n \in U$ for $n \in \mathbb{N}$ and by the boundedness of U, $x = t_n x_n \to 0$ as $n \to \infty$. But clearly $x \to x$, whence $x = 0$, which is a contradiction and completes the proof. □

Remark 3. Under the assumptions of Theorem 2, the generalized Minkowski functional is a generalized norm in X.

Let's note the following useful

Lemma 1. *Let $U \subset X$ be a convex set and $0 \in U$. Then*

$$\alpha U \subset U \tag{8}$$

for all $0 \leq \alpha \leq 1$.

The simple proof of this Lemma is omitted here.

Next we prove

Lemma 2. *Let U be as in Theorem 1. If, moreover, U does not contain half-lines, then*

$$p(x) = 0 \quad \Rightarrow \quad x = 0.$$

Proof. For the contrary, suppose that $x \neq 0$. By the definition of $p(x)$ for every $\varepsilon > 0$, there exists a $0 < t < \varepsilon$ such that $x \in tU$. Take $r > 0$ and $\varepsilon < \frac{1}{r}$. Clearly, $\frac{x}{t} \in U$. Furthermore,

$$rx = \frac{x}{t}(tr) = \alpha \frac{x}{t}, \qquad \text{where } \alpha = tr < 1.$$

By Lemma 1, $rx = \alpha\frac{x}{t} \in \alpha U \subset U$, which means that there exists an $x \neq 0$ such that for every $r > 0$, $rx \in U$ what contradicts the assumptions on U. This yields our statement. □

We have also

Lemma 3. *Let U be as in Theorem 1. Then*

$$[p(x) = 0 \ \Rightarrow \ x = 0] \quad \Rightarrow \quad U \text{ does not contain half-lines.} \tag{9}$$

Proof. For the contrary, suppose that there exists an $x \neq 0$ such that for every $r > 0$ we have $rx \in U$. Hence

$$x \in \frac{1}{r}U \quad \text{for} \quad r > 0$$

and therefore

$$\frac{1}{r} \in \{t > 0 : x \in tU\}$$

which implies that $p(x) = 0$. From (9) we get $x = 0$, which is a contradiction. Eventually, one gets the implication (9) and this ends the proof. □

Therefore, Lemmas 2 and 3, we can rewrite as the following

Proposition 1. *Let the assumptions of Theorem 1 be satisfied. Then the generalized Minkowski functional p for U is a generalized norm iff U does not contain half-lines.*

3 Properties of the Generalized Minkowski Functionals

In this part we start with the following

Theorem 3. *Let X be a linear topological (Hausdorff) space over $\mathbb{R}$ and let $f: X \to \mathbb{R}^+$ be any function with properties:*

$$f(\alpha x) = |\alpha| f(x) \text{ for all } x \in X \text{ and } \alpha \in \mathbb{R}, \tag{10}$$

$$f(x + y) \leq f(x) + f(y) \text{ for all } x, y \in X. \tag{11}$$

Define

$$U := \{x \in X : f(x) < 1\}. \tag{12}$$

Then

a) U is a symmetric set,
b) U is a convex (nonempty) set,
c) $f = p$, i.e. f is the generalized Minkowski functional of U.

Proof. The conditions a) and b) follow directly from the definition (12) and properties (10) and (11), respectively. To prove c), assume that $x \in U$, thus $f(x) < \infty$. Therefore, for $t > 0$

$$x \in tU = tf^{-1}([0,1)) \Leftrightarrow \frac{x}{t} \in f^{-1}([0,1))$$
$$\Leftrightarrow f\left(\frac{x}{t}\right) \in [0,1) \Leftrightarrow \frac{1}{t}f(x) \in [0,1) \Leftrightarrow f(x) \in [0,t),$$

i.e. $x \in tU \Leftrightarrow f(x) \in [0,t)$ for $t > 0$.

Thus

$$A_x = \{t > 0 : x \in tU\} = \{t > 0 : f(x) \in [0,t)\},$$

whence

$$p(x) = \inf A_x = \inf\{t > 0 : f(x) \in [0,t)\} = f(x).$$

Now let $f(x) = \infty$. For the contrary, assume that $p(x) < \infty$. Then by the definition of p,

$$\{t > 0 : x \in tU\} \neq \emptyset,$$

which implies that there exists $t > 0$ such that $x \in tU$, thus also

$$\frac{x}{t} \in U = f^{-1}([0,1)),$$
$$\frac{1}{t}f(x) \in [0,1)$$

and finally $f(x) \in [0,t)$ which is impossible. This completes the proof. □

The next result reads as follows.

Theorem 4. *Let X, f, U be as in the Theorem 3. If U is sequentially bounded, then $f = p$ is a generalized norm.*

Proof. Assume that $f(x) = p(x) = 0$. One has

$$p(x) = \inf\{t > 0 : x \in tU\} = 0,$$

therefore, for every $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$, there exists $0 < t_n < \frac{1}{n}$ such that $x \in t_n U$, i.e. $x = t_n u_n$, where $u_n \in U$ for $n \in \mathbb{N}$. Since U is bounded

$$x = t_n u_n \to 0 \quad \text{as} \quad n \to \infty,$$

thus $x = 0$, as claimed. □

4 Continuity of the Generalized Minkowski Functionals

Let's note the following

Theorem 5. *Let the assumptions of Theorem 1 be satisfied. Then*

$$p \text{ is continuous at zero} \quad \Rightarrow \quad 0 \in \mathit{int}\, U. \tag{13}$$

Proof. From the assumption, for $0 < \varepsilon < 1$, there exists a neighbourhood V of zero such that

$$p(u) < \varepsilon \quad \text{for} \quad u \in V.$$

But $p(u) < 1$ for $u \in V$, whence by Lemma 1 $u \in U$, and therefore, $V \subset U$, which proves the implication (13). □

We have also

Theorem 6. *Let the assumptions of Theorem 1 be satisfied. Then*

$$0 \in \mathit{int}\, U \quad \Rightarrow \quad p \text{ is continuous at zero.} \tag{14}$$

Proof. Let U_0 be a neighbourhood of zero such that $U_0 \subset U$. For the contrary, suppose that there exists an $\varepsilon_0 > 0$ such that for every neighbourhood V of zero there exists an $x \in V$ with $p(x) \geq \varepsilon_0$. Take $V = V_n = \frac{1}{n} U_0$, $n \in \mathbb{N}$ (clearly V_n is a neighbourhood of zero). Then there exists an $x_n \in \frac{1}{n} U_0 \subset \frac{1}{n} U$, such that

$$p(x_n) \geq \varepsilon_0 \quad \text{for} \quad n \in \mathbb{N}. \tag{15}$$

Take n such that $\frac{1}{n} < \varepsilon_0$. Thus, one has

$$p(x_n) = \inf\{t > 0 \colon x_n \in tU\} \leq \frac{1}{n},$$

i.e.

$$p(x_n) \leq \frac{1}{n} < \varepsilon_0$$

which contradicts the inequality (15) and completes the proof. □

We have even more.

Theorem 7. *Let the assumptions of Theorem 1 be satisfied. Then*

$$0 \in \mathit{int}\, U \quad \Rightarrow \quad p \text{ is continuous.} \tag{16}$$

Proof. First of all, observe that since there exists a neighbourhood V of zero, contained in U, then for $x \in X$

$$\frac{1}{n} x \in V \quad \text{for} \quad n > n_0,$$

and hence

$$A_x = \{t > 0 \colon x \in tU\} \neq \emptyset \quad \text{for all } x \in X.$$

Therefore, we have $p(x) < \infty$ for any $x \in X$. Since p is also convex and, by Theorem 6, p is continuous at zero, then by the famous theorem of Bernstein–Doetsch (see e.g. [1]), p is continuous in X, which ends the proof. □

Remark 4. To see that the condition $0 \in \mathit{int}\, U$ is essential in Theorems 5 and 6, the reader is referred to Example 1.

Eventually, taking into account Theorems 5 and 7, we can state the following useful result about the continuity of the generalized Minkowski functionals.

Proposition 2. *Under the assumptions of Theorem 1, the equivalence*

$$p \text{ is continuous} \quad \Leftrightarrow \quad 0 \in \mathit{int}\, U \tag{17}$$

holds true.

5 Kolmogorov Type Result

Let X be a linear space (over $\mathbb{R}$ or $\mathbb{C}$—the set of all complex numbers) and a generalized metric space. We say that X is a generalized linear-metric space, if the operations of addition and multiplication by constant are continuous, i.e. if $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$ and $tx_n \to tx$ (with respect to a generalized metric in X).

For example, if generalized metric is introduced by a generalized norm, then we get a generalized linear-metric space.

We shall prove the following.

Theorem 8. *Let* (X, ϱ) *be a generalized linear-metric space over* $\mathbb{R}$*. Suppose that* $U \subset X$ *is an open, convex and sequentially bounded set. Then there exists a generalized norm* $\|\cdot\|$ *such that the generalized metric induced by this norm is equivalent to a generalized metric* ϱ*.*

Proof. Take a point $x_0 \in U$, then

$$V := (U - x_0) \cap (x_0 - U)$$

is an open, convex, symmetric and sequentially bounded subset of X (the details we omit here).

Define

$$\|x\| := \begin{cases} \inf\{t > 0 : x \in tV\}, & x \in X, \text{ if } A_x \neq \emptyset, \\ \infty, & \text{if } A_x = \emptyset. \end{cases} \tag{18}$$

By Theorem 2 we see that this function is a generalized norm.

At first we shall show the implication:

$$\varrho(x_n, 0) \to 0 \quad \Rightarrow \quad \|x_n\| \to 0. \tag{19}$$

To this end, take $\varepsilon > 0$. Then the set εV is also open: for it, because $f(x) = \frac{1}{\varepsilon}x$, $x \in X$, is a continuous function and

$$f^{-1}(V) = \varepsilon V,$$

we see that also εV is open. Therefore, $x_n \in \varepsilon V$ for $n > n_0$ and consequently

$$\|x_n\| < \varepsilon \quad \text{for} \quad n > n_0$$

i.e. (19) is satisfied.

Conversely, assume that $\|x_n\| \to 0$ as $n \to \infty$. By the definition (18) for every $\varepsilon_n = \frac{1}{n}$, $n > n_0$, there exists $t_n > 0$ such that

$$\|x_n\| \le t_n < \|x_n\| + \varepsilon_n \quad \text{and} \quad x_n \in t_n V.$$

Let $\varepsilon_n \to 0$, then $t_n \to 0$ as $n \to \infty$. Also $\frac{x_n}{t_n} \in V$, but since V is bounded, then

$$t_n\left(\frac{x_n}{t_n}\right) = x_n \to 0 \quad \text{as} \quad n \to \infty$$

in the generalized metric ϱ thus $\varrho(x_n, 0) \to 0$, which ends the proof. □

Remark 5. If ϱ is a metric, from Theorem 7, we get the Kolmogorov result (see [4]).

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A Network Design Model Under Uncertainty

E. D'Amato, E. Daniele, and L. Mallozzi

In Honor of Constantin Carathéodory

Abstract In this paper we present a cooperative game theoretical model for the well-known problem of network design. There is a multi-commodity network flow problem for each subset of players, who optimize the design of the network. Each player receives a return for shipping his commodity, and we consider the possibility to have uncertainty in this return. A cooperative game under interval uncertainty is presented for the model, and the existence of core solutions and approximate core solutions is investigated.

1 Introduction

Design and formation of networks is a topical matter because of the increased importance that issues as peer-to-peer connections, job-scheduling on computer cluster, and traffic routing are gaining.

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Regardless of the specific kind of network, a first distinction could be done by observing the behavior of the agents (or users, player) that operate within the network.

They could operate in a selfish, or noncooperative way, when they build or maintain a large network, by paying for available edges, independently of the other agents. In these cases, it is possible to measure the inefficiency, or suboptimality, by means of the price of anarchy (POA) based on the ratio between the objective functions of the worst Nash equilibrium case and the optimal solution [8, 13, 21].

On the other hand, a more realistic behavior for the agents would consider the change of form coalitions by means of strategic actions that lead to major profit for all the members of the consortium. In addition to that, a pragmatic policy from external authorities could be included, such as incentives to agents' cooperation [4].

Having this in mind, a network design model could be regarded as a cooperative game model [23] which has been shown to offer a general description for decision-making process and economic interaction of players, such as those exhibited in, for example, general welfare and procure-production games, as well as investment-production and activity selection games. For these activity optimization games, the natural condition involving the complementarity among the decision variables of the players causes the game to possess a convex property (also denoted as supermodularity), which usefulness for problems involving the design of a network to accommodate multi-commodity flows as been reported in [25].

This case represents the situation for which, through the considered network, the flow units are not all of the same commodities, thus requiring a balance at node level between inflow and outflow for each commodity separately. To overcome the single-commodity case, it required the introduction for each player of a specific origin-destination pair (denoted with *OD* in the following, also called source-sink pair).

We focus on network design games, namely, cooperative games, where players share the profit of shipping some commodity from a given origin to a given destination. The profit is the revenue minus the cost of installing infrastructures on edges, in order to ship the commodity. Since the players may use different paths, there is the possibility to cooperate and design the optimal network satisfying the requests of all the players and minimizing the cost.

As proved in literature [19, 22, 25], under suitable assumptions, the game is convex and the core is nonempty. So the proposed solution concept could be the core. One could consider also other solution concepts, as the Nucleolus or the Shapley value.

In our paper, we introduce uncertainty in the network design game. More precisely, as it happened in real situations, players do not know exactly the revenue they get from shipping, but they can estimate a possible lower bound of the revenue and a possible upper bound. We present the cooperative interval-valued network design game.

Several papers about cooperative interval-valued games have appeared recently [1–3, 5–7]. There is a cooperative game with an interval-valued characteristic function, i.e., the worth of a coalition is not a real number, but a compact interval of real numbers.

This means that one observes a lower bound and an upper bound of the worth of the considered coalitions. This is very important, for example, from a computational and algorithmic point of view where the numerical quantities are determined in terms of lower bounds and upper bounds.

The definition of interval-valued cooperative games has been given in [3], together with several notions of balancedness and cores by using selections of such games. In [1, 2, 5, 6], the authors investigate several interval solutions and convexity properties for the class of interval-valued games.

In this paper we study the cooperative interval-valued network design game, discussing its properties and some existence results for interval core solutions, as well as some computational procedures to evaluate numerically the solutions. In Sect. 2 the crisp model, i.e., the model without uncertainty, is presented, while in Sect. 3 the model is considered under uncertainty: existence and computational procedure of the solutions are discussed together with some illustrative examples. In the concluding section, we summarize the major results and address some future research developments.

2 The Network Design Model

Let us consider a set of players $N = \{1, \ldots, n\}$ $(n \in \mathbb{N})$ and a graph $G = (V, E)$ where $V = \{1, \ldots, k\}$ is the finite set of vertexes or nodes and $E = \{1, \ldots, m\}$ the set of directed edges (k, m are natural numbers). Each player $i \in N$ has to ship $h_i > 0$ units of a commodity i between a given ordered pair of nodes (o_i, d_i) with $o_i, d_i \in V$, for any $i \in N$. We denote $h = (h_1, \ldots, h_n)$ and $OD = \big((o_1, d_1), \ldots, (o_n, d_n)\big)$ the vectors of $\mathbb{R}^n$ and $\mathbb{R}^{2n}$, respectively.

From the shipment, player i receives a return r_i. The initial capacity of each edge of E for accommodating shipments of the players' commodities is set at zero, and there is an investment cost $c_j(x)$ for installing x units of capacity on edge $j \in E$. Any coalition $S \subseteq N$ of players could construct capacities on the edges of E to create a capacitated network in which the requirements of any player of S are satisfied (admissible network). Coalition S chooses the admissible network of minimum cost. Define for any player $i \in N$ the set

$$P_i = \{\text{path connecting } o_i \text{ and } d_i\}$$

and for any edge $j \in E$ the set

$$Q_j = \{\text{path of edges from } E \text{ including } j\}.$$

A path is the union of consecutive edges (ijk is the path given by edge i, then edge j, then edge k).

For each player i in the coalition S, fix a path $p_i \in P_i$; then we consider the quantity:

$$\sum_{j\in E} c_j \left(\sum_{i:i\in S, p_i\in Q_j} h_i \right)$$

that represents the sum of the costs of each edge j by considering all the players of coalition S that are using that edge j when they choose the paths $p_i \in P_i$, $\forall i \in S$.

We denote by $r = (r_1, \ldots, r_n)$ the revenue profile vector ($r_i > 0$) and $IC = \{c_1, \ldots, c_m\}$ the installing cost functions ($c_j : [0, +\infty[\to [0, +\infty[$, $c_j(0) = 0$, c_j increasing in $[0, +\infty[$). We call the tuple (N, G, h, OD, r, IC) a network design situation.

Definition 1. Given a network design situation (N, G, h, OD, r, IC), we define the network design cooperative game $< N, v >$ where $N = \{1, \ldots, n\}$ is the set of the players and $v : 2^N \to \mathbb{R}$ is the characteristic function such that $v(\emptyset) = 0$ and for each coalition $S \subseteq N$ the worth of the coalition is given by

$$v(S) = \sum_{i\in S} r_i - c(S)$$

being $c(S)$ the cost of the coalition S defined as

$$c(S) = \min_{p_i:p_i\in P_i, \forall i\in S} \sum_{j\in E} c_j \left(\sum_{i:i\in S, p_i\in Q_j} h_i \right).$$

A natural solution concept for this cooperative game is the core [19, 22, 24]. The core $\mathscr{C}(v)$ of the cooperative game $< N, v >$ is defined by

$$\mathscr{C}(v) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N), \ \sum_{i\in S} x_i \geq v(S), \forall\ S \subseteq N\}.$$

The core of a network design cooperative game may be empty as in the following example [25].

Example 1. Let us consider a network design situation (N, G, h, OD, r, IC) where $N = \{1, 2, 3\}$, $V = \{1, 2, 3, 4, 5, 6, 7\}$, and $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (with $G = (V, E)$), $h = (1, 1, 1)$, $OD = \big((1, 7), (3, 7), (5, 7)\big)$, $r = (3, 2, 4)$, $c_j(x) = \sqrt{x}, j \in E$. In this case (see Fig. 1)

$$P_1 = \{18, 67, 123457, 1239\},\ P_2 = \{2167, 28, 3457, 39\},\ P_3 = \{57, 49, 432167, 4328\}$$

$$Q_1 = \{18, 123457, 1239, 2167, 432167\},\ Q_2 = \{123457, 1239, 2167, 28, 432167, 4328\},$$

$$Q_3 = \{123457, 1239, 3457, 39, 432167, 4328\},\ Q_4 = \{123457, 3457, 49, 432167, 4328\},$$

$$Q_5 = \{123457, 3457, 57\},\ Q_6 = \{67, 2167, 432167\},$$

$$Q_7 = \{67, 123457, 2167, 3457, 57, 432167\},\ Q_8 = \{18, 28, 4328\},\ Q_9 = \{1239, 39, 49\}$$

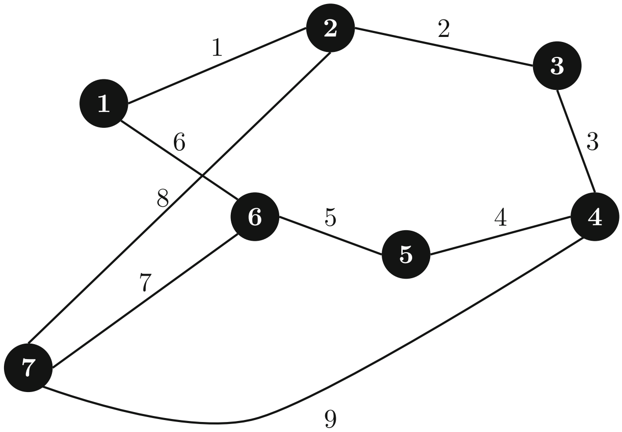

Fig. 1 Scheme for the network described into Example 1 on page 84

and it is easy to compute

$$c(\{1\}) = c(\{2\}) = c(\{3\}) = 2$$
$$c(\{1,2\}) = c(\{2,3\}) = c(\{1,3\}) = 2 + \sqrt{2}$$
$$c(\{1,2,3\}) = 4 + \sqrt{2}.$$

The characteristic function is

$$v(\{1\}) = 1, v(\{2\}) = 0, v(\{3\}) = 2$$
$$v(\{1,2\}) = 3 - \sqrt{2}, v(\{2,3\}) = 4 - \sqrt{2}, v(\{1,3\}) = 5 - \sqrt{2}$$
$$v(\{1,2,3\}) = 5 - \sqrt{2}$$

and the core of this game is empty. If we consider $c_j(x) = x^2, j \in E$, the characteristic function is

$$v(\{1\}) = 1, v(\{2\}) = 0, v(\{3\}) = 2$$
$$v(\{1,2\}) = 1, v(\{2,3\}) = 2, v(\{1,3\}) = 3$$
$$v(\{1,2,3\}) = 3$$

and the vector $(1, 0, 2)$ is in the core.

In the special case where the following assumption is satisfied

$$\text{each } P_i \text{ consists of a single path } p_i' \qquad (H)$$

the characteristic function is

$$v(S) = \sum_{i \in S} r_i - \sum_{j \in E} c_j \left(\sum_{i: i \in S, p'_i \in Q_j} h_i \right),$$

by assuming concave cost functions $c_j, j \in E$, the cooperative game is a convex game and there exist core solutions ([25] part (a) of Lemma 2.6.2, Theorem 2.6.4, and part (b) of Lemma 2.6.1). Recall that a cooperative game $< N, v >$ is convex if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \ \forall S, T \in 2^N$$

and if the game is convex, the core is nonempty [19, 22, 24].

Remark 1. Let us observe that without hypothesis (H) the network design situation, given the installing cost functions IC and without revenue, is nothing but the congestion situation of Mallozzi [15] and Monderer [17], studied from a noncooperative point of view: there exists for such games a pure Nash equilibrium, because they are potential games.

2.1 A Computational Procedure for the Network Design Game

A numerical procedure to compute core solutions, given a network design situation (N, G, h, OD, r, IC), can be divided in two phases:

- 2^n combinatorial optimization problems must be solved to find the shortest paths for players, considering all the feasible coalitions;
- once collected all the coalition values, the core can be computed by using a constrained optimization procedure.

In a network design situation, the weight of each edge is a function of the number of players passing through it. In this work, an optimization algorithm based on ant colony paradigm (AC) to solve the shortest path with variable weight on edges has been developed [11, 16, 20].

Ant colony algorithm is a simulated artificial version of ant social behavior. Ants' organization and their related interaction rules are replied in a computer system, and the ability to find shortest paths (or equivalent cost functions) in a well-defined environment is exploited in a general way.

Let us consider the minimization of a typical N-P hard combinatorial problem. Artificial ants will construct solutions making random steps on a graph $G = (V, E)$ where V represents the set of nodes and E the set of edges.

A pheromone trail τ_j and a heuristics η_j can be associated to each edge j whose endpoints are the nodes a_j and b_j. The pheromone trail represents a sort of shared memory for the ant colony, spread over the chosen paths, while the heuristics η_j is an immutable information. In most cases, η_j is strictly connected to the cost function, giving to the ants a valuable support in choosing the best component routes.

The AC algorithm can be seen as a succession of several procedures. Analogously to other natural algorithms, epochs represent the time base of evolution. For each epoch, ants concurrently build solutions moving themselves on the construction graph, on the basis of the pheromone trails and heuristic information. At each construction step, the ants choose the nodes to switch on, via a probabilistic choice biased on a proportional-random rule:

$$p_j^k = \frac{[\tau_j]^\alpha [\eta_j]^\beta}{\sum_{l \in L} [\tau_l]^\alpha [\eta_l]^\beta}, \tag{1}$$

where p_j^k is the probability of transition on edge j for the k-th ant via node b_j, τ is the pheromone matrix, η is the heuristic information matrix that depends on the specific problem, exponents α and β are parameters used to bias the influence of pheromone trail, and L is the set of edges connected to node a_j.

An artificial ant is equipped with the following abilities/features:

- Path exploration in searching for the best solution;
- A general-purpose local memory used for: (1) partial route storage, (2) feasible solution construction, (3) heuristics and objective function evaluation support, and (4) reverse path reconstruction;
- An initial state and one or more termination rules. Typically, an initial state consists of a void or a single-element array.

In the state $x_r = \{x_{r-1}, j\}$, the ant moves to a node j of its neighborhood $N^k(x_r)$ if none of termination rules is verified. When one of termination criteria is met, the ant stops. This rule may be varied if a different behavior is preferred for the ant, i.e., the construction of infeasible solutions is allowed or not. This is simply obtained modifying the heuristics.

The ant chooses a move using either heuristics or pheromone information previously deposited onto the tracks by the colony; at the end of its journey, ant upgrades the pheromone track associated to the connection edges, adding an amount of pheromone that depends on the length of the path.

It should be remarked that all the ants move in parallel and independently of one another, except for the pheromone-tracking phase, that is performed in a synchronous mode. This circumstance can be seen as a sort of shared learning, where each ant does not adapt itself but adapt the representation of the problem for the other ones.

Concerning the core solutions, we solve an optimization problem for the payoff function:

$$x_1 + \cdots + x_n - v(N)$$

constrained with $2^n - 2$ conditions of coalitional core properties

$$\sum_{i \in S} x_i \geq v(S), \forall\, S \subset N.$$

3 The Model Under Uncertainty

Let us now recall basic definitions of interval analysis [18]. Let IR be the set of real intervals $\mathrm{IR} = \{[\underline{I}, \bar{I}] \subset \mathbb{R}, \underline{I}, \bar{I} \in \mathbb{R}, \underline{I} \leq \bar{I}\}$. For any $I \in \mathrm{IR}$, the values $\underline{I}, \bar{I}$ are called lower and upper bound, respectively, of the interval I. Two intervals are equal if their respective bounds coincide. For any real number $k \in \mathbb{R}$, we also denote $k = [k, k]$.

Let $I, J \in \mathrm{IR}$; we consider the following partial order $I \succeq J$ iff $\underline{I} \geq \underline{J}$ and $\bar{I} \geq \bar{J}$. If $I = [a, a],\ a \in \mathbb{R}$ and $J = [b, b],\ b \in \mathbb{R}$, we have that $I \succeq J \Longleftrightarrow a \geq b$. Remark that $\succeq$ is a partial order relation in IR: for example, $[0, 2] \succeq [-1, 2]$, but $[-2, 3]$ and $[-1, 2]$ are not comparable with respect to $\preceq$.

The sum operation is defined in IR as follows:

$$I + J = [\underline{I} + \underline{J}, \bar{I} + \bar{J}].$$

The product of an interval $I \in \mathrm{IR}$ times a nonnegative real number $\alpha \geq 0$ is defined as follows:

$$\alpha I = [\alpha \underline{I}, \alpha \bar{I}].$$

We denote by IR_+ the set of intervals $I \succeq [0, 0]$ and by IR^n_+ the Cartesian product $\mathrm{IR}^n_+ = \{(A_1, \ldots, A_n), A_i \in \mathrm{IR}_+, i = 1, \ldots, n\}$. Interval analysis has been used in several applicative contexts (see, e.g., [12, 14, 18]); we shall consider this approach in a game theoretical setting.

A cooperative interval game is an ordered pair $< N, w >$ where $N = \{1, \ldots, n\}$ is the set of the players and $w : 2^N \rightarrow \mathrm{IR}$ is the characteristic function such that $w(\emptyset) = [0, 0]$. We denote by $w(S) = [\underline{w}(S), \bar{w}(S)]$ the worth of the coalition S. The border games $< N, \underline{w} >$, $< N, \bar{w} >$ and the length game $< N, |w| >$ are classical TU-games (transferable utility) with characteristic functions $\underline{w}$, $\bar{w}$ and $|w| = \bar{w} - \underline{w}$, respectively.

A cooperative interval game $< N, w >$ is convex if it is supermodular, i.e.

$$w(S \cup T) + w(S \cap T) \succeq w(S) + w(T),\ \forall S, T \in 2^N,$$

and the length game $< N, |w| >$ is also supermodular (convex) in the classical definition, i.e.

$$|w|(S \cup T) + |w|(S \cap T) \geq |w|(S) + |w|(T),\ \forall S, T \in 2^N.$$

A cooperative interval game $< N, w >$ is size monotonic if $< N, |w| >$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for any $S, T \in 2^N$ with $S \subset T$.

If $|I| \geq |J|$ we define the difference interval $I - J = [\underline{I} - \underline{J}, \bar{I} - \bar{J}]$.

The definition of interval-valued cooperative games has been given in [3], together with several notions of balancedness and cores by using selections of such games. In [1, 2, 5, 6], the authors investigate several interval solutions for the class of interval-valued games and convexity properties of such games.

As in the crisp case, we are interested in the core solution concept for the interval game, namely, the interval core that is defined by

$$\mathscr{C}(w) = \left\{ (I_1, \ldots, I_n) \in \mathrm{IR}^n : \sum_{i \in N} I_i = w(N),\ \sum_{i \in S} I_i \succeq w(S), \forall\ S \subseteq N \right\}.$$

The interval core of an interval network design cooperative game may be empty as in the following example.

Let us consider (N, G, h, OD, R, IC) a network design situation as in the previous section except for the revenue vector that in this case will be an interval vector, namely, we assume that from the shipment player i receives a return that is between a lower assigned bound $\underline{r}_i$ and an upper bound $\bar{r}_i$, and $R = \left([\underline{r}_1, \bar{r}_1], \ldots, [\underline{r}_n, \bar{r}_n]\right)$. We compute the cost $c(S)$ of any coalition S as in the previous section. We call the tuple (N, G, h, OD, R, IC) an interval network design situation.

Definition 2. Given an interval network design situation (N, G, h, OD, R, IC), we define the interval network design cooperative game $< N, w >$ where $N = \{1, \ldots, n\}$ is the set of the players and $w : 2^N \to \mathbb{R}$ is the characteristic function such that $w(\emptyset) = [0, 0]$, and for each coalition $S \subseteq N$, the worth of the coalition is given by

$$w(S) = \sum_{i \in S} [\underline{r}_i, \bar{r}_i] - c(S) = \left[\sum_{i \in S} \underline{r}_i - c(S), \sum_{i \in S} \bar{r}_i - c(S) \right]$$

being $c(S)$ the cost of the coalition S defined as

$$c(S) = \min_{p_i : p_i \in P_i, \forall i \in S} \sum_{j \in E} c_j \left(\sum_{i : i \in S, p_i \in Q_j} h_i \right)$$

Note that the operations in the definition of the interval characteristic function are well defined, since we can always subtract a real number from an interval.

We can prove the existence of interval core solutions under suitable assumptions specified in the following proposition.

Proposition 1. *Let (N, G, h, OD, R, IC) be an interval network design situation where $c_j, j \in E$ are concave cost functions. Under assumption (H), the interval core of the interval cooperative game $< N, w >$ is not empty.*

Proof. The border games $< N, \underline{w} >$ and $< N, \bar{w} >$, where $\underline{w}(S) = \sum_{i \in S} \underline{r}_i - c(S)$ and $\bar{w}(S) = \sum_{i \in S} \bar{r}_i - c(S)$, are convex games as remarked in Sect. 2. Let us consider $(\underline{x}_1, \ldots, \underline{x}_n) \in \mathscr{C}(\underline{w})$ and $(\bar{x}_1, \ldots, \bar{x}_n) \in \mathscr{C}(\bar{w})$: the interval vector $(I_1, \ldots, I_n) = ([\underline{x}_1, \bar{x}_1], \ldots, [\underline{x}_n, \bar{x}_n])$ is in the interval core.

Nevertheless, the interval core can be nonempty if assumption (H) and concavity of the cost functions are violated as in the following example.

Example 2. Let us consider the interval network design situation (N, G, h, OD, R, IC) where $N = \{1, 2, 3\}$, $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $h = (1, 1, 1)$, $OD = \big((1, 7), (3, 7), (5, 7)\big)$, $c_j(x) = \sqrt{x}, j \in E$, and R is a vector of real intervals $R = \big([2, 3.5], [1.5, 2], [2, 4]\big)$. By using the interval algebra, the characteristic function is

$$
\begin{aligned}
w(\{1\}) &= [0, 1.5], w(\{2\}) = [-0.5, 0], w(\{3\}) = [0, 2]\\
w(\{1, 2\}) &= [1.5 - \sqrt{2}, 7.5 - \sqrt{2}], w(\{2, 3\}) = [1.5 - \sqrt{2}, 4 - \sqrt{2}],\\
w(\{1, 3\}) &= [2 - \sqrt{2}, 5.5 - \sqrt{2}]\\
w(\{1, 2, 3\}) &= [1.5 - \sqrt{2}, 5.5 - \sqrt{2}]
\end{aligned}
$$

and the interval core of this game is empty. If we consider $c_j(x) = x^2, j \in E$, the characteristic function is

$$
\begin{aligned}
w(\{1\}) &= [0, 1.5], w(\{2\}) = [-0.5, 0], w(\{3\}) = [0, 2]\\
w(\{1, 2\}) &= [-0.5, 1.5], w(\{2, 3\}) = [-0.5, 2], w(\{1, 3\}) = [0, 3.5]\\
w(\{1, 2, 3\}) &= [-0.5, 3.5]
\end{aligned}
$$

and the interval vector $\big([0, 1.5], [-0.5, 0], [0, 2]\big)$ is in the interval core $\mathscr{C}(w)$.

3.1 A Computational Procedure for the Interval Network Design Game

The computational procedure presented in Sect. 2 can be easily modified for an interval network design cooperative game. By using the definition of order relation between intervals, we consider two inequalities between real numbers to compare two intervals. The rest is similar to the procedure used in the crisp case.

Example 3. Let (N, G, h, OD, R, IC) be an interval network design situation (see Fig. 2) where $N = \{1, \ldots, 4\}$, $V = \{1, \ldots, 12\}$ and $E = \{1, \ldots, 14\}$ $(G = (V, E))$, $h_i = 1, i \in N$, and

$$
\begin{aligned}
OD &= \big((1, 3), (8, 10), (9, 12), (10, 12)\big),\\
R &= \big([4, 4.5], [4, 4], [3, 3.5], [5, 5]\big),
\end{aligned}
$$

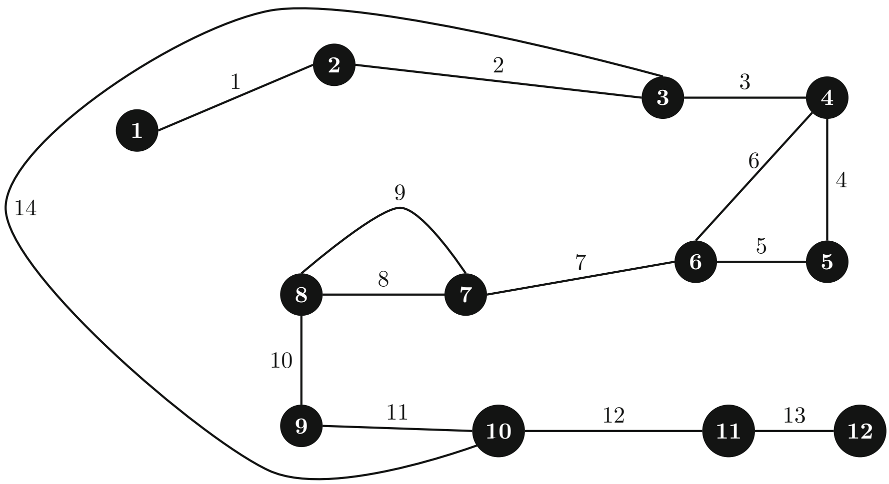

Fig. 2 Scheme for the network described into Example 3 on page 90

$c_j(x) = x, j \in E$. In this case, one can compute numerically $w(N) = [7, 8]$, and we have the following interval core

$$\mathscr{C}(w) = \{[2, 2.5], [2, 2], [0, 0.5], [3, 3]\}.$$

In the case where $c_j(x) = \sqrt{x}, j \in E$, the interval core is the following:

$$\mathscr{C}(w) = \{[2, 2.5], [2, 2.57], [1.76, 1.69], [3, 3]\}.$$

Note that the worth of the grand coalition in this case is $w(N) = [8.76, 9.76]$.

4 Conclusions

In this paper we considered a network design problem for which a cooperative game has been defined and core solutions have been considered. We presented the game in the case where uncertainty may affect the data of the network design problem, in line with previous papers [6, 14]. In particular, we consider the possibility that the revenue of each player R_i can be evaluated with an error, and we know a lower $\underline{r}_i$ and an upper bound $\bar{r}_i$ of the value, $R_i = [\underline{r}_i, \bar{r}_i]$. By using the interval algebra, we have defined the interval network design cooperative game, and interval core solutions have been considered for it. In analogous way, the interval game can be defined also in the case we have interval uncertainty on the revenue as well as on the costs, i.e. $C(S) = [\underline{c}(S), \bar{c}(S)]$, assuming that $|\sum_{i \in S} R_i| \geq |C(S)|$ for any coalition S. The case where this assumption is not satisfied needs different interval algebra tools and will be investigated in the future. Moreover, from a computational point of view, as done for noncooperative game models [9, 10], the multiplicity of the core solutions will be also studied.

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Optimal Rational Approximation Number Sets: Application to Nonlinear Dynamics in Particle Accelerators

Nicholas J. Daras and Michael N. Vrahatis

In Honor of Constantin Carathéodory

Abstract We construct optimal multivariate vectors of rational approximation numbers with common denominator and whose coordinate decimal expansion string of digits coincides with the decimal expansion digital string of a given sequence of mutually irrational numbers as far as possible. We investigate several numerical examples and we present an application in Nuclear Physics related to the beam stability problem of particle beams in high-energy hadron colliders.

1 Introduction

A central problem in number theory is how to construct "optimal" rational approximants to irrational numbers [49]. In spite of its simple formulation, the fraction that is closer to an irrational number than any other rational approximant with a smaller denominator depends strongly on the denominator of the convergent of the continued fraction expansion of the irrational number [28], and thus, from a numerical point of view, one might expect that the "optimal" rational approximants lack most of their practical usefulness appeal.

There is another independent reason advocating for this expectation. It is perhaps surprising that the "innocent" generalization of this problem to the *simultaneous rational approximants to several mutually irrational numbers* is considered a

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difficult, essentially unsolved problem in number theory. The precise generalization of rational approximants to a single irrational number is to define a sequence:

$$\left\{\sigma^{(k)} = \left(\sigma_1^{(k)}, \sigma_2^{(k)}, \dots, \sigma_n^{(k)}\right) \in \mathbb{Q}^n,\ \ k = 1, 2, \dots\right\}$$

of ordered sets of n rational numbers:

$$\sigma_j^{(k)} = \left(p_j^{(k)} / r^{(k)}\right),\ \ j = 1, 2, \dots, n,$$

each set with *common* denominator $r^{(k)} \in \mathbb{Z} \setminus \{0\}$, which converges to a given n-vector of mutually irrational numbers $a = (a_1, a_2, \dots, a_n) \in \mathbb{A}^n$. In this direction, S. Kim and S. Ostlund gave ordered sets of two rational approximants to pairs of mutually irrational numbers [29]. The ordered sets of two rational approximants generated by their algorithm are in fact the best ordered pairs of rational approximants relative to a criterion of weak convergence [29]. However, their algorithm does not always give "optimal" simultaneous rational approximation to mutually irrational numbers for any $n > 2$. For $n \geqslant 2$, the only well-known efficient approximation method reveals the Jacobi–Perron classical algorithm (JPA) [7]. Under general enough circumstances, this inductive algorithm generates sequences of optimal n-vectors containing mutually rational numbers with common denominator and approximating the given n-vector of irrational numbers. The approximation method is convergent, but the resultant construction depends strongly on the associated function defining the algorithm's transformation. So, numerators and (common) denominator in Jacobi–Perron rational approximation n-vectors are completely determined by this function, and no freedom is left.

The principal aim of the paper at hand is to show how rational approximation theory can be cleared of its strong dependence on "optimal" approximants and reconnected to original ideas of *numerical approximation*. To do so, we will investigate multivariate vectors of rational approximation numbers—the so-called Optimal Rational Approximation NUmber Sets or simply ORANUS—whose decimal expansion string of digits coincides with the decimal expansion digital string of mutually irrational numbers as far as possible. More precisely, we will look at a direct numerical construction of simultaneous rational approximations with *arbitrary common denominator*. *The advantage of these approximants over Jacobi–Perron approximants lies in the completely free choice of the common denominator which may lead to a better approximation.*

The paper is organized as follows. Section 2 gives a concise overview of JPA's classical applications. Section 3 recalls basic results of rational approximation to analytic functions, while Sect. 4 develops and analyzes the multivariate rational approximation method of the paper. In Sects. 5 and 6, we study the efficacy of the method, and in Sect. 7 we give an application related to the beam stability problem in circular particle accelerators. Finally, Sect. 8 summarizes and gives concluding remarks.

2 The Jacobi–Perron Algorithm

Let $\langle a^{(k)} \rangle \equiv \left(a^{(0)}, a^{(1)}, \ldots, a^{(k)}, \ldots\right)$ be a sequence of vectors in $\mathbb{R}^n$. Let also $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be any mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that

$$\Phi\left(a^{(k)}\right) = b^{(k)} = \left(b_1^{(k)}, b_2^{(k)}, \ldots, b_n^{(k)}\right) \Longrightarrow a_1^{(k)} \neq b_1^{(k)} \ \ (k = 0, 1, 2, \ldots).$$

Definition 1.

(i) The sequence $\langle a^{(k)} \rangle$ is called a *Jacobi–Perron algorithm* (in short JPA) of the vector $a^{(0)} \in \mathbb{R}^n$, if there exists a T-transformation of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that:

(a) $T\left(a^{(k)}\right) = a^{(k+1)}$ and

(b) $T\left(a^{(k)}\right) = \left(a_1^{(k)} - b_1^{(k)}\right)^{-1} \left(a_2^{(k)} - b_2^{(k)}, \ldots, a_n^{(k)} - b_n^{(k)}, 1\right)$.

In such a case, we shall call the sequence $\left\langle \Phi\left(a^{(k)}\right)\right\rangle = \langle b^{(k)} \rangle$ a *T-function*.

(ii) The T-function $\left\langle \Phi\left(a^{(k)}\right)\right\rangle = \langle b^{(k)} \rangle$ is said to be *P-bounded*, if there is a constant C independent of k and satisfying Perron's conditions $0 < 1/b_n^{(k)} \leqslant C$ and $0 \leqslant b_i^{(k)}/b_n^{(k)} \leqslant C$ (for any $i = 1, 2, \ldots, n$ and $k = 0, 1, \ldots$).

(iii) Define numbers $d_i^{(j)}$ as follows:

$$d_i^{(j)} = \begin{cases} d_i^{(j)} = \delta_{i,j}, & i,j = 0, 1, \ldots, n \\ d_i^{(n+1+k)} = \sum_{j=0}^{n} b_j^{(k)} d_i^{(k+j)} \ (b_0^{(k)} = 1), & i = 0, 1, \ldots, n;\ k = 0, 1, \ldots \end{cases}$$

where $\delta_{i,j}$ denotes the Krönecker's delta. Then:

(a) The JPA of the vector $a^{(0)}$ is said to be *convergent*, if

$$a_i^{(0)} = \lim_{k\to\infty} d_i^{(k)}/d_0^{(k)},$$

whenever $i = 1, 2, \ldots, n$.

(b) The JPA of the vector $a^{(0)}$ is said to be *ideally convergent*, if the sequences $\left\langle d_i^{(k)} - a_i^{(0)} d_0^{((k)}\right\rangle$ $(i = 1, 2, \ldots, n)$ are all null sequences. □

Notation 1. *It is clear that a JPA which is ideally convergent is also convergent if and only if* $\left|d_0^{(j)}\right| > 1$, *for any* $j > j_0$. *It follows from* $\left|d_i^{(k)} - a_i^{(0)} d_0^{(k)}\right| < \varepsilon$ *for* $j > j_0(\varepsilon)$ *that*

$$\left|a_i^{(0)} - \frac{d_i^{(k)}}{d_0^{(k)}}\right| < \frac{\varepsilon}{\left|d_0^{(k)}\right|} < \varepsilon \quad (i = 1, 2, \ldots, n).$$ □

Example 1. Let us consider the special case:

$$\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right] := \left(\left[a_1^{(k)}\right], \left[a_2^{(k)}\right], \ldots, \left[a_n^{(k)}\right]\right), \quad k = 0, 1, \ldots,$$

where $[x]$ denotes the integer part of x. For $n = 2$, the JPA with the T-function $\Phi\left(a^{(k)}\right) = b^{(k)} \equiv \left[a^{(k)}\right]$ becomes the Euclidean algorithm and yields the expansion of any real number by simple continued fractions. If the JPA of a vector $a^{(0)} \in \mathbb{R}^n$ is associated with the T-function $\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right]$, then $\left|d_0^{(j)}\right| > 1$ for any $j \geqslant n+1$ so that ideal convergence here always implies convergence. But, as the reader can easily verify, the JPA of a $a^{(0)} \in \mathbb{R}^n$ with the associated T-function $\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right]$ is always convergent, since in this case $a_n^{(k)} = \left(a_1^{(k-1)} - b_1^{(k-1)}\right)^{-1}$ $(k = 1, 2, \ldots)$ so that $b_n^{(k)} \geqslant 1$, and it is also easily verified that $0 \leqslant \left(b_i^{(k)}/(b_n^{(k)}\right) < 1$ for any $i = 1, 2, \ldots, n$ and $k = 0, 1, 2 \ldots$. But we can also always achieve that $b_n^{(0)} \geqslant 1$; thus, if $\left[a_n^{(0)}\right] = -l < 0$, then, substituting $a_n^{(0)'} = a_n^{(0)} + l + 1$, we obtain $b_n^{(0)'} = 1$; the same holds for $a_i^{(0)}$ $(i = 1, 2, \ldots, n-1)$. Since the JPA of $a^{(0)} \in \mathbb{R}^n$ with the T-function $\Phi\left(a^{(k)}\right) \equiv \left[a^{(k)}\right]$ is convergent, we obtain for a rational approximation of the $a_i^{(0)}$:

$$a_i^{(0)} = \lim_{k\to\infty} \frac{d_i^{(k)}}{d_0^{(k)}}, \quad \text{whenever} \quad i = 1, 2, \ldots n. \qquad \square$$

With the notation of Definition 1, the main convergence criterion of JPA can be stated as follows.

Theorem 1 ([7]). *The JPA of the vector $a^{(0)} \in \mathbb{R}^n$ is convergent if its T-function $\left\langle\Phi\left(a^{(k)}\right)\right\rangle = \langle b^{(k)}\rangle$ is P-bounded.* $\square$

3 Rational Approximation to Analytic Functions

Let $F(z) = \sum_{v=0}^{\infty} a_v^{(F)} z^v$ be a function analytic in the open disk $\Delta(0;\varrho) = \{z \in \mathbb{C} : |z| < \varrho\}$, and let Λ_F be the $\mathbb{C}$-linear defined on the space $\mathbb{P}(\mathbb{C})$ of all analytic polynomials by $\Lambda_F(x^v) := a_v^{(F)}$ $(v = 0, 1, 2, \ldots)$. By density, Λ_F extends on the space $\mathcal{O}(\overline{\Delta(0;[1/\varrho])})$ of the functions which are analytic in an open neighborhood of the closed disk $\overline{\Delta(0;[1/\varrho])}$ and holds that $F(z) = \Lambda_F\big((1-xz)^{-1}\big)$ for any $z \in \Delta(0;\varrho)$. If $p_k(x,z)$ is the unique Hermite polynomial with degree at most k, which interpolates $(1-xz)^{-1}$ at the $(k+1)$ points $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$, then the expression $\Lambda_F(p_k(x,z))$ is a function with numerator and denominator degrees at most k and $k+1$, respectively. In fact, by setting $V_{k+1}(x) = \gamma \prod_{i=0}^{k}(x - \pi_i)$

($\gamma \in \mathbb{C} \setminus \{0\}$), it is easily seen that $W_k(z) = \Lambda_F([V_{k+1}(x) - V_{k+1}(z)]/[x - z])$ is a polynomial in z of degree at most k, and we obtain

$$\Lambda_F(p_k(x,z)) = \left\{\widehat{W}_k(z)/\widehat{V}_{k+1}(z)\right\} := \left\{z^k W_k(z^{-1})/z^{k+1} V_{k+1}(z^{-1})\right\}.$$

This rational function, denoted by $(k/(k+1))_F(z)$, is characterized by the property that $F(z) - (k/(k+1))_F(z) = O\left(z^{k-1}\right)$ (as $z \to 0$) and is known as a *Padé-type approximant* to the Taylor series $\sum_{\upsilon=0}^{\infty} a_\upsilon^{(F)} z^\upsilon$, whereas every polynomial $V_{k+1}(x)$ is called a *generating polynomial* of this approximation [12–15, 25].

Remark 1. It is possible to construct Padé-type approximants with various degrees in the numerator and in the denominators [12, 13]. □

A natural problem connected with the choice of the generating polynomials is the convergence of such a sequence of rational approximants. It has been completely solved by M. Eiermann.

Theorem 2 ([18, 24]). *If the generating polynomials $V_k(x)$ satisfy*

$$\lim_{k\to\infty} \left[V_k(x)/V_k\left(z^{-1}\right)\right] = 0$$

uniformly on any compact subset of an open set $\Omega \subset \mathbb{C}^2$ containing $\{(x,0) : x \in \mathbb{C}\}$, then there holds

$$\lim_{k\to\infty} (k/(k+1))_F(z) = F(z),$$

uniformly on every compact subset of

$$\left\{z \in \mathbb{C} : \lim_{k\to\infty} \left[V_k\left(\xi^{-1}\right)/V_k\left(z^{-1}\right)\right] = 0, \quad \forall\, \xi \in \mathbb{C} \setminus \Delta(0;\varrho)\right\}. \qquad \square$$

A second reasonable question concerns the "optimal" choice of the interpolation points $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$ (equivalently, of the poles of the rational approximants). Some attempts to solve this problem have been made by Magnus [31]. A general answer is given in the following results.

Theorem 3 ([19]). *Let $z \in \Delta(0;\varrho) \setminus \{0\}$ and let k be a positive integer. If $\tilde{p}_k(x,z)$ is the unique polynomial of degree at most k, which interpolates $(1 - xz)^{-1}$ at the $(k+1)$ roots $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_k \in \mathbb{C}$ of the generating polynomial $\widetilde{V}_{k+1}(x) = \widetilde{V}_{k+1}^{(z)}(x) = x^{k+1} + 1/z(z^k - 1)x^k$, then*

(i) $|F(z) - \Lambda_F(\tilde{p}_k(x,z))| \leqslant |F(z) - \Lambda_F(p_k(x,z))|$, *for any Hermite polynomial $p_k(x,z) \in \mathbb{P}(\mathbb{C})$ in x and any function F analytic in the open disk $\Delta(0;\varrho)$.*

(ii) If moreover k is even and $0 < \varepsilon < \delta < \varrho$, then

$$\left\|F(z)-\Lambda_F(\tilde{p}_k(x,z))\right\|_2^{\delta,\varepsilon} \leqslant \left\|F(z)-\Lambda_F(p_k(x,z))\right\|_2^{\delta,\varepsilon},$$

for any Hermite polynomial $p_k(x,z) \in \mathbb{P}(\mathbb{C})$ in x and any function F analytic in the open disk $\Delta(0;\varrho)$. Here, we have used the notation $\left\|g(z)\right\|_2^{\delta,\varepsilon} := \left(\int_{\delta\leqslant|z|<\varepsilon} |g(z)|^2\, dz\right)^{1/2}$. □

In spite of these results, there is no analogous possibility to determine an "optimal" uniform choice for the interpolation system $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$, since a minimum for the uniform norm:

$$\left\|F(z)-(k/(k+1))_F\,(z)\right\|_\infty^{\delta,\varepsilon} := \sup_{\delta\leqslant|z|<\varepsilon} \left|F(z)-(k/(k+1))_F\,(z)\right|,$$

of the error on a compact ring $\overline{\Delta(0;\delta,\varepsilon)} = \{z \in \mathbb{C} : \delta \leqslant |z| \leqslant \varepsilon\}$ is obtained at the limit points $\pi_v = \infty$ $(v = 0, 1, \ldots, k-i)$ and $\pi_v = 0$ $(v = k-i+1, \ldots, k)$ for any $i = 0, 1, \ldots, k+1$. In particular, the only feasible optimal interpolation system is given by $\pi_0 = \pi_1 = \cdots = \pi_k = 0$ [19].

Remark 2. Another way to indicate "optimal" choice of the interpolation points $\pi_0, \pi_1, \ldots, \pi_k \in \mathbb{C}$ is by exactness properties of formal orthogonal polynomials [1–4, 6, 9, 11, 16, 20–23, 27, 30, 32, 33, 38, 41, 48, 50, 51]. □

4 ORANUS

In this section, we will show how to construct simultaneous rational approximants to several mutually irrational numbers.

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{A}^n$ be an irrational n-vector. Suppose that the irrational coordinates a_j of a are expressed in the decimal system:

$$a_j = a_j^{(0)}.a_j^{(1)}a_j^{(2)}a_j^{(3)}\ldots a_j^{(m)}\ldots = a_j^{(0)} + \left(a_j^{(1)}/10\right) + \cdots + \left(a_j^{(m)}/10^m\right) + \cdots$$

$(a_j^{(0)} \in \mathbb{N}$ and $0 \leqslant a_j^{(v)} \leqslant 9$ whenever $v = 1, 2, \ldots$ and $j = 1, 2, \ldots, n)$. The associated power series with integral coefficients

$$f_j(z) = \sum_{v=0}^{\infty} a_j^{(v)} z^v = a_j^{(0)} + a_j^{(1)} z + a_j^{(2)} z^2 + \cdots + a_j^{(m)} z^m + \cdots$$

converges uniformly on any compact subset of the open unit disk $\Delta(0;1)$.

We will approximate $f_j(z)$ by using rational approximants. To do so, let us consider the $\mathbb{C}$-linear functional defined on the space of all analytic polynomials by $\Lambda_{f_j}(x^v) := a_j^{(v)}$ whenever $v = 0, 1, \ldots$ If $V_{k+1}(x) = \gamma \prod_{v=0}^{k}(x-\pi_v)$ $(\gamma \in \mathbb{C}\setminus\{0\})$ for some suitably chosen complex numbers $\pi_0, \pi_1, \ldots, \pi_k$, the rational map:

$$\left((k/(k+1))_{f_1}(z), \ldots, (k/(k+1))_{f_n}(z)\right) = \left(\frac{\widehat{W}_k^{(1)}(z)}{\widehat{V}_{k+1}(z)}, \ldots, \frac{\widehat{W}_k^{(n)}(z)}{\widehat{V}_{k+1}(z)}\right),$$

is a *vector rational* approximant to the map $\left(f_1(z), f_2(z), \ldots, f_n(z)\right)$ with generating polynomial $V_{k+1}(x) = \gamma \prod_{v=0}^{k}(x - \pi_v)$. In particular, for $z = 10^{-1}$, we obtain an ordered set of n rational numbers:

$$\begin{aligned}&\left((k/(k+1))_{f_1}(10^{-1}), \ldots, (k/(k+1))_{f_n}(10^{-1})\right)\\ &= \left(pt_1 := \frac{\widehat{W}_k^{(1)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}, \ldots, pt_n := \frac{\widehat{W}_k^{(n)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}\right)\end{aligned}$$

with *arbitrary common denominator* $\widehat{V}_{k+1}(10^{-1})$ and approximating $a = (a_1, a_2, \ldots, a_n)$ in the sense that

$$\left(a_1 - \frac{\widehat{W}_k^{(1)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}, \ldots, a_n - \frac{\widehat{W}_k^{(n)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}\right) = \left(\underbrace{O\left(10^{-k-1}\right), \ldots, O\left(10^{-k-1}\right)}_{n-\text{times}}\right).$$

This means that the decimal coordinate expression:

$$\left(pt_1^{(0)}.\,pt_1^{(1)} \ldots pt_1^{(m_1)}, pt_2^{(0)}.pt_2^{(1)} \ldots pt_2^{(m_2)}, \ldots, pt_n^{(0)}.pt_n^{(1)} \ldots pt_n^{(m_n)}\right)$$

of $(pt_1, pt_2, \ldots, pt_n)$ $(= (pt_{1,k}, pt_{2,k}, \ldots, pt_{n,k}))$ agrees with the decimal coordinate expression:

$$\left(a_1^{(0)}.\,a_1^{(1)} \ldots a_1^{(m)} \ldots, a_2^{(0)}.a_2^{(1)} \ldots a_2^{(m)} \ldots, \ldots, a_n^{(0)}.\,a_n^{(1)} \ldots a_n^{(m)} \ldots\right)$$

of $(a_1, a_2, \ldots, a_n)$ up to the k first decimal digits:

$$\begin{aligned}&\left(pt_1^{(0)}.\,pt_1^{(1)} \ldots pt_1^{(k)}, pt_2^{(0)}.pt_2^{(1)} \ldots pt_2^{(k)}, \ldots, pt_n^{(0)}.pt_n^{(1)} \ldots pt_n^{(k)}\right)\\ &= \left(a_1^{(0)}.\,a_1^{(1)} \ldots a_1^{(k)}, a_2^{(0)}.a_2^{(1)} \ldots a_2^{(k)}, \ldots, a_n^{(0)}.a_n^{(1)} \ldots a_n^{(k)}\right).\end{aligned}$$

Definition 2. The vector:

$$\left(pt_1 := \frac{\widehat{W}_k^{(1)}}{\widehat{V}_{k+1}}, \ldots, pt_n := \frac{\widehat{W}_k^{(n)}}{\widehat{V}_{k+1}},\right) := \left(\frac{\widehat{W}_k^{(1)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}, \ldots, \frac{\widehat{W}_k^{(n)}(10^{-1})}{\widehat{V}_{k+1}(10^{-1})}\right)$$

is said to be an *ORANUS (Optimal Rational Approximation Number Set)* to the irrational vector a. It will be denoted by $(\text{ORANUS}/k)_a$. The polynomial:

$$V_{k+1}(x) = \gamma \prod_{v=0}^{k}(x - \pi_v) \quad (\gamma \in \mathbb{C} \setminus \{0\})$$

is called the *generating polynomial* of this approximation. □

We are thus in position to formulate a first theoretical method for approximating irrational vectors:

Framework for Approximating Irrational Vectors by ORANUS

1. Given an irrational n-vector:

$$a = \left(\underbrace{a_1^{(0)}.\,a_1^{(1)}a_1^{(2)}a_1^{(3)}\ldots}_{a_1},\ \underbrace{a_2^{(0)}.\,a_2^{(1)}a_2^{(2)}a_2^{(3)}\ldots}_{a_2},\ \ldots,\ \underbrace{a_n^{(0)}.\,a_n^{(1)}a_n^{(2)}a_n^{(3)}\ldots}_{a_n} \right)$$

2. Let $k \in \mathbb{N}$.
3. Let $\pi_0, \pi_1, \ldots, \pi_k$ be arbitrarily chosen complex numbers.
4. Put $V_{k+1}(x) = \gamma \prod_{v=0}^{k}(x - \pi_v)$ $(\gamma \in \mathbb{C} \setminus \{0\})$.
5. Set $\widehat{V}_{k+1} := 10^{-(k+1)} V_{k+1}(10)$.
6. For each $j = 1, 2, \ldots, n$ and each $v = 0, 1, 2, \ldots$, define

$$\Lambda_j(x^v) := a_j^{(v)}$$

and

$$\widehat{W}_k^{(j)} := 10^{-k} \Lambda_j \left(\frac{V_{k+1}(x) - V_{k+1}(10)}{x - 10} \right).$$

7. The ordered set:

$$(\text{ORANUS}/k)_a := \left(\frac{\widehat{W}_k^{(1)}}{\widehat{V}_{k+1}},\ \frac{\widehat{W}_k^{(2)}}{\widehat{V}_{k+1}},\ \ldots,\ \frac{\widehat{W}_k^{(n)}}{\widehat{V}_{k+1}} \right)$$

is a rational n-vector of n rational numbers:

$$pt_1 := \frac{\widehat{W}_k^{(1)}}{\widehat{V}_{k+1}},\ pt_2 := \frac{\widehat{W}_k^{(2)}}{\widehat{V}_{k+1}},\ \ldots,\ pt_n := \frac{\widehat{W}_k^{(n)}}{\widehat{V}_{k+1}}$$

with common denominator $\widehat{V}_{k+1}$. The decimal coordinate expansion:

$$\left(pt_1^{(0)}.pt_1^{(1)}\ldots pt_1^{(m_1)},\ pt_2^{(0)}.pt_2^{(1)}\ldots pt_2^{(m_2)},\ \ldots,\ pt_n^{(0)}.pt_n^{(1)}\ldots pt_n^{(m_n)} \right)$$

of $(\text{ORANUS}/k)_a$ matches the decimal coordinate expansion:

$$\left(a_1^{(0)}.a_1^{(1)}a_1^{(2)}a_1^{(3)}\ldots,\ a_2^{(0)}.a_2^{(1)}a_2^{(2)}a_2^{(3)}\ldots,\ \ldots,\ a_n^{(0)}.a_n^{(1)}a_n^{(2)}a_n^{(3)}\ldots \right)$$

of a up to the k first decimal digits:

$$pt_1^{(0)}.pt_1^{(1)}\dots pt_1^{(k)} = a_1^{(0)}.a_1^{(1)}\dots a_1^{(k)},$$
$$pt_2^{(0)}.pt_2^{(1)}\dots pt_2^{(k)} = a_2^{(0)}.a_2^{(1)}\dots a_2^{(k)},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$pt_n^{(0)}.pt_n^{(1)}\dots pt_n^{(k)} = a_n^{(0)}.a_n^{(1)}\dots a_n^{(k)}.$$

8. End.

5 Convergence of ORANUS' Sequences

A main question on ORANUS' asymptotic behavior is the "optimal" choice of the interpolation points $\pi_0, \pi_1, \dots, \pi_k$. We will study this question in the next section. Another problem connected with the choice of the π_v's is the problem of convergence of a given vector $a = (a_1, a_2, \dots, a_n)$ of n irrational numbers. According to Theorem 2, we have:

Theorem 4. *If the generating polynomials* $V_k(x) = \gamma \prod_{v=0}^{k-1}(x-\pi_v)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ *satisfy*

$$\lim_{k\to\infty} \left[V_k(x)/V_k\left(z^{-1}\right)\right] = 0,$$

uniformly on any compact subset of an open set $\Omega \subset \mathbb{C}^2$ *containing*

$$\mathbb{C} \times \{0\} \cup \Delta(0;1) \times \{10^{-1}\},$$

then it holds

$$\lim_{k\to\infty} (\mathrm{ORANUS}/k)_a = \lim_{k\to\infty} \left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1}, \dots, \widehat{W}_k^{(n)}/\widehat{V}_{k+1}\right) = (a_1, \dots, a_n). \qquad \square$$

We shall now give some examples of ORANUS' sequences and apply Theorem 4 to these special cases.

Example 2. Let the generating polynomials $V_k(x) = \gamma \prod_{v=0}^{k-1}(x-\pi_v)$ $(\gamma \in \mathbb{C} \setminus \{0\})$ have the form:

$$V_k(x) = (x-\beta)^k \ \ (\beta \in \mathbb{C},\ k = 0, 1, 2, \dots).$$

If the complex number β is chosen so that

$$|10-\beta| \geqslant \sup_{|\xi|\leqslant 1} |\xi-\beta|,$$

then

$$\lim_{k\to\infty}(\text{ORANUS}/k)_a = \lim_{k\to\infty}\left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1},\ldots,\widehat{W}_k^{(n)}/\widehat{V}_{k+1}\right) = (a_1,\ldots,a_n). \qquad \square$$

Example 3. Assume that the generating polynomials are given by:

$$V_k(x) = \prod_{i=0}^{k-1}(x-\beta_i) \ \ \left(\beta_i \in \mathbb{C},\ i = 0,1,2,\ldots,k-1\right)$$

i.e., the zeroes of V_k do not depend on k. Further, suppose the limit $b = \lim_{k\to\infty}\beta_i$ exists, thus:

(i) If $b = 0$, then $\lim_{k\to\infty}(\text{ORANUS}/k)_a = (a_1,\ldots,a_n)$.
(ii) If $b \neq 0$ and $\Re(b) < 5$, then $\overline{\lim}_{k\to\infty}(\text{ORANUS}/k)_a = (a_1,\ldots,a_n)$. □

Example 4. Assume that the generating polynomials have the form:

$$V_k(x) = \prod_{i=0}^{k-1}(x-b_i)^{k-1}, \quad k = 1,2,\ldots.$$

Suppose further that the sequence $(b_i,\ i = 0,1,2,\ldots)$ has k limit points $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ approached cyclically, i.e.:

$$\lim_{s\to\infty}\beta_{sk+j} = \gamma_j, \quad j = 0,1,\ldots,k-1.$$

We do not require that the limit points $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ are distinct. Define

$$q_k(x) = \prod_{i=0}^{k-1}(x-\gamma_i).$$

For each positive number ρ, let $\mathcal{L}_\rho$ denote the interior of the lemniscates with foci $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ and radius ρ, i.e., the set of all points z satisfying the inequality:

$$|q_k(z)| < \rho.$$

If

$$10 \notin \overline{\mathcal{L}}_{\rho_0} \ \left(\text{with } \rho_0 = \sup_{|\xi|\leqslant 1}|q_k(\xi)|\right),$$

then

$$\lim_{k\to\infty}(\mathrm{ORANUS}/k)_a = \lim_{k\to\infty}\left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1}, \ldots, \widehat{W}_k^{(n)}/\widehat{V}_{k+1}\right) = (a_1, \ldots, a_n). \qquad \square$$

Example 5. We choose the zeros of the Chebyshev polynomials as $\pi_0, \pi_1, \ldots, \pi_k$, i.e.:

$$V_k(x) = \prod_{i=0}^{k-1}\left(x - \cos\left[\frac{2i+1}{2(k+1)}\pi\right]\right), \quad k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty}(\mathrm{ORANUS}/k)_a = \lim_{k\to\infty}\left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1}, \ldots, \widehat{W}_k^{(n)}/\widehat{V}_{k+1}\right) = (a_1, \ldots, a_n). \qquad \square$$

6 Best Choice of ORANUS

A natural question which now arises is the "optimal" choice of an ORANUS. In the present section, we discuss this problem. Let

$$a = (a_1, \ldots, a_n) = \left(\underbrace{a_1^{(0)}.\,a_1^{(1)}a_1^{(2)}a_1^{(3)}\ldots}_{a_1}, \ \ldots, \ \underbrace{a_n^{(0)}.\,a_n^{(1)}a_n^{(2)}a_n^{(3)}\ldots}_{a_n}\right) \in \mathbb{A}^n.$$

As usually, for any $j = 1, 2, \ldots, n$ and $v = 0, 1, \ldots$, we set $\Lambda_j(x^v) := a_j^{(v)}$ and $\widehat{W}_k^{(j)} := 10^{-k}\Lambda_j([V_{k+1}(x) - V_{k+1}(10)]/[x-10])$, where $V_{k+1}(x) = \gamma\prod_{i=0}^{k}(x - \pi_i)$ ($\gamma \in \mathbb{C}\setminus\{0\}$), the $\pi_0, \pi_1, \ldots, \pi_k$ being arbitrarily chosen complex numbers. Obviously, the coordinate absolute differences $\left|a_j - \left[\widehat{W}_k^{(n)}/\widehat{V}_{k+1}\right]\right|$, with $\widehat{V}_{k+1} = 10^{-(k+1)}V_{k+1}(10)$, are the *coordinate absolute errors* of the considered approximation. The above asked question can be rephrased in terms of the coordinate absolute errors as follows. Given a $k \in \mathbb{N}$, find an

$$(\widetilde{\mathrm{ORANUS}/k})_a = \left(\widetilde{\widehat{W}_k^{(1)}}/\widetilde{\widehat{V}_{k+1}}, \quad \widetilde{\widehat{W}_k^{(2)}}/\widetilde{\widehat{V}_{k+1}}, \quad \ldots \quad, \widetilde{\widehat{W}_k^{(n)}}/\widetilde{\widehat{V}_{k+1}}\right),$$

minimizing all coordinate absolute errors, in the sense that

$$\left|a_j - \widetilde{\widehat{W}_k^{(j)}}/\widetilde{\widehat{V}_{k+1}}\right| \left|a_j - \widehat{W}_k^{(j)}/\widehat{V}_{k+1}\right|, \quad j = 1, 2, \ldots, n,$$

whenever

$$\left(\widehat{W}_k^{(1)}/\,\widehat{V}_{k+1},\quad \widehat{W}_k^{(2)}/\,\widehat{V}_{k+1},\quad \ldots\quad ,\widehat{W}_k^{(n)}/\,\widehat{V}_{k+1}\right),$$

is an ORANUS to a.

Application of Theorem 3 shows that the generating polynomial:

$$\widetilde{\widehat{V}_{k+1}}(x) = x^{k+1} + 10\left(10^{-k} - 1\right)x^k,\ \ k \in \mathbb{N}$$

leads to the ORANUS:

$$\left(\widetilde{\widehat{W}_k^{(1)}}/\,\widetilde{\widehat{V}_{k+1}},\quad \widetilde{\widehat{W}_k^{(2)}}/\,\widetilde{\widehat{V}_{k+1}},\quad \ldots\quad ,\widetilde{\widehat{W}_k^{(n)}}/\,\widetilde{\widehat{V}_{k+1}}\right),$$

minimizing all coordinate absolute errors. Especially, since

$$\widetilde{\widehat{V}_{k+1}} = 10^{-(k+1)}\,\widetilde{\widehat{V}_{k+1}}(10) = 10^{-k+2}$$

and

$$\begin{aligned}\widetilde{\widehat{W}_k^{(j)}} &= 10^{-k}\Lambda_j\left(\frac{x^{k+1} + 10\left(10^{-k} - 1\right)x^k - 10^{k+1} - 10\left(10^{-k} - 1\right)10^k}{x - 10}\right)\\ &= 10^{-k}\Lambda_j\left(\frac{\left[x^{k+1} - 10^{k+1}\right] + 10\left(10^{-k} - 1\right)\left[x^k - 10^k\right]}{x - 10}\right),\end{aligned}$$

we infer that

$$\frac{\widetilde{\widehat{W}_k^{(j)}}}{\widetilde{\widehat{V}_{k+1}}} = \frac{1}{100}\Lambda_j\left(\frac{\left[x^{k+1} - 10^{k+1}\right] + 10\left(10^{-k} - 1\right)\left[x^k - 10^k\right]}{x - 10}\right).$$

So, we have obtained the following:

Theorem 5. *An "optimal" ORANUS to an n-vector of mutually irrational numbers $a = (a_1, a_2, \ldots, a_n) \in \mathbb{A}^n$ is given by:*

$$\left(\widehat{\mathrm{ORANUS}/k}\,\right)_a = (\Lambda_1, \ldots, \Lambda_n)\left(\frac{\left[x^{k+1} - 10^{k+1}\right] + 10\left(10^{-k} - 1\right)\left[x^k - 10^k\right]}{10^2\left[x - 10\right]}\right). \qquad \square$$

7 Application

A very important application of the abovementioned work can be in the field of the beam stability problem in circular accelerators like the large hadron collider (LHC) machine at the European Organization for Nuclear Research (CERN). The LHC is considered "one of the great engineering achievements of mankind" and the largest and highest-energy particle accelerator in the world. It remains one of the largest and most complicate experimental machine ever constructed and is expected to address some of the still unsolved questions of science. Particularly, the application at hand concerns the stability of particle beams in high-energy hadron colliders, where symplectic mappings naturally arise due to the periodically repeated (and of very brief duration) effects of beam-beam collisions or beam passage through magnetic focusing elements [5, 10, 17, 39, 43, 44, 46, 47]. The main open problems in such mappings (particularly in the n-dimensional case, $n \geqslant 2$) concern the long-term stability of orbits, which can slowly diffuse away from the origin through thin chaotic layers, leading, e.g., to particle loss in the storage rings of an accelerator or, in similar cases, stars escaping from a galaxy [47]. A well-studied and widely applied such mapping is the following *symplectic mapping T*:

$$T: \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \cos\omega_1 & -\sin\omega_1 & 0 & 0 \\ \sin\omega_1 & \cos\omega_1 & 0 & 0 \\ 0 & 0 & \cos\omega_2 & -\sin\omega_2 \\ 0 & 0 & \sin\omega_2 & \cos\omega_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + x_1^2 - x_3^2 \\ x_3 \\ x_4 - 2x_1x_3 \end{pmatrix}, \tag{1}$$

which describes the (instantaneous) effect experienced by a hadronic particle as it passes through a magnetic focusing element of the FODO cell type [8, 34–36, 42, 44, 46, 47]. The coordinates x_1 and x_3 represent the particle's deflections from the ideal (circular) orbit, in the horizontal and vertical directions, respectively, and x_2, x_4 are the associated "momenta," while ω_1, ω_2 are related to the accelerator's betatron frequencies (or "tunes") q_x, q_y by:

$$\omega_1 = 2\pi q_x, \qquad \omega_2 = 2\pi q_y$$

and constitute the main parameters that can be varied by an experimentalist [47].

In general, the accurate computation of periodic orbits and the knowledge of their stability properties play a central role for studying the behavior of various such mappings. We say that $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)$ is a *fixed point of a mapping T of order p* or a *periodic orbit of period p*, if:

$$x^\star = T^p(x^\star) \equiv \underbrace{T\Big(T\big(\cdots T\big(T(x^\star)\big)\cdots\big)\Big)}_{p \text{ times}}, \quad p = 1, 2, 3, \ldots .$$

For efficient methods of computing periodic orbits, we refer the interested reader to [35, 37, 44–47].

In previous papers of ours [46, 47], we have studied the structure and breakdown of invariant tori of the 4-D symplectic mapping (1) which, as we have already mentioned, arises in a realistic application related to the beam stability problem in circular particle accelerators. Our original goal was to examine the structure of tori by approximating them with sequences of periodic orbits whose *rational* rotation numbers converge to the pair of *irrational* rotation numbers of an invariant torus. Particularly, the sequence of rational rotation numbers:

$$\left(\sigma_1^{(k)},\, \sigma_2^{(k)}\right) = \left(\frac{p_k}{r_k},\, \frac{q_k}{r_k}\right), \quad k = 0, 1, 2, \ldots, \tag{2}$$

have been taken to converge, as $k \to \infty$, to a pair of incommensurate irrationals. In the problem studied in [46, 47], we have chosen the example:

$$\left(\sigma_1^{(k)},\, \sigma_2^{(k)}\right) \underset{n\to\infty}{\longrightarrow} (\sigma_1, \sigma_2) = \left(\frac{\sqrt{5}-1}{2},\, \sqrt{2}-1\right) = (0.61803\ldots, 0.41421\ldots)\,. \tag{3}$$

The choice of σ_1, σ_2 is arbitrary, but it may be useful for comparison purposes with the 2-D case [26]. Next, by selecting linear frequencies $q_x = 0.61903$, $q_y = 0.4152$, we approximated the (σ_1, σ_2)-invariant torus by periodic orbits characterized by the rotation numbers of the Jacobi–Perron sequence [7, 40] which are recursively obtained from the relation:

$$s_{k+1} = l_{k+1}s_k + m_{k+1}s_{k-1} + s_{k-2}, \quad k = 0, 1, \ldots\ ,$$

$(s_k = p_k, q_k, r_k)$, with the integers l_k, m_k determined as follows:

$$\left(s_1^{(k+1)},\, s_2^{(k+1)}\right) = \left(\left\{\frac{1}{s_2^{(k)}}\right\},\, \left\{\frac{s_1^{(k)}}{s_2^{(k)}}\right\}\right), \left(l_{k+1},\, m_{k+1}\right) = \left(\left[\frac{1}{s_2^{(k)}}\right],\, \left[\frac{s_1^{(k)}}{s_2^{(k)}}\right]\right),$$

where $[x]$ and $\{x\}$ denote to the integer and fractional part of the number x, respectively, and $\left(s_1^0,\, s_2^0\right) = \left(\sigma_1,\, \sigma_2\right), (p_0, q_0, r_0) = (0,\, 0,\, 1), (p_{-1}, q_{-1}, r_{-1}) = (1,\, 0,\, 0), (p_{-2}, q_{-2}, r_{-2}) = (0,\, 1,\, 0)$.

In Table 1 we exhibit Jacobi–Perron approximates to the irrationals (3), up to $k = 16$, cf. (2). Notice that the convergence is rather slow, as one might expect of quadratic irrationals, like σ_1, σ_2.

Using the approach of the paper at hand, we can construct simultaneous rational approximants to the given pair of irrational numbers:

$$\sigma_1 = \frac{\sqrt{5}-1}{2} \cong 0.6180339887499\ldots \text{ and } \sigma_2 = \sqrt{2}-1 \cong 0.4142135623731\ldots$$

In fact, let us define the corresponding two $\mathbb{C}$-linear functionals Λ_1 and Λ_2 as they are exhibited in Table 2.

Table 1 Rational approximants of the Jacobi–Perron algorithm of the quadratic irrationals $\sigma_1 = \left(\sqrt{5}-1\right)/2$ and $\sigma_2 = \sqrt{2}-1$ and the period p of the corresponding periodic orbit [47]

k	p_k	q_k	$p_k/r_k - \sigma_1$	$q_k/r_k - \sigma_2$	$p = r_k$
4	1	1	-0.11803399	$0.85786438 \times 10^{-1}$	2
5	3	2	0.13196601	$0.85786438 \times 10^{-1}$	4
6	3	2	$-0.18033989 \times 10^{-1}$	$-0.14213562 \times 10^{-1}$	5
7	91	61	$-0.31691239 \times 10^{-2}$	$-0.20514002 \times 10^{-2}$	148
8	94	63	$0.38706388 \times 10^{-3}$	$0.26012184 \times 10^{-3}$	152
9	755	506	$0.31162960 \times 10^{-3}$	$0.20085204 \times 10^{-3}$	1221
10	846	567	$-0.64668078 \times 10^{-4}$	$-0.42634689 \times 10^{-4}$	1369
11	940	630	$-0.19524582 \times 10^{-4}$	$-0.12378941 \times 10^{-4}$	1521
12	8181	5483	$0.63527172 \times 10^{-5}$	$0.41606759 \times 10^{-5}$	13237
13	9027	6050	$-0.30396282 \times 10^{-6}$	$-0.22538829 \times 10^{-6}$	14606
14	9967	6680	$-0.21167340 \times 10^{-5}$	$-0.13716371 \times 10^{-5}$	16127
15	37142	24893	$0.18932888 \times 10^{-6}$	$0.12549818 \times 10^{-6}$	60097
16	83311	55836	$0.13587919 \times 10^{-6}$	$0.87478537 \times 10^{-7}$	134800

Table 2 The corresponding two $\mathbb{C}$-linear functionals Λ_1 and Λ_2

Λ_1	Λ_2
$\Lambda_1(1) = 0$	$\Lambda_2(1) = 0$
$\Lambda_1(x) = 6$	$\Lambda_2(x) = 4$
$\Lambda_1(x^2) = 1$	$\Lambda_2(x^2) = 1$
$\Lambda_1(x^3) = 8$	$\Lambda_2(x^3) = 4$
$\Lambda_1(x^4) = 0$	$\Lambda_2(x^4) = 2$
$\Lambda_1(x^5) = 3$	$\Lambda_2(x^5) = 1$
$\Lambda_1(x^6) = 3$	$\Lambda_2(x^6) = 3$
$\Lambda_1(x^7) = 9$	$\Lambda_2(x^7) = 5$
$\Lambda_1(x^8) = 8$	$\Lambda_2(x^8) = 6$
$\Lambda_1(x^9) = 8$	$\Lambda_2(x^9) = 2$
$\Lambda_1(x^{10}) = 7$	$\Lambda_2(x^{10}) = 3$
$\Lambda_1(x^{11}) = 4$	$\Lambda_2(x^{11}) = 7$
$\Lambda_1(x^{12}) = 9$	$\Lambda_2(x^{12}) = 3$
$\Lambda_1(x^{13}) = 9$	$\Lambda_2(x^{13}) = 1$
$\vdots$	$\vdots$

(a) Let us now choose

$$k = 3 \text{ and } \pi_0 = \pi_1 = \pi_2 = 0,\ \pi_3 = -\tfrac{i}{2}.$$

The corresponding generating polynomial is

$$V_4(x) = x^4 + i\frac{x^3}{2},$$

and therefore

$$\widehat{W}_3^{(1)} = 10^{-3} \Lambda_1 \left(\frac{V_4(x) - V_4(10)}{x - 10} \right) = 10^{-3} \left(680 + i \frac{121}{2} \right),$$

$$\widehat{W}_3^{(2)} = 10^{-3} \Lambda_2 \left(\frac{V_4(x) - V_4(10)}{x - 10} \right) = 10^{-3} \left(440 + i \frac{81}{2} \right),$$

$$\widehat{V}_4 = 10^{-4} V_4(10) = 10^{-4} \left(10^4 + i \frac{10^3}{2} \right).$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\text{ORANUS}/3)_\sigma := \left(\frac{\widehat{W}_3^{(1)}}{\widehat{V}_4}, \frac{\widehat{W}_3^{(2)}}{\widehat{V}_4} \right)$$
$$= (0.6813216957606 + i0.313216957606,$$
$$0.4409226932688 + i0.0184538653367).$$

(b) Similarly, for $k = 3$, we can choose the zeros of the Tchebycheff polynomial:

$$T_4(x) = \cos(4 \arccos x)$$

divided by $\sqrt{\pi}$ as interpolation nodes, i.e.:
$\pi_0 = \frac{1}{\sqrt{\pi}} \cos\left(\frac{\pi}{7}\right)$, $\pi_1 = \frac{1}{\sqrt{\pi}} \cos\left(\frac{3\pi}{7}\right)$, $\pi_2 = \frac{1}{\sqrt{\pi}} \cos\left(\frac{5\pi}{7}\right)$ and $\pi_3 = \frac{1}{\sqrt{\pi}} \cos(\pi)$.
The generating polynomial is

$$V_4(x) = x^4 + 0.282x^3 - 0.318x^2 - 0.067x - 0.012$$

and we have

$$\widehat{W}_3^{(1)} = 10^{-3} \Lambda_1 \left(\frac{V_4(x) - V_4(10)}{x - 10} \right) = 633.293 \times 10^{-3},$$

$$\widehat{W}_3^{(2)} = 10^{-3} \Lambda_2 \left(\frac{V_4(x) - V_4(10)}{x - 10} \right) = 424.29 \times 10^{-3},$$

$$\widehat{V}_4 = 10^{-4} V_4(10) = 10314.458 \times 10^{-4}.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\text{ORANUS}/3)_\sigma := \left(\frac{\widehat{W}_3^{(1)}}{\widehat{V}_4}, \frac{\widehat{W}_3^{(2)}}{\widehat{V}_4} \right)$$
$$= (0.6139866971197, \ \ 0.4113546247413).$$

(c) Continuing, we can choose $k = 12$ and

$$\pi_0 = \pi_1 = \cdots = \pi_{11} = 0, \quad \pi_{12} = -1.$$

Then the generating polynomial is

$$V_{13}(x) = x^{13} + x^{12}$$

and therefore

$$\widehat{W}_{12}^{(1)} = 10^{-12} \Lambda_1 \left(\frac{V_{13}(x) - V_{13}(10)}{x - 10} \right) = 679837387579 \times 10^{-12},$$

$$\widehat{W}_{12}^{(2)} = 10^{-12} \Lambda_2 \left(\frac{V_{13}(x) - V_{13}(10)}{x - 10} \right) = 455634918610 \times 10^{-12},$$

$$\widehat{V}_{13} = 10^{-13} V_{13}(10) = 11000000000000 \times 10^{-13}.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\text{ORANUS}/12)_\sigma := \left(\frac{\widehat{W}_{12}^{(1)}}{\widehat{V}_{13}}, \quad \frac{\widehat{W}_{12}^{(2)}}{\widehat{V}_{13}} \right)$$
$$= (0.6180339887082, \quad 0.4142130169182).$$

(d) Next, we may choose $k = 6$ and

$$\pi_0 = \pi_1 = \cdots = \pi_6 = 0.$$

This choice implies

$$V_7(x) = x^7$$

and therefore

$$\widehat{W}_6^{(1)} = 10^{-6} \Lambda_1 \left(\frac{V_7(x) - V_7(10)}{x - 10} \right) = 618053 \times 10^{-6},$$

$$\widehat{W}_6^{(2)} = 10^{-6} \Lambda_2 \left(\frac{V_7(x) - V_7(10)}{x - 10} \right) = 414213 \times 10^{-6},$$

$$\widehat{V}_7 = 10^{-7} V_7(10) = 1.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\text{ORANUS}/6)_\sigma := \left(\frac{\widehat{W}_6^{(1)}}{\widehat{V}_7}, \ \frac{\widehat{W}_6^{(2)}}{\widehat{V}_7} \right) = (0.618033, \ \ 0.414213) .$$

e) Finally, let $k = 4$ and let $V_5(x)$ be the Legendre polynomial:

$$V_5(x) = \frac{1}{8} \left(63x^5 - 70x^3 + 15x\right) .$$

Then

$$\widehat{W}_4^{(1)} = 10^{-4} \Lambda_1 \left(\frac{V_5(x) - V_5(10)}{x - 10} \right) = 48133 \times 10^{-4},$$

$$\widehat{W}_4^{(2)} = 10^{-4} \Lambda_2 \left(\frac{V_5(x) - V_5(10)}{x - 10} \right) = 32259.5 \times 10^{-4},$$

$$\widehat{V}_5 = 10^{-5} V_4(10) = 778768.75 \times 10^{-5}.$$

Thus, if $\sigma = (\sigma_1, \sigma_2)$, then

$$(\text{ORANUS}/4)_\sigma := \left(\frac{\widehat{W}_4^{(1)}}{\widehat{V}_5}, \ \frac{\widehat{W}_4^{(2)}}{\widehat{V}_5} \right)$$

$$= (0.6180750062198, \ \ 0.4142372174025) .$$

Summarizing, it should also be noted that, for small values of k (this means a few interpolation points $\pi_0, \pi_1, \ldots, \pi_k$ and a low-degree generating polynomial $V_{k+1}(x)$), we can achieve good rational approximations.

We can group the above numerical results in Table 3, which are directly comparable with those exhibited in Table 1.

8 Epilogue and Synopsis

In the paper at hand, we investigated multivariate rational approximation numbers whose decimal expansion string of digits coincides with the decimal expansion digital string of mutually irrational numbers as far as possible. The main advantage of these approximants over Jacobi–Perron approximants lies in the completely free choice of their common denominator which may lead to a better and increasingly rapid approximation.

Table 3 Optimal rational approximation number sets of the quadratic irrationals $\sigma_1 = \left(\sqrt{5}-1\right)/2$ and $\sigma_2 = \sqrt{2}-1$

k	$V_{k+1}(x)$	$\left\lvert \sigma_1 - \left(\widehat{W}_k^{(1)}/\widehat{V}_{k+1}\right)\right\rvert$	$\left\lvert \sigma_2 - \left(\widehat{W}_k^{(2)}/\widehat{V}_{k+1}\right)\right\rvert$	$\left\lvert \sigma - (\text{ORANUS}/k)_\sigma \right\rvert$
3	$x^4 + i\frac{x^3}{2}$ (interpolation points $\pi_0 = \pi_1 = \pi_2 = 0$, $\pi_3 = -i/2$)	0.3195468610246	0.0010539228191	0.3195485990331
3	$x^4 + 0.282x^3 - 0.318x^2 - 0.067x - 0.012$ (interpolation points $\pi_0 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{\pi}{7}\right)$, $\pi_1 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{3\pi}{7}\right)$, $\pi_2 = \frac{1}{\sqrt{\pi}}\cos\left(\frac{5\pi}{7}\right)$ and $\pi_3 = \frac{1}{\sqrt{\pi}}\cos(\pi)$, the zeros of the Tchebycheff polynomial $T_4(x) = \cos(4\arccos x)$)	$0.0040472916302 \approx 0.40472916 \times 10^{-2}$	$0.0028589376318 \approx 0.28589376 \times 10^{-2}$	$0.0049552087645 \approx 0.49552088 \times 10^{-2}$
12	$x^{13} + x^{12}$ (interpolation points $\pi_0 = \pi_1 = \cdots = \pi_{11} = 0$, $\pi_{12} = -1$)	$0.0000000000417 = 0.417 \times 10^{-10}$	$0.0000005454549 = 0.5454549 \times 10^{-6}$	$0.0000005477226 = 0.5477226 \times 10^{-6}$
6	x^7 (interpolation points $\pi_0 = \pi_1 = \cdots = \pi_6 = 0$)	$0.000000987499 = 0.987499 \times 10^{-6}$	$0.0000005623731 = 0.5623731 \times 10^{-6}$	$0.0000011401754 = 0.11401754 \times 10^{-5}$
4	$\frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$ (Legendre polynomial)	$0.00004101747 = 0.4101747 \times 10^{-4}$	$0.000023655029 = 0.23655029 \times 10^{-4}$	$0.0002400801949 = 0.2400801949 \times 10^{-3}$

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A Characterization Theorem for the Best L_1 Piecewise Monotonic Data Approximation Problem

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In Honor of Constantin Carathéodory

Abstract Let a sequence of n univariate data that include random errors be given. We consider the problem of calculating a best L_1 approximation to the data subject to the condition that the first differences of the approximated values have at most $k-1$ sign changes, where k is a prescribed integer. The choice of the positions of sign changes by considering all possible combinations of positions can be of magnitude n^{k-1}, so that it is not practicable to test each one separately. We provide a theorem that decomposes the problem into the best L_1 monotonic approximation (case $k = 1$) problems to disjoint sets of adjacent data. The decomposition allows the development of a dynamic programming procedure that provides a highly efficient calculation of the solution.

1 Introduction

The purpose of this paper is to present a characterization theorem for the following data approximation problem. Let $\{\phi_i : i = 1, 2, \ldots, n\}$ be measurements of the real function values $\{f(x_i) : i = 1, 2, \ldots, n\}$, where the abscissae $\{x_i : i = 1, 2, \ldots, n\}$ are in strictly ascending order. If the measurements are contaminated by random errors, then it is likely that the sequence of the first differences $\{\phi_{i+1} - \phi_i : i = 1, 2, \ldots, n-1\}$ contains far more sign changes than the sequence $\{f(x_{i+1}) - f(x_i) : i = 1, 2, \ldots, n-1\}$. Therefore, for some integer k that is much smaller than n, we

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seek numbers $\{y_i : i = 1, 2, \ldots, n\}$ that make least the sum of absolute changes to the measurements so that the sequence $\{y_{i+1} - y_i : i = 1, 2, \ldots, n-1\}$ changes sign at most $k-1$ times. We regard the original measurements and the approximated data as n-vectors $\boldsymbol{\phi}$ and $\boldsymbol{y}$. The constraints on $\boldsymbol{y}$ allow at most k sections of monotonic components, alternately increasing and decreasing, while without loss of generality, we suppose that the first monotonic section is increasing.

Hence we denote by $Y(k, n)$ the set of n-vectors $\boldsymbol{y}$ whose components satisfy the piecewise monotonicity constraints

$$\left.\begin{array}{l} y_{t_{j-1}} \leq y_{t_{j-1}+1} \leq \cdots \leq y_{t_j},\ j \text{ is odd} \\ y_{t_{j-1}} \geq y_{t_{j-1}+1} \geq \cdots \geq y_{t_j},\ j \text{ is even} \end{array}\right\}, \tag{1}$$

where the integers $\{t_j : j = 1, 2, \ldots, k-1\}$ satisfy the conditions

$$1 = t_0 \leq t_1 \leq \cdots \leq t_k = n, \tag{2}$$

and the optimization calculation seeks a vector $\boldsymbol{y}^*$ in $Y(k, n)$ that minimizes the objective function

$$||\boldsymbol{y} - \boldsymbol{\phi}||_1 = \sum_{i=1}^{n} |y_i - \phi_i|. \tag{3}$$

We call $\boldsymbol{y}^*$ a best L_1 piecewise monotonic, or optimal, approximation to $\boldsymbol{\phi}$ and note that it need not be unique. Since $\{t_j : j = 1, 2, \ldots, k-1\}$ are also variables of the optimization problem, there exist about n^{k-1} combinations of these integers in order to find a combination that gives an optimal approximation, which makes an exhaustive search prohibitively expensive.

However, the following property is considered by Demetriou [5]. It is that if the integers $\{t_j : j = 1, 2, \ldots, k-1\}$ are optimal, then an optimal piecewise monotonic approximation is made up of separate optimal monotonic sections between adjacent t_j. In Sect. 2 we present a characterization theorem that gives an equivalent formulation of the problem where the main unknowns are the $\{t_j\}$. The important consequence of this theorem is that it allows a dynamic programming procedure to calculate the required least value of the objective function and obtain a solution in at most $n^3 + O(kn^2)$ computer operations [5]. In Sect. 3 we give a brief summary.

Two related calculations are studied by Demetriou and Powell [7] and Cullinan and Powell [3], which instead of (3) minimize the sum of squares $||\boldsymbol{y} - \boldsymbol{\phi}||_2^2 = \sum_{i=1}^{n}(y_i - \phi_i)^2$ and the supremum norm $||\boldsymbol{y} - \boldsymbol{\phi}||_\infty = \max_{1 \leq i \leq n} |y_i - \phi_i|$, respectively, subject to the same constraints on $\boldsymbol{y}$.

2 The Theorem

The reformulation of the problem of Sect. 1 makes use of the highly useful property of an optimal approximation to be stated. Indeed, if the n-vector $\boldsymbol{y}$ is optimal and if $\{t_j : j = 1, 2, \ldots, k-1\}$ are the associated integer variables, then $\boldsymbol{y}$ consists of separate optimal monotonic increasing and monotonic decreasing sections of components between adjacent optimal integer variables that can be calculated independently of each other. This result is given in the next lemma.

Lemma 1. *Let the integer variables* $\{t_j : j = 1, 2, \ldots, k-1\}$ *be associated with a best* L_1 *approximation* $\mathbf{y}$ *from* $Y(k,n)$ *to* $\boldsymbol{\phi}$*. Then the components* $\{y_i : i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$ *have the values that solve the problem*

$$\alpha(t_{j-1}, t_j) = \min_{y_{t_{j-1}} \le y_{t_{j-1}+1} \le \cdots \le y_{t_j}} \sum_{i=t_{j-1}}^{t_j} |y_i - \phi_i|, \tag{4}$$

if j is odd, and solve the problem

$$\beta(t_{j-1}, t_j) = \min_{y_{t_{j-1}} \ge y_{t_{j-1}+1} \ge \cdots \ge y_{t_j}} \sum_{i=t_{j-1}}^{t_j} |y_i - \phi_i|, \tag{5}$$

if j is even. Further, the interpolation equations

$$y_{t_j} = \phi_{t_j}, \; j = 1, 2, \ldots, k-1 \tag{6}$$

are satisfied.

Proof. For a proof see Lemma 2 of Demetriou [5]. □

A solution to problem (4) always exists, but it need not be unique. A solution is called a best L_1 monotonic increasing approximation to $\{\phi_i : i = t_{j-1}, t_{j-1} + 1, \ldots, t_j\}$. The calculation of this approximation can be solved as a linear programming problem (for a general reference, see Barrodale and Roberts [1]), but there are special algorithms that are more efficient than general algorithms (see Cullinan and Powell [3], Menéndez and Salvador [8], and Stout [10]).

The following characterization theorem gives necessary and sufficient conditions for the vector $\boldsymbol{y}^*$ to be a best L_1 piecewise monotonic approximation from $Y(k,n)$ to $\boldsymbol{\phi}$. It provides a decomposition of the problem of Sect. 1 into at most best L_1 monotonic approximation problems to disjoint subsets of adjacent data.

Theorem 1. *Let* $\mathbf{y}^*$ *be an n-vector that minimizes the objective function* (3) *subject only to the constraints* (1)*, where* $\{t_j : j = 1, 2, \ldots, k-1\}$ *are any integers that satisfy the conditions* (2)*. The vector* $\mathbf{y}^*$ *minimizes* (3) *subject to* $\mathbf{y} \in Y(k,n)$ *if and only if the equation*

$$\sum_{j=1,\, j \text{ odd}}^{k} \alpha(t_{j-1}, t_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(t_{j-1}, t_j)$$
$$= \min_{1=s_0 \le s_1 \le \cdots \le s_k = n} \left\{ \sum_{j=1,\, j \text{ odd}}^{k} \alpha(s_{j-1}, s_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(s_{j-1}, s_j) \right\} \tag{7}$$

holds.

Proof. We prove first that if $\boldsymbol{y}^*$ is optimal, namely, if $\boldsymbol{y}^*$ minimizes (3) subject to $\boldsymbol{y} \in Y(k,n)$, then Eq. (7) is obtained. In the statement of the theorem, we let $\{t_j : j = 1, 2, \ldots, k-1\}$ be the values of the integer variables associated with $\boldsymbol{y}^*$. In view of Lemma 1, the sequence $\{y_i^* : \; i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$ is a best L_1 monotonic increasing approximation to the data $\{\phi_i : \; i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$ if j is odd and a best L_1 monotonic decreasing approximation if j is even, because otherwise we can reduce $||\boldsymbol{y}^* - \boldsymbol{\phi}||_1$ by replacing $\{y_i^* : \; i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$ by a best L_1 monotonic approximation to the data $\{\phi_i : \; i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$, which preserves $\boldsymbol{y}^* \in Y(k,n)$. Thus we obtain the relations

$$\sum_{i=t_{j-1}}^{t_j} |y_i^* - \phi_i| = \begin{cases} \alpha(t_{j-1}, t_j), \; j \text{ odd} \\ \beta(t_{j-1}, t_j), \; j \text{ even.} \end{cases} \tag{8}$$

Furthermore, in view of (6), we have the equations $y_{t_j}^* = \phi_{t_j}, \; j = 1, 2, \ldots, k-1$. Hence and from (8), the left-hand side of the expression (7) has the value $||\boldsymbol{y}^* - \boldsymbol{\phi}||_1$, or pertaining to notation

$$\sum_{j=1,\, j \text{ odd}}^{k} \alpha(t_{j-1}, t_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(t_{j-1}, t_j) = ||\boldsymbol{y}^* - \boldsymbol{\phi}||_1. \tag{9}$$

Consequently the value $||\boldsymbol{y}^* - \boldsymbol{\phi}||_1$ implies the bound on the right-hand side of (7),

$$\min_{1=s_0 \le s_1 \le \cdots \le s_k = n} \left\{ \sum_{j=1,\, j \text{ odd}}^{k} \alpha(s_{j-1}, s_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(s_{j-1}, s_j) \right\} \le ||\boldsymbol{y}^* - \boldsymbol{\phi}||_1. \tag{10}$$

It follows that Eq. (7) is satisfied, provided that we can establish the inequality

$$||\boldsymbol{z}^* - \boldsymbol{\phi}||_1 \le \min_{1=s_0 \le s_1 \le \cdots \le s_k = n} \left\{ \sum_{j=1,\, j \text{ odd}}^{k} \alpha(s_{j-1}, s_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(s_{j-1}, s_j) \right\}, \tag{11}$$

where $\boldsymbol{z}^*$ is any solution of the calculation.

Let $\{s_j : \; j = 0, 1, \ldots, k\}$ be any integers that satisfy the conditions $1 = s_0 \le s_1 \le \cdots \le s_k = n$, and let $\boldsymbol{y}^-$ be the n-vector that gives the terms of the expression

$$||\boldsymbol{y}^- - \boldsymbol{\phi}||_1 = \alpha(s_0, s_1) + \sum_{j=2,\, j \text{ odd}}^{k} \alpha(s_{j-1}+1, s_j) + \sum_{j=2,\, j \text{ even}}^{k} \beta(s_{j-1}+1, s_j), \tag{12}$$

where we define $\alpha(i,j)$ and $\beta(i,j)$ to be zero if $j < i$. Hence, we obtain the inequality

$$\alpha(s_0, s_1) + \sum_{j=2,\, j \text{ odd}}^{k} \alpha(s_{j-1}+1, s_j) + \sum_{j=2,\, j \text{ even}}^{k} \beta(s_{j-1}+1, s_j)$$
$$\leq \sum_{j=1,\, j \text{ odd}}^{k} \alpha(s_{j-1}, s_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(s_{j-1}, s_j). \tag{13}$$

As $\boldsymbol{y}^-$ is in $Y(k,n)$ and $\boldsymbol{z}^*$ is optimal, we have $||\boldsymbol{z}^* - \boldsymbol{\phi}||_1 \leq ||\boldsymbol{y}^- - \boldsymbol{\phi}||_1$. The last inequality, (12) and (13), imply inequality (11). We deduce from (9)–(11) that Eq. (7) is true.

In order to complete the proof of the theorem, we let the integers $\{t_j : j = 0, 1, \ldots, k\}$ satisfy the conditions (2) and Eq. (7). It suffices to construct a vector that minimizes (3) subject only to the constraints (1), which is an optimal approximation to $\boldsymbol{\phi}$. In other words, if $\hat{\boldsymbol{y}}$ is this vector, then we prove that $\hat{\boldsymbol{y}}$ minimizes (3) subject to $\boldsymbol{y} \in Y(k,n)$.

As a consequence of Lemma 1, if the values $\{t_j : j = 1, 2, \ldots, k-1\}$ are optimal, then the least value of the objective function (3) in $Y(k,n)$ is achieved, and it is the expression

$$\sum_{j=1,\, j \text{ odd}}^{k} \alpha(t_{j-1}, t_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(t_{j-1}, t_j). \tag{14}$$

We define the n-vector $\boldsymbol{\psi}$ whose components occur in the definition of $\alpha(t_{j-1}, t_j - 1)$ when j is odd in $[1,k]$ and in the definition of $\beta(t_{j-1}, t_j - 1)$ when j is even in $[1,k]$. It follows that either $\psi_{t_j-1} \geq \psi_{t_j}$ or $\psi_{t_j-1} \leq \psi_{t_j}$ when j is odd and similarly when j is even, which implies that $\boldsymbol{\psi}$ satisfies the piecewise monotonicity constraints. Since the expression of the left-hand side of (7) is an upper bound on the sum

$$\sum_{j=1,\, j \text{ odd}}^{k-1} \alpha(t_{j-1}, t_j - 1) + \sum_{j=1,\, j \text{ even}}^{k-1} \beta(t_{j-1}, t_j - 1) + \delta(t_{k-1}, t_k), \tag{15}$$

where δ stands for α if k is odd and for β if k is even, we obtain the inequality

$$\sum_{j=1,\, j \text{ odd}}^{k} \alpha(t_{j-1}, t_j - 1) + \sum_{j=1,\, j \text{ even}}^{k} \beta(t_{j-1}, t_j - 1) + \delta(t_{k-1}, t_k)$$
$$\leq \sum_{j=1,\, j \text{ odd}}^{k} \alpha(t_{j-1}, t_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(t_{j-1}, t_j). \tag{16}$$

Since the expression of the left-hand side of (16) is a value of the objective function that is achieved by the piecewise monotonic vector $\boldsymbol{\psi}$, while the expression of the right-hand side has the value $||z^* - \boldsymbol{\phi}||_1$, where z^* is any solution of the calculation, inequality (16) should hold as an equality. Therefore, we have obtained the relations

$$\left.\begin{aligned} \alpha(t_{j-1}, t_j - 1) &= \alpha(t_{j-1}, t_j),\ j \text{ odd} \\ \beta(t_{j-1}, t_j - 1) &= \beta(t_{j-1}, t_j),\ j \text{ even} \end{aligned}\right\},\tag{17}$$

where we let $t_k - 1 = t_k$.

Taking account of (17), we are going to construct an n-vector that satisfies the constraints (1) and Eq. (6). Remembering the monotonicity of the components of $\boldsymbol{\psi}$ on the interval $[t_{j-1}, t_j - 1]$, when $j \in [1, k-1]$ is odd, we let $q = t_j - 1$, and we let p be an integer such that $t_{j-1} \leq p \leq q < t_j$ and $\psi_{p-1} < \psi_p = \psi_{p+1} = \cdots = \psi_q = \eta^*$, except that we ignore the inequality $\psi_{p-1} < \psi_p$ if $p = t_{j-1}$, where η^* is a real number that minimizes the expression $\sum_{i=p}^{q} |\eta - \phi_i|$. Then, η^* is either a single value or it belongs to a closed interval of the real line, say it is $I(p, q)$. Note that η^* is the median of $\{\phi_p, \phi_{p+1}, \ldots, \phi_q\}$. In the following consideration, we assume that $\eta^* \in I(p, q)$, because the case when η^* is a single value is treated similarly. Further, we let $\{\psi_i^{(\alpha)} : i = t_{j-1}, t_{j-1} + 1, \ldots, t_j\}$ be components that occur in $\alpha(t_{j-1}, t_j)$. In order to calculate $\alpha(t_{j-1}, t_j)$ from $\alpha(t_{j-1}, t_j - 1)$, if $t_j > 1$, we find that $\inf I(p, q) > \phi_{t_j}$ would give $\alpha(t_{j-1}, t_j) > \alpha(t_{j-1}, t_j - 1)$, which contradicts the first line of (17). Thus $\inf I(p, q) \leq \phi_{t_j}$ and we proceed by considering the following two cases. In the case of $\inf I(p, q) \leq \phi_{t_j} \leq \sup I(p, q)$, it is straightforward to prove that ϕ_{t_j} minimizes the sum $\sum_{i=p}^{q+1} |\eta - \phi_i|$, which implies $\psi_p^{(\alpha)} = \psi_{p+1}^{(\alpha)} = \cdots = \psi_q^{(\alpha)} = \psi_{q+1}^{(\alpha)} = \phi_{t_j}$ and $\sum_{i=p}^{q+1} |\phi_{t_j} - \phi_i| = \sum_{i=p}^{q} |\eta^* - \phi_i|$. Since the components $\{\psi_i^{(\alpha)} = \psi_i : i = t_{j-1}, t_{j-1} + 1, \ldots, p - 1\}$ are allowed by conditions $y_{t_{j-1}} \leq y_{t_{j-1}+1} \leq \cdots \leq y_{p-1}$ and the inequalities $\psi_{p-1} < \inf I(p, q) \leq \phi_{t_j}$ hold, it follows from the definitions of $\{\psi_i : i = t_{j-1}, t_{j-1} + 1, \ldots, t_j - 1\}$ and $\{\psi_i^{(\alpha)} : i = t_{j-1}, t_{j-1} + 1, \ldots, t_j\}$ that the components

$$\psi_i^{(\alpha)} = \begin{cases} \psi_i,\ i = t_{j-1}, t_{j-1} + 1, \ldots, p - 1 \\ \phi_{t_j},\ i = p, p + 1, \ldots, t_j \end{cases}\tag{18}$$

can occur in $\alpha(t_{j-1}, t_j)$ while the first line of (17) is preserved. In the case of $\sup I(p, q) < \phi_{t_j}$, we let $\psi_{t_j}^{(\alpha)} = \phi_{t_j}$, which implies $\psi_{t_j - 1} < \psi_{t_j}^{(\alpha)}$ and $|\psi_{t_j}^{(\alpha)} - \phi_{t_j}| = 0$. Hence and since the components $\{\psi_i^{(\alpha)} = \psi_i : i = t_{j-1}, t_{j-1} + 1, \ldots, t_j - 1\}$ are allowed by conditions $y_{t_{j-1}} \leq y_{t_{j-1}+1} \leq \cdots \leq y_{t_j - 1}$, it follows that the components

$$\psi_i^{(\alpha)} = \begin{cases} \psi_i,\ i = t_{j-1}, t_{j-1} + 1, \ldots, t_j - 1 \\ \phi_{t_j},\ i = t_j \end{cases}\tag{19}$$

must occur in $\alpha(t_{j-1}, t_j)$ while the first line of (17) is preserved. Both the cases, (18) and (19), give the relations

$$\psi^{(\alpha)}_{t_j-1} \leq \psi^{(\alpha)}_{t_j} = \phi_{t_j},\ j \text{ odd} \tag{20}$$

and similarly we deduce from the second line of the Eq. (17) that

$$\psi^{(\beta)}_{t_j-1} \geq \psi^{(\beta)}_{t_j} = \phi_{t_j},\ j \text{ even}, \tag{21}$$

where we let $\{\psi^{(\beta)}_i : i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$ be components that occur in $\beta(t_{j-1}, t_j)$, j even.

Next, we continue the proof by establishing the relations

$$\left.\begin{array}{l} \phi_{t_j} = \psi^{(\beta)}_{t_j} \geq \psi^{(\beta)}_{t_j+1},\ j \text{ odd} \\ \phi_{t_j} = \psi^{(\alpha)}_{t_j} \leq \psi^{(\alpha)}_{t_j+1},\ j \text{ even} \end{array}\right\}. \tag{22}$$

To this end, we will first show that $\beta(t_j, t_{j+1}) = \beta(t_j + 1, t_{j+1})$. If we remove the condition $y_{t_j} \geq y_{t_j+1}$ from the calculation of $\beta(t_j, t_{j+1})$, where j is odd, the minimum value of $\sum_{i=t_j}^{t_{j+1}} |y_i - \phi_i|$ subject to $y_{t_j+1} \geq y_{t_j+2} \geq \cdots \geq y_{t_{j+1}}$ is not greater than before, and it is equal to $\beta(t_j + 1, t_{j+1})$. Hence we have the inequality $\beta(t_j, t_{j+1}) \geq \beta(t_j + 1, t_{j+1})$. Strict inequality would imply $\phi_{t_j} < \psi^{(\beta)}_{t_j}$, where without loss of generality, we assume that $\psi^{(\beta)}_{t_j}$ has the smallest value if an option exists. Let q be the greatest integer such that $t_j < q \leq t_{j+1}$ and $\psi^{(\beta)}_{t_j} = \psi^{(\beta)}_{t_j+1} = \cdots = \psi^{(\beta)}_q$, $\psi^{(\beta)}_{t_j}$ being the median of $\{\phi_i : i = t_j, t_j + 1, \ldots, q\}$. In view of the median properties, either there exists an integer κ such that $t_j < \kappa \leq q$ and $\psi^{(\beta)}_\kappa < \phi_\kappa$ or $\psi^{(\beta)}_{t_j} = \phi_{t_j+1} = \phi_{t_j+2} = \cdots = \phi_q$. Now, if $\psi^{(\beta)}_\kappa < \phi_\kappa$, we can increase $\psi^{(\beta)}_\kappa$ to ϕ_κ, which reduces the value of $||\boldsymbol{z}^* - \boldsymbol{\phi}||_1$ and yet, remembering the monotonicity of $\{\psi^{(\alpha)}_i : i = t_{j-1}, t_{j-1}+1, \ldots, t_j\}$, the relations (20) and the inequality $\phi_{t_j} < \psi^{(\beta)}_{t_j}$, allows the inequalities

$$\psi^{(\alpha)}_{t_{j-1}} \leq \cdots \leq \psi^{(\alpha)}_{t_j} = \phi_{t_j} < \psi^{(\beta)}_{t_j} = \cdots = \psi^{(\beta)}_{\kappa-1} < \psi^{(\beta)}_\kappa. \tag{23}$$

Hence by changing t_j to the integer κ, we can restore the conditions (1) and preserve the relations (20). Since these remarks contradict the assumption $\beta(t_j, t_{j+1}) > \beta(t_j + 1, t_{j+1})$, the equation $\beta(t_j, t_{j+1}) = \beta(t_j + 1, t_{j+1})$ follows. A similar contradiction is derived if $\psi^{(\beta)}_{t_j} = \phi_{t_j+1} = \phi_{t_j+2} = \cdots = \phi_q$ upon replacing t_j by $t_j + 1$. Hence and by an argument similar to that given in the paragraph ensuing (17), we obtain the first line of (22), and similarly we can establish the second line of (22). We deduce from (20)–(22) that the equations

$$\psi^{(\alpha)}_{t_j} = \phi_{t_j} = \psi^{(\beta)}_{t_j},\ j \in [1, k-1] \tag{24}$$

hold. Therefore by letting the components of the vector $\boldsymbol{\psi}$ be

$$\psi_{t_j} = \phi_{t_j},\ j = 1, 2, \ldots, k-1 \tag{25}$$

and

$$\psi_i = \begin{cases} \psi_i^{(\alpha)},\ i = t_{j-1}+1, t_{j-1}+2, \ldots, t_j - 1,\ j \text{ odd} \\ \psi_i^{(\beta)},\ i = t_{j-1}+1, t_{j-1}+2, \ldots, t_j - 1,\ j \text{ even}, \end{cases} \tag{26}$$

in view of (24), (25), and the monotonicity properties of $\boldsymbol{\psi}^{(\alpha)}$ and $\boldsymbol{\psi}^{(\beta)}$ on the intervals $[t_{j-1}, t_j]$, we have the bounds

$$\left.\begin{array}{l} \psi_{t_j-1} \le \psi_{t_j} \ge \psi_{t_j+1},\ j \text{ odd} \\ \psi_{t_j-1} \ge \psi_{t_j} \le \psi_{t_j+1},\ j \text{ even} \end{array}\right\}, \tag{27}$$

for all integers j in $[1, k-1]$.

Thus, we have constructed a vector $\boldsymbol{\psi}$ that satisfies the constraints (1) and provides the least value of $||\boldsymbol{y} - \boldsymbol{\phi}||_1$ in the set $Y(k, n)$. This construction allows also the interpolation conditions (25). However, since $\{t_j : \ j = 1, 2, \ldots, k-1\}$ are known, $\boldsymbol{\psi}$ is a solution of the linear programming problem that minimizes (3) subject only to (1). Hence, if $\hat{\boldsymbol{y}}$ is any solution of the linear programming problem that satisfies the same constraints as $\boldsymbol{\psi}$, then

$$||\hat{\boldsymbol{y}} - \boldsymbol{\phi}||_1 = ||\boldsymbol{\psi} - \boldsymbol{\phi}||_1,$$

which shows that $\hat{\boldsymbol{y}}$ is optimal as required. The proof of the theorem is complete. □

Theorem 1 is important in both theory and practice. It reduces the combinatorial optimization problem of Sect. 1 to the equivalent formulation (7) that can be solved by dynamic programming (for a general reference on dynamic programming, see Bellman [2]). The procedure of Demetriou and Powell [7] is quite efficient for this calculation, and subsequently we give a brief description. For any integers $m \in [1, k]$ and $t \in [1, n]$, we let $Y(m, t)$ be the set of t-vectors $\boldsymbol{z}$ with m monotonic sections, we introduce the notation

$$\gamma(m, t) = \min_{z \in Y(m,t)} \sum_{i=1}^{t} |z_i - \phi_i|$$

and we consider the expression

$$\gamma(m, t) = \min_{1=s_0 \le s_1 \le \cdots \le s_k = t} \left\{ \sum_{j=1,\, j \text{ odd}}^{k} \alpha(s_{j-1}, s_j) + \sum_{j=1,\, j \text{ even}}^{k} \beta(s_{j-1}, s_j) \right\}. \tag{28}$$

Theorem 1 implies that if $\{t_j\}$ are optimal and k is odd, then the integer t_{k-1} satisfies the equation

$$\gamma(k-1, t_{k-1}) + \alpha(t_{k-1}, n) = \min_{1\le s\le n} [\gamma(k-1, s) + \alpha(s, n)], \ k \text{ odd}. \tag{29}$$

We see that the least value of the right-hand side of (29) can be found in $O(n)$ computer operations provided that the sequences $\{\gamma(k-1, s) : \ s = 1, 2, \ldots, n\}$ and $\{\alpha(s, n) : \ s = 1, 2, \ldots, n\}$ are available. Therefore in order to calculate $\gamma(k, n)$, which is the least value of (3), we begin with the values $\gamma(1, t) = \alpha(1, t)$, for $t = 1, 2, \ldots, n$ and proceed by applying the formulae

$$\gamma(m, t) = \begin{cases} \min\limits_{1\le s\le t} [\gamma(m-1, s) + \alpha(s, t)], \ m \text{ odd} \\ \min\limits_{1\le s\le t} [\gamma(m-1, s) + \beta(s, t)], \ m \text{ even}, \end{cases} \tag{30}$$

and storing $\tau(m, t)$, which is the value of s that minimizes the right-hand term of expression (30), for $t = 1, 2, \ldots, n$, for each $m = 2, 3, \ldots, k$. Then $\gamma(k, n)$ can be found in $O(kn^2)$ computer operations in addition to the numerical work required to calculate the numbers $\alpha(s, t)$ and $\beta(s, t)$. At the end of the calculation, $m = k$ occurs, and the value $\tau(k, n)$ is the integer t_{k-1} that is required in Eq. (29) if k is odd and analogously if k is even. Hence, we set $t_0 = 1$ and $t_k = n$, and we obtain the sequence of optimal values $\{t_j : \ j = 1, 2, \ldots, k-1\}$ by the backward formula

$$t_{j-1} = \tau(j, t_j), \text{ for } j = k, k-1, \ldots, 2. \tag{31}$$

Finally, the components of $\boldsymbol{y}^*$ are obtained by independent monotonic approximation calculations between adjacent $\{t_j\}$.

Further considerations on improved versions of this calculation, including the development of a relevant Fortran software package, are given by the author in [5, 6]. This software is freely available through the CPC Program Library (http://www.cpc.cs.qub.ac.uk/).

3 Summary

We considered the data smoothing problem that obtains a best L_1 approximation to a set of noisy data subject to the condition that the first differences of the approximated values have at most $k-1$ sign changes, where k is a prescribed integer. Since there are about $O(n^{k-1})$ combinations of positions of sign changes, an exhaustive search for finding the solution is not practicable. Based on the important property that the components of the required approximation consist of sections of separate optimal monotonic approximations, we provided a characterization theorem that reduces the problem to solving a sequence of separate best L_1

monotonic approximation problems to subranges of data, where each monotonic problem can be solved as a special linear program. This reformulation allows a dynamic programming procedure to search for optimal values of the sign change positions, which provides a highly efficient calculation of the solution.

Besides the benefits that we derive from the use of the L_1-norm (see, e.g., Rice [9]) in our approximation calculation, an advantage of the piecewise monotonicity constraints to data approximation is that it gives properties that occur in a wide range of underlying functions. Hence many useful applications in science, engineering, and social sciences may be developed. In addition, the author has written a software package that it would be very useful if it tried on real problems.

Acknowledgements The theorem was motivated by an analogous theorem on the least squares case that had been suggested by my supervisor, the late Professor M.J.D. Powell of Cambridge University. I am most grateful to him for his invaluable advice and guidance [4].

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Supermeasures Associated to Some Classical Inequalities

Silvestru Sever Dragomir

In Honor of Constantin Carathéodory

Abstract In this paper we give some examples of supermeasures that are naturally associated to classical inequalities such as Jensen's inequality, Hölder's inequality, Minkowski's inequality, Cauchy–Bunyakovsky–Schwarz's inequality, Čebyšev's inequality, Hermite–Hadamard's inequalities and the definition of convexity property. As a consequence of monotonic nondecreasing property of these supermeasures, some refinements of the above inequalities are also obtained.

1 Introduction

Let Ω be a nonempty set. A subset $\mathscr{A}$ of the power set 2^{Ω} is called an *algebra* if the following conditions are satisfied:

(1) Ω is in $\mathscr{A}$;
(2) $\mathscr{A}$ is closed under complementation, namely, if $A \in \mathscr{A}$, then $\Omega \setminus A \in \mathscr{A}$;
(3) $\mathscr{A}$ is closed under union, i.e. if $A, B \in \mathscr{A}$ then, $A \cup B \in \mathscr{A}$.

By applying *de Morgan*'s laws, it follows that $\mathscr{A}$ is closed under intersection, namely, if $A, B \in \mathscr{A}$, then $A \cap B \in \mathscr{A}$. It also follows that the empty set $\emptyset$ belongs to $\mathscr{A}$. Elements of the algebra are called measurable sets. An ordered pair $(\Omega, \mathscr{A})$, where Ω is a set and $\mathscr{A}$ is a algebra over Ω, is called a *measurable space*.

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The function $\mu : \mathscr{A} \to [0, \infty)$ is called a *measure* [*submeasure* (*supermeasure*)] on $\mathscr{A}$ if:

(a) For all $A \in \mathscr{A}$, we have $\mu(A) \geq 0$ (nonnegativity);
(aa) We have $\mu(\emptyset) = 0$ (null empty set);
(aaa) For any $A, B \in \mathscr{A}$ with $A \cap B = \emptyset$, we have

$$\mu(A \cup B) = [\leq (\geq)]\, \mu(A) + \mu(B), \tag{1}$$

i.e. μ is *additive* [*subadditive* (*superadditive*)] on $\mathscr{A}$.

For μ as above, we denote by

$$\mathscr{A}_\mu := \{A \in \mathscr{A} \mid \mu(A) > 0\}.$$

If $\mathscr{A}_\mu = \mathscr{A} \setminus \{\emptyset\}$, then we say that μ is *positive* on $\mathscr{A}$.

Let $A, B \in \mathscr{A}$ with $A \subset B$, then $B = A \cup (B \setminus A)$, $A \cap (B \setminus A) = \emptyset$ and $B \setminus A \in \mathscr{A}$. If μ is additive (superadditive), then

$$\mu(B) = \mu(A \cup (B \setminus A)) = (\geq)\, \mu(A) + \mu(B \setminus A) \geq \mu(A)$$

showing that μ is *monotonic nondecreasing* on $\mathscr{A}$.

In this paper we give some examples of supermeasures that are naturally associated to classical inequalities such as Jensen's inequality, Hölder's inequality, Minkowski's inequality, Cauchy–Bunyakovsky–Schwarz's inequality, Čebyšev's inequality, Hermite–Hadamard's inequalities and the definition of convexity property. As a consequence of monotonic nondecreasing property of these supermeasures, some refinements of the above inequalities are also obtained.

2 The Case of Jensen's Inequality

Let $(\Omega, \mathscr{A}, \nu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathscr{A}$ of parts of Ω and a countably additive and positive measure ν on $\mathscr{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a ν-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$, consider the *Lebesgue space*

$$L_w(\Omega, \nu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x) |f(x)|\, d\nu(x) < \infty\}.$$

For simplicity of notation, we write everywhere in the sequel $\int_\Omega w d\nu$ instead of $\int_\Omega w(x)\, d\nu(x)$.

Let also

$$\mathscr{A}_\nu := \{A \in \mathscr{A} \mid \mu(A) > 0\}.$$

For a ν-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) > 0$ for ν -a.e. $x \in \Omega$, we consider the functional $J(\cdot, w; \Phi, f) : \mathscr{A}_\nu \to [0, \infty)$ defined by [8]

$$J(A,w;\Phi,f) := \int_A w(\Phi\circ f)\,d\nu - \Phi\left(\frac{\int_A wfd\nu}{\int_A wd\nu}\right)\int_A wd\nu \geq 0, \tag{2}$$

where $\Phi : [m,M] \to \mathbb{R}$ is a continuous convex function on the interval of real numbers $[m,M]$, $f : \Omega \to [m,M]$ is ν-measurable and such that $f, \Phi\circ f \in L_w(\Omega,\nu)$.

Theorem 1. *Let $\Phi : [m,M] \to \mathbb{R}$ be a continuous convex function on the interval of real numbers $[m,M]$, $f : \Omega \to [m,M]$ is ν-measurable and such that $f, \Phi\circ f \in L_w(\Omega,\nu)$. Then the functional $J(\cdot,w;\Phi,f)$ defined by (3) is a supermeasure on $\mathscr{A}_\nu$.*

Proof. Let $A, B \in \mathscr{A}_\nu$ with $A\cap B = \emptyset$. Observe that

$$\begin{aligned} &J(A\cup B,w;\Phi,f) \\ &= \int_A w(\Phi\circ f)\,d\nu + \int_B w(\Phi\circ f)\,d\nu \\ &\quad - \Phi\left(\frac{\int_A wfd\nu + \int_B wfd\nu}{\int_A wd\nu + \int_B wd\nu}\right)\left(\int_A wd\nu + \int_B wd\nu\right) \\ &= \int_A w(\Phi\circ f)\,d\nu + \int_B w(\Phi\circ f)\,d\nu \\ &\quad - \Phi\left(\frac{\int_A wd\nu \frac{\int_A wfd\nu}{\int_A wd\nu} + \int_B wd\nu \frac{\int_B wfd\nu}{\int_B wd\nu}}{\int_A wd\nu + \int_B wd\nu}\right)\left(\int_A wd\nu + \int_B wd\nu\right) \\ &=: L. \end{aligned} \tag{3}$$

By convexity of the function $\Phi : [m,M] \to \mathbb{R}$ and since

$$\frac{\int_A wfd\nu}{\int_A wd\nu}, \frac{\int_B wfd\nu}{\int_B wd\nu} \in [m,M]$$

we have

$$\begin{aligned} &\Phi\left(\frac{\int_A wd\nu \frac{\int_A wfd\nu}{\int_A wd\nu} + \int_B wd\nu \frac{\int_B wfd\nu}{\int_B wd\nu}}{\int_A wd\nu + \int_B wd\nu}\right) \\ &\qquad \leq \frac{\int_A wd\nu\Phi\left(\frac{\int_A wfd\nu}{\int_A wd\nu}\right) + \int_B wd\nu\Phi\left(\frac{\int_B wfd\nu}{\int_B wd\nu}\right)}{\int_A wd\nu + \int_B wd\nu}. \end{aligned}$$

Therefore by (3) we have

$$\begin{aligned} L \geq &\int_A w(\Phi\circ f)\,d\nu + \int_B w(\Phi\circ f)\,d\nu \\ &- \frac{\int_A wd\nu\Phi\left(\frac{\int_A wfd\nu}{\int_A wd\nu}\right) + \int_B wd\nu\Phi\left(\frac{\int_B wfd\nu}{\int_B wd\nu}\right)}{\int_A wd\nu + \int_B wd\nu}\left(\int_A wd\nu + \int_B wd\nu\right) \end{aligned} \tag{4}$$

$$= \int_A w(\Phi \circ f)\,d\nu - \int_A w d\nu \Phi\left(\frac{\int_A wf d\nu}{\int_A w d\nu}\right)$$
$$+ \int_B w(\Phi \circ f)\,d\nu - \int_B w d\nu \Phi\left(\frac{\int_B wf d\nu}{\int_B w d\nu}\right)$$
$$= J(A, w; \Phi, f) + J(B, w; \Phi, f).$$

Making use of (3) and (4), we conclude that

$$J(A \cup B, w; \Phi, f) \geq J(A, w; \Phi, f) + J(B, w; \Phi, f)$$

for any $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$, which shows that $J(\cdot, w; \Phi, f)$ is a supermeasure on $\mathscr{A}$.

For some Jensen's inequality-related functionals and their properties, see [1, 2, 4, 6, 7, 10, 11, 18, 19, 21].

Let $\Phi : [m, M] \to \mathbb{R}$ be a continuous convex function on the interval of real numbers $[m, M]$, $x = (x_i)_{i \in \mathbb{N}}$ a sequence of real numbers with $x_i \in [m, M]$, $i \in \mathbb{N}$ and $w = (w_i)_{i \in \mathbb{N}}$ a sequence of positive real numbers.

Let $\Omega = \mathbb{N}$ and $\mathscr{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathscr{P}_f(\mathbb{N})$, we have from the above results that the sequence

$$J_n(w; \Phi, x) := \left(\sum_{i=0}^{n} w_i\right)^{-1} \left[\frac{\sum_{i=0}^{n} w_i \Phi(x_i)}{\sum_{i=0}^{n} w_i} - \Phi\left(\frac{\sum_{i=0}^{n} w_i x_i}{\sum_{i=0}^{n} w_i}\right)\right],$$

is *monotonic nondecreasing*, namely,

$$\left(\sum_{i=0}^{n+1} w_i\right)^{-1} \left[\frac{\sum_{i=0}^{n+1} w_i \Phi(x_i)}{\sum_{i=0}^{n+1} w_i} - \Phi\left(\frac{\sum_{i=0}^{n+1} w_i x_i}{\sum_{i=0}^{n+1} w_i}\right)\right] \geq \left(\sum_{i=0}^{n} w_i\right)^{-1} \left[\frac{\sum_{i=0}^{n} w_i \Phi(x_i)}{\sum_{i=0}^{n} w_i} - \Phi\left(\frac{\sum_{i=0}^{n} w_i x_i}{\sum_{i=0}^{n} w_i}\right)\right] \tag{5}$$

for any $n \in \mathbb{N}$ and

$$J_n(w; \Phi, x) \geq \max_{0 \leq i \neq j \leq n} \left\{ (w_i + w_j)^{-1} \left[\frac{w_i \Phi(x_i) + w_j \Phi(x_j)}{w_i + w_j} - \Phi\left(\frac{w_i x_i + w_j x_j}{w_i + w_j}\right)\right]\right\}. \tag{6}$$

We also have for $n \geq 1$ that

$$\left(\sum_{i=0}^{2n} w_i\right)^{-1}\left[\frac{\sum_{i=0}^{2n} w_i \Phi\left(x_i\right)}{\sum_{i=0}^{2n} w_i} - \Phi\left(\frac{\sum_{i=0}^{2n} w_i x_i}{\sum_{i=0}^{2n} w_i}\right)\right] \tag{7}$$

$$\geq \left(\sum_{i=0}^{n} w_{2i}\right)^{-1}\left[\frac{\sum_{i=0}^{n} w_{2i} \Phi\left(x_{2i}\right)}{\sum_{i=0}^{n} w_{2i}} - \Phi\left(\frac{\sum_{i=0}^{n} w_{2i} x_{2i}}{\sum_{i=0}^{n} w_{2i}}\right)\right]$$

$$+ \left(\sum_{i=0}^{n-1} w_{2i+1}\right)^{-1}\left[\frac{\sum_{i=0}^{n-1} w_{2i+1} \Phi\left(x_{2i+1}\right)}{\sum_{i=0}^{n-1} w_{2i+1}} - \Phi\left(\frac{\sum_{i=0}^{n-1} w_{2i+1} x_{2i+1}}{\sum_{i=0}^{n-1} w_{2i+1}}\right)\right]$$

and

$$\left(\sum_{i=0}^{2n+1} w_i\right)^{-1}\left[\frac{\sum_{i=0}^{2n+1} w_i \Phi\left(x_i\right)}{\sum_{i=0}^{2n} w_i} - \Phi\left(\frac{\sum_{i=0}^{2n+1} w_i x_i}{\sum_{i=0}^{2n} w_i}\right)\right] \tag{8}$$

$$\geq \left(\sum_{i=0}^{n} w_{2i}\right)^{-1}\left[\frac{\sum_{i=0}^{n} w_{2i} \Phi\left(x_{2i}\right)}{\sum_{i=0}^{n} w_{2i}} - \Phi\left(\frac{\sum_{i=0}^{n} w_{2i} x_{2i}}{\sum_{i=0}^{n} w_{2i}}\right)\right]$$

$$+ \left(\sum_{i=0}^{n} w_{2i+1}\right)^{-1}\left[\frac{\sum_{i=0}^{n} w_{2i+1} \Phi\left(x_{2i+1}\right)}{\sum_{i=0}^{n} w_{2i+1}} - \Phi\left(\frac{\sum_{i=0}^{n} w_{2i+1} x_{2i+1}}{\sum_{i=0}^{n} w_{2i+1}}\right)\right].$$

3 The Case of Hölder's Inequality

Let $(\Omega, \mathscr{A}, \nu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathscr{A}$ of parts of Ω and a countably additive and positive measure ν on $\mathscr{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a ν-measurable function $w : \Omega \to \mathbb{C}$, with $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$, consider the α-Lebesgue space

$$L_w^{\alpha}(\Omega, \nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_{\Omega} w\,|f|^{\alpha}\,d\nu < \infty\},$$

for $\alpha \geq 1$.

The following inequality is well known in the literature as *Hölder's inequality*

$$\left|\int_{\Omega} wfg d\nu\right| \leq \left(\int_{\Omega} w\,|f|^{\alpha}\,d\nu\right)^{1/\alpha}\left(\int_{\Omega} w\,|g|^{\beta}\,d\nu\right)^{1/\beta} \tag{9}$$

where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $f \in L_w^{\alpha}(\Omega, \nu)$, $g \in L_w^{\beta}(\Omega, \nu)$.

We consider the functional $H_{\alpha,\beta}(\cdot, w; f, g) : \mathscr{A}_\nu \to [0, \infty)$ defined by

$$H_{\alpha,\beta}(A, w; f, g) = \left(\int_A w|f|^\alpha d\nu\right)^{1/\alpha} \left(\int_A w|g|^\beta d\nu\right)^{1/\beta} - \left|\int_A wfgd\nu\right|. \tag{10}$$

We have:

Theorem 2. *Let $f \in L_w^\alpha(\Omega, \nu)$, $g \in L_w^\beta(\Omega, \nu)$ where $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then the functional $H_{\alpha,\beta}(\cdot, w; f, g) : \mathscr{A}_\nu \to [0, \infty)$ defined by (10) is a supermeasure.*

Proof. Let $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$. Observe that

$$\begin{aligned}
&H_{\alpha,\beta}(A \cup B, w; f, g) \qquad (11)\\
&= \left(\int_{A\cup B} w|f|^\alpha d\nu\right)^{1/\alpha} \left(\int_{A\cup B} w|g|^\beta d\nu\right)^{1/\beta} - \left|\int_{A\cup B} wfgd\nu\right| \\
&= \left(\int_A w|f|^\alpha d\nu + \int_B w|f|^\alpha d\nu\right)^{1/\alpha} \left(\int_A w|g|^\beta d\nu + \int_B w|g|^\beta d\nu\right)^{1/\beta} \\
&\quad - \left|\int_A wfgd\nu + \int_B wfgd\nu\right| \\
&= \left(\left[\left(\int_A w|f|^\alpha d\nu\right)^{1/\alpha}\right]^\alpha + \left[\left(\int_B w|f|^\alpha d\nu\right)^{1/\alpha}\right]^\alpha\right)^{1/\alpha} \\
&\quad \times \left(\left[\left(\int_A w|g|^\beta d\nu\right)^{1/\beta}\right]^\beta + \left[\left(\int_B w|g|^\beta d\nu\right)^{1/\beta}\right]^\beta\right)^{1/\beta} \\
&\quad - \left|\int_A wfgd\nu + \int_B wfgd\nu\right| \\
&:= U.
\end{aligned}$$

By the elementary inequality

$$(a^\alpha + b^\alpha)^{1/\alpha} \left(c^\beta + d^\beta\right)^{1/\beta} \geq ac + bd,$$

where $a, b, c, d \geq 0$ and $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and the triangle inequality for modulus, we have

$$U \geq \left(\int_A w|f|^\alpha d\nu\right)^{1/\alpha} \left(\int_A w|f|^\beta d\nu\right)^{1/\beta} \tag{12}$$

$$+\left(\int_B w\,|f|^{\alpha}\,d\nu\right)^{1/\alpha}\left(\int_B w\,|f|^{\beta}\,d\nu\right)^{1/\beta}-\left|\int_A wfgd\nu\right|-\left|\int_B wfgd\nu\right|$$
$$=H_{\alpha,\beta}\,(A,w;f,g)+H_{\alpha,\beta}\,(B,w;f,g)\,.$$

By (11) and (12), we get the desired result.

For some Hölder's inequality-related functionals and their properties, see [3, 5, 20].

Let $\Omega = \mathbb{N}$ and $\mathscr{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathscr{P}_f(\mathbb{N})$, we have from the above results that the sequence

$$H_{n,\alpha,\beta}\,(w;x,y):=\left(\sum_{i=0}^{n}w_i\,|x_i|^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{n}w_i\,|\,y_i|^{\beta}\right)^{1/\beta}-\left|\sum_{i=0}^{n}w_ix_iy_i\right| \tag{13}$$

is monotonic nondecreasing and

$$\begin{aligned}&H_{n,\alpha,\beta}\,(w;x,y)\\&\quad\geq\max_{0\leq i\neq j\leq n}\left\{\left[\left(w_i\,|x_i|^{\alpha}+w_j\,|x_j|^{\alpha}\right)^{1/\alpha}\left(w_i\,|\,y_i|^{\beta}+w_j\,|\,y_j|^{\beta}\right)^{1/\beta}\right.\right.\\&\qquad\left.\left.-\left|w_ix_iy_i+w_jx_jy_j\right|\right]\right\}.\end{aligned} \tag{14}$$

We also have for $n\geq 1$ that

$$\begin{aligned}&\left(\sum_{i=0}^{2n}w_i\,|x_i|^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{2n}w_i\,|\,y_i|^{\beta}\right)^{1/\beta}-\left|\sum_{i=0}^{2n}w_ix_iy_i\right|\\&\geq\left(\sum_{i=0}^{n}w_{2i}\,|x_{2i}|^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{n}w_{2i}\,|\,y_{2i}|^{\beta}\right)^{1/\beta}-\left|\sum_{i=0}^{n}w_{2i}x_{2i}y_{2i}\right|\\&\quad+\left(\sum_{i=0}^{n-1}w_{2i+1}\,|x_{2i+1}|^{\alpha}\right)^{1/\alpha}\left(\sum_{i=0}^{n-1}w_{2i+1}\,|\,y_{2i+1}|^{\beta}\right)^{1/\beta}-\left|\sum_{i=0}^{n-1}w_{2i+1}x_{2i+1}y_{2i+1}\right|.\end{aligned} \tag{15}$$

4 The Case of Minkowski's Inequality

Let $(\Omega,\mathscr{A},\nu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathscr{A}$ of parts of Ω and a countably additive and positive measure ν on $\mathscr{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a ν-measurable function $w:\Omega\to\mathbb{C}$, with $w(x)\geq 0$ for ν-a.e. (almost every) $x\in\Omega$, consider the r-Lebesgue space

$$L_w^r(\Omega,\nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w\,|f|^r\,d\nu < \infty\},$$

for $r \geq 1$.

The following inequality is well known in the literature as Minkowski's inequality:

$$\left(\int_\Omega w\,|f+g|^r\,d\nu\right)^{1/r} \leq \left(\int_\Omega w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_\Omega w\,|g|^r\,d\nu\right)^{1/r} \tag{16}$$

for any $f,g \in L_w^r(\Omega,\nu)$.

Consider the functional $M_r(\cdot,w;f,g) : \mathscr{A}_\nu \to [0,\infty)$ defined by

$$M_r(A,w;f,g) := \left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r - \int_A w\,|f+g|^r\,d\nu. \tag{17}$$

Theorem 3. *Let $f,g \in L_w^r(\Omega,\nu)$ for $r \geq 1$. Then the functional $M_r(\cdot,w;f,g) : \mathscr{A}_\nu \to [0,\infty)$ defined by (17) is a supermeasure.*

Proof. Let $A,B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$. Observe that

$$\begin{aligned}
&M_r(A\cup B,w;f,g) \qquad (18)\\
&= \left[\left(\int_{A\cup B} w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_{A\cup B} w\,|g|^r\,d\nu\right)^{1/r}\right]^r - \int_{A\cup B} w\,|f+g|^r\,d\nu\\
&= \left[\left(\int_A w\,|f|^r\,d\nu + \int_B w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_A w\,|g|^r\,d\nu + \int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r\\
&\quad - \int_A w\,|f+g|^r\,d\nu - \int_B w\,|f+g|^r\,d\nu\\
&= \left[\left(\left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|f|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right.\\
&\quad \left. + \left(\left[\left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right]^r\\
&\quad - \int_A w\,|f+g|^r\,d\nu - \int_B w\,|f+g|^r\,d\nu\\
&=: V.
\end{aligned}$$

By the elementary inequality

$$(a^r + b^r)^{1/r} + (c^r + d^r)^{1/r} \geq [(a+c)^r + (b+d)^r]^{1/r}$$

that holds for $a, b, c, d \geq 0$ and $r \geq 1$, we have

$$\left[(a^r + b^r)^{1/r} + (c^r + d^r)^{1/r}\right]^r \geq (a+c)^r + (b+d)^r . \tag{19}$$

Applying the inequality (19), we have

$$\begin{aligned}
&\left[\left(\left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|f|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right. \\
&\quad \left. + \left(\left[\left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r + \left[\left(\int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r\right)^{1/r}\right]^r \\
&\quad \geq \left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r \\
&\qquad + \left[\left(\int_B w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r
\end{aligned} \tag{20}$$

for $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$.

On making use of (20), we then have

$$\begin{aligned}
V &\geq \left[\left(\int_A w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_A w\,|g|^r\,d\nu\right)^{1/r}\right]^r - \int_A w\,|f+g|^r\,d\nu \\
&\quad + \left[\left(\int_B w\,|f|^r\,d\nu\right)^{1/r} + \left(\int_B w\,|g|^r\,d\nu\right)^{1/r}\right]^r - \int_B w\,|f+g|^r\,d\nu \\
&= M_r\,(A, w; f, g) + M_r\,(B, w; f, g)\,,
\end{aligned}$$

for $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$.

This completes the proof.

For some Minkowski's inequality-related functionals and their properties, see [3, 5, 20].

Let $x = (x_i)_{i \in \mathbb{N}}$ and $x = (y_i)_{i \in \mathbb{N}}$ be sequences of complex numbers and $w = (w_i)_{i \in \mathbb{N}}$ a sequence of positive real numbers. Let $\Omega = \mathbb{N}$ and $\mathscr{P}_f\,(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathscr{P}_f\,(\mathbb{N})$, we have from the above results that the sequence

$$
\begin{aligned}
&M_{n,r}(w;x,y) \\
&:= \left[\left(\sum_{i=0}^{n} w_i |x_i|^r \right)^{1/r} + \left(\sum_{i=0}^{n} w_i |y_i|^r \right)^{1/r} \right]^r - \sum_{i=0}^{n} w_i |x_i + y_i|^r
\end{aligned} \tag{21}
$$

is monotonic nondecreasing and

$$
\begin{aligned}
&M_{n,r}(w;x,y) \\
&\geq \max_{0\leq i\neq j\leq n} \left\{ \left(\left[\left(w_i |x_i|^r + w_j |x_j|^r \right)^{1/r} + \left(w_i |y_i|^r + w_j |y_j|^r \right)^{1/r} \right]^r \right.\right. \\
&\qquad \left.\left. - w_i |x_i + y_i|^r - w_j |x_j + y_j|^r \right) \right\}.
\end{aligned} \tag{22}
$$

We have the inequality

$$
\begin{aligned}
&\left[\left(\sum_{i=0}^{2n} w_i |x_i|^r \right)^{1/r} + \left(\sum_{i=0}^{2n} w_i |y_i|^r \right)^{1/r} \right]^r - \sum_{i=0}^{2n} w_i |x_i + y_i|^r \\
&\geq \left[\left(\sum_{i=0}^{n} w_{2i} |x_{2i}|^r \right)^{1/r} + \left(\sum_{i=0}^{n} w_{2i} |y_{2i}|^r \right)^{1/r} \right]^r - \sum_{i=0}^{n} w_{2i} |x_{2i} + y_{2i}|^r \\
&\quad + \left[\left(\sum_{i=0}^{n-1} w_{2i+1} |x_{2i+1}|^r \right)^{1/r} + \left(\sum_{i=0}^{n-1} w_{2i+1} |y_{2i+1}|^r \right)^{1/r} \right]^r \\
&\quad - \sum_{i=0}^{n-1} w_{2i+1} |x_{2i+1} + y_{2i+1}|^r.
\end{aligned} \tag{23}
$$

5 The Case of Cauchy–Bunyakovsky–Schwarz's Inequality

Consider the Hilbert space

$$
L_w^2(\Omega, \nu) := \{f : \Omega \to \mathbb{C},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w |f|^2 d\nu < \infty\}.
$$

The following inequality is well known in the literature as *Cauchy–Bunyakovsky–Schwarz's (CBS) inequality*:

$$
\left| \int_\Omega wfg d\nu \right| \leq \left(\int_\Omega w |f|^2 d\nu \right)^{1/2} \left(\int_\Omega w |g|^2 d\nu \right)^{1/2} \tag{24}
$$

where $f \in L_w^2(\Omega, \nu)$, $g \in L_w^2(\Omega, \nu)$.

We consider the functional $H(\cdot, w; f, g) : \mathscr{A}_\nu \to [0,\infty)$ defined by

$$H(A, w; f, g) = \left(\int_A w|f|^2 d\nu\right)^{1/2} \left(\int_A w|g|^2 d\nu\right)^{1/2} - \left|\int_A wfgd\nu\right|. \tag{25}$$

Taking into account that $H(A, w; f, g) = H_{\alpha,\beta}(A, w; f, g)$ for $\alpha = \beta = 2$, see (10), we have:

Theorem 4. *Let $f \in L^2_w(\Omega,\nu)$, $g \in L^2_w(\Omega,\nu)$, then the functional $H(\cdot, w; f, g) : \mathscr{A}_\nu \to [0,\infty)$ defined by (25) is a supermeasure.*

Now, consider the functional $L(\cdot, w; f, g) : \mathscr{A}_\nu \to [0,\infty)$ defined by

$$L(A, w; f, g) = \int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2. \tag{26}$$

Theorem 5. *Let $f \in L^2_w(\Omega,\nu)$, $g \in L^2_w(\Omega,\nu)$. Then for any $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$, we have*

$$\begin{aligned} &L(A \cup B, w; f, g) \\ &\quad \geq L(A, w; f, g) + L(B, w; f, g) \\ &\quad + \left(\det\begin{bmatrix} \left(\int_A w|f|^2 d\nu\right)^{1/2} & \left(\int_A w|g|^2 d\nu\right)^{1/2} \\ \left(\int_B w|f|^2 d\nu\right)^{1/2} & \left(\int_B w|g|^2 d\nu\right)^{1/2} \end{bmatrix}\right)^2. \end{aligned} \tag{27}$$

Proof. Let $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$. Then we have

$$\begin{aligned} &L(A \cup B, w; f, g) \\ &= \int_{A\cup B} w|f|^2 d\nu \int_{A\cup B} w|f|^2 d\nu - \left|\int_{A\cup B} wfgd\nu\right|^2 \\ &= \left(\int_A w|f|^2 d\nu + \int_B w|f|^2 d\nu\right)\left(\int_A w|g|^2 d\nu + \int_B w|g|^2 d\nu\right) \\ &\quad - \left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2 \\ &= \int_A w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu \\ &\quad + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu \\ &\quad - \left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2. \end{aligned} \tag{28}$$

Observe that

$$\begin{aligned}
&\left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2 \qquad (29)\\
&= \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2 + 2\operatorname{Re}\left(\int_A wfgd\nu \overline{\int_B wfgd\nu}\right)\\
&\leq \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2 + 2\left|\int_A wfgd\nu \overline{\int_B wfgd\nu}\right|\\
&= \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2 + 2\left|\int_A wfgd\nu\right|\left|\int_B wfgd\nu\right|\\
&\leq \left|\int_A wfgd\nu\right|^2 + \left|\int_B wfgd\nu\right|^2\\
&\quad + 2\left(\int_A w|f|^2 d\nu\right)^{1/2}\left(\int_A w|g|^2 d\nu\right)^{1/2}\\
&\quad \times \left(\int_B w|f|^2 d\nu\right)^{1/2}\left(\int_B w|g|^2 d\nu\right)^{1/2},
\end{aligned}$$

where for the last inequality, we used the (CBS) inequality twice.

Making use of (28) and (29), we get

$$\begin{aligned}
L(A \cup B, w; f, g) &\geq L(A, w; f, g) + L(B, w; f, g)\\
&\quad + \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu\\
&\quad - 2\left(\int_A w|f|^2 d\nu\right)^{1/2}\left(\int_A w|g|^2 d\nu\right)^{1/2}\\
&\quad \times \left(\int_B w|f|^2 d\nu\right)^{1/2}\left(\int_B w|g|^2 d\nu\right)^{1/2}
\end{aligned}$$

and the inequality (27) is proved.

Corollary 1. *Let $f \in L_w^2(\Omega, \nu)$, $g \in L_w^2(\Omega, \nu)$. The functional $L(\cdot, w; f, g) : \mathscr{A}_\nu \to [0, \infty)$ defined by (26) is a supermeasure.*

Let $f \in L_w^2(\Omega, \nu)$, $g \in L_w^2(\Omega, \nu)$. We can also consider the functional $Q(\cdot, w; f, g) : \mathscr{A}_\nu \to [0, \infty)$ defined by

$$\begin{aligned}
Q(A, w; f, g) &= \left[\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2\right]^{1/2} \qquad (30)\\
&= \sqrt{L(A, w; f, g)}.
\end{aligned}$$

Theorem 6. *Let* $f \in L^2_w(\Omega,\nu)$, $g \in L^2_w(\Omega,\nu)$, *then the functional* $H(\cdot, w; f, g) : \mathscr{A}_\nu \to [0,\infty)$ *defined by (30) is a supermeasure.*

Proof. Let $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$. Then we have

$$
\begin{aligned}
&Q^2(A \cup B, w; f, g) \qquad (31)\\
&= \int_{A\cup B} w|f|^2 d\nu \int_{A\cup B} w|f|^2 d\nu - \left|\int_{A\cup B} wfgd\nu\right|^2\\
&= \left(\int_A w|f|^2 d\nu + \int_B w|f|^2 d\nu\right)\left(\int_A w|g|^2 d\nu + \int_B w|g|^2 d\nu\right)\\
&\quad - \left|\int_A wfgd\nu + \int_B wfgd\nu\right|^2\\
&= \int_A w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu\\
&\quad + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu\\
&\quad - \left|\int_A wfgd\nu\right|^2 - \left|\int_B wfgd\nu\right|^2 - 2\,\mathrm{Re}\left(\int_A wfgd\nu \int_B \overline{wfg}d\nu\right)\\
&= Q^2(A, w; f, g) + Q^2(B, w; f, g)\\
&\quad + \int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu\\
&\quad - 2\,\mathrm{Re}\left(\int_A wfgd\nu \overline{\int_B wfgd\nu}\right).
\end{aligned}
$$

On the other hand, by the arithmetic mean-geometric mean inequality, we have

$$
\begin{aligned}
&\int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu \qquad (32)\\
&\quad \geq 2\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu}.
\end{aligned}
$$

By the CBS integral inequality and the properties of modulus, we also have

$$
\begin{aligned}
&\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu} \qquad (33)\\
&\quad \geq \left|\int_A wfgd\nu\right|\left|\int_B wfgd\nu\right| = \left|\int_A wfgd\nu\right|\left|\int_B \overline{wfg}d\nu\right|\\
&\quad = \left|\int_A wfgd\nu \int_B \overline{wfg}d\nu\right| \geq \mathrm{Re}\left(\int_A wfgd\nu \int_B \overline{wfg}d\nu\right).
\end{aligned}
$$

By (32) and (33), we have

$$\begin{aligned}
&\int_A w|f|^2 d\nu \int_B w|g|^2 d\nu + \int_B w|f|^2 d\nu \int_A w|g|^2 d\nu \\
&\quad - 2\operatorname{Re}\left(\int_A wfgd\nu \int_B \overline{wfg}d\nu\right) \\
&\quad \geq 2\left(\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu \int_B w|f|^2 d\nu \int_B w|g|^2 d\nu}\right. \\
&\quad\quad \left. - \left|\int_A wfgd\nu\right| \left|\int_B wfgd\nu\right|\right) \\
&\quad \geq 0.
\end{aligned} \tag{34}$$

By the elementary inequality

$$(ab - cd)^2 \geq \left(a^2 - c^2\right)\left(b^2 - d^2\right), \ d, b, c, d \in \mathbb{R},$$

we have

$$\begin{aligned}
&\left(\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu}\sqrt{\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu} - \left|\int_A wfgd\nu\right|\left|\int_B wfgd\nu\right|\right)^2 \\
&\geq \left(\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2\right) \\
&\quad \times \left(\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu - \left|\int_B wfgd\nu\right|^2\right)
\end{aligned}$$

and since the term in the left bracket is nonnegative, by taking the square root, we get

$$\begin{aligned}
&\sqrt{\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu}\sqrt{\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu} \\
&\quad - \left|\int_A wfgd\nu\right|\left|\int_B wfgd\nu\right| \\
&\quad\quad \geq \left(\int_A w|f|^2 d\nu \int_A w|g|^2 d\nu - \left|\int_A wfgd\nu\right|^2\right)^{1/2} \\
&\quad\quad\quad \times \left(\int_B w|f|^2 d\nu \int_B w|g|^2 d\nu - \left|\int_B wfgd\nu\right|^2\right)^{1/2} \\
&\quad\quad = Q(A, w; f, g)\, Q(B, w; f, g).
\end{aligned} \tag{35}$$

Finally, by (31)–(35), we have

$$Q^2(A\cup B,w;f,g)\geq Q^2(A,w;f,g)+Q^2(B,w;f,g)+2Q(A,w;f,g)\,Q(B,w;f,g)$$
$$=(Q(A,w;f,g)+Q(B,w;f,g))^2$$

and the superadditivity of the mapping $Q(\cdot,w;f,g)$ is proved.

For some CBS's inequality-related functionals and their properties, see [3, 5, 12–14].

Let $x=(x_i)_{i\in\mathbb{N}}$ and $x=(y_i)_{i\in\mathbb{N}}$ be sequences of complex numbers and $w=(w_i)_{i\in\mathbb{N}}$ a sequence of positive numbers. Let $\Omega=\mathbb{N}$ and $\mathscr{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathscr{P}_f(\mathbb{N})$, we have from the above results that the sequences

$$H_n(w;x,y) := \left(\sum_{i=0}^n w_i|x_i|^2\right)^{1/2}\left(\sum_{i=0}^n w_i|y_i|^2\right)^{1/2}-\left|\sum_{i=0}^n w_ix_iy_i\right| \tag{36}$$

and

$$L_n(w;x,y):=\sum_{i=0}^n w_i|x_i|^2\sum_{i=0}^n w_i|y_i|^2-\left|\sum_{i=0}^n w_ix_iy_i\right|^2 \tag{37}$$

are monotonic nondecreasing and

$$H_n(w;x,y) \geq \max_{0\leq i\neq j\leq n}\left[\left(w_i|x_i|^2+w_j|x_j|^2\right)^{1/2}\left(w_i|y_i|^2+w_j|y_j|^2\right)^{1/2}-\left|w_ix_iy_i+w_jx_jy_j\right|\right] \tag{38}$$

and

$$L_n(w;x,y) \geq \max_{0\leq i\neq j\leq n}\left[\left(w_i|x_i|^2+w_j|x_j|^2\right)\left(w_i|y_i|^2+w_j|y_j|^2\right)-\left|w_ix_iy_i+w_jx_jy_j\right|^2\right] \tag{39}$$

for any $p\geq 1$.

Finally, the sequence

$$Q_n(w;x,y) := \left[\sum_{i=0}^{n} w_i |x_i|^2 \sum_{i=0}^{n} w_i |y_i|^2 - \left| \sum_{i=0}^{n} w_i x_i y_i \right|^2 \right]^{1/2} \tag{40}$$

is also monotonic nondecreasing and we have the bound

$$\begin{aligned} &Q_n(w;x,y) \\ &\geq \max_{0\leq i\neq j\leq n} \left[\left(w_i |x_i|^2 + w_j |x_j|^2 \right) \left(w_i |y_i|^2 + w_j |y_j|^2 \right) - \left| w_i x_i y_i + w_j x_j y_j \right|^2 \right]^{1/2}. \end{aligned} \tag{41}$$

6 The Case of Čebyšev's Inequality

We say that the pair of measurable functions (f,g) is *synchronous* on Ω if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \tag{42}$$

for ν-a.e. $x, y \in \Omega$. If the inequality reverses in (42), the functions are called *asynchronous* on Ω.

If (f,g) are synchronous on Ω and $f,\ g,\ fg \in L_w(\Omega,\nu)$, then the following inequality, that is known in the literature as *Čebyšev's Inequality,* holds

$$\int_\Omega w d\nu \int_\Omega wfg d\nu \geq \int_\Omega wf d\nu \int_\Omega wg d\nu, \tag{43}$$

where $w(x) \geq 0$ for ν-a.e. (almost every) $x \in \Omega$.

We consider the *Čebyšev functional* $C(\cdot, w; f, g) : \mathscr{A}_\nu \to [0,\infty)$ defined by

$$C(A, w; f, g) := \int_A w d\nu \int_A wfg d\nu - \int_A wf d\nu \int_A wg d\nu. \tag{44}$$

The following result is known in the literature as *Korkine's identity:*

$$\begin{aligned} &C(A, w; f, g) \\ &= \frac{1}{2} \int_A \int_A w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y). \end{aligned} \tag{45}$$

The proof is obvious by developing the right side of (45) and using Fubini's theorem.

Theorem 7. *Let (f,g) be synchronous on Ω and $f,\ g,\ fg \in L_w(\Omega,\nu)$. Then the Čebyš ev functional defined by (44) is a supermeasure on $\mathscr{A}_\nu$.*

Proof. Let $A, B \in \mathscr{A}_\nu$ with $A \cap B = \emptyset$. Then by (45), we have

$$C(A \cup B, w; f, g) = \frac{1}{2} \int_{A\cup B} \int_{A\cup B} w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y).$$

Since

$$(A \cup B) \times (A \cup B) = (A \times A) \cup (B \times A) \cup (A \times B) \cup (B \times A)$$

then

$$\int_{A\cup B} \int_{A\cup B} = \int_A \int_A + \int_B \int_A + \int_A \int_B + \int_B \int_B .$$

Therefore

$$C(A \cup B, w; f, g) = C(A, w; f, g) + C(B, w; f, g) + \int_A \int_B w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y) \quad (46)$$

since by symmetry reasons,

$$\int_A \int_B w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y) = \int_B \int_A w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y).$$

Now, since (f, g) are synchronous on Ω, then

$$\int_A \int_B w(x) w(y) (f(x) - f(y)) (g(x) - g(y)) d\nu(x) d\nu(y) \geq 0$$

and by (46) we get

$$C(A \cup B, w; f, g) \geq C(A, w; f, g) + C(B, w; f, g)$$

that proves the statement.

For some Čebyšev's inequality-related functionals and their properties, see [3, 15].

Let $x = (x_i)_{i\in\mathbb{N}}$ and $x = (y_i)_{i\in\mathbb{N}}$ be synchronous sequences of real numbers and $w = (w_i)_{i\in\mathbb{N}}$ a sequence of positive real numbers. Let $\Omega = \mathbb{N}$ and $\mathscr{P}_f(\mathbb{N})$ be the algebra of finite parts of natural numbers $\mathbb{N}$. By the monotonicity property of supermeasure on $\mathscr{P}_f(\mathbb{N})$, we have from the above results that the sequence

$$C_n(w;x,y) := \sum_{i=0}^{n} w_i \sum_{i=0}^{n} w_i x_i y_i - \sum_{i=0}^{n} w_i x_i \sum_{i=0}^{n} w_i y_i \geq 0$$

is monotonic nondecreasing and we have the bound

$$C_n(w;x,y) \geq \frac{1}{2} \max_{0\leq i\neq j\leq n} \left\{ w_i w_j \left(x_i - x_j\right)\left(y_i - y_j\right)\right\}.$$

7 The Case of Hermite–Hadamard Inequalities

Let I be an interval consisting of more than one point and $f : I \to \mathbb{R}$ a convex function. If $a, b \in I$ with $a < b$, then we have the well-known *Hermite–Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{47}$$

For some classical results on Hermite–Hadamard inequality, see the monograph online [16].

Suppose $f : I \to \mathbb{R}$ and for $f \in L[a,b]$ define the functionals

$$H([a,b];f) := \int_a^b f(t)\,dt - (b-a)f\left(\frac{a+b}{2}\right)$$

and

$$L([a,b];f) := \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt.$$

We have the following result concerning the properties of these mappings as functions of interval [17]:

Theorem 8. *Let $f : I \to \mathbb{R}$ be a convex function. Then*

(i) For all $a, b, c \in I$ with $a \leq c \leq b$, we have

$$0 \leq H([a,c];f) + H([c,b];f) \leq H([a,b];f) \tag{48}$$

and

$$0 \leq L([a,c];f) + L([c,b];f) \leq L([a,b];f), \tag{49}$$

i.e. the functionals $H(\cdot;f)$ and $L(\cdot;f)$ are superadditive as functions of interval;

(ii) For all $[c, d] \subseteq [a, b] \subseteq I$*, we have*

$$0 \leq H([c, d]; f) \leq H([a, b]; f) \tag{50}$$

and

$$0 \leq L([c, d]; f) \leq L([a, b]; f), \tag{51}$$

i.e. the functionals $H(\cdot; f)$ *and* $L(\cdot; f)$ *are monotonic nondecreasing as functions of interval.*

Proof. (i) Let $c \in [a, b]$ and put $\alpha := (c - a) / (b - a)$, $\beta := (b - c) / (b - a)$. We have $\alpha + \beta = 1$ with $\alpha, \beta \geq 0$ and by the convexity of f, we have with $x = (a + c) / 2$, $y = (b + c) / 2 \in I$ that

$$\begin{aligned}
&\frac{c-a}{b-a} f\left(\frac{a+c}{2}\right) + \frac{b-c}{b-a} f\left(\frac{b+c}{2}\right) \\
&\quad = \alpha f(x) + \beta f(y) \geq f(\alpha x + \beta y) \\
&\quad = f\left(\frac{c-a}{b-a} \cdot \frac{a+c}{2} + \frac{b-c}{b-a} \cdot \frac{b+c}{2}\right) = f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&H([a, b]; f) - H([a, c]; f) - H([c, b]; f) \\
&\quad = (c - a) f\left(\frac{a+c}{2}\right) + (b - c) f\left(\frac{b+c}{2}\right) - (b - a) f\left(\frac{a+b}{2}\right) \geq 0
\end{aligned}$$

and the second part of (48) is proved.

Further, since f is convex on $[a, b]$, we have for all $c \in [a, b]$ that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & c & b \\ f(a) & f(c) & f(b) \end{bmatrix} \geq 0,$$

that is

$$f(a)(b - c) - f(c)(b - a) + f(b)(c - a) \geq 0.$$

Because

$$\begin{aligned}
&L([a,b];f) - L([a,c];f) - L([c,b];f)\\
&\quad = \frac{f(a)+f(b)}{2}(b-a) - \frac{f(a)+f(c)}{2}(c-a) - \frac{f(c)+f(b)}{2}(b-c)\\
&\quad = \frac{1}{2}\left[f(a)(b-c) - f(c)(b-a) + f(b)(c-a)\right]
\end{aligned}$$

we have therefore that the second part of (49) holds also.

The first parts of (48) and (49) are obvious by (47) inequality.

(ii) Follows by (i) and we omit the details.

For an arbitrary function $f : I \to \mathbb{R}$, we introduce the mapping

$$S([a,b];f) := (b-a)\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]$$

where $a, b \in I$ and $a < b$.

We have [17]:

Theorem 9. *Let $f : I \to \mathbb{R}$ a convex function. Then*

(i) For all $a, b, c \in I$ with $a \le c \le b$, we have

$$0 \le S([a,c];f) + S([c,b];f) \le S([a,b];f) \tag{52}$$

i.e. the functional $S(\cdot;f)$ is superadditive as function of interval;

(ii) For all $[c,d] \subseteq [a,b] \subseteq I$, we have

$$0 \le S([c,d];f) \le S([a,b];f) \tag{53}$$

i.e. the functional $H(\cdot;f)$ is monotonic nondecreasing as function of interval.

The proof is immediate from Theorem 8 observing that

$$S([a,b];f) = H([a,b];f) + L([a,b];f).$$

8 The Case of Convex Functions Defined on Intervals

Consider a convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ defined on the interval I of the real line $\mathbb{R}$ and two distinct numbers $a, b \in I$ with $a < b$. We denote by $[a,b]$ the closed segment defined by $\{(1-t)a + tb, t \in [0,1]\}$. We also define the functional of interval

$$\Psi_f([a,b];t) := (1-t)f(a) + tf(b) - f((1-t)a + tb) \geq 0 \tag{54}$$

where $a, b \in I$ with $a < b$ and $t \in [0,1]$.

We have [9]:

Theorem 10. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I. Then for each $a, b \in I$ with $a < b$ and $c \in [a,b]$, we have*

$$(0 \leq)\, \Psi_f([a,c];t) + \Psi_f([c,b];t) \leq \Psi_f([a,b];t) \tag{55}$$

for each $t \in [0,1]$, i.e. the functional $\Psi_f(\cdot;t)$ is superadditive as a function of interval.

If $[c,d] \subset [a,b]$, then

$$(0 \leq)\, \Psi_f([c,d];t) \leq \Psi_f([a,b];t) \tag{56}$$

for each $t \in [0,1]$, i.e., the functional $\Psi_f(\cdot;t)$ is nondecreasing as a function of interval.

Proof. Let $c = (1-s)a + sb$ with $s \in (0,1)$. For $t \in (0,1)$, we have

$$\Psi_f([c,b];t) = (1-t)f((1-s)a+sb) + tf(b) - f((1-t)[(1-s)a+sb]+tb)$$

and

$$\Psi_f([a,c];t) = (1-t)f(a) + tf((1-s)a+sb) - f((1-t)a + t[(1-s)a+sb])$$

giving that

$$\begin{aligned} &\Psi_f([a,c];t) + \Psi_f([c,b];t) - \Psi_f([a,b];t) \\ &\quad = f((1-s)a+sb) + f((1-t)a+tb) \\ &\qquad - f((1-t)(1-s)a + [(1-t)s+t]b) - f((1-ts)a+tsb). \end{aligned} \tag{57}$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \leq s_1$ and $t_2 \leq s_2$, we have that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} \leq \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2}. \tag{58}$$

Indeed, since φ is convex on I, then for any $a \in I$, the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$\psi(t) := \frac{\varphi(t) - \varphi(a)}{t - a}$$

is monotonic nondecreasing where it is defined. Utilising this property repeatedly, we have

$$\frac{\varphi(t_1)-\varphi(t_2)}{t_1-t_2} \le \frac{\varphi(s_1)-\varphi(t_2)}{s_1-t_2} = \frac{\varphi(t_2)-\varphi(s_1)}{t_2-s_1} \le \frac{\varphi(s_2)-\varphi(s_1)}{s_2-s_1} = \frac{\varphi(s_1)-\varphi(s_2)}{s_1-s_2}$$

which proves the inequality (58).

Consider the function $\varphi : [0,1] \to \mathbb{R}$ given by $\varphi(t) := f((1-t)a + tb)$. Since f is convex on I, it follows that φ is convex on $[0,1]$. Now, if we consider for given $t, s \in (0,1)$

$$t_1 := ts < s =: s_1 \text{ and } t_2 := t < t + (1-t)s =: s_2,$$

then we have

$$\varphi(t_1) = f((1-ts)a + tsb), \varphi(t_2) = f((1-t)a + tb)$$

giving that

$$\frac{\varphi(t_1)-\varphi(t_2)}{t_1-t_2} = \frac{f((1-ts)a+tsb) - f((1-t)a+tb)}{t(s-1)}.$$

Also

$$\varphi(s_1) = f((1-s)a + sb), \varphi(s_2) = f((1-t)(1-s)a + [(1-t)s + t]b)$$

giving that

$$\frac{\varphi(s_1)-\varphi(s_2)}{s_1-s_2} = \frac{f((1-s)a+sb) - f((1-t)(1-s)a + [(1-t)s+t]b)}{t(s-1)}.$$

Utilising the inequality (58) and multiplying with $t(s-1) < 0$, we deduce the inequality

$$f((1-ts)a + tsb) - f((1-t)a + tb) \ge f((1-s)a + sb) - f((1-t)(1-s)a + [(1-t)s+t]b). \tag{59}$$

Finally, by (57) and (59), we get the desired result (55).

Applying repeatedly the superadditivity property, we have for $[c,d] \subset [a,b]$ that

$$\Psi_f([a,c];t) + \Psi_f([c,d];t) + \Psi_f([d,b];t) \le \Psi_f([a,b];t)$$

giving that

$$0 \le \Psi_f([a,c];t) + \Psi_f([d,b];t) \le \Psi_f([a,b];t) - \Psi_f([c,d];t)$$

which proves (56).

For $t = \frac{1}{2}$, we consider the functional

$$\Psi_f([a,b]) := \Psi_f\left([a,b];\frac{1}{2}\right) = \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Psi_f(\cdot,\cdot;t)$. We are able then to state the following:

Corollary 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $a,b \in I$. Then we have the bounds*

$$\inf_{c\in[a,b]}\left[f\left(\frac{a+c}{2}\right) + f\left(\frac{c+b}{2}\right) - f(c)\right] = f\left(\frac{a+b}{2}\right) \tag{60}$$

and

$$\sup_{c,d\in[a,b]}\left[\frac{f(c)+f(d)}{2} - f\left(\frac{c+d}{2}\right)\right] = \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right). \tag{61}$$

Proof. By the superadditivity of the functional $\Psi_f(\cdot,\cdot)$, we have for each $c \in [a,b]$ that

$$\begin{aligned}\frac{f(a)+f(b)}{2} &- f\left(\frac{a+b}{2}\right)\\ &\ge \frac{f(a)+f(c)}{2} - f\left(\frac{a+c}{2}\right) + \frac{f(c)+f(b)}{2} - f\left(\frac{c+b}{2}\right),\end{aligned}$$

which is equivalent to

$$f\left(\frac{a+c}{2}\right) + f\left(\frac{c+b}{2}\right) - f(c) \ge f\left(\frac{a+b}{2}\right). \tag{62}$$

Since the equality case in (62) is realised for either $c = a$ or $c = b$, we get the desired bound (60).

The bound (61) is obvious by the monotonicity of the functional $\Psi_f(\cdot)$ as a function of interval.

Consider now the following functional:

$$\Gamma_f([a,b];t) := f(a) + f(b) - f((1-t)a + tb) - f((1-t)b + ta),$$

where, as above, $f : I \subset a \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, while $t \in [0,1]$.

We notice that

$$\Gamma_f([a,b];t) = \Gamma_f([a,b];1-t)$$

and

$$\Gamma_f([a,b];t) = \Psi_f([a,b];t) + \Psi_f([a,b];1-t) \geq 0$$

for any $a, b \in I$ with $a < b$ and $t \in [0,1]$.

Therefore, we can state the following result as well:

Corollary 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $t \in [0,1]$. The functional $\Gamma_f(\cdot;t)$ is superadditive and monotonic nondecreasing as a function of interval.*

In particular, if $c \in [a,b]$, then we have the inequality

$$\begin{aligned}&\frac{1}{2}[f((1-t)a + tb) + f((1-t)b + ta)] \\ &\quad \leq \frac{1}{2}[f((1-t)a + tc) + f((1-t)c + ta)] \\ &\qquad + \frac{1}{2}[f((1-t)c + tb) + f((1-t)b + tc)] - f(c)\end{aligned} \tag{63}$$

Also, if $c, d \in [a,b]$, then we have the inequality

$$\begin{aligned}&f(a) + f(b) - f((1-t)a + tb) - f((1-t)b + ta) \\ &\quad \geq f(c) + f(d) - f((1-t)c + td) - f((1-t)c + td)\end{aligned} \tag{64}$$

for any $t \in [0,1]$.

Perhaps the most interesting functional we can consider from the above is the following one:

$$\begin{aligned}\Theta_f([a,b]) &:= \frac{f(a) + f(b)}{2} - \int_0^1 f((1-t)a + tb)\,dt \\ &= \frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds \geq 0,\end{aligned} \tag{65}$$

which is related to the second Hermite–Hadamard inequality.

We observe that

$$\Theta_f([a,b]) = \int_0^1 \Psi_f([a,b];t)\,dt = \int_0^1 \Psi_f([a,b];1-t)\,dt. \tag{66}$$

Utilising this representation, we can state the following result as well:

Corollary 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $t \in [0,1]$. The functional $\Theta_f(\cdot,\cdot)$ is superadditive and monotonic nondecreasing as a function of interval. Moreover, we have the bounds*

$$\inf_{c\in[a,b]}\left[\frac{1}{c-a}\int_a^c f(s)\,ds + \frac{1}{b-c}\int_c^b f(s)\,ds - f(c)\right] = \frac{1}{b-a}\int_a^b f(s)\,ds \tag{67}$$

and

$$\sup_{c,d\in[a,b]}\left[\frac{f(c)+f(d)}{2} - \frac{1}{c-d}\int_d^c f(s)\,ds\right] \tag{68}$$
$$= \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds.$$

For extension of this section's results in the case of convex functions defined on intervals incorporated in convex sets in linear spaces, see [9].

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Remarks on Solutions of a Functional Equation Arising from an Asymmetric Switch

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In Honor of Constantin Carathéodory

Abstract During the last five decades, various functional equations possessing a certain structure popped up in many modern disciplines like queuing theory, communication and networks. There is no universal solution methodology available for them, and the closed-form solutions are known only in some particular cases. We address several issues concerning solutions of one of such functional equations, arising in a model of the clocked buffered switch that has been represented as a queueing system.

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1 Introduction

The 2×2 switch is a well-known device used in data-processing networks for routing messages from one node to another. The switch is simply to handle the messages from one node to the other. Its function has been modelled as a two-server, time-slotted, queueing system. In this way, a two-place functional equation (FE) has been obtained in [5], in which the unknowns are generating functions of the switch distribution. A solution of the functional equation arising from such an asymmetric switch has been described there, but not in a closed form as it is given in terms of infinite products.

Similar equations appear in [7, 9, 11, 13–15, 18, 19]. Some of them have been solved by techniques involving some tools from the theory of boundary value problems (extensive treatments of such techniques can be found in, e.g. [4, 6, 12]) and Rouché's theorem. Usually, the solutions obtained in these ways are not closed-form solutions.

We address several issues concerning solutions of one of such functional equations that was introduced in [5]. In particular, we present a description of solution in the symmetric case of the equation (obtained by assuming the full symmetry in the system parameters). It is achieved by a reduction to the Riemann–Hilbert boundary value problem through some conformal mapping.

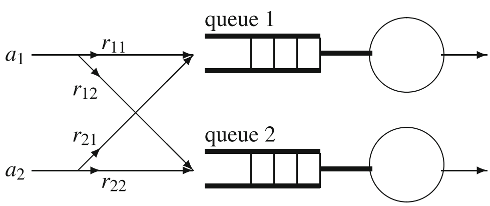
2 The Functional Equation and Its Solution

The equation arises in [5] from a description of a 2×2 clocked buffered switch, illustrated in Fig. 1; the message handling process of this switch is modelled as a two-server, time-slotted, queueing process with the state space of the pairs of numbers of messages (x_n, y_n) present at the servers at the end of a time slot.

The probability generating function (PGF) $f : \overline{D} \to \mathbb{C}$ (D denotes the open unit disc in the complex plane $\mathbb{C}$) of the two-dimensional distribution characterizing the system fulfils the two-place functional equation

$$(xy - \phi(x,y))f(x,y) = \phi(x,y)\big[(y-1)f(x,0) + (x-1)f(0,y) + (x-1)(y-1)f(0,0)\big], \tag{1}$$

Fig. 1 Asymmetric 2×2 switch modelled as a queueing system



where

$$\phi(x,y) = [1 - a_1 + a_1(r_{11}x + r_{12}y)][1 - a_2 + a_2(r_{21}x + r_{22}y)], \tag{2}$$

a_1, a_2 are the probability that the arrival stream generated at the start of a slot is of stream $1, 2$, respectively, and $r_{i,j}$ is the probability that i-arrival joins the queue of the j-th service facility, for $i, j = 1, 2$. Equation (1) has been solved with the analytic continuation technique, by locating the zeros and poles of the unknown functions $f(x,0), f(0,y)$, which eventually has been described by the formulas

$$f(x,0) = f(1,0)\frac{P^{(I)}(1)P^{(II)}(1)}{P^{(I)}(x)P^{(II)}(x)}\frac{A^{(I)}(x)A^{(II)}(x)}{A^{(I)}(1)A^{(II)}(1)},$$
$$f(0,y) = f(0,1)\frac{Q^{(I)}(1)Q^{(II)}(1)}{Q^{(I)}(y)Q^{(II)}(y)}\frac{\Gamma^{(I)}(y)\Gamma^{(II)}(y)}{\Gamma^{(I)}(1)\Gamma^{(II)}(1)},$$

for all $x, y \in \overline{D}$, where

$$f(1,0) = 1 - a_2 r_{22} - a_1 r_{12},$$
$$f(0,1) = 1 - a_1 r_{11} - a_2 r_{21},$$
$$f(0,0) = (1 - a_2 r_{22} - a_1 r_{12})\frac{Q^{(I)}(1)Q^{(II)}(1)}{\Gamma^{(I)}(1)\Gamma^{(II)}(1)},$$

and the functions $P^{(I)}$, $P^{(II)}$, $Q^{(I)}$, $Q^{(II)}$, $A^{(I)}$, $A^{(II)}$, $\Gamma^{(I)}$ and $\Gamma^{(II)}$ are defined in [5] by some infinite products.

3 General Observations

A very important information that we use in solving Eq. (1) is the fact that the unknown function f is a PGF of a sequence of nonnegative real numbers $p_{m,n}$ $(m, n = 0, 1, 2, \ldots)$ with the normalization condition

$$\sum_{m,n=0}^{\infty} p_{m,n} = 1; \tag{3}$$

so it has the following form

$$f(x,y) = \sum_{m,n=0}^{\infty} p_{m,n} x^m y^n, \qquad x, y \in \overline{D}. \tag{4}$$

Clearly, f is analytic with respect to either variable separately in the closed unit disc $\overline{D}$ (i.e. it is analytic at every point of D and continuous at every point of $\overline{D}$). Moreover, condition (3) means that

$$f(1,1) = 1. \tag{5}$$

The next important information is the observation following from the form of Eq. (1) that if $xy - \phi(x, y) = 0$ for some $x, y \in \overline{D}$, then also

$$\phi(x, y)\big[(y-1)f(x, 0) + (x-1)f(0, y) + (x-1)(y-1)f(0, 0)\big] = 0. \tag{6}$$

So, the problem of finding a solution of the original functional equation (1) is now reduced to the issue of solving the conditional functional equation

$$\phi(x, y)\big[(y-1)f(x, 0) + (x-1)f(0, y) + (x-1)(y-1)f(0, 0)\big] = 0, \tag{7}$$
$$x, y \in \overline{D},\ xy = \phi(x, y).$$

Write

$$\mathscr{K} := \{(x, y) \in \overline{D}^2 : \phi(x, y) = xy\},$$
$$\mathscr{K}_0 := \{x \in \overline{D} : (x, 0) \in \mathscr{K}\} = \{x \in \overline{D} : \phi(x, 0) = 0\},$$
$$\mathscr{K}^0 := \{y \in \overline{D} : (0, y) \in \mathscr{K}\} = \{y \in \overline{D} : \phi(0, y) = 0\}.$$

It is obvious that the function $f(x, y) \equiv 0$ is a solution to (1). We will present several other observations concerning solutions to the equation, which are analytic or (only) continuous. We assume all the time that the following four hypotheses are satisfied:

(a) $r_{11} \neq 0$ or $a_1 \neq 1$.
(b) $r_{21} \neq 0$ or $a_2 \neq 1$.
(c) $r_{12} \neq 0$ or $a_1 \neq 1$.
(d) $r_{22} \neq 0$ or $a_2 \neq 1$.

Certainly, we should supply some comments on those hypotheses. So, to this end suppose, for instance, that (a) does not hold. Then $\mathscr{K}_0 = \overline{D}$ and

$$\phi(x, y) = r_{12}y[1 - a_2 + a_2(r_{21}x + r_{22}y)], \qquad x, y \in \overline{D}.$$

Let $f : \overline{D}^2 \to \mathbb{C}$ be a solution to (1). Clearly, (1) implies that

$$\begin{aligned}(x - r_{12}[1 - a_2 + a_2(r_{21}x + r_{22}y)])f(x, y) \\ = r_{12}[1 - a_2 + a_2(r_{21}x + r_{22}y)]\big[(y-1)f(x, 0) \\ + (x-1)f(0, y) + (x-1)(y-1)f(0, 0)\big], \qquad x, y \in \overline{D},\end{aligned} \tag{8}$$

whence, with $y \to 0$ we get

$$xf(x, 0) = 0, \qquad x \in \overline{D}, \tag{9}$$

which means that $f(x,0) \equiv 0$. So, f fulfils the functional equation

$$\begin{aligned}(xy - \phi(x,y))\, f(x,y) \\ = \ \phi(x,y)\big[(x-1)f(0,y) + (x-1)(y-1)f(0,0)\big].\end{aligned} \tag{10}$$

It is easily seen that the assumption that one of the hypotheses (b)–(d) is not fulfilled leads to a similar equation.

We are not going to study those equations here. Let us now return to the general case of (1).

We have the following simple observation.

Theorem 1. *If an analytic (continuous, respectively) function* $f : \overline{D}^2 \to \mathbb{C}$ *is a solution to Eq.* (1)*, then there exist analytic (continuous, resp.) functions* $\xi, \eta : \overline{D} \to \mathbb{C}$ *such that* $\xi(0) = \eta(0)$*,*

$$(y-1)\xi(x) + (x-1)\eta(y) + (x-1)(y-1)\xi(0) = 0 \tag{11}$$

for $(x,y) \in \mathscr{K}$ *and*

$$f(x,y) = \frac{\phi(x,y)\big[(y-1)\xi(x) + (x-1)\eta(y) + (x-1)(y-1)\xi(0)\big]}{xy - \phi(x,y)}, \qquad (x,y) \in \overline{D}^2 \setminus \mathscr{K}. \tag{12}$$

Moreover, if hypotheses (a)–(d) hold, then every analytic (continuous, resp.) function $f : \overline{D}^2 \to \mathbb{C}$ *fulfilling* (12)*, with some analytic (continuous, resp.) functions* $\xi, \eta : \overline{D} \to \mathbb{C}$ *such that* $\xi(0) = \eta(0)$ *and condition* (11) *is satisfied, is a solution to Eq.* (1)*.*

Proof. The reasonings are simple and straightforward, but for the convenience of readers, we present them here. We consider only the “analytic case”; the “continuous case” is analogous.

First assume that an analytic function $f : \overline{D} \to \mathbb{C}$ is a solution to Eq. (1). Write

$$\xi(x) := f(x,0), \qquad \eta(x) := f(0,y), \qquad x \in \overline{D}.$$

Clearly, $\xi(0) = f(0,0) = \eta(0)$. Next, take $(x,y) \in \mathscr{K}$. Then $xy - \phi(x,y) = 0$, whence

$$\begin{aligned} 0 &= (xy - \phi(x,y))f(x,y) \\ &= \phi(x,y)\big[(y-1)f(x,0) + (x-1)f(0,y) + (x-1)(y-1)f(0,0)\big] \\ &= \phi(x,y)\big[(y-1)\xi(x) + (x-1)\eta(y) + (x-1)(y-1)\xi(0)\big]. \end{aligned} \tag{13}$$

It is easily seen that in the case $0 \neq xy = \phi(x,y)$, we get (11). So, it remains to consider the case $0 = xy = \phi(x,y)$. But it is possible when $x = 0$ or $y = 0$ and only for finitely many points. So, by the continuity of f, (11) hold for all $(x,y) \in \mathscr{K}$. Finally, observe that (12) results directly from (1).

Now, assume that hypotheses (a)–(d) are fulfilled. Let a function $f : \overline{D} \to \mathbb{C}$ be analytic and (12) hold for some analytic functions $\xi, \eta : \overline{D} \to \mathbb{C}$ such that $\xi(0) = \eta(0)$ and (11) is valid for all $(x, y) \in \mathscr{K}$. We show that f is a solution to Eq. (1).

It is easily seen that each of the sets $\mathscr{K}_0$ and $\mathscr{K}^0$ has at most two elements (in view of (a)–(d)). Hence, by the continuity of f and ξ and η, we obtain that $\xi(x) = f(x, 0)$ and $\eta(x) = f(0, x)$ for each $x \in \overline{D}$.

Take $x, y \in \overline{D}$. If $(x, y) \in \mathscr{K}$, then $xy - \phi(x, y) = 0$ and (11) holds, and consequently

$$\begin{aligned}&(xy - \phi(x, y))f(x, y)\\&\quad = 0 = \phi(x, y)\big[(y - 1)\xi(x) + (x - 1)\eta(y) + (x - 1)(y - 1)\xi(0)\big]\\&\quad = \phi(x, y)\big[(y - 1)f(x, 0) + (x - 1)f(0, y) + (x - 1)(y - 1)f(0, 0)\big].\end{aligned}$$

If $(x, y) \notin \mathscr{K}$, then (12) implies (1). □

4 The Issue of Uniqueness of Solutions

One of the important issues in solving a functional equation is how many solutions it has in a given class of functions. Let us try to address that question in the case of continuous solutions of Eq. (1), under the assumption that hypotheses (a)–(d) are valid.

So, suppose that $f : \overline{D} \to \mathbb{C}$ is a continuous solution to (1). Then, as we can see from the proof of Theorem 1, the functions $\xi, \eta : \overline{D} \to \mathbb{C}$, given by:

$$\xi(x) := f(x, 0), \qquad \eta(x) := f(0, x), \qquad x \in \overline{D},$$

satisfy condition (11), which implies that

$$\frac{\xi(x)}{x - 1} + \frac{\xi(0)}{2} + \frac{\eta(y)}{y - 1} + \frac{\eta(0)}{2} = 0, \qquad (x, y) \in \mathscr{K}, x \neq 1, y \neq 1. \tag{14}$$

Write

$$\xi_0(x) := \frac{\xi(x)}{x - 1} + \frac{\xi(0)}{2}, \qquad \eta_0(x) := -\frac{\eta(y)}{y - 1} - \frac{\eta(0)}{2}, \qquad x \in \overline{D}, x \neq 1. \tag{15}$$

Then, by (14),

$$\xi_0(x) = \eta_0(y), \qquad (x, y) \in \mathscr{K}, x \neq 1, y \neq 1. \tag{16}$$

Let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function such that $F(\xi_0(0)) = F(\eta_0(0))$, $\xi_1 := F \circ \xi_0$ and $\eta_1 := F \circ \eta_0$. Then $\xi_1(0) = \eta_1(0)$ and (16) yields

$$\xi_1(x) = \eta_1(y), \qquad (x, y) \in \mathscr{K}, x \neq 1, y \neq 1. \tag{17}$$

Define $\xi_2, \eta_2 : \overline{D} \setminus \{1\} \to \mathbb{C}$ by

$$\xi_2(x) := (x-1)(\xi_1(x) + z_0), \qquad \eta_2(x) := (1-x)(\eta_1(x) + z_0), \tag{18}$$
$$x \in \overline{D}, x \neq 1,$$

with a fixed $z_0 \in \mathbb{C}$. It is easily seen that (17) and (18) imply that

$$\frac{\xi_2(x)}{x-1} + \frac{\xi_2(0}{2} + \frac{\eta_2(y)}{y-1} + \frac{\eta_2(0}{2} = 0, \qquad (x, y) \in \mathscr{K}, x \neq 1, y \neq 1, \tag{19}$$

which means that

$$(y-1)\xi_2(x) + (x-1)\eta_2(y) + (x-1)(y-1)\eta_2(0) = 0,$$
$$(x, y) \in \mathscr{K}, x \neq 1, y \neq 1. \tag{20}$$

If F is such that ξ_2 and η_2 have finite limits at 1, then we can extend them to that point and have

$$(y-1)\xi_2(x) + (x-1)\eta_2(y) + (x-1)(y-1)\eta_2(0) = 0, \qquad (x, y) \in \mathscr{K}. \tag{21}$$

Consequently, by Theorem 1, each continuous function $f_2 : \overline{D}^2 \to \mathbb{C}$ such that

$$f_2(x, y) := \frac{(y-1)\xi_2(x) + (x-1)\eta_2(y) + (x-1)(y-1)\eta_2(0)}{xy - \phi(x, y)}, \tag{22}$$
$$(x, y) \in \overline{D}^2 \setminus \mathscr{K},$$

is a solution to Eq. (1). If we choose the function F in a suitable way, then the values of f_2 at the points of $\mathscr{K}$ are uniquely determined by (22), because that set is nowhere dense in $\overline{D}^2$.

5 The Symmetric Case

Assume full symmetry in the system under study, i.e. let

$$a_1 = a_2 =: a, \qquad r_{ij} = 1/2, \qquad i, j = 1, 2.$$

So, condition (2) takes the form

$$\phi(x, y) = \left[1 - a + \frac{a(x + y)}{2}\right]^2 \tag{23}$$

and

$$\mathscr{K} = \left\{(x, y) \in \overline{D} : \left[1 - a + \frac{a(x + y)}{2}\right]^2 = xy\right\}.$$

Write

$$\mathscr{K}_1 := \{(x, y) \in \mathscr{K} : (x - 1)(y - 1) \neq 0\}.$$

Then (7) implies that

$$\frac{f(x, 0)}{x - 1} + \frac{f(0, y)}{y - 1} + f(0, 0) = 0, \qquad (x, y) \in \mathscr{K}_1. \tag{24}$$

Introduce the function

$$F(x) := \frac{f(x, 0)}{x - 1} + \frac{1}{2}f(0, 0).$$

Then Eq. (24) can be rewritten as

$$F(x) + F(y) = 0, \qquad (x, y) \in \mathscr{K}_1, \tag{25}$$

where the function F is analytic in $\overline{D}$, except possibly at 1, where a simple pole may occur.

Now, we have reduced the issue of solving the symmetric case of the main functional equation to the problem of finding solutions to (25).

6 Boundary Value Problem

Now, we will show that the issue of solving (25) can be reduced to a boundary value problem. Certainly, it can be done in several different ways. We will present a very simple and natural example of such reduction.

Namely, let

$$L_0 := \{x \in \overline{D} : (x, \overline{x}) \in \mathscr{K}\},$$

where $\overline{x}$ is the complex conjugate of x. Using this special set, we can derive from Eq. (25) the following condition

$$F(x) + F(\overline{x}) = 0, \qquad x \in L_0, x \neq 1. \tag{26}$$

Since the coefficients a_{ij} defining the function f are real numbers (see (4)), we have

$$f(\overline{x}) = \overline{f(x)}, \qquad x \in \overline{D}.$$

Consequently (26) yields

$$\Re F(x) = 0, \qquad x \in L_0, x \neq 1, \tag{27}$$

where $\Re z$ stands for the real part of the complex number z.

In this way we have obtained the boundary value problem to determine a function F, which is analytic, except possibly at a simple pole at 1, and satisfies the following two conditions:

1. $\Re F(z) = 0$ for $z \in L_0 \setminus \{1\}$.
2. $\lim_{x \to 1}(x-1)F(x) = f(1,0) = 1 - a$.

The equality $f(1,0) = 1 - a$ has been derived from (1) (with $x = y$) and the normalization condition: $f(1,1) = 1$ (which is a consequence of the fact that f is a PGF).

In order to solve the boundary value problem, described above, we follow the classical approach (see, e.g. [18]) as follows: let Λ (see Fig. 2) be the conformal function (cf., e.g. [1, 2, 8, 17]) that maps the unit disc onto the region bounded by the curve L, given by

$$\begin{aligned} L &= \{(x,\overline{x}) : x \in L_0\} \\ &= \{(x,\overline{x}) : x\overline{x} - \phi(x,\overline{x}) = 0\} \\ &= \{(x,\overline{x}) : |x|^2 = [1 - a + a\Re\, x]^2\}, \end{aligned}$$

with the normalization conditions $\Lambda(0) = 0$, $\Lambda(1) = 1$. Such a mapping exists by the Riemann mapping theorem, because the curve L encloses a simply connected domain. It is easily seen that actually the curve L is an ellipse. Let Ω denote the function inverse to Λ.

We obtain a relatively simple Riemann–Hilbert boundary value problem with a pole, for $H := F \circ \Lambda$ on the unit circle D. Actually, it is a Dirichlet problem with a pole (see [6, Chap. 1]): to determine the function H, which is analytic on D, continuous on $\overline{D} \setminus \{1\}$ and such that

$$\Re H(w) = 0, \qquad w \in \overline{D} \setminus \{1\},$$

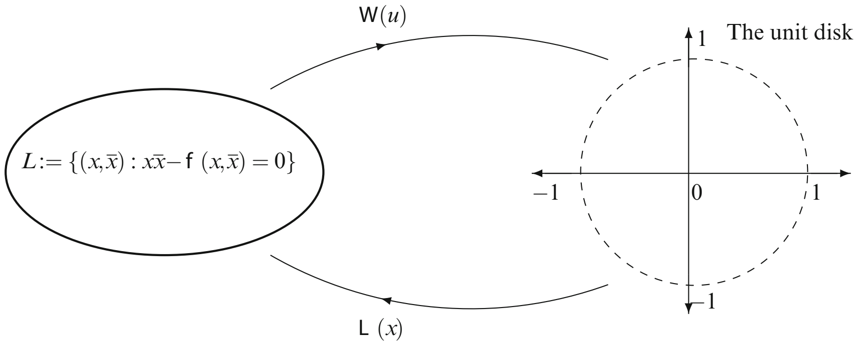


Fig. 2 The conformal mapping from L onto the unit disc and its inverse

$$\lim_{w \to 1} (w - 1)H(w) = \frac{1-a}{\Lambda_x(1)},$$

where

$$\Lambda_x := \frac{d\Lambda}{dx}.$$

The solution of this boundary value problem is (see, e.g. [6, Chap. 1])

$$H(w) = \frac{1}{2}\,\frac{1-a}{\Lambda_x(1)}\,\frac{w+1}{w-1}, \qquad w \in D,$$

which means that

$$F(x) = H(\Omega(x)) = \frac{1}{2}\,\frac{1-a}{\Lambda_x(1)}\,\frac{\Omega(x)+1}{\Omega(x)-1}$$

inside the curve L. Substitution in the original equation finally yields

$$f(x, y) \equiv (1-a)\Omega_x(1)\,\frac{(x-1)(y-1)\phi(x, y)}{(\Omega(x)-1)(\Omega(y)-1)}\,\frac{\Omega(x)\Omega(y)-1}{xy-\phi(x, y)}. \tag{28}$$

The above formula represents a possible solution of the original equation. It should be yet validated; for instance, it should be checked if it satisfies the normalization condition, i.e. the equality

$$f(1, 1) = 1.$$

Let us mention yet that it is well known (see, e.g. [10, 16]) that the conformal mappings used to map an ellipsoid region to the unit disc can be explicitly expressed in terms of the Jacobi elliptic functions (see, e.g. [3]).

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Hyers–Ulam Stability of Wilson's Functional Equation

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In Honor of Constantin Carathéodory

Abstract Given a unitary character $\mu : G \to \mathbf{C}$ and an involution σ of a group G, we study the Hyers–Ulam–Rassias stability of Wilson's functional equations:

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y),\ x, y \in G,$$
$$f(xy) + \mu(y)f(x\sigma(y)) = 2g(x)f(y),\ x, y \in G.$$

As a consequence, we find the superstability of d'Alembert's functional equation:

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y),\ x, y \in G.$$

1 Introduction

The following version of d'Alembert's functional equation

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)g(y),\ x, y \in G \tag{1}$$

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respectively, Wilson's functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y),\ x, y \in G \tag{2}$$

is a generalization of classical d'Alembert's functional equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y),\ x, y \in G \tag{3}$$

respectively of Wilson's classical functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y),\ x, y \in G, \tag{4}$$

where $\mu : G \to \mathbf{C} \setminus \{0\}$ is a character and σ is an involution of G. The nonzero solutions of Eq. (3) are the normalized traces of certain representation of G on $\mathbf{C}^2$. The result was obtained by Davison via his work [13] on the pre-d'Alembert functional equation on monoids.

Several authors are interested to study d'Alembert's and Wilson's functional equations on abelian groups, we refer, for example, to the monograph by Aczél [1]. There has been quite a development of the theory of d'Alembert's functional equation on non-abelian groups. For more details we refer the reader to [2, 13–18, 20, 31–35, 41].

The continuous, complex-valued solutions of d'Alembert equation (1) with $\sigma(x) = x^{-1}, x \in G$ are obtained by Stetkaer [35].

The basic link between Wilson's and d'Alembert's functional equations was due to Corovei [12]: Given any group G and the pair (f, g) is a solution of Wilson's functional equation (4) such that $f \neq 0$, then g satisfies d'Alembert long functional equation: $g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y)$ for all $x, y \in G$.

In [37], Stetkaer upgraded the Corovei result and got the following strong result: if (f, g) is a solution of Wilson's functional equation, (4) such that $f \neq 0$, then g satisfies d'Alembert's short functional equation (3). Bouikhalene and Elqorachi [9] extended Stetkaer result's [37] to Wilson's functional equation (2) as follows:

Proposition 1 ([9]). *If μ is an involutive multiplicative automorphism of a monoid G such that $\mu(x\sigma(x)) = 1$ for all $x \in G$ and $(f, g) : G \to \mathbf{C}$ is a solution of Wilson's functional equation (2) such that $f \neq 0$, then g is a solution of d'Alembert's short functional equation (1).*

The stability of d'Alembert's functional equation (3) and Wilson's functional equation (4) and other functional equations has been investigated by several authors; see, for example, [3, 4, 8, 10, 11, 19, 21–24, 26–30, 36, 42].

In [7], Baker et al. introduced the superstability of the exponential equation $f : X \to \mathbf{R}, f(x + y) = f(x)f(y)$, where X is a vector space. The result was generalized by Baker [6], by replacing the vector space by a semigroup and $\mathbf{R}$ by a normed algebra in which the norm is multiplicative. A different generalization of Baker's result [7] was given by Székelyhidi [38–40].

Badora and Ger [5] have improved the superstability of the d'Alembert equation (3) in the abelian group under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$.

Kim [25] investigated the stability problem of d'Alembert's and Wilson's functional equations with involution in the abelian groups and improved the superstability of d'Alembert's equation under the condition

$$|f(x+y)+f(x+\sigma(y))-2f(x)f(y)| \le \begin{cases} \varphi(x), \\ \varphi(y) \text{ and } \varphi(x). \end{cases} \tag{5}$$

In [11], Chung and Sahoo extended the superstability result obtained in [25].

Recently, Bouikhalene and Elqorachi [9] studied the Hyers–Ulam stability of Wilson's equation (2) and obtained the superstability of d'Alembert's equation (1).

In this paper, we investigate the Hyers–Ulam–Rassias stability of Wilson's functional equations (2) and

$$f(xy)+\mu(y)f(x\sigma(y)) = 2g(x)f(y),\ x,y \in G \tag{6}$$

on groups G. As an application, we obtain the superstability of the d'Alembert's functional equation (1).

Throughout this paper G denote a group with identity element e, $\mathbf{C}$ the set of complex numbers. Usually we write the group operation multiplicatively, but if the group is abelian, we mainly use $+$. We let $\sigma : G \to G$ be an involutive anti-automorphism, that is, $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$, and $\mu : G \to \mathbf{C}^*$ be a fixed unitary character on G which satisfies $\mu(x\sigma(x)) = 1$ for all $x \in G$ and $\varphi : G \to \mathbf{R}^+$ a mapping. f_μ^o and f_μ^e denote the odd and even parts of f, respectively, i.e., $f_\mu^o(x) = \dfrac{f(x)-\mu(x)f(\sigma(x))}{2}$, $f_\mu^e(x) = \dfrac{f(x)+\mu(x)f(\sigma(x))}{2}$ for all $x \in G$.

2 Main Result

In the following theorem, we give the solutions of d'Alembert's functional equation (1). The proof of this theorem is based on the same computations used in the [35, Proposition 5.2 and Theorem 6.1] for the particular case $\sigma(x) = x^{-1}$ for all $x \in G$.

Theorem 1. *Let G an abelian group. Let $\mu : G \to \mathbf{C}^*$ be a character on G and $g : G \to \mathbf{C}$ be a nonzero solution of d'Alembert's functional equation (1). Then there exists a character χ of G such that*

$$g = \frac{\chi + \mu\chi\circ\sigma}{2}. \tag{7}$$

The character χ in decomposition (7) of g is unique, except χ can be replaced by $\mu\chi\circ\sigma$.

Conversely, any function g of the form (7), where χ is a character, is a nonzero abelian solution of (1).

The following theorem gives the solutions of Wilson's functional equation (2) on abelian group.

Theorem 2. *Let $f, g : G \to \mathbf{C}$ be solutions of Wilson's functional equation*

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y) \tag{8}$$

for all $x, y \in G$ and such that $f \neq 0$. Then

(1) g satisfies Eq. (1).
(2) When G is an abelian group, then there exists a a character χ of G such that

$$g = \frac{\chi + \mu\chi \circ \sigma}{2}. \tag{9}$$

(3) If $\chi \neq \mu\chi \circ \sigma$, then f has the form

$$f = \lambda_1\chi + \lambda_2\mu\chi \circ \sigma \tag{10}$$

for some constants $\lambda_1, \lambda_2 \in \mathbf{C}$.
(4) If $\chi = \mu\chi \circ \sigma$, f has the form

$$f = \chi(\delta + a(x)) \tag{11}$$

where $a : G \to \mathbf{C}$ is an additive function such that $a(\sigma(x)) = -a(x)$ and δ is a constant in $\mathbf{C}$.

Proof. (1) See Proposition 1.
(2) If $\chi \neq \mu\chi \circ \sigma$. Putting $x = e$ in (8), we get $f_\mu^e = f(e)g$. In view of Stetkaer [35, Theorem 5.1] and Theorem 1, there exists a character χ of G and a constant $c_1 \in \mathbf{C}$ such that $f_\mu^e = c_1\frac{\chi+\mu\chi\circ\sigma}{2}$.

By simple computations we verify that f_μ^o satisfies Wilson's functional equation with g unchanged, that is,

$$f_\mu^o(x + y) + \mu(y)f_\mu^o(x + \sigma(y)) = 2f_\mu^o(x)g(y) \tag{12}$$

for all $x, y \in G$. Interchanging the roles of x and y in (12) and using the invariance formula $f_\mu^o(y) = -\mu(y)f_\mu^o(\sigma(y)),\ y \in G$, we obtain that the pair (f_μ^o, g) satisfies the sine addition formula:

$$f_\mu^o(xy) = f_\mu^o(x)g(y) + f_\mu^o(y)g(x)\ x, y \in G. \tag{13}$$

So, by Theorem 5.1 [35], there exists a character χ of G and a constant $c_2 \neq 0$ such that $f_\mu^o = c_2(\chi - \mu\chi \circ \sigma)$. Since $f = f_\mu^e + f_\mu^o$, the form of f is given by (10) where $\lambda_1 = \frac{c_1}{2} + c_2$ and $\lambda_2 = \frac{c_1}{2} - c_2$. Now, if $\chi = \mu\chi \circ \sigma$, then $f_\mu^e = \delta\chi$ where $\delta = c_1$. We have $\frac{f_\mu^o}{g}$ is an additive function which satisfies $(\frac{f_\mu^o}{g})(x) = \frac{1}{2}(\frac{f_\mu^o}{g})(x - \sigma(x))$ and the rest of the proof is obvious.

Theorem 3. *Let G be a group, let σ an involution of G, and let μ be a unitary character of G such that $\mu(x\sigma(x)) = 1$ for all $x \in G$. Suppose that $f, g : G \to \mathbf{C}$ satisfy the inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)g(y)| \leq \varphi(y) \tag{14}$$

for all $x, y \in G$. Under these assumptions the following statements hold:

(1) If f is bounded and $f \neq 0$, then g satisfies

$$|g(x)| \leq 1 + \frac{\varphi(x)}{2M} \tag{15}$$

for all $x, y \in G$, where $M = \sup_{x \in G} |f(x)|$.

(2) If f is unbounded, then g is a solution of d'Alembert's long equation:

$$g(xy) + \mu(y)g(x\sigma(y)) + g(yx) + \mu(y)g(\sigma(y)x) = 4g(x)g(y), \ x, y \in G. \tag{16}$$

Furthermore, if there exists a sequence $(z_n) \in G$ satisfying

$$\frac{|g(z_n)|}{1 + \varphi(z_n) + \varphi(z_n x) + \varphi(x z_n) + \varphi(x\sigma(z_n)) + \varphi(\sigma(z_n)x)} \to \infty \tag{17}$$

as $n \to \infty$ for all $x \in G$. Then, the pair (f, g) satisfies Wilson's functional equation (2) and g satisfies d'Alembert's short functional equation (1).

Proof. First, assume that f is bounded and $f \neq 0$. By using triangle inequality and (14), we have

$$|2f(x)g(y)| \leq \varphi(y) + |f(xy)| + |\mu(y)||f(x\sigma(y))| \leq 2M + \varphi(y), \tag{18}$$

where $M = \|f\|_\infty$. Dividing both sides of (18) by $2M$, we get (15). Now, assume that f is unbounded. Choosing a sequence $(z_n)_{n \in \mathbf{N}} \in G$ such that $|f(z_n)| \to \infty$ as $n \to \infty$. By using triangle inequality and inequality (14), for all $x, y, z_n \in G$, one has

$$\begin{aligned}
&2|f(z_n)||g(xy) + \mu(y)g(x\sigma(y)) + g(yx) + \mu(y)g(\sigma(y)x) - 4g(x)g(y)| \\
&\leq |f(z_n xy) + \mu(xy)f(z_n\sigma(y)\sigma(x)) - 2f(z_n)g(xy)| \\
&\quad + |\mu(y)f(z_n x\sigma(y)) + \mu(y)\mu(x\sigma(y))f(z_n y\sigma(x)) - 2\mu(y)f(z_n)g(x\sigma(y))| \\
&\quad + |f(z_n yx) + \mu(yx)f(z_n\sigma(x)\sigma(y)) - 2f(z_n)g(yx)| \\
&\quad + |\mu(y)f(z_n\sigma(y)x) + \mu(y)\mu(\sigma(y)x)f(z_n\sigma(x)y) - 2\mu(y)f(z_n)g(\sigma(y)x)| \\
&\quad + |f(z_n xy) + \mu(y)f(z_n x\sigma(y)) - 2f(z_n x)g(y)| \\
&\quad + |f(z_n yx) + \mu(x)f(z_n y\sigma(x)) - 2f(z_n y)g(x)| \\
&\quad + |\mu(y)f(z_n\sigma(y)x) + \mu(y)\mu(x)f(z_n\sigma(y)\sigma(x)) - 2\mu(y)f(z_n\sigma(y))g(x)|
\end{aligned}$$

$$+|\mu(x)f(z_n\sigma(x)y) + \mu(x)\mu(y)f(z_n\sigma(x)\sigma(y)) - 2\mu(x)f(z_n\sigma(x))g(y)|$$
$$+2|g(y)||f(z_nx) + \mu(x)f(z_n\sigma(x)) - 2f(z_n)g(x)|$$
$$+2|g(x)||f(z_ny) + \mu(y)f(z_n\sigma(y)) - 2f(z_n)g(y)|.$$
$$\leq \varphi(xy) + |\mu(y)|\varphi(x\sigma(y)) + \varphi(yx) + |\mu(y)|\varphi(\sigma(y)x) + \varphi(y) + \varphi(x) + |\mu(y)|\varphi(x)$$
$$+|\mu(x)|\varphi(y) + 2|g(y)|\varphi(x) + 2|g(x)|\varphi(y).$$

Moreover, since f is unbounded, we conclude that g is a solution of Eq. (16).

Assume that (17) holds. By using inequality (14) and triangle inequality for every $x, y, z_n \in G$, we have

$$2|g(z_n)||f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)g(y)|$$
$$\leq |f(xyz_n) + \mu(z_n)f(xy\sigma(z_n)) - 2f(xy)g(z_n)|$$
$$+ |\mu(y)f(x\sigma(y)z_n) + \mu(y)\mu(z_n)f(x\sigma(y)\sigma(z_n)) - 2\mu(y)f(x\sigma(y))g(z_n)|$$
$$+ |f(xyz_n) + \mu(yz_n)f(x\sigma(z_n)\sigma(y)) - 2f(x)g(yz_n)|$$
$$+ |\mu(z_n)f(xy\sigma(z_n)) + \mu(z_n)\mu(y\sigma(z_n))f(xz_n\sigma(y)) - 2\mu(z_n)f(x)g(y\sigma(z_n))|$$
$$+ |\mu(z_n)f(x\sigma(z_n)y) + \mu(z_n)\mu(\sigma(z_n)y)f(x\sigma(y)z_n) - 2\mu(z_n)f(x)g(\sigma(z_n)y)|$$
$$+ |\mu(z_ny)f(x\sigma(y)\sigma(z_n)) + f(xz_ny) - 2f(x)g(z_ny)|$$
$$+ |\mu(z_n)f(x\sigma(z_n)y) + \mu(z_n)\mu(y)f(x\sigma(z_n)\sigma(y)) - 2\mu(z_n)f(x\sigma(z_n))g(y)|$$
$$+ |f(xz_ny) + \mu(y)f(xz_n\sigma(y)) - 2f(xz_n)g(y)|$$
$$+ 2|f(x)||g(yz_n) + \mu(z_n)g(y\sigma(z_n)) + g(z_ny) + \mu(z_n)g(\sigma(z_n)y) - 4g(y)g(z_n)|$$
$$+ 2|g(y)||f(xz_n) + \mu(z_n)f(x\sigma(z_n)) - 2f(x)g(z_n)|$$
$$\leq \varphi(z_n) + |\mu(y)|\varphi(z_n) + \varphi(yz_n) + |\mu(z_n)|\varphi(y\sigma(z_n)) + |\mu(z_n)|\varphi(\sigma(z_n)y)$$
$$+ \varphi(z_ny) + |\mu(z_n)|\varphi(y) + \varphi(y) + 2|f(x)| \times 0 + 2|g(y)|\varphi(z_n).$$

By using (17), we get that the pair (f, g) satisfies Wilson's functional equation (2), and this completes the proof of theorem.

As a consequence of Theorem 3, we obtain the superstability of d'Alembert's functional equation (1).

Corollary 1. *Let G be a group, let σ an involution of G, and let μ be a unitary character of G such that $\mu(x\sigma(x)) = 1$ for all $x \in G$. Suppose that $f, g : G \to \mathbf{C}$ satisfy the inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)f(y)| \leq \varphi(y) \tag{19}$$

for all $x, y \in G$. Then either f satisfies

$$|f(x)| \leq \frac{1 + \sqrt{1 + 2\varphi(x)}}{2} \tag{20}$$

for all $x \in G$ or f satisfies d'Alembert's short functional equation (1).

As a direct consequence of Theorem 3, we get the results obtained in [5, 9, 11, 25].

Corollary 2. *Let $\delta \geq 0$, G a group, μ a unitary character of G, and σ an involution of G such that $\mu(x\sigma(x)) = 1$ for all $x \in G$. Suppose that the pair $f, g : G \to \mathbf{C}$ satisfies*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)g(y)| \leq \delta \tag{21}$$

for all $x, y \in G$. Under these assumptions the following statements hold:

(1) If f is unbounded, then g satisfies d'Alembert's short functional equation (1).
(2) If g is unbounded and $f \neq 0$, then the pair (f, g) satisfies Wilson's functional equation (2) and g satisfies the d'Alembert's short functional equation (1).

Proof. (2) If g is unbounded and $f \neq 0$, then by simple computations we get f unbounded, so the proof of (2) follows from Theorem 3.

(1) Assume that f is unbounded and then from Theorem 3, g satisfies d'Alembert's long functional equation. But we want to prove that g satisfies d'Alembert's short functional equation. In this case we refer to the proof given by Bouikhalene and Elqorachi [9, Proposition 3.7].

Corollary 3. *Let G be an abelian group, let σ an involution of G, and let μ be a unitary character of G such that $\mu(x + \sigma(x)) = 1$ for all $x \in G$. Let $f, g : G \to \mathbf{C}$ satisfy the functional inequality*

$$|f(x + y) + \mu(y)f(x + \sigma(y)) - 2f(x)g(y)| \leq \varphi(y) \tag{22}$$

for all $x, y \in G$. Then,

(1) if f is a nonzero bounded function, g satisfies

$$|g(x)| \leq 1 + \frac{\varphi(x)}{2M} \tag{23}$$

for all $x, y \in G$, where $M = \|f\|_\infty$.

(2) If f is unbounded, then g has the form

$$g(x) = \frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2}, \quad x \in G, \tag{24}$$

where χ is a character of G. Assume that their exists a sequence $(z_n)_{n\in\mathbf{N}} \in G$ which satisfies (17). Then,

(i) if $\chi \neq \mu(\chi \circ \sigma)$, f has the form

$$f(x) = \lambda_1\chi(x) + \lambda_2\mu(x)\chi(\sigma(x)),\ x \in G \tag{25}$$

where $\lambda_1, \lambda_2 \in \mathbf{C}$,

(ii) if $\chi = \mu(\chi \circ \sigma)$, f has the form

$$f(x) = \chi(x)(\delta + a(x)),\ x \in G \tag{26}$$

where $a : G \to \mathbf{C}$ is an additive function such that $a(\sigma(x)) = -a(x)$ for all $x \in G$ and $\delta \in \mathbf{C}$.

The following theorem is a generalization of the result obtained in [11].

Theorem 4. *Let G be an abelian group, let σ an involution of G, and let μ be a unitary character of G such that $\mu(x + \sigma(x)) = 1$ for all $x \in G$. Suppose that $f, g : G \to \mathbf{C}$ be unbounded functions such that*

$$|f(x + y) + \mu(y)f(x + \sigma(y)) - 2f(x)g(y)| \leq \varphi(x) \tag{27}$$

for all $x, y \in G$. Then there exist $\lambda_1, \lambda_2 \in \mathbf{C}$ and a character $\chi : G \to \mathbf{C}$ for which $\chi \neq \mu\chi \circ \sigma$ such that

$$f(x) = \lambda_1\chi(x) + \lambda_2\mu(x)\chi(\sigma(x)), \tag{28}$$

$$\left|g(x) - \frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2}\right| \leq \inf_{y\in G} \frac{\varphi(y)}{2|f(y)|} \tag{29}$$

for all $x \in G$, or else there exists $\delta \in \mathbf{C}$, a character $\chi : G \to \mathbf{C}$ for which $\chi = \mu\chi \circ \sigma$, and an additive $a : G \to \mathbf{C}$ such that $a(\sigma(x)) = -a(x)$ for all $x \in G$ and

$$f(x) = \chi(x)(\delta + a(x)), \tag{30}$$

$$|g(x) - \chi(x)| \leq \inf_{y\in G} \frac{\varphi(y)}{2|f(y)|} \tag{31}$$

for all $x \in G$. Furthermore if there exists a sequence $(z_n)_{n\in\mathbf{N}} \in G$ such that

$$\frac{|f(z_n)|}{1 + \varphi(z_n)} \to \infty \tag{32}$$

as $n \to \infty$. Then either there exist $\lambda_1, \lambda_2 \in \mathbf{C}$ and a character $\chi : G \to \mathbf{C}$ such that $\chi \neq \mu\chi \circ \sigma$ and

$$f(x) = \lambda_1 \chi(x) + \lambda_2 \mu(x)\chi(\sigma(x)), \; g(x) = \frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2} \tag{33}$$

for all $x \in G$, or else there exist $\delta \in \mathbf{C}$, a character $\chi : G \to \mathbf{C}$ satisfying $\chi = \mu\chi \circ \sigma$ and an additive function $a : G \to \mathbf{C}$ such that

$$f(x) = \chi(x)(\delta + a(x)), \; g(x) = \chi(x) \tag{34}$$

for all $x \in G$.

Proof. The proof is closely related to the one used in [11, Theorem 2.7]. Let $(z_n)_{n \in \mathbf{N}} \in G$ be a sequence such that $|g(z_n)| \to \infty$. Replacing y by z_n in (27), dividing the result obtained by $2|g(z_n)|$, and letting $n \to \infty$, we have

$$f(x) = \lim_{n \to \infty} \frac{f(x + z_n) + \mu(z_n) f(x + \sigma(z_n))}{2g(z_n)} \tag{35}$$

for all $x, y \in G$. Replacing y, respectively, by $y + z_n$ and by $\sigma(y) + z_n$ in (27), using triangle inequality, dividing the result by $2|g(z_n)|$ and, after rearranging the terms, we obtain

$$\begin{aligned} \Bigg| & \frac{f(x + y + z_n) + \mu(z_n) f(x + y + \sigma(z_n))}{2g(z_n)} \\ & + \mu(y) \frac{f(x + \sigma(y) + z_n) + \mu(z_n) f(x + \sigma(y) + \sigma(z_n))}{2g(z_n)} \\ & - 2f(x) \frac{g(z_n + y) + \mu(y) f(z_n + \sigma(y))}{2g(z_n)} \Bigg| \leq \frac{\varphi(x) + |\mu(y)|\varphi(x)}{2g(z_n)} \end{aligned}$$

for all $x, y, z_n \in G$. Letting $n \to \infty$ in the last inequality and using (35), we can see that the limit

$$k(y) = \lim_{n \to \infty} \frac{g(z_n + y) + \mu(y) f(z_n + \sigma(y))}{2g(z_n)} \tag{36}$$

exists for all $y \in G$ and the pair (f, k) satisfies the following functional equation

$$f(x + y) + \mu(y) f(x + \sigma(y)) = 2f(x)k(y) \tag{37}$$

for all $x, y \in G$. Thus, k satisfies d'Alembert's functional equation (1). Substituting (37) into (27) and taking the infimum of the right hand side of the result obtained, we get

$$|g(x) - k(x)| \leq \inf_{y\in G} \frac{\varphi(y)}{2|f(y)|} \tag{38}$$

for all $x \in G$. Assume that there exist a sequence $(z_n)_{n\in \mathbf{N}} \in G$ which satisfies the condition (32). By using (38) we get $k(x) = g(x)$ for all $x \in G$ and then the function g satisfies d'Alembert's functional equation (1). So, we get the rest of the proof.

Lemma 1 describes all solutions of Wilson's functional equation (6) on abelian group. Its proof is elementary.

Lemma 1. *Let G be an abelian group, let σ an involution of G, and let μ be a unitary character of G. The solutions $f, g : G \to \mathbf{C}$ of Wilson's functional equation*

$$f(x + y) + \mu(y)f(x + \sigma(y)) = 2g(x)f(y),\ x, y \in G \tag{39}$$

can be described as follows:

(i) if $f(e) = 0$, then $f = 0$ and g arbitrary,
(ii) if $f(e) \neq 0$, then there exists a character χ of G such that

$$g(x) = \frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2},\ f(x) = f(e)\left(\frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2}\right) \tag{40}$$

for all $x \in G$.

The following lemma is useful for the proof of Theorem 5.

Lemma 2. *Let G be a group, let σ an involution of G, and μ be a unitary character of G. Assume that $f, g : G \to \mathbf{C}$ satisfies the inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x) \tag{41}$$

for all $x, y \in G$ such that $f(e) \neq 0$. Then,

$$|g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)h(y)| \leq \psi(x) + \frac{\psi(xy) + |\mu(y)|\psi(x\sigma(y))}{2} \tag{42}$$

for all $x, y \in G$ and where $h = \frac{f}{f(e)}$, $\psi = \frac{\varphi}{|f(e)|}$.

Proof. By dividing inequality (41) by $|f(e)|$, we get

$$|h(xy) + \mu(y)h(x\sigma(y)) - 2g(x)h(y)| \leq \psi(x) \tag{43}$$

for all $x, y \in G$. Taking $y = e$ in inequality (43) we get

$$|h(x) - g(x)| \leq \frac{\psi(x)}{2} \tag{44}$$

for all $x \in G$. In virtue of inequality (44) and (43), we obtain

$$\begin{aligned}
&|g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)h(y)| \\
&\quad \leq |h(xy) + \mu(y)h(x\sigma(y)) - 2g(x)h(y)| \\
&\qquad + |(h-g)(xy)| + |\mu(y)||(h-g)(x\sigma(y))| \\
&\quad \leq \psi(x) + \frac{\psi(xy) + |\mu(y)|\psi(x\sigma(y))}{2}
\end{aligned}$$

for all $x, y \in G$.

Now, we are ready to prove the stability and superstability of Wilson's functional equation (6) and d'Alembert's functional equation (1), respectively.

Theorem 5. *Let G be a group. Let σ an involution of G and μ be a unitary character of G such that $\mu(x\sigma(x)) = 1$ for all $x \in G$. Suppose that $f, g : G \to \mathbf{C}$ satisfy the functional inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x) \tag{45}$$

for all $x, y \in G$ with $f(e) \neq 0$. Then the pair (f, g) satisfies one of the following statements:

(i) If f is a nonzero bounded function, then g satisfies

$$|g(x)| \leq 1 + \frac{\varphi(x)}{2M} \tag{46}$$

for all $y \in G$, where $M = \|f\|_\infty$.

(ii) If there exists a sequence $(z_n)_{n\in\mathbf{N}} \in G$ such that

$$\frac{|g(z_n)|}{1 + \varphi(z_n) + \varphi(z_n y) + \varphi(z_n\sigma(y)) + \varphi(z_n x) + \varphi(z_n\sigma(x)) + \varphi(xz_n) + \varphi(x\sigma(z_n))} \to \infty \tag{47}$$

as $n \to \infty$ for all $x, y \in G$, then $h = \frac{f}{f(e)}$ satisfies the d'Alembert's long functional equation (16), and if f is unbounded, then (f, g) satisfies Wilson's functional equation (6). Furthermore, if $f \neq 0$, then g satisfies the d'Alembert's short functional equation (1).

Proof. Let f be a nonzero bounded function. Using triangle inequality (45) and dividing both sides of the result by $2M$, we get inequality (46).

Putting $h = \frac{f}{f(e)}$; $\psi = \frac{\varphi}{|f(e)|}$. Dividing both sides of inequality (45) by $|f(e)|$, we get

$$|h(xy) + \mu(y)h(x\sigma(y)) - 2g(x)h(y)| \leq \psi(x) \tag{48}$$

for all $x, y \in G$. In what follows we will show that h satisfies d'Alembert's long functional equation (16). By using the following decomposition

$$
\begin{aligned}
&2|g(z_n)||h(xy) + \mu(y)h(x\sigma(y)) + h(yx) + \mu(y)h(\sigma(y)x) - 4h(x)h(y)| \\
&\quad \leq |h(z_nxy) + \mu(xy)h(z_n\sigma(y)\sigma(x)) - 2g(z_n)h(xy)| \\
&\qquad +|\mu(y)h(z_nx\sigma(y)) + \mu(y)\mu(x\sigma(y))h(z_ny\sigma(x)) - 2\mu(y)g(z_n)h(x\sigma(y))| \\
&\qquad +|h(z_nyx) + \mu(yx)h(z_n\sigma(x)\sigma(y)) - 2g(z_n)h(yx)| \\
&\qquad +|\mu(y)h(z_n\sigma(y)x) + \mu(y)\mu(\sigma(y)x)h(z_n\sigma(x)y) - 2\mu(y)g(z_n)h(\sigma(y)x)| \\
&\qquad +|h(z_nxy) + \mu(y)h(z_nx\sigma(y)) - 2g(z_nx)h(y)| \qquad (49) \\
&\qquad +|h(z_nyx) + \mu(x)h(z_ny\sigma(x)) - 2g(z_ny)h(x)| \\
&\qquad +|\mu(y)h(z_n\sigma(y)x) + \mu(y)\mu(x)h(z_n\sigma(y)\sigma(x)) - 2\mu(y)g(z_n\sigma(y))h(x)| \\
&\qquad +|\mu(x)h(z_n\sigma(x)y) + \mu(x)\mu(y)h(z_n\sigma(x)\sigma(y)) - 2\mu(x)g(z_n\sigma(x))h(y)| \\
&\qquad +2|h(y)||g(z_nx) + \mu(x)g(z_n\sigma(x)) - 2g(z_n)h(x)| \\
&\qquad +2|h(x)||g(z_ny) + \mu(y)g(z_n\sigma(y)) - 2g(z_n)h(y)|,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&2|g(z_n)||h(xy) + \mu(y)h(x\sigma(y)) + h(yx) + \mu(y)h(\sigma(y)x) - 4h(x)h(y)| \\
&\quad \leq 4\psi(z_n) + \psi(z_n\sigma(y)) + \psi(z_n\sigma(x)) + \psi(z_nx) + \psi(z_ny) \\
&\qquad +2|F(y)|(\psi(z_n) + \frac{\psi(z_nx) + |\mu(x)|\psi(z_n\sigma(x))}{2}) \\
&\qquad +2|h(x)|(\psi(z_n) + \frac{\psi(z_ny) + |\mu(y)|\psi(z_n\sigma(y))}{2}) \qquad (50)
\end{aligned}
$$

for all $z_n, x, y, \in G$. Since g satisfies (47), then we have

$$\frac{|g(z_n)|}{1 + \psi(z_n) + \psi(z_ny) + \psi(z_n\sigma(y)) + \psi(z_nx) + \psi(z_n\sigma(x)) + \psi(xz_n) + \psi(x\sigma(z_n))} \to \infty \qquad (51)$$

as $n \to \infty$ for all $x, y \in G$. Thus, by using (51), inequality (50) shows that h satisfies the functional equation (16). Now, we will show that the pair (f, g) is a solution of Wilson's functional equation (6). Assume that f is unbounded. The conditions (51) and (44) imply that

$$\frac{|h(z_n)|}{1 + \psi(z_n) + \psi(z_ny) + \psi(z_n\sigma(y)) + \psi(z_nx) + \psi(z_n\sigma(x)) + \psi(xz_n) + \psi(x\sigma(z_n))} \to \infty \qquad (52)$$

as $n \to \infty$ holds too. By using (48), triangle inequality, we have

$$\begin{aligned}
&2|h(z_n)||g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)h(y)| \qquad (53)\\
&\leq |h(xyz_n) + \mu(z_n)h(xy\sigma(z_n)) - 2g(xy)h(z_n)|\\
&\quad +|\mu(y)h(x\sigma(y)z_n) + \mu(y)\mu(z_n)h(x\sigma(y)\sigma(z_n)) - 2\mu(y)g(x\sigma(y))h(z_n)|\\
&\quad +|h(xyz_n) + \mu(yz_n)h(x\sigma(z_n)\sigma(y)) - 2g(x)h(yz_n)|\\
&\quad +|\mu(z_n)h(xy\sigma(z_n)) + \mu(z_n)\mu(y\sigma(z_n))h(xz_n\sigma(y)) - 2\mu(z_n)g(x)h(y\sigma(z_n))|\\
&\quad +|\mu(z_n)h(x\sigma(z_n)y) + \mu(z_n)\mu(\sigma(z_n)y)h(x\sigma(y)z_n) - 2\mu(z_n)g(x)h(\sigma(z_n)y)|\\
&\quad +|\mu(z_ny)h(x\sigma(y)\sigma(z_n)) + h(xz_ny) - 2g(x)h(z_ny)|\\
&\quad +|\mu(z_n)h(x\sigma(z_n)y) + \mu(z_n)\mu(y)h(x\sigma(z_n)\sigma(y)) - 2\mu(z_n)g(x\sigma(z_n))h(y)|\\
&\quad +|h(xz_ny) + \mu(y)h(xz_n\sigma(y)) - 2g(xz_n)h(y)|\\
&\quad +2|g(x)||h(yz_n) + \mu(z_n)h(y\sigma(z_n)) + h(z_ny) + \mu(z_n)h(\sigma(z_n)y) - 4h(y)h(z_n)|\\
&\quad +2|h(y)||g(xz_n) + \mu(z_n)g(x\sigma(z_n)) - 2g(x)h(z_n)|\\
&\leq \psi(xy) + |\mu(y)|\psi(x\sigma(y)) + \psi(x) + |\mu(z_n)|\psi(x) + |\mu(z_n)|\psi(x)\\
&\quad +\psi(x) + |\mu(z_n)|\psi(x\sigma(z_n)) + \psi(xz_n) + 2|g(x)| \times 0\\
&\quad +2|h(y)|(\psi(x) + \frac{\psi(xz_n) + |\mu(z_n)|\psi(x\sigma(z_n))}{2})
\end{aligned}$$

for all $z_n, x, y \in G$. By using (52) we conclude that the pair (g, h) satisfies Wilson's functional equation

$$g(xy) + \mu(y)g(x\sigma(y)) = 2g(x)h(y), \; x, y \in G. \qquad (54)$$

From Proposition 1, h satisfies d'Alembert's short functional equation (1), so we have $f(x) = \mu(x)f(\sigma(x))$ for all $x \in G$. Putting $x = e$ in (41), we get

$$|f(y)(1 - g(e))| \leq \frac{\varphi(e)}{2} \qquad (55)$$

for all $y \in G$. Since f is assumed to be unbounded, we get $g(e) = 1$.

Substituting x by e in (54), we have the μ-even part of g is equal to h, that is, $h(y) = \frac{g(y)+\mu(y)g(\sigma(y))}{2}$. Hence $\frac{g(y)+\mu(y)g(\sigma(y))}{2}$ is a solution of the d'Alembert's short functional equation (1).

Writing $g = g^e_\mu + g^o_\mu$ with $g^e_\mu = h$ and using (48), we obtain

$$|g^o_\mu(x)f(y)| \leq \varphi(x)$$

for all $x, y \in G$. Since f is unbounded, then we get $g^o_\mu(x) = 0$ for all $x \in G$. By a simple computation, we find that (f, g) satisfies the Wilson's functional equation (6). This completes the proof of the theorem.

In the following corollary, we present the abelian case of Theorem 5.

Corollary 4. *Let G be an abelian group, σ an involution on G, and μ a unitary character of G. Assume that $f, g : G \to \mathbf{C}$ satisfies the functional inequality*

$$|f(x+y) + \mu(y)f(x+\sigma(y)) - 2g(x)f(y)| \leq \varphi(x) \tag{56}$$

for all $x, y \in G$ with $f(e) \neq 0$. Then the pair (f, g) satisfies one of the following statements:

(i) if f is bounded, then g satisfies

$$|g(y)| \leq 1 + \frac{\varphi(y)}{2M} \tag{57}$$

for all $y \in G$, where $M = \sup_{x\in G} |f(x)|$,

(ii) if there exists a sequence $(z_n)_{n\in\mathbf{N}} \in G$ such that (47), then there exists a character χ of G such that

$$g(x) = \frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2} \tag{58}$$

for all $x \in G$, and if f is unbounded, then f has the form

$$f(x) = f(e)\left(\frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2}\right),\ x \in G. \tag{59}$$

The following corollary follows easily from Theorem 5.

Corollary 5. *Let G be a group. Let σ an involution of G and μ be a unitary character of G such that $\mu(x\sigma(x)) = 1$ for all $x \in G$. Suppose that the pair (f, g) satisfies*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)f(y)| \leq \varphi(x),\ x, y \in G. \tag{60}$$

Then either f satisfies $|f(x)| \leq \frac{1+\sqrt{1+2\varphi(x)}}{2}$ for all $x \in G$ or f satisfies d'Alembert's short functional equation (1).

Next, we will prove a stability theorem where the control function is a function of the variable y. We start with the following elementary lemma.

Lemma 3. *Let G be a group. Let σ an involution of G and μ a unitary character of G. Assume that $f, g : G \to \mathbf{C}$ satisfies the inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(y) \tag{61}$$

for all $x, y \in G$ *with* $f(e) \neq 0$. *Then*

$$|g(xy) + \mu(y)g(x\sigma(y)) - 2g(x)h(y)| \leq \psi(y) + \psi(e) \tag{62}$$

for all $x, y \in G$, *and where* $h = \frac{f}{f(e)}$, $\psi = \frac{\varphi}{|f(e)|}$.

Now we prove the following result.

Theorem 6. *Let* G *be a group. Let* σ *an involution of* G *and* μ *be a unitary character of* G *such that* $\mu(x\sigma(x)) = 1$ *for all* $x \in G$. *Let* $f, g : G \to \mathbf{C}$ *satisfies the functional inequality*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(y) \tag{63}$$

for all $x, y \in G$. *Then the pair* (f, g) *satisfies one of the following statements:*

(i) If $f(e) = 0$, *then*

$$|f(x)| \leq \frac{\varphi(e)}{2}, \qquad |g(x)| \leq \inf_{y \in G} \frac{\varphi(y) + \varphi(e)}{2|f(y)|} \tag{64}$$

for all $x \in G$.

(ii) If $f(e) \neq 0$ *and* g *is bounded, then* f *satisfies*

$$|f(y)| \leq |f(e)| + \frac{\varphi(y) + \varphi(e)}{2M} \tag{65}$$

for all $y \in G$, *where* $M = \sup_{x \in G} |g(x)|$.

(iii) If $f(e) \neq 0$ *and* g *is unbounded, then* $h = \frac{f}{f(e)}$ *satisfies d'Alembert's long functional equation (16), and if there exists a sequence* $(z_n)_{n \in \mathbf{N}} \in G$ *such that*

$$\frac{|f(z_n)|}{1 + \varphi(z_n) + \varphi(yz_n) + \varphi(z_n y) + \varphi(y\sigma(z_n)) + \varphi(\sigma(z_n)y) + \varphi(xz_n) + \varphi(x\sigma(z_n))} \to \infty \tag{66}$$

as $n \to \infty$ *for all* $x, y \in G$, *then the pair* (f, g) *satisfies Wilson's functional equation (6). Furthermore, if* $f \neq 0$, g *satisfies d'Alembert's short functional equation (1).*

Proof. Putting $y = e$ in (63) and dividing the result by 2, we get

$$|f(x) - g(x)f(e)| \leq \frac{\varphi(e)}{2} \tag{67}$$

for all $x \in G$. First, assume that $f(e) = 0$. From (67) we have

$$|f(x)| \leq \frac{\varphi(e)}{2} \tag{68}$$

for all $x \in G$. Using the triangle inequality, (63) and (68), dividing the resulting by $2|f(y)|$, and taking the infimum of the right hand side, we have the second inequality of (64). Now, we assume that $f(e) \neq 0$ and g is bounded. In view of Lemma 3 and inequality (62), we obtain (65). By using the decomposition (49), we have

$$\begin{aligned} &2|g(z_n)||h(xy) + \mu(y)h(x\sigma(y)) + h(yx) + \mu(y)h(\sigma(y)x) - 4h(x)h(y)| \\ &\le \psi(xy) + |\mu(y)|\psi(x\sigma(y)) + \psi(xy) + |\mu(y)|\psi(\sigma(y)x) + \psi(y) + \psi(x) \\ &\quad + |\mu(y)|\psi(x) + |\mu(x)|\psi(y) + 2(|h(y)| + |h(x)|)(\psi(x) + \psi(e)) \end{aligned}$$

for all $z_n, x, y \in G$. Since g is unbounded, then h satisfies d'Alembert's long functional equation (16). The decomposition (53) with (66) implies that the pair (g, h) satisfies Wilson's functional equation (2). By following the same arguments of the proof of Theorem 5, we get the rest of the proof.

The following corollaries are a direct consequence of Theorem 6.

Corollary 6. *Let G be a group. Let σ an involution of G and μ be a unitary character of G such that $\mu(x\sigma(x)) = 1$ for all $x \in G$. Suppose that the pair $f, g : G \to \mathbf{C}$ satisfies*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2g(x)f(y)| \le \varphi(x) \text{ and } \varphi(y) \tag{69}$$

for all $x, y \in G$. Then the pair (f, g) satisfies the following statements:

(i) If $f(e) = 0$, then

$$|f(x)| \le \frac{\varphi(e)}{2}, \qquad |g(x)| \le \inf_{y\in G} \frac{\varphi(y) + \varphi(e)}{2|f(y)|} \tag{70}$$

for all $x \in G$.

(ii) If $f(e) \neq 0$ and g is bounded, then f satisfies

$$|f(y)| \le |f(e)| + \frac{\varphi(y) + \varphi(e)}{2M} \tag{71}$$

for all $y \in G$, where $M = \sup_{x\in G} |g(x)|$.

(iii) If $f(e) \neq 0$ and g is unbounded, then $h = \frac{f}{f(e)}$ satisfies d'Alembert's long functional equation (16).

(iv) If f is unbounded and $f(e) \neq 0$, then the pair (f, g) satisfies Wilson's functional equation (6). Furthermore, if $f \neq 0$, g satisfies d'Alembert's short functional equation (1).

Corollary 7. *Let G be an abelian group, σ be an involution of G, and μ a unitary character onG. Suppose that $f, g : G \to \mathbf{C}$ satisfy the functional inequality*

$$|f(x + y) + \mu(y)f(x + \sigma(y)) - 2g(x)f(y)| \le \varphi(y) \tag{72}$$

for all $x, y \in G$ *with* $f(e) \neq 0$*. Then the pair* (f, g) *satisfies one of the following statements:*

(i) If $f(e) = 0$*, then*

$$|f(x)| \leq \frac{\varphi(e)}{2}, \qquad |g(x)| \leq \inf_{y \in G} \frac{\varphi(y) + \varphi(e)}{2|f(y)|} \tag{73}$$

for all $x \in G$*,*

(ii) if $f(e) \neq 0$ *and g is bounded, then f satisfies*

$$|f(y)| \leq |f(e)| + \frac{\varphi(y) + \varphi(e)}{2M} \tag{74}$$

for all $y \in G$*, where* $M = \sup_{x \in G} |g(x)|$*,*

(iii) if $f(e) \neq 0$ *and g is unbounded, then there exists a character* χ *of G such that*

$$f(x) = f(e)\left(\frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2}\right), \tag{75}$$

$$|g(x) - \frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2}| \leq \frac{\varphi(e)}{2|f(e)|} \tag{76}$$

for all $x \in G$*. In particular, if there exists a sequence* $(z_n)_{n \in \mathbf{N}} \in G$ *satisfying (66), then g has the form*

$$g(x) = \frac{\chi(x) + \mu(x)\chi(\sigma(x))}{2}, \; x \in G. \tag{77}$$

Corollary 8. *Let G be a group. Let* σ *an involution of G and* μ *be a unitary character of G such that* $\mu(x\sigma(x)) = 1$ *for all* $x \in G$*. Suppose that the pair* (f, g) *satisfies*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)f(y)| \leq \varphi(y), \; x, y \in G. \tag{78}$$

Then either f satisfies $|f(x)| \leq \frac{1+\sqrt{1+2\varphi(x)}}{2}$ *for all* $x \in G$ *or f satisfies d'Alembert's short functional equation (1).*

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The General Sampling Theory by Using Reproducing Kernels

Hiroshi Fujiwara and Saburou Saitoh

In Honor of Constantin Carathéodory

Abstract We would like to propose a new method for the sampling theory which represents the functions by a finite number of point data in a very general reproducing kernel Hilbert space function space. The result may be looked as an ultimate sampling theorem in a reasonable sense. We shall give numerical experiments also as its evidences.

1 Introduction

The Shannon sampling theorem with many related mathematicians is famous and we have great references, see, for example, [7–9, 13]. However, as we see in the typical Shannon theorem, the sampling points will be determined with strictly strong conditions; this viewpoint will be a popular understanding. One more very important point is as we see from the typical Shannon theorem, the function space, in the Shannon sampling theorem, the Paley–Wiener space will contain very and very bad functions such that for any given finite point set, functions taking any given values exist. So, practically, the sampling theorem is valid among some good functions only—not mathematically, but numerically, when we calculate and look for the functions by computers. We will give this meaning more clearly in this paper. So,

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in the sampling theorem, we wish to select sampling points following the function property that we are concerning, because we wish to collect some useful function information from the sampling points. In this sense the sampling theory will have a weak point, because the sampling points are determined by the function spaces, as in the Paley–Wiener space. We shall give a new sampling theorem overcoming this weak point that may be looked as an ultimate sampling theorem whose essential tool is introduced in [2].

The construction of the paper is as follows: In Sect. 2, we shall fix the essences of the reproducing kernel Hilbert spaces for our purpose. In Sect. 4, as the basic method, we shall introduce the Aveiro discretization method which means the general sampling theory and a general discretization for a general linear analytical problems. In Sect. 5, as the basic reproducing kernel Hilbert spaces, we introduce Sobolev reproducing kernel Hilbert spaces and the Paley–Wiener spaces. In Sect. 6, we give numerical experiments as the main results of this paper. In Sect. 7, for the practical calculation in the Aveiro discretization, we shall introduce the support vector machine method as a new approach by Mo and Qian [10], and in Sect. 8, we give the conclusion of the paper.

2 Preliminaries and the Basic Starting Points

First, we shall recall the reproducing kernel Hilbert spaces for their essences [11, 12] for our purpose.

Let $\mathscr{H}$ be a Hilbert (possibly finite-dimensional) space, and consider E to be an abstract set and $\mathbf{h}$ a Hilbert $\mathscr{H}$-valued function on E. Then, we are able to consider the linear transform

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathscr{H}}, \quad \mathbf{f} \in \mathscr{H}, \tag{1}$$

from $\mathscr{H}$ into the linear space $\mathscr{F}(E)$ comprising all the complex-valued functions on E. In order to investigate the linear mapping (1), we form a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathscr{H}} \quad \text{on} \quad E \times E. \tag{2}$$

Then, we obtain the following fundamental results:

Proposition 2.1. (I)*The range of the linear mapping* (1) *by* $\mathscr{H}$ *is characterized as the reproducing kernel Hilbert space* $H_K(E)$ *admitting the reproducing kernel* $K(p,q)$ *whose characterization is given by the two properties:* (i) $K(\cdot,q) \in H_K(E)$ *for any* $q \in E$ *and,* (ii) *for any* $f \in H_K(E)$ *and for any* $p \in E$, $(f(\cdot), K(\cdot.p))_{H_K(E)} = f(p)$.

(II) *In general, we have the inequality*

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathscr{H}}.$$

Here, for any member f *of* $H_K(E)$*, there exists a uniquely determined* $\mathbf{f}^* \in \mathscr{H}$ *satisfying*

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathscr{H}} \quad \text{on} \quad E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathscr{H}}. \tag{3}$$

(III) *In general, we have the inversion formula in* (1) *in the form*

$$f \mapsto \mathbf{f}^* \tag{4}$$

in (*II*) *by using the reproducing kernel Hilbert space* $H_K(E)$.

The inversion (4) is, in general, very difficult and delicate problems; see the details and the history, for example, [2, 11, 12].

The next result will show that a reproducing kernel Hilbert space is a good and natural function space:

Proposition 2.2. *For a Hilbert space H comprising of functions* $\{f(p)\}$ *on a set E, the space admits a reproducing kernel if and only if, for any point* $q \in E, f \to f(q)$ *is a bounded linear functional on H. If a function sequence* $\{f_n\}$ *converges to f in the space H, then it converges to the function, point wisely on E. Furthermore, on a subset of E where* $K(p, p)$ *is bounded, its convergence is uniform.*

We shall call a complex-valued function $k(p,q)$ on a set $E \times E$ a **positive definite quadratic form function** (or, a positive semi-definite matrix) on the set E when it satisfies the property: for an arbitrary function $X(p)$ on E that is zero on E except for a finite number of points of E, $\sum_{p,q} \overline{X(p)} X(q) k(p,q) \geq 0$.

As we can see simply, a reproducing kernel $K(p,q)$ on E is a positive definite quadratic form function on E, and indeed, its converse statement is very important:

Proposition 2.3. *For any positive definite quadratic form function* $K(p,q)$ *on E, there exists a uniquely determined reproducing kernel Hilbert space* $H_K(E)$ *admitting the reproducing kernel* $K(p,q)$ *on E.*

For a general reproducing kernel Hilbert space, we see that the space, in general, contains a great number of functions in the following sense:

For any large number of points $\{p_j\}_{j=1}^n$ of the set E, we shall assume that, without loss of generality, $\{K(p, p_j)\}_{j=1}^n$ are linearly independent in $H_K(\equiv H_K(E))$. Then, for any given values $\{\alpha_j\}_{j=1}^n$, there exists a uniquely determined member $f \in H_K$ satisfying

$$f(p_j) = \alpha_j, \quad j = 1, 2, 3, .., n, \tag{5}$$

as follows:

$$f(p) = \sum_{j=1}^{n} C_j K(p, p_j) \tag{6}$$

where the constants $\{C_j\}_{j=1}^n$ are determined by the equations:

$$\sum_{j=1}^{n} C_j K(p_{j'}, p_j) = \alpha_{j'}, \quad j' = 1, 2, \ldots, n. \tag{7}$$

Note that the functions f satisfying (5) are, in general, not uniquely determined, but the function f given by (6) has the minimum norm among the functions f satisfying (5).

For any finite number of points $\{p_j\}_{j=1}^n$ and any given values $\{\alpha_j\}_{j=1}^n$, there exists a function $f \in H_K$ satisfying (5) certainly. However, for many points $\{p_j\}_{j=1}^n$ and bad values $\{\alpha_j\}_{j=1}^n$, the calculations (7) solving the Eq. (7) will be difficult numerically and practically. The difficulty to calculate (7) will depend on the given data and the function space H_K. We looked such phenomena for the Paley–Wiener spaces in some cases [1, 2]; however, to represent such deep and delicate phenomena exactly will be difficult. However, we may expect that the smoothness property of functions may be reflected to some properties on a large point set. We shall propose such method in the next section how to catch such property more clearly.

The goodness of a function in the reproducing kernel H_K to solve the Eq. (7) may be given by:

(1) The number of the points $\{p_j\}$ in (5)
(2) The distributions of the coefficients $\{C_j\}_{j=1}^n$ (the solutions of the Eq. (6)) and of the given values $\{\alpha_j\}_{j=1}^n$ in (5)

and

(3) The distribution of the points $\{p_j\}_{j=1}^n$ on the set E.

The factors (1) and (2) may be considered in a general setting; however, (3) will depend on the reproducing kernel Hilbert space H_K.

For the bad functions in the above sense, we will not be able to catch the functions practically by computers from data on a finite number of points effectively.

3 General Linear Problems and Reproducing Kernels

In general linear problems, for many cases we can formulate them as follows. As a physical problem, the function space is written by a basic function family $\{\psi_j\}_{j=1}^n$ with constants $\{C_j\}_{j=1}^n$ as follows:

$$f(p) = \sum_{j=1}^{n} C_j \psi_j(p)$$

and we wish to determine the function f satisfying the data in the form (5). For this problem, we can consider the reproducing kernel in the form

$$K(p,q) = \sum_{j=1}^{n} \psi_j(p)\overline{\psi_j(q)}.$$

The important viewpoint is that here the functions $\{\psi_j(p)\}_{j=1}^n$ may be considered as *arbitrary* functions. Therefore, following a physical property, we can consider the basis $\{\psi_j(p)\}_{j=1}^n$ and we can apply the theory of reproducing kernels for the basic problem.

4 Aveiro Discretization Method

Our new idea in [2] is based on the approximate realization of the abstract Hilbert space H_K in Proposition 2.1 by taking a finite number of points of E, **because, in general, the reproducing kernel Hilbert space H_K has a complicated structure**.

By taking a finite number of points $\{p_j\}_{j=1}^n$, we set

$$K(p_j, p_{j'}) := a_{jj'}. \tag{8}$$

Then, if the matrix $A := \left(a_{jj'}\right)$ is positive definite, then, the corresponding norm in H_A comprising the vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is determined by

$$\|\mathbf{x}\|_{H_A}^2 = \mathbf{x}^* \widetilde{A} \mathbf{x},$$

where $\widetilde{A} = \overline{A^{-1}} = \left(\widetilde{a_{jj'}}\right)$ (see [11, p. 250]). This property may, however, be confirmed directly and easily.

When we approximate the reproducing kernel Hilbert space H_K by the vector space H_A, then the following proposition is derived:

Proposition 4.1. *In the linear mapping*

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathscr{H}}, \quad \mathbf{f} \in \mathscr{H} \tag{9}$$

for

$$E_n = \{p_1, p_2, \ldots, p_n\},$$

the minimum norm inverse $\mathbf{f}_{A_n}$ *satisfying*

$$f(p_j) = (\mathbf{f}, \mathbf{h}(p_j))_{\mathscr{H}}, \quad \mathbf{f} \in \mathscr{H} \tag{10}$$

is given by

$$\mathbf{f}_{A_n} = \sum_{j=1}^{n}\sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \tag{11}$$

where $\widetilde{a_{jj'}}$ are assumed the elements of the complex conjugate inverse of the positive definite Hermitian matrix A_n constituted by the elements

$$a_{jj'} = (\mathbf{h}(p_{j'}), \mathbf{h}(p_j))_{\mathscr{H}}.$$

Here, the positive definiteness of A_n is a basic assumption.

The following proposition deals with the convergence of our approximate inverses in Proposition 4.1.

Proposition 4.2. *Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of distinct points on E, that is the positive definiteness in Proposition 4.1 for any n and a uniqueness set for the reproducing kernel Hilbert space H_K; that is, for any $f \in H_K$, if all $f(p_j) = 0$, then $f \equiv 0$. Then, in the space $\mathscr{H}$*

$$\lim_{n\to\infty} \mathbf{f}^*_{A_n} = \mathbf{f}^* \tag{12}$$

for the given inverse $\mathbf{f}^*$ which is given by (4) satisfying (3) in Proposition 2.1.

The result is a surprisingly simple and pleasant result; indeed, we can obtain directly the ultimate realization of the reproducing kernel Hilbert spaces and the ultimate sampling theory that are very simpler than the known derivation (cf. [11, pp. 92–96]):

Proposition 4.3 (Ultimate Realization of Reproducing Kernel Hilbert Spaces). *In our general situation and for a uniqueness set $\{p_j\}_{j=1}^{\infty}$ of the set E satisfying the linearly independence in Proposition 4.1, we obtain*

$$\|f\|^2_{H_K} = \|\mathbf{f}^*\|^2_{\mathscr{H}} = \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}\overline{f(p_{j'})}. \tag{13}$$

Here, the limit is determined as the nondecreasing sequence.

Proposition 4.4 (Ultimate Sampling Theory). *In our general situation and for a uniqueness set $\{p_j\}_{j=1}^{\infty}$ of the set E satisfying the linearly independence in Proposition 4.1, we obtain*

$$f(p) = \lim_{n\to\infty} (\mathbf{f}_{A_n}, \mathbf{h}(p))_{\mathscr{H}} = \lim_{n\to\infty} \left(\sum_{j=1}^{n}\sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \mathbf{h}(p) \right)_{\mathscr{H}} \tag{14}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n}\sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}} K(p, p_{j'}).$$

Now, our basic idea is as follows: Fujiwara's multiple-precision arithmetic environment and high computer ability will be able to calculate the norm (13) for many cases and practical cases; see the case of numerical and real inversion formula of the Laplace transform that is a famous difficult case [1, 2, 4–6].

5 Sobolev Spaces and Paley–Wiener Spaces

In order to give numerical experiments, we shall introduce the typical reproducing kernel Hilbert spaces, Sobolev Hilbert spaces.

Let $m > \frac{n}{2}$ be an integer. Denote by ${}_mC_\nu$ the binomial coefficient. Then we have

$$W^{m,2}(\mathbb{R}^n) = H_{K_m}(\mathbb{R}^n), \tag{15}$$

where $W^{m,2}(\mathbb{R}^n)$ denotes the Sobolev space whose norm is given by

$$\| F\|_{W^{m,2}(\mathbb{R}^n)} = \sqrt{\sum_{\nu=0}^{m} {}_mC_\nu \left(\sum_{\alpha\in\mathbb{Z}^n_+,\, |\alpha|\le\nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}} \left| \frac{\partial^\nu F(x)}{\partial x^\nu} \right|^2 dx \right)}, \tag{16}$$

and

$$K_m(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}} \frac{\exp(i(x-y)\cdot\xi)}{(1+|\xi|^2)^m}\, d\xi. \tag{17}$$

In particular, we note that:

If $m > \frac{n}{2}$, then $W^{m,2}(\mathbb{R}^n)$ is embedded into $\mathrm{BC}(\mathbb{R}^n)$.

A generalization of the above spaces is given by:

Let $s > \frac{n}{2}$. Define

$$K_s(x, y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} \exp(i(x-y)\cdot\xi)\, d\xi. \tag{18}$$

Then we have

$$H_{K_s}(\mathbb{R}^n) = H_s(\mathbb{R}^n), \tag{19}$$

where the norm is given by

$$\|f\|_{H_s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathscr{F}f(\xi)|^2\, d\xi \right)^{\frac{1}{2}}. \tag{20}$$

The simplest example is given by the following:

The space $H_1(\mathbb{R})$ is made up of absolutely continuous functions F on $\mathbb{R}$ with the norm

$$\|F\|_{H_1(\mathbb{R})} := \sqrt{\int_{\mathbb{R}} (F(x)^2 + F'(x)^2)dx}. \tag{21}$$

The Hilbert space $H_1(\mathbb{R})$ admits the reproducing kernel

$$K_1(x, y) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+\xi^2} \exp(i(x-y)\xi)d\xi = \frac{1}{2}e^{-|x-y|}. \tag{22}$$

Note that if the factor $(1 + |\xi|^2)^{-s}$ is replaced by the characteristic function on $(-\pi/h, +\pi/h)^n$, $h > 0$ on $\mathbb{R}^n$, then the space becomes the Paley–Wiener space comprising of entire functions of exponential type.

Indeed, for $n = 1$, we shall consider the integral transform, for the functions F $L_2(-\pi/h, +\pi/h)$, $h > 0$ as

$$f(z) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} F(t)e^{-izt}dt. \tag{23}$$

In order to identify the image space following the theory of reproducing kernels, we form the reproducing kernel

$$\begin{aligned} K_h(z, \overline{u}) &= \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-izt}\overline{e^{-iut}}dt \\ &= \frac{1}{\pi(z-\overline{u})} \sin \frac{\pi}{h}(z-\overline{u}). \end{aligned} \tag{24}$$

The image space of (23) is called the Paley–Wiener space $W\left(\frac{\pi}{h}\right)$ comprised of all analytic functions of exponential type satisfying, for some constant C and as $z \to \infty$

$$|f(z)| \le C \exp\left(\frac{\pi|z|}{h}\right)$$

and

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

From the identity

$$K_h(jh, j'h) = \frac{1}{h}\delta(j, j')$$

(the Kronecker's δ), since $\delta(j, j')$ is the reproducing kernel for the Hilbert space ℓ^2, from the general theory of integral transforms and the Parseval's identity, we have the isometric identities in (23)

$$\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} |F(t)|^2 dt = h\sum_{j=-\infty}^{\infty} |f(jh)|^2 = \int_{\mathbb{R}} |f(x)|^2 dx.$$

That is, the reproducing kernel Hilbert space H_{K_h} with $K_h(z, \overline{u})$ is characterized as a space comprising the Paley–Wiener space $W\left(\frac{\pi}{h}\right)$ and with the norm squares above. Here we used the well-known result that $\{jh\}_{j=-\infty}^{\infty}$ is a unique set for the Paley–Wiener space $W\left(\frac{\pi}{h}\right)$; that is, $f(jh) = 0$ for all j implies $f \equiv 0$. Then, the reproducing property of $K_h(z, \overline{u})$ states that

$$\begin{aligned} f(x) = (f(\cdot), K_h(\cdot, x))_{H_{K_h}} &= h\sum_{j=-\infty}^{\infty} f(jh)K_h(jh, x) \\ &= \int_{\mathbb{R}} f(\xi)K_h(\xi, x)d\xi. \end{aligned}$$

In particular, on the real line x, this representation is the sampling theorem which represents the whole data $f(x)$ in terms of the discrete data $\{f(jh)\}_{j=-\infty}^{\infty}$. For a general theory for the sampling theory and error estimates for some finite points $\{hj\}_j$, see [11].

6 Numerical Experiments

In this section we show some numerical examples in order to look our principle from the quantitative standpoint. Applications of the proposed sampling theorem are compared in three function spaces: the Sobolev spaces $H_1(\mathbb{R})$, $H_2(\mathbb{R})$, and the Paley–Wiener space $W(\pi)$. We recall that these function spaces admit reproducing kernels

$$K_1(x, y) = \frac{1}{2}e^{-|x-y|} \in C,$$

$$K_2(x, y) = \frac{1}{4}e^{-|x-y|}\big(1 + |x - y|\big) \in C^2,$$

and

$$K_h(x,y) = \frac{1}{\pi(x-y)} \sin \pi(x-y) \in C^{\omega} \quad \text{(analytic)},$$

respectively. All computations presented in the following examples are processed in 600 decimal digit precision by multiple-precision arithmetic library *exflib* [3]. We discuss instability of numerical procedures in the latter of this section.

Example 1. We set the target function as

$$f(x) = e^{-x^2/\pi} \sin \pi x.$$

Sampling points $\{p_j\}$ are uniformly randomly distributed in the interval $[-5, 5]$ by using the random-number generator function in the standard C library.

Figures 1, 2, and 3 show numerical reconstructions by the proposed formula in $H_1(\mathbb{R})$, $H_2(\mathbb{R})$, and $W(\pi)$, respectively. In figures, cross points ($\times$) are sampling values, dotted curves are the target function f, and solid curves are numerically reconstructed functions f_{A_n}.

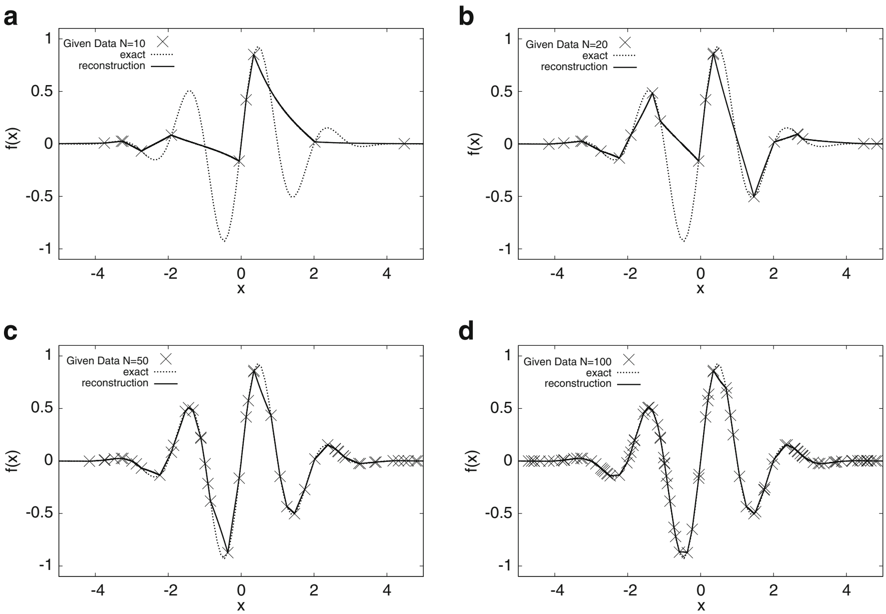

Fig. 1 Reconstruction of an analytic function (Example 1) in $H_1(\mathbb{R})$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 20$. (**c**) Number of sampling points $n = 50$. (**d**) Number of sampling points $n = 100$

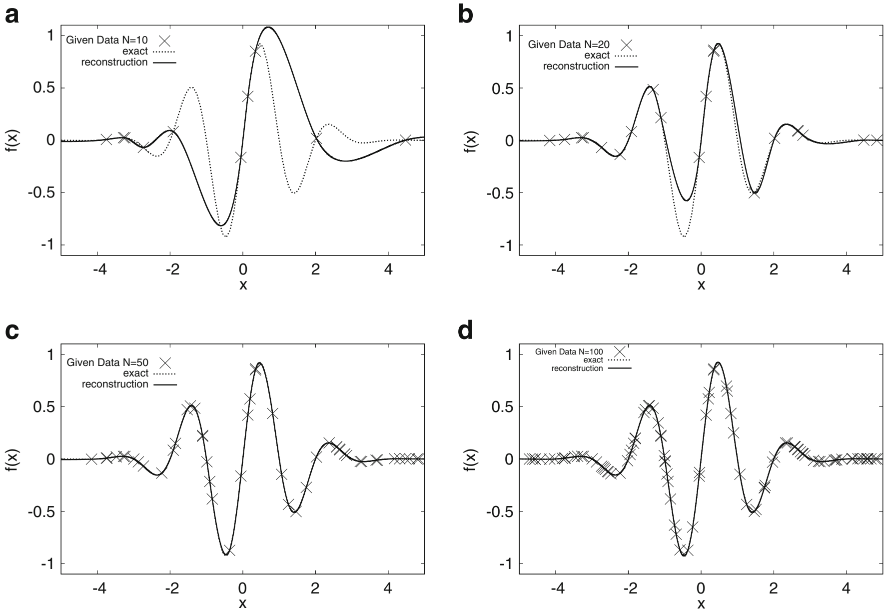

Fig. 2 Reconstruction of an analytic function (Example 1) in $H_2(\mathbb{R})$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 20$. (**c**) Number of sampling points $n = 50$. (**d**) Number of sampling points $n = 100$

We note that the target function f belongs to $H_1(\mathbb{R})$, $H_2(\mathbb{R})$, and $W(\pi)$; thus f_{A_n} in these settings give good approximations to f as the number of sampling points n is sufficiently large (Proposition 4.2).

Example 2. We consider a piecewise linear function

$$f(x) = \begin{cases} x - 1, & 1 \leq x < 2, \\ 3 - x, & 2 \leq x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Sampling points $\{p_j\}$ are chosen in the same manner as Example 1. Figures 4, 5, and 6 show numerical reconstructions in the space $H_1(\mathbb{R})$, $H_2(\mathbb{R})$, and $W(\pi)$, respectively. We give a remark that f belongs to $H_1(\mathbb{R})$, hence reconstruction gives a good approximation shown in Fig. 4. On the other hand, f does not belong to $W(\pi)$; thus reconstructed f_{A_n} has serious oscillations shown in Fig. 6, and it diverges as n tends to large (Fig. 7).

Example 3. We apply the proposed procedure to the characteristic function $\chi_{[1,3]}(x)$. Figures 8, 9, and 10 show numerically reconstructed f_{A_n}. Since the target function is discontinuous and it does not belong to the considered function spaces, oscillation

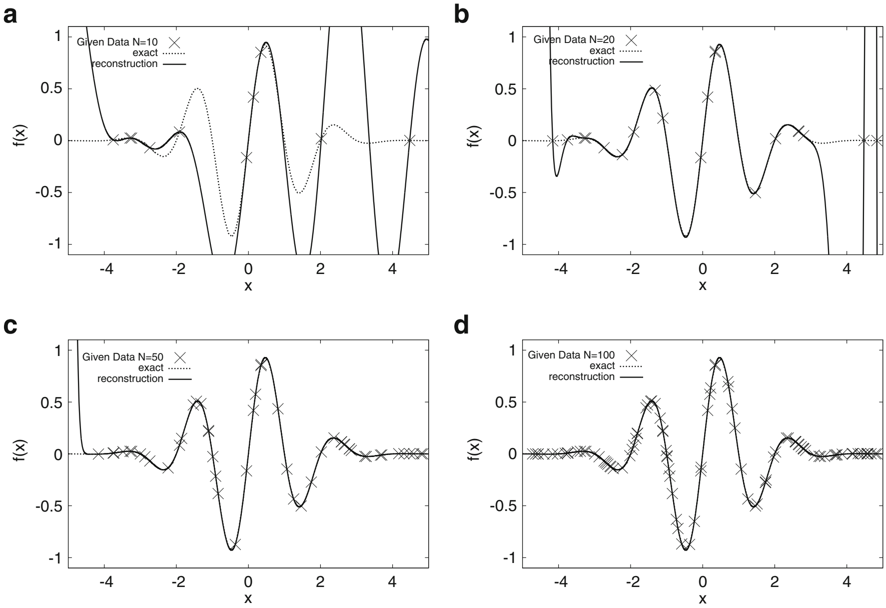

Fig. 3 Reconstruction of an analytic function (Example 1) in $W(\pi)$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 20$. (**c**) Number of sampling points $n = 50$. (**d**) Number of sampling points $n = 100$

appears in f_{A_n} as Figs. 9d and 10. More precisely, reconstruction in $W(\pi)$ grows rapidly as n tends to increase as shown in Fig. 11.

Finally we give a remark on the instability of the proposed algorithm.

The procedure includes calculating the inversion of the matrix constituted by elements (8). Figure 12 shows the condition numbers of the matrix in 2-norm for some reproducing kernels. The reproducing kernel K_1 is not differentiable and K_2 is twice continuously differentiable, and condition numbers corresponding to K_2 are larger than those of K_1 from Fig. 12a. Moreover, the reproducing kernel of the Paley–Wiener space is analytic, and corresponding condition numbers shown in Fig. 12b are larger than those of K_2. These numerical results indicate that smoothness of a function space causes instability of the numerical procedure.

Since the procedure is numerically unstable, the influence of rounding errors is serious in its numerical computations. Figure 13 shows numerical results with various computational precisions. Figure 13a, b are results by the standard double precision (16 decimal digits) and 50 decimal digits. The influence of rounding errors appears due to lack of computational precision. Multiple-precision arithmetic enables us to reduce the influence of rounding errors and to obtain reliable numerical results shown in Fig. 13c, d. It also gives a quantitative viewpoint to check the influence of rounding errors.

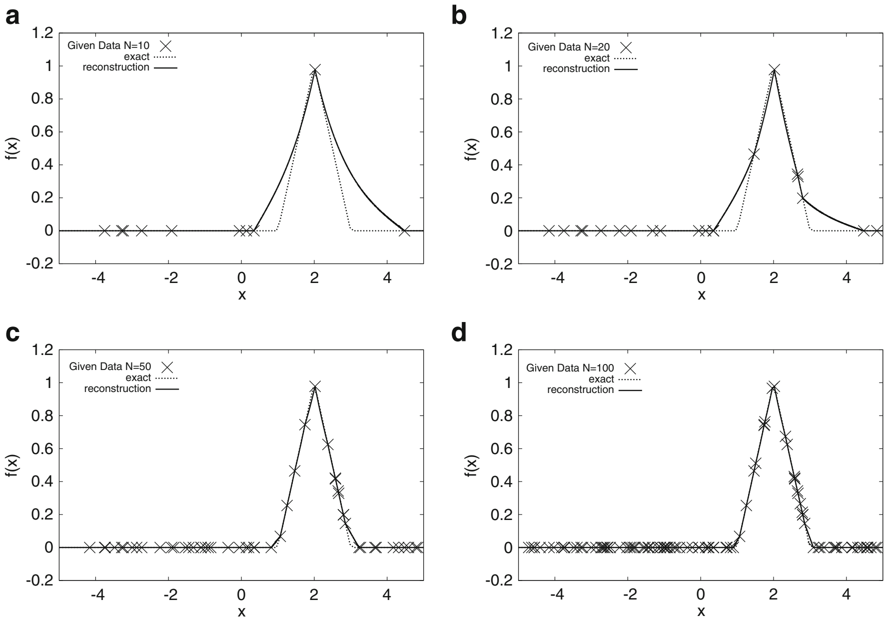

Fig. 4 Numerical reconstructions for Example 2 in $H_1(\mathbb{R})$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 20$. (**c**) Number of sampling points $n = 50$. (**d**) Number of sampling points $n = 100$

The results imply that 200 decimal digit precision is not enough, and almost 300 decimal digits are required to obtain reliable profiles of reconstructed functions by overcoming the instability.

7 Applications of the Support Vector Machine Method

For the extremal problem (5)–(7), Mo and Qian [10] applied the support vector machine method and compared with other typical numerical methods in analytical problems generally; that is, the Aveiro discretization method was compared with the classical methods: the finite difference method (FDM) and the finite element method (FEM). Furthermore, they referred to the artificial neural networks (ANNs) to solve partial differential equations in the discretization method. Meanwhile, support vector machine (SVM), developed by V. Vapnik and his coworkers in 1995 [14], is based on statistical learning theory which seeks to minimize an upper bound of the generalization error consisting of the sum of the training error and a confidence interval. This principle is different from the commonly used empirical risk minimization (ERM) principle which only minimizes the training error. Based on this, SVMs usually achieve higher generalization performance than ANNs

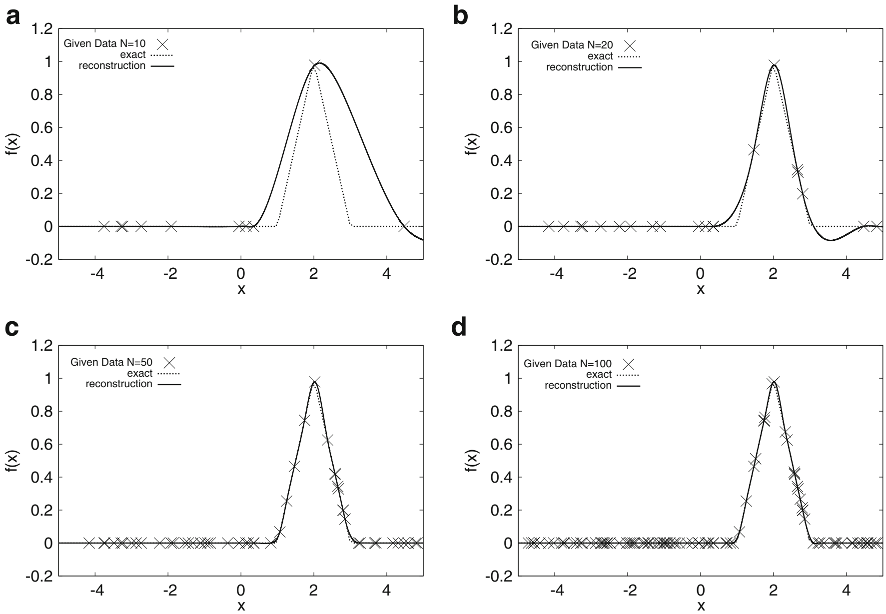

Fig. 5 Numerical reconstructions for Example 2 in $H_2(\mathbb{R})$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 20$. (**c**) Number of sampling points $n = 50$. (**d**) Number of sampling points $n = 100$

which implement ERM principle. As a consequence, SVMs can be used wherever that ANNs can, and usually achieve better results. Another key characteristic of SVM is that training SVM is equivalent to solve a linearly constrained quadratic programming problem so that the solution of SVM is unique and global, unlike ANNs' training which requires nonlinear optimization with the possibility of getting stuck into local minima.

Based on the above viewpoints of Mo and Qian [10], the extremal problem (5)–(7) will be very important and may be related to the support vector machine, directly. Our numerical calculation for the introduced algorithms requested a great computer power as in the Fujiwara's multiple-precision arithmetic strategy introduced, but the application of the SVMs to the problem may be applied with a usual level computer power and furthermore may be dealt with erroneous data. Therefore, the application of the support vector machine to the Aveiro discretization method will be very interested. The paper [10] is the first challenge for this approach. See [10] for the detail algorithm and numerical experiments.

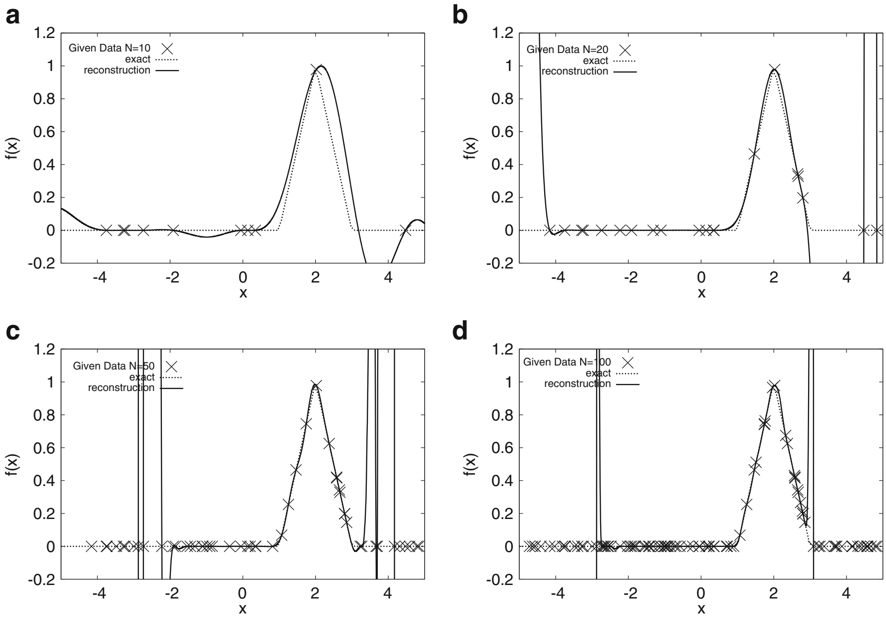

Fig. 6 Numerical reconstructions for Example 2 in $W(\pi)$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 20$. (**c**) Number of sampling points $n = 50$. (**d**) Number of sampling points $n = 100$

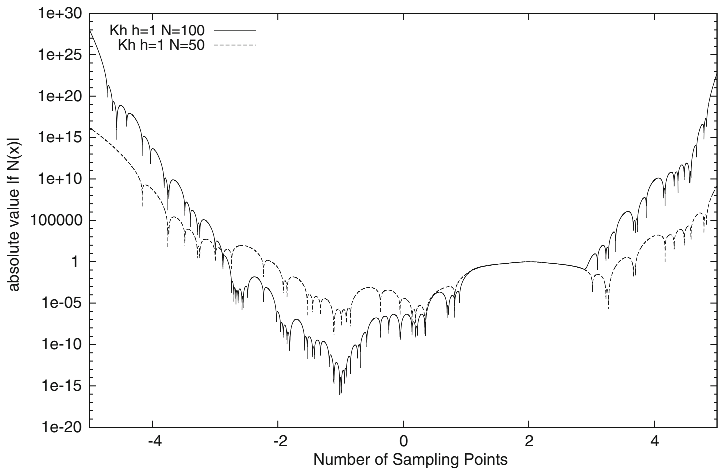

Fig. 7 Profiles of $|f_n(x)|$ $(n = 50, 100)$ in Example 2 in $W(\pi)$

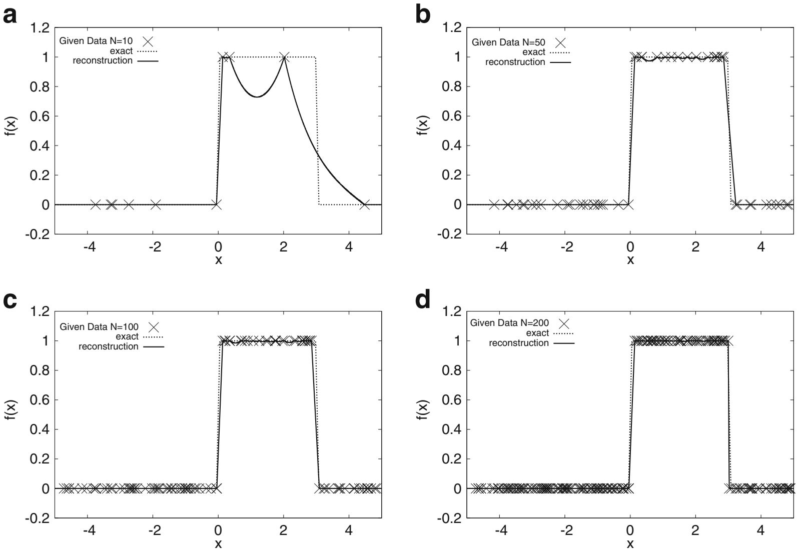

Fig. 8 Reconstruction of a discontinuous (step) function (Example 3) in $H_1(\mathbb{R})$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 50$. (**c**) Number of sampling points $n = 100$. (**d**) Number of sampling points $n = 200$

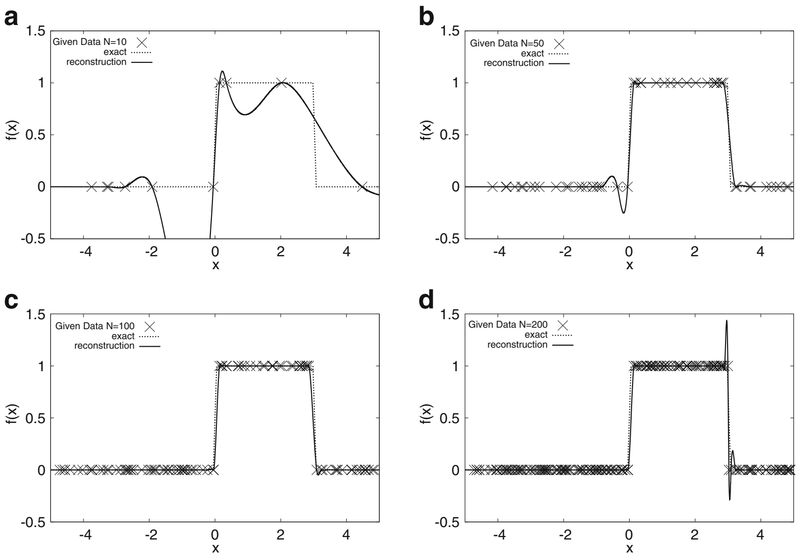

Fig. 9 Reconstruction of a discontinuous (step) function (Example 3) in $H_2(\mathbb{R})$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 50$. (**c**) Number of sampling points $n = 100$. (**d**) Number of sampling points $n = 200$

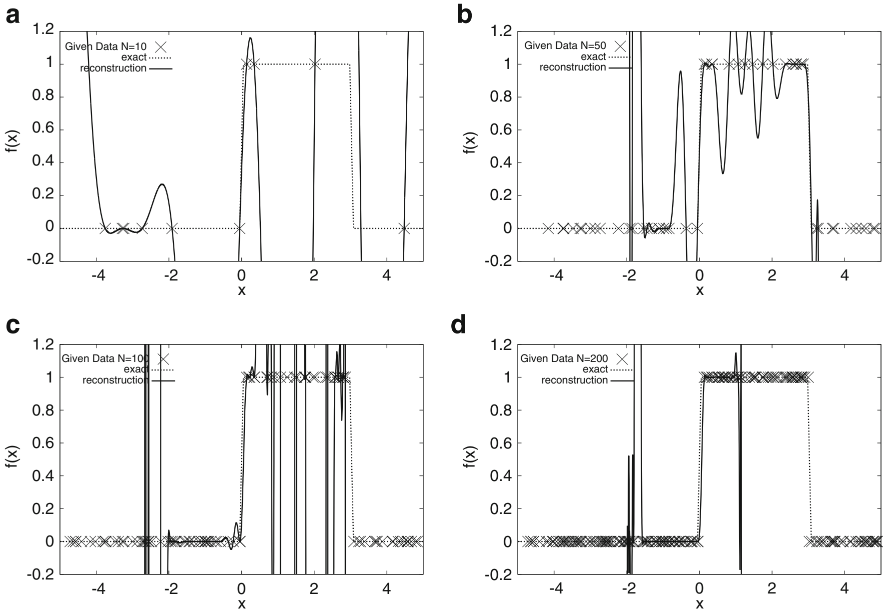

Fig. 10 Reconstruction of a discontinuous function (Example 3) in $W(\pi)$. (**a**) Number of sampling points $n = 10$. (**b**) Number of sampling points $n = 50$. (**c**) Number of sampling points $n = 100$. (**d**) Number of sampling points $n = 200$

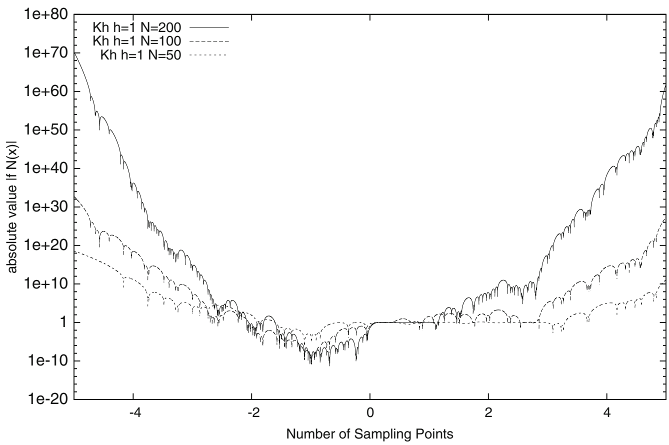

Fig. 11 Profiles of $|f_n(x)|$ ($n = 50$, 100, and 200) in Example 3 in $W(\pi)$

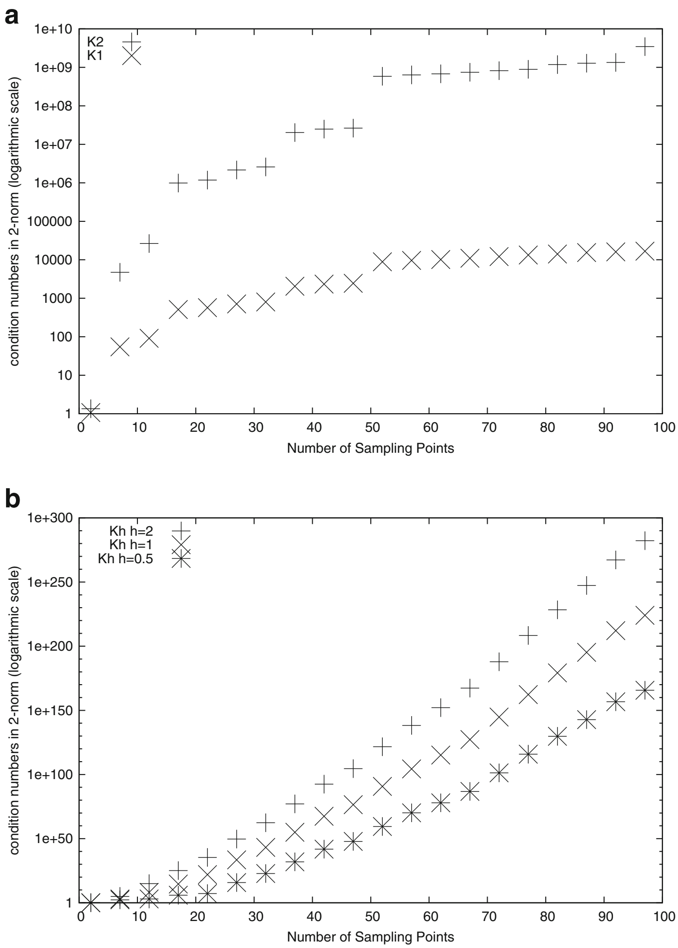

Fig. 12 Condition numbers of the matrix $\left(K(p_i, p_j)\right)$ in 2-norm. (**a**) K_1 and K_2. (**b**) K_h

8 Conclusion

We showed a general sampling theorem and the concrete numerical experiments for the simplest and typical examples. We gave the sampling theorem in the Sobolev Hilbert spaces with numerical experiments. For the Sobolev Hilbert spaces, sampling theorems seem to be a new concept.

For the typical Paley–Wiener spaces, the sampling points are automatically determined as the common sense; however, in our general sampling theorem, we can select the sampling points freely, and so, case by case, following some a priori information of a considering function, we can take the effective sampling points. We showed these properties by the concrete examples.

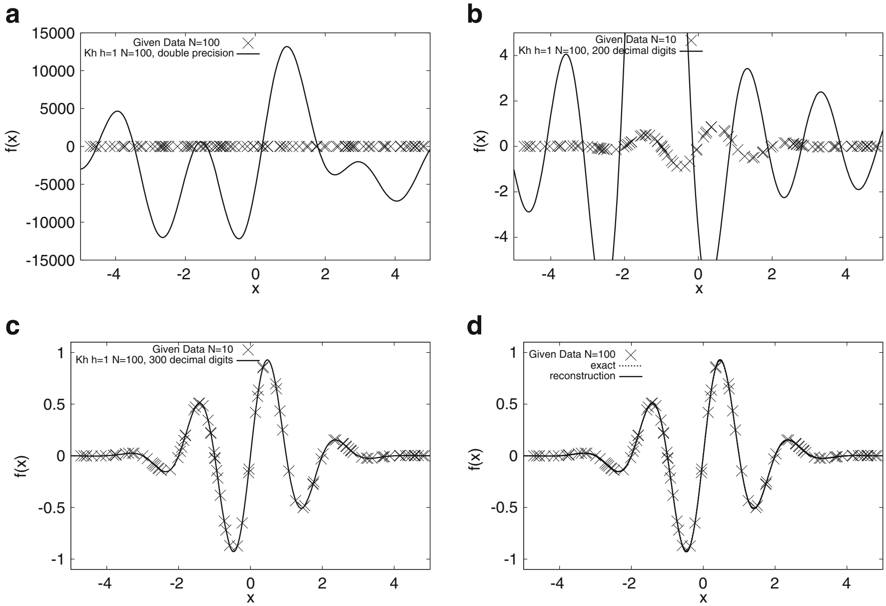

Fig. 13 Influence of rounding errors in numerical reconstruction by various computational precision in $W(\pi)$ with $n = 100$. (**a**) The standard double precision. (**b**) 200 decimal digits. (**c**) 300 decimal digits. (**d**) 600 decimal digits

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Kronecker's Products and Kronecker's Sums of Operators

Michael Gil'

In Honor of Constantin Carathéodory

Abstract This chapter is a survey of recent results of the author on operators on tensor products of Hilbert and Euclidean spaces. We derive norm estimates for the resolvents of Kronecker's products of operators, Kronecker's sums of operators, and operator pencils on tensor products of Hilbert spaces. By these estimates, we investigate bounds for spectra of perturbed operators. Applications of our results to matrix differential and integro-differential operators are also discussed.

1 Introduction and Notations

1.1 Introduction

The present paper is a survey of the recent results of the author on operators on tensor products of Hilbert spaces.

Operators on tensor products of Hilbert (in particular, Euclidean) spaces arise in various problems of pure and applied mathematics, for instance, in the theories of matrix equations [27], system theory [1], and quantum mechanics [34], as well as in the theories of differential [11], partial integral and integro-differential operators [10], dynamical systems [33], etc.

In the finite dimensional case, tensor products of spaces are a classical notion of the multilinear algebra [24]. The theory of linear operators on tensor products

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of Euclidean spaces is well developed; see the well-known books [23, 25, 34]. The infinite-dimensional case is investigated considerably less than the finite dimensional one.

The classical results on operators on tensor products of Hilbert spaces are presented in [2, 3, 31, 32]. In particular, Brown and Pearcy [2] investigated the spectrum of the Kronecker (tensor) product of operators and Ichinose [26] explored the spectrum of the Kronecker (tensor) sum of operators. The recent results can be found, in particular, in the papers [28–30] and references therein. In [28] the author investigates the invariant subspaces of operators on multiple tensor products. In the paper [30] the authors prove that the weak (strong, uniform) convergence of sequences of Hilbert space operators is preserved by tensor products. In the case of convergence to zero, it is shown that the boundedness of one sequence and the weak (strong, uniform) convergence to zero of the other one suffice to ensure the convergence of their tensor products to zero in the same topology and that the converse holds for power sequences. They also show that a tensor product of operators is a unilateral shift if and only if it coincides with a tensor product of a unilateral shift and an isometry. In the paper [29] the authors investigate the problem of transferring the Weyl and Browder theorems from operators to their tensor product.

In the present paper, we establish norm estimates for the resolvents of Kronecker's products of operators, Kronecker's sums of operators, and operator pencils on tensor products of Hilbert spaces. By these estimates, we investigate bounds for the spectra of perturbed operators. Applications of our results to matrix differential and integro-differential operators are also discussed.

Here are a few words about the contents. The paper consists of 13 sections.

In Sect. 2 we collected some results on matrices and operators which are systematically used in the following sections.

Sections 3–6 are concerned with the Kronecker products of operators; in particular, norm estimates for the resolvents of Kronecker's products of operators in Euclidean and Hilbert spaces are suggested.

Sections 7–10 are devoted to the resolvents and spectrum perturbations of Kronecker's sums of operators.

Section 11 deals with operator pencils on tensor products of Hilbert spaces. The results obtained in Sect. 11 enable us to investigate in Sects. 12 and 13 the spectra of differential operators with matrix coefficients and integro-differential operators, respectively.

1.2 Notations

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be separable Hilbert spaces with scalar products $\langle .,.\rangle_1$ and $\langle .,.\rangle_2$, respectively, and the norms $\|.\|_j = \sqrt{\langle .,.\rangle_j}$ $(j = 1, 2)$ and the unit operator $I = I_j$.

The tensor product $\mathscr{H} = \mathscr{E}_1 \otimes \mathscr{E}_2$ of $\mathscr{E}_1$ and $\mathscr{E}_2$ is defined by the following way. Consider the collection of all formal finite sums of the form

$$u = \sum_j y_j \otimes h_j \quad (y_j \in \mathscr{E}_1, h_j \in \mathscr{E}_2)$$

with the understanding that

$$\lambda(y \otimes h) = (\lambda y) \otimes h = y \otimes (\lambda h), (y + y_1) \otimes h = y \otimes h + y_1 \otimes h,$$
$$y \otimes (h + h_1) = y \otimes h + y \otimes h_1 \;\; (y, y_1 \in \mathscr{E}_1; h, h_1 \in \mathscr{E}_2; \lambda \in \mathbf{C}).$$

On that collection define the scalar product as

$$\langle y \otimes h, y_1 \otimes h_1 \rangle_{\mathscr{H}} = \langle y, y_1 \rangle_1 \; \langle h, h_1 \rangle_2 \;\; (y, y_1 \in \mathscr{E}_1, h, h_1 \in \mathscr{E}_2)$$

and take the norm $\|.\|_{\mathscr{H}} = \sqrt{\langle ., . \rangle_{\mathscr{H}}}$. Then $\mathscr{H}$ is the completion of the considered collection in the norm $\|.\|_{\mathscr{H}}$. Besides $I_{\mathscr{H}} = I$ denotes the unit operator in $\mathscr{H}$.

From the theory of tensor products, we only need elementary facts which can be found in [3].

For a linear operator A in $\mathscr{H}$, $\sigma(A)$ is the spectrum; Dom (A) is the domain; $R_\lambda(A) := (A - I\lambda)^{-1}$ is the resolvent; $\lambda_k(A)$ $(k = 1, 2, \cdots)$ are the eigenvalues with their multiplicities; co(A) is the closed convex hull of $\sigma(A)$; A^* is the adjoint one; Im $A = (A - A^*)/2i$ is the imaginary Hermitian component; $r_s(A)$ denotes the (upper) spectral radius; $r_{\text{low}}(A)$ is the lower spectral radius: $r_{\text{low}}(A) = \inf\ |\sigma(A)|$; $\alpha\ (A) = \sup\ \text{Re}\ \sigma(A)$, $\beta\ (A) = \inf\ \text{Re}\ \sigma(A)$, and $\rho(A, \lambda) := \inf_{t \in \sigma(A)} |t - \lambda|$ is the distance between $\sigma(A)$ and a $\lambda \in \mathbf{C}$; $\lambda_k(A)$ are the eigenvalues taken with their multiplicities; and $\|A\| = \|A\|_{\mathscr{H}}$ means the operator norm.

By SN_p we denote the Schatten–von Neumann ideal of operators K with the finite norm $N_p(K) := [\text{Trace}(KK^*)]^{p/2}$ $(p \geq 1)$. So SN_2 is the Hilbert–Schmidt ideal.

A linear operator V is said to be *quasinilpotent if* $\sigma(V) = \{0\}$. *V is called a Volterra operator if it is quasinilpotent and compact.*

$L(\mathscr{E}_1, \mathscr{E}_2)$ denotes the set of all bounded operators acting from a space $\mathscr{E}_1$ into a space $\mathscr{E}_2$. $L(\mathscr{E})$ denotes the set of all bounded operators in a space $\mathscr{E}$.

Lemma 1. *If $x_n \to x_0$ in $\mathscr{E}_1$ and $y_n \to y_0$ in $\mathscr{E}_2$, then $x_n \otimes y_n \to x_0 \otimes y_0$.*

Proof. With obvious notations, for an $\epsilon > 0$, suppose $\|x - x_0\| \leq \epsilon$ and $\|y - y_0\| \leq \epsilon$. The identity

$$x \otimes y - x_0 \otimes y_0 = x \otimes (y - y_0) + (x - x_0) \otimes y_0$$

yields the inequalities

$$\|x \otimes y - x_0 \otimes y_0\| \leq \|x \otimes (y - y_0)\| + \|(x - x_0) \otimes y_0\|$$
$$= \|x\| \|y - y_0\| + \|x - x_0\| \|y_0\| \leq \epsilon(\|x\| + \|y_0\|).$$

This proves the lemma. Q.E.D.

2 Preliminaries

2.1 Resolvents of Finite-Dimensional Operators

As it is well-known, by Schur's theorem [25], for any operator A in $\mathbf{C}^n$, there is an orthogonal normal basis (Schur's basis) $\{e_k\}_{k=1}^n$, in which A is represented by a triangular matrix. That is,

$$Ae_k = \sum_{j=1}^{k} a_{jk} e_j \text{ with } a_{jk} = (Ae_k, e_j) \quad (j = 1, \ldots, n),$$

and $a_{jj} = \lambda_j(A)$. So $A = D_A + V_A$ $(\sigma(A) = \sigma(D_A))$ with a normal (diagonal) matrix D_A defined by $D_A e_j = \lambda_j(A) e_j$ $(j = 1, \ldots, n)$ and a nilpotent (strictly upper-triangular) matrix V_A defined by $V_A e_k = a_{1k} e_1 + \cdots + a_{k-1,k} e_{k-1}$ $(k = 2, \ldots, n)$, $V_A e_1 = 0$. So D_A and V_A are the diagonal part and nilpotent part of A, respectively. The Schur basis is not unique.

In the sequel $|A| = |A|_{Sb}$ *means the operator, whose entries in some of its fixed Schur basis are the absolute values of the entries of operator* A *in that basis. We will call* $|A|$ *the absolute value of* A (with respect to the Schur basis), i.e.,

$$|A|e_k = \sum_{j=1}^{k} |a_{jk}| e_j \quad (j = 1, \ldots, n).$$

We write $A \geq 0$ if all the entries of A are nonnegative and $A \geq B$ if $A - B \geq 0$. If A is normal, then $|A| \leq r_s(A) I$.

The smallest integer $\nu_A \leq n$, *such that* $|V_A|^{\nu_A} = 0$, *will be called the nilpotency index of* A.

In the following lemma A, V_A and $(A - \lambda I)^{-1}$ are considered in the same Schur basis.

Lemma 2. *Let* A *be an* $n \times n$*-matrix. Then*

$$|(A - \lambda I)^{-1}| \leq \sum_{j=0}^{\nu_A - 1} \frac{1}{\rho^{j+1}(A, \lambda)} |V_A|^j \quad (\lambda \notin \sigma(A)),$$

where $\rho(A, \lambda) = \min_{k=1,\ldots,n} |\lambda - \lambda_k(A)|$ *and* V_A *is the nilpotent part of* A.

Proof. Due to the triangular representation, we have

$$A - \lambda I = D_A + V_A - \lambda I = (D_A - \lambda I)(I - Q_\lambda),$$

where $Q_\lambda = (D_A - \lambda I)^{-1} V_A$ is nilpotent, since V_A in Schur's basis is a nilpotent triangular matrix and $(D_A - \lambda I)^{-1}$ is a diagonal one. Hence,

$$(A - \lambda I)^{-1} = (I - Q_\lambda)^{-1}(D_A - \lambda I)^{-1} \quad (\lambda \notin \sigma(D_A)).$$

But $\sigma(A) = \sigma(D_A)$ and $|(D_A - \lambda I)^{-1}| \leq \frac{1}{\rho(D_A,\lambda)} I$ and therefore $|Q_\lambda| \leq \frac{1}{\rho(D_A,\lambda)} |V_A|$. So $Q_\lambda^{\nu_A} = 0$ and thus,

$$(A - \lambda I)^{-1} = (D_A - \lambda I)^{-1} \sum_{k=0}^{\nu_A - 1} Q_\lambda^k \quad (\lambda \notin \sigma(A)).$$

This proves the required result. Q.E.D.

Lemma 3. *For any nilpotent operator $V \in L(\mathbf{C}^n)$, we have*

$$\|V^j\| \leq \frac{N_2^j(V)}{\sqrt{j!}} \quad (j = 1, \ldots, n-1).$$

For the proof, see [4, Corollary 2.5.2],

Put

$$g(A) \;=\; (N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2},$$

$$u(A) := \left[2N^2(\text{Im } A) - 2\sum_{k=1}^{n} |\text{Im } \lambda_k(A)|^2 \right]^{1/2},$$

and

$$\vartheta(A) = \left[\text{Tr } (A^*A - I) - \sum_{k=1}^{n} (|\lambda_k(A)|^2 - 1) \right]^{1/2},$$

where $\lambda_k(A)$ are the eigenvalues of A with their multiplicities. Due to Theorem 2.3.1 and Lemma 7.15.2 from the book [4], the following result is true.

Lemma 4. *Let $A \in L(\mathbf{C}^n)$. Then*

$$N_2(V_A) = u(A) = g(A) = \vartheta(A).$$

Lemma 5. *One has*

$$\|(A - \lambda I)^{-1}\| \leq \sum_{j=0}^{\nu_A - 1} \frac{g^j(A)}{\sqrt{j!}\,\rho^{j+1}(A,\lambda)} \quad (\lambda \notin \sigma(A)).$$

Proof. Due to Lemma 3,

$$\| |V_A|^j \| \le \frac{N_2^j(|V_A|)}{\sqrt{j!}} = \frac{N_2^j(V_A)}{\sqrt{j!}} \quad (j = 1, \ldots, \nu_A - 1).$$

Now the previous lemma yields the required result. Q.E.D.

The later lemma is a slight refinement of Corollary 2.1.2 from [4].
The following relations are checked in [4, Sect. 1.5]:

$$g^2(A) \le N_2^2(A) - |\text{Trace } A^2|, g(A) \le \frac{1}{\sqrt{2}} N_2(A - A^*)$$

and $g(e^{ia}A + zI_H) = g(A)$ $(a \in \mathbf{R}, z \in \mathbf{C})$; if A is a normal matrix: $A^*A = AA^*$, then $g(A) = 0$. If A_1 and A_2 have a joint Schur's basis (in particular, they commute), then $g(A_1 + A_2) \le g(A_1) + g(A_2)$. In addition, by the inequality between the geometric and arithmetic mean values,

$$\left(\frac{1}{n} \sum_{k=1}^{n} |\lambda_k(A)|^2 \right)^n \ge \left(\prod_{k=1}^{n} |\lambda_k(A)| \right)^2.$$

Hence $g^2(A) \le N_2^2(A) - n(\det A)^{2/n}$.

2.2 *Resolvents of Infinite-Dimensional Operators*

In this subsection $\mathscr{H}$ is an infinite-dimensional separable Hilbert space and $A \in L(\mathscr{H})$.

2.2.1 Resolvents of Hilbert–Schmidt Operators

Theorem 1. *Let* $A \in \mathrm{SN}_2$ *and*

$$g(A) = \left[N_2^2(A) - \sum_{k=1}^{\infty} |\lambda_k(A)|^2 \right]^{1/2}.$$

Then

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A, \lambda)} \quad (\lambda \notin \sigma(A)).$$

Moreover,

$$\|R_\lambda(A)\| \le \frac{1}{\rho(A,\lambda)} \exp\left[\frac{1}{2} + \frac{g^2(A)}{2\rho^2(A,\lambda)}\right] \quad (\lambda \notin \sigma(A)).$$

This result is due to Theorems 6.4.1 and 6.4.2 from [4].

In the infinite-dimensional case, the relations

$$g^2(A) \le N_2^2(A) - |\mathrm{Trace}\, A^2|, g(A) \le \frac{1}{\sqrt{2}} N_2(A - A^*)$$

are also valid. If A is a normal operator: $A^*A = AA^*$, then $g(A) = 0$. If A_1 and A_2 are commuting operators, then $g(A_1 + A_2) \le g(A_1) + g(A_2)$.

2.2.2 Resolvents of Schatten–von Neumann Operators

Theorem 2. *Let the condition $A \in \mathrm{SN}_{2p}$ hold for an integer $p \ge 2$. Then*

$$\|R_\lambda(A)\| \le \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{(2N_{2p}(A))^{pk+m}}{\rho^{pk+m+1}(A,\lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(A)).$$

Moreover,

$$\|R_\lambda(A)\| \le e^{1/2} \sum_{m=0}^{p-1} \frac{(2N_{2p}(A))^m}{\rho^{m+1}(A,\lambda)} \exp\left[\frac{(2N_{2p}(A))^{2p}}{2\rho^{2p}(A,\lambda)}\right] \quad (\lambda \notin \sigma(A)).$$

The proof of this theorem can be found in [4, Sects. 6.7 and 6.8]. Put

$$\theta_j^{(p)} = \frac{1}{\sqrt{[j/p]!}},$$

where $[x]$ means the integer part of a real number x. Now the previous theorem implies

Corollary 1. *For an integer $p \ge 2$, let $A \in \mathrm{SN}_{2p}$. Then*

$$\|R_\lambda(A)\| \le \sum_{j=0}^{\infty} \frac{\theta_j^{(p)}(2N_{2p}(A))^j}{\rho^{j+1}(A,\lambda)} \quad (\lambda \notin \sigma(A)).$$

Since the condition $A \in \mathrm{SN}_{2p}$ implies $A - A^* \in \mathrm{SN}_{2p}$, one can utilize additional estimates, presented in Sect. 2.2.4 below. The next result is proved in [4, Theorem 7.7.1].

2.2.3 Operators with Hilbert–Schmidt Hermitian Components

Theorem 3. *Let $A - A^* \in \mathrm{SN}_2$. Then with the notation*

$$u(A) = \left[2N_2^2(\mathrm{Im}\ A) - 2\sum_{k=1}^{\infty}(\mathrm{Im}\ \lambda_k)^2\right]^{1/2}$$

we have

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{u^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)} \quad (\lambda \notin \sigma(A)).$$

Moreover,

$$\|R_\lambda(A)\| \le \frac{1}{\rho(A,\lambda)} \exp\left[\frac{1}{2} + \frac{u^2(A)}{2\rho^2(A,\lambda)}\right] \quad (\lambda \notin \sigma(A)).$$

If operators $A, B \in L(\mathscr{H})$ commute, and $A - A^*, B - B^* \in \mathrm{SN}_2$, then $u(A+B) \le u(A) + u(B)$.

Furthermore, if a unitary operator U commutes with A, and

$$\mathrm{Im}\ (UA) := (UA - (UA)^*)/2i \in \mathrm{SN}_2,$$

then the previous theorem is valid with

$$u(UA) := 2\left[N_2^2((UA)_I) - \sum_{k=1}^{\infty}(\mathrm{Im}\ (\lambda_k(UA))^2\right]$$

instead of $u(A)$. One can take the operator U defined by the multiplication by e^{it} for a real t. Then

$$u^2(e^{it}A)) = 2N_2^2(\mathrm{Im}\ (e^{it}A)) - 2\sum_{k=1}^{\infty}(\mathrm{Im}\ (e^{it}\lambda_k(A)))^2.$$

2.2.4 Operators with Neumann–Schatten Hermitian Components

Assume that

$$\mathrm{Im}\, A := (A - A^*)/2i \in \mathrm{SN}_{2p} \text{ for some integer } p > 1. \tag{1}$$

Put

$$\beta_p := \begin{cases} 2(1 + ctg\ (\frac{\pi}{4p})\ ) & \text{if } p = 2^{m-1},\ m = 1, 2, \dots \\ 2(1 + \frac{2p}{\exp(2/3)\ln 2}) & \text{otherwise} \end{cases}.$$

Theorem 4. *Let condition (1) hold. Then*

$$\|R_\lambda(A)\| \le \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{(\beta_p N_{2p}(A_I))^{kp+m}}{\rho^{pk+m+1}(A,\lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(A)).$$

Moreover,

$$\|R_\lambda(A)\| \le e^{1/2} \sum_{m=0}^{p-1} \frac{(\beta_p N_{2p}(A_I))^m}{\rho^{m+1}(A,\lambda)} \exp\left[\frac{(\beta_p N_{2p}(A_I))^{2p}}{\rho^{2p}(A,\lambda)}\right] \quad (\lambda \notin \sigma(A)).$$

For the proof, see Theorems 7.9.1 and 7.9.2 from [4].

2.2.5 Operators Close to Unitary Ones

Assume that A has a regular point on the unit circle and $AA^* - I \in \mathrm{SN}_1$. Put

$$\vartheta(A) = \left[\mathrm{Tr}\ (A^*A - I) - \sum_{k=1}^{\infty}(|\lambda_k(A)|^2 - 1)\right]^{1/2},$$

where $\lambda_k(A), k = 1, 2, \dots$ are the nonunitary eigenvalues with their multiplicities, that is, the eigenvalues with the property $|\lambda_k(A)| \neq 1$.

Theorem 5. *Let $AA^* - I \in \mathrm{SN}_1$ and A have a regular point on the unit circle. Then*

$$\|R_\lambda(A)\| \le \sum_{k=0}^{\infty} \frac{\vartheta^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)} \quad (\lambda \notin \sigma(A)).$$

Moreover,

$$\|R_\lambda(A)\| \le \frac{1}{\rho(A,\lambda)} \exp\left[\frac{1}{2} + \frac{\vartheta^2(A)}{2\rho^2(A,\lambda)}\right] \quad (\lambda \notin \sigma(A)).$$

For the proof, see [4, Theorem 7.15.1].

If A is a normal operator, then $\vartheta(A) = 0$. Let A have the unitary spectrum only. That is, $\sigma(A)$ lies on the unit circle. Then $\vartheta(A) = [\mathrm{Tr}\ (A^*A - I)]^{1/2}$. Moreover, if the condition

$$\sum_{k=1}^{\infty}(|\lambda_k(A)|^2-1)\geq 0$$

holds, then

$$\text{Tr }(A^*A-I)=\sum_{k=1}^{\infty}(s_k^2(A)-1)\geq\sum_{k=1}^{\infty}(|\lambda_k(A)|^2-1)=\text{Tr }(D^*D-I)\geq 0$$

and therefore, $\vartheta(A)\leq[\text{Tr }(A^*A-I)]^{1/2}$.

2.3 Spectral Variations

Definition 1. Let A and B be linear operators in $\mathscr{H}$. Then the quantity

$$sv_A(B):=\sup_{\mu\in\sigma(B)}\ \inf_{\lambda\in\sigma(A)}|\mu-\lambda|$$

is called the spectral variation of a B with respect to A. In addition,

$$\text{hd}(A,B):=\max\{sv_A(B),sv_B(A)\}$$

is the Hausdorff distance between the spectra of A and B.

We will need the following technical lemma.

Lemma 6. *Let A_1 and A_2 be linear operators in $\mathscr{H}$ with the same domain and $q:=\|A_1-A_2\|<\infty$. In addition, let*

$$\|R_\lambda(A_1)\|\leq F\left(\frac{1}{\rho(A_1,\lambda)}\right)\quad(\lambda\notin\sigma(A_1)),$$

where $F(x)$ is a monotonically increasing continuous function of a nonnegative variable x, such that $F(0)=0$ and $F(\infty)=\infty$. Then $sv_{A_1}(A_2)\leq z(F,q)$, where $z(F,q)$ is the unique positive root of the equation $1=qF(1/z)$.

For the proof, see [4, Lemma 8.4.2].

Consider the scalar equation

$$\sum_{k=1}^{\infty}a_k z^k=1, \tag{2}$$

where the coefficients a_k $(k = 1, 2, \ldots)$ have the property

$$\gamma_0 := 2 \max_k \sqrt[k]{|a_k|} < \infty.$$

We need the following:

Lemma 7. *Any root z_0 of Eq. (2) satisfies the estimate $|z_0| \geq 1/\gamma_0$.*

For the proof, see [4, Lemma 8.3.1].

Lemma 8. *The unique positive root z_a of the equation*

$$\sum_{j=0}^{p-1} \frac{1}{y^{j+1}} \exp\left[\frac{1}{2}\left(1 + \frac{1}{y^{2p}}\right)\right] = a \quad (a = \text{const} > 0; p = 1, 2, \ldots) \tag{3}$$

satisfies the inequality $z_a \leq \delta_p(a)$, where

$$\delta_p(a) := \begin{cases} pe/a & \text{if } a \leq pe, \\ [\ln (a/p)]^{-1/2p} & \text{if } a > pe \end{cases}.$$

The proof of this result can be found in [4, Lemma 8.3.2].

Corollary 2. *The unique positive root $z(c, d)$ of the equation*

$$\frac{c}{y} \exp\left[\frac{b^2}{y^2}\right] = 1 \quad (c, d = \text{const} > 0) \tag{4}$$

satisfies the inequality $z(b, c) \leq \delta(b, c)$, where

$$\delta(b, c) := \begin{cases} ce^{1/2} & \text{if } b\sqrt{2} \leq ce^{1/2}, \\ b\sqrt{2}\left(\ln\left(\frac{b\sqrt{2e}}{c}\right)\right)^{-1/2} & \text{if } b\sqrt{2} > ce^{1/2} \end{cases}.$$

Indeed substitute $y = b\sqrt{2}x$ into (4), then we obtain

$$\frac{c}{b\sqrt{2}} \frac{1}{x} \exp\left[1/(2x^2)\right] = 1.$$

Hence we obtain Eq. (3) with

$$a = \sqrt{2e}\, \frac{b}{c}.$$

Now the previous lemma implies the result.

We need also the following lemma from [12, Lemma 1.6.5] .

Lemma 9. *The unique positive root z_0 of the equation*

$$ze^z = a \quad (a = \text{const} > 0)$$

satisfies the estimate

$$z_0 \geq \ln \left[\frac{1}{2} + \sqrt{\frac{1}{4} + a} \right].$$

If, in addition, the condition $a \geq e$ holds, then $z_0 \geq \ln a - \ln \ \ln a$.

For small a we asymptotically have $\sqrt{1 + 4a} \approx 1 + 2a$ and

$$z_0 \geq \ln \left[\frac{1}{2}(1 + \sqrt{1 + 4a}) \right] \approx \ln [1 + a] \approx a.$$

2.4 Additional Perturbation Results

Let X be a Banach space with a norm $\|.\|_X$.

Lemma 10. *Let A and $\tilde{A}$ be linear operators in X with the same dense domain* $\text{Dom}(A)$*, and the operators $C = \tilde{A} - A$ and $Z := \tilde{A}C - CA$ be bounded. In addition, let $\lambda \in \mathbf{C}$ be a regular point of both operators A and $\tilde{A}$. Then*

$$R_\lambda(\tilde{A}) - R_\lambda(A) = R_\lambda(\tilde{A}) Z R_\lambda^2(A) - C R_\lambda^2(A). \tag{5}$$

For the proof, see Lemma 3.1 from [14]. The result similar to the latter lemma for bounded operators has been proved in [13].

Denote

$$\eta(A, C) := \sup_{0 \leq t \leq 1} t \|(AC - CA + tC^2) R_\lambda^2(A)\|_X.$$

Lemma 11. *Let A and $\tilde{A}$ be linear operators in X with the same dense domain* $\text{Dom}(A)$*, and the operators $C = \tilde{A} - A$ and $\tilde{A}C - CA$ be bounded. In addition, let $\lambda \in \mathbf{C}$ be a regular point of A and $\eta(A, C, \lambda) < 1$. Then $\lambda \notin \sigma(\tilde{A})$ and identity (5) holds. Moreover,*

$$\|R_\lambda(\tilde{A})\|_X \leq \frac{\|R_\lambda(A) - C R_\lambda^2(A)\|_X}{1 - \eta(A, C, \lambda)}.$$

For the proof, see Lemma 3.2 from [14].

It is clear that $\eta(A, C, \lambda) \le \zeta_X^2(A, C)\|R_\lambda^2(A)\|_X$, where

$$\zeta_X(A, C) := \sqrt{\|AC - CA\|_X + \|C^2\|_X}.$$

Now the previous lemma yields the following result.

Corollary 3. *Let A and $\tilde{A}$ satisfy the hypothesis of Lemma 11. Let $\lambda \notin \sigma(A)$ and $\zeta_X(A, C)\|R_\lambda(A)\|_X < 1$. Then $\lambda \notin \sigma(\tilde{A})$ and relation (5) holds.*

Remark 1. The Hilbert identity $R_z(\tilde{A}) - R_z(A) = -R_z(\tilde{A})(\tilde{A} - A)R_z(A)$ implies that $\lambda \notin \sigma(\tilde{A})$ provided $\lambda \notin \sigma(A)$ and

$$\|C\|_X\|R_\lambda(A)\|_X < 1. \tag{6}$$

Thus Corollary 3 improves (6), provided

$$\|AC - CA\|_X + \|C^2\|_X < \|C\|_X^2.$$

This inequality holds, for example, if A and $\tilde{A}$ are commuting and C is non-normal.

About perturbations of finite and infinite-dimensional operators with simple spectra via condition numbers, see [16, 18, 21, 22].

We need also the following result proved in [15].

Theorem 6. *Let $A \in \mathrm{SN}_2$. Then*

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^2 \le \left[N_2^4(A) - \frac{1}{2}N_2^2(A^*A - AA^*)\right]^{1/2}.$$

3 Basic Properties of Kronecker's Products of Operators

Let $\mathscr{E}_1, \mathscr{E}_2$ be separable Hilbert spaces and $\mathscr{H} = \mathscr{E}_1 \otimes \mathscr{E}_2$, again.

Definition 2. The Kronecker product of $A \in L(\mathscr{E}_1)$ and $B \in L(\mathscr{E}_2)$ denoted by $A \otimes B$ is defined by

$$(A \otimes B)(f_1 \otimes f_2) = (Af_1) \otimes (Bf_2) \quad (f_l \in \mathscr{E}_l, l = 1, 2).$$

Some very basic properties of the Kronecker product are stated in the following lemma.

Lemma 12. *With obvious notations:*

(a) $(A + A_1) \otimes B = A \otimes B + A_1 \otimes B$; $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$; $(\lambda A \otimes B) = \lambda(A \otimes B)$; $(A \otimes \lambda B) = \lambda(A \otimes B)$ $(\lambda \in \mathbf{C})$.

(b) $I_1 \otimes I_2 = I_{\mathscr{H}}$.

(c) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
(d) $(A \otimes B)^* = A^* \otimes B^*$.
(e) $\|A \otimes B\| = \|A\|\|B\|$.
(f) *$A \otimes B$ is invertible if and only if A and B are both invertible,*

in which case $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Proof.

(a) For example,

$$((A + A_1) \otimes B)(x \otimes y) = (Ax + A_1x) \otimes By = Ax \otimes By + A_1x \otimes By =$$

$$(A \otimes B)(x \otimes y) + (A_1 \otimes B)(x \otimes y).$$

Similarly the other relations from (a) can be proved.

(b) $(I \times I)(x \otimes y) = x \otimes y = I(x \otimes y)$.

(c) $(A \otimes B)(C \otimes D)(x \otimes y) = (A \otimes B)(Cx \otimes Dy) = (ACx \otimes BDy) = [(AC) \otimes (BD)](x \otimes y)$.

(d) $\langle (A \otimes B)^*(x \otimes y), u \otimes v\rangle = \langle x \otimes y, (Au \otimes Bv)\rangle = \langle x, Au\rangle_1 \langle y, Bv\rangle_2 = \langle A^*x, u\rangle_1 \langle B^*y, v\rangle_2 = \langle A^*x \otimes B^*y, u \otimes y\rangle = \langle (A^* \otimes B^*)(x \otimes y), u \otimes y\rangle$.

(e) We have

$$\|(A \otimes B)(x \otimes y)\| = \|Ax \otimes By\| = \|Ax\|\|By\| \leq \|A\|\|x\|\|B\|\|y\|.$$

So $\|A \otimes B\| \leq \|A\|\|B\|$. To prove the reverse inequality, choose sequences $x_n \in \mathscr{E}_1, y_n \in \mathscr{E}_2$, $\|x_n\| = \|y_n\| = 1$, and $\|Ax_n\| \to \|A\|$, $\|By_n\| \to \|B\|$. Then

$$\|Ax_n \otimes By_n\| = \|Ax_n\|\|By_n\| \to \|A\|\|B\|.$$

Since

$$\|x_n \otimes y_n\| = \|x_n\|\|y_n\| = 1,$$

we have

$$\|(A \otimes B)(x_n \otimes y_n)\| \leq \|A \otimes B\|$$

whence

$$\|A\|\|B\| \leq \|A \otimes B\|.$$

on passing to the limit.

(f) If A and B are invertible, then $(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1} = I$ and similarly $(A^{-1} \otimes B^{-1})(A \otimes B) = I$. Conversely, suppose $A \otimes B$ is invertible. Since

$$A \otimes B = (A \otimes I)(I \otimes B),$$

it follows that $A \otimes I$ and $I \otimes B$ are also invertible, so it will suffice to show that the invertibility of $A \otimes I$ implies that of A (the proof for B is similar). We know that $A \otimes I$ and $A^* \otimes I$ are bounded from below, and it will suffice to show that A and $A*$ are bounded from below. Thus we are reduced to showing that the boundedness from below of $A \otimes I$ implies that of A. By supposition, there exists an $\epsilon > 0$ such that $\|(A \otimes I)u\| \geq \epsilon\|u\|$ for all $u \in \mathscr{H}$. Then, $\|(A \otimes I)x \otimes y\| \geq \epsilon\|x\|\|y\|$ for all $x \in \mathscr{E}_1, y \in \mathscr{E}_2$, that is, $\|Ax \otimes y\| \geq \epsilon\|x\|\|y\|$ whence $\|Ax\| \geq \epsilon\|x\|$ (choose any nonzero y, and then cancel). Similarly, $\|A^*x\| \geq \epsilon\|x\|$. Q.E.D.

Corollary 4. *If $A \in L(\mathscr{E}_1)$ and $B \in L(\mathscr{E}_2)$ are either (a) unitary, (b) self-adjoint, (c) positive definite, or (d) normal, then so is $A \otimes B$.*

Indeed, for example, let A and B be unitary. Then $(A \otimes B)^*(A \otimes B) = A^*A \otimes B^*B = I$ and similarly $(A \otimes B)(A \otimes B)^* = I$.

Similarly the other assertions can be proved.

Theorem 7. *The spectrum of $A \otimes B$ is*

$$\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B) = \{ts : t \in \sigma(A), s \in \sigma(B)\}.$$

For the proof, see [2]. In the finite dimensional case, see [25].

If A and B are finite dimensional operators, then from the latter theorem, it follows Trace $(A \otimes B) =$ Trace (A) Trace (B). Hence we easily get

Corollary 5. *Let $A, B \in \mathrm{SN}_1$. Then* Trace $(A \otimes B) =$ Trace (A) Trace (B) .

If A and B are finite dimensional operators, then

$$(A \otimes B)^*(A \otimes B) = A^*A \otimes B^*B$$

and

$$[(A \otimes B)^*(A \otimes B)]^p = (A^*A)^p \otimes (B^*B)^p \quad (p > 0);$$

we arrive at the relation

$$\text{Trace } [(A \otimes B)^*(A \otimes B)]^p = \text{Trace } (A^*A)^p \text{ Trace } (B^*B)^p.$$

We thus obtain

Corollary 6. *Let $A, B \in \mathrm{SN}_p$ $(1 \leq p < \infty)$. Then $N_p(A \otimes B) = N_p(A)N_p(B)$.*

4 Norm Estimates for Resolvents of Kronecker's Products in Finite Dimensional Spaces

4.1 Statement of the Result

In this subsection $\mathscr{E}_l = \mathbf{C}^{n_l}$. So $\mathscr{H} = \mathbf{C}^{n_1} \otimes \mathbf{C}^{n_2}$ and $A_l \in L(\mathbf{C}^{n_l})$ $(n_l < \infty; l = 1, 2)$. Recall that

$$g(A_l) = \left[N_2^2(A_l) - \sum_{k=1}^{n_l} |\lambda_k(A_l)|^2 \right]^{1/2}$$

and denote

$$\zeta_j(A_1, A_2) := \sum_{k_1, k_2 = 0}^{j} C^j_{k_1, k_2, j-k_1-k_2} r_s^{k_1}(A_1) r_s^{k_2}(A_2) \frac{g^{j-k_1}(A_1) g^{j-k_2}(A_2)}{\sqrt{(j-k_1)!(j-k_2)!}},$$

where

$$C^j_{k_1, k_2, k_3} = \frac{j!}{k_1! k_2! k_3!}.$$

Note that $C^j_{k_1, k_2, j-k_1-k_2} = 0$ if $k_1 + k_2 > j$. So one can write

$$\zeta_j(A_1, A_2) = \sum_{0 \le k_1 + k_2 \le j} C^j_{k_1, k_2, j-k_1-k_2} r_s^{k_1}(A_1) r_s^{k_2}(A_2) \frac{g^{j-k_1}(A_1) g^{j-k_2}(A_2)}{\sqrt{(j-k_1)!(j-k_2)!}}.$$

If A_1 is normal, then $g(A_1) = 0$ and with $g^0(A_1) = 1$ we have

$$C^j_{k_2, j, -k_2} = \frac{j!}{k_2! j! (-k_2)!} = 0 \text{ for } k_2 > 0 \text{ and } C^j_{0, j, 0} = \frac{j!}{0! j! 0!} = 1.$$

Thus, in this case

$$\zeta_j(A_1, A_2) = \frac{r_s^j(A_1) g^j(A_2)}{\sqrt{j!}} \quad (j = 0, 1, 2, \ldots).$$

If both A_1 and A_2 are normal, then $\zeta_0(A_1, A_2) = 1$ and $\zeta_j(A_1, A_2) = 0$ for $j \ge 1$.

Put $\nu(n_1, n_2) = n_1 + n_2 + \min\{n_1, n_2\} - 2$.

Theorem 8. *Let* $A_l \in L(\mathbf{C}^{n_l})$ $(l = 1, 2)$. *Then*

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\nu(n_1, n_2) - 1} \frac{\zeta_j(A_1, A_2)}{\rho^{j+1}(A_1 \otimes A_2, \lambda)} \quad (\lambda \notin \sigma(A_1 \otimes A_2)), \tag{7}$$

where

$$\rho(A_1 \otimes A_2, \lambda) := \inf_{t \in \sigma(A_1), s \in \sigma(A_2)} |ts - \lambda|.$$

With $k_3 = j - k_1 - k_2$ we have

$$\frac{1}{(j-k_1)!(j-k_2)!} = \frac{1}{(k_1+k_3)!(k_2+k_3)!} \le \frac{1}{k_1!k_2!k_3!} = \frac{j!}{j!k_1!k_2!k_3!} \le \frac{3^j}{j!},$$

since

$$(a+b+c)^j = \sum_{k_1+k_2+k_3=j} a^{k_1} b^{k_2} c^{k_3} C^j_{k_1,k_2,k_3} \quad (a,b,c = \text{const}).$$

Thus

$$\begin{aligned}\zeta_j(A_1,A_2) &\le \frac{3^{j/2}}{\sqrt{j!}} \sum_{k_1,k_2=0}^{j} C^j_{k_1,k_2,j-k_1-k_2} r_s^{k_1}(A_1) r_s^{k_2}(A_2) g^{j-k_1}(A_1) g^{j-k_2}(A_2) \\ &= \frac{3^{j/2}}{\sqrt{j!}} (g(A_1)r_s(A_2) + g(A_2)r_s(A_1) + g(A_1)g(A_2))^j.\end{aligned}$$

Now Theorem 8 implies

Corollary 7. *Let* $A_l \in L(\mathbf{C}^{n_l})$ $(l = 1,2)$. *Then for all* $\lambda \notin \sigma(A_1 \otimes A_2)$,

$$\begin{aligned}&\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \\ &\quad \le \sum_{j=0}^{\nu(n_1,n_2)-1} \frac{3^{j/2}(g(A_1)r_s(A_2) + g(A_2)r_s(A_1) + g(A_1)g(A_2))^j}{\sqrt{j!}\rho^{j+1}(A_1 \otimes A_2, \lambda)}.\end{aligned}$$

4.2 *Proof of Theorem 8*

We need a few technical lemmas.

Lemma 13. *Let* W_k $(k = 1,2,3)$ *be mutually commuting nilpotent operators in* $\mathbf{C}^n$ *and* $W_k^{j_k} = 0$ *for an integer* $j_k \le n$. *Then*

$$(W_1 + W_2)^{j_1+j_2-1} = 0 \tag{8}$$

and

$$(W_1 + W_2 + W_3)^{j_1+j_2+j_3-2} = 0. \tag{9}$$

Proof. Put $\hat{W}_2 = W_1 + W_2$. We have

$$\hat{W}_2^{j_1+j_2-1} = \sum_{k=0}^{j_1+j_2-1} \binom{j_1+j_2-1}{k} W_1^{j_1+j_2-k-1} W_2^k,$$

where $\binom{j}{k}$ are the binomial coefficients. Thus either $j_1+j_2-k-1 \geq j_1$ or $k \geq j_2$. This proves (8). Furthermore, put $\hat{W}_3 = W_1 + W_2 + W_3 = \hat{W}_2 + W_3$. Since $\hat{W}_2^{j_1+j_2-1} = 0$, replacing in our arguments W_1 by $\hat{W}_2$ and W_2 by W_3, we obtain $\hat{W}_3^{j_1+j_2+j_3-2} = 0$ as claimed. Q.E.D.

Let $V_l \in L(\mathbf{C}^{n_l})$ be nilpotent operators and $B_l \in L(\mathbf{C}^{n_l})$ be normal ones $(l = 1, 2)$. In addition, V_1 and B_1 have a joint Schur's basis $\left\{e_k^{(1)}\right\}_{k=1}^{n_1}$. Similarly, V_2 and B_2 have a joint Schur's basis $\left\{e_k^{(2)}\right\}_{k=1}^{n_2}$.

Consider the operator

$$W = V_1 \otimes B_2 + B_1 \otimes V_2 + V_1 \otimes V_2. \tag{10}$$

Then the Schur basis of W is defined as $\{e_j^{(1)} \otimes e_k^{(2)}\}_{j=1,\ldots,n_1;k=1,\ldots,n_2}$. Recall that the absolute value of a matrix with respect to a Schur's basis is defined in Sect. 2.1. Besides, we put $|A_1 \otimes A_2| = |A_1| \otimes |A_2|$. So if

$$W\left(e_j^{(1)} \otimes e_k^{(2)}\right) = \sum_{j_1=1}^{j} \sum_{k_1=1}^{k} w_{j_1 j k_1 k} \left(e_{j_1}^{(1)} \otimes e_{k_1}^{(2)}\right),$$

then

$$|W|\left(e_j^{(1)} \otimes e_k^{(2)}\right) = \sum_{j_1=1}^{j} \sum_{k_1=1}^{k} |w_{j_1 j k_1 k}| \left(e_{j_1}^{(1)} \otimes e_{k_1}^{(2)}\right),$$

and therefore

$$|W| \leq |V_1| \otimes |B_2| + |B_1| \otimes |V_2| + |V_1| \otimes |V_2|.$$

Lemma 14. *Let W be defined by (10). Then $W^{\nu(n_1,n_2)} = 0$ and*

$$|W|^j \leq \sum_{k_1+k_2+k_3=j} C^j_{k_1,k_2,k_3} r_s^{k_1}(B_1) r_s^{k_2}(B_2)(|V_1|^{k_2+k_3} \otimes |V_2|^{k_1+k_3}) \quad (j < \nu(n_1, n_2)).$$

Proof. In a Schur's basis of W, we have

$$|W| \leq r_s(B_1)I \otimes |V_2| + |V_1| \otimes r_s(B_2)I + |V_1| \otimes |V_2|.$$

Put $W_1 = r_s(B_1)I \otimes |V_2|, W_2 = |V_1| \otimes r_s(B_2)I,$ and $W_3 = |V_1| \otimes |V_2|$. Since they commute and $|V_l|^{n_l} = 0$, due to (9) we have $|W|^{\nu(n_1,n_2)} = 0$. In addition, we can write

$$(W_1 + W_2 + W_3)^j = \sum_{k_1+k_2+k_3=j} C^j_{k_1,k_2,k_3} W_1^{k_1} W_2^{k_2} W_3^{k_3}$$
$$= \sum_{k_1+k_2+k_3=j} C^j_{k_1,k_2,k_3} r_s^{k_1}(B_1) r_s^{k_2}(B_2)(I \otimes |V_2|^{k_1}) (|V_1|^{k_2} \otimes I)(|V_1|^{k_3} \otimes |V_2|^{k_3})$$
$$= \sum_{k_1+k_2+k_3=j} C^j_{k_1,k_2,k_3} r_s^{k_1}(B_1) r_s^{k_2}(B_2)(|V_1|^{k_2+k_3} \otimes |V_2|^{k_2+k_3})$$

as claimed. Q.E.D.

From Lemma 3 we have

$$\|V_l^j\| \leq \frac{N_2^j(V_l)}{\sqrt{j!}} \quad (j = 1, \dots, n_l - 1).$$

Thus the previous lemma implies

$$\||W|^j\| \leq \sum_{k_1+k_2+k_3=j} C^j_{k_1,k_2,k_3} r_s^{k_1}(B_1) r_s^{k_2}(B_2) \frac{N_2^{k_2+k_3}(V_1) N_2^{k_1+k_3}(V_2)}{\sqrt{(k_1+k_3)!(k_2+k_3)!}}.$$

Moreover, taking into account that $k_3 = j - k_1 - k_2$, from the previous lemma, we get

Corollary 8. *Let W be defined by (10). Then $W^{\nu(n_1,n_2)} = 0$ and*

$$\||W|^j\| \leq \sum_{k_1,k_2=0}^{j} C^j_{k_1,k_2,j-k_1-k_2} r_s^{k_1}(B_1) r_s^{k_2}(B_2) \frac{N_2^{j-k_1}(V_1) N_2^{j-k_2}(V_2)}{\sqrt{(j-k_2)!(j-k_1)!}}$$

for $j < \nu(n_1, n_2)$.

Furthermore, we need the triangular representations of $A_l \in L(E_l)$:

$$A_l = D_l + V_l \quad (\sigma(A_l) = \sigma(D_l);\ l = 1, 2),$$

where D_l and V_l are the diagonal and nilpotent parts of A_l, respectively.
Consequently,

$$T := A_1 \otimes A_2 = D_T + V_T, \text{ where } D_T = D_1 \otimes D_2, \text{ and}$$
$$V_T = V_1 \otimes D_2 + D_1 \otimes V_2 + V_1 \otimes V_2. \tag{11}$$

Lemma 15. *One has*

$$|||V_T|^j\| \le \zeta_j(A_1, A_2) \ \ (j < \nu(n_1, n_2)) \text{ and } V_T^{\nu(n_1,n_2)} = 0.$$

Proof. Making use of the previous corollary with $B_l = D_l$, we have $W = V_T$ and therefore, $V_T^{\nu(n_1,n_2)} = 0$, $N_2(V_l) = g(A_l)$, and

$$|||V_T|^j\| \le \sum_{k_1,k_2=0}^{j} C^j_{k_1,k_2,j-k_1-k_2} r_s^{k_1}(A_2) r_s^{k_2}(A_1) \frac{g^{j-k_2}(A_1) g^{j-k_1}(A_2)}{\sqrt{(j-k_2)!(j-k_1)!}} = \zeta_j(A_1, A_2)$$

as claimed. Q.E.D.

Proof of Theorem 8. From Lemma 2 it follows:

$$\|(T - \lambda I)^{-1}\| \le \sum_{j=0}^{\nu_T - 1} \frac{1}{\rho^{j+1}(T, \lambda)} |||V_T|^j\|.$$

Now the previous lemma implies the assertion of the theorem. Q.E.D.

4.3 Additional Estimates for the Resolvent

Take into account that $D_l^* V_l$ is nilpotent. So $\text{Trace}(D_l^* V_l) = \text{Trace}(D_l V_l^*) = 0$. Now from (11) it follows:

$$\begin{aligned}
\text{Trace } (V_T^* V_T) &= \text{Trace } (V_1^* \otimes D_2^* + D_1^* \otimes V_2^* + V_1^* \otimes V_2^*) \\
&\quad\ (V_1 \otimes D_2 + D_1 \otimes V_2 + V_1 \otimes V_2) \\
&= \text{Trace } (V_1^* V_1 \otimes D_2^* D_2 + D_1^* D_1 \otimes V_2^* V_2 + V_1^* V_1 \otimes V_2^* V_2) \\
&= N_2^2(V_1) N_2^2(D_2) + N_2^2(V_2) N_2^2(D_1) + N_2^2(V_1) N_2^2(V_2).
\end{aligned}$$

Consequently,

$$g^2(T) = \text{Trace } (V_T^* V_T) = g^2(A_1) N_2^2(D_2) + g^2(A_2) N_2^2(D_1) + g^2(A_1) g^2(A_2) = g^2(A_1, A),$$

where

$$g^2(A_1, A) = g^2(A_1)\tau^2(A_2) + g^2(A_2)\tau^2(A_1) + g^2(A_1) g^2(A_2)$$

with

$$\tau(A_l) = \left[\sum_{k=1}^{n_l} |\lambda_k(A_l)|^2\right]^{1/2}.$$

Recall that $\tau(A_l) \le N_2(A_l)$. In addition, due to Theorem 6,

$$\tau^2(A_l) \le \left[N_2^4(A_l) - \frac{1}{2}N_2^2(A_l^*A_l - A_lA_l^*)\right]^{1/2}.$$

Now Lemmas 2 and 15 imply

Lemma 16. *Let* $A_l \in L(\mathbf{C}^{n_l})$ $(l = 1, 2)$. *Then*

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\nu(n_1,n_2)-1} \frac{g^j(A_1, A_2)}{\sqrt{j!}\rho^{j+1}(A_1 \otimes A_2, \lambda)} \quad (\lambda \notin \sigma(A_1 \otimes A_2)).$$

Remark 2. Due to Lemma 3, $g(A) = u(A) = \vartheta(A)$. So throughout this section, one can replace $g(A)$ by $u(A)$ or $\vartheta(A)$.

About other norm estimates for resolvents of operators on tensor products in finite dimensional spaces and their applications to the two-parameter problem, matrix equations, and differential equations, see [19, 20].

5 Resolvents of Kronecker's Products in Infinite-Dimensional Spaces

In this section $\mathscr{E}_1$ and $\mathscr{E}_2$ can be infinite-dimensional spaces. As above $A_l \in L(\mathscr{E}_l)$ $(l = 1, 2)$.

5.1 Products with Hilbert–Schmidt Operators

Assume that

$$A_l \in \mathrm{SN}_2 \quad (l = 1, 2). \tag{12}$$

Recall that $g(A)$ is defined in Sect. 2.2.1. Again put

$$\zeta_j(A_1, A_2) := \sum_{k_1,k_2=0}^{j} C^j_{k_1,k_2,j-k_1-k_2} r_s^{k_1}(A_1) r_s^{k_2}(A_2) \frac{g^{j-k_1}(A_1) g^{j-k_2}(A_2)}{\sqrt{(j-k_1)!(j-k_2)!}}.$$

Theorem 9. *Let condition (12) hold. Then*

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\infty} \frac{\zeta_j(A_1, A_2)}{\rho^{j+1}(A_1 \otimes A_2, \lambda)} \quad (\lambda \notin \sigma(A_1 \otimes A_2)),$$

where $\rho(A_1 \otimes A_2, \lambda) = \inf_{s \in \sigma(A_1), t \in \sigma(A_2)} |ts - \lambda|$.

This result is due to Theorem 8 with $n_1, n_2 \to \infty$. Q.E.D.

In particular, if A_1 is normal, then

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\infty} \frac{r_s^j(A_1) g^j(A_2)}{\rho^{j+1}(A_1 \otimes A_2, \lambda)\sqrt{j!}}.$$

If both A_1 and A_2 are normal, then

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \frac{1}{\rho(A_1 \otimes A_2, \lambda)}.$$

So Theorem 9 is sharp.

Moreover, from Corollary 9 we get

Corollary 9. *Let condition (12) hold. Then for all* $\lambda \notin \sigma(A_1 \otimes A_2)$,

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\infty} \frac{3^{j/2}(g(A_1) r_s(A_2) + g(A_2) r_s(A_1) + g(A_1) g(A_2))^j}{\sqrt{j!}\rho^{j+1}(A_1 \otimes A_2, \lambda)} .$$

In addition, Lemma 16 implies

Corollary 10. *Let condition (12) hold. Then*

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\infty} \frac{g^j(A_1, A_2)}{\sqrt{j!}\rho^{j+1}(A_1 \otimes A_2, \lambda)} \quad (\lambda \notin \sigma(A_1 \otimes A_2)),$$

where

$$g(A_1, A_2) = [g^2(A_1)\tau^2(A_2) + g^2(A_2)\tau^2(A_1) + g^2(A_1) g^2(A_2)]^{1/2}$$

with

$$\tau(A_l) = \left[\sum_{k=1}^{\infty} |\lambda_k(A_l)|^2\right]^{1/2}.$$

That result is sharper than Corollary 9, provided

$$g(A_1, A_2) < 3^{1/2}(g(A_1) r_s(A_2) + g(A_2) r_s(A_1) + g(A_1) g(A_2)).$$

As in the finite dimensional case, we can use the Weyl inequality $\tau(A_l) \leq N_2(A_l)$. Moreover, Theorem 6 implies

$$\tau^2(A_l) \leq \left[N_2^4(A_l) - \frac{1}{2} N_2^2(A_l^* A_l - A_l A_l^*) \right]^{1/2}.$$

5.2 Products with Schatten–von Neumann Operators

Let the condition

$$A_1, A_2 \in \mathrm{SN}_{2p} \tag{13}$$

hold for an integer $p \geq 2$. Take into account that $N_{2p}(A_1 \otimes A_2) = N_{2p}(A_1)N_{2p}(A_2)$. Then Theorem 2 implies

Theorem 10. *Let condition (13) hold. Then*

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \sum_{m=0}^{p-1} \sum_{k=0}^{\infty} \frac{(2N_{2p}(A_1)N_{2p}(A_2))^{pk+m}}{\rho^{pk+m+1}(A_1 \otimes A_2, \lambda)\sqrt{k!}} \quad (\lambda \notin \sigma(A_1 \otimes A_2)).$$

In addition,

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}}$$
$$\leq e^{1/2} \sum_{m=0}^{p-1} \frac{(2N_{2p}(A_1)N_{2p}(A_2))^m}{\rho^{m+1}(A, \lambda)} \exp\left[\frac{(2N_{2p}(A_1)N_{2p}(A_2))^{2p}}{2\rho^{2p}(A_1 \otimes A_2, \lambda)} \right] \quad (\lambda \notin \sigma(A_1 \otimes A_2)).$$

Remark 3. Since condition (13) implies

$$\mathrm{Im}\,(A_1 \otimes A_2) = (\mathrm{Im}\,A_1) \otimes A_2 + A_1 \otimes (\mathrm{Im}\,A_2) \in \mathrm{SN}_{2p} \quad (\mathrm{Im}\,A = (A - A^*)/2i),$$

we have

$$N_{2p}(\mathrm{Im}\,(A_1 \otimes A_2)) \leq N_{2p}(\mathrm{Im}\,A_1)N_{2p}(A_2) + N_{2p}(A_1)N_{2p}(\mathrm{Im}\,A_2).$$

So according to Theorem 4 in the previous theorem, one can replace $N_{2p}(A_1)N_{2p}(A_2)$ by

$$\beta_p\,(N_{2p}(\mathrm{Im}\,A_1)N_{2p}(A_2) + N_{2p}(A_1)N_{2p}(\mathrm{Im}\,A_2)).$$

This replacement improves Theorem 10 for operators close to self-adjoint ones.

5.3 Kronecker's Products with Non-compact Operators

Recall that $u(A)$ and $\vartheta(A)$ are defined in Sects. 2.2.3 and 2.2.4, respectively. Put

$$\hat{u}(A) = \begin{cases} g(A) & \text{if } A \in \mathrm{SN}_2, \\ u(A) & \text{if } \operatorname{Im} A \in \mathrm{SN}_2, \\ \vartheta(A) & \text{if } A^*A - I \in \mathrm{SN}_1 \end{cases}$$

and

$$\hat{\zeta}_j(A_1, A_2) := \sum_{k_1,k_2=0}^{j} C^j_{k_1,k_2,j-k_1-k_2} r_s^{k_1}(A_2) r_s^{k_2}(A_1) \frac{\hat{u}^{j-k_2}(A_1)\hat{u}^{j-k_1}(A_2)}{\sqrt{(j-k_2)!(j-k_1)!}}.$$

Theorem 11. *For each $l = 1, 2$, let one of the following conditions hold:*

$$(a) \quad A_l \in \mathrm{SN}_2, \ (b) \ \operatorname{Im} A_l \in \mathrm{SN}_2, \ \textit{or}$$

$$(c) \ A_l^* A_l - I \in \mathrm{SN}_1 \textit{ and } A_l \textit{ has a regular point on the unit circle }. \tag{14}$$

Then

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\infty} \frac{\hat{\zeta}_j(A_1, A_2)}{\rho^{j+1}(A_1 \otimes A_2, \lambda)} \quad (\lambda \notin \sigma(A_1 \otimes A_2)).$$

Proof. Letting $n_1, n_2 \to \infty$ in Theorem 8 and taking into account Remark 2, we get the required assertion. Q.E.D.

If both A_1 and A_2 are normal, then from Theorem 11 it follows,

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \frac{1}{\rho(A_1 \otimes A_2, \lambda)}.$$

As it is shown in Sect. 4.1,

$$\frac{1}{(j-k_1)!(j-k_2)!} \le \frac{3^j}{j!} \quad (k_1 + k_2 \le j).$$

Hence, we have

$$\hat{\zeta}_j(A_1, A_2) \le \frac{\xi^j(A_1, A_2)}{\sqrt{j!}},$$

where

$$\xi_j(A_1, A_2) := \frac{3^{j/2}(\hat{u}(A_1)r_s(A_2) + \hat{u}(A_2)r_s(A_1) + \hat{u}(A_1)\hat{u}(A_2))^j}{\sqrt{j!}}.$$

So we arrive at

Corollary 11. *Under the hypothesis of Theorem 11, for all $\lambda \notin \sigma(A_1 \otimes A_2)$, one has*

$$\|(A_1 \otimes A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{\infty} \frac{\xi^j(A_1, A_2)}{\sqrt{j!}\ \rho^{j+1}(A_1 \otimes A_2, \lambda)}.$$

6 Spectrum Perturbations of Kronecker's Products

Let $\tilde{A}$ be an arbitrary bounded operator in $\mathscr{H} = \mathscr{E}_1 \otimes \mathscr{E}_2$ and $A_l \in L(\mathscr{E}_l), l = 1, 2$. Denote $q = \|A_1 \otimes A_2 - \tilde{A}\|$. Then by virtue of Corollary 11, we arrive at the following result.

Theorem 12. *For each $l = 1, 2$, let A_l satisfy one of the conditions (14). Then $sv_{A_1 \otimes A_2}(\tilde{A}) \le x_1(q)$, where $x_1(q)$ is the unique positive root of the equation*

$$1 = q \sum_{k=0}^{\infty} \frac{\xi^k}{\sqrt{k!} x^{k+1}} \quad (\xi = \xi(A_1, A_2)).$$

By the Schwarz inequality,

$$\left(\sum_{k=0}^{\infty} \frac{(\sqrt{2}\xi)^k}{\sqrt{2^k k!} x^{k+1}} \right)^2 \le \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{k=0}^{\infty} \frac{2^k \xi^{2k}}{k! x^{2k+2}} = \frac{2}{x^2} e^{2(\xi/x)^2}.$$

So $1 \le 2q^2 \frac{1}{x^2} e^{2(\xi/x)^2}$. Hence $x_1(q) \le x_0(q)$, where $x_0(q)$ is the unique positive root of the equation

$$2q^2 \frac{1}{x^2} e^{2(\xi/x)^2} = 1. \tag{15}$$

Substitute into this equation the equality $y = 2(\xi/x)^2$. Then we get

$$ye^y = \frac{\xi^2}{q^2}.$$

Due to Lemma 9, the unique positive root y_0 of the latter equation satisfies the inequality

$$y_0 \geq \ln\left[1/2 + \sqrt{1/4 + \xi^2/q^2}\right].$$

Consequently,

$$x_0(q) \leq \delta(\xi, q), \text{ where } \delta(\xi, q) := \frac{\sqrt{2}\xi}{\ln^{1/2}\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\xi^2}{q^2}}\right]}.$$

Now Theorem 12 yields

Corollary 12. *For each $l = 1, 2$, let one of the conditions (14) hold and $\tilde{A}$ be a bounded operator in $\mathscr{H}$. Then for any $\tilde{\mu} \in \sigma(\tilde{A})$, there is a $\mu \in \sigma(A_1 \otimes A_2)$, such that $|\mu - \tilde{\mu}| \leq \delta(\xi, q)$ ($\xi = \xi(A_1, A_2)$).*

Note that for small ξ, we have asymptotically

$$\delta(\xi, q) \approx \sqrt{2}q.$$

Moreover, rewrite (15) as

$$\sqrt{2}q\frac{1}{x}e^{(\xi/x)^2} = 1.$$

Then Corollary 2 implies $x_0(q) \leq \hat{\delta}(\xi, q)$, where

$$\hat{\delta}(\xi, q) := \begin{cases} q\sqrt{2e} & \text{if } \xi \leq qe^{1/2}, \\ \sqrt{2}(\ln(\sqrt{e}\xi/q))^{-1/2}\xi & \text{if } \xi > qe^{1/2} \end{cases}.$$

Now Theorem 12 yields

Corollary 13. *For each $l = 1, 2$, let one of the conditions (14) hold and $\tilde{A}$ be a bounded operator in $\mathscr{H}$. Then for any $\tilde{\mu} \in \sigma(\tilde{A})$, there is a $\mu \in \sigma(A_1 \otimes A_2)$, such that $|\mu - \tilde{\mu}| \leq \hat{\delta}(\xi, q)$ ($\xi = \xi(A_1, A_2)$).*

About the spectrum perturbations of finite dimensional Kronecker's products, see [17, Sect. 5].

7 Basic Properties of Kronecker's Sums of Operators

Definition 3. The Kronecker sum of $A_1 \in L(\mathscr{E}_1)$ and $A_2 \in L(\mathscr{E}_2)$ denoted by $A_1 \oplus A_2$ is defined by $A_1 \oplus A_2 = A_1 \otimes I_2 + I_1 \otimes A_2$.

Some of the basic operations of Kronecker's sums in Hilbert spaces are summarized in the next lemma. Its proof is straightforward.

Lemma 17. *For every* $\alpha, \beta \in \mathbf{C}$, $A, A_1 \in L(E_1)$, *and* $B, B_1 \in L(E_2)$*:*

(a) $(\alpha + \beta)(A \oplus B) = \alpha A \oplus \beta B + \beta A \oplus \alpha B$
(b) $(A_1 + A) \oplus (B_1 + B) = A_1 \oplus B_1 + A \oplus B$
(c) $(A_1 \oplus B_1)(A \oplus B) = A_1 \otimes B + A \otimes B_1 + A_1 A \oplus B_1 B$
(d) $(A \oplus B)^* = A^* \oplus B^*$
(e) $\|A \oplus B\| \leq \|A\| + \|B\|$

By this lemma

$$(A \oplus B)^*(A \oplus B) = (A^* \oplus B^*)(A \oplus B) = A^* \otimes B + A \otimes B^* + A^*A \oplus B^*B$$

and

$$(A \oplus B)(A \oplus B)^* = (A \oplus B)(A^* \oplus B^*) = A \otimes B^* + A^* \otimes B + AA^* \oplus BB^*.$$

Thus $A \oplus B$ is normal if A and B are normal.

If $\dim E_l = n_l < \infty, l = 1, 2$, then

$$\begin{aligned} N_p(A \oplus B) &\leq N_p(A \otimes I_2) + N_p(I_1 \otimes B) \\ &= N_p(A)N_p(I_2) + N_p(I_1)N_p(B) = \sqrt[p]{n_2}\, N_p(A) + \sqrt[p]{n_1}\, N_p(B). \end{aligned}$$

Theorem 13. *Let* $A_l \in L(E_l)$. *Then*

$$\sigma(A_1 \oplus A_2) = \{s + t : s \in \sigma(A_1), t \in \sigma(A_2)\}.$$

For the proof, see [26].

8 Resolvents of Kronecker's Sums with Finite-Dimensional Operators

Let $A_l \in L(\mathbf{C}^{n_l})$ $(n_l < \infty, l = 1, 2)$. Put

$$\tau_j(A_1, A_2) := \sum_{k=0}^{j} \binom{j}{k} \frac{g^k(A_1) g^{j-k}(A_2)}{\sqrt{(j-k)!k!}}, \text{ where } \binom{j}{k} = \frac{j!}{k!(j-k)!}.$$

Theorem 14. *Let* $A_l \in L(\mathbf{C}^{n_l})$. *Then*

$$\|(A_1 \oplus A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \sum_{j=0}^{n_1+n_2-2} \frac{\tau_j(A_1, A_2)}{\rho^{j+1}(A_1 \oplus A_2, \lambda)} \quad (\lambda \notin \sigma(A_1 \oplus A_2)).$$

Proof. Put $A = A_1 \oplus A_2$. Let D_l and V_l be the diagonal and nilpotent parts of A_l, respectively. Then

$$A = (D_1 + V_1) \otimes I_2 + I_1 \otimes (D_2 + V_2) = D_A + V_A$$

with

$$V_A := V_1 \otimes I_2 + I_1 \otimes +V_2, \quad D_A = D_1 \otimes I_2 + I_1 \otimes D_2.$$

It is not hard to check that V_A is nilpotent. Clearly, A, D_A, and V_A have the joint invariant subspaces. So D_A and V_A is the normal part and nilpotent one of A, respectively.

Recall that $|A_l|$ is the absolute value of A_l with respect to its fixed Schur basis. Besides, we put $|A_1 \oplus A_2| = |A_1| \oplus |A_2|$.

Let $W_1 = V_1 \otimes I_2$ and $W_2 = I_1 \otimes V_2$. Since they commute, we have

$$V_A^j = (W_1 + W_2)^j = \sum_{k=0}^{j} \binom{j}{k} W_1^k W_2^{j-k}.$$

Thus we can write

$$|V_A|^j \leq \sum_{k=0}^{j} \binom{j}{k} |W_1|^k |W_2|^{j-k}.$$

Due to (8) $\nu_A \leq n_1 + n_2 - 1$. Obviously,

$$\| |V_A|^j \|_{\mathscr{H}} \leq \sum_{k=0}^{j} \binom{j}{k} \| |V_1|^k \|_1 \| |V_2|^{j-k} \|_2 \quad (j < \nu_A).$$

Making use of Lemma 3, we obtain $\| |V_l|^j \|_l \leq \frac{N_2^j(V_l)}{\sqrt{j!}}$. Thus

$$\| |V_A|^j \|_{\mathscr{H}} \leq \sum_{k=0}^{j} \binom{j}{k} \frac{N_2^k(V_1) N_2^{j-k}(V_2)}{\sqrt{(j-k)!k!}}.$$

Hence, applying the equality $g(A_l) = N_2(V_l)$, we get

$$\| |V_A|^j \|_{\mathscr{H}} \leq \sum_{k=0}^{j} \binom{j}{k} \frac{g^k(A_1) g^{j-k}(A_2)}{\sqrt{(j-k)!k!}} = \tau_j(A_1, A_2).$$

From Lemma 2 it follows

$$\|(A_1 \oplus A_2 - \lambda I)^{-1}\| \le \sum_{j=0}^{\nu_A - 1} \frac{1}{\rho^{j+1}(A,\lambda)} \| |V_A|^j \|$$
$$\le \sum_{j=0}^{n_1+n_2-2} \frac{\tau_j(A_1,A_2)}{\rho^{j+1}(A,\lambda)}$$

as claimed. Q.E.D.

Furthermore, since

$$\frac{j!}{(j-k)!k!} \le 2^j \quad (k \le j),$$

we obtain

$$\tau_j(A_1,A_2) \le \frac{2^{j/2}}{\sqrt{j}} \sum_{k=0}^{j} \binom{j}{k} g^k(A_1)\hat{g}^{j-k}(A_2) = \frac{2^{j/2}(g(A_1)+g(A_2))^j}{\sqrt{j}}.$$

So we arrive at

Corollary 14. *Let* $A_l \in L(\mathbf{C}^{n_l}), l = 1, 2$. *Then*

$$\|(A_1 \oplus A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \sum_{j=0}^{n_1+n_2-2} \frac{2^{j/2}(g(A_1)+g(A_2))^j}{\rho^{j+1}(A_1 \oplus A_2, \lambda)\sqrt{j}} \quad (\lambda \notin \sigma(A_1 \oplus A_2)).$$

Remark 4. Due to Remark 2, $g(A) = u(A) = \vartheta(A)$. So in Theorem 14 and Corollary 14, one can replace $g(A)$ by $u(A)$ or $\vartheta(A)$.

9 Resolvents of Kronecker's Sums with Infinite-Dimensional Operators

In this section $\mathscr{H} = \mathscr{E}_1 \otimes \mathscr{E}_2$, where $\mathscr{E}_1, \mathscr{E}_2$ are separable Hilbert spaces, again.

Due to Theorem 13, $\rho(A_1 \oplus A_2, \lambda) = \inf_{s \in \sigma(A_1), t \in \sigma(A_2)} |t + s - \lambda|$.

Recall that $\hat{u}(A)$ is defined in Sect. 5. Put

$$\hat{\tau}_j(A_1,A_2) := \sum_{k=0}^{j} \binom{j}{k} \frac{\hat{u}^k(A_1)\hat{u}^{j-k}(A_2)}{\sqrt{(j-k)!k!}}.$$

Theorem 15. *For each $l = 1, 2$, let one of the conditions (14) hold. Then*

$$\|(A_1 \otimes A_2 - \lambda I_H)^{-1}\|_H \leq \sum_{j=0}^{\infty} \frac{\hat{\tau}_j(A_1, A_2)}{\rho^{j+1}(A_1 \otimes A_2, \lambda)} \quad (\lambda \notin \sigma(A_1 \otimes A_2)).$$

Proof. Letting $n_1, n_2 \to \infty$ in Theorem 14 and taking into account Remark 4, we get the required assertion. Q.E.D.

In particular, if A_1 is normal, then

$$\hat{\tau}_j(A_1, A_2) = \frac{\hat{u}^j(A_2)}{\sqrt{j}}$$

and thus

$$\|(A_1 \oplus A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \sum_{j=0}^{\infty} \frac{\hat{u}^j(A_2)}{\sqrt{j!}\rho^{j+1}(A_1 \oplus A_2, \lambda)}.$$

If both A_1 and A_2 are normal, then

$$\|(A_1 \oplus A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \frac{1}{\rho(A_1 \oplus A_2, \lambda)}.$$

So Theorem 15 is sharp.

As it was mentioned,

$$\frac{j!}{(j-k)!k!} \leq 2^j \quad (k \leq j),$$

and thus we obtain

$$\hat{\tau}_j(A_1, A_2) \leq \frac{\hat{g}^j(A_1, A_2)}{\sqrt{j}},$$

where

$$\hat{g}(A_1, A_2) := 2^{j/2}(\hat{u}(A_1) + \hat{u}(A_2)).$$

Due to the previous theorem, we arrive at

Corollary 15. *For each $l = 1, 2$, let one of the conditions (14) hold. Then*

$$\|(A_1 \oplus A_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \sum_{j=0}^{\infty} \frac{\hat{g}^j(A_1, A_2)}{\rho^{j+1}(A_1 \oplus A_2, \lambda)\sqrt{j}} \quad (\lambda \notin \sigma(A_1 \oplus A_2)).$$

10 Spectrum Perturbations of Kronecker's Sums

Let $\tilde{A}$ be an arbitrary bounded operator in $\mathscr{H} = \mathscr{E}_1 \otimes \mathscr{E}_2$. Denote

$$q_+ = \|A_1 \oplus A_2 - \tilde{A}\|.$$

Then by virtue of Lemma 6 and Corollary 15, we arrive at the following result.

Theorem 16. *For each $l = 1, 2$, let A_l satisfy one of the conditions (14). Then $sv_{A_1 \oplus A_2}(\tilde{A}) \le y_1(q_+)$, where $y_1(q_+)$ is the unique positive root of the equation*

$$1 = q_+ \sum_{k=0}^{\infty} \frac{\hat{g}^k}{\sqrt{k!}x^{k+1}} \quad (\hat{g} = \hat{g}(A_1, A_2)).$$

Repeating the arguments of Sect. 6, by the Schwarz inequality, we get $y_1(q_+) \le y_0(q_+)$, where $y_0(q_+)$ is the unique positive root of the equation

$$2q_+^2 \frac{1}{x^2} e^{2(\hat{g}/x)^2} = 1.$$

Hence, applying Lemma 9, we get

$$y_0(q_+) \le \delta(\hat{g}, q_+) := \frac{\sqrt{2}\hat{g}}{\ln^{1/2}\left[1/2 + \sqrt{1/4 + \hat{g}^2/(q_+^2)}\right]}.$$

Now Theorem 16 yields

Corollary 16. *For each $l = 1, 2$, let one of the conditions (14) hold and $\tilde{A}$ be a bounded operator in $\mathscr{H}$. Then for any $\tilde{\mu} \in \sigma(\tilde{A})$, there is a $\mu \in \sigma(A_1 \oplus A_2)$, such that $|\mu - \tilde{\mu}| \le \delta(\hat{g}, q_+)$ $(\hat{g} = \hat{g}(A_1, A_2))$.*

Note that for small $\hat{g}$, we have asymptotically

$$\delta(\hat{g}, q_+) \approx \sqrt{2}\ q_+.$$

Moreover, replacing in Corollary 13 ξ by $\hat{g}$ and q by q_+, we get

Corollary 17. *For each $l = 1, 2$, let one of the conditions (5.3) hold and $\tilde{A}$ be a bounded operator in H. Then for any $\tilde{\mu} \in \sigma(\tilde{A})$, there is a $\mu \in \sigma(A_1 \oplus A_2)$, such that $|\mu - \tilde{\mu}| \le \hat{\delta}(\hat{g}, q_+)$ $(\hat{g} = \hat{g}(A_1, A_2))$, where*

$$\hat{\delta}(\hat{g}, q_+) := \begin{cases} q_+\sqrt{2e} & \text{if } \hat{g} \le q_+ e^{1/2}, \\ \hat{g}\sqrt{2}(\ln\,(\hat{g}\sqrt{e}/q_+))^{-1/2}\hat{g} & \text{if } \hat{g} > q_+ e^{1/2} \end{cases}.$$

11 Resolvents of Operator Pencils on Tensor Products of Spaces

11.1 Preliminaries

In this section $\mathscr{X}$ and $\mathscr{Y}$ are separable Hilbert spaces with scalar products $\langle .,.\rangle_{\mathscr{X}}$ and $\langle .,.\rangle_{\mathscr{Y}}$, respectively, and norms $\|.\|_{\mathscr{X}} = \sqrt{\langle .,.\rangle_{\mathscr{X}}}, \|.\|_{\mathscr{Y}} = \sqrt{\langle .,.\rangle_{\mathscr{Y}}}$ and $\mathscr{H} = \mathscr{X} \otimes \mathscr{Y}$ with the scalar product defined by

$$\langle x \otimes y, x_1 \otimes y_1\rangle_{\mathscr{H}} = \langle y, y_1\rangle_{\mathscr{Y}} \ \langle x, x_1\rangle_{\mathscr{X}} \quad (y, y_1 \in \mathscr{Y}; x, x_1 \in \mathscr{X})$$

and the norm $\|.\|_{\mathscr{H}} = \sqrt{\langle .,.\rangle_{\mathscr{H}}}$. Let $B_k \quad (k = 0, \ldots, m-1, m < \infty)$ be bounded operators acting in $\mathscr{Y}$, $B_m = I_{\mathscr{Y}} = I$ the unit operator. In addition, S is a positive-defined Self-adjoint operator acting in $\mathscr{X}$. Our main object in this section is the operator

$$B(S) = \sum_{k=0}^{m} B_k \otimes S^k \text{ with the domain Dom } (B(S)) = \text{Dom } (S^m) \otimes \mathscr{Y}.$$

Below we present the relevant examples. Recall that the operator polynomial

$$\sum_{k=0}^{m} B_k \lambda^k \quad (\lambda \in \mathbf{C})$$

is called an operator pencil (of a scalar argument λ). Following this definition we will call $B(S)$ *an operator pencil of an operator argument* S.

Let E_s $(s \in \sigma(S))$ be the orthogonal resolution of the identity of S:

$$S = \int_{\sigma(S)} s dE_s$$

and

$$B(s) := s^m I_{\mathscr{Y}} + \sum_{k=0}^{m-1} B_k s^k \quad (s \geq 0).$$

Then it is not hard to see that

$$B(S) = \int_{\sigma(S)} B(s) \otimes dE_s,$$

where the integral for $h = x \otimes y, g = x_1 \otimes y_1$ with $x \in \text{Dom}\,(S^m), x_1 \in \mathscr{X}; y, y_1 \in \mathscr{Y}$ is defined by

$$\langle B(S)h, g\rangle_{\mathscr{H}} = \int_{\sigma(S)} \langle B(s)y, y_1\rangle_{\mathscr{Y}} d\langle E_s x, x_1\rangle_{\mathscr{X}}$$

in the Lebesgue–Stieltjes sense and is linearly extended to the whole Dom (S^m).

Put $S_0 = S \otimes I_{\mathscr{Y}}$.

Lemma 18. *Let $\lambda \in \mathbf{C}$ be a regular point of $B(s)$ for all $s \in \sigma(S)$ and*

$$\theta(\nu, \lambda) := \sup_{s \in \sigma(S)} \|s^{\nu}(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} < \infty \tag{16}$$

for a $\nu \in [0, m)$. Then λ is a regular point of $B(S)$,

$$(B(S) - \lambda I_{\mathscr{H}})^{-1} = \int_{\sigma(S)} (B(s) - \lambda I_{\mathscr{Y}})^{-1} \otimes dE_s,$$

and

$$\|S_0^{\nu}(B(S) - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \theta(\nu, \lambda).$$

Proof. Put

$$Z(\lambda) = \int_{\sigma(S)} (B(s) - \lambda I_{\mathscr{Y}})^{-1} \otimes dE_s.$$

Clearly,

$$\begin{aligned}
(B(S) - \lambda I_{\mathscr{H}})Z(\lambda) &= \int_{\sigma(S)} (B(s) - \lambda I_{\mathscr{Y}}) \otimes dE_s \int_{\sigma(S)} (B(s_1) - \lambda I_{\mathscr{Y}})^{-1} \otimes dE_{s_1} \\
&= \int_{\sigma(S)} (B(s) - \lambda I_{\mathscr{Y}})(B(s) - \lambda I_{\mathscr{Y}})^{-1} \otimes dE_s \\
&= \int_{\sigma(S)} I_{\mathscr{Y}} \otimes dE_s = I_{\mathscr{Y}} \otimes I_{\mathscr{X}} = I_{\mathscr{H}}.
\end{aligned}$$

Similarly $Z(\lambda)(B(S) - \lambda I_{\mathscr{H}}) = I_{\mathscr{H}}$. Moreover,

$$\begin{aligned}
&\langle S_0^{\nu}(B(S) - \lambda I_{\mathscr{H}})^{-1}h, S_0^{\nu}(B(S) - \lambda I_{\mathscr{H}})^{-1}h\rangle_{\mathscr{H}} \\
&\quad = \int_{\sigma(S)} \langle s^{\nu}(B(s) - \lambda I_{\mathscr{Y}})^{-1}y, s^{\nu}(B(s) - \lambda I_{\mathscr{Y}})^{-1}y\rangle_{\mathscr{Y}} d\langle E_s x, x\rangle_{\mathscr{X}}
\end{aligned}$$

for an $h = x \otimes y$, with $x \in \mathscr{X}$, $y \in \mathscr{Y}$. Hence,

$$\langle S_0^{\nu}(B(S)-\lambda I_{\mathscr{H}})^{-1}h, S_0^{\nu}(B(S)-\lambda I_{\mathscr{H}})^{-1}h\rangle_{\mathscr{H}} \le \theta^2(\nu,\lambda)\|y\|^2_{\mathscr{Y}} \int_{\sigma(S)} d(E_s x, x)_{\mathscr{X}}$$

$$= \theta^2(\nu,\lambda)\|y\|^2_{\mathscr{Y}}\|x\|^2_{\mathscr{X}}.$$

Extending linearly this inequality, we prove the lemma. Q.E.D.

For any $\lambda \in \mathbf{C}$, set $\rho(B(s),\lambda) := \inf_{z\in\sigma(B(s))}|z-\lambda|$. Now Lemma 18 implies

Corollary 18. *Assume that there is a monotonically increasing continuous function F defined on the positive half line independent of s, such that $F(0) = 0$, $F(\infty) = \infty$, and for all $s \in \sigma(S)$, the inequality*

$$\|(B(s)-\lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} \le F(1/\rho(B(s),\lambda)) \tag{17}$$

holds, provided

$$\rho(B(S),\lambda) := \inf_{s\in\sigma(S)} \rho(B(s),\lambda) > 0.$$

Then $\|(B(S)-\lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le F(1/\rho(B(S),\lambda))$.

11.2 Bounded Perturbations of Pencils

From Corollary 18 we easily have

Corollary 19. *Let $\tilde{A}$ be a linear operator in $\mathscr{H}$, such that* Dom $(\tilde{A})$ = Dom (S_0^m). *Let the conditions (17) and*

$$q_0 = \|\tilde{A} - B(S)\|_{\mathscr{H}} < \infty \tag{18}$$

hold. If, in addition,

$$q_0 F(1/\rho(B(S),\lambda)) < 1,$$

then λ is a regular point for $\tilde{A}$, and

$$\|(\tilde{A}-\lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \frac{F(1/\rho(B(S),\lambda))}{1-q_0F(1/\rho(B(S),\lambda))}.$$

Due to the previous corollary, for any $\tilde{\mu} \in \sigma(\tilde{A})$, there is some $s_0 \in \sigma(S)$, such that

$$q_0 F(1/\rho(B(s_0),\tilde{\mu})) \ge 1.$$

This means that for some $\mu(B(s_0)) \in \sigma(B(s_0))$, we have

$$q_0 F\left(\frac{1}{|\mu(B(s_0)) - \tilde{\mu}|}\right) \geq 1.$$

Since F is monotone, it follows

Theorem 17. *Let conditions (17) and (18) hold. Then for any $\tilde{\mu} \in \sigma(\tilde{A})$, there is an $s_0 \in \sigma(S)$ and a $\mu(B(s_0)) \in \sigma(B(s_0))$, such that $|\mu(B(s_0)) - \tilde{\mu}| \leq x(F, q_0)$, where $x(F, q_0)$ is the unique positive root of the equation $q_0 F(1/x) = 1$.*

According to the definition of the spectral variation, put

$$sv_{B(S)}(\tilde{A}) := \sup_{\mu \in \sigma(\tilde{A})} \inf_{\lambda \in \sigma(B(S))} |\mu - \lambda|.$$

Then the previous theorem can be reformulated as the following:

Corollary 20. *Let conditions (17) and (18) hold. Then $sv_{B(S)}(\tilde{A}) \leq x(F, q_0)$.*

Note that

$$r_{\text{low}}(B(S)) := \inf |\sigma(B(S))| = \inf_{s \in \sigma(S)} \inf |\sigma(B(s))|.$$

Now Theorem 17 implies

Corollary 21. *Under the hypothesis of Theorem 17, one has $r_{\text{low}}(\tilde{A}) \geq r_{\text{low}}(B(S)) - x(F, q_0)$ and therefore $\tilde{A}$ is invertible, provided $r_{\text{low}}(B(S)) > x(F, q_0)$.*

We will say that *operator $\tilde{A}$ is stable if* $\beta(\tilde{A}) = \inf \text{ Re } \sigma(\tilde{A}) > 0$. Note that $\beta(B(S)) = \inf_{s \in \sigma(S)} \inf \text{ Re } \sigma(B(s))$.

Corollary 22. *Under the hypothesis of the previous theorem, one has $\beta(\tilde{A}) \geq \beta(B(S)) - x(F, q_0)$, and therefore $\tilde{A}$ is stable, provided $\beta(B(S)) > x(F, q_0)$.*

11.3 Unbounded Perturbations of Pencils

We need the following simple lemma.

Lemma 19. *Let condition (16) hold and $\tilde{A}$ be a linear operator in $\mathscr{H}$ satisfying the conditions* Dom $(\tilde{A})$ = Dom (S_0^m) *and*

$$q_\nu := \|(\tilde{A} - B(S))S_0^{-\nu}\|_{\mathscr{H}} < \infty. \tag{19}$$

If, in addition, $q_\nu \theta(\nu, \lambda) < 1$, then λ is a regular point for $\tilde{A}$, and

$$\|(\tilde{A} - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \frac{\theta(0, \lambda)}{1 - q_\nu \theta(\nu, \lambda)}.$$

Proof. Since

$$(B(S)-\lambda I)^{-1}-(\tilde{A}-\lambda I)^{-1}=(B(S)-\lambda I)^{-1}(\tilde{A}-B(S))(B(S)-\lambda I)^{-1}$$
$$=(\tilde{A}-\lambda I)^{-1}(\tilde{A}-B(S))S_0^{-\nu}S_0^{\nu}(B(S)-\lambda I)^{-1}$$

it is not hard to check that λ is regular for $\tilde{A}$, provided

$$q_\nu\|S_0^\nu(B(S)-\lambda I)^{-1}\|_{\mathscr{H}}<1.$$

Besides,

$$\|(\tilde{A}-\lambda I)^{-1}\|_{\mathscr{H}}\le\frac{\|(B(S)-\lambda I)^{-1}\|_{\mathscr{H}}}{1-q_\nu\|S_0^\nu(B(S)-\lambda I)^{-1}\|_{\mathscr{H}}}.$$

Now Lemma 18 yields the required result. Q.E.D.

Corollary 23. *Let $B(S)$ be invertible and the conditions (16) with $\lambda=0$, (19), and $q_\nu\theta(\nu,0)<1$ hold. Then $\tilde{A}$ is also invertible.*

11.4 Pencils with Matrix Coefficients

In this subsection $\mathscr{Y}=\mathbf{C}^n$ and B_k $(k=0,\dots,m-1)$ are $n\times n$-matrices. We need the inequality

$$\|(B(s)-\lambda I_{\mathscr{Y}})^{-1}\|_{C^n}\le\sum_{j=0}^{n-1}\frac{(\sqrt{2}N_2(\text{Im } B(s)))^j}{\sqrt{j!}\rho^{j+1}(B(s),\lambda)}\quad(\lambda\notin\sigma(B(s))),\tag{20}$$

which is due to Lemma 5 and the inequality $g(A)\le\sqrt{2}N_2(\text{Im } A)$.

11.4.1 Bounded Perturbations and the Spectral Variation

In this subsubsection we do not assume that S is invertible, i.e., it can be $r_{\text{low}}(S)=0$.

Assume that the conditions

$$B_k=B_k^*\quad(k=1,\dots,m-1)\tag{21}$$

hold. Then $\text{Im } B(s)=\text{Im } (B_0)$ and due to (20)

$$\|(B(s)-\lambda I_{\mathscr{Y}})^{-1}\|_n\le\sum_{j=0}^{n-1}\frac{(\sqrt{2}N_2(\text{Im } B_0))^j}{\sqrt{j!}\rho^{j+1}(B(s),\lambda)}.$$

Let $\tilde{A}$ satisfy the condition (18). Then Lemma 6 implies $sv_{B(S)}(\tilde{A}) \le \tilde{r}(q_0)$, where $\tilde{r}_0(q_0)$ is the unique positive root of the equation

$$1 = q_0 \sum_{k=0}^{n-1} \frac{(\sqrt{2}N_2(\text{Im } B_0))^k}{y^{k+1}\sqrt{k!}},$$

which is equivalent to the algebraic equation

$$y^n = q_0 \sum_{k=0}^{n-1} \frac{(\sqrt{2}N_2(\text{Im } B_0))^k y^{n-k-1}}{\sqrt{k!}}. \tag{22}$$

We thus arrive at

Corollary 24. *Let $\mathscr{H} = \mathscr{X} \otimes \mathbf{C}^n$ and conditions (18) and (21) hold. Then $sv_{B(S)}(\tilde{A}) \le \tilde{r}(q_0)$, where $\tilde{r}_0(q_0)$ is the unique positive root of (22).*

Consider the algebraic equation

$$z^n = \sum_{k=0}^{n-1} a_k z^{n-k}.$$

Recall the well-known inequality for the roots $r_k, k \le n$ of that equation:

$$\max_j |r_j| \le 2 \max_{k=1,\dots,n-1} \sqrt[k+1]{|a_k|}$$

cf. [4, Corollary 1.6.2].

Substituting

$$x = \frac{y}{\sqrt{2}N_2(\text{Im } B_0)}$$

into (22) and applying the just mentioned inequality, we arrive at the inequality $\tilde{r}(q_0) \le \delta_n(q_0, B_0)$, where

$$\delta_n(q_0, B_0) = 2 \begin{cases} q_0^{1/n}(\sqrt{2}N_2(\text{Im } B_0))^{1-1/n} & \text{if } q_0 \le \sqrt{2}N_2(\text{Im } B_0), \\ q_0 & \text{if } q_0 \ge \sqrt{2}N_2(\text{Im } B_0). \end{cases}$$

So we arrive at

Corollary 25. *Under the hypothesis of Corollary 24, we have $sv_A(\tilde{A}) \le \delta_n(q_0, B_0)$.*

11.4.2 Unbounded Perturbations of Pencils with Matrix Coefficients

Let m_0 $(0 \le m_0 \le m-1)$ be *the smallest integer, such that*

$$B_k = B_k^* \ (k = m_0 + 1, \dots, m-1). \tag{23}$$

So if for all $k \ge 1$, B_k are self-adjoint, then $m_0 = 0$. If all the B_k $(k \le m-1)$ are non-self-adjoint, then $m_0 = m-1$.

In this subsubsection it is assumed that $\inf_{s\in\sigma(S)} |s| > 0$. Take $\nu \ge m_0$. Note that the case $\nu = m_0 = 0$ is considered in the previous subsubsection. Put

$$\rho_\nu(\lambda) := \inf_{s\in\sigma(S),k=1,\dots,n} s^{-\nu}|\lambda - \lambda_k(B(s))|.$$

Due to (20) we have

$$\|s^\nu(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_n = \|(s^{-\nu}B(s) - s^{-\nu}\lambda I_{\mathscr{Y}})^{-1}\|_n \le \sum_{j=0}^{n-1} \frac{(s^{-\nu}\sqrt{2}N_2(\text{Im}\, B(s)))^j}{\sqrt{j!}\rho_\nu^{j+1}(\lambda)},$$

provided $\rho_\nu(\lambda) > 0$. But

$$\text{Im}\, B(s) = \sum_{k=0}^{m_0} s^k \text{Im}\, B_k$$

and therefore,

$$s^{-\nu}\sqrt{2}N_2(\text{Im}\, B(s)) \le \chi_\nu \quad (s \in \sigma(S)),$$

where

$$\chi_\nu := \sqrt{2}\sum_{k=0}^{m_0} N_2(\text{Im}\, B_k) r_{\text{low}}^{k-\nu}(S) \quad (\nu \ge m_0).$$

Consequently,

$$\|s^\nu(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_n \le \eta_n(\nu, \lambda),$$

where

$$\eta_n(\nu, \lambda) := \sum_{j=0}^{n-1} \frac{\chi_\nu^j}{\sqrt{j!}\rho_\nu^{j+1}(\lambda)}.$$

Now Lemma 19 implies

Theorem 18. *Let $\mathscr{H} = \mathbf{C}^n \otimes \mathscr{Y}$. For a $\lambda \in \mathbf{C}, \lambda \neq \infty$, let the conditions (23) and $\rho_\nu(\lambda) > 0$ with $\nu \geq m_0$ be fulfilled. Then λ is a regular point of $B(S)$ and*

$$\|S_0^\nu (B(S) - \lambda I)^{-1}\|_{\mathscr{H}} \leq \eta_n(\nu, \lambda).$$

This theorem and Lemma 18 yield

Corollary 26. *Let $q_\nu = \|(\tilde{A} - B(S))S_0^{-\nu}\|_{\mathscr{H}} < \infty$ and $q_\nu \eta_n(\nu, \lambda) < 1$. Then λ is a regular point for $\tilde{A}$, and*

$$\|(\tilde{A} - I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \frac{\eta_n(0, \lambda)}{1 - q_\nu \eta_n(\nu, \lambda)}.$$

11.5 Pencils with Hilbert–Schmidt Hermitian Components

If

$$\text{Im } B(s) \in \text{SN}_2 \quad (s \geq 0), \tag{24}$$

then due to Theorem 4, we have

$$\|(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} \leq \frac{1}{\rho(B(s), \lambda)} \exp\left[\frac{1}{2} + \frac{(\sqrt{2} N_2(\text{Im } B(s)))^2}{2\rho^2(B(s), \lambda)}\right] \quad (\lambda \notin \sigma(B(s))). \tag{25}$$

First consider bounded perturbations. Besides, it can be $r_{\text{low}}(S) = 0$. Again assume that condition (21) holds. Then $\text{Im } B(s)) = \text{Im } (B_0)$ and due to (25) we have

$$\|(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} \leq \frac{1}{\rho(B(s), \lambda)} \exp\left[\frac{1}{2} + \frac{(\sqrt{2} N_2(\text{Im } B_0))^2}{2\rho^2(B(s), \lambda)}\right] \quad (\text{Im } B_0 \in \text{SN}_2).$$

Let $\tilde{A}$ satisfy condition (18) and $\text{Im } (B_0) \in \text{SN}_2$. Then Corollary 20 implies $sv_{B(S)}(\tilde{A}) \leq r_2(q_0)$, where $r_2(q_0)$ is the unique positive root of the equation

$$1 = \frac{q_0}{x} \exp\left[\frac{1}{2} + \frac{(\sqrt{2} N_2(\text{Im } B_0))^2}{2x^2}\right]. \tag{26}$$

Substituting

$$x = \frac{y}{\sqrt{2} N_2(\text{Im } B_0)}$$

into (26) and applying Lemma 8, we obtain $r_2(q_0) \le \delta_2(q_0)$, where

$$\delta_2(q_0) := \begin{cases} q_0 e & \text{if } \sqrt{2}N_2(\text{Im } B_0) \le q_0 e, \\ \sqrt{2}N_2(\text{Im } B_0)[\ln (\sqrt{2}N_2(\text{Im } B_0)/q_0)]^{-1/2} & \text{if } \sqrt{2}N_2(\text{Im } B_0) \ge q_0 e. \end{cases}$$

We thus arrive at

Corollary 27. *Let the conditions (18), (21), and* Im $(B_0) \in \text{SN}_2$ *hold. Then* $sv_{B(S)}(\tilde{A}) \le r_2(q_0) \le \delta_2(q_0)$.

Consider now unbounded perturbations. Let the conditions (23) and $\rho_\nu(\lambda) > 0$ hold for a $\nu \ge m_0$. Due to (25) we have

$$\|s^\nu(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} = \|(s^{-\nu}B(s) - s^{-\nu}\lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}}$$
$$\le \frac{1}{\rho_\nu(\lambda)} \exp\left[\frac{1}{2} + \frac{(\sqrt{2}N_2(s^{-\nu}B(s)))^2}{2\rho_\nu^2(\lambda)}\right].$$

But Im $B(s) = \text{Im } B_0 + s\text{Im } B_1 + \ldots + s^{m_0}\text{Im } B_{m_0}$ and therefore,

$$s^{-\nu}\sqrt{2}N_2(\text{Im } B(s)) \le \chi_\nu \quad (s \in \sigma(S)).$$

Recall that

$$\chi_\nu := \sqrt{2}\sum_{k=0}^{m_0} N_2(\text{Im } B_k) r_{\text{low}}^{k-\nu}(S) \quad (\nu \ge m_0).$$

Consequently,

$$\|s^\nu(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} \le \eta(\text{SN}_2, \nu, \lambda),$$

where

$$\eta(\text{SN}_2, \nu, \lambda) := \frac{1}{\rho_\nu(\lambda)} \exp\left[\frac{1}{2} + \frac{\chi_\nu^2}{2\rho_\nu^2(\lambda)}\right].$$

Now Lemma 18 implies

Theorem 19. *For a* $\lambda \in \mathbf{C}, \lambda \ne \infty$, *let the conditions (23),* Im $(B_k) \in \text{SN}_2$ $(k = 0, \ldots, m_0)$, *and* $\rho_\nu(\lambda) > 0$ *with* $\nu \ge m_0$ *be fulfilled. Then* λ *is a regular point of* $B(S)$ *and*

$$\|S_0^\nu(B(S) - \lambda I)^{-1}\|_{\mathscr{H}} \le \eta(\text{SN}_2, \nu, \lambda).$$

This theorem yields

Corollary 28. *Let $B(S)$ satisfy the hypothesis of the previous theorem and condition (19) be fulfilled. In addition, let $q_\nu \eta(\mathrm{SN}_2, \nu, \lambda) < 1$. Then λ is a regular point for $\tilde{A}$, and*

$$\|(\tilde{A} - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \frac{\eta(\mathrm{SN}_2, 0, \lambda)}{1 - q_\nu \eta(\mathrm{SN}_2, \nu, \lambda)}.$$

11.6 Pencils with Schatten–von Neumann Hermitian Components

Assume that

$$\mathrm{Im}\, B(s) \in \mathrm{SN}_{2p} \quad (s \geq 0) \tag{27}$$

for an integer $p \geq 2$. Recall that

$$\beta_p := 2\left(1 + \frac{2p}{\exp(2/3)\ \ln 2}\right).$$

According to Theorem 4, for any regular λ of $B(s)$,

$$\|(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} \leq \sum_{j=0}^{p-1} \frac{(\beta_p N_{2p}(\mathrm{Im}\, B(s))^j}{\rho^{j+1}(B(s), \lambda)} \exp\left[\frac{1}{2} + \frac{(\beta_p N_{2p}(\mathrm{Im}\, B(s))^{2p}}{2\rho^{2p}(B(s), \lambda)}\right]. \tag{28}$$

Let us begin with bounded perturbations. Besides, it can be $r_{\mathrm{low}}(S) = 0$. Again assume that condition (21) holds. Then $\mathrm{Im}\, B(s) = \mathrm{Im}\, (B_0)$ and with $\mathrm{Im}\, (B_0) \in \mathrm{SN}_{2p}$ due to (28), we have

$$\|(B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} \leq \sum_{j=0}^{p-1} \frac{(\beta_p N_{2p}(\mathrm{Im}\, B_0))^j}{\rho^{j+1}(B(S), \lambda)} \exp\left[\frac{1}{2} + \frac{(\beta_p N_{2p}(\mathrm{Im}\, B_0))^{2p}}{2\rho^{2p}(B(S), \lambda)}\right].$$

Let $\tilde{A}$ satisfy condition (18). Then Theorem 17 implies $sv_{B(S)}(\tilde{A}) \leq \hat{r}_{2p}(q_0)$, where $\hat{r}_{2p}(q_0)$ is the unique positive root of the equation

$$1 = q_0 \sum_{j=0}^{p-1} \frac{(\beta_p N_{2p}(\mathrm{Im}\, B_0))^j}{x^{j+1}} \exp\left[\frac{1}{2} + \frac{(\beta_p N_{2p}(\mathrm{Im}\, B_0))^{2p}}{2x^{2p}}\right]. \tag{29}$$

Substituting

$$x = \frac{y}{\beta_p N_{2p}(\mathrm{Im}\, B_0)}$$

into (29) and applying Lemma 8, we obtain $\hat{r}_{2p}(q_0) \le \hat{\delta}_{2p}(q_0)$, where

$$\hat{\delta}_{2p}(q_0) := \begin{cases} q_0 pe & \text{if } \beta_p N_{2p}(\text{Im } B_0) \le q_0 ep, \\ \frac{\beta_p N_{2p}(\text{Im } B_0)}{[\ln (\beta_p N_{2p}(\text{Im } B_0)/(pq_0))]^{1/2p}} & \text{if } \beta_p N_{2p}(\text{Im } B_0) \ge q_0 ep. \end{cases}$$

We thus arrive at

Corollary 29. *Let the conditions (18), (21), and* Im $(B_0) \in \text{SN}_{2p}$ *hold. Then*

$$sv_{B(S)}(\tilde{A}) \le \hat{r}_{2p}(q_0) \le \hat{\delta}_{2p}(q_0).$$

Furthermore, consider unbounded perturbations, assuming that the conditions (23) and $\rho_\nu(\lambda) > 0$ hold for a $\nu \ge m_0$. Clearly

$$\beta_p s^{-\nu} N_{2p}(\text{Im } B(s)) \le \chi(p, \nu, m_0) := \beta_p \sum_{k=0}^{m_0} N_{2p}(\text{Im } B_k) r_{\text{low}}^{k-\nu}(S) \quad (s \ge r_{\text{low}}(S) > 0).$$

Due to (28) we have

$$\|s^\nu (B(s) - \lambda I_{\mathscr{Y}})^{-1}\|_{\mathscr{Y}} \le \hat{\eta}(\text{SN}_{2p}, \nu, \lambda) := \sum_{j=0}^{p-1} \frac{\chi^j(p, \nu, m_0)}{\rho_\nu^{j+1}(\lambda)} \exp\left[\frac{1}{2} + \frac{\chi^{2p}(p, \nu, m_0)}{2\rho_\nu^{2p}(\lambda)}\right].$$

Now Lemma 18 implies

Theorem 20. *For a $\lambda \in \mathbf{C}, \lambda \ne \infty$, let the conditions (23),* Im $(B_k) \in \text{SN}_{2p}$ *$(k = 0, \dots, m_0)$, and $\rho_\nu(\lambda) > 0$ with $\nu \ge m_0$ be fulfilled. Then λ is a regular point of $B(S)$ and*

$$\|S_0^\nu (B(S) - \lambda I)^{-1}\|_{\mathscr{H}} \le \hat{\eta}(\text{SN}_{2p}, \nu, \lambda).$$

This theorem and Lemma 19 yield

Corollary 30. *Let B(S) satisfy the hypotheses of the previous theorem and condition (19) be fulfilled. In addition, let $q_\nu \hat{\eta}(\text{SN}_{2p}, \nu, \lambda) < 1$. Then λ is a regular point for $\tilde{A}$, and*

$$\|(\tilde{A} - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \frac{\hat{\eta}(\text{SN}_{2p}, 0, \lambda)}{1 - q_\nu \hat{\eta}(\text{SN}_{2p}, \nu, \lambda)}.$$

For more details on pencils see [5–9].

12 Differential Operators with Matrix Coefficients

In this section we apply the results of Sect. 11.4 to matrix differential operators. Besides, $\mathscr{X} = L^2(0, 1)$, $\mathscr{Y} = \mathbf{C}^n$, and $\mathscr{H} = \mathscr{X} \otimes \mathscr{Y} = L^2([0, 1], \mathbf{C}^n)$.

12.1 A Second-Order Matrix Operator Without a Damping Term

On the domain

$$\text{Dom}\ (\tilde{A}) = \{u \in \mathscr{H} : u^{''} \in \mathscr{H};\ u(0) = u(1) = 0\}$$

consider the operator

$$\tilde{A} = -\frac{d}{dx}a(x)\frac{d}{dx} + C(x) \quad (x \in (0,1)), \tag{30}$$

where $a(x)$ is a scalar positive function having a continuous derivative and $C(x)$ is a variable-bounded $n \times n$-matrix.

To apply our above results, take $S = -\frac{d}{dx}a(x)\frac{d}{dx}$ with

$$\text{Dom}\ (S) = \{u \in L^2(0,1) : u^{''} \in L^2(0,1);\ u(0) = u(1) = 0\}.$$

In addition $S_0 = S \otimes I_{C^n}$ with Dom $(S_0) =$ Dom $(\tilde{A})$ and $B(S) = S_0 + I_{\mathscr{X}} \otimes B_0$ with some constant $n \times n$-matrix B_0. Then $q_0 = \|B(S) - \tilde{A}\| = \sup_x \|C(x) - B_0\|_{C^n}$ and

$$\sigma(B(S)) = \{\lambda_k(S) + \lambda_j(B_0) : k = 1,2,\dots\ ; j = 1,\dots,n\}.$$

In the considered case, $B(s) = sI_n + B_0$. Due to Corollaries 24 and 25,

$$sv_{B(S)}(\tilde{A}) \le \tilde{r}(q_0) \le \delta_n(q_0, B_0).$$

Thus, the operator $\tilde{A}$ defined by (30) is invertible, provided either

$$r_{\text{low}}(B(S)) = \inf\{|\lambda_k(S) + \lambda_j(B_0)| : k = 1,2,\dots\ ; j = 1,\dots,n\} > \tilde{r}(q_0)$$

or $r_{\text{low}}(B(S)) > \delta_n(q_0, B_0)$. Moreover, that operator is stable, provided either

$$\beta(B(S)) = \inf\{\lambda_k(S) + \text{Re}\ \lambda_j(B_0) : k = 1,2,\dots\ ; j = 1,\dots,n\} > \tilde{r}(q_0)$$

or $\beta(B(S)) > \delta_n(q_0, B_0)$.

12.2 A Second-Order Matrix Operator with a Damping Term

On the same domain

$$\{u \in \mathscr{H} : u^{''} \in \mathscr{H};\ u(0) = u(1) = 0\},$$

consider the operator

$$\tilde{A}_1 = -\frac{d^2}{dx^2} + C_1(x)\frac{d}{dx} + C_0(x) \quad (x \in (0,1)), \tag{31}$$

where $C_1(x)$ and $C_0(x)$ are variable $n \times n$-matrices defined and smooth on [0, 1].

In this subsection we take $S = -\frac{d^2}{dx^2}$ with the same domain as in the previous subsection. We have $\sigma(S) = \{(\pi k)^2 : k = 1, 2, \dots\}$. So $r_{\text{low}}(S) = \pi^2$, and $\|S^{-1/2}\|_{\mathscr{X}} = 1/\pi$. Let $e_k(x) = \sqrt{2}\sin(\pi k x)$. Taking $\nu = 1/2$, we obtain

$$S^{-1/2}h = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\langle h, e_k\rangle_{\mathscr{X}} e_k \ \ (h \in \mathscr{X}).$$

Hence,

$$\left(\frac{d}{dx}S^{-1/2}h\right)(x) = \sqrt{2}\sum_{k=1}^{\infty}\langle h, e_k\rangle_{\mathscr{X}}\cos(\pi k x).$$

So

$$\left\langle \frac{d}{dx}S^{-1/2}h, w\right\rangle_{\mathscr{X}} = \sum_{k=1}^{\infty}\langle h, e_k\rangle_{\mathscr{X}}\langle\sqrt{2}\cos(\pi k x), w\rangle_{\mathscr{X}} \ \ (h, w \in \mathscr{X})$$

and by the Schwarz inequality,

$$\left|\left\langle \frac{d}{dx}S^{-1/2}h, w\right\rangle_{\mathscr{X}}\right|^2 \le \sum_{k=1}^{\infty}|\langle h, e_k\rangle_{\mathscr{X}}|^2\sum_{k=1}^{\infty}|\langle\sqrt{2}\cos(\pi k x), w\rangle_{\mathscr{X}}|^2 \le \|h\|_{\mathscr{X}}^2\|w\|_{\mathscr{X}}^2.$$

Consequently,

$$\left\|\frac{d}{dx}S^{-1/2}\right\|_{\mathscr{X}} = \sup\left\{\left\langle \frac{d}{dx}S^{-1/2}h, w\right\rangle_{\mathscr{X}} : h, w \in \mathscr{X}; \|h\|_{\mathscr{X}}^2 = \|w\|_{\mathscr{X}}^2 = 1\right\} \le 1,$$

and

$$\left\|\frac{d}{dx}S_0^{-1/2}\right\|_{\mathscr{H}} \le 1. \text{ Thus } \ q_{1/2} = \|(B(S) - \tilde{A}_1)S_0^{-1/2}\|_{\mathscr{H}} \le \hat{q},$$

where

$$\hat{q} := \sup_x \|(C_0(x) - B_0)S_0^{-1/2}\|_{C^n} + \sup_x \|C_1(x)\|_{C^n} \le \frac{1}{\pi}\sup_x \|C_0(x) - B_0\|_{C^n} + \sup_x \|C_1(x)\|_{C^n}.$$

In the considered case, $m_0 = 0$, $\chi_{1/2} = \frac{1}{\pi}N_2(\text{Im}\, B_0)$, $B(s) = sI_{C^n} + B_0$ and

$$\gamma_{1/2}(\lambda) = \sup_{k=1,2,\dots} \frac{\pi k}{\rho(B(\pi^2 k^2), \lambda)} = \sup_{k=1,2,\dots;j=1,\dots,n} \frac{\pi k}{|\pi^2 k^2 + \lambda_j(B_0) - \lambda|}.$$

In addition,

$$\eta_n(1/2, \lambda) := \sum_{j=0}^{n-1} \frac{\chi_{1/2}^j}{\sqrt{j!}\rho_{1/2}^{j+1}(\lambda)}.$$

Let $\lambda \notin \sigma(B(S))$ and $\hat{q}\eta_n(1/2, \lambda) < 1$. Then by Corollary 26, λ is regular for the operator $\tilde{A}_1$, defined by (31), and

$$\|(\tilde{A}_1 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \le \frac{\eta_n(0, \lambda)}{1 - \hat{q}\eta_n(1/2, \lambda)}.$$

12.3 A Higher-Order Matrix Differential Operator

On the domain

$$\text{Dom } (\tilde{A}_2) = \{u \in \mathscr{H} = L^2([0, 1], \mathbf{C}^n) : u^{(k)} \in \mathscr{H}, : \ k = 1, \dots, 2m,$$
$$u^{(2j)}(0) = u^{(2j)}(1) = 0, \ j = 0, \dots, m-1\},$$

consider the operator

$$\tilde{A}_2 = \sum_{k=0}^{m} (-1)^k C_k(x) \frac{d^{2k}}{dx^{2k}} \quad (C_m(x) \equiv I_{C^n}, \ x \in (0, 1)), \tag{32}$$

where $C_k(x)$ are matrix-valued functions defined and smooth on [0, 1]. Again take

$$B(s) = s^m I + \sum_{k=0}^{m-1} B_k s^k$$

with constant matrices B_k. Take S as in the previous subsection. As it was mentioned, $\lambda_k(S) = \pi^2 k^2$ and $k = 1, 2, \dots$. Without loss of the generality, take $m_0 = m - 1$. So for $\nu = m - 1$ we have

$$\chi_{m-1} = \sqrt{2} \sum_{k=0}^{m-1} N_2(\text{Im } B_k) \pi^{2(k+1-m)}.$$

Under the consideration,

$$q_{m-1} = \|(B(S) - \tilde{A}_2)S_0^{-m+1}\| \leq \sum_{j=0}^{m-1} \sup_x \|B_j - C_j(x)\|_n \pi^{2(j-m+1)},$$

$$\rho_{m-1}(\lambda) = \inf_{j=1,2,\ldots,k=1,\ldots,n} \frac{(\pi^2 j^2)^{m-1}}{|\lambda - \lambda_k(B(\pi^2 j^2)|},$$

and

$$\eta_n(m-1,\lambda) = \sum_{j=0}^{n-1} \frac{\chi_{m-1}^j}{\sqrt{j!}\rho_{m-1}^{j+1}(\lambda)}.$$

Let $\lambda \notin \sigma(A)$ and

$$q_{m-1}\eta_n(m-1,\lambda) < 1.$$

Then by Corollary 26, λ is regular for the operator $\tilde{A}_2$, defined by (32), and

$$\|(\tilde{A}_2 - \lambda I_{\mathscr{H}})^{-1}\|_{\mathscr{H}} \leq \frac{\eta_n(0,\lambda)}{1 - q_{m-1}\eta_n(m-1,\lambda)}.$$

13 Integro-Differential Operators

Let $\Omega = [0,1] \times [a,b]$, where $[a,b]$ is a finite or infinite real segment. Take $\mathscr{X} = L^2(0,1), \mathscr{Y} = L^2(a,b)$, and $\mathscr{H} = \mathscr{X} \otimes \mathscr{Y} = L^2(\Omega)$.

Let $K_0(x,y,s)$ be a real function defined on $[0,1] \times [a,b]^2$, such that the operator

$$u \to \int_a^b K_0(x,y,s)u(x,s)ds \quad (x \in (0,1); y \in (a,b))$$

is bounded in $\mathscr{H}$. Consider the operator $\tilde{A}$ defined on the domain

$$\text{Dom}\,(\tilde{A})$$
$$= \{u(x,y) \in L^2(\Omega) : \frac{\partial^2}{\partial x^2}u(x,y) \in L^2(\Omega) : \ u(0,y) = u(1,y) = 0, 0 < x < 1; a \leq y \leq b\}$$

by the expression

$$(\tilde{A}u)(x,y) = -\frac{\partial^2}{\partial x^2}u(x,y) + \int_a^b K_0(x,y,s)u(x,s)ds \quad (x \in (0,1); y \in (a,b)). \tag{33}$$

Take $S = -\frac{\partial^2}{\partial x^2}$ with the domain

$$\text{Dom }(S) = \{u \in L^2(0,1) : u'' \in L^2(0,1);\ u(0) = u(1) = 0\}.$$

So the operator $S_0 = S \otimes I_{\mathscr{Y}}$ is defined on Dom $(S_0) =$ Dom $(\tilde{A})$ by

$$(S_0 u)(x,y) = -\frac{\partial^2 u(x,y)}{\partial x^2}.$$

Let

$$(B_0 w)(y) = \int_a^b K_1(y,s)w(s)ds \quad (w \in \mathscr{Y};\ y \in [a,b]),$$

where $K_1(y,s)$ is a real function defined on $[a,b]^2$ and satisfying the condition

$$N_2^2(\text{Im } B_0) = \int_a^b \int_a^b (K_1(y,s) - K_1(s,y))^2 ds < \infty.$$

So

$$((I_{\mathscr{X}} \otimes B_0)u)(x,y) = \int_a^b K_1(y,s)u(x,s)ds \quad (u \in \mathscr{H};\ x,y \in \Omega).$$

Take $B(S) = S_0 + I_{\mathscr{X}} \otimes B_0$ and assume that $q_0 = \|I_{\mathscr{X}} \otimes B_0 - \tilde{K}\|_{\mathscr{H}} < \infty$, where $\tilde{K}$ is defined by

$$(\tilde{K}u)(x,y) = \int_a^b K_0(x,y,s)u(x,s)ds \quad (x,y \in \Omega).$$

We have $\sigma(S) = \{(\pi k)^2 : k = 1,2,\ldots\}$ and

$$\sigma(B(S)) = \{(\pi k)^2 + t : k = 1,2,\ldots\ ; t \in \sigma(B_0)\}.$$

In the considered case, $B(s) = sI + B_0$. Due to Corollary 27,

$$sv_{B(S)}(\tilde{A}) \le r_2(q_0) \le \delta_2(q_0).$$

Thus, the operator $\tilde{A}$ defined by (33) is invertible, provided either

$$r_{\text{low}}(B(S)) = \inf\{|\pi^2 k^2 + t| : k = 1,2,\ldots\ ; t \in \sigma B_0)\} > r_2(q_0)$$

or $r_{\text{low}}(B(S)) > \delta_2(q_0, B_0)$.

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Effective Conductivity and Critical Properties of a Hexagonal Array of Superconducting Cylinders

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In Honor of Constantin Carathéodory

Abstract Effective conductivity of a 2D composite corresponding to the regular hexagonal arrangement of superconducting disks is expressed in the form of a long series in the volume fraction of ideally conducting disks. According to our calculations based on various resummation techniques, both the threshold and critical index are obtained in good agreement with expected values. The critical amplitude is in the interval $(5.14, 5.24)$ that is close to the theoretical estimation 5.18. The next-order (constant) term in the high-concentration regime is calculated for the first time, and the best estimate is equal to -6.229. The final formula is derived for the effective conductivity for arbitrary volume fraction.

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1 Introduction

We consider a two-dimensional composite corresponding to the regular hexagonal lattice arrangement of ideally conducting (superconducting) cylinders of concentration x embedded into the matrix of a conducting material. The studies of the effective conductivity $\sigma(x)$ of regular composites were pioneered by Maxwell [33] and Rayleigh [40]. The results of this fundamental research remained limited to the lowest orders in x. Their work was extended in [39], resulting in rather good numerical solutions valid in much broader concentration intervals.

The effective conductivity $\sigma(x)$ is an analytic function in x. In general case of a two-phase composite, the so-called contrast parameter should be also included into consideration explicitly; see, e.g., [12]. We are interested in the case of high-contrast regular composites, when the conductivity of the inclusions is much larger than the conductivity of the host. That is, the highly conducting inclusions are replaced by the ideally conducting inclusions with infinite conductivity. In this case, the contrast parameter is equal to unity and remains implicit. The conductivity of the matrix is normalized by unity as well. From the phase interchange theorem [30], it follows that in two dimensions, the superconductivity problem is dual to the conductivity problem and the superconductivity critical index is equal to the conductivity index.

Our study is restricted to the two-dimensional case which is still interesting, both for practical [3, 9] and physical reasons [39, 44]. Composite materials often consist of a uniform background-host reinforced by a large number (high concentration) of unidirectional rod- or fiber-like inclusions with high conductivity [3].

On the other hand, two-dimensional regular hexagonal-arrayed composites [3] much closer resemble the two-dimensional random composites, than their respective 3D counterparts do [44]. The tendency to order in the two-dimensional random system of disks is a crucial feature in the theory of composites at high concentrations.

Most strikingly, it appears that the maximum volume fraction of $\frac{\pi}{\sqrt{12}} \approx 0.9069$ is attained both for the regular hexagonal array of disks and for random (irregular) 2D composites [44].

A numerical study of the 2D hexagonal case can be found in [39]. Their final formula (1)

$$\sigma(x) = 1 - \frac{2x}{\frac{0.075422x^6}{1-1.06028x^{12}} + x - 1}, \tag{1}$$

compares rather well with numerical data of [39]. Note that (1) diverges with critical exponent $s = 1$, as $x \to 0.922351$. This property on one hand makes the formula more accurate in the vicinity of a true critical point but, on the other hand, makes any comparison in the critical region meaningless. It remains rather accurate till $x = 0.85$, where the error is 0.47 %. For $x = 0.905$, the error is 52 %. Expression (1) was derived using only terms up to the 12th order in concentration. The expansion of (1) is characterized by a rather regular behavior of the coefficients,

$$\sigma^{reg}(x) = 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 \\ +2.15084x^7 + 2.30169x^8 + 2.45253x^9 + 2.60338x^{10} \\ +2.75422x^{11} + 2.90506x^{12} + O(x^{13}). \quad (2)$$

One can, in principle, collect the higher-order terms as well. However, such derivation of additional terms cannot be considered as consistent since it relies on the agreement with numerical results. It turns out, though, that (2) compares well with our results shown below; see (6). Except an immediate vicinity of the critical point, analytic-numeric approach of [3] is in a good agreement with the numerical results of [39].

In a different limit of high concentrations, Keller [29] suggested a constructive asymptotic method for regular lattices, leading to very transparent, inverse square-root formula for the square array [29]. Berlyand and Novikov [10] extended Keller's method to the hexagonal array,

$$\sigma \simeq \frac{\sqrt[4]{3}\pi^{3/2}}{\sqrt{2}} \frac{1}{\sqrt{\frac{\pi}{\sqrt{12}} - x}}. \quad (3)$$

Thus, the critical amplitude A (pre-factor) is equal to $A \approx 5.18$.

We will examine below this result for the critical amplitude from the perspective of resummation techniques suggested before for square regular arrays [21]. By analogy with square lattice [35], we expect a constant correction in the asymptotic regime,

$$\sigma \simeq \frac{\sqrt[4]{3}\pi^{3/2}}{\sqrt{2}} \frac{1}{\sqrt{\frac{\pi}{\sqrt{12}} - x}} + B, \quad (4)$$

where the correction term B cannot be found in the literature, to the best of our knowledge. It will be calculated below by different methods.

With account for such correction, the final universal formula valid for all possible concentrations from 0 to $\frac{\pi}{\sqrt{12}}$ has the form

$$\sigma(x) = a(x)\,\frac{P(x)}{Q(x)}, \quad (5)$$

where

$$a(x) = \frac{36.1415}{\sqrt{\frac{\pi}{\sqrt{12}} - x}} + 15.9909\sqrt{\frac{\pi}{\sqrt{12}} - x} - 45.685 + 2.46148x,$$

$$P(x) = (0.939152 + x)(1.38894 - 2.16685x + x^2) \\ \times(2.55367 - 0.836613x + x^2)(2.08347 + 2.12786x + x^2)$$

and

$$Q(x) = (1.01215 + x)(1.61369 - 2.31669x + x^2) \times(6.51762 - 0.173965x + x^2)(4.88614 + 3.28716x + x^2).$$

The rest of the paper is organized as follows: in Sect. 2, we describe the essentials of the long series derivation. Section 3 applies various methods to the critical point calculation and compares the obtained results. In Sect. 4, the critical index and amplitude A are calculated. In Sect. 5, the most accurate formula for all volume fractions is derived, comparing the obtained predictions to numerical data. The amplitude B is calculated. Section 6 is concerned with interpolation with Padé approximants. Section 7 returns to discussion of the ansatz for construction of the starting approximation. Section 8 gives unified approach to the square and hexagonal lattices. Section 9 considers Dirichlet summation to extract the asymptotic behavior of series coefficients. Section 10 derives the asymptotic formula by the use of the lubrication theory. Section 11 considers random composites related to the hexagonal lattice. Finally, Sect. 12 concludes with a discussion of obtained results.

2 Series for Hexagonal Array of Superconducting Cylinders

We proceed to the case of a hexagonal lattice of inclusions, where rather long expansions in concentration will be presented and analyzed systematically. The coefficients a_n in the expansion of $\sigma(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ are expressed through elliptic functions by exact formulae from [36, 37]. Below, this expansion is presented in the truncated numerical form,

$$\begin{aligned}\sigma(x) = 1 &+ 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 \\ &+2.1508443464271876x^7 + 2.301688692854377x^8 \\ &+2.452533039281566x^9 + 2.6033773857087543x^{10} \\ &+2.754221732135944x^{11} + 2.9050660785631326x^{12} \\ &+3.0674404324522926x^{13} + 3.2411917947659736`x^{14} \\ &+3.426320165504177x^{15} + 3.6228255446669055x^{16} \\ &+3.8307079322541555x^{17} + 4.049967328265928x^{18} \\ &+4.441422739726373x^{19} + 4.845994396051242x^{20} \\ &+5.264540375940583x^{21} + 5.69791875809444x^{22} \\ &+6.146987621212864x^{23} + 6.6126050439959x^{24} \\ &+7.135044602470776x^{25} + 7.700073986554016x^{26} \\ &+O(x^{27}).\end{aligned} \tag{6}$$

The first 12 coefficients of (6) and the Taylor expansions of (1) coincide. The next coefficients can be calculated by exact formulae from [36, 37]. This requires the use of the double precision and perhaps a powerful computer, not a standard laptop.

Since we are dealing with the limiting case of perfectly conducting inclusions when the conductivity of inclusions tends to infinity, the effective conductivity is also expected to tend to infinity as a power law, as the concentration x tends to the maximal value x_c for the hexagonal array,

$$\sigma(x) \simeq A(x_c - x)^{-s} + B. \tag{7}$$

The critical superconductivity index (exponent) s is believed to be $1/2$ for all lattices [10]. For the sake of exploring how consistent are various resummation techniques, we will calculate the index. The critical amplitudes A and B are unknown nonuniversal parameters to be calculated below as well.

The problem of interest can be formulated mathematically as follows: given the polynomial approximation (6) of the function $\sigma(x)$, to estimate the convergence radius x_c of the Taylor series $\sigma(x)$ and to determine critical index s and amplitudes A, B of the asymptotically equivalent approximation (7) near $x = x_c$.

When such extrapolation problem is solved, we proceed to solve an interpolation problem of matching the two asymptotic expressions for the conductivity and derive interpolation formula for all concentrations.

3 Critical Point

3.1 Padé Approximants

Probably the simplest and direct way to extrapolate is to apply the Padé approximants $P_{n,m}(x)$, which are nothing else but the ratio of the two polynomials $P_n(x)$ and $P_m(x)$ of the order n and m, respectively. The coefficients are derived directly from the coefficients of the given power series [6, 41] from the requirement of asymptotic equivalence to the given series or function $f(x)$. When there is a need to stress the last point, we simply write *PadeApproximant*$[f[x], n, m]$.

In order to estimate the position of a critical point, let us apply the diagonal Padé approximants,

$$P_{1,1}(x) = \frac{m_1 x + 1}{n_1 x + 1}, \quad P_{2,2}(x) = \frac{m_2 x^2 + m_1 x + 1}{n_2 x^2 + n_1 x + 1}, \ldots \tag{8}$$

Padé approximants locally are the best rational approximations of power series. Their poles determine singular points of the approximated functions [6, 41]. Calculations with Padé approximants are straightforward and can be performed with *Mathematica*®. They do not require any preliminary knowledge of the critical index, and we have to find the position of a simple pole. In the theory of periodic

2D composites [8, 22, 42], their application is justifiable rigorously away from the square-root singularity and from the high-contrast limit.

There is a convergence within the approximations for the critical point generated by the sequence of Padé approximants, corresponding to their order increasing:

$x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 - n.a.$, $x_5 - n.a.$, $x_6 = 0.945958$, $x_7 = 0.945929$, $x_8 = 0.947703$, $x_9 = 0.946772$, $x_{10} = 0.942378$, $x_{11} = 0.945929$, $x_{12} = 0.945959$, $x_{13} = 0.920878$.

The main body of the approximations is well off the exact value. The percentage error given by the last/best approximant in the sequence equals to 1.5413 %. If only the first row of the Padé table is studied [41], then the best estimate is equal to 0.929867, close to the estimates with the diagonal sequence.

We suggest that further increase in accuracy is limited by triviality or "flatness" of the coefficient values in six starting orders of (6). Consider another sequence of approximants, when diagonal Padé approximants are multiplied with Clausius–Mossotti-type expression,

$$P_1^t(x) = \frac{(1-x)}{(1+x)} \frac{(1+m_1 x)}{(1+n_1 x)};$$

$$P_2^t(x) = \frac{(1-x)}{(1+x)} \frac{(1+m_1 x + m_2 x^2)}{(1+n_1 x + n_2 x^2)}, \ldots \tag{9}$$

The transformation which lifts the flatness does improve convergence of the sequence of approximations for the threshold,

$x_7 = 0.94568$, $x_8 - n.a.$, $x_9 = 0.948299$, $x_{10} = 0.9287$, $x_{11} = 0.945681$, $x_{12} = 0.89793$, $x_{13} = 0.903517$. The percentage error given by the last approximant in the sequence equals -0.373 %.

In order to judge the quality of the latter estimate, let us try the highly recommended $D - Log$ Padé method [6], which also does not require a preliminary knowledge of the critical index value. One has to differentiate *Log* of (6), apply the diagonal Padé approximants, and define the critical point as the position of the pole nearest to the origin. The best estimate obtained this way is $x_{12} = 0.919304$, with percentage error of 1.368 %. One can also estimate the value of critical index as a residue [6] and obtain rather disappointing value of 0.73355.

3.2 Corrected Threshold

An approach based on the Padé approximants produces the expressions for the cross properties from "left to right," extending the series from the dilute regime of small x to the high-concentration regime of large x. Alternatively, one can proceed from "right to left," i.e., extending the series from the large x (close to x_c) to small x [16, 21, 49].

We will first derive an approximation to the high-concentration regime and then extrapolate to the less concentrated regime. There is an understanding that physics of a 2D high-concentration, regular, and irregular composites is related to the so-called necks, certain areas between closely spaced disks [9, 10, 29].

Assume also that the initial guess for the threshold value is available from previous Padé estimates and is equal to $x_6 = 0.945958$.

The simplest way to proceed is to look for the solution in the whole region $[0, x_c)$, in the form which extends asymptotic expression from [34], $\sigma = \alpha_1(x_c - x)^{-1/2} + \alpha_2$. This approximation works well for the square lattice of inclusions [21].

In the case of hexagonal lattice, we consider its further extension, with higher-order term in the expansion

$$\sigma = \alpha_1(x_6 - x)^{-s} + \alpha_2 + \alpha_3(x_6 - x)^s, \tag{10}$$

where index s is considered as another unknown. All unknowns can be obtained from the three starting nontrivial terms of (6), namely, $\sigma \simeq 1 + 2x + 2x^2 + 2x^3$. Thus, the parameters equal $\alpha_1 = 2.24674$, $\alpha_2 = -1.43401$, $\alpha_3 = 0.0847261$, $s = 0.832629$.

Let us assume that the true solution σ may be found in the same form but with exact yet unknown threshold X_c,

$$\Sigma = \alpha_1(X_c - x)^{-s} + \alpha_2 + \alpha_3(X_c - x)^s. \tag{11}$$

The expression (11) may be inverted and X_c expressed explicitly,

$$X_c = 2^{-1/s}\left(\frac{-\sqrt{(\alpha_2 - \Sigma)^2 - 4\alpha_1\alpha_3} - \alpha_2 + \Sigma}{\alpha_3}\right)^{1/s} + x. \tag{12}$$

Formula (12) is a formal expression for the threshold, since $\Sigma(x)$ is also unknown. We can use for Σ the series in x, so that instead of a true threshold, we have an effective threshold, $X_c(x)$, given in the form of a series in x. For the concrete series (6), the following expansion follows,

$$\begin{aligned} X_c(x) = x_6 &+ 0.0134664x^4 + 0.00883052x^5 \\ &+ 0.00647801x^6 - 0.0709217x^7 + 0.0032732x^8 \\ &+ 0.00244442x^9 + 0.00594779x^{10} + 0.00482187x^{11} \\ &+ 0.00413887x^{12} + \cdots, \end{aligned} \tag{13}$$

which should become a true threshold X_c as $x \to X_c$.

Moreover, let us apply resummation procedure to the expansion (13) using the diagonal Padé approximants. Finally, let us define the sought threshold X_c^* self-consistently from the following equations dependent on the approximant order,

$$X_c^* = P_{n,n}(X_c^*), \tag{14}$$

meaning simply that as we approach the threshold, the RHS of (14) should become the threshold. Since the diagonal Padé approximants of the nth order are defined for an even number of terms $2n$, we will also have a sequence of $X_{c,n}^*$.

Solving Eq. (14), we obtain $X^*_{c,4} = 0.930222$, $X^*_{c,5} = 0.855009$, $X^*_{c,6} = 0.9483$, $X^*_{c,7} = 0.932421$, $X^*_{c,8} = 0.946773$, $X^*_{c,9} = 0.941391$, $X^*_{c,10} = 0.94682$, $X^*_{c,11} = 0.932752$, $X^*_{c,12} = 0.907423$, $X^*_{c,13} = 0.903303$. The last two estimates for the threshold are good.

3.3 Threshold with Known Critical Index

Also, one can pursue a slightly different strategy, assuming that critical index is known ($s = 1/2$) and is incorporated into initial approximation. Recalculated parameters equal $\alpha_1 = 5.12249$, $\alpha_2 = -5.74972$, $\alpha_3 = 1.52472$. For the series (6), the following expansion follows,

$$\begin{aligned} X_c(x) = x_6 - 0.082561x^3 + 0.0282108x^4 - 0.000383173x^5 \\ +0.0228241x^6 - 0.0649593x^7 + 0.01561635x^8 \\ -0.00911151x^9 + 0.01874715x^{10} + 0.00688507x^{11} \\ +0.0169516x^{12} + \cdots . \end{aligned} \tag{15}$$

Let us apply resummation procedure to the expansion (15) using super-exponential approximants $E^*(x)$ [48]. Finally, let us define the sought threshold X^*_c self-consistently,

$$X^*_c = 0.945958 - 0.082561x^3 E^*(X^*_c). \tag{16}$$

Since the super-exponential approximants are defined as E^*_k for arbitrary number of terms k, we will also have a sequence of $X^*_{c,k}$. For example,

$$\begin{aligned} E^*_1 &= e^{-0.341697x}, \\ E^*_2 &= e^{-0.341697e^{0.157266x}x}, \\ E^*_3 &= e^{-0.341697e^{0.157266e^{5.28382x}x}x}, \ldots , \end{aligned} \tag{17}$$

and so on iteratively. Solving Eq. (16), we obtain $X^*_{c,1} = 0.901505$, $X^*_{c,2} = 0.903321$, $X^*_{c,3} = 0.945958$, $X^*_{c,4} = 0.903404$, $X^*_{c,5} = 0.916641$, $X^*_{c,6} = 0.903412$, $X^*_{c,7} = 0.903556$, $X^*_{c,8} = 0.903412$, $X^*_{c,9} = 0.903412$.

There is a convergence in the sequence of approximations for the threshold. The percentage error achieved for the last point is equal to $-0.384537\,\%$.

The method of corrected threshold produces good results based only on the starting 12 terms from the expansion (6), in contrast with the Padé-based approximations, requiring all available terms to gain similar accuracy. The task of extracting the threshold, a purely geometrical quantity, from the solution of the physical

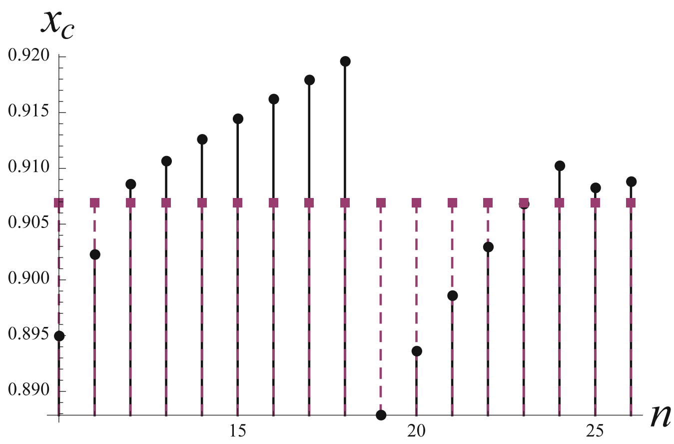

Fig. 1 x_c calculated by ratio method, compared with the exact threshold

problem is not trivial and is relevant to similar attempts to find the threshold for random systems from the expressions for some physical quantities [44].

Instead of the super-exponential approximants, one can exactly as above apply the diagonal Padé approximants,

$$X^*_{c,n} = P_{n,n}(X^*_c). \tag{18}$$

Solving Eq. (18), we obtain $X^*_{c,3} = 0.908188$, $X^*_{c,4} = 0.889169$, $X^*_{c,5} = 0.889391$, $X^*_{c,6} = 0.887983$, $X^*_{c,7} = 0.899495$, $X^*_{c,11} = 0.903011$, $X^*_{c,12} = 0.90296$, $X^*_{c,13} = 0.9057$. The last value is closest to the exact result.

Ratio method [6] also works well. It evaluates the threshold through the value of index and ratio of the series coefficients, $x_{c,n} = \frac{\frac{s-1}{n}+1}{\frac{a_n}{a_{n-1}}}$. The last point gives rather good estimate, $x_{c,26} = 0.908801$, despite of the oscillations in the dependence on n, as seen in Fig. 1.

4 Critical Index and Amplitude

The standard way to proceed with critical index calculations when the value of the threshold is known can be found in [6, 18]. One would first apply the following transformation:

$$z = \frac{x}{x_c - x} \Leftrightarrow x = \frac{zx_c}{z+1}, \tag{19}$$

to the series (6) in order to make application of the different approximants more convenient.

Then, to such transformed series $M_1(z)$, apply the $D{-}Log$ transformation and call the transformed series $M(z)$. In terms of $M(z)$, one can readily obtain the sequence of approximations s_n for the critical index s,

$$s_n = \lim_{z\to\infty}(zPadeApproximant[M[z], n, n+1]). \tag{20}$$

Unfortunately, in the case of (6), this approach fails. There is no discernible convergence at all within the sequence of s_n. Also, even the best result $s_{12} = 0.573035$ is far off the expected 0.5. Failure of the standard approach underscores the need for new methods.

4.1 Critical Index with $D - Log$ Corrections

Let us look for a possibility of improving the estimate for the index along the same lines as were already employed in the case of a square lattice of inclusions [21], by starting to find a suitable starting approximation for the conductivity and critical index.

Mind that one can derive the expressions for conductivity from "left to right," i.e., extending the series from small x to large x. Alternatively, one can proceed from "right to left," i.e., extending the series from large x (close to x_c) to small x [16, 21, 49]. Let us start with defining reasonable "right-to-left" zero approximation, which extends the form used in [21, 34] for the square arrays.

The simplest way to proceed is to look for the solution in the whole region $[0, x_c)$. As the formal extension of the expansion,

$$\sigma^{r-l} = \alpha_1(x_c - x)^{-s} + \alpha_2 + \alpha_3(x_c - x)^s + \alpha_4(x_c - x)^{2s}, \tag{21}$$

All parameters in (21) will be obtained by matching it asymptotically with the truncated series $\sigma_4 = 1 + 2x + 2x^2 + 2x^3 + 2x^4$, with the following result,

$$\sigma_4^{r-l}(x) = \frac{4.69346}{(0.9069 - x)^{0.520766}} - 5.86967$$
$$+2.53246(0.9069 - x)^{0.520766} - 0.526588(0.9069 - x)^{1.04153}. \tag{22}$$

We present below a concrete scheme for calculating both critical index and amplitude, based on the idea of corrected approximants [17]. We will attempt to correct the value of $s_0 = 0.520766$ for the critical index by applying $D - Log$ Padé approximation to the remainder of series (6).

Let us divide the original series (6) by $\sigma_4^{r-l}(x)$ given by (22), apply to the newly found series transformation (19), then apply $D - Log$ transformation, and call the transformed series $K(z)$. Finally, one can obtain the following sequence of the Padé approximations for the corrected critical index,

$$s_n = s_0 + \lim_{z\to\infty}(zPadeApproximant[K[z], n, n+1]). \tag{23}$$

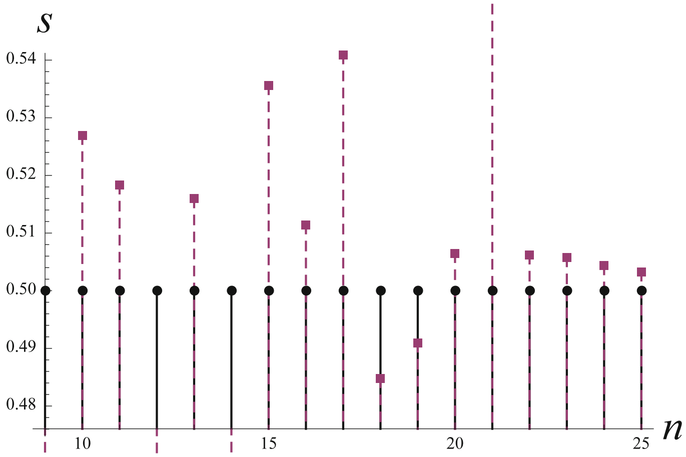

Fig. 2 Critical index s is calculated by $D - Log$ correction method and compared with the exact value

The following "corrected" sequence of approximate values for the critical index can be calculated readily: $s_4 = 0.522573$, $s_5 = 0.518608$, $s_6 = 0.554342$, $s_7 = 0.281015$, $s_8 = -0.209639$, $s_9 = 0.279669$, $s_{10} = 0.527055$, $s_{11} = 0.518543$, and $s_{12} = 0.488502$. The last two estimates surround the correct value.

Generally, one would expect that with adding more terms to the expansion (6), quality of estimates for s would improve. As was briefly discussed above, formula (1) can be expanded in arbitrary order in x, generating more terms in expansion (2). Such procedure, of course, is not a rigorous derivation of true expansion but can be used for illustration of the convergent behavior of s_n with largerv n (Fig. 2).

If $\gamma_n(z) = PadeApproximant[K[z], n, n+1]$, then

$$\sigma_n^*(x) = \sigma_4^{r-l}(x) \exp\left(\int_0^{\frac{x}{x_c - x}} \gamma_n(z)\, dz\right), \tag{24}$$

and one can compute numerically corresponding amplitude,

$$A_n = \lim_{x \to x_c} (x_c - x)^{s_n} \sigma_n^*(x), \tag{25}$$

with $A_0 = 4.693$. Expressions of the type (24) have more general form than suggested before in [14, 18, 20], based on renormalization methods.

Convergence for the index above is expected to be supplemented by convergence in the sequence of approximate values for critical amplitude, but results are still a bit scattered to conclude about the amplitude value. For the last two approximations, we find $A_{11} = 4.80599$, $A_{12} = 5.38288$, signaling possibility of a larger value than

4.82, originating from multiplication of the critical amplitude for the square lattice by $\sqrt{3}$, as suggested by O'Brien [39].

To improve the estimates for amplitude A, assume that the value of critical index $s = 1/2$ is given and construct $\gamma_n(z)$ to satisfy the correct value at infinity. There is now a good convergence for the amplitude, i.e., in the highest orders, $A_{10} = 5.09584$, $A_{11} = 5.1329$, $A_{12} = 5.14063$. Corresponding expression for the approximant

$$\gamma_{12}(z) = \frac{b_1(z)}{b_2(z)}, \tag{26}$$

where

$$\begin{aligned} b_1(z) = &-0.079533z^4 - 0.745717z^5 - 2.5712z^6 \\ &-4.16091z^7 - 2.88816z^8 + 0.36028z^9 \\ &+1.74741z^{10} + 0.951728z^{11} - 0.0792987z^{12}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} b_2(z) = &\ 1 + 14.3691z + 94.745z^2 + 380.2z^3 \\ &+1037.51z^4 + 2036.14z^5 + 2961.45z^6 \\ &+3238.1z^7 + 2667.9z^8 + 1641.88z^9 \\ &+739.461z^{10} + 235.321z^{11} + 48.8016z^{12} \\ &+3.81868z^{13}. \end{aligned} \tag{28}$$

Corresponding effective conductivity can be obtained numerically,

$$\sigma_{12}^*(x) = \sigma_4^{r-l}(x) \exp\left(\int_0^{\frac{x}{x_c - x}} \gamma_{12}(z)\, dz\right), \tag{29}$$

The maximum error is at $x = 0.905$ and equals 0.4637 %. It turns out that formula (29) is good.

5 Critical Amplitude and Formula for All Concentrations

For practical applications, we suggest below the particular resummation schemes, leading to the analytical expressions for the effective conductivity.

5.1 Correction with Padé Approximants

Let us ensure the correct critical index already in the starting approximation for σ^{r-l}, so that all parameters in (30) are obtained by matching it asymptotically with the truncated series $\sigma_3 = 1 + 2x + 2x^2 + 2x^3$,

$$\sigma_3^{r-l}(x) = \frac{5.09924}{(0.9069 - x)^{1/2}} - 6.67022 \\ +3.04972(0.9069 - x)^{1/2} - 0.649078(0.9069 - x). \tag{30}$$

To extract corrections to the critical amplitude, we divide the original series (6) by (30), apply to the new series transformation (19), call the newly found series $G[z]$, and finally build a sequence of the diagonal Padé approximants, so that the amplitudes are expressed by the formula ($\alpha_1 = 5.09924$),

$$A_n = \alpha_1 \lim_{z\to\infty} (PadeApproximant[G[z], n, n]), \tag{31}$$

leading to several reasonable estimates $A_7 = 5.26575$, $A_{11} = 5.23882$, $A_{12} = 5.25781$, $A_{13} = 5.25203$. Complete expression for the effective conductivity corresponding to A_{11} can be reconstructed readily,

$$\sigma_{11}^*(x) = \sigma_3^{r-l}(x) C_{11}(x), \tag{32}$$

where $C_{11}(x) = \frac{c_1(x)}{c_2(x)}$,

$$c_1(x) = 1.15947 + 1.13125x + 1.12212x^2 \\ +1.1167x^3 + 3.8727x^4 + 0.824247x^5 \\ -2.62954x^6 + 1.19135x^7 + 1.21923x^8 \\ +1.42832x^9 + 1.0608x^{10} + 1.53443x^{11}; \tag{33}$$

and

$$c_2(x) = 1.15947 + 1.13125x + 1.12212x^2 \\ +1.1167x^3 + 3.86892x^4 + 0.849609x^5 \\ -2.58112x^6 + 1.11709x^7 + 1.18377x^8 \\ +1.36969x^9 + 1.06062x^{10} + x^{11}. \tag{34}$$

Formula (32) is practically as good as (29). Maximum error is at the point $x = 0.905$ and equals 0.563 %.

5.2 *Padé Approximants: Standard Scheme*

Our second suggestion for the conductivity formula valid for all concentrations is based on the following conventional considerations [7]. Let us calculate the critical amplitude A. To this end, let us again apply transformation (19) to the original series (6) to obtain transformed series $M_1(z)$ and then apply to $M_1(z)$ another transformation to get yet another series, $T(z) = M_1(z)^{-1/s}$, in order to get rid of the square-root behavior at infinity. In terms of $T(z)$, one can readily obtain the sequence of approximations A_n for the critical amplitude A,

$$A_n = x_c^s \lim_{z\to\infty} (zPadeApproximant[T[z], n, n+1])^{-s}; \tag{35}$$

There are only few reasonable estimates for the amplitude, $A_6 = 4.55252$, $A_{11} = 4.49882$, $A_{12} = 4.64665$, and $A_{13} = 4.68505$. The last value is the best if compared with the conjectured in [39], $A = 4.82231$.

Following the prescription, the effective conductivity can be easily reconstructed in terms of the Padé approximant (corresponding to A_{12}) and compared with the numerical results in the whole region of concentrations. The maximum error is at $x = 0.905$ and equals $-5.67482\,\%$. On the other hand, if the conjectured value A_b is enforced at infinity, through the two-point Padé approximant, the results improve, and the maximum error at the same concentration is $-3.18511\,\%$. Corresponding formula for all concentrations, which also respects 24 terms from the series $T[z]$, is given as follows:

$$\sigma_p^*(x) = \frac{1.02555}{\sqrt{0.9069 - x}} \sqrt{\frac{V_1(x)}{V_2(x)}}, \tag{36}$$

where

$$\begin{aligned} V_1(x) = & -0.927562 - 0.877939x + 0.0406992x^2 \\ & +0.0440014x^3 + 0.0414973x^4 + 0.0436199x^5 \\ & +0.319848x^6 + 0.0110109x^7 - 0.122646x^8 \\ & +0.0351069x^9 + 0.0439523x^{10} + 0.0380654x^{11} \\ & +1.01499x^{12} + x^{13} \end{aligned} \tag{37}$$

and

$$\begin{aligned} V_2(x) = & -1.07571 + 2.09854x - 2.17187x^2 \\ & +2.23064x^3 - 2.3122x^4 + 2.374x^5 \\ & -2.1397x^6 + 1.87791x^7 - 1.78516x^8 \\ & +1.86446x^9 - 1.94838x^{10} + 2.03264x^{11} \\ & -x^{12} \end{aligned} \tag{38}$$

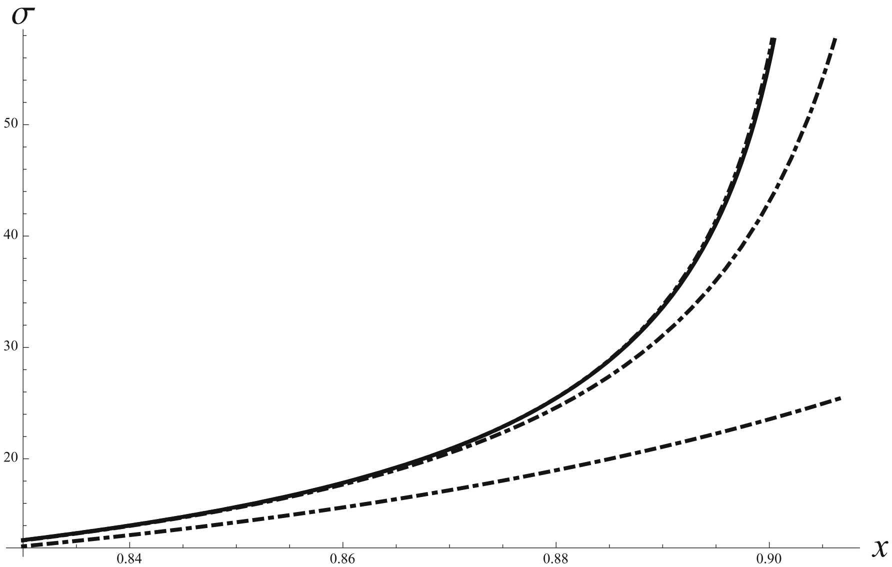

Fig. 3 Our formula (44) (*solid*) is compared with the standard Padé approximant (36) (*dotted*) and rational approximation (1) (*dashed*). The series (6) is shown with *dashed line*

Various expressions are shown in Fig. 3. Note that significant deviations of the corrected Padé formula (44) and of the standard Padé formula (36) from the reference rational expression (1) start around $x = 0.85$. All formulae start to depart from the original series around $x = 0.8$. The two formulae, (44) and (36), happen to be very close to each other almost everywhere, except in the immediate vicinity of the critical point.

5.3 Accurate Final Formula

According to our calculations, based on various resummation techniques applied to the series (6), we conclude that the critical amplitude is in the interval from 5.14 to 5.24, by 6–9 % higher than following naively to O'Brien's 4.82.

Below, we present an exceptionally accurate and more compact formula for the effective conductivity (32) valid for all concentrations.

Let us start with modified expression (30) taking into account also the O'Brien suggestion already in the starting approximation for the amplitude in σ^{r-l}. All remaining parameters in (30) are obtained by matching it asymptotically with the truncated series $\sigma_2 = 1 + 2x + 2x^2$,

$$\sigma_2^{r-l}(x) = \frac{4.82231}{(0.9069 - x)^{1/2}} - 5.79784$$
$$+2.13365(0.9069 - x)^{1/2} - 0.328432(0.9069 - x). \qquad (39)$$

Repeating the procedure developed in Sect. 5.1, we receive several reasonable estimates for the critical amplitude, $A_7 = 5.18112$, $A_{11} = 5.15534$, $A_{12} = 5.19509$, $A_{13} = 5.18766$.

Complete expression for the effective conductivity corresponding to the first estimate for the amplitude is given as follows:

$$\sigma_7^*(x) = \sigma_2^{r-l}(x) F_7(x), \tag{40}$$

and $F_7(x) = \frac{f_1(x)}{f_2(x)}$, where

$$\begin{aligned} f_1(x) = {} & 52.0141 + 10.3198x - 38.8957x^2 + 4.70555x^3 \\ & + 4.89777x^4 + 4.6887x^5 + 0.476241x^6 + 7.49464x^7, \end{aligned} \tag{41}$$

and

$$\begin{aligned} f_2(x) = {} & 52.0141 + 10.3198x - 38.8957x^2 + 2.17078x^3 \\ & + 5.80088x^4 + 6.03946x^5 + 1.80866x^6 + x^7. \end{aligned} \tag{42}$$

The formulae predict a sharp increase from $\sigma_7^*(0.906) = 166.708$ to $\sigma_7^*(0.9068) = 513.352$, in the immediate vicinity of the threshold, where other approaches [3, 9, 39] fail to produce an estimate. At the largest concentration $x = 0.9068993$ mentioned in [39], the conductivity is very large, 8375.34. This formula (40) after slight modifications can be written in the form (5).

Asymptotic expression can be extracted from the approximant (40),

$$\sigma^* \simeq \frac{5.18112}{\sqrt{0.9069 - x}} - 6.229231. \tag{43}$$

Even closer agreement with numerical results of [39] is achieved with approximant corresponding to A_{13}.

$$\sigma_{13}^*(x) = \sigma_2^{r-l}(x) F_{13}(x), \tag{44}$$

where $F_{13}(x) = \frac{f_1(x)}{f_3(x)}$,

$$\begin{aligned} f_1(x) = {} & 1.49313 + 1.30576x + 0.383574x^2 + 0.467713x^3 \\ & + 0.471121x^4 + 0.510435x^5 + 0.256682x^6 \\ & + 0.434917x^7 + 0.813868x^8 + 0.961464x^9 \\ & + 0.317194x^{10} + 0.377055x^{11} - 1.2022x^{12} - 0.931575x^{13}; \end{aligned} \tag{45}$$

and

$$\begin{aligned} f_3(x) = &\ 1.49313 + 1.30576x + 0.383574x^2 + 0.394949x^3 \\ &+0.44785x^4 + 0.503394x^5 + 0.303285x^6 \\ &+0.271498x^7 + 0.732764x^8 + 0.827239x^9 \\ &+0.25509x^{10} + 0.239752x^{11} - 1.26489x^{12} - x^{13}. \end{aligned} \tag{46}$$

It describes even more accurately than (40) the numerical data in the interval from $x = 0.85$ up to the critical point. The maximum error for the formula (44) is truly negligible, $-0.042\,\%$.

Asymptotic expression can be extracted from the approximant (44),

$$\sigma^* \simeq \frac{5.18766}{\sqrt{0.9069 - x}} - 6.2371. \tag{47}$$

5.3.1 Role of Randomness

For random two-dimensional composite, we obtained, recently [15], the following closed-form expression for the effective conductivity:

$$\begin{aligned} &\sigma^*(x) = 0.121708 f^*_{0,r}(x) \\ &\times \exp\left(\frac{(0.64454x - 1.38151)x + 0.72278}{(x-0.9069)^2\sqrt{\frac{x(x+0.435329)+0.3582}{(x-0.9069)^2}}} - 0.815613\sinh^{-1}\left(\frac{2.0171(x+0.494058)}{x-0.9069}\right)\right), \end{aligned} \tag{48}$$

with

$$f^*_{0,r}(x) = \frac{(0.419645x + 1)^{3.45214}}{\sqrt{1 - 1.10266x}}. \tag{49}$$

Closed-form expression for the effective conductivity of the regular hexagonal array of disks is given by (44). Since the two expressions are defined in the same domain of concentrations, a comparison can explicitly quantify the role of randomness (irregularity) of the composite. In order to estimate an enhancement factor due to randomness, we use the ratio of (48) to (44). In particular, the enhancement factor at $x = 0.906$ is equal to 104.593. In Fig. 4, such an enhancement factor is shown in the region of high concentrations. Enhancement factor with respect to numerical results of Ref. [39] is shown with “fat” dots.

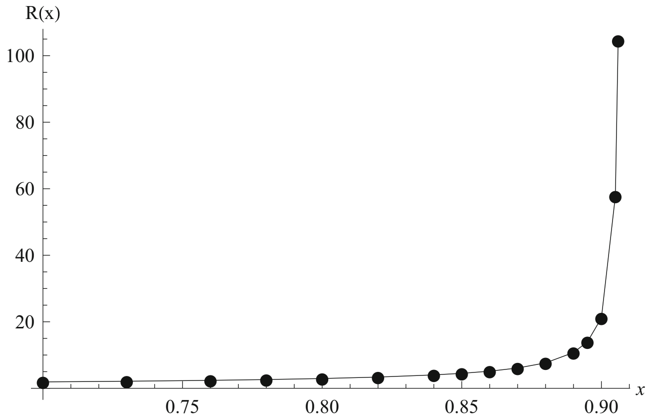

Fig. 4 Ratio $R(x) = \frac{\sigma^*(x)}{\sigma^*_{13}(x)}$ of the effective conductivity for the random composite to the effective conductivity of the hexagonal regular lattice calculated with (48) and (44), respectively

6 Interpolation with High-Concentration Padé Approximants

When two expansions (6) and (43) are available, the problem of reconstruction greatly simplifies and can be solved upfront in terms of Padé approximants.

This approach requires as an input at least two parameters from weak- and strong-coupling (high-concentration) regimes, including the value of amplitude $A = 5.18112$ from (43). Similar problem for random composites was considered in [4].

Assume that the next-order term, $B = -6.22923$ from (43), is known in advance. The high-concentration limit, in terms of z-variable (19), the strong-coupling limit, is simply

$$\sigma \simeq \frac{A}{\sqrt{x_c}}\sqrt{z} + B + O(z^{-1/2}). \tag{50}$$

The Padé approximants all conditioned to give a constant value as $z \to 0$ are given below,

$$\begin{aligned} p_{2,1}(z) &= \frac{\beta\sqrt{z}\left(1+\beta_1\frac{1}{\sqrt{z}}+\frac{\beta_2}{z}\right)}{1+\beta_3\frac{1}{\sqrt{z}}}, \\ p_{3,2}(z) &= \frac{\beta\sqrt{z}\left(1+\beta_1\frac{1}{\sqrt{z}}+\frac{\beta_2}{z}+\beta_3 z^{-3/2}\right)}{1+\beta_5\frac{1}{\sqrt{z}}+\frac{\beta_6}{z}}, \\ p_{4,3}(z) &= \frac{\beta\sqrt{z}\left(1+\beta_1\frac{1}{\sqrt{z}}+\frac{\beta_2}{z}+\beta_3 z^{-3/2}+\frac{\beta_4}{z^2}\right)}{1+\beta_5\frac{1}{\sqrt{z}}+\frac{\beta_6}{z}+\beta_7 z^{-3/2}}. \end{aligned} \tag{51}$$

The unknowns in (51) will be obtained by the asymptotic conditioning to (50) and (6). In all orders, $\beta = \frac{A}{\sqrt{x_c}}$. Explicitly, in original variables, the following expressions transpire,

$$p_{2,1}(x) = \frac{\sqrt{\frac{x}{0.9069-x}}+\frac{4.9348}{0.9069-x}-4.11284}{\sqrt{\frac{x}{0.9069-x}}+1.32856},$$

$$p_{3,2}(x) = \frac{\left(0.608173\sqrt{\frac{x}{0.9069-x}}+1.26563\right)x+0.677749\sqrt{\frac{x}{0.9069-x}}+1.13282}{-\left(0.747325\sqrt{\frac{x}{0.9069-x}}+1\right)x+0.677749\sqrt{\frac{x}{0.9069-x}}+1.13282}, \tag{52}$$

$$p_{4,3}(x) = \frac{5.4414\sqrt{z(x)}\left(1+3.76414\frac{1}{\sqrt{z(x)}}+\frac{7.73681}{z(x)}+1.97396z(x)^{-3/2}+\frac{3.76815}{z(x)^2}\right)}{1+4.90893\frac{1}{\sqrt{z(x)}}+\frac{10.7411}{z(x)}+20.504z(x)^{-3/2}}.$$

The approximants are strictly nonnegative and respect the structure of (6), e.g., for small x,

$$p_{4,3}(x) \simeq 1 + 2x + 2x^2 + O(x^3), \tag{53}$$

since all lower-order powers generated by square roots are suppressed by design. But in higher order, emerging integer powers of roots should be suppressed again and again, to make sure that only integer powers of x are present. As $x \to x_c$,

$$p_{4,3}(x) \simeq A(x_c - x)^{-1/2} + B + O((x_c - x)^{1/2}), \tag{54}$$

and only integer powers of a square root appear in higher orders. Both $p_{3,2}(x)$ and $p_{4,3}(x)$ give good estimates for the conductivity, from below and above, respectively. Their simple arithmetic average works better than each of the approximants. The bounds hold till the very core of the high-concentration regime, till $x = 0.906$.

Particularly clear form is achieved for the resistivity, an inverse of conductivity, $r(z) = (p(z))^{-1}$, e.g.,

$$r_{3,4}(z) = \frac{3.76815 + 1.97396\sqrt{z} + 0.902145z + 0.183776z^{3/2}}{3.76815 + 1.97396\sqrt{z} + 7.73681z + 3.76414z^{3/2} + z^2}. \tag{55}$$

With the variable $X = \sqrt{z}$, the resistivity problem is reduced to studying the sequence of Padé approximants $R_n = r_{n,n+1}(X)$, $n = 1, 2 \ldots l/2$, with $X \in [0, \infty)$, and analogy with the Stieltjes truncated moment problem [1, 13, 31] is complete as long as the resistivity expands at $X \to \infty$ in the Laurent polynomial with the sign-alternating coefficients, coinciding with the "Stieltjes moments" μ_k (see, e.g., [45, 46], where the original work of Stieltjes is explained very clearly).

The moments formally define corresponding Stieltjes integral as $X \to \infty$,

$$\int_0^\infty \frac{d\phi(u)}{u + X} \sim \sum_{k=0}^{l} (-1)^k \mu_k X^{-k-1} + O(X^{-l}), \tag{56}$$

l is even [13], and $\mu_k = \int_0^\infty u^k \, d\phi(u)$. Approximant $R_n(X)$ should match (56) asymptotically.

The Stieltjes moment problem can possess a unique solution or multiple solutions, dependent on the behavior of the moments, in contrast with the problem of moments for the finite interval [8, 22, 42], which is solved uniquely if the solution exists [47]. The role of variable is played by the contrast parameter, while in our case of a high-contrast composite, the variable is X.

In our setup, there are just two moments available, and resistivity is reconstructed using also a finite number of coefficients in the expansion at small X. That is, the reduced (truncated) two-point Padé approximation is considered, also tightly related to the moment problem [24, 26, 31]. In fact, even pure interpolation problem can be presented as a moment problem. We obtain here upper and lower bounds for resistivity (conductivity) in a good agreement with simulations [39].

It does seem interesting and nontrivial that the effective resistivity (conductivity) can be presented in the form of a Stieltjes integral [45–47], when the variable (19) is used.

6.1 Independent Estimation of the Amplitude B

We intend to re-calculate the amplitude B independent on previous estimates. Start with the choice of the simplest approximant as zero approximation,

$$p_{1,0}(z) = \beta\sqrt{z}\left(\frac{\frac{1}{\sqrt{z}}}{\beta} + 1\right). \tag{57}$$

$$p_{1,0}(x) = 5.4414\sqrt{\frac{x}{0.9069 - x}} + 1, \tag{58}$$

The way how we proceeded above was to look for multiplicative corrections to some plausible "zero-order" approximate solution. We can also look for additive corrections in a similar fashion. To this end, subtract (58) from (2) to get some new series $g(x)$. Change the variable $x = y^2$ to bring the series to a standard form. The diagonal Padé approximants to the series $g(y)$ are supposed to give a correction to the value of 1, suggested by (58). To calculate the correction, one has to find the value of the corresponding approximant as $y \to \sqrt{x_c}$. The following sequence of approximations for the amplitude B can be calculated now readily,

$$B_n = 1 + PadeApproximant[g(y \to \sqrt{x_c}), n, n]. \tag{59}$$

The sequence of approximations is shown in Fig. 5.

There is clear saturation of the results for larger n, and $B_{26} = -5.94966$. One can reconstruct the expression for conductivity corresponding to B_{26} in additive form

$$\sigma^*_{26}(x) = p_{1,0}(x) + F_{26}(x), \tag{60}$$

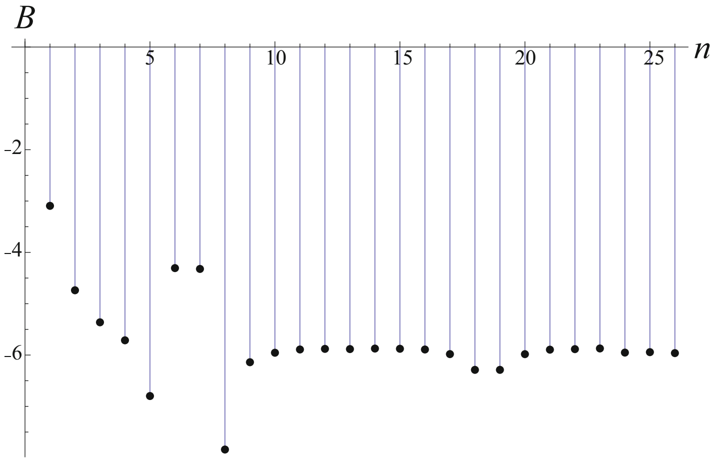

Fig. 5 Sequence of approximations B_n calculated from (59)

where $F_{26}(x) = \frac{F_2(x)}{F_6(x)}$,

$$
\begin{aligned}
F_2(x) = {} & -5.71388\sqrt{x} \\
& -1.5564x - 0.358877x^{3/2} - 2.18519x^2 + 0.0918426x^{5/2} - 1.59468x^3 \\
& -0.149418x^{7/2} - 1.47691x^4 - 0.366848x^{9/2} - 1.49733x^5 - 0.56432x^{11/2} \\
& -1.58738x^6 + 0.21344x^{13/2} - 1.31081x^7 - 0.366156x^{15/2} + 15.1037x^8 \\
& -15.1703x^{17/2} - 6.38227x^9 + 0.576004x^{19/2} - 2.5147x^{10} + 0.715526x^{21/2} \\
& -1.53752x^{11} + 0.28655x^{23/2} - 1.19851x^{12} + 5.9511x^{25/2} + 0.558011x^{13}
\end{aligned} \qquad (61)
$$

and

$$
\begin{aligned}
F_6(x) = {} & 1 + 0.622415\sqrt{x} - 0.27066x + 0.294568x^{3/2} - 0.00182913x^2 \\
& +0.0875453x^{5/2} + 0.0832152x^3 + 0.098912x^{7/2} + 0.114562x^4 + 0.116471x^{9/2} \\
& +0.133685x^5 + 0.133737x^{11/2} - 0.0205003x^6 + 0.0451001x^{13/2} \\
& +0.134146x^7 - 2.77482x^{15/2} + 1.6976x^8 + 3.12806x^{17/2} - 0.87267x^9 \\
& +0.16541x^{19/2} - 0.179645x^{10} + 0.0404152x^{21/2} + 0.000620289x^{11} \\
& +0.0514796x^{23/2} - 0.986857x^{12} - 0.58415x^{25/2} + 0.388415x^{13}.
\end{aligned} \qquad (62)
$$

The maximum error for the formula (60) is very small, 0.0824 %, only slightly inferior compared with (44). The amplitude B is firmly in the interval $(5.95, 6.22)$, according to our best two formulae.

7 Discussion of the Ansatz ((21),(30))

In the case of a square lattice of inclusions [2, 21, 27, 28, 39], we looked for the solution in a simple form,

$$\sigma_1^{r-l} = \alpha_1 (x_c - x)^{-1/2} + \alpha_2, \quad x_c = \frac{\pi}{4}, \tag{63}$$

and obtained the unknowns from the two starting terms of the corresponding series,

$$\sigma \simeq 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots \tag{64}$$

Then, $\alpha_1 = \frac{\pi^{3/2}}{2} \approx 2.784$, $\alpha_2 = (1 - \pi)$, same form as obtained asymptotically in [34], with exactly the same value for the leading amplitude as obtained in [29].

Formula (63), despite its asymptotic nature, turned out to be rather accurate in the whole region of concentrations. We try to understand below why it is so.

Let us subtract the approximant (63) from the series (64) and apply to the new series transformation (19). Then we apply to such transformed series another procedure, intended to find corrections to the values of amplitudes α_1 and α_2. Such task is nontrivial, especially when one is interested in analytical solutions. It can be solved using the general form of root approximants derived in [16, 49],

$$\sigma_{\text{add}} = b_0 z^2 \left((b_1 z + 1)^{s_1} + b_2 z^2 \right)^{s_2} \tag{65}$$

under asymptotic condition

$$\sigma_{\text{add}} \simeq d_1 \sqrt{z} + d_2, \quad \text{as } z \to \infty. \tag{66}$$

Elementary power counting gives $s_1 = 3/2$, $s_2 = -3/4$. All other unknowns can now be determined uniquely in a standard fashion from the condition of asymptotic equivalence as $z \to 0$. Final expression

$$\sigma_{\text{add}} = \frac{0.0556033 x^2}{(0.785398\ - x)^2 \left(\frac{3.69302 x^2}{(0.785398 - x)^2} + \left(\frac{1.98243 x}{0.785398 - x} + 1 \right)^{3/2} \right)^{3/4}}, \tag{67}$$

can be re-expanded in the vicinity of x_c with the result

$$\sigma_{\text{add}} \simeq \frac{0.0184973}{\sqrt{0.785398\ - x}} - 0.0118315 + O(\sqrt{x_c - x}), \tag{68}$$

indicating only small corrections to the values of amplitudes. Such asymptotic stability of all amplitudes additionally justifies the ansatz, and final corrected expression $\sigma^{\text{sq}} = \sigma_1^{r-l} + \sigma_{\text{add}}$ appears to be just slightly larger than (63). Note that modified Padé approximants as described above are only able to produce additive corrections in the form $\sigma_{\text{add}} \simeq d\sqrt{z} + O(\frac{1}{\sqrt{z}})$ as $z \to \infty$.

In the case of hexagonal lattice, such simple proposition as (63) does not appear to be stable in the sense described above. We have to try lengthier expressions of the same type, such as (30).

If we simply expand (30), we find the fifth-order coefficient $a_5 = 1.99674$, in excellent agreement with exact value of 2.

Let us consider (30) as an initial approximation for the critical index calculation and calculate corrections by $d - Log$-Pade technique. As expected, the calculated values of the corrections are small and when all terms from the expansion are utilized are equal just (−0.0028).

Additive correction in the form $\sigma_{\text{add}} = b_0 z^4 \left(b_2 z^2 + (b_1 z + 1)^{3/2}\right)^{-7/4}$, or

$$\sigma_{\text{add}}(x) = \frac{0.00220821x^4}{(0.9069 - x)^4 \left(\frac{21.8184x^2}{(0.9069-x)^2} + \left(\frac{3.48493x}{0.9069-x} + 1\right)^{3/2}\right)^{7/4}}, \tag{69}$$

leads to the very small, almost negligible asymptotic corrections to the ansatz (30). For example, the leading amplitude changes to the value of 5.09925. Such asymptotic stability of all amplitudes justifies the ansatz. Of course, it also appears to be reasonable when compared with the whole body of numerical results. The lower bound= 5.0925, and the upper bound= 5.298, can be found directly from the corresponding corrected Padé sequences.

8 Square and Hexagonal United

From the physical standpoint, there is no qualitative difference between the properties of hexagonal and square lattice arrangements of inclusions. Therefore, one might expect that a single expression exists for the effective conductivity of the two cases.

Mathematically, one is confronted with the following problem: for the functions of two variables $\sigma_{\text{sq}}(x, x_c^{\text{sq}})$ and $\sigma_{\text{hex}}(x, x_c^{\text{hex}})$, to find the transformation or relation which connects the two functions. (Here, $x_c^{\text{hex}} \equiv x_c$.)

The problem is really simplified due to similarity of the leading asymptotic terms in the dilute and highly concentrated limits. On general grounds, one can expect that up to some simply behaving "correcting" function of a properly chosen nondimensional concentration, the two functions are identical. Below, we do not solve the problem from the first principles but address it within the limits of some accurate approximate approach.

We intend to express σ_{sq} and σ_{hex} in terms of the corresponding nondimensional variables, $Z_{\text{sq}} = \frac{x}{x_c^{\text{sq}} - x}$ and $Z_{\text{hex}} = \frac{x}{x_c^{\text{hex}} - x}$, respectively. Each of the variables is in the range between 0 and ∞.

Then, we formulate a new ansatz which turns to be good both for square and hexagonal lattices,

$$\sigma^g = \alpha_0 + \alpha_1 \sqrt[4]{1 + \alpha_2 Z + \alpha_3 Z^2}. \tag{70}$$

One can obtain the unknowns from the three starting terms of the corresponding series, which happen to be identical for both lattices under investigation.

Then, the method of Subsection 5.1, when the ansatz (70) is corrected through application of the Padé approximants, is applied. Emerging diagonal Padé sequences for critical amplitudes are convergent for both lattices, and good results are simultaneously achieved in the same order, employing 22 terms from the corresponding expansions.

We select from the emerging sequences only approximants which are also holomorphic functions. Not all approximants generated by the procedure are holomorphic. The holomorphy of diagonal Padé approximants in a given domain implies their uniform convergence inside this domain (A.A. Gonchar, see [23]).

Corresponding corrective Padé approximants, $\mathrm{Cor}_{11}^{\mathrm{hex}}$, $\mathrm{Cor}_{11}^{\mathrm{sq}}$, are given below in a closed form. For the hexagonal lattice,

$$\sigma_{\mathrm{hex}}^*(Z) = \sigma^{g,\mathrm{hex}}(Z)\mathrm{Cor}_{11}^{\mathrm{hex}}(Z), \tag{71}$$

and for the square lattice,

$$\sigma_{\mathrm{sq}}^*(Z) = \sigma^{g,\mathrm{sq}}(Z)\mathrm{Cor}_{11}^{\mathrm{sq}}(Z). \tag{72}$$

The initial approximation for the hexagonal lattice,

$$\sigma^{q,\mathrm{hex}}(Z) = -6.44154 + 7.44154 \sqrt[4]{0.265686Z^2 + 0.974959Z + 1}, \tag{73}$$

and for the square lattice,

$$\sigma^{g,\mathrm{sq}}(Z) = -1.79583 + 2.79583 \sqrt[4]{1.41167Z^2 + 2.24734Z + 1}. \tag{74}$$

Correction term for the hexagonal lattice has the following form,

$$\mathrm{Cor}_{11}^{\mathrm{hex}}(Z) = \frac{\mathrm{cor}_1^{\mathrm{hex}}(Z)}{\mathrm{cor}_2^{\mathrm{hex}}(Z)}, \tag{75}$$

and for the square lattice,

$$\mathrm{Cor}_{11}^{\mathrm{sq}}(Z) = \frac{\mathrm{cor}_1^{\mathrm{sq}}(Z)}{\mathrm{cor}_2^{\mathrm{sq}}(Z)}. \tag{76}$$

Numerators and denominators of these expressions are given by polynomials,

$$\begin{aligned}\mathrm{cor}_1^{\mathrm{hex}}(Z) = 1 &+ 11.8932Z + 64.7366Z^2 + 213.169Z^3 \\ &+474.557Z^4 + 755.98Z^5 + 884.496Z^6 + 760.227Z^7 \\ &+468.277Z^8 + 196.502Z^9 + 51.5454Z^{10} + 7.29645Z^{11}, \end{aligned} \quad (77)$$

$$\begin{aligned}\mathrm{cor}_2^{\mathrm{hex}}(Z) = 1 &+ 11.8932Z + 64.7366Z^2 + 213.169Z^3 \\ &+474.565Z^4 + 756.051Z^5 + 884.793Z^6 + 760.882Z^7 \\ &+469.097Z^8 + 197.06Z^9 + 51.7143Z^{10} + 7.12936Z^{11}, \end{aligned} \quad (78)$$

$$\begin{aligned}\mathrm{cor}_1^{\mathrm{sq}}(Z) = 1 &+ 12.211Z + 66.2975Z^2 + 212.904Z^3 + 451.409Z^4 \\ &+664.782Z^5 + 693.726Z^6 + 511.717Z^7 + 259.861Z^8 \\ &+84.9746Z^9 + 15.2213Z^{10} + 0.73003Z^{11} \end{aligned} \quad (79)$$

$$\begin{aligned}\mathrm{cor}_2^{\mathrm{sq}}(Z) = 1 &+ 12.211Z + 66.2975Z^2 + 212.904Z^3 + 451.492Z^4 \\ &+665.308Z^5 + 694.974Z^6 + 513.153Z^7 + 260.53Z^8 \\ &+84.8414Z^9 + 14.9845Z^{10} + 0.706023Z^{11}. \end{aligned} \quad (80)$$

The ratio of final expressions for the conductivity of corresponding lattices, $\frac{\sigma^*_{\mathrm{hex}}(Z_{\mathrm{hex}})}{\sigma^*_{\mathrm{sq}}(Z_{\mathrm{sq}})}$, can be plotted (as $Z_{\mathrm{hex}} = Z_{\mathrm{sq}} = Z$), as shown in Fig. 6.

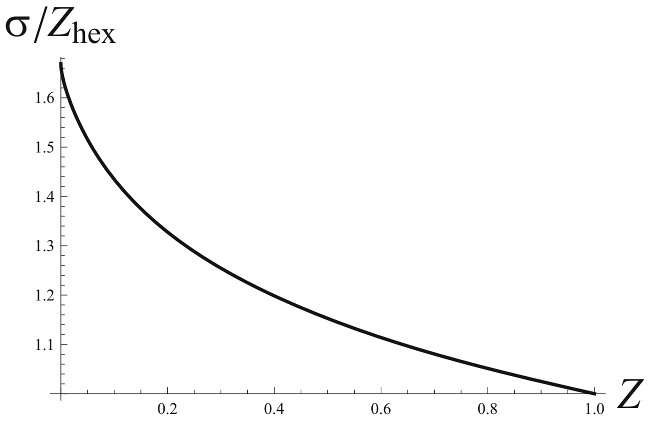

Fig. 6 The ratio of final expressions for the conductivity of corresponding lattices, $\frac{\sigma^*_{\mathrm{hex}}(Z_{\mathrm{hex}})}{\sigma^*_{\mathrm{sq}}(Z_{\mathrm{sq}})}$, can be plotted (as $Z_{\mathrm{hex}} = Z_{\mathrm{sq}} = Z$)

It turns out that the ratio is bounded function of Z and changes monotonously from 1 ($Z = 1$) to 1.7352 ($Z \to \infty$). The last number is very close to the O'Brien suggestion, that the ratio should be equal to $\sqrt{3} \approx 1.73205$ [39] and is simply $\frac{A^{\text{hex}}}{A^{\text{sq}}}\sqrt{\frac{x_c^{\text{sq}}}{x_c^{\text{hex}}}}$. Here, $A^{\text{hex}} = 5.20709$ and $A^{\text{sq}} = 2.79261$ are the critical amplitudes.

We also bring here an accurate and compact enough expression for the effective conductivity of the square lattice of inclusions, employing 10 terms from the corresponding expansion for the square lattice,

$$\sigma^*_{\text{sq}}(x) = \sigma^{\text{sq}}(x)\text{Cor}_5^{\text{sq}}(x), \tag{81}$$

where

$$\sigma^{\text{sq}}(x) = 2.79583\sqrt[4]{\frac{x(2.62925x + 3.10816) + 9.8696}{(\pi - 4x)^2}} - 1.79583, \tag{82}$$

and the correction is given as follows,

$$\begin{aligned}&\text{Cor}_5^{\text{sq}}(x)\\ &= \frac{x(x(x(x(14.1698x-2.6844)+9.16247)-12.0988)+14.5921)+32.5445}{x(x(x(x(1.x+4.34651)+9.16247)-12.0988)+14.5921)+32.5445}.\end{aligned} \tag{83}$$

It works rather well in the whole interval of concentrations, with maximum error of 0.088 %.

As $x \to x_c^{\text{sq}}$, the following expansion for the conductivity follows readily,

$$\sigma^*_{\text{sq}}(x) \simeq \frac{2.78007}{\sqrt{x - \frac{\pi}{4}}} - 1.84856 + \cdots . \tag{84}$$

The leading critical amplitude equals 2.78, and the next-order amplitude is equal to −1.849. Both amplitudes are in a good agreement with [34].

From the formula (81), one can readily obtain the higher-order coefficients, not employed in the derivation. That is, for small x, the following expansion follows:

$$\begin{aligned}\sigma^*_{\text{sq}}(x) \simeq\ &1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2.61166x^5\\ &+3.22331x^6 + 3.83497x^7 + 4.44662x^8 + 5.27206x^9 + 6.28456x^{10}\\ &+7.64531x^{11} + 9.29831x^{12} + 11.4116x^{13} + 13.971x^{14} + 17.2054x^{15}\\ &+21.169x^{16} + 26.1655x^{17} + 32.3365x^{18} + 40.094x^{19} + 49.7181x^{20}\\ &+61.7964x^{21} + 76.8317x^{22} + 95.6911x^{23} + 119.229x^{24} + 148.753x^{25}\\ &+185.673x^{26}.\end{aligned} \tag{85}$$

The computed values of the coefficients are in a fairly good agreement with the original series (6).

9 Dirichlet Summation: Large-*n* Behavior of Series Coefficients

We will try to evaluate how the coefficients of the series behave at large n. From the practical viewpoint, it is beneficial to have such information (if available) to be included into resummation procedure. The so-called Borel summation is known to render field-theoretical calculations more consistent. With a similar goal, we employ the ordinary Dirichlet's series, defined conventionally $\phi(c) = \sum_{n=1}^{\infty} a_n n^{-c}$, where a_n stands for the coefficients of the original series.

The essential difference distinguishes the general theory of Dirichlet's series from the simpler theory of power series. The region of convergence of a power series is determined by the position of the nearest singular points of the function which it represents. The circle of convergence extends up to the nearest singular point. No such simple relation holds in the general case of Dirichlet's series. When convergent in a portion of the plane, they only may represent a function regular all over the plane or in a wider region of it.

However, in an important case relevant to our study, the line of convergence necessarily contains at least one singularity. It is covered by the following theorem:

Theorem 10 [25]. If all the coefficients of the series are positive or zero, then the real point of the line of convergence is a singular point of the function represented by the series.

We conjecture, following [32], that for large n, the sum-function of coefficients, $S_n = a_1 + a_2 + \cdots + a_n$, behaves as follows:

$$S_n \simeq \delta n^{c_1} \log^{\varepsilon}(n). \tag{86}$$

Then, Dirichlet's series can be written explicitly in the form [32],

$$\phi(c) = \delta c \Gamma(\varepsilon + 1) (c - c_1)^{-\varepsilon - 1} + g(c), \tag{87}$$

where $g(c)$ stands for the regular part and δ is a parameter. This expression is valid at $c > c_1$, where the Dirichlet's series are convergent.

In order to return to the physical region of variables x and conductivity, let us apply the following transformation:

$$c(x) = \frac{x_c(x + x_c)}{x_c - x}, \tag{88}$$

with the inverse

$$x(c) = \frac{x_c c - x_c{}^2}{x_c + c}, \tag{89}$$

with $c_1 = x_c$.

The singular part of the conductivity after such transformation is expressed in the form

$$\sigma_s(x) = \frac{2^{-\varepsilon-1}\delta\Gamma(\varepsilon+1)(x+x_c)\left(\frac{xxc}{x_c-x}\right)^{-\varepsilon}}{x}, \tag{90}$$

and we should also set $\varepsilon = -1/2$. Parameter δ is simply connected with the critical amplitude A, $\delta = \frac{A}{\sqrt{2\pi x_c}}$.

Finally,

$$\sigma_s(x) = \frac{A\sqrt{\frac{xx_c}{x_c-x}}(x+x_c)}{2xx_c}. \tag{91}$$

This expression should also be regularized at small x, so that

$$\sigma_{s,r}(x) = \sigma_s(x) - \frac{A}{2\sqrt{x}}. \tag{92}$$

Close to critical point it can be expanded,

$$\sigma_{s,r}(x) \simeq \frac{A}{\sqrt{x_c-x}} - \frac{A}{2\sqrt{x_c}} - \frac{A(-x+x_c)}{4{x_c}^{3/2}} + O((x_c-x)^{3/2}). \tag{93}$$

After extracting the singular part from the series, the regular part expands for small x into the following expression (only few low-order terms are shown)

$$g(x) \simeq 1 - \frac{3A\sqrt{x}}{4x_c} + 2x - \frac{7Ax^{3/2}}{16{x_c}^2} + 2x^2 + O(x^{5/2}), \tag{94}$$

which is an expansion in $\sqrt{x}$. To this expansion, we apply the diagonal Padé approximants. The presence of fractional powers can be easily taken into account by change of variables, $x = y^2$, leading to doubling the number of approximants which can be constructed, compared with series of only integer powers.

For example, in the lowest orders, in addition to a standard polynomial ratio with integer highest power, $\frac{-0.137912x-3.89399\sqrt{x}+1}{-0.460544x+0.391414\sqrt{x}+1}$, there is another ratio $\frac{-3.81871\sqrt{x}+1}{0.4667\sqrt{x}+1}$, with fractional highest power, which can be considered as a diagonal Padé approximant too. Only the former-type polynomial ratios will be presented below, since the latter-type ratios do not bring better results in the current context.

Our goal now is to calculate the second, constant term in expansion close to x_c, denoted above as B. The correction to the constant term in the expansion emerges directly from the Padé approximant calculated at $x = x_c$,

$$B_n = -\frac{A}{2\sqrt{x_c}} + PadeApproximant[g(x \to x_c), n, n]. \tag{95}$$

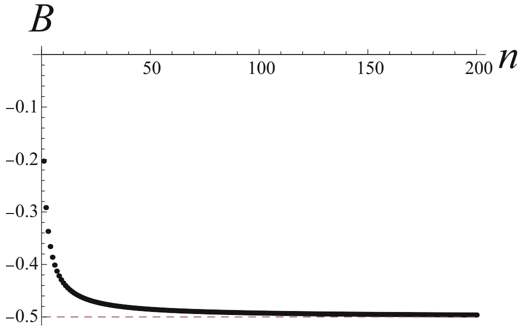

Fig. 7 The results for B_n are shown and compared with the exact value

9.1 Test

Consider some text example, when the value of B is known in advance, and the methodology described above can be tested against it.

The following test function will be considered,

$$f(x) = \frac{1}{8\left(\sqrt{1-x}+1\right) - 4\left(\sqrt{1-x}+2\right)x}. \tag{96}$$

Its low- and high-concentration expansions are similar to the corresponding expansions for conductivity. The coefficients a_i can be obtained in arbitrary order from Taylor expansion and threshold $x_c = 1$. Critical characteristics such as index $s = \frac{1}{2}$ and amplitudes $A = \frac{1}{4}$ and $B = -\frac{1}{2}$, ($B = -\frac{1}{2}$) can be recovered from the expansion in the vicinity of x_c.

The results for B_n are shown in Fig. 7 for very large n. The result for sought amplitude appears to be quite accurate, $B_{200} = -0.49628$, and the monotonous convergence of results should be noted.

9.2 Hexagonal Lattice

Following the same procedure, we obtain several reasonable estimates for the amplitude B: $B_5 = -6.40157$, $B_6 = -6.28506$, $B_7 = -6.27028$, $B_8 = -6.33762$, $B_{11} = -6.29595$, $B_{12} = -6.29695$, and $B_{13} = -6.29842$.

Explicitly in the *seven*th order,

$$\sigma_7^D = \sigma_{s,r}(x) + PadeApproximant[g[x], 7, 7]. \tag{97}$$

Corresponding expressions for the singular part of solution,

$$\sigma_{s,r}(x) = \frac{\pi\left(3\sqrt{\frac{1}{\pi-2\sqrt{3}x}}\left(2\sqrt{3}x+\pi\right) - 3\sqrt{\pi}\right)}{2\sqrt{2}3^{3/4}\sqrt{x}}, \tag{98}$$

and for the regular part $g(x) = \frac{G_1(x)}{G_2(x)}$ given by the Padé approximant, we find

$$G_1(x) = 23.7835 - 88.5524\sqrt{x} - 39.6443x + 71.3743x^{3/2}$$
$$+36.2957x^2 - 12.3254x^{5/2} - 5.54733x^3 - 1.28303x^{7/2}$$
$$-4.81208x^4 - 1.00508x^{9/2} - 4.36713x^5 - 1.3826x^{11/2}$$
$$-6.04028x^6 + 4.92363x^{13/2} + 0.518137x^7; \tag{99}$$

$$G_2(x) = 23.7835 + 13.3695\sqrt{x} - 29.9175x - 18.0154x^{3/2}$$
$$+8.21262x^2 + 6.49253x^{5/2} + 0.387922x^3 + 1.2539x^{7/2}$$
$$+0.50647x^4 + 0.689113x^{9/2} + 0.505114x^5 + 0.554584x^{11/2}$$
$$-1.31685x^6 - 0.498839x^{13/2} + x^7. \tag{100}$$

The maximum error for the formula (97) is small, just $-0.1602\,\%$.

The formulae predict the following values: $\sigma_7^D(0.906) = 166.494$, $\sigma_7^D(0.9068) = 512.7472$, and $\sigma_7^D(0.9068993) = 8376.58$. These values are very close to the predictions already presented above.

We conclude that our conjecture concerning the large-n behavior of the sum-function of the coefficients is in good agreement with available numerical data. Also, the estimates for B, which stem from the conjecture, are close to other estimates from the present paper.

Algorithms and mathematical methods used above are based on asymptotic power series for the effective conductivity and various resummation techniques to ensure their convergence. Such approach is typical for computational science of composite materials. It appears to be complementary to classic methodology based on direct solutions of PDEs.

10 Application of Lubrication Theory

To find the effective conductivity in a classic way, one has to consider the local problem for Laplace equation describing regular hexagonal lattice of cylindrical inclusions. Such a study can be based on the lubrication theory [11], applicable for an asymptotic regime of large, ideally conducting inclusions. It has to be applied in conjunction with some averaging technique to derive effective conductivity. It is expedient first to consider inclusions with finite conductivity λ and then to consider the limit $\lambda \to \infty$.

The main idea of the lubrication theory consists in replacing the original boundary problem with another, corresponding to a simpler geometry (see Fig. 8). That is, the original hexagonal elementary cell is replaced by a circle of radii b.

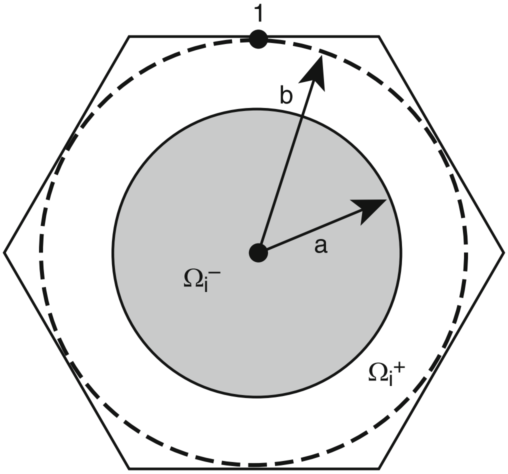

Fig. 8 The hexagonal cell with the disk of the radius a is approximated by the circle cell of the radius b

Using so-called "fast" variables (ξ, η) and the corresponding local polar coordinates (r, θ), we arrive at the following problem (for details, see [27, 28]):

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad r < a, \ a < r < b, \tag{101}$$

$$u^+ = u^-, \ \frac{\partial u^+}{\partial r} - \lambda \frac{\partial u^-}{\partial r} = (\lambda - 1)(\cos\theta + \sin\theta), \quad r = a, \tag{102}$$

$$u = 0, \quad r = b, \tag{103}$$

where a is the radius of inclusions. For definiteness, the external flux is taken in such a way that the macroscopic flow is presented by the potential $u_0 = (x_1, x_2)$ and the flux by the vector $(1, 1)$ (for details, see [27]). The problems (101)–(103) have the solution

$$u = \begin{cases} N_1 r\cos\theta + N_2 r\sin\theta, & r \le a, \\ (M_1 r + \frac{K_1}{r})\cos\theta + (M_2 r + \frac{K_2}{r})\sin\theta, & a \le r \le b, \end{cases} \tag{104}$$

where the constants are determined by the boundary conditions

$$N_1 = N_2 = \frac{(\lambda - 1)(b^2 - a^2)}{[b^2 + a^2 - \lambda(b^2 - a^2)]},$$

$$M_1 = M_2 = -\frac{(\lambda - 1)a^2}{[b^2 + a^2 - \lambda(b^2 - a^2)]},$$

$$K_1 = K_2 = -\frac{(\lambda - 1)a^2 b^2}{[b^2 + a^2 - \lambda(b^2 - a^2)]}. \tag{105}$$

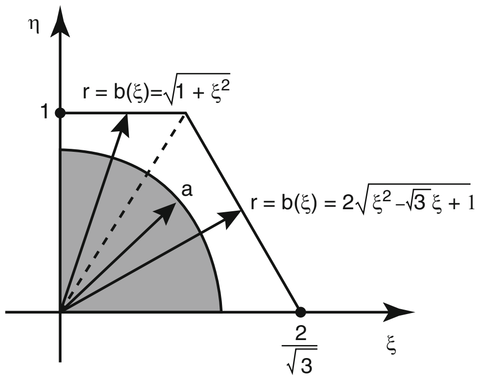

Fig. 9 Approximation of the hexagonal cell by the circle cell of the variable radius $b(\xi)$

According to the lubrication approach [2, 11], let us consider an external contour for the cell, as a circle of varying radii

$$b(\xi) = \begin{cases} 2\sqrt{\xi^2 - \sqrt{3}\xi + 1}, & 0 \le \theta < \frac{\pi}{3}, \\ \sqrt{\xi^2 + 1}, & \frac{\pi}{3} \le \theta \le \frac{\pi}{2}, \end{cases} \tag{106}$$

Integration is conducted over the quarter of the elementary cell, shown in Fig. 9. Following general prescriptions of the averaging method, we derive averaged coefficient

$$\sigma = \frac{1}{|\Omega|}\left[\int_{\Omega_i^+}\left(1 + \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) d\xi d\eta + \lambda \int_{\Omega_i^-}\left(1 + \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) d\xi d\eta\right], \tag{107}$$

where $|\Omega| = 2\sqrt{3}$. The integration is performed to satisfy also the relation (106); in particular, $b(\xi)$ is considered as a corresponding function of varying radius.

After some transformations, we receive the following expression for the effective conductivity (or thermal conductivity) as the function of the inclusion size a:

$$\sigma(a) = \frac{\left(2\sqrt{3}a^2\right)\tan^{-1}\left(\frac{\sqrt{3}}{3\sqrt{1-a^2}}\right)}{\sqrt{1-a^2}} + 1$$
$$+\frac{1}{3}\left(\sqrt{3}a^2\right)\left(\frac{\pi}{4} - \frac{3}{2}\sin^{-1}\left(\frac{\sqrt{3}}{3a}\right)\right) + \frac{4\sqrt{3}a^2}{3\sqrt{1-a^2}}$$
$$\times\left(\tan^{-1}\left(\frac{\left(\sqrt{3}a - \sqrt{3a^2-1}\right)\sqrt{1-a^2}}{a+1}\right)\right.$$

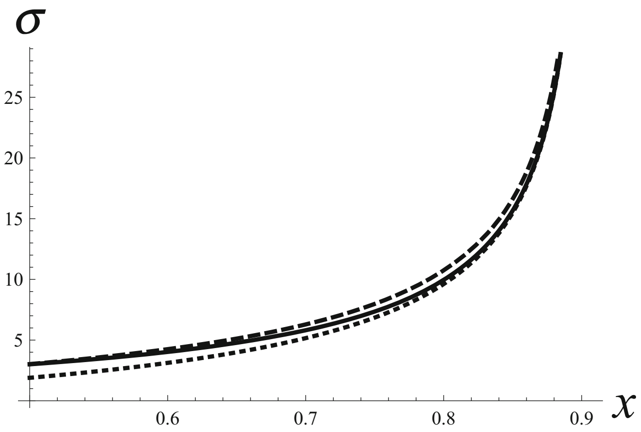

Fig. 10 σ calculated by formulae (45) (*solid line*), by (47) (*dotted line*), and by (110) (*dashed line*)

$$-\frac{1}{4}\tan^{-1}\left(\frac{2\left(-\sqrt{3a^2-1}+\sqrt{3}a-1\right)\sqrt{1-a^2}}{\sqrt{3a^2-1}\left(\sqrt{3}a+a-2\right)+\left(1+\sqrt{3}\right)a\left(1-\sqrt{3}a\right)+2}\right)$$
$$-\frac{1}{8}\tan^{-1}\left(\frac{2\sqrt{1-a^2}}{a}\right)\Bigg)-\frac{1}{4}a^2\log\left(\frac{3a^2+2\sqrt{3\left(3a^2-1\right)}+2}{4-3a^2}\right) \quad (108)$$

As the inclusion size tends to its limiting value, $a \to 1$, the leading term in the conductivity of the ideally conducting inclusions can be found in the familiar form:

$$\sigma_0 \simeq \frac{\sqrt{\frac{3}{2}}\pi}{\sqrt{1-a}}. \quad (109)$$

When expressed in terms of volume fraction of inclusions (109), it coincides with Keller's formula (3).

The first (constant) correction term to the formulae (108) can be also obtained, leading to "shifted" expression for the conductivity in the critical region,

$$\sigma_1 \simeq \sigma_0 - 5.10217. \quad (110)$$

Formula (108) works rather well, with accuracy less than 2 %, for concentrations as low as $x \simeq 0.82$. Its predictions for concentrations very close to x_c are also very near to predictions from other formulae given above (see Fig. 10). Formula (108) becomes invalid for $x \leq 0.3023$.

In the case of a square lattice of inclusions, lubrication theory gives the following asymptotic result [5]:

$$\sigma(x) \simeq \frac{\pi^{3/2}}{2\sqrt{\frac{\pi}{4} - x}} - 1. \tag{111}$$

Formula (111) should be compared with the more accurate result of [34, 35],

$$\sigma(x) \simeq \frac{\pi^{3/2}}{2\sqrt{\frac{\pi}{4} - x}} - \pi + 1. \tag{112}$$

It appears that lubrication theory assumptions, concerning reduction of the elementary cell to a circle, work better for the hexagonal lattice than for the square lattice. In both cases, the correction term is overestimated.

Classic approach to PDE's solution thus is limited to high-concentration asymptotic regime with strong interactions between inclusions.

On the other side, the whole well-developed family of self-consistent methods which include Maxwell's approach, effective medium approximations, differential schemes, etc., is valid only for a dilute composites when interactions between inclusions do not matter [38].

In contrast, computational methods of the present paper are applicable everywhere.

Let us derive an interpolation formula by matching the two limiting expressions, (6) and (110). The method of sewing the two limiting behaviors together will be chosen to employ the main idea of Sect. 9. First, we assume that the high-concentration formula (110) holds everywhere and then derive an additive correction in the form of the diagonal Padé approximants in such a way that also the low-concentration limit (6) is respected. It turns out that such approach not only generates another good interpolation formula but also calculates an additive correction to the amplitude B. Technically, one should only replace the expression (91) with (110) and extract it from the (6), leading to the new series $g(x)$ and to corresponding approximations to the sought amplitude,

$$B_n = -5.10217 + PadeApproximant[g(x \to x_c), n, n]. \tag{113}$$

We receive several reasonable estimates for the amplitude B: $B_5 = -6.37811$, $B_6 = -6.29179$, $B_7 = -6.28019$, $B_8 = -6.42952$, $B_{11} = -6.29702$, $B_{12} = -6.29908$, and $B_{13} = -6.32249$. These results are only slightly higher than estimates obtained above in Sect. 9. Interpolation formula corresponding to B_7 is as accurate as its counterpart suggested in Sect. 9.

11 Random Composite from Hexagonal Representative Cell

In the present paper, the numerical computations for random composites are performed for the hexagonal representative cell. The number of inclusions per cell can be taken arbitrarily large; hence, the shape of the cell does somewhat influence the final result.

The hexagonal lattice serves as the domain Q, where random composite is generated as a probabilistic distribution of disks of radius r (particles), by means of some Monte Carlo algorithm (protocol) [12]. Below, we outline two different protocols (algorithms) systematically described in [12].

Algorithm 1, random sequential addition (RSA). The first random point is randomly distributed in Q. The second point is randomly distributed in Q with exception of the small circular region of radius r surrounding the first point. Hence, the distribution of the second random point is conditional and depends on the first random point. More points, up to some number N, can be generated, conditioned that circular regions around all previous points are excluded from Q. This joint random variable for all points correctly determines sought probabilistic distribution. But the computer simulations work only up to concentrations as high as 0.5773, hence is the main RSA limitation. To overcome the limitation and to penetrate the region of larger concentrations, one has to apply some extrapolation technique.

Algorithm 2, random walks (RW) employed also in [15]. N-random points are generated, at first being put onto the nodes of the hexagonal array. Let each point move in a randomly chosen direction with some step. Thus each center obtains new complex coordinate. This move is repeated many times, without particles overlap. If particle does overlap with some previously generated, it remains blocked at this step. After a large number of walks, the obtained locations of the centers can be considered as a sought statistical realization, defining random composite.

RW protocol can be applied for arbitrary concentrations including those very close to x_c, which stands also for the maximum volume fraction of random composites. At $x = xc = \frac{\pi}{\sqrt{12}}$, we arrive to the regular hexagonal array of disks.

The effective conductivity of random composite is also expected to tend to infinity as a power law, as the concentration x tends to the maximal value x_c,

$$\sigma(x) \simeq A(x_c - x)^{-s}. \tag{114}$$

The superconductivity critical exponent s believed to be close to $\frac{4}{3} \approx 1.3$ [43], much different from the regular case. The critical amplitude A is an unknown nonuniversal parameter. We demonstrate below that s depends on the protocol and suggest a simple way to decrease the dependence on protocol. Still, more studies are needed with different protocols.

Algorithm 2 allows to obtain the following power series in concentration,

$$\sigma^{\mathrm{RW}} = 1 + 2x + 2x^2 + 4.23721x^3 + 6.8975x^4. \tag{115}$$

The higher-order polynomial representations fail to give a non-zero value for the fourth-order coefficient.

Reasonable estimate for the critical index s can be obtained already from the $D-Log$ formula combined with the transformation (19), (20). Namely, the result is $s_2 = 1.43811$, and for the amplitude, we obtain $A = 1.21973$.

The algorithm 1 produced the following series in concentration [15],

$$\sigma^{\mathrm{RSA}} = 1 + 2x + 2x^2 + 5.00392x^3 + 6.3495x^4. \tag{116}$$

The coefficients on x^k ($k = 5, 6, 7, 8$) vanish in (116) with the precision 10^{-10}.

A good estimate for the critical index s can be obtained already from the $D-Log$ formula (20). The results are $s_2 = 1.28522$ for the critical index, and $A = 1.57678$ for the amplitude.

Ideally, we would like to have s and A to be evaluated independent on protocol, but can hope only that combining two different protocols can decrease the dependence of s on protocols, because errors of the two protocols can compensate.

Assume that both schemes should lead to the same index, amplitude, and threshold. Let us form a simple product,

$$\sigma^J = \sqrt{\sigma^{\mathrm{RW}}\sigma^{\mathrm{RSA}}}. \tag{117}$$

11.1 D-Log Estimates

Apply now the $D - Log$ technique combined with the transformation (19), to the series (117). The result is $s = s_2 = 1.34715$, better than for each of the individual components.

Simple addition of (116) and (115) also leads to a good estimate 1.34888, by the $D - Log$ technique.

A slightly better result is achieved for the geometrical mean of the series,

$$\sigma^M = \frac{2\sigma^{\mathrm{RW}}\sigma^{\mathrm{RSA}}}{\sigma^{\mathrm{RW}} + \sigma^{\mathrm{RSA}}}, \tag{118}$$

and $s = s_2 = 1.34542$. The coefficients in the expansion for small x,

$$\sigma^M \simeq 1 + 2x + 2x^2 + 4.62056x^3 + 6.6235x^4, \tag{119}$$

are formed as a compromise between the two algorithms.

The effective conductivity can be reconstructed [18, 20, 33], from an effective critical index (or β-function). After some calculations, we obtain

$$\sigma_*^M(x) = 3.24319e^{0.441389\tan^{-1}\left(2.18756+\frac{2.43087}{x-0.9069}\right)} \times \left(\frac{0.9069}{0.9069 - x} - 0.515166\right)^{1.33609}\left(\frac{x(x+0.0245056)+0.176696}{(0.9069\ - x)^2}\right)^{0.00466513} \tag{120}$$

Also, the critical amplitude evaluates as 1.423. Equation (120) works as good as any other formula for the effective conductivity obtained in [15].

11.2 "Single Pole" Approximation

The critical index can be estimated also from a standard representation for the derivative

$$B_a(x) = \partial_x \log(\sigma^M(x)) \simeq \frac{s}{x_c - x}, \tag{121}$$

as $x \to x_c$, thus defining critical index as the residue in the corresponding single pole [6].

Outside of the immediate vicinity of the critical point, a diagonal Padé approximant is assumed for the residue estimation [6], but such approach fails in the case under study. Let us use another representation, in the form of a factor approximant [20],

$$B_a(x) = \frac{2(b_2x + 1)^{s_2}}{1 - \frac{x}{x_c}}, \tag{122}$$

with the following values for parameters $b_2 = 7.84091$, $s_2 = -0.140629$, found for the series (118).

Formula (122) leads to the simple expression for the critical index

$$s = 2x_c(b_2x_c + 1)^{s_2}, \tag{123}$$

and to the value $s = 1.35129$. The effective conductivity can be reconstructed as follows,

$$\sigma^*(x) = \exp\left(\frac{2\pi\left((b_2x+1)^{s_2+1}\,{}_2F_1\left(1, s_2+1; s_2+2; \frac{2\sqrt{3}(b_2x+1)}{\pi b_2+2\sqrt{3}}\right) - {}_2F_1\left(1, s_2+1; s_2+2; \frac{2\sqrt{3}}{\pi b_2+2\sqrt{3}}\right)\right)}{\left(\pi b_2+2\sqrt{3}\right)(s_2+1)}\right), \tag{124}$$

through the hypergeometric function F_1. The "single pole" approximation (121) is in fact equivalent to the particular case of the hypergeometric function.

For the RSA-series (116), the same approach gives $s = 1.31786$, while for the RW-series (115), $s = 1.37978$. The difference between the two algorithms is small compared to all others methods employed for the index estimations.

11.3 Corrected Index: Scheme 1

We follow below the general idea of Refs. [21, 33], also explained in Sect. 4.1. At first, one should obtain an approximate solution explicitly as a factor approximant [19, 50]. Then, we attempt to correct the form of the initial approximation with additional factor, originated from the part of series which did not participate in the formation of the initial approximation, following literally the way leading to (24).

The simplest factor approximant can be calculated,

$$f_0(x) = \frac{(x+1)^{1.04882}}{(1-1.10266x)^{0.862622}}. \tag{125}$$

Such approximant satisfies the two nontrivial starting terms from the series (119) and incorporates the accepted value of the threshold x_c. It predicts for the critical index the value $s_0 = 0.862622$.

In the next step, we attempt to correct s_0 using the $D - Log$-correction approach [21, 33], as described also in Sect. 4.1. Let us form the following ratio, $\frac{\sigma^M}{f_0(x)}$. Repeating the same steps that lead to the corrected expression for the index (23), we obtain the corrected value $s_2 = 1.32067$. The corresponding amplitude is equal to 1.48267.

The conductivity can be reconstructed in a closed form. Calculating corresponding integral with β-function[18, 20] $P_{2,3}(z)$,

$$P_{2,3}(z) = \frac{5.71085z^2}{12.4679z^3 + 10.3351z^2 + 4.31945z + 1}, \tag{126}$$

we obtain

$$\sigma_2^*(x) = 1.77719\frac{(x+1)^{1.04882}}{(1-1.10266x)^{0.862622}} e^{-0.465101\tan^{-1}\left(\frac{2.16258x+0.451103}{0.9069-x}\right)} \times \left(\frac{0.545059x+0.412586}{0.9069-x}\right)^{0.444818} \left(\frac{x^2+0.0241854x+0.18073}{(0.9069-x)^2}\right)^{0.00661298}. \tag{127}$$

11.4 Corrected Index: Scheme 2

Let us start from the initial approximation (125) and recast it more generally as

$$f_0(x) = \left(1 - \frac{x}{x_c}\right)^{-s_0} R(x), \tag{128}$$

where $R(x)$ stands for the regular part of (125). In what follows, we attempt to correct $f_0(x)$ differently than above, assuming instead of s_0 some functional dependence $S(x)$.

As $x \to x_c$, $S(x) \to s_c$, the corrected value. The function $S(x)$ will be designed in such a way that it smoothly interpolates between the initial value s_0 and the sought value s_c. The corrected functional form for the conductivity is now

$$f^*(x) = \left(1 - \frac{x}{x_c}\right)^{-S(x)} R(x). \tag{129}$$

From (129), one can express $S(x)$ but only formally since $f^*(x)$ is not known. But we can use its asymptotic form (119), express $S(x)$ as a series, and apply some resummation procedure (e.g., Padé technique). Finally, calculate the limit of the approximants as $x \to x_c$.

In what follows, the ratio $C(x) = \frac{\sigma^M(x)}{R(x)}$ stands for an asymptotic form of the singular part of the solution, and as $x \to 0$,

$$S(x) \simeq \frac{\log(C(x))}{\log\left(1 - \frac{x}{x_c}\right)}, \tag{130}$$

which can be easily expanded in powers x, around the value of s_0. It appears that one can construct a single meaningful Padé approximant,

$$S(x) = \frac{4.91072x^2 + 0.703479x + 0.862622}{3.00966x^2 + 0.815512x + 1}, \tag{131}$$

and find the corrected index, $s_c = S(x_c) = 1.31426$. Now, we also possess a complete expression for conductivity (129).

Scheme 2, due to its simplicity, can always lead to the analytical expression. But Scheme 1 seems to be the most flexible. It also turns out to be weakly dependent on the starting approximation $f_0(x)$. Indeed, if another starting approximation is considered,

$$f_0(x) = \frac{(2x+1)^{0.355391}}{(1 - 1.10266x)^{1.16919}}, \tag{132}$$

the corrected index remains good, $s_3 = 1.31094$.

The conductivity again can be reconstructed in a closed form. Calculating corresponding integral with β-function $P_{3,4}(z)$,

$$P_{3,4}(z) = \frac{9.85652z^3 + 4.06592z^2}{69.5337z^4 + 35.9326z^3 + 20.6483z^2 + 6.17673z + 1}, \tag{133}$$

we obtain a rather lengthy expression,

$$\begin{aligned}
&\sigma_3^*(x) = 1.05615\frac{(2x+1)^{0.355391}}{(1-1.10266x)^{1.16919}} \\
&\times \exp\left(0.0615685\tan^{-1}\left(2.23114 - \frac{2.10073}{0.9069-x}\right) - 0.0789727\tan^{-1}\left(4.693 - \frac{5.46758}{0.9069-x}\right)\right) \\
&\times \left(\frac{x(1-0.245701)+0.138583}{(0.9069-x)^2}\right)^{0.081481} / \left(\frac{x(x+0.415099)+0.099469}{(0.9069-x)^2}\right)^{0.0106051}.
\end{aligned} \tag{134}$$

The form of expressions (127), (134) is unlikely to be guessed as an independent approximant.

12 Conclusion

Based on estimates for the critical amplitudes A and B, we derived an accurate and relatively compact formula for the effective conductivity (5) (see also asymptotically equivalent formula (40)) valid for all concentrations, including the most interesting regime of very high concentrations. For the high-concentration limit, in addition to the amplitude value of 5.18112, we deduce also that the next-order (constant) term B equals −6.22923. It is possible to extract more coefficients in the high-concentration expansion based on the formula (70). Dirichlet summation is suggested to extract an arbitrary large-n behavior of the coefficients.

When two expansions around different points (6) and (43) are available, the problem of reconstruction can be solved in terms of high-concentration Padé approximants, implying that the effective resistivity (conductivity) can be presented in the form of a Stieltjes integral, in terms of the variable $X = \sqrt{\frac{x}{x_c - x}}$. Such Padé approximants give tight lower and upper bounds for the conductivity, valid up to the very high x.

Such properties as the superconductivity critical index and threshold for conductivity can be calculated from the series (6). In the case of truncated series, the standard Padé approximants are not able to describe the correct asymptotic behavior in the high-concentration limit, where in addition to the leading critical exponent also a nontrivial sub-leading exponent(s) plays the role [16, 49]. On the other hand, when such a nontrivial asymptotic behavior is treated separately with different type of approximants, the Padé approximants are able to account for the correction. Such patchwork approximations appear to be more accurate and powerful than approximating conventionally with a single type of approximants.

A simple functional relation between the effective conductivity of the hexagonal and square lattices is suggested, expressed in terms of some bounded monotonous function of a nondimensional concentration of inclusions. Getting an accurate formula in this case means that correct asymptotic behavior (43) indeed can be extracted from the series (6), and together they determine the behavior in the whole interval with good accuracy. Neglecting the high-concentration regime dominated by necks is not admissible.

We also considered a classic approach based on lubrication theory and concluded that it can be applied strictly within the high-concentration asymptotic regime. In contrary, the celebrated Maxwell's approach, effective medium approximations, and differential schemes are valid only for dilute composites [38]. Computational approach and results of the present paper are applicable everywhere.

We conclude that approach based on the long power series for the effective conductivity as a function of particle volume fraction can be consistently applied in the important case of highly conducting (superconducting) inclusions. Based on our investigation, we put forward the final formula (5), for the effective conductivity of the hexagonal array.

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A Survey on Durrmeyer-Type Operators

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In Honor of Constantin Carathéodory

Abstract It is well known that discretely defined operators such as Bernstein polynomials, Baskakov operators, Szász–Mirakyan, Meyer-König–Zeller operators, Stancu operators, Jain operators, and their modified versions are not possible to approximate Lebesgue integrable functions. In the year 1930, Kantorovich proposed integral modification of the Bernstein polynomials. The more general integral modification of the Bernstein polynomial was given by Durrmeyer in the year 1967. After this several Durrmeyer-type modifications of different operators have been introduced and their approximation properties have been discussed. In the present note, to the best of our knowledge, we present the different Durrmeyer-type operators introduced in the last five decades. We also propose some open problems in the end.

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1 Introduction

The basic result in the field of approximation by linear positive operators starts with the well-known theorem due to Weierstrass. There are many proofs available in the literature on this important theorem. The most common proof is based on the Bernstein polynomials [8], which for $f \in C[0,1]$ is defined as

$$Ł_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n).$$

For detailed study on the Bernstein polynomials, we refer the readers to the book due to Lorentz [33]. In the year 1930, Kantorovich [30] introduced the integral modification of these operators as

$$\boldsymbol{K}_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt, x \in [0,1].$$

The operators $\boldsymbol{K}_n$ are defined over a larger class of functions, e.g., $f \in L_p[0,1]$, $p \geq 1$. In the year 1967, Durrmeyer [10], to approximate functions in $L_p[0,1], p \geq 1$, introduced another important modification of the Bernstein polynomials. But no much work has been done on these operators for next few years. After the work of Derriennic [9], it became a topic of interest among researchers, and in the last four decades, many researchers proposed Durrmeyer-type generalizations of different operators and developed their approximation properties.

In the present note, we just mention different Durrmeyer-type operators considered in the last five decades, to the best of our knowledge. There are many other summation-integral-type operators having defined at $f(0)$, of genuine type and the q analogues, etc.; we present here only the Durrmeyer variants.

2 Durrmeyer Operators

In the year 1967, Durrmeyer [10] introduced the integral modification of the Bernstein polynomials in order to approximate Lebesgue integrable functions on the interval [0, 1] as

$$A_n(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)dt, x \in [0,1], \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

These operators were studied by Derriennic [9], who obtained some direct results in ordinary and simultaneous approximation. Later Agrawal–Gupta [5] applied linear combinations to these operators and estimated some direct results in order to achieve a faster rate of convergence.

Two years later after the work of Derriennic [9], Prasad et al. [37] proposed Durrmeyer-type hybrid operators containing Szász and Baskakov basis functions in summation and integration respectively to approximate Lebesgue integrable functions on $[0, \infty)$ as

$$B_n(f,x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} v_{n,k}(t)f(x+t)dt, x \in [0,\infty), \tag{2}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, v_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}.$$

While studying on this topic, Gupta [14] observed that there were a lot of errors in the results of Prasad et al. [37] concerning asymptotic approximation and error estimations. In [14] improved results have been presented.

After a gap of 2 years, in 1985 Mazhar–Totik [35] proposed Szász–Durrmeyer operators as

$$C_n(f,x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt, x \in [0,\infty), \tag{3}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

Also in the same year, Kasana et al. [31] also introduced Szász–Durrmeyer operators and estimated some direct results.

In the same year in 1985, Sahai–Prasad [38] proposed independently the Baskakov–Durrmeyer operators as

$$D_n(f,x) = (n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} v_{n,k}(t)f(t)dt, x \in [0,\infty), \tag{4}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

The operators (4) were termed as modified Lupas operators by Sahai and Prasad [38]. In the year 1991, Sinha et al. [39] improved the results of Sahai and Prasad [38] and termed the operators (4) as modified Baskakov operators.

In 1986 Chen proposed Meyer-König–Zeller–Durrmeyer operators in the following form:

$$E_n(f,x) = \sum_{k=0}^{\infty} \frac{(n+k+1)(n+k+3)}{(n+1)} \breve{m}_{n,k}(x) \int_0^1 \breve{m}_{n,k}(t) f(t)dt, x \in [0,1], \quad (5)$$

where

$$\breve{m}_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

Heilmann [27] in the year 1987 gave general Durrmeyer operators, which include the above three operators (1), (3), and (4) as special cases. She introduced the following form

$$F_{n,c}(f,x) = (n-c) \sum_{k=0}^{\infty} p_{n,k}(x;c) \int_I p_{n,k}(t;c) f(t)dt, \quad (6)$$

where

$$p_{n,k}(x;c) = \frac{(-x)^k}{k!} \phi_{n,c}^{(k)}(x).$$

One has the following special cases:

(a) If $\phi_{n,c}(x) = e^{-nx}, (c=0), x \in I \equiv [0,\infty)$, we get Szász–Durrmeyer operators [see (3)].
(b) In case $\phi_{n,c}(x) = (1+cx)^{-n/c}, (c>0), x \in I \equiv [0,\infty)$, we get the Baskakov–Durrmeyer operators [see (4)].
(c) In case $\phi_{n,c}(x) = (1-x)^n, (c=-1), x \in I \equiv [0,1]$, with finite sum from 0 to n, we get the Bernstein–Durrmeyer-type operators [see (1)].

In 1988 Guo [13] introduced a Durrmeyer variant of Meyer-König and Zeller operators on $x \in [0,1]$ as

$$G_n(f,x) = \sum_{k=1}^{\infty} \frac{(n+k-1)(n+k-3)}{(n-2)} m_{n,k+1}(x) \int_0^1 m_{n-2,k-1}(t) f(t)dt, x \in [0,1] \quad (7)$$

where

$$m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n.$$

Guo [13] studied the rate of convergence for functions of bounded variation. Later Gupta [16] gave a sharp estimate than the one established by Guo [13].

Another mixed hybrid operator in the year 1993 was proposed by Gupta and Srivastava [23] with Baskakov and Szász basis functions in summation and integration, respectively, as

$$H_n(f,x) = n\sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt, x \in [0,\infty), \tag{8}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}.$$

In [23] some direct results in simultaneous approximation have been discussed.

In the year 1994, Gupta [15] proposed the modification of the Baskakov operators with weights of Beta basis functions in order to approximate Lebesgue integrable functions of positive real axis as

$$I_n(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)dt, x \in [0,\infty), \tag{9}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}}.$$

Gupta [15] observed that this form of integral modification of the Baskakov operators is capable of providing better approximation. He established an asymptotic formula and error estimation in simultaneous approximation for the operators I_n.

After a year in 1995, Gupta and Ahmad [19] considered the reverse form of (9) by taking the Beta basis in summation and the Baskakov basis in integration, and they considered the following operators:

$$J_n(f,x) = \frac{n-1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} v_{n,k}(t)f(t)dt, x \in [0,\infty), \tag{10}$$

where

$$b_{n,k}(x) = \frac{1}{B(k+1,n)}\frac{x^k}{(1+x)^{n+k+1}}, v_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}.$$

They also considered the simultaneous approximation and established the Voronovskaja-kind formula and error estimation.

Also Gupta and Srivastava [24] in 1995 considered Beta–Szász operators as

$$K_n(f,x) = \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t)dt, x \in [0,\infty), \tag{11}$$

where

$$b_{n,k}(x) = \frac{1}{B(k+1,n)} \frac{x^k}{(1+x)^{n+k+1}}, s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}.$$

Later Gupta [17] extended this definition, and while studying the rate of convergence for absolutely continuous functions having a derivative coinciding a.e. with a function of bounded variation, Gupta [17] considered the form of the operators (11) as

$$K_{n,r}(f,x) = \begin{cases} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t)dt, & r=0 \\ \frac{(n+r-1)!}{(n-1)!n^r} \sum_{k=0}^{\infty} b_{n+r,k}(x) \int_0^{\infty} s_{n,k+r}(t) f(t)dt, & r>0, \end{cases}$$

where $b_{n,k}(x)$and$s_{n,k}(t)$ are same as considered in (11).

In the same year, i.e., in 1995, Gupta–Sahai–Srivastava [26] considered the Durrmeyer-type Szász–Beta operators to approximate Lebesgue integrable functions on $[0,\infty)$ as

$$L_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t)dt, x \in [0,\infty), \tag{12}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, b_{n,k}(x) = \frac{1}{B(k+1,n)} \frac{x^k}{(1+x)^{n+k+1}}.$$

In [26] authors established an asymptotic formula and error estimation in simultaneous approximation for the operators L_n.

Next year in 1996, Gupta and Srivastava [25] proposed the general form of the operators (3) and (9) and suggested the following form while presenting global direct results:

$$M_{n,c}(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x;c) \int_I b_{n,k}(t;c) f(t)dt, \tag{13}$$

where

$$p_{n,k}(x;c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x), b_{n,k}(t;c) = \frac{-(-t)^k}{k!}\phi_{n,c}^{(k+1)}(t).$$

One has the following special cases:

(a) If $\phi_{n,c}(x) = e^{-nx}, (c = 0), x \in I \equiv [0, \infty)$, we get Szász–Durrmeyer operators (3).
(b) In case $\phi_{n,c}(x) = (1 + cx)^{-n/c}, (c > 0), x \in I \equiv [0, \infty)$, we get the Baskakov-Beta operators (9).
(c) In case $\phi_{n,c}(x) = (1 - x)^n, (c = -1), x \in I \equiv [0, 1]$, with a finite sum from 0 to n, we get the Bernstein–Durrmeyer-type polynomials.

In the year 1999, Agratini [4] proposed the Durrmeyer variant of the Lupas operators to approximate integrable functions on $[0, \infty)$ as

$$N_n(f, x) = \sum_{k=0}^{\infty} \left(\int_0^{\infty} l_{n,k}(t)dt \right)^{-1} l_{n,k}(x) \int_0^{\infty} l_{n,k}(t)f(t)dt, \tag{14}$$

where

$$l_{n,k}(x) = 2^{-nx} \binom{nx + k - 1}{k} 2^{-k}.$$

It was observed in [4] that

$$\left(\int_0^{\infty} l_{n,k}(t)dt \right)^{-1} \simeq \frac{1}{n2^k.k!} \sum_{i=0}^{k} (-1)^{k-i} \frac{s_{k,i}.i!}{2^{i+1}},$$

with $s_{k,i}$ representing the Stirling numbers of first kind. While studying integral modification of Lupas operators, Agratini [4] estimated some direct results on Lupas-Kantorovich operators, but till date no proper direct results are available in the literature for the operators N_n due to its complicated form.

Abel–Gupta–Ivan [2] in the year 2003 proposed the Durrmeyer variant of Meyer-König and Zeller operators. For functions $f \in L_1[0, 1]$, they defined the operators as

$$O_n(f, x) = \sum_{k=0}^{\infty} m_{n,k}(x) \int_0^1 q_{n,k}(t)f(t)dt, \tag{15}$$

where

$$m_{n,k}(x) = \binom{n + k - 1}{k} x^k (1 - x)^n, q_{n,k}(t) = n \binom{n + k}{k} t^k (1 - t)^{n-1}.$$

Li [32] in the year 2005 proposed the modified form of Baskakov-Beta operators as

$$P_{n\alpha}(f, x) = \sum_{k=0}^{\infty} v_{n,k,\alpha}(x) \int_0^{\infty} b_{n,k,\alpha}(t)f(t)dt, x \in [0, \infty), \tag{16}$$

where

$$v_{n,k,\alpha}(x)=\frac{\Gamma(n/\alpha+k)}{k!\Gamma(n/\alpha)}\frac{(\alpha x)^k}{(1+\alpha x)^{n/\alpha+k}},\ b_{n,k,\alpha}(t)=\frac{1}{B(k+1,n/\alpha)}\frac{(\alpha t)^k}{(1+\alpha t)^{n/\alpha+k+1}}.$$

Here the Baskakov basis function is given by

$$v_{n,k}(x) = \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}\frac{x^k}{(1+x)^{n+k}}$$

which is modified with $\alpha > 0$ by

$$\begin{aligned} v_{n,k,\alpha}(x) &= \frac{n(n+\alpha)(n+2\alpha)\cdots(n+(k-1)\alpha)}{k!}\frac{x^k}{(1+\alpha x)^{n/\alpha+k}} \\ &= \frac{\Gamma(n/\alpha+k)}{k!\Gamma(n/\alpha)}\frac{(\alpha x)^k}{(1+\alpha x)^{n/\alpha+k}}. \end{aligned}$$

For further studies on the operators $P_{n\alpha}(f,x)$, we refer the readers [20]. Abel et al. [2] investigated local approximation properties of the above Durrmeyer variant of Meyer-König and Zeller operators. They derived sharp estimates of the first and second central moments. The other results include the rate of convergence by first modulus of continuity, the Voronovskaja-type formula, and also the rate of convergence for a Bézier variant of these operators.

Mihesan [36] introduced the generalization of the Baskakov operators depending on a nonnegative constant a, independent of n as

$$M_n^a(f,x) = \sum_{k=0}^{\infty} e^{-\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}}f(k/n), x \geq 0$$

where $P_k(n,a)=\sum_{i=0}^{k}\binom{k}{i}(n)_i a^{k-i}$, $(n)_0=1$, and $(n)_i = n(n+1)\cdots(n+i-1)$. In the year 2011, Erençin [11] proposed a Durrmeyer-type generalization of the operators $M_n^a(f,x)$ but with weights of Beta basis functions. For a nonnegative constant a, independent of n, the operators considered in [11] are defined as follows:

$$Q_n^a(f,x) = \sum_{k=0}^{\infty} e_{n,k}^a(x)\int_0^{\infty} b_{n,k}(t)f(t)dt, x \geq 0, \tag{17}$$

where

$$e_{n,k}^a(x) = e^{-\frac{ax}{1+x}}\frac{P_k(n,a)}{k!}\frac{x^k}{(1+x)^{n+k}},\ b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}}.$$

It can be observed that in case $a = 0$ these operators reduce to the Baskakov-Beta operators defined by I_n above. Erençin [11] obtained some direct results for these operators. Later Agrawal et al. [7] extended the work and they study simultaneous approximation results.

Agrawal et al. [6] considered the weights of Szász basis functions in integration while introducing the Durrmeyer variant of the operators $M_n^a(f, x)$ as

$$R_n^a(f, x) = n \sum_{k=0}^{\infty} e_{n,k}^a(x) \int_0^1 s_{n,k}(t) f(t) dt, x \geq 0, \tag{18}$$

where

$$e_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \frac{P_k(n, a)}{k!} \frac{x^k}{(1+x)^{n+k}}, s_{n,k}(t) = e^{-nt} \frac{(nt)^k}{k!}.$$

Agrawal et al. [6] established the rate of convergence in ordinary and simultaneous approximation for the operators R_n^a.

Recently Gupta [18] considered the Durrmeyer-type integral modification of Abel–Ivan operators [1] (which is the modified form of Jain–Pethe operators [29]) with the weights of Baskakov or Szász basis functions. For $c = c_n > \beta$ $(n = 0, 1, 2, \ldots)$ for certain constant $\beta > 0$, the operators are defined as

$$S_{n,c,d}(f, x) = (n - d) \sum_{k=0}^{\infty} p_{n,k}^c(x) \int_0^{\infty} b_{n,k}^d(t) f(t) dt, x \geq 0, \tag{19}$$

where

$$p_{n,k}^c(x) = \left(\frac{c}{1+c}\right)^{ncx} \frac{(ncx)_k}{k!} (1+c)^{-k}, b_{n,k}^d(t) = \frac{(-t)^k}{k!} \phi_{n,d}^{(k)}(t)$$

with $(a)_k$ rising factorial and for $\phi_{n,d}(t) = e^{-nt}, d = 0$ Szász basis function, for $\phi_{n,d}(t) = (1+t)^{-n}, d > 0$ Baskakov basis functions can be obtained. Gupta [18] obtained some direct estimates for these operators. Later Govil–Gupta–Soybaş [12] estimated the rate of convergence for functions having derivatives of bounded variation.

Stancu [40] introduced a sequence of linear positive operators $S_n^{(\alpha)} : C[0, 1] \to C[0, 1]$ depending on a nonnegative parameter α given by

$$S_n^{(\alpha)}(f, x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{v=0}^{k-1}(x + v\alpha) \prod_{\mu=0}^{n-k-1}(1 - x - \mu\alpha)}{\prod_{\lambda=0}^{n-1}(1 + \lambda\alpha)} f(k/n), x \in [0, 1].$$

In case $\alpha = 0$, these operators reduce to the well-known Bernstein polynomials. Here the basis function is the Polya distribution. Lupaş–Lupaş [34] considered the case $\alpha = 1/n$, which is important for approximation point of view. Recently Gupta and Rassias [22] considered the Durrmeyer-type modification of these operators with weights of Bernstein basis functions as

$$T_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_0^1 p_{n,k}(t)f(t)dt, x \in [0,1], \tag{20}$$

where

$$p_{n,k}^{(1/n)}(x) = \binom{n}{k}\frac{2(n!)}{(2n)!}(nx)_k(n-nx)_{n-k}, p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k}.$$

Gupta and Rassias [22] established some direct results for these operators in ordinary approximation.

With the aim to generalize the classical Szász operators Jain [28], for $0 \le \beta < 1$ proposed the operators

$$J_n^{\beta}(f,x) = \sum_{k=0}^{\infty}\frac{nx(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)}f(k/n), \qquad x \in [0,\infty).$$

Very recently Gupta and Greubel [21] for $0 \le \beta < 1$ proposed the following Durrmeyer variant of the operators $J_n^{\beta}(f,x)$ in order to approximate integrable functions as

$$U_n^{\beta}(f,x) = \sum_{k=0}^{\infty}\left(\int_0^{\infty} L_{n,k}^{(\beta)}(t)\,dt\right)^{-1} L_{n,k}^{(\beta)}(x)\int_0^{\infty} L_{n,k}^{(\beta)}(t)f(t)\,dt, x \ge 0 \tag{21}$$

where the basis function is defined as

$$L_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)}.$$

As a special case $\beta = 0$, these operators reduce to the Szász–Mirakyan–Durrmeyer operators defined in (3) above. These operators have complicated representation, and for higher-order moments, still recurrence relation may be considered as an open problem.

Stancu [40] introduced a sequence of positive linear operators depending on the parameters α and β, satisfying the condition $0 \le \alpha \le \beta$, so-called Bernstein–Stancu operators, as

$$B_{n,\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)p_{n,k}(x),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and showed that as classical Bernstein operators $B_n(f;x)$, $B_{n,\alpha,\beta}(f;x)$ also converges to $f(x)$ uniformly on [0, 1] for $f \in C[0,1]$. Very recently Acar et al. [3] considered a new type of Bernstein–Durrmeyer operators, with the purpose of approximating Lebesgue integrable function on the mobile subinterval of [0, 1] for $0 \le \alpha \le \beta$ as

$$V_{n,\alpha,\beta}(f,x) = (n+1)\left(\frac{n+\beta}{n}\right)^{2n+1} \sum_{k=0}^{n} \bar{p}_{n,k}(x) \int_{\frac{\alpha}{n+\beta}}^{\frac{n+\alpha}{n+\beta}} \bar{p}_{n,k}(t) f(t)\, dt, \tag{22}$$

where

$$\bar{p}_{n,k}(x) = \binom{n}{k}\left(x - \frac{\alpha}{n+\beta}\right)^k \left(\frac{n+\alpha}{n+\beta} - x\right)^{n-k}.$$

For these operators authors [3] represented the operators in terms of hypergeometric series and established local and global approximation results for these operators in terms of modulus of continuity. In the last section, better error estimation for the operators using King-type approach has been discussed.

3 Open Problems

While defining the integral modification of the generalized Baskakov operators $M_n^a(f,x)$, Erençin [11] and Agrawal et al. [6] considered the Beta and Szász basis functions, respectively, in integration, which are not the usual Durrmeyer variants of $M_n^a(f,x)$ [see (17) and (18)]. One can consider the same basis in integral, i.e., the basis $e_{n,k}^a(x)$, and the operators for nonnegative parameter a, takes the form

$$\begin{aligned} W_n^a(f,x) &= \sum_{k=0}^{\infty}\left(\int_0^\infty e_{n,k}^a(t)dt\right)^{-1} e_{n,k}^a(x)\int_0^\infty e_{n,k}^a(t)f(t)dt \\ &= \sum_{k=0}^{\infty} e_{n,k}^a(x)\frac{< e_{n,k}^a(t), f(t) >}{< e_{n,k}^a(t), 1 >}, x \ge 0, \end{aligned} \tag{23}$$

where

$$e_{n,k}^a(x) = e^{-\frac{ax}{1+x}} \frac{x^k}{(1+x)^{n+k}} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (n)_j\, a^{k-j}.$$

In approximation theory to check the convergence theorem, the basic Korovkin's conditions must be satisfied, i.e., $W_n^a(t^i,x) \to x^i,, i = 0,1,2$ for sufficiently large n. In the case of operators $W_n^a(f,x)$, the ratio of the term

$$\frac{< e^a_{n,k}(t), t^i >}{< e^a_{n,k}(t), 1 >}$$

is an infinite series and does not have a finite form. So it would be difficult to even have first- and second-order moments, and it may be considered as an open problem for the readers.

In (19) the basis in integration is different. In case of a same basis function for $c = c_n > \beta$ $(n = 0, 1, 2, \ldots)$ for certain constant $\beta > 0$, the operators can be considered as

$$X_{n,c}(f, x) = \sum_{k=0}^{\infty} p^c_{n,k}(x) \left(p^c_{n,k}(t)dt\right)^{-1} \int_0^{\infty} p^c_{n,k}(t)f(t)dt, x \geq 0, \tag{24}$$

where

$$p^c_{n,k}(x) = \left(\frac{c}{1+c}\right)^{ncx} \frac{(ncx)_k}{k!}(1+c)^{-k}$$

with $(a)_k = a(a+1)\cdots(a+k-1)$. Actually the basis function $p^c_{n,k}(x)$ is good for a discrete case, and integral modification of the form (24) seems not an appropriate representation as far as approximation properties are concerned. It is also open for readers to find the $Y_{n,c}(t^i, x)$ for $i = 1, 2$.

Just like other Durrmeyer operators based on Polya distribution as considered in (20) with different bases in integration above, one can define with the same basis under an integral sign as

$$Y_n(f, x) = \sum_{k=0}^{n} p^{(1/n)}_{n,k}(x) \left(\int_0^1 p^{(1/n)}_{n,k}(t)dt\right)^{-1} \int_0^1 p^{(1/n)}_{n,k}(t)f(t)dt, x \in [0, 1], \tag{25}$$

where

$$p^{(1/n)}_{n,k}(x) = \binom{n}{k} \frac{2(n!)}{(2n)!}(nx)_k(n-nx)_{n-k}.$$

It has the same problems as discussed above for the cases (23) and (24).

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On the Imaginary Part of the Nontrivial Zeros of the Riemann Zeta Function

Mehdi Hassani

In Honor of Constantin Carathéodory

Abstract Based on the recent improved upper bound for the argument of the Riemann zeta function on the critical line, we obtain explicit sharp bounds for γ_n, where $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots$ are consecutive ordinates of nontrivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function. Among several bounds, we show validity of the double-side inequality:

$$\frac{2\pi n}{\log n}\Big(1 + \frac{11}{12}\frac{\log\log n}{\log n}\Big) \leq \gamma_n \leq \frac{2\pi n}{\log n}\Big(1 + \frac{23}{12}\frac{\log\log n}{\log n}\Big),$$

for each $n \geq 3598$.

1 Introduction

The Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) > 1$ and extended by analytic continuation to the complex plane with a simple pole at $s = 1$. It is known [7, 13] that

$$N(T) := \sum_{\substack{0<\gamma\leq T \\ \zeta(\beta+i\gamma)=0}} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi\mathrm{e}} + O(\log T), \tag{1}$$

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in which the term $O(\log T)$ comes from the approximation of the function $S(T)$, which is defined traditionally by

$$S(T) = \frac{1}{\pi} \arg \zeta\Big(\frac{1}{2} + it\Big),$$

where the argument is determined via continuous variation along the line segments connecting 2, $2 + iT$ and $\frac{1}{2} + iT$, with taking the argument of $\zeta(s)$ at $s = 2$ to be zero. If T is an ordinate of a zero of $\zeta(s)$, then we set

$$S(T) = \frac{1}{2} \lim_{\varepsilon \to 0} \Big(S(T + \varepsilon) + S(T - \varepsilon)\Big).$$

Indeed, the approximation of the function $N(T)$ related strongly to the approximation of $S(T)$. More precisely, for $T \geq 1$ it is known (see [15]) that

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi \mathrm{e}} - \frac{7}{8} \right| \leq |S(T)| + \mathscr{E}(T), \tag{2}$$

where

$$\mathscr{E}(T) = \frac{1}{4\pi} \arctan \frac{1}{2T} + \frac{T}{4\pi} \log\Big(1 + \frac{1}{4T^2}\Big) + \frac{1}{3\pi T}.$$

Conditional approximations of $S(T)$, assuming the Riemann hypothesis (RH), have the form

$$|S(T)| \leq (C + o(1)) \frac{\log T}{\log\log T},$$

where C is an effective constant and $o(1)$ refers to a quantity which tends to 0 as T grows (see [8]). The best known of such conditional approximations asserts that

$$|S(T)| \leq \frac{1}{4} \frac{\log T}{\log\log T} + O\Big(\frac{\log T \log\log\log T}{(\log\log T)^2}\Big),$$

assuming RH, for T sufficiently large (see [3]). Unconditional approximations of $S(T)$ usually take the form

$$|S(T)| \leq a \log T + b \log\log T + c, \tag{3}$$

for $T \geq T_0$ where a, b, c, and T_0 are computable constants. The following table summarizes some known values of a, b, c, and T_0 for which the approximation (3) is valid.

We let $\gamma_1 = \min\{\gamma > 0 : \zeta(\beta + i\gamma) = 0\} \cong 14.134725142$, and more generally, we set $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots$ to be consecutive ordinates of the imaginary parts of non-real zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. The approximate relation (1) implies that

$$\gamma_n \sim \frac{2\pi n}{\log n},$$

as $n \to \infty$. It is possible to combine the relations (2) and (3) to get an explicit bound for the function $N(T)$ and then utilize it to obtain

$$\gamma_n \geq \frac{2\pi n}{\log n}\left(1 + \frac{\lambda}{2\pi}\frac{\log\log n}{\log n}\right), \tag{4}$$

for $n \geq n_\lambda$, and

$$\gamma_n \leq \frac{2\pi n}{\log n}\left(1 + \frac{\eta}{2\pi}\frac{\log\log n}{\log n}\right), \tag{5}$$

for $n \geq n_\eta$. Recently, based on the Rosser's explicit bound for $N(T)$ (Theorem 19 of [12]), and unwitting of the recent results of Trudgian [14, 15], we obtained (see [5]) some bounds for γ_n as in (4) and (5). Our intention in writing this note is to obtain some new and shaper explicit bounds based on the most recent result of Trudgian [15]. More precisely, we prove the following.

Theorem 1. *Assume that we choose pairs λ and n_λ from Table 1, and also we choose pairs η and n_η from Table 3. Then, the inequalities (4) and (5) are valid, respectively, for $n \geq n_\lambda$ and for $n \geq n_\eta$.*

Table 1 Some known values of a, b, c, and T_0 for which the inequality (3) holds

Author	Date	a	b	c	T_0
Von Mangoldt [16]	1905	0.432	1.917	12.204	28.588
Grossmann [4]	1913	0.291	1.787	6.137	50
Backlund [1]	1914	0.275	0.979	7.446	200
Backlund [2]	1918	0.137	0.443	4.35	200
Rosser [11]	1939	1.12	0	9.5	1450
Rosser [12]	1941	0.137	0.443	1.588	1467
Trudgian [14]	2012	0.17	0	1.998	e
Trudgian [15]	2014	0.112	0.278	2.510	e

Table 2 Some values of λ and n_λ for which the inequality (4) is valid for $n \geq n_\lambda$

λ	$n_\lambda \cong$	λ	$n_\lambda \cong$
π	39.2	5.8	281689.7
$3\pi/2$	368.2	5.85	762802.6
$5\pi/3$	2076.1	5.9	2734673.6
$7\pi/4$	9382.8	5.95	15505047.6
$9\pi/5$	37888.1	5.99	111747280.4
$11\pi/6$	144857.3	6	209209097.8

Table 3 Some values of η and n_η for which the inequality (5) is valid for $n \geq n_\eta$

η	$n_\eta \cong$	η	$n_\eta \cong$
5π	1332.6	$13\pi/4$	990145917795.7
4π	327934.9	3π	5349674936546058248.1
$23\pi/6$	1950923.5	$14\pi/5$	319868073000142693339662257 1377.7
$7\pi/2$	485148930.1	$11\pi/4$	1921157257024623128845528138047 1122965.9

Corollary 1. *For each $n \geq 2$ we have*

$$\gamma_n \geq \frac{2\pi n}{\log n}\Big(1 + \frac{11}{12}\frac{\log\log n}{\log n}\Big), \tag{6}$$

and also, for each $n \geq 3598$

$$\gamma_n \leq \frac{2\pi n}{\log n}\Big(1 + \frac{23}{12}\frac{\log\log n}{\log n}\Big). \tag{7}$$

Remark 1. We let n'_λ and n'_η be the least positive integers, for which the inequalities (4) and (5) are valid, respectively. The values of n_λ and n_η in the above tables are not optimal, in the sense of $n'_\lambda \leq n_\lambda$ and $n'_\eta \leq n_\eta$. It is possible, by computation, to determine the values of n'_λ and n'_η for given λ and η. For example, in the above corollary, indeed we have $n'_{\lambda=11\pi/6} = 2$ and $n'_{\eta=23\pi/6} = 3598$. The truth of Corollary 1.1 of [5] asserts that

$$\gamma_n = \frac{2\pi n}{\log n}\Big(1 + (1 + o(1))\frac{\log\log n}{\log n}\Big),$$

as $n \to \infty$. Assume that $\delta > 0$ is given. Considering the values of $n'_{\lambda=2\pi-\delta}$ and $n'_{\eta=2\pi-\delta}$ we guess that

$$\mathscr{B}(\delta) := \frac{n'_{\lambda=2\pi-\delta}}{n'_{\eta=2\pi-\delta}} \to 0,$$

rapidly.

Our argument to obtain the above results is similar to what we have developed in [5], as well as we will do several computations running over the numbers γ_n, all of which have been done by using Maple software and are based on the tables of zeros of the Riemann zeta function due to Odlyzko [9].

2 Some Preliminary Results

Proposition 1. *For each* $T \geq \mathrm{e}$ *we have* $|N(T) - F(T)| \leq R(T)$, *with*

$$F(T) = \frac{T}{2\pi}\log\frac{T}{2\pi\mathrm{e}} + \frac{7}{8}, \qquad \text{and} \qquad R(T) = \frac{14}{125}\log T + \frac{139}{500}\log\log T + \frac{2571}{1000}.$$

Proof. We note that the function $\mathscr{E}(T)$ is strictly decreasing for $T > 0$. Thus, for $T \geq \mathrm{e}$, we have $\mathscr{E}(T) \leq \mathscr{E}(\mathrm{e}) < \frac{61}{1000}$. We consider the relation (2), and we recall the known (see [15]) approximation:

$$|S(T)| \leq \frac{14}{125}\log T + \frac{139}{500}\log\log T + \frac{251}{100},$$

which is valid for $T \geq \mathrm{e}$, to get the result.

The above result gives lower and upper bounds for the zero-counting function $N(T)$, in terms of $T\log T$, T, $\log T$, and $\log\log T$. To accomplish our method, we need to modify these bounds to include only the terms $T\log T$ and T. Thus, we reform the statement of the above proposition as follows. For the whole text, we set

$$\ell = \frac{1}{2}, \qquad \text{and} \qquad u = \frac{1}{4}.$$

Lemma 1. *Let*

$$L(T) = \frac{1}{2\pi}T\log T - \ell T, \qquad \text{and} \qquad U(T) = \frac{1}{2\pi}T\log T - uT. \tag{8}$$

Then, $U(T)$ *and* $L(T)$ *are strictly increasing for* $T \geq \mathrm{e}^{2\pi u - 1} \cong 1.769676$ *and* $T \geq \mathrm{e}^{2\pi\ell - 1} \cong 8.512985$, *respectively. Also, for* $T \geq \gamma_1 - 10^{-5}$, *we have*

$$L(T) \leq N(T) \leq U(T). \tag{9}$$

Proof. Monotonicity of the functions $U(T)$ and $L(T)$ is straightforward. To get the inequalities in (9), we utilize Proposition 1 to write

$$F(T) - R(T) \leq N(T) \leq F(T) + R(T),$$

for $T \geq \mathrm{e}$. An easy calculus computation implies that for $T \geq 20.28$, we have $F(T) + R(T) \leq U(T)$ and consequently $N(T) \leq U(T)$. Since $\gamma_2 \cong 21.022039639$, thus for $T < 20.28$, we have $N(T) \leq 1$. On the other hand, since $U(T)$ is strictly increasing for $T \geq \mathrm{e}^{2\pi u-1} \cong 1.769676$, thus $U(T) \geq U(\gamma_1 - 10^{-5}) > 2.4 > 1 \geq N(T)$. This gives the right-hand side of (9).

Similarly, we observe that for $T \geq 52.16$ the inequality $L(T) \leq F(T)-R(T)$, and consequently $L(T) \leq N(T)$, are valid. Further, we note that $\gamma_{11} \cong 52.970321478$. Considering monotonicity of $L(T)$, and validity of the inequalities $L(\gamma_n) < N(\gamma_n) = n$ for $1 \leq n \leq 11$, we deduce the left-hand side of (9) for $T < 52.9$, too. This completes the proof.

The following lemma, which its statement and its proof are similar to the Lemma 2.2 of [5], transfers lower and upper bounds for $N(T)$ to bounds for γ_n in terms of inverses of mentioned bounds for $N(T)$.

Lemma 2. *Assume that $L(T)$ and $U(T)$ are defined as in (8), and let us denote by $L^{-1}(T)$ and $U^{-1}(T)$ their inverses, respectively. Then, for each integer $n \geq 1$, we have*

$$U^{-1}(n) \leq \gamma_n \leq L^{-1}(n). \tag{10}$$

The above lemma gives lower and upper bounds for γ_n, but in terms of $L^{-1}(n)$ and $U^{-1}(n)$. As the following lemma asserts, it is possible to write these inverse functions in terms of the Lambert W function $W(x)$, which is defined by the relation $W(x)\mathrm{e}^{W(x)} = x$ for $x \in [-\mathrm{e}^{-1}, +\infty)$. The following result, which is indeed Lemma 2.3 of [5], allows us to find the above required inverses.

Lemma 3. *Assume that a and b are some positive real numbers, and let*

$$f(T) = \frac{1}{a} T \log T - bT.$$

We denote the inverse function of f by f^{-1}. Then, for $T \geq \mathrm{e}^{ab-1}$, the function f is strictly increasing and we have

$$f^{-1}(T) = \frac{aT}{W(a\mathrm{e}^{-ab}T)}. \tag{11}$$

In particular, as $T \to +\infty$, we obtain $f^{-1}(T) \sim \frac{aT}{\log T}$.

The last asymptotic for $f^{-1}(T)$ in the above lemma comes from the fact that Lambert W function has the asymptotic expansion $W(x) = \log x + O(\log \log x)$ as $x \to \infty$, (see [10], page 111). Albeit, to get desired explicit bounds concerning γ_n, we need some explicit bounds for the Lambert W function. The following proposition, which is Theorem 2.8 of [6], offers such sharp bounds.

Proposition 2. *Assume that $\alpha > 0$ is real, and let*

$$\omega_\alpha(x) := \log x - \log\log x + \alpha \frac{\log \log x}{\log x}.$$

Then, for each $x \geq \mathrm{e}$ we have

$$\omega_{\frac{1}{2}}(x) \leq W(x) \leq \omega_{\frac{\mathrm{e}}{\mathrm{e}-1}}(x), \tag{12}$$

with equality only for $x = \mathrm{e}$.

3 Proofs of Theorem 1 and Corollary 1

To prove (4) we let $c_u = 2\pi \mathrm{e}^{-2\pi u}$. By applying the truth of Lemma 3, considering the left-hand side of (10), and considering the right-hand side of (12), we obtain

$$\gamma_n \geq U^{-1}(n) = \frac{2\pi n}{W(c_u n)} \geq \frac{2\pi n}{\omega_{\frac{\mathrm{e}}{\mathrm{e}-1}}(c_u n)} := g(n),$$

for $c_u n \geq \mathrm{e}$ and $n \geq 1$ or equivalently for $n \geq \max\{\frac{\mathrm{e}}{c_u}, 1\} \geq 3$. We let

$$h(n) := \frac{g(n) - \frac{2\pi n}{\log n}}{\frac{n \log\log n}{\log^2 n}}.$$

Now, we note that the function $h : (\mathrm{e}, +\infty) \longrightarrow (-\infty, 2\pi)$ defined by $h(n)$ is continuous and strictly increasing. Moreover, we have

$$\lim_{n\to \mathrm{e}^+} h(n) = -\infty, \qquad \text{and} \qquad \lim_{n\to +\infty} h(n) = 2\pi.$$

Therefore, for any real $\lambda \in (-\infty, 2\pi)$, there exists unique $n_\lambda \in (\mathrm{e}, +\infty)$ such that $h(n) \geq \lambda$ for $n \geq n_\lambda$ with equality only for $n = n_\lambda$. Hence, for $n \geq n_\lambda$ we obtain

$$\gamma_n \geq \frac{2\pi n}{\log n} + \lambda \frac{n \log\log n}{\log^2 n},$$

which is indeed (4). Table 2 contains some of our computational results, including some values of λ and related n_λ.

To prove (5) we let $c_\ell = 2\pi \mathrm{e}^{-2\pi\ell}$. We use the truth of Lemma 3, the right-hand side of (10), and the left-hand side of (12), to get

$$\gamma_n \leq L^{-1}(n) = \frac{2\pi n}{W(c_\ell n)} \leq \frac{2\pi n}{\omega_{\frac{1}{2}}(c_\ell n)} := v(n),$$

for $c_\ell n \geq \mathrm{e}$ and $n \geq 1$ or equivalently for $n \geq \max\{\frac{\mathrm{e}}{c_\ell}, 1\} \geq 11$. We let

$$z(n) := \frac{v(n) - \frac{2\pi n}{\log n}}{\frac{n \log\log n}{\log^2 n}}.$$

We note that the function $y(n) = \frac{1}{z(n)}$ defined over the interval $(\frac{1}{c_\ell}, +\infty)$ is continuous and satisfies the limit relation $\lim_{n\to\infty} y(n) = \frac{1}{2\pi}$. Moreover, it is strictly increasing for $n \geq 4$, and $y(n) < 0$ for $\frac{1}{c_\ell} < n \leq 5$. Hence, there exists unique n_0 with $n_0 > 5$ such that $y(n_0) = 0$. By computation, we observe that $n_0 \cong 5.312502$. Now, we note that the function $z : (n_0, +\infty) \longrightarrow (2\pi, +\infty)$ defined by $z(n)$ is continuous and strictly decreasing, and

$$\lim_{n\to n_0^+} z(n) = +\infty, \qquad \text{and} \qquad \lim_{n\to+\infty} z(n) = 2\pi.$$

Therefore, for any $\eta \in (2\pi, +\infty)$, there exists unique $n_\eta \in (n_0, +\infty)$ such that $z(n) \leq \eta$ for $n \geq n_\eta$ with equality only for $n = n_\eta$, and consequently, for $n \geq n_\eta$ we get

$$\gamma_n \leq \frac{2\pi n}{\log n} + \eta \frac{n \log\log n}{\log^2 n},$$

which is indeed (5). Table 3 includes some values of η and related values of n_η.

To get (6) we utilize Theorem 1, with $\lambda = 11\pi/6$, which gives (6) for $n \geq 144858$. For $2 \leq n \leq 144857$ we confirm validity of it by computation. Also, to prove (7) we apply Theorem 1 with $\eta = 23\pi/6$, which gives (7) for $n \geq 1950924$. For $3598 \leq n \leq 1950923$ we confirm validity of it by computation.

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The Ubiquitous Lambert Function and its Classes in Sciences and Engineering*

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In Honor of Constantin Carathéodory

Abstract The Lambert-W function satisfies the functional equation $W(x)e^{W(x)} = x$. This function shares with many other functions the remarkable property of appearing frequently in the solutions of a variety of mathematical problems stemming from diverse scientific domains that range from chemistry and mechanical engineering to computer science and network design. In this paper we survey some recent research results on various, diverse problems whose solutions involve the Lambert-W function as well as a natural extension there of, called Hyper Lambert functions.

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Our approach relies on highlighting the intuition behind the appearance of the Lambert function in the solution to these problems rather than providing fine details of the solution process, which can be found in the original research works. Our hope is that our work will be useful to other researchers working in similar problems serving as a guideline for uncovering a solution to their research problem that possibly involves the Lambert-W function or it extensions.

1 Introduction

The *Lambert-W* function is defined as the function $W(x)$ satisfying the following functional equation (for functional equations see in [17]):

$$W(x)e^{W(x)} = x. \tag{1}$$

By differentiating Equation (1), we obtain the first order derivative of the Lambert-W function:

$$\frac{dW(x)}{dx} = \frac{W(x)}{x(1 + W(x))}. \tag{2}$$

The Lambert-W function has appeared in the solutions of problems stemming from numerous diverse application domains such as electrical and mechanical engineering, computer network protocols and network design, chemistry and physics as well as biology. The definitive work on the Lambert-W function and its applications is the excellent paper [4]. There, the authors give basic and advanced properties of the Lambert-W function and then describe the results of numerous papers where this function appears in the solution of problems from a variety of scientific disciplines, also delineating the solution process as well as how the Lambert-W function arises in the solution.

Since the publication of [4] more works have appeared in which the Lambert function plays a central role in the solutions of the problems they study, some other problems have solutions that involve a natural extension of the Lambert-W function called *Hyper Lambert* functions. Our work comes as an extension to [4] that covers the more recent research results involving the Lambert-W function and its extensions. We, therefore, refer the reader to the excellent coverage of the mathematical properties of the function in [4] and concentrate on describing the target papers' results and the intuition behind the derivation of the solutions that involve the Lambert-W function and its extensions.

Thus, in Section 2 we study a differential equation that was derived in [6] in the theoretical analysis of a randomized, key agreement scheme for distributed agents. A special case of this differential equation can be transformed into a form that leads to a solution involving the Lambert-W function while the more general case is not amenable to such a transformation. However, we describe a strategy followed in [7, 8] that leads to a solution based on a generalization of the Lambert-W function, the *Hyper Lambert* functions, which were defined in [5].

In Section 3 we study two *Resource Allocations Problems* in Wireless Networks. The first one is the *Subcarrier Allocation problem in Wireless Mesh Networks* and the second one is the *Power Allocation problem in Relay Networks*. We define the problems and show (see [9]) how the solutions to the derived mathematical expressions involve the Lambert-W function.

In Section 4 we show how the Lambert-W function is involved in the solution of a problem stemming from the problem of estimating bounds on the function estimating the distribution of primes and the n-th prime number (see [11]). Moreover, in this section we show how the Lambert-W function was used in the analysis of the performance of the quadratic sieve algorithm (see [13]).

In Section 5 we present from [15] the derivation of a solution of a system of Dynamic Differential Equations that describe glucose-insulin concentration dynamics in blood as described in [14].

Finally, in Section 6 we summarize our findings and attempt to justify the ubiquity of the Lambert function based on the surveyed results and its applicability outreach.

2 Solution of an ODE arising in a Key Agreement Scheme

In this section, the derivation of Lambert-W function closed-form solution of an ordinary differential equation that arose in the analysis of a key-agreement scheme, appeared in [7, 8], is presented. In [6], the following ordinary differential equation arose in the analysis of a key agreement scheme:

$$\frac{dy(t)}{dt} = \frac{1}{1-t}\left[y(t)^{k-\lceil \frac{k}{2} \rceil+1}(1-y(t))^{\lceil \frac{k}{2} \rceil}\binom{k-1}{\lceil \frac{k}{2} \rceil - 1}\right]. \tag{3}$$

The differential equation given by (3) is *separable* and, thus, its solution can be found by solving the following equation for y (see, e.g., [1, 2]), with $w(y) = \frac{1}{y(t)^{k-\lceil \frac{k}{2} \rceil+1}(1-y(t))^{\lceil \frac{k}{2} \rceil}\binom{k-1}{\lceil \frac{k}{2} \rceil-1}}$ and $g(t) = \frac{1}{1-t}$:

$$\int \frac{dy}{w(y)} = \int g(t)dt + C. \tag{4}$$

From (4), we obtain the following:

$$\frac{1}{\binom{k-1}{\lceil \frac{k}{2} \rceil-1}} \int \frac{dy}{y^{k-\lceil \frac{k}{2} \rceil+1}(1-y)^{\lceil \frac{k}{2} \rceil}} = -\ln(1-t) + C. \tag{5}$$

For $k = 2$, Equation (5) becomes as follows:

$$\frac{dy(t)}{dt} = -y(t)^3 + y(t)^2 - y(t). \tag{6}$$

This is an instance of the *Abel Equation of the First Kind*, which has the following general form:

$$\frac{dy(t)}{dt} = f_3(t)y(t)^3 + f_2(t)y(t)^2 + f_1(t)y(t) + f_0(t) \tag{7}$$

and was solved in [6] based on a standard methodology for this type of differential equations (see, e.g., [1, 2]).

However, the connection with the $W(x)$ function becomes more apparent by solving Equation (5) for $k = 2$ directly, without resorting to the methodology for solving the general Abel equation of the first kind. After computing the integral on the left-hand side of (5), for $k = 2$, and exponentiating both sides of the equality we obtain the following equation:

$$\left(1 - \frac{1}{y(t)}\right) e^{1/y(t)} = (1-t)e^{-C}. \tag{8}$$

Setting $z(t) = 1/y(t) - 1$ and multiplying both sides with $-e^{-1}$ we obtain from (8) the following:

$$ze^z = -(1-t)e^{-C-1}. \tag{9}$$

Comparing (9) with the definition of the Lambert function, we conclude that its solution is given by the following:

$$y(t) = \frac{1}{z(t)+1} = \frac{1}{W[-(1-t)e^{-C-1}]+1}. \tag{10}$$

However, for $k > 2$ the differential equation defined by Equation (3) does not appear to have a closed-form solution based on the Lambert function. Fortunately, based on [7, 8], a natural generalization of the Lambert function enables its solution in a form similar to (10).

The departure point is the computation of the integral on the right-hand side of (5) for a general value of k. The computation is based on the partial fraction analysis of the integrand into fractions of the form $\frac{1}{y^i}$ and $\frac{1}{(1-y)^i}$. As it was shown in [8], this integral is equal to the following expression:

$$\frac{1}{A_{y,1}} \left[\sum_{i=2}^{\lceil \frac{k}{2} \rceil + \delta_k} \frac{A_{y,i}}{(i-1)y^{i-1}} - \sum_{i=2}^{\lceil \frac{k}{2} \rceil} \frac{A_{1-y,i}}{(i-1)(1-y)^{i-1}} \right]$$

with the $A_{y,i}, A_{1-y,i}$ constants dependent on k only and $\delta_k = 0$ if k is odd while $\delta = 1$ if k is even. We set

$$\Delta = \frac{1}{A_{y,1}} \left[\sum_{i=2}^{\lceil \frac{k}{2} \rceil + \delta_k} \frac{A_{y,i}}{(i-1)y^{i-1}} - \sum_{i=2}^{\lceil \frac{k}{2} \rceil} \frac{A_{1-y,i}}{(i-1)(1-y)^{i-1}} \right].$$

$$S = \frac{\binom{k-1}{\lceil \frac{k}{2} \rceil - 1}}{A_{y,1}} \left(-\ln(1-t) + C\right). \tag{11}$$

Then, after integrating and exponentiating, Equation (5) can be rewritten as

$$\frac{y}{1-y}e^{-\Delta} = e^{S} \Rightarrow \left(1-\frac{1}{y}\right)e^{\Delta} = e^{-S}. \tag{12}$$

As in the case $k = 2$, we set $z = \frac{1}{y} - 1$, and obtain the following, where we have also multiplied by the factor e^{-1} for similarity with (9):

$$\begin{aligned} ze^{\Delta_z - 1} &= -e^{-S+1} \\ &= -e^{-1}e^{\frac{\binom{k-1}{\lceil \frac{k}{2}\rceil - 1}}{A_{y,1}}[\ln(1-t)-C]} \\ &= -e^{-1}[(1-t)e^{-C}]^{\frac{\binom{k-1}{\lceil \frac{k}{2}\rceil - 1}}{A_{y,1}}}. \end{aligned} \tag{13}$$

where Δ_z is Δ with y replaced by $\frac{1}{z+1}$.

We observe that for $k > 2$ it is not possible to transform (13) into a form suitable for the application of the definition of the W function since, as remarked in [5], the most general form of equation that can be cast into (1) is $az^m e^{bz^n} = g(t)$, with m, n integers, a, b complex numbers, and $g(t)$ a complex function of t.

However, in [5] a generalization of the W function was proposed, which helped to write an explicit solution of (13) using this class of generalized Lambert functions. In what follows, we will describe this class, based on [5].

Definition 1. Let I be an index set and $f_i : \mathbb{C} \to \mathbb{C}$ be arbitrary complex functions not vanishing identically. Assuming $m, n \in \mathbb{N}$ such that $m \geq n$ we define $F_{n,m}(z) : \mathbb{N}^2 \times \mathbb{C} \to \mathbb{C}$ as follows:

$$F_{n,m}(z) = \begin{cases} e^{z} & \text{if } n = 1 \\ e^{f_{m-(n-1)}(z)F_{n-1,m}(z)} & \text{if } n > 1. \end{cases} \tag{14}$$

Definition 2. Let f_i be as in Definition 1 and $z \in \mathbb{C}$. Then we define the function G as $G(f_1(z), f_2(z), \ldots, f_k(z); z) = zF_{k+1,k+1}(z)$.

Definition 3. Let f_i be as in Definition 1, G as in Definition 2, and $y \in \mathbb{C}$. Then the function $HW(\{f_i\}_{i\in I}; y)$ is the function which satisfies the following equation:

$$G(\{f_i\}_{i\in I}; HW(\{f_i\}_{i\in I}; y)) = y. \tag{15}$$

Actually, Definition 1 defines a stack of exponents composed of $n-1$ functions from the index set I, namely the functions from $f_{m-(n-1)}(z)$ to $f_{m-1}(z)$.

For instance, let $m = 5, n = 3$ with index set

$$I = \{f_1(z), f_2(z), f_3(z), f_4(z), f_5(z), \ldots\}.$$

Then

$$F_{3,5} = e^{f_3(z)e^{f_4(z)e^{z}}}.$$

Based on the general Definition 1, Definition 2 describes a function which "stacks" a given index set I containing k functions while the whole stack is also multiplied by the variable z. For instance, for $k = 3$ we have the following:

$$G(f_1(z), f_2(z), f_3(z); z) = zF_{4,4} = ze^{f_1(z)e^{f_2(z)e^{f_3(z)e^{z}}}}.$$

Finally, Definition 3 defines the generalization of the Lambert function, the Hyper Lambert function through a functional equation. Observe that Equation (15) reduces to the definition of the Lambert function for $k = 1$ and $f_1(z) = \frac{z}{e^z}$.

We will, now, solve (13) using the class of Hyper Lambert functions, as defined by (15) in Definition 3. Setting $f(z) = \frac{\Delta}{e^z}$ and using the functions defined in Definition 3 we conclude that the solution to (13) is given by

$$z(t) = HW(\frac{\Delta_z - 1}{e^z}; -e^{-1}[(1-t)e^{-C}]^{\frac{(\lceil\frac{k}{2}\rceil^{k-1} - 1)}{A_{y,1}}}). \tag{16}$$

Since $z(t) = \frac{1}{y(t)} - 1$ and, thus, $y(t) = \frac{1}{z(t)+1}$, we conclude that

$$y(t) = \frac{1}{HW(\frac{\Delta_z - 1}{e^z}; -e^{-1}[(1-t)e^{-C}]^{\frac{(\lceil\frac{k}{2}\rceil^{k-1} - 1)}{A_{y,1}}}) + 1}. \tag{17}$$

3 Closed-form Solutions of Resource Allocation Problems in Wireless Networks

Many emerging wireless applications require the involvement of a large number of small and low cost wireless devices distributed over a wide geographic area. However, the deployment of such applications is obstructed by hostile environments as well as the existing resource constraints of the devices, such as limited battery power and shortage of communication bandwidth, which is shared among large numbers of devices. These obstructions can be resolved if the users would agree to share their local resources and cooperate for the transmission of their messages. This is the basic idea of the *cooperative communications* concept which has become very popular in wireless communications as a means of overcoming resource constraints.

Under the various resource constraints of different network topologies, cooperation protocols require certain optimization procedures for managing the allocation of a resource (e.g. bandwidth). These optimization procedures lead to certain algebraic expressions that need to be optimized. Usually, the optimization solution is obtained through numerical methods rather than through closed-form solutions, which are much harder to derive due to the complexity of the expressions. In [9], however, the studied optimization problem for resource allocation on various topologies was amenable to a closed-form solution involving the Lambert-W function. In this section we present how the Lambert-W function appears in the obtained closed-form solutions for two problems (see [9]): (i) the subcarrier allocation problem in Wireless Mesh Networks, and (ii) the power allocation in relay networks.

In what follows, we focus on these two problems and their solutions involving the Lambert-W function.

3.1 The Subcarrier Allocation problem in Wireless Mesh Networks

A *Wireless Mesh Network* (WMN) (initially deployed for military services) is a type of wireless adhoc network where each node is capable of relaying messages for other nodes in the network. The coverage area of the network is the coverage area of the nodes which, thus, act as a single network. Actually, each node relays messages using routing protocols which means that a transmitted message can reach its destination through a multi-hop path. If the nodes can, also, move during their lifetime, then the WMN can be considered as a *Mobile adhoc Network* (MANET). Typically, in such a network, there are three types of devices: the mesh clients (MCs), the mesh routers (MRs) and the gateways which may, also, be connected to the internet.

The authors in [9] considered a WMN consisting of one MR, which is also a gateway, and M MCs. The network system consists of S subcarriers where each subcarrier has a bandwidth B. If s_j is the number of subcarriers assigned to the j-th MC, where $1 \le j \le M$, then the following constraint holds:

$$\sum_{j=1}^{M} s_j \le S. \tag{18}$$

It is also assumed that the MR knows only the average channel gain of all outgoing links at the j-th MC, denoted by G_j. An MC j has a minimum transmission rate R_j and a limit p_j on the transmission power which is uniformly distributed to the s_j allocated subcarriers, i.e. the transmission power per link is $\frac{p_j}{s_j}$. If σ_n^2 is the power of the thermal noise and Γ is the Signal to Noise Ratio (SNR) gap related to the required Bit Error Rate (BER) then the MR determines for every MC the approximate rate by the equation ([9])

$$r_j(s_j) = s_j\, B\, log_2(1 + d_j), \tag{19}$$

where $d_j = \frac{a_j}{s_j}$ and $a_j = \frac{G_j p_j}{\Gamma \sigma_n^2}$. In addition to the constraint (18), the authors added to their subcarrier allocation model the constraint of the minimum rate. This is used for OFDMA-based WMNs and it is given by

$$r_j(s_j) \ge R_j. \tag{20}$$

Thus, the subcarrier allocation problem for an MR, as it is formulated in ([9]), consists in the objective function

$$\max_{s_j} \sum_{j=1}^{M} r_j(s_j), \tag{21}$$

subject to constraints (18) and (20).

Based on the *Lagrange dual approach* ([3]), the authors in [9] provided the Lagrangian of the above optimization problem as

$$L(s_j, u, v) = \sum_{j=1}^{M} r_j(s_j) + \sum_{j=1}^{M} u_j \left(r_j(s_j) - R_j\right) + v \left(\sum_{j=1}^{M} s_j - N\right), \tag{22}$$

where $u = (u_1, \ldots, u_M)$ while the Lagrange dual function is

$$g(u, v) = \max_{s_j} L(s_j, u, v). \tag{23}$$

Subsequently, for fixed (u, v), the derivative of $L(s_j, u, v)$ with respect to s_j is taken and set to zero

$$\ln\left(\frac{a_j + s_j}{s_j}\right) - \frac{a_j}{a_j + s_j} - \frac{v}{\frac{B}{\ln 2}(1 + u_j)} = 0. \tag{24}$$

By setting $q(u_j, v) = \frac{v}{\frac{B}{\ln 2}(1+u_j)}$ and $z = \frac{a_j+s_j}{s_j}$, Equation (24) becomes

$$\ln(z) + z^{-1} + (-1 - q(u_j, v)) = 0. \tag{25}$$

After some algebraic manipulation of Equation (25), the following equation is obtained:

$$-z^{-1}\, e^{-z^{-1}} = -e^{-1-q(u_j,v)}. \tag{26}$$

It is obvious that the left-hand side of Equation (26) has the form of $f(z) = ze^z$, where its inverse function is the Lambert-W function. Thus, it can be written as

$$-z^{-1} = W\left(-e^{-1-q(u_j,v)}\right). \tag{27}$$

Substituting z in Equation (27) and solving for s_j, the optimal value for the number of subcarriers for the j-th MC is given by

$$s_j^* = \frac{-a_j W(-e^{(-1-q(u_j,v))})}{1 + W(-e^{(-1-q(u_j,v))})}. \tag{28}$$

It remains to derive the optimal values u_j^* and v^*. These values are derived by substituting s_j with s_j^* in Equation (22) and then solving the dual problem, which is now $\min_{u_j,v} L(u_j, v)$. The problem is solved in [9] by differentiating $L(u_j, v)$ (using the first derivative of the Lambert-W function given by Equation (2)), and setting the first derivative to zero. This leads to the following equation:

$$f(u_j^*) = \frac{R_j}{a_j}, \tag{29}$$

where $f(u_j^*) = rw_j\, h(u_j^*)$ with $rw_j = \frac{w_j^*}{1+w_j^*}$, $w_j^* = W(-e^{1-q(u_j^*,v)})$ and

$$h(u_j^*) = \left(\frac{B}{\ln 2} + \frac{v}{(1+u_j^*)(1+w_j^*)^2}\right)\ln(-w_j^*) + \\ + \frac{v}{(1+u_j^*)(1+w_j^*)} + \frac{v^2}{(1+u_j^*)^2(1+w_j^*)^2}.$$

Subsequently, the optimal u_j^* is given by

$$u_j^* = f^{-1}\left(\frac{R_j}{a_j}\right). \tag{30}$$

since the inverse function of f can be proved that it exists.

Having found the optimal value u_j^*, the optimal value for v^* is found by substituting u_j with u_j^* in Equation (28) and using constraint (18). Specifically, the following equation is obtained:

$$\sum_{j=1}^{M} \frac{-a_j W\left(-e^{(-1-q(u_j^*,v))}\right)}{1 + W\left(-e^{(-1-q(u_j^*,v))}\right)} = S. \tag{31}$$

If $t(v) = \sum_{j=1}^{M} \frac{-a_j W\left(-e^{(-1-q(u_j^*,v))}\right)}{1+W\left(-e^{(-1-q(u_j^*,v))}\right)}$, then the optimal value v^* is given by

$$v^* = t^{-1}(S) \tag{32}$$

since in [9] it was proved that the inverse function of t exists.

3.2 *Power Allocation problem in Relay Networks*

A *relay network* is a network topology widely used by wireless networks. Its infrastructure consists of relay stations (RS), operated by a service provider, that can support multihop communication protocols by relaying messages to stations, which can be either relay stations or mobile stations, on behalf of other stations (see Figure 1).

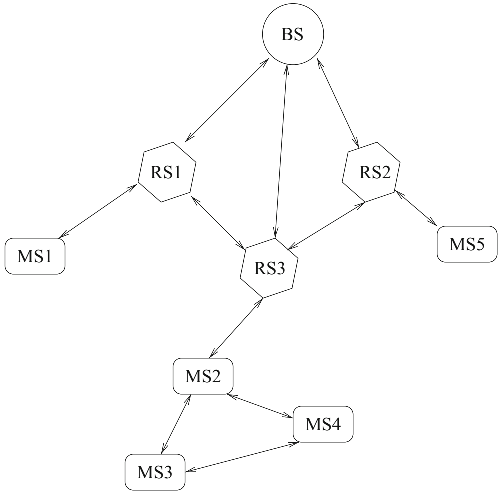


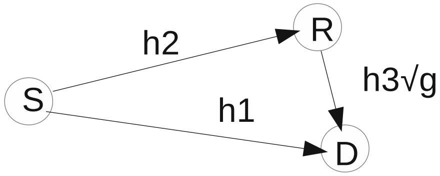
Fig. 1 A simple Relay Network

Relay Stations are not directly connected to wired network infrastructure. Instead, they communicate only to base stations (BS) which may, in turn, be connected to fixed networks. For formulating the power allocation problem, it is assumed that a station that relays messages can transmit and receive, simultaneously, on the same allocated channel. Cooperative communication protocols on relay network topologies share network resources among users based on the channel state information (CSI) at each network node.

There are two relay network operating modes, the *receiver cooperation mode* where the relay station is near to the receiver and the *transmitter cooperation mode*, where the relay station is near to the transmitter. In [9], the receiver cooperation mode is used for the formulation of the problem. In this case, the receiver has accurate knowledge of CSI but the transmitter does not have such knowledge because of the absence of a fast CSI feedback by the receiver. Thus, the authors in [9] use the *ergodic capacity* (or Shannon Capacity, see in [10]) in order to characterize the transmission rate of the channel.

Figure 2 presents the concept of the receiver cooperation mode, where h_1, h_2 and h_3 are the gains of the respective channels. These are independent identically distributed and normalized random variables that have unit variance. Thus, the channel power gain is characterized by independent identically distributed, exponential random variables with unit mean. In the mode of receiver cooperation, $g = d^{-a}$ is the average channel power gain between the receiver and the relay station, where d is their distance and a is a path-loss attenuation exponent. In the receiver cooperation mode, we assume the existence of independent, identically distributed, zero mean circularly symmetric, complex Gaussian additive noise random variables with unit variance z and z_1 in the receiver and the relay station respectively. Thus, if x and x_1 are the messages transmitted by the transmitter and the relay station respectively,

Fig. 2 Receiver Cooperation Mode



then the received messages y_1 and y at the relay station and the receiver respectively can be expressed by the following expressions:

$$\begin{aligned} y_1 &= h_2 x + z_1 \\ y &= h_1 x + \sqrt{g} h_3 x_1 + z \end{aligned} \tag{33}$$

The total power P is allocated between the transmitter and the relay station which means that $E[|x|^2] \leq b\, P$ and $E[|x_1|^2] \leq (1-b)\, P$, where b is a parameter to be optimized based on the partial knowledge of CSI and g. Assuming $P \gg 1$, the authors in [9] provide an upper bound to the ergodic capacity $C_{erg}(b)$ given by

$$C_{erg} = \max_{0 \leq b \leq 1} \min(A(b), D(b)) \tag{34}$$

where $A(b) = E[\log_2(bP(g_1 + g_2))]$, $D(b) = E[\log_2(bPg_1 + (1-b)gPg_3))]$ and $g_i = |h_i|^2$.

The two terms in the min function, which is the objective function, can be written as

$$A(b) = \log_2(P) + \log_2(b) + \log_2(e^{1-\gamma}) \tag{35}$$

$$D(b) = \log_2(P) + \frac{g(1-b)\log_2(g(1-b)) - b\log_2(b)}{g(1-b) - b} - \log_2(e^{\gamma}) \tag{36}$$

where γ is Euler's constant. Since A is an increasing function of b and D is a decreasing function of b, the optimal value b^* for the maximization problem can be given by solving equation $A(b) = D(b)$ for b. By equating $A(b)$ and $D(b)$, the following equation is derived:

$$\ln(b^*) + 1 = \frac{g(1-b^*)\ln(g(1-b^*)) - b^* \ln(b^*)}{g(1-b^*) - b^*}. \tag{37}$$

Equation (37) can be rewritten as,

$$\frac{-b^*}{g(1-b^*)} e^{\frac{-b^*}{g(1-b^*)}} = -\frac{1}{e}. \tag{38}$$

This is in the form of $f(z) = ze^z$ and, consequently, leads to the following equation:

$$W(-\frac{1}{e}) = \frac{-b^*}{g(1-b^*)}. \tag{39}$$

Thus, the optimal value for b is given by

$$b^* = \frac{W(-\frac{1}{e})}{W(-\frac{1}{e})g - 1}. \tag{40}$$

4 Primes Distribution Upper and Lower Bounds

In [11], the author used the Lambert-W function to study the problem of estimating the distribution of prime numbers from a new perspective. More specifically, the author presented some new applications of the Lambert-W function on the problem of deriving bounds for the prime number counting function $\pi(n)$ and the function giving the n-th prime number p_n.

It is known that the n-th prime, p_n, is greater than $n\ln(n)$ i.e. $p_n > n\ln(n)$ ([12]) and that $n \geq p_{\pi(n)}$. Consequently, it holds

$$n \geq p_{\pi(n)} > \pi(n)\ln(\pi(n)). \tag{41}$$

Since $\pi(n)\ln(\pi(n))$ is monotonically increasing function of n, for $n \geq 2$ it holds

$$\frac{n}{\pi(n)} > \ln(\pi(n)) \Rightarrow e^{\frac{n}{\pi(n)}} > \pi(n) \Rightarrow e^{\frac{n}{\pi(n)}}\frac{n}{\pi(n)} > n. \tag{42}$$

Inequality (42) has the form $y < f(z) = ze^z$, where $y = n$ and $z = \frac{n}{\pi(n)}$. Based on the fact that the Lambert-W function is monotonically increasing for $n \geq 2$, the above inequality can be written as follows:

$$W(n) < \frac{n}{\pi(n)} \Rightarrow \pi(n) < \frac{n}{W(n)}. \tag{43}$$

The Lambert-W function satisfies the equality $W(n)e^{W(n)} = n$ and, consequently, Inequality (43) can lead to the following upper bound for the function $\pi(n)$:

$$\pi(n) < e^{W(n)}. \tag{44}$$

With respect to deriving a lower bound for the function $\pi(n)$, the author in [11] used the following inequalities:

$$\begin{aligned} \ln(n) &\leq \frac{n^\epsilon}{\epsilon\, e}\ \forall\epsilon > 0, \\ p_n &< n\ln(n\ln(n))\ \forall n \geq 6. \end{aligned} \tag{45}$$

Since the function $n\ln(n\ln(n))$ is monotonically increasing in $n \in (1,\infty)$, $n < p_{\pi(n)+1}$, and based on Inequalities (45), the following inequality is derived:

$$n < (\pi(n)+1)\ln\left(\frac{(\pi(n)+1)^{1+\epsilon}}{\epsilon\, e}\right)$$
$$< (\pi(n)+1)(1+\epsilon)\ln\left(\frac{(\pi(n)+1)}{(\epsilon\, e)^{\frac{1}{1+\epsilon}}}\right). \tag{46}$$

Inequality (46) can be rewritten so as to obtain an inequality in which a function of the form $f(z) = ze^z$ is on its one side. We, first, obtain the inequality:

$$\frac{n}{(\pi(n)+1)}\frac{1}{1+\epsilon} < \ln\left((\pi(n)+1)\,(\epsilon\, e)^{-\frac{1}{1+\epsilon}}\right) \tag{47}$$

which gives

$$e^{\frac{n}{(\pi(n)+1)}\frac{1}{(1+\epsilon)}} < (\pi(n)+1)\,(\epsilon\, e)^{-\frac{1}{1+\epsilon}}. \tag{48}$$

By multiplying both sides of Inequality (48) with $\frac{n}{\pi(n)+1}\frac{1}{1+\epsilon}$, an inequality where a function of the form ze^z appears on left side is derived:

$$\frac{n}{(\pi(n)+1)}\frac{1}{(1+\epsilon)}e^{\frac{n}{(\pi(n)+1)}\frac{1}{(1+\epsilon)}} < \frac{n}{1+\epsilon}\,(\epsilon\, e)^{-\frac{1}{1+\epsilon}}. \tag{49}$$

Thus, a lower bound for $\pi(n)$ can be obtained using the Lambert W function, as follows:

$$\frac{n}{(\pi(n)+1)}\frac{1}{(1+\epsilon)} < W\left(\frac{n}{1+\epsilon}\,(\epsilon\, e)^{-\frac{1}{1+\epsilon}}\right) \tag{50}$$

$$\Rightarrow \frac{\frac{n}{(1+\epsilon)}}{W\left(\frac{n}{1+\epsilon}\,(\epsilon\, e)^{-\frac{1}{1+\epsilon}}\right)} - 1 < \pi(n). \tag{51}$$

In [11], upper and lower bounds for $\pi(n)$ were given for various values of ϵ, using the closed forms of (51) and (44). Moreover, in [11] lower and upper bounds for p_n were given which could be derived similarly using the inequality $n > \frac{p_n}{\ln(p_n)}$. More specifically, the following lower and upper bounds for p_n were given:

$$-(n-1)\,W_{-1}\left(-\frac{e^{\frac{3}{2}}}{n-1}\right) < p_n < -n\,W_{-1}\left(-\frac{1}{n}\right). \tag{52}$$

Here, $W_{-1}(x)$ is the real branch of the Lambert W function defined on $x \in [-\frac{1}{e}, 0)$. The lower bound holds for $n \geq 14$ while the upper bound holds for $n \geq 4$.

In addition to the above lower and upper bounds for the prime's distribution, another interesting result where the Lambert-W function is involved appears in [13] (Chapter 3). There, a closed-form estimate is presented for a lower bound on the time complexity of the Quadratic Sieve algorithm, using the Lambert-W function. Let n be an odd composite integer which is not a power of any natural number. The Quadratic Sieve algorithm provides nontrivial prime factors of n and it is based on locating $\pi(B)$ B-smooth integers in the range $[1,\ldots,X]$, where $X = 2n^{\frac{1}{2}+\epsilon}$ and $B = X^{\frac{1}{u}}$ for $u \geq 1$. An integer m is a B-smooth integer if all its prime factors are less than or equal to B. Since $\ln(\ln(B))$ steps are required to test an integer in $m \in [1,\ldots,X]$ whether it is B-smooth or not, the total time for collecting $\pi(B)$ B-smooth integers is given by

$$t(X,B,u) = \pi(B)\ln(\ln(B))\ \frac{X}{\psi(X,X^{\frac{1}{u}})}, \tag{53}$$

where $\psi(X,X^{\frac{1}{u}}) = |\{m|m \in [1,\ldots,X]\ and\ X^{\frac{1}{u}} - smooth\}|$. By setting $\pi(B)\ln(\ln(B)) = X^{\frac{1}{u}}$ and $\frac{X}{\psi(X,X^{\frac{1}{u}})} = u^u$ the following equation is obtained:

$$\bar{t}(X,u) = X^{\frac{1}{u}}\ u^u. \tag{54}$$

The optimal performance of the algorithm can be achieved by finding a u that minimizes the objective function $\bar{t}(X,u)$. This is equivalent to finding a u that minimizes the logarithm of the objective function, which is given by

$$\ln(\bar{t}(X,u)) = \frac{1}{u}\ln(X) + u\ln(u). \tag{55}$$

The first order derivative of $\ln(\bar{t}(X,u))$ is given by

$$\frac{d\ \ln(\bar{t}(X,u))}{du} = -\frac{1}{u^2}\ln(X) + \ln(u) + 1. \tag{56}$$

By setting the above first order derivative of $\ln(\bar{t}(X,u))$ to zero we obtain

$$\begin{aligned}
\ln(X) = u^2(\ln(u)+1) &\Rightarrow \ln(X) = e^{2\ln(u)}\ln(u\,e)\\
&\Rightarrow \ln(X)\,e^2 = e^{2\ln(u)+2}\ln(u\,e)\\
&\Rightarrow \ln(X)\,e^2 = e^{2\ln(u\,e)}\ln(u\,e)\\
&\Rightarrow 2\ln(X)\,e^2 = 2\ln(u\,e)\,e^{2\ln(u\,e)}.
\end{aligned} \tag{57}$$

The Equation (57) is of the form $z = ve^v$ where $z = 2\ln(X)\,e^2$ and $v = 2\ln(u\,e)$. Consequently, we can solve it for u as follows:

$$
\begin{aligned}
W\left(2\ln(X)\, e^2\right) = 2\ln(u\, e) &\Rightarrow \frac{1}{2}W\left(2\ln(X)\, e^2\right) = \ln(u\, e) \\
&\Rightarrow \frac{1}{2}W\left(2\ln(X)\, e^2\right) = \ln(u) + 1 \\
&\Rightarrow \frac{1}{2}W\left(2\ln(X)\, e^2\right) - 1 = \ln(u) \\
&\Rightarrow u = e^{\frac{1}{2}W\left(2\ln(X)\, e^2\right)-1}. \qquad (58)
\end{aligned}
$$

Thus, the right-hand side of Equation (58) provides an expression based on the Lambert-W function for the calculation of the optimal value of u.

5 A Solution of a System of DEs arising in a Glucose-Insulin Dynamic System

Time-Delay Systems (TDS) occur often in engineering, biology, chemistry, physics, and ecology (see [16]). They can be represented by *Delay Differential Equations* (DDEs) which have been extensively studied over the past decades [19]. Time delays in systems can limit and degrade their performance while they may induce instability. They actually lead to an infinite number of roots of the characteristic equation, making systems difficult to be analyzed with classical methods, especially in checking controller stability (see [16, 18]). DDEs are a type of differential equations where the time derivatives, at the current time instance, depend on the solution, and possibly its derivatives, at previous time instances [21].

An analytic approach to obtain a complete solution of a system of DDEs based on the Lambert-W function was presented in [20]. The solution has an analytical form expressed in terms of the parameters of the DDEs. The advantage of this approach lies in the fact that the form of the obtained solution of a system of DDEs is analogous to the general solution form of a system of ODEs. This approach was applied in [15] to a system of DDEs that represents the glucose-insulin dynamics presented in [14]. The glucose-insulin dynamic system can be represented by the following differential equations ([15]):

$$
\begin{aligned}
\frac{dG(t)}{dt} &= -b_1 G(t) - \frac{b_4 I(t) G(t)}{\alpha G(t) + 1} + b_7 \\
\frac{dI(t)}{dt} &= -b_2 I(t) + b_6 G(t-\tau). \qquad (59)
\end{aligned}
$$

where $G(t)$ and $I(t)$ are the glucose and insulin concentrations, respectively, in blood at time t with initial conditions $G(t) \equiv G(0)$ for all t and $I(t) \equiv I(0)$ for all $t \epsilon [-\tau, 0]$, with τ denoting the time after which the pancreas responds to changes of glucose in blood. The parameters α and b_i for $i = 1, \ldots, 7$ are described in [15].

If (G^*, I^*) is the interior equilibrium point, then by substituting $G(t)$ and $I(t)$ with $u_1(t) + G^*$ and $u_2(t) + I^*$ respectively in Equations (59), the following linear time-delay system is obtained:

$$\frac{du_1(t)}{dt} = -\left(b_1 + \frac{b_4 I^*}{(aG^* + 1)^2}\right) u_1(t) - \frac{b_4 G^*}{aG^* + 1} u_2(t)$$
$$\frac{du_2(t)}{dt} = -b_2 u_2(t) + b_6 u_1(t - \tau) \tag{60}$$

The coefficients of the linear system (60) are functions of the dynamic system parameters and are organized in matrices as follows:

$$A = -\begin{bmatrix} b_1 + \frac{b_4 I^*}{(aG^*+1)^2} & \frac{b_4 G^*}{aG^*+1} \\ 0 & b_2 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ b_6 & 0 \end{bmatrix}.$$

Let $y(t) = \{G(t), I(t)\}^T$ be the 2×1 state vector. The glucose-insulin linear dynamic system can be expressed in the state space as follows:

$$\frac{dy(t)}{dt} = Ay(t) + By(t - \tau), \; t > 0$$
$$y(t) = y(0), \; t = 0$$
$$y(t) = g(t), \; t \epsilon [-\tau, 0). \tag{61}$$

Instead of a simple initial condition, as in ODEs, two initial conditions need to be specified for DDEs: a preshape function $g(t)$ for $-\tau \le t < 0$, where τ denotes the time-delay, and an initial state, $y(0)$, at $t = 0$. Function $g(t)$ actually determines the system's behavior before the pancreas reaction. Let a solution of (61) be of the form

$$y(t) = e^{St} C \tag{62}$$

where S is a 2×2 matrix and C is a 2×1 constant vector (see [20]). By substituting $y(t)$ with $e^{St}C$ into (61) the following equation is obtained:

$$Se^{St}C - Ae^{St}C - Be^{S(t-\tau)}C = (S - A - Be^{-S\tau})e^{St}C = 0. \tag{63}$$

Because the matrix S is an inherent characteristic of the system, and independent of initial conditions, it follows that for any arbitrary initial condition and for every time instance t, the following equation holds:

$$S - A - Be^{(-S\tau)} = 0 \Rightarrow S - A = Be^{(-S\tau)}. \tag{64}$$

By multiplying both sides of Equation (64) with $\tau e^{(S-A)\tau}$, the following equation is obtained,

$$(S - A)\tau e^{(S-A)\tau} = B\tau e^{-A\tau}. \tag{65}$$

The function on the left-hand side of Equation (65) has the form $f(x) = xe^x$ where $x = (S-A)\tau$. Consequently, the Lambert-W function can be used in order to obtain a closed-form for S as follows:

$$(S-A)\tau = W(B\tau e^{-A\tau}). \tag{66}$$

Then the solution matrix, S, is obtained by solving (66) for S:

$$S = \frac{W(B\tau e^{-A\tau})}{\tau} + A. \tag{67}$$

The matrix Lambert-W function, $W_k(H_k)$, is complex valued, with a complex argument H_k, and it has an infinite number of branches $W_k(H_k)$, $k = 0, \pm 1, \pm 2, \ldots, \pm\infty$ (see [20, 22]). It satisfies the following condition:

$$W_k(H_k)e^{W_k(H_k)} = H_k. \tag{68}$$

Corresponding to each branch, k, of the Lambert-W function, denoted by W_k, there is a solution Q_k from

$$W_k(H_k)e^{W_k(H_k)+A\tau} = B\tau. \tag{69}$$

and for $H_k = B\tau Q_k$.

The Jordan canonical form $J_k = diag(J_{k_1}(\lambda_1), J_{k_2}(\lambda_2), \ldots, J_{k_p}(\lambda_p))$, where $J_{k_i}(\lambda_i)$ is a 2×2 Jordan block, is computed from $H_k = Z_k J_k {Z_k}^{-1}$. Thus, the matrix Lambert-W function can be computed as follows (see [15]):

$$W_k(H_k) = Z_k\, D\, {Z_k}^{-1} \tag{70}$$

where $D = diag(W(J_{k_1}(\lambda_1)), W(J_{k_2}(\lambda_2)), ..., W(J_{k_p}(\lambda_p)))$, and

$$W_k(J_{k_1}(\lambda_i)) = \begin{bmatrix} W_k(\lambda_i) & \frac{dW_k(\lambda_i)}{d\lambda_i} \\ 0 & W_k(\lambda_i) \end{bmatrix}. \tag{71}$$

The obtained Q_k can be substituted into (67) and the characteristic root S_k is obtained:

$$S_k = \frac{W_k(B\tau Q_k)}{\tau} + A. \tag{72}$$

Finally, by substituting S_k into (62), a homogeneous solution to (61) is obtained:

$$y(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{S_k t} \tag{73}$$

where the coefficient matrix C_k is determined numerically from the preshape function $g(t)$ and the initial state $y(0)$.

6 Conclusions

The Lambert-W function is remarkable in that, since its inception, it keeps appearing frequently in the solutions of diverse problems from, almost, every scientific field, from chemistry and physics to computer science and mechanical engineering.

The list of problems in whose solutions the Lambert-W function appears is, virtually, endless. The most authoritative survey paper on the Lambert-W paper (see [4]) as well as ours can only give a glimpse of the vast Lambert-W landscape in science. Thus, our survey paper took some of the latest problems involving the Lambert-W function as a departure point and attempted to identify patterns and techniques that have been, successfully, employed in transforming a given expression appearing in their solutions into one that resembles (after suitable substitutions) the definition of the Lambert-W function.

As we demonstrated in this survey paper, the general technique involves a sequence of steps that attempt to gather an unknown to one of the sides of an equality relation. Then, if the gathered expression has the general form (see [4]) $az^m e^{bz^n} = g(t)$, with m, n integers, a, b complex numbers, and $g(t)$ a complex function of t, a solution can be derived by suitable substitutions of expressions into the equation that defines the Lambert-W function.

We, also, saw that there may be problems in which expressions arise that may look like a Lambert-W definition which, however, defy a direct solution involving this function. Then a natural generalization of the Lambert function, called the *Hyper Lambert* functions may provide the solution, as we saw. Thus, a "closed-form" solution can be derived using these functions in cases where the Lambert-W function definition cannot be applied. We note that the Lambert-W function cannot be written in terms of elementary functions (see [4]) and, thus, its definition, if included in these functions, enriches the class of equations solvable in "closed-form" with the Hyper Lambert functions providing a broader class of such equations.

On the other hand, however, a more fundamental question arises beyond the technical issues involved in "unearthing" a Lambert-W like definition out of a given expression: "Why is the Lambert-W function ubiquitous? What makes it appearing in so many diverse application domains and problems?" Our intuition, which has been formed out of our research as well as after surveying relevant results in the bibliography, is that the definition of the Lambert-W function must, in fact, describe a generic dynamic process which may be inherent in a wide range of changing, with time, physical systems. It is not a coincidence that most cases where the Lambert-W function arises model a dynamic system, of some kind, that changes according to certain laws. And the expressions deriving the evolution of such changing processes appear to involve exponential functions which are the bases of the definition of the Lambert-W function.

A complete coverage of the applications of the Lambert-W function is rather impossible today since numerous applications have been studied whose solutions involve it. In this survey paper we concentrated on how one can manipulate a given expression and transform it into a form that resembles the definition of the Lambert-W function, hoping that this work will prove useful to other researchers, of any scientific domain, who work in the study of dynamic systems such as the ones we considered.

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A Computational Approach to the Unwrappings of the Developable Surfaces

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In Honor of Constantin Carathéodory

Abstract We present a closed-formula description for the unwrappings of the developable surfaces based on information obtained mainly by plane intersection curves on them. Among other applications, this description is suitable for computational implementation in CAD systems, for the construction of visual pictures of unwrappings onto planes, and for the computer graphic modeling of the approximations of real-object surfaces via the developable ones.

1 Motivation and Method

The surfaces that can be flattened onto a plane without any stretching or tearing are called developable. This means that they map isometrically onto some plane image by a kind of real-life unwrapping, and this makes them suitable candidates for using in many applications like computer graphics and manufacture, among others.

The developable surfaces have been widely used in engineering [6] and sheet metal work or texture mappings [2, 10] but less so in computer graphics or in CAD calculations. There have been proposed many techniques for modeling developable surfaces [1, 3, 5, 12, 13, 15] mainly via piecewise polynomial surface approximations, running so to the risk of violating the surfaces' isometric properties during flattenings due to the algebraic manipulation of the polynomials.

The developable surfaces are also ruled, meaning they are generated by lines called *rulings*. In this article, we deal with C^2 developable surfaces. These moreover

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admit a single tangent plane along any given ruling, and they can be flattened out at least locally on any chosen tangent plane in a real-like manner, the same way a wrapping is unwrapped from an ice cream cone and is put flat on a table. For this reason, we are going to call these plane images of the developable surfaces as unwrappings rather than as flattenings or developments.

We advocate a description for the unwrappings of these surfaces based on most cases on information extracted from plane intersection curves on them. This description is presented in Sect. 3 in a closed-formula form rather than as a numerical solution of a system of differential equations (as is done, e.g., in [11, 14] where cone approximations of object surfaces are considered). The exact equations derived here enhance a deeper understanding of the unwrappings and are also suitable enough for computational implementation and for use in applications like computer graphic modeling of real-life surfaces approximated by piecewise continuous developable surfaces. This description of the unwrappings is also suitable for use in a variety of applications based on CAD calculations such as the construction of visual illustrations of the unwrapped plane images of the developable surfaces (the so-called developments of these surfaces in descriptive geometry) and for the computation of areas of regions and lengths of curves in such surfaces.

Although there exists an infinity of distinct kinds of developable surfaces, under some very mild assumptions, they all consist of consecutive pieces, each being of just one of a handful of well-known types which can thus be thought of as building blocks for these surfaces. The analytic equations which we relate in Sect. 3 describe an unwrapping of some region of a regular point of such a developable surface depending on the type of the building block of the surface around the given point. These results arise from work done in [7] where the main goal was to establish a connection between the centers of curvature of a plane intersection curve and of its unwrapping at the point under consideration, and the proofs will not be repeated here.

First we need to clarify the meaning of an unwrapping as this is not an established notion in the literature. This is done in Sect. 2 where we also provide all other necessary related definitions and get better acquainted with the developable surfaces and their isometries.

2 Definitions and Preliminary Results

A *ruled surface* is a C^1 parametrized surface $F : x(u, v) = \alpha(u) + vw(u),\ u, v \in \mathbb{R}$ where $a(u)$ is a regular curve and the vector $w(u)$ is normalized to unit length $|w(u)| = 1, \forall u$. F is generated by a one-parameter family of lines $\ell_u : y(v) = \alpha(u) + vw(u)$ called the *rulings* of F. The curve $\alpha(u)$ does not completely determine by itself the positions of the generating lines in space, but it does so with the help of the vectors $w(u)$. In any case, this curve directs in a way the positions in space through which the generating lines pass, and so we call it the *directrix* of F. Although not necessary, the directrix is usually conveniently considered as a plane curve. This definition of a ruled surface is more or less standard in the literature and allows it to contain singular points, that is, points at which $x_u \times x_v = 0$. This way generalized cones and tangent surfaces of curves are not excluded from the class of ruled surfaces. A ruled surface is called a *generalized cone* whenever it is not (part of) a plane and all its rulings pass through a common point p called the vertex of F. Then the cone can be parametrized as $F : x(u, v) = p + v\alpha(u),\ u, v \in \mathbb{R}$ and contains p as a singular point. A ruled surface is called the *tangent surface* of a regular parametrized curve $y(s)$ whenever $y(s)$ is its directrix, and at each one of its points, the ruling is determined by the tangent vector at that point. We insist that s is an arc length parameter and demand that there exist no inflection points in the directrix. The tangent surface of $y(s)$ is parametrized as $F : x(s, v) = y(s) + vy'(s)$ and contains all points of its directrix usually as singular ones.

The subclass of the C^2, regular, ruled surfaces with a constant tangent plane along any ruling are the surfaces called *developable*. This definition is also more or less standard in the literature. Notice that the definitions of the ruled and of the developable surfaces permit self-intersections, and so generalized cones, tangent surfaces of curves, and generalized cylinders are not automatically excluded from the class of developable surfaces. Actually, the first two of them are indeed developable away from their singular points, as are for all their points the planes and the *generalized cylinders* which are defined as ruled surfaces $F : x(u, v) = y(u) + vg, |g| = 1$ with rulings of a fixed direction g.

We intend to work only with tame developable surfaces, which we propose to define as follows:

Definition 1. A developable surface $F : x(u, v) = \alpha(u) + vw(u), u \in I, v \in \mathbb{R}$ is called *tame* whenever there exists a partition of I so that the part of F corresponding to each subinterval of the partition is just a plane, a generalized cylinder, a generalized cone, or the tangent surface of a curve.

Restricting attention to tame developable surfaces is not a real loss since in [4] it is shown that all developable surfaces are tame provided some very mild assumptions hold: it suffices that the derivative of $w(u)$ and of the striction line of F have no clustering point for their zeros, where the striction line of F is the unique curve $\beta(u)$ of F so that $\beta' \cdot w' = 0, \forall u$ [4, 8], defined on those parts of F corresponding to the subintervals of I for which $w'(u) \neq 0$ throughout.

To proceed we need an exact definition for the unwrappings of a developable surface.

To begin with, let us observe that their parametrization $F : x(u, v)$ implies [4, 9] that any such surface has a Gauss curvature equal to 0, and then by Mindings Theorem [9] and assuming high enough differentiability (at least C^3), we know that all regular points A on the surface can map some neighborhood U of them isometrically to a plane (Fig. 1). In [8] this is proved for C^2 developable surfaces, and our results in the next section prove this fact anew. We can considerably improve this isometry f by making it map the neighborhood U on the tangent plane T of F at A (a triviality) and more importantly so that it fixes the ruling through A. Indeed:

First compose with a translation τ of T so that A is mapped onto itself. The ruling ℓ through A is a usual line, which makes it a geodesic of F, and so it has to be mapped by τf onto a geodesic of T, that is again to a line. Now any point

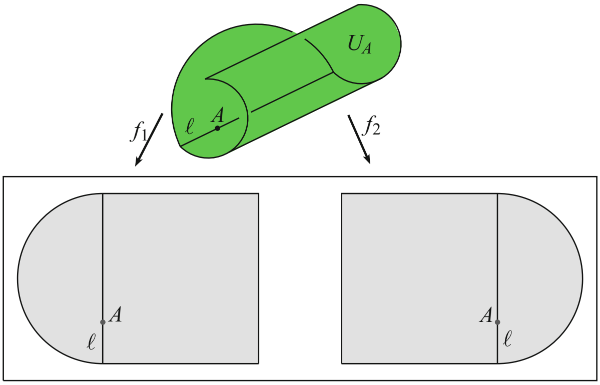

Fig. 1 f_1 and f_2 are two isometries of U_A into the same plane

on ℓ has to be mapped on an equally distant point from A on the image line $\tau f(\ell)$, thus composing if necessary with a rotation r of T the resulting isometry $r\tau f$ fixes pointwise ℓ as wanted.

The isometries of developable surfaces onto tangent planes that fix some ruling are usually referred to as *developments* in the literature.

A natural observation is that if f is any development of some neighborhood U around a point A of a developable surface F fixing the ruling ℓ, then its composition f_{sym} with the reflection on T with respect to ℓ is another development of U into T fixing ℓ. It is intuitively clear that the two mappings f, f_{sym} are the only two possible developments of U fixing ℓ and that only one of them looks like an "unwrapping" of U on the plane T. This one sends all nearby points of U on the two sides of ℓ to the "correct" half plane of T with respect to ℓ. We suggest that we express this rigorously as follows:

Definition 2. For an open, connected subset U of a developable surface F, we call *unwrapping* of U on the tangent plane T of the surface along a ruling ℓ, any development f, i.e., isometry (or isometric image if we wish) of U on T which fixes all points of ℓ inside U and for which the tangent vector of any curve $y(t)$ on U coincides with that of its image curve $f(y(t))$ at their common points $y(t_0) = f(y(t_0))$ on ℓ. For a curve γ on U, we call $f(\gamma)$ as the *unwrapping* of γ by f.

Some topological considerations can convince us [7] that there exist exactly two developments fixing a ruling (Fig. 2) and that exactly one of them is an unwrapping:

Proposition 1. *(a) For any regular point A on a C^2 tame developable surface F, there exists an open neighborhood U_A around A for which there exist exactly two developments of U_A on the tangent plane T of F fixing the ruling ℓ through A. The images of the two developments are symmetric with respect to ℓ, and exactly one of them is the unique unwrapping f of U_A on T along the ruling ℓ. (b) If V is another open neighborhood around A, then any unwrapping of V on T along the ruling ℓ coincides with that of U_A in an open neighborhood of A common to both U_A and V. (c) An isometry f from U_A to T which fixes ℓ pointwise is an unwrapping of U_A if and*

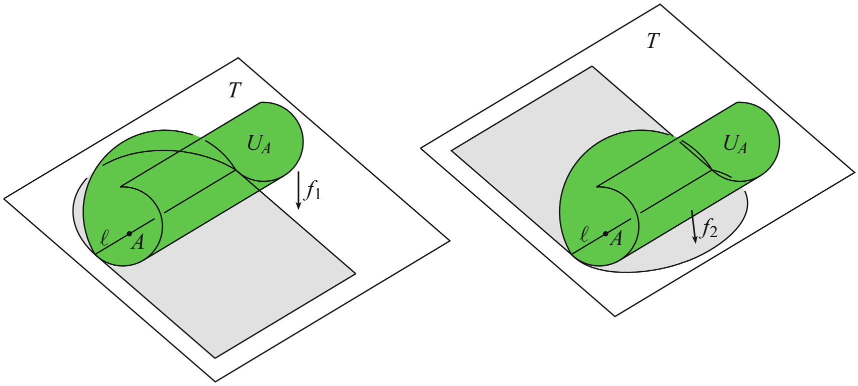

Fig. 2 f_1 and f_2 are the two developments of U_A on T along ℓ (fixing the ruling ℓ pointwise). f_1 is the unique unwrapping among them

only if the tangent vector of either parameter curve of $F : x(u, v) = \alpha(u) + vw(u)$ *coincides with that of its image curve at their common points on* ℓ.

3 A Synoptic Description of the Unwrappings

In this section we consider open neighborhoods U_A of regular points A on tame developable surfaces F, and given a parametrization of U_A, we provide an analytic expression of the unwrapping f of U_A on the tangent plane T of F which fixes the ruling ℓ through A.

First, recall that by Definition 1, each side of U_A with respect to ℓ is part of a plane, cylinder, cone, or the tangent plane of a curve. Nevertheless, it will be no harm to assume that the whole neighborhood U_A belongs entirely to the same plane, cylinder, cone, or the tangent surface of a curve. Since the first case is trivial, we shall not consider it any further. For the remaining three cases, the details are as follows:

3.1 Unwrapping Whenever U_A Is Part of the Tangent Surface of a Curve

As initial data let $x(s, v) = y(s) + vy'(s)$ be a given parametrization of U_A in some Cartesian coordinate system where s = arc length parameter of the regular curve $c : y(s)$, and let $\ell : y(s_0) + vy'(s_0)$ be the ruling of U_A through A.

Call $M = y(s_0)$ the point of c on ℓ. Denote differentiation with respect to s by primes and set $t(s) = y'(s)$ the unit tangent vector of c, $n(s)$ a choice for the principal unit normal vector of c.

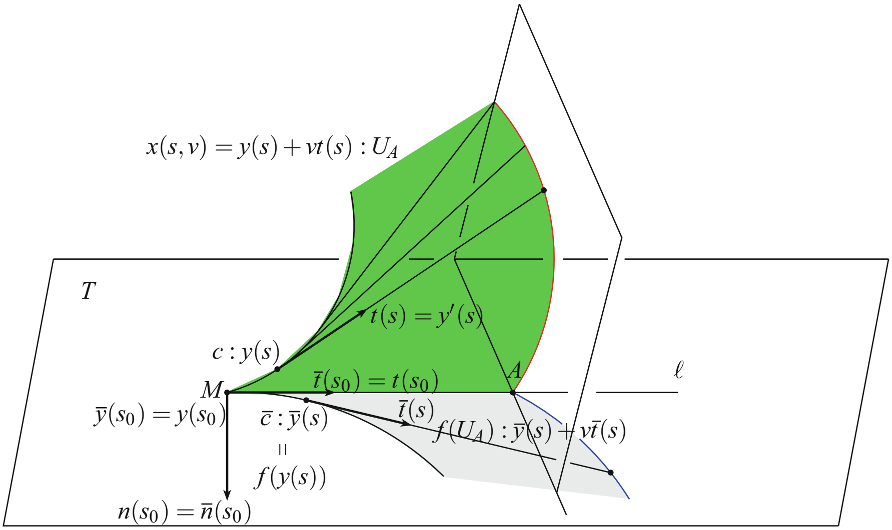

Fig. 3 Unwrapping of U_A whenever it is part of the tangent surface of a curve

On T consider the curve $\bar{c} : \bar{y}(s)$ for which s is a natural parameter, $\bar{y}(s_0) = y(s_0) = M$, $\bar{t}(s_0) = t(s_0)$ the tangent vector at s_0, $\bar{n}(s_0) = n(s_0)$ the principal normal vector at s_0, and $\bar{k}(s) = k(s)$ at any s, where $\bar{k}(s)$ denotes the signed curvature of $\bar{c}$.

Call $\bar{t}(s)$ = the unit vector of $\bar{c}$ at s and shrink U_A if necessary so that the v-parameter curve does not intersect ℓ in U_A for $s \neq s_0$ in some interval of values around s_0. Consider s close enough to s_0 so that $(\overline{y_1}', \overline{y_2}') \cdot (-\overline{y_2}'', \overline{y_1}'') \neq 0$ where $\bar{y} = (\overline{y_1}, \overline{y_2})$ (such an interval of values for s indeed exists), and if necessary shrink U_A so that this condition holds inside U_A. Then the unwrapping f of U_A on T along ℓ (Fig. 3) is given by

$$f(x(s,v)) = f\,(y(s) + vt(s)) = \bar{y}(s) + v\bar{t}(s).$$

3.2 Unwrapping Whenever U_A Is Part of a Generalized Cylinder

As initial data, let $x(u,v) = y(u) + vg$ be a given parametrization of U_A in some Cartesian coordinate system where u is not necessary an arc length parameter of the curve $c : y(u)$, but where $|g| = 1$, $A = y(u_0)$ for some $u_0 \in \mathbb{R}$, and let $\ell : y(u_0) + vg, v \in \mathbb{R}$ be the ruling of U_A through A.

Consider an arbitrary plane $\rho : a_1x_1 + a_2x_2 + a_3x_3 = a_0$ so that $A \in \rho$, $T \cap \rho \neq \ell$, $|(a_1, a_2, a_3)| = 1$ (thus, $e = (a_1, a_2, a_3)$ is a unit vector normal to ρ) and call $\gamma = U_A \cap \rho$ the section curve of U_A with ρ.

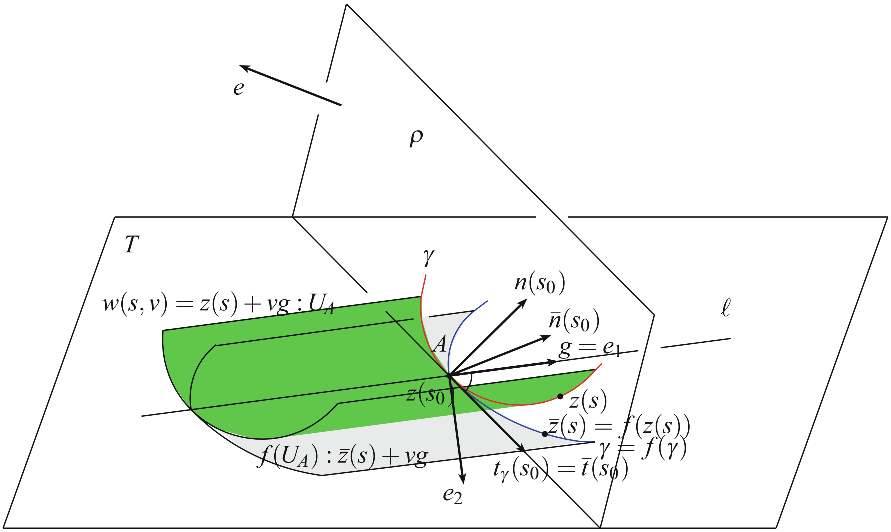

Fig. 4 Unwrapping of U_A whenever it is part of a generalized cylinder

Then u parametrizes γ as $\gamma : z(u) = y(u) + \dfrac{a_0 - y(u) \cdot e}{g \cdot e} g$. For some interval of values of u around u_0, it is $\frac{dz}{du} \neq 0$ and shrinking U_A if necessary so that $\frac{dz}{du} \neq 0$ for all u for which $z(u) \in U_A$ reparametrize γ as $\gamma : z(s)$ by arc length s. Now reparametrize U_A as $U_A : w(s, v) = z(s) + vg$ and let $A = z(s_0)$ and $\ell : z(s_0) + vg$ be the ruling of U_A through A.

Denote differentiation with respect to s by primes and set $t_\gamma = z'$ the unit tangent vector of γ. Introduce a Cartesian coordinate system Ax_1x_2 on T with $e_1 = g$ as the unit vector along the positive semiaxis Ax_1, and choose arbitrarily one of the two unit normal vectors to e_1 as the second unit vector e_2 of Ax_1x_2.

Consider the angle $\phi(s) = \angle(g, t_\gamma(s))$ and the curve $\overline{\gamma} : \overline{z}(s) = \int_{s_0}^{s} (\cos\phi(\sigma)e_1 + \sin\phi(\sigma)e_2)d\sigma + z(s_0)$ on T, and call $\overline{t}$ = the unit tangent vector of $\overline{\gamma}$. The vector $t_\gamma(s_0)$ is equal to $\overline{t}(s_0)$ or to its symmetric with respect to the line ℓ, depending on the choice of e_2. If necessary, change the direction of e_2 so that $\overline{t}(s_0) = t_\gamma(s_0)$.

Consider s close enough to s_0 so that $g \cdot (-\overline{z_2}', \overline{z_1}') \neq 0$ where $\overline{z} = (\overline{z_1}, \overline{z_2})$ (such an interval of values for s indeed exists), and if necessary shrink U_A so that this condition holds inside U_A. Then the unwrapping f of U_A on T along ℓ (Fig. 4) is given by

$$f(w(s, v)) = f\,(z(s) + vg) = \overline{z}(s) + vg.$$

3.3 Unwrapping Whenever U_A Is Part of a Generalized Cone

As initial data, let $x(u, v) = vy(u)$ be a given parametrization of U_A in a Cartesian coordinate system with origin at the vertex of the cone where u is not necessary an arc length parameter of the curve $c : y(u)$, $A = v_0 y(u_0)$ for some $u_0 \in \mathbb{R}$, $v_0 \in \mathbb{R}^*$ and let $\ell : vy(u_0)$, $v \in \mathbb{R}$ be the ruling of U_A through A.

Consider an arbitrary plane $\rho : a_1x_1 + a_2x_2 + a_3x_3 = a_0$ so that $A \in \rho$, $T \cap \rho \neq \ell$, $|(a_1, a_2, a_3)| = 1$ (thus, $e = (a_1, a_2, a_3)$ is a unit vector normal to ρ) and call $\gamma = U_A \cap \rho$ the section curve of U_A with ρ.

For some interval of values u around u_0, it is $y(u)\cdot e \neq 0$. Shrink U_A (if necessary) so that this relation holds for all u for which $z(u) \in U_A$ and parametrize γ by u as $\gamma : z(u) = \dfrac{a_0}{y(u)\cdot e} y(u)$. Reparametrize U_A as $U_A : h(u, v) = vz(u)$ and let $A = z(u_0)$.

For some interval of values of u around u_0, it is $\frac{dz}{du} \neq 0$ and shrinking U_A (if necessary) so that $\frac{dz}{du} \neq 0$ for all u for which $z(u) \in U_A$) reparametrize γ by arc length as $\gamma : z(s)$. Now reparametrize U_A as $U_A : w(s, v) = vz(s)$, and let $A = z(s_0)$, $\ell : vz(s_0)$ the ruling of U_A through A.

Denote differentiation with respect to s by primes and set $t_\gamma = z'$ the unit tangent vector of γ. Introduce a Cartesian coordinate system Ox_1x_2 on T with origin the vertex O of the cone, $e_1 = \dfrac{z(s_0)}{|z(s_0)|}$ as the unit vector of the positive semiaxis Ox_1, and choose arbitrarily one of the two unit normal vectors to e_1 as the unit vector e_2 of the positive semiaxis Ox_2.

Consider the angle $\theta(s) = \angle(z(s), t_\gamma(s))$. The vector $z'(s_0)$ is equal to $\cos(\theta(s_0))e_1 + \sin(\theta(s_0))e_2$ or to $\cos(\theta(s_0))e_1 - \sin(\theta(s_0))e_2$ depending on the choice of e_2. If necessary, change the direction of e_2 so that $z'(s_0) = \cos(\theta(s_0))e_1 + \sin(\theta(s_0))e_2$.

Consider s close enough to s_0 so that $\dfrac{|z(s_0)|}{2} < |z(s)|$ and $|\sin(\theta(z))| < \dfrac{3\pi|y(z_0)|}{2}$ (such an interval of values for s indeed exists), and if necessary shrink U_A so that these conditions hold inside U_A. Then the unwrapping f of U_A on T along ℓ (Fig. 5) is given by

$$f(w(s, v)) = v|z(s)| \left(\cos\left(\int_{s_0}^{s} \frac{\sin(\theta)}{|z(s)|} d\sigma \right) e_1 + \sin\left(\int_{s_0}^{s} \frac{\sin(\theta)}{|z(s)|} d\sigma \right) e_2 \right).$$

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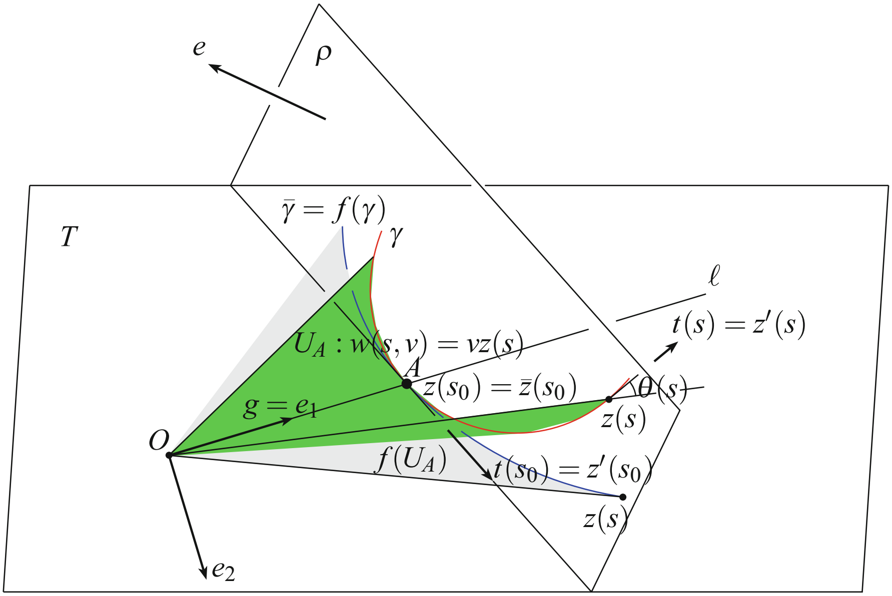

Fig. 5 Unwrapping of U_A whenever it is part of a generalized cone

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Multiple Weighted Orlicz Spaces and Applications

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In Honor of Constantin Carathéodory

Abstract In 2014, author Kuang (Handbook of Functional Equations: Functional Inequalities, vol. 95, pp. 273–280. Springer, Berlin, 2014) introduced the new multiple weighted Orlicz spaces; they are generalizations of the variable exponent Lebesgue spaces. In this paper, we consider further definitions and properties of these spaces and establish some new interest inequalities on these new spaces. They are significant generalizations of many known results.

1 Introduction

Throughout this paper, we write

$$\|f\|_{p,\omega} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p \omega(x) dx \right)^{1/p},$$

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) : x_k \geq 0, 1 \leq k \leq n\},$$

$$L^p(\omega) = \{f : f \text{ is measurable, and} \|f\|_{p,\omega} < \infty\}; \|x\| = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2}.$$

If the measurable function $p : \mathbb{R}^n \to [1, \infty)$ as exponential function, by $L^{p(\cdot)}(\mathbb{R}^n)$, we denote the Banach function space of the measurable function $f : \mathbb{R}^n \to \mathbb{R}^1$ such that

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$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\} < \infty.$$

For the basic properties of spaces $L^{p(\cdot)}(\mathbb{R}^n)$, we refer to [1–6]. For example, the generalized Hölder inequality: For all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \|f\|_{p(\cdot)}\|g\|_{q(\cdot)}, \tag{1}$$

where $1 < p(x) < \infty, \frac{1}{p(x)} + \frac{1}{q(x)} = 1, x \in \mathbb{R}^n, p_- = \text{essinf}\{p(x) : x \in \mathbb{R}^n\}$, $p_+ = \text{esssup}\{p(x) : x \in \mathbb{R}^n\}, 1 < p_- \le p_+ < \infty$.

The variable Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$ is the space of all measurable functions f satisfying that $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and its weak derivatives

$$D^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N^n \text{ with } |\alpha| = \sum_{j=1}^{n} \alpha_j \le k.$$

$W^{k,p(\cdot)}(\mathbb{R}^n)$ is Banach space with the norm defined by

$$\|f\|_{p(\cdot),k} = \sum_{|\alpha|\le k} \|D^\alpha f\|_{p(\cdot)}.$$

Here $D^\alpha f = f$ if $\alpha = (0, 0, \ldots, 0)$.

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ are of interest for their applications to fluid dynamics, elasticity, modeling problems in physics, and to the study calculus of variations and partial differential equations with nonstandard growth conditions. In the past 20 years, the theory of these spaces has made progress rapidly (see, e.g., [7–14] and the references cited therein). It is well known that the Orlicz spaces are the generalizations of L^p spaces and play an important role in mathematical physics. In 2014, the author Kuang [15] introduced the new multiple weighted Orlicz spaces; they are generalizations of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n_+)$. The aim of this paper is to consider further definitions and properties of these spaces and establish some new interest inequalities on these new spaces. They are significant generalizations of many known results.

2 Multiple Weighted Orlicz Spaces

Definition 1 (See [16–19]). We call φ a Young's function if it is a nonnegative increasing convex function on $(0, \infty)$ with $\varphi(0) = 0, \varphi(u) > 0, u > 0$, and

$$\lim_{u\to 0} \frac{\varphi(u)}{u} = 0, \lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty.$$

To Young's function φ, we can associate its convex conjugate function denoted by $\psi = \varphi^*$ and defined by

$$\psi(v) = \varphi^*(v) = \sup\{uv - \varphi(u) : u \geq 0\}, v \geq 0.$$

We note that $\psi = \varphi^*$ is also a Young's function and $\psi^* = (\varphi^*)^* = \varphi$. From the definition of $\psi = \varphi^*$, we get Young's inequality

$$uv \leq \varphi(u) + \psi(v), u, v > 0. \tag{2}$$

Let φ^{-1} be inverse function of φ, we have

$$v \leq \varphi^{-1}(v)\psi^{-1}(v) \leq 2v, v \geq 0. \tag{3}$$

Definition 2 (See [15]). Let φ be a Young's function on $(0, \infty)$; for any measurable function f and nonnegative weight function ω on $\mathbb{R}^n_+$, the multiple weighted Luxemburg norm is defined as follows:

$$\|f\|_{\varphi,\omega} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n_+} \varphi\left(\frac{|f(x)|}{\lambda}\right)\omega(x)dx \leq 1\right\}. \tag{4}$$

The multiple weighted Orlicz space is defined as follows:

$$L_\varphi(\omega) = \{f : \|f\|_{\varphi,\omega} < \infty\}. \tag{5}$$

In particular, if $\varphi(u) = u^{p(x)}$, then $L_\varphi(\omega)$ is the weighted variable exponent Lebesgue spaces $L^{p(\cdot)}(\omega)$; if the exponents $p(x)$ and $q(x)$ are constants, for example, $\varphi(u) = u^p, 1 < p < \infty$, then $L_\varphi(\omega)$ is the weighted Lebesgue spaces $L^p(\omega)$ on $\mathbb{R}^n_+$; if $\varphi(u) = u(\log(u + c))^q, q \geq 0, c > 0$, then $L_\varphi(\omega)$ is the weighted spaces $L(\omega)(\log L(\omega))^q$ on $\mathbb{R}^n_+$.

Definition 3 ([16, 18]). We call the Young's function φ on $(0, \infty)$ sub-multiplicative, if

$$\varphi(uv) \leq \varphi(u)\varphi(v) \tag{6}$$

for all $u, v \geq 0$.

Remark 1. If φ satisfies (6), then φ also satisfies Orlicz ∇_2 condition; that is, there exists a constant $c > 1$ such that

$$\varphi(2u) \leq c\varphi(u)$$

for all $u \geq 0$.

We also define the weighted Orlicz sequence space $l_\varphi(\omega)$.

Definition 4. Let $x = (x_1, x_2, \ldots, x_n, \ldots), \forall x_k \in \mathbb{R}^1$, $\varphi : (0, \infty) \to (0, \infty)$ be a Young's function, $\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots)$ with $\omega_k = \omega(k) \geq 0 (\forall k \in N)$ be a weight sequence, the weighted Luxemburg norm is defined as follows:

$$\|x\|_{\varphi,\omega} = \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty} \varphi\left(\frac{|x_k|}{\lambda}\right)\omega_k \le 1\right\}. \tag{7}$$

The weighted Orlicz sequence space is defined as follows:

$$l_\varphi(\omega) = \{x = (x_1, x_2, \ldots, x_n, \ldots) : \|x\|_{\varphi,\omega} < \infty\}.$$

In particular, if $\varphi(u) = u^{p(x)}$, then $l_\varphi(\omega)$ is the weighted variable exponent Lebesgue sequence space $l^{p(\cdot)}(\omega)$; if the exponent $p(x)$ is constant, for example , $\varphi(u) = u^p, 1 < p < \infty$, then $l_\varphi(\omega)$ is the weighted Lebesgue sequence spaces $l^p(\omega)$:

$$l^p(\omega) = \{x = (x_1, x_2, \ldots, x_n, \ldots) : \|x\|_{p,\omega} < \infty\},$$

where $\|x\|_{p,\omega} = (\sum_{k=1}^{\infty} |x_k|^p \omega_k)^{1/p}$.

Definition 5. Let $x = (x_1, x_2, \ldots, x_n), x_j = (x_{j,1}, \ldots, x_{j,m}, \ldots), 1 \le j \le n, \forall x_{j,k_j} \in \mathbb{R}^1$, $\varphi : (0,\infty) \to (0,\infty)$ be a Young's function, $\omega : N^n \to (0,\infty)$ be a weight sequence, the weighted Luxemburg norm is defined as follows :

$$\|x\|_{\varphi,\omega} = \inf\left\{\lambda > 0 : \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \varphi\left(\frac{|x_{1,k_1}| + \cdots + |x_{n,k_n}|}{\lambda}\right)\omega(k_1, \ldots, k_n) \le 1\right\}.$$

The multiple weighted Orlicz sequence space is defined as follows:

$$l_\varphi(\omega) = \{x = (x_1, x_2, \ldots, x_n), x_j = (x_{j,1}, \ldots, x_{j,m}, \ldots), 1 \le j \le n : \|x\|_{\varphi,\omega} < \infty\}.$$

3 Some New Basic Inequalities on the Multiple Weighted Orlicz Spaces

In this section, we establish some new basic inequalities on these new spaces.

Theorem 1. *Let the conjugate Young's functions φ, ψ on $(0,\infty)$ be sub-multiplicative and $K(\|x\|, \|y\|)$ be a nonnegative measurable function on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ and satisfies*

$$K(\|x\|, \|y\|) = \|y\|^{-\lambda_2} K(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}, 1), \tag{8}$$

where λ_1 and λ_2 are real numbers and $\lambda_1 \times \lambda_2 \neq 0$. Let $f \in L_\varphi(\omega_1)$, $g \in L_\psi(\omega_2)$ and $\|f\|_{\varphi,\omega_1} > 0, \|g\|_{\psi,\omega_2} > 0$, where

$$\omega_1(x) = \|x\|^{-\lambda_1 + \frac{(n\lambda_1)}{\lambda_2}}, \omega_2(y) = \|y\|^{-\lambda_2 + \frac{(n\lambda_2)}{\lambda_1}}.$$

If

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_2} \times \int_0^\infty K(u,1)\psi^{-1}(u)u^{-\left(\frac{n}{\lambda_2}\right)}du < \infty; \tag{9}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_1} \times \int_0^\infty K(u,1)\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right) \times u^{\left(\frac{n}{\lambda_1}\right)-1}du < \infty, \tag{10}$$

then

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} K(\|x\|,\|y\|) \times f(x)g(y)dxdy \le c(\varphi,\psi)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}, \tag{11}$$

where $c(\varphi,\psi) = c_1 + c_2$ *is defined by (9) and (10).*

In particular, if $n = 1$, in Theorem 1, then

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy \le (c_1 + c_2)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}, \tag{12}$$

where

$$c_1 = \frac{1}{\lambda_2} \times \int_0^\infty K(u,1)\psi^{-1}(u)u^{-\left(\frac{1}{\lambda_2}\right)}du < \infty, \tag{13}$$

$$c_2 = \frac{1}{\lambda_1} \times \int_0^\infty K(u,1)\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right) \times u^{\left(\frac{1}{\lambda_1}\right)-1}du < \infty. \tag{14}$$

Theorem 2. *Let the conjugate Young's functions* φ, ψ *on* $(0,\infty)$ *be sub-multiplicative and* $K(\|x\|,\|y\|)$ *be a nonnegative measurable function on* $\mathbb{R}_+^n \times \mathbb{R}_+^n$ *and satisfies*

$$K(\|x\|,\|y\|) = \|y\|^{-\lambda}K(\|x\|\cdot\|y\|^{-1},1), \tag{15}$$

where λ *is real number. Let* $f \in L_\varphi(\omega)$, $g \in L_\psi(\omega)$ *and* $\|f\|_{\varphi,\omega} > 0, \|g\|_{\psi,\omega} > 0$ *, where* $\omega(x) = \|x\|^{n-\lambda}$ *. If*

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \times \int_0^\infty K(u,1)\psi^{-1}(u)u^{\lambda-n-1}du < \infty, \tag{16}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \times \int_0^\infty K(u,1)\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)u^{n-1}du < \infty, \tag{17}$$

then

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} K(\|x\|,\|y\|)f(x)g(y)dxdy \le c(\varphi,\psi)\|f\|_{\varphi,\omega}\|g\|_{\psi,\omega}, \tag{18}$$

where $c(\varphi,\psi) = c_1 + c_2$ *is defined by (16) and (17).*

In particular, if $n = 1$ in Theorem 2, then

$$\int_0^\infty \int_0^\infty K(x, y)f(x)g(y)dxdy \le (c_1+_2)\|f\|_{\varphi,\omega}\|g\|_{\psi,\omega}, \tag{19}$$

where $\omega(x) = x^{1-\lambda}$, and

$$c_1 = \int_0^\infty K(u, 1)\psi^{-1}(u)u^{\lambda-2}du < \infty, \tag{20}$$

$$c_2 = \int_0^\infty K(u, 1)\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)du < \infty. \tag{21}$$

4 Proofs of Theorems

We require the following Lemmas to prove our results:

Lemma 1 ([20]). *If* $a_k, b_k, p_k > 0$*,* $1 \le k \le n$*,* f *be a measurable function on* $(0,\infty)$*, then*

$$\int_{\mathbb{R}^n_+} f\left(\sum_{k=1}^n \left(\frac{x_k}{a_k}\right)^{b_k}\right) x_1^{p_1-1}\cdots x_n^{p_n-1}dx_1\cdots dx_n$$
$$= \frac{\prod_{k=1}^n a_k^{p_k}}{\prod_{k=1}^n b_k} \times \frac{\prod_{k=1}^n \Gamma(\frac{p_k}{b_k})}{\Gamma\left(\sum_{k=1}^n \frac{p_k}{b_k}\right)} \times \int_0^\infty f(t)t^{\left(\sum_{k=1}^n \frac{p_k}{b_k}-1\right)}dt.$$

We get the following Lemma 2 by taking $a_k = 1, b_k = 2, p_k = 1, 1 \le k \le n$, in Lemma 1.

Lemma 2. *Let* f *be a measurable function on* $(0,\infty)$*, then*

$$\int_{\mathbb{R}^n_+} f(\|x\|^2)dx = \frac{\pi^{n/2}}{2^n\Gamma(n/2)}\int_0^\infty f(t)t^{(n/2)-1}dt, \tag{22}$$

Proof (Proof of Theorem 1). Applying (3) and Young's inequality (2), we obtain

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} K(\|x\|, \|y\|) \times f(x)g(y)dxdy$$
$$\le \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \{|f(x)|\varphi^{-1}(K(\|x\|, \|y\|))\}\{|g(y)|\psi^{-1}(K(\|x\|, \|y\|))\}dxdy$$
$$= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \{|f(x)|\varphi^{-1}(K(\|x\|, \|y\|))\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2}))\}$$

$$
\begin{aligned}
&\times \left\{ |g(y)|\psi^{-1}(K(\|x\|, \|y\|)) \frac{1}{\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}))} \right\} dxdy \\
&\leq \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \varphi\{|f(x)|\varphi^{-1}(K(\|x\|, \|y\|))\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}))\} dxdy \\
&\quad + \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \psi \left\{ |g(y)|\psi^{-1}(K(\|x\|, \|y\|)) \frac{1}{\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}))} \right\} dxdy \\
&= I_1 + I_2. \qquad (23)
\end{aligned}
$$

Since φ on $(0, \infty)$ is sub-multiplicative, we have

$$
\begin{aligned}
&\varphi\{|f(x)|\varphi^{-1}(K(\|x\|, \|y\|)) \times \varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}))\} \\
&\quad \leq \varphi(|f(x)|)\varphi\{\varphi^{-1}(K(\|x\|, \|y\|)) \times \varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}))\} \\
&\quad \leq \varphi(|f(x)|)K(\|x\|, \|y\|) \times \psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}). \qquad (24)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
I_1 &\leq \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \varphi(|f(x)|) \times K(\|x\|, \|y\|) \times \psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}) dxdy \\
&= \int_{\mathbb{R}_+^n} \varphi(|f(x)|) \left\{ \int_{\mathbb{R}_+^n} \|y\|^{-\lambda_2} K(\|x\|^{\lambda_1}\|y\|^{-\lambda_2}, 1)\psi^{-1}(\|x\|^{\lambda_1}\|y\|^{-\lambda_2}) dy \right\} dx. \qquad (25)
\end{aligned}
$$

By (22), we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+^n} \|y\|^{-\lambda_2} K(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}, 1) \times \psi^{-1}(\|x\|^{\lambda_1} \cdot \|y\|^{-\lambda_2}) dy \\
&\quad = \frac{\pi^{n/2}}{2^n \Gamma(n/2)} \int_0^\infty t^{-\frac{\lambda_2}{2}} K\left(\|x\|^{\lambda_1} \cdot t^{-\frac{\lambda_2}{2}}, 1\right) \times \psi^{-1}\left(\|x\|^{\lambda_1} \cdot t^{-\frac{\lambda_2}{2}}\right) t^{(n/2)-1} dt. \qquad (26)
\end{aligned}
$$

Let $u = \|x\|^{\lambda_1} \cdot t^{-\frac{\lambda_2}{2}}$, and by (25), (26) and (9), we get

$$
\begin{aligned}
I_1 &\leq \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_2} \int_{\mathbb{R}_+^n} \int_0^\infty \varphi(|f(x)|)\|x\|^{-\lambda_1 + \frac{n\lambda_1}{\lambda_2}} K(u, 1)\psi^{-1}(u) u^{-\frac{n}{\lambda_2}} dudx \\
&= \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_2} \left\{ \int_0^\infty K(u, 1)\psi^{-1}(u) u^{-\frac{n}{\lambda_2}} du \right\} \left\{ \int_{\mathbb{R}_+^n} \varphi(|f(x)|)\|x\|^{\left(-\lambda_1 + \frac{n\lambda_1}{\lambda_2}\right)} dx \right\} \\
&= c_1 \int_{\mathbb{R}_+^n} \varphi(|f(x)|)\omega_1(x) dx. \qquad (27)
\end{aligned}
$$

Similarly, we have

$$\psi\left\{|g(y)|\psi^{-1}(K(\|x\|,\|y\|))\times\frac{1}{\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2}))}\right\}$$
$$\le\psi(|g(y)|)K(\|x\|,\|y\|)\times\psi\left\{\frac{1}{\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2}))}\right\}$$
$$=\psi(|g(y)|)\|y\|^{-\lambda_2}K(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2},1)\psi\left\{\frac{1}{\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2}))}\right\}. \tag{28}$$

By (22), we have

$$\int_{\mathbb{R}^n_+}\|y\|^{-\lambda_2}K(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2},1)\times\psi\left\{\frac{1}{\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2}))}\right\}dx$$
$$=\frac{\|y\|^{-\lambda_2}\pi^{n/2}}{2^n\Gamma(n/2)}\int_0^\infty K(t^{\frac{\lambda_1}{2}}\|y\|^{-\lambda_2},1)\psi\left\{\frac{1}{\varphi^{-1}(\psi^{-1}(t^{(\frac{\lambda_1}{2})}\|y\|^{-\lambda_2}))}\right\}t^{\frac{n}{2}-1}dt. \tag{29}$$

Let $u=t^{\left(\frac{\lambda_1}{2}\right)}\cdot\|y\|^{-\lambda_2}$,and by (28), (29), and (10), we get

$$I_2=\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\psi\left\{|g(y)|\times\psi^{-1}(K(\|x\|,\|y\|))\times\frac{1}{\varphi^{-1}(\psi^{-1}(\|x\|^{\lambda_1}\cdot\|y\|^{-\lambda_2}))}\right\}dxdy$$
$$\le\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_1}\int_0^\infty K(u,1)\times\psi\left\{\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right\}u^{(\frac{n}{\lambda_1})-1}du$$
$$\times\int_{\mathbb{R}^n_+}\psi(|g(y)|)\|y\|^{\left(-\lambda_2+\frac{n\lambda_2}{\lambda_1}\right)}dy$$
$$=c_2\int_{\mathbb{R}^n_+}\psi(|g(y)|)\omega_2(y)dy. \tag{30}$$

Thus, by (27) and (30), we obtain

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}K(\|x\|,\|y\|)\times f(x)g(y)dxdy$$
$$\le c_1\int_{\mathbb{R}^n_+}\varphi(|f(x)|)\omega_1(x)dx+c_2\int_{\mathbb{R}^n_+}\psi(|g(y)|)\omega_2(y)dy.$$

It follows that

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} K(\|x\|, \|y\|) \times \left(\frac{f(x)}{\|f\|_{\varphi,\omega_1}}\right)\left(\frac{g(y)}{\|g\|_{\psi,\omega_2}}\right) dxdy$$
$$\leq c_1 \int_{\mathbb{R}_+^n} \varphi\left(\frac{|f(x)|}{\|f\|_{\varphi,\omega_1}}\right)\omega_1(x)dx + c_2 \int_{\mathbb{R}_+^n} \psi\left(\frac{|g(y)|}{\|g\|_{\psi,\omega_2}}\right)\omega_2(y)dy$$
$$\leq c_1 + c_2 = c(\varphi, \psi).$$

Hence,

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} K(\|x\|, \|y\|) \times f(x)g(y)dxdy \leq c(\varphi, \psi)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}.$$

The proof is complete.

We can similarly prove Theorem 2; the details will be omitted.

5 Some Applications

As applications, a large number of known and new results have been obtained by proper choice of kernel K. In this section we present some model applications which display the importance of our results.

Corollary 1. *Let $f, g, \varphi, \psi, \lambda_1, \lambda_2, \omega_1$, and ω_2 satisfy the conditions of Theorem 1; then*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{\log\left(\frac{\|x\|^{\lambda_1}}{\|y\|^{\lambda_2}}\right)}{\|x\|^{\lambda_1} - \|y\|^{\lambda_2}} f(x)g(y)dxdy \leq (c_1 + c_2)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}, \quad (31)$$

where

$$c_1 = \frac{\pi^{n/2}\lambda_1}{2^{n-1}\Gamma(n/2)\lambda_2}\int_0^\infty \frac{\log u}{u^{\lambda_1} - 1}\psi^{-1}(u)u^{-\left(\frac{n}{\lambda_2}\right)}du < \infty; \quad (32)$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\int_0^\infty \frac{\log u}{u^{\lambda_1} - 1}\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)u^{\left(\frac{n}{\lambda_1}\right)-1}du < \infty. \quad (33)$$

If $n = 1$ in Corollary 1 , then

$$\int_0^\infty\int_0^\infty \frac{\log(\frac{x^{\lambda_1}}{y^{\lambda_2}})}{x^{\lambda_1} - y^{\lambda_2}} f(x)g(y)dxdy \leq (c_1 + c_2)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}, \quad (34)$$

where

$$c_1 = \frac{\lambda_1}{\lambda_2}\int_0^\infty \frac{\log u}{u^{\lambda_1}-1}\psi^{-1}(u)u^{-\left(\frac{1}{\lambda_2}\right)}du < \infty; \quad (35)$$

$$c_2 = \int_0^\infty \frac{\log u}{u^{\lambda_1}-1}\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)u^{\left(\frac{1}{\lambda_1}\right)-1}du < \infty. \quad (36)$$

Corollary 2. *Let $f, g, \varphi, \psi, \lambda$, and ω satisfy the conditions of Theorem 2. If*

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\int_0^\infty \frac{1}{|u-1|^\lambda}\psi^{-1}(u)u^{\lambda-n-1}du < \infty;$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\int_0^\infty \frac{1}{|u-1|^\lambda}\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)u^{n-1}du < \infty,$$

then

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{f(x)g(y)}{|\|x\|-\|y\||^\lambda}dxdy \le (c_1+c_2)\|f\|_{\varphi,\omega}\|g\|_{\psi,\omega}. \quad (37)$$

If $n = 1$ in Corollary 2, then

$$\int_0^\infty\int_0^\infty \frac{1}{|x-y|^\lambda}f(x)g(y)dxdy \le (c_1+c_2)\|f\|_{\varphi,\omega}\|g\|_{\psi,\omega}, \quad (38)$$

where $\omega(x) = x^{1-\lambda}$, and

$$c_1 = \int_0^\infty \frac{1}{|u-1|^\lambda}\psi^{-1}(u)u^{\lambda-2}du < \infty, \quad (39)$$

$$c_2 = \int_0^\infty \frac{1}{|u-1|^\lambda}\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)du < \infty. \quad (40)$$

We obtain the following Corollaries 3 and 4 by taking $\varphi(u) = u^{p(x)}, \psi(v) = v^{q(x)}$, in Theorems 1 and 2, where $1 < p(x) < \infty, \frac{1}{p(x)} + \frac{1}{q(x)} = 1, x \in \mathbb{R}^n_+$ and $p_- = \operatorname{essinf}\{p(x) : x \in \mathbb{R}^n_+\}, p_+ = \operatorname{esssup}\{p(x) : x \in \mathbb{R}^n_+\}, 1 < p_- \le p_+ < \infty$.

Corollary 3. *Let $K, \lambda_1, \lambda_2, \omega_1$, and ω_2 satisfy the conditions of Theorem 1. If $f \in L^{p(\cdot)}(\omega_1), g \in L^{q(\cdot)}(\omega_2)$, then*

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} K(\|x\|, \|y\|)f(x)g(y)dxdy \le c(p,q)\|f\|_{p(\cdot),\omega_1}\|g\|_{q(\cdot),\omega_2}, \quad (41)$$

where

$$c(p,q) = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \left\{ \frac{1}{\lambda_2} \left(\int_0^1 K(u,1) u^{\left(\frac{1}{q+} - \frac{n}{\lambda_2}\right)} du + \int_1^\infty K(u,1) u^{\left(\frac{1}{q-} - \frac{n}{\lambda_2}\right)} du \right) \right.$$
$$\left. + \frac{1}{\lambda_1} \left(\int_0^1 K(u,1) u^{\left(-\frac{1}{p-} + \frac{n}{\lambda_1} - 1\right)} du + \int_1^\infty K(u,1) u^{\left(-\frac{1}{p+} + \frac{n}{\lambda_1} - 1\right)} du \right) \right\}. \tag{42}$$

In particular, if $n = 1$, in Corollary 3, then

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y) dxdy \le c(p,q) \|f\|_{p(\cdot),\omega_1} \|g\|_{q(\cdot),\omega_2}, \tag{43}$$

where $\omega_1(x) = x^{-\lambda_1(1-\lambda_2^{-1})}$, $\omega_2(y) = y^{-\lambda_2(1-\lambda_1^{-1})}$, and

$$c(p,q) = \frac{1}{\lambda_2} \left(\int_0^1 K(u,1) u^{\left(\frac{1}{q+} - \frac{1}{\lambda_2}\right)} du + \int_1^\infty K(u,1) u^{\left(\frac{1}{q-} - \frac{1}{\lambda_2}\right)} du \right)$$
$$+ \frac{1}{\lambda_1} \left(\int_0^1 K(u,1) u^{-\left(\frac{1}{p-} - \frac{1}{\lambda_1} + 1\right)} du + \int_1^\infty K(u,1) u^{-\left(\frac{1}{p+} - \frac{1}{\lambda_1} + 1\right)} du \right). \tag{44}$$

If $K(x,y) = \frac{\log\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)}{x^{\lambda_1} - y^{\lambda_2}}$ in (43), then

$$\int_0^\infty \int_0^\infty \frac{\log\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)}{x^{\lambda_1} - y^{\lambda_2}} f(x) g(y) dxdy \le c(p,q) \|f\|_{p(\cdot),\omega_1} \|g\|_{q(\cdot),\omega_2}, \tag{45}$$

where

$$c(p,q) = \frac{\lambda_1}{\lambda_2} \left(\int_0^1 \frac{\log u}{u^{\lambda_1} - 1} u^{\left(\frac{1}{q+} - \frac{1}{\lambda_2}\right)} du + \int_1^\infty \frac{\log u}{u^{\lambda_1} - 1} u^{\left(\frac{1}{q-} - \frac{1}{\lambda_2}\right)} du \right)$$
$$+ \left(\int_0^1 \frac{\log u}{u^{\lambda_1} - 1} u^{-\left(\frac{1}{p-} - \frac{1}{\lambda_1} + 1\right)} du + \int_1^\infty \frac{\log u}{u^{\lambda_1} - 1} u^{-\left(\frac{1}{p+} - \frac{1}{\lambda_1} + 1\right)} du \right). \tag{46}$$

Corollary 4. *Let λ and ω satisfy the conditions of Corollary 2. If $f \in L^{p(\cdot)}(\omega)$, $g \in L^{q(\cdot)}(\omega)$, and $\max\{n - \frac{1}{p+}, n - \frac{1}{q+}\} < \lambda < 1, n > \max\{\frac{1}{p-}, \frac{1}{q-}\}$, then*

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \frac{f(x) g(y)}{|\|x\| - \|y\||^\lambda} dxdy \le (c_1 + c_2) \|f\|_{p(\cdot),\omega} \|g\|_{q(\cdot),\omega}, \tag{47}$$

where $\omega(x) = \|x\|^{n-\lambda}$ *, and*

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \left\{ B\left(1-\lambda, \lambda + \frac{1}{q_+} - n\right) + B\left(1-\lambda, n - \frac{1}{q_-}\right) \right\}, \quad (48)$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \left\{ B\left(1-\lambda, n - \frac{1}{p_-}\right) + B\left(1-\lambda, \lambda + \frac{1}{p_+} - n\right) \right\}, \quad (49)$$

and $B(\alpha, \beta)$ *is the Beta function.*

We obtain the following Corollaries 5 and 6 by taking $\varphi(u) = u^p, \psi(v) = v^q$, $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, in Corollaries 3 and 4:

Corollary 5. *Let* $K, \lambda_1, \lambda_2, \omega_1$*, and* ω_2 *satisfy the conditions of Theorem 1. If* $f \in L^p(\omega_1), g \in L^q(\omega_2)$*, then*

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} K(\|x\|, \|y\|) f(x) g(y) dx dy \leq c(p,q) \|f\|_{p,\omega_1} \|g\|_{q,\omega_2}, \quad (50)$$

where

$$c(p,q) = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \left\{ \frac{1}{\lambda_2} \int_0^\infty K(u,1) u^{\left(\frac{1}{q} - \frac{n}{\lambda_2}\right)} du + \frac{1}{\lambda_1} \int_0^\infty K(u,1) u^{\left(-\frac{1}{p} + \frac{n}{\lambda_1} - 1\right)} du \right\}. \quad (51)$$

In particular, if $n = 1$, in Corollary 5, then

$$\int_0^\infty \int_0^\infty K(x,y) f(x) g(y) dx dy \leq c(p,q) \|f\|_{p,\omega_1} \|g\|_{q,\omega_2}, \quad (52)$$

where $\omega_1(x) = x^{-\lambda_1(1-\lambda_2^{-1})}$, $\omega_2(y) = y^{-\lambda_2(1-\lambda_1^{-1})}$, and

$$c(p,q) = \frac{1}{\lambda_2} \int_0^\infty K(u,1) u^{\left(\frac{1}{q} - \frac{1}{\lambda_2}\right)} du + \frac{1}{\lambda_1} \int_0^\infty K(u,1) u^{-\left(\frac{1}{p} - \frac{1}{\lambda_1} + 1\right)} du. \quad (53)$$

If $K(x,y) = \frac{\log\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)}{x^{\lambda_1} - y^{\lambda_2}}$ in (52), and $\frac{1}{\lambda_1}\left(\frac{1}{q} - \frac{1}{\lambda_2} + 1\right) > 0$, $\frac{1}{\lambda_1}\left(\frac{1}{\lambda_1} - \frac{1}{p}\right) > 0$, then

$$\int_0^\infty \int_0^\infty \frac{\log\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)}{x^{\lambda_1} - y^{\lambda_2}} f(x) g(y) dx dy \leq c(p,q) \|f\|_{p,\omega_1} \|g\|_{q,\omega_2},$$

where

$$c(p,q)=\frac{1}{\lambda_1\lambda_2}\left(\frac{\pi}{\sin\frac{1}{\lambda_1}\left(\frac{1}{q}-\frac{1}{\lambda_2}+1\right)\pi}\right)^2+\frac{1}{\lambda_1^2}\left(\frac{\pi}{\sin\left(\frac{p-\lambda_1}{p\lambda_1^2}\right)\pi}\right)^2. \tag{54}$$

In particular, if $\lambda_1=\lambda_2=1$, then

$$\int_0^\infty\int_0^\infty \frac{\log(x/y)}{x-y}f(x)g(y)dxdy\le 2\left(\frac{\pi}{\sin(\pi/p)}\right)^2\|f\|_p\|g\|_q.$$

Corollary 6. *Let* $f\in L^p(\omega), g\in L^q(\omega)$, $\omega(x)=\|x\|^{n-\lambda}$, $1<p<\infty, (1/p)+(1/q)=1$, *and* $\max\{n-(1/p), n-(1/q)\}<\lambda<1$, *then*

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+}\frac{1}{|\|x\|-\|y\||^\lambda}f(x)g(y)dxdy\le (c_1+c_2)\|f\|_{p,\omega}\|g\|_{q,\omega},$$

where

$$c_1=\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{B\left(1-\lambda,\lambda+\frac{1}{q}-n\right)+B\left(1-\lambda,n-\frac{1}{q}\right)\right\},$$
$$c_2=\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{B\left(1-\lambda,n-\frac{1}{p}\right)+B\left(1-\lambda,\lambda+\frac{1}{p}-n\right)\right\}.$$

In particular, $n=1$, then

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{|x-y|^\lambda}dxdy\le c(p,q)\|f\|_{p,\omega}\|g\|_{q,\omega},$$

where $\omega(x)=x^{1-\lambda}$, and

$$c(p,q)=B\left(1-\lambda,\lambda-\frac{1}{p}\right)+B\left(1-\lambda,\frac{1}{p}\right)+B\left(1-\lambda,\frac{1}{q}\right)+B\left(1-\lambda,\lambda-\frac{1}{q}\right). \tag{55}$$

Equation (55) reduces to Theorem 4.5 in [18]. Hence, Corollary 6 is the multidimensional generalization of the corresponding results in [18].

We can obtain the corresponding series form of the above results, such as by taking f and g :

$$f(x)=a_m(m-1\le x<m); g(y)=b_n(n-1\le y<n),$$

and by Theorems 1 and 2 and Corollaries 1–6, we get

Corollary 7. *Let $K, \varphi, \psi, \lambda_1$, and λ_2 satisfy the conditions of Theorem 1. If $a = \{a_m\} \in l_\varphi(\omega_1)$, $b = \{b_n\} \in l_\psi(\omega_2)$, $\omega_1(m) = m^{-\lambda_1 + (\frac{\lambda_1}{\lambda_2})}$, $\omega_2(n) = n^{-\lambda_2 + (\frac{\lambda_2}{\lambda_1})}$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)a_m b_n \le c(\varphi,\psi)\|a\|_{\varphi,\omega_1}\|b\|_{\psi,\omega_2}, \tag{56}$$

where $c(\varphi,\psi) = c_1 + c_2$ is defined by (13) and (14).

If $K(m,n) = \frac{\log\left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)}{m^{\lambda_1} - n^{\lambda_2}}$ in (56), then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log\left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)}{m^{\lambda_1} - n^{\lambda_2}} a_m b_n \le c(\varphi,\psi)\|a\|_{\varphi,\omega_1}\|b\|_{\psi,\omega_2}, \tag{57}$$

where $c(\varphi,\psi) = c_1 + c_2$ is defined by (35) and (36) .

Corollary 8. *Let $\lambda_1, \lambda_2, p(\cdot)$, and $q(\cdot)$ satisfy the conditions of Corollary 3. If $a = \{a_m\} \in l^{p(\cdot)}(\omega_1)$, $b = \{b_n\} \in l^{q(\cdot)}(\omega_2)$, $\omega_1(m) = m^{-\lambda_1 + (\frac{\lambda_1}{\lambda_2})}$, $\omega_2(n) = n^{-\lambda_2 + (\frac{\lambda_2}{\lambda_1})}$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)a_m b_n \le c(p,q)\|a\|_{p(\cdot),\omega_1}\|b\|_{q(\cdot),\omega_2}, \tag{58}$$

where $c(p,q)$ is defined by (44).

If $K(m,n) = \frac{\log\left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)}{m^{\lambda_1} - n^{\lambda_2}}$ in (58) , then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log\left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)}{m^{\lambda_1} - n^{\lambda_2}} a_m b_n \le c(p,q)\|a\|_{p(\cdot),\omega_1}\|b\|_{q(\cdot),\omega_2}, \tag{59}$$

where $c(p,q)$ is defined by (46).

If $\lambda_1 = \lambda_2 = 1$ in (59), then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log(\frac{m}{n})}{m-n} a_m b_n \le c(p,q)\|a\|_{p(\cdot)}\|b\|_{q(\cdot)},$$

where

$$c(p,q) = \left(\int_0^1 \frac{\log u}{u-1} u^{\left(\frac{1}{q+}-1\right)} du + \int_1^\infty \frac{\log u}{u-1} u^{\left(\frac{1}{q-}-1\right)} du\right) + \left(\int_0^1 \frac{\log u}{u-1} u^{-\left(\frac{1}{p-}\right)} du + \int_1^\infty \frac{\log u}{u-1} u^{-\left(\frac{1}{p+}\right)} du\right).$$

Corollary 9. *Let $K, \lambda_1, \lambda_2, \omega_1$, and ω_2 satisfy the conditions of Corollary 5. If $a = \{a_m\} \in l^p(\omega_1), b = \{b_n\} \in l^q(\omega_2)$,$1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)a_m b_n \leq \|a\|_{p,\omega_1}\|b\|_{q,\omega_2}, \tag{60}$$

where $c(p,q)$ is defined by (53).

If $K(m,n) = \frac{\log\left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)}{m^{\lambda_1}-n^{\lambda_2}}$ in (60), and $\frac{1}{\lambda_1}\left(\frac{1}{q} - \frac{1}{\lambda_2} + 1\right) > 0, \frac{1}{\lambda_1}\left(\frac{1}{\lambda_1} - \frac{1}{p}\right) > 0$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log\left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)}{m^{\lambda_1} - n^{\lambda_2}} a_m b_n \leq c(p,q)\|a\|_{p,\omega_1}\|b\|_{q,\omega_2}, \tag{61}$$

where $c(p,q)$ is defined by (54). If $\lambda_1 = \lambda_2 = 1$ in (61), then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\log(m/n)}{m-n} a_m b_n \leq 2\left(\frac{\pi}{\sin(\pi/p)}\right)^2 \|a\|_p\|b\|_q.$$

Corollary 10. *Let K, φ, ψ, and λ satisfy the conditions of Theorem 2. If $a = \{a_m\} \in l_\varphi(\omega)$, $b = \{b_n\} \in l_\psi(\omega)$, $\omega(m) = m^{1-\lambda}$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)a_m b_n \leq c(\varphi,\psi)\|a\|_{\varphi,\omega}\|b\|_{\psi,\omega}, \tag{62}$$

where $c(p,q) = c_1 + c_2$ is defined by (20) and (21) .

If $K(m,n) = \frac{1}{|m-n|^\lambda}$ in (62), then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{|m-n|^\lambda} a_m b_n \leq c(\varphi,\psi)\|a\|_{\varphi,\omega}\|b\|_{\psi,\omega}, \tag{63}$$

where $c(\varphi,\psi) = c_1 + c_2$ is defined by (39) and (40).

Corollary 11. *Let $\lambda, p(\cdot)$, and $q(\cdot)$ satisfy the conditions of Corollary 4. If $a = \{a_m\} \in l^{p(\cdot)}(\omega)$, $b = \{b_n\} \in l^{q(\cdot)}(\omega)$, $\omega(n) = n^{1-\lambda}$, and $\max\{1 - \frac{1}{q_+}, 1 - \frac{1}{p_+}\} < \lambda < 1$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{|m-n|^\lambda} a_m b_n \leq (c_1 + c_2)\|a\|_{p(\cdot),\omega}\|b\|_{q(\cdot),\omega},$$

where

$$c_1 = B\left(1-\lambda, \lambda + \frac{1}{q_+} - 1\right) + B\left(1-\lambda, 1 - \frac{1}{q_-}\right),$$
$$c_2 = B\left(1-\lambda, 1 - \frac{1}{p_-}\right) + B\left(1-\lambda, \lambda + \frac{1}{p_+} - 1\right).$$

Corollary 12. *Let λ, p, q, and ω satisfy the conditions of Corollary 6; if $a = \{a_m\} \in l^p(\omega)$, $b = \{b_n\} \in l^q(\omega)$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{|m-n|^{\lambda}} a_m b_n \leq c(p,q)\|a\|_{p,\omega}\|b\|_{q,\omega},$$

where $c(p,q)$ is defined by (55).

6 Further Results

When the integral kernels $K(\|x\|, \|y\|)$ do not satisfy the conditions (8) and (15), we can similarly prove the corresponding results. For example,

Theorem 3. *Let the conjugate Young's functions φ, ψ on $(0,\infty)$ be sub-multiplicative. Let $f \in L_{\varphi}(\omega)$, $g \in L_{\psi}(\omega)$, and $\|f\|_{\varphi,\omega} > 0, \|g\|_{\psi,\omega} > 0$, where $\omega(x) = \|x\|^{n-\lambda}$ and λ, α, β are real numbers. If*

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^{\infty} \frac{1}{|u-1|^{\lambda-\alpha-\beta}} \psi^{-1}(u) u^{\lambda-n-1} du < \infty, \tag{64}$$
$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^{\infty} \frac{1}{|u-1|^{\lambda-\alpha-\beta}} \psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right) u^{n-1} du < \infty, \tag{65}$$

then

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{f(x)g(y)}{\|x\|^{\alpha}\cdot\|y\|^{\beta}|\|x\|-\|y\||^{\lambda-\alpha-\beta}} dxdy \leq c(\varphi,\psi)\|f\|_{\varphi,\omega}\|g\|_{\psi,\omega}, \tag{66}$$

where $c(\varphi,\psi) = c_1 + c_2$ is defined by (64) and (65).

In particular, if $n = 1$ in Theorem 3, then

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x^{\alpha}y^{\beta}|x-y|^{\lambda-\alpha-\beta}} dxdy \leq (c_1+c_2)\|f\|_{\varphi,\omega}\|g\|_{\psi,\omega},$$

the corresponding series form is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\alpha}n^{\beta}|m-n|^{\lambda-\alpha-\beta}} \leq \|a\|_{\varphi,\omega}\|b\|_{\psi,\omega},$$

where $\omega(x) = x^{1-\lambda}$ and

$$c_1 = \int_0^{\infty} \frac{1}{|u-1|^{\lambda-\alpha-\beta}} \psi^{-1}(u)u^{\lambda-2}du < \infty, \tag{67}$$

$$c_2 = \int_0^{\infty} \frac{1}{|u-1|^{\lambda-\alpha-\beta}} \psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right) du < \infty. \tag{68}$$

In the following , we still set $1 < p(x) < \infty, \frac{1}{p(x)} + \frac{1}{q(x)} = 1, x \in \mathbb{R}^n_+$, and $p_- = \text{essinf}\{p(x) : x \in \mathbb{R}^n_+\}, p_+ = \text{esssup}\{p(x) : x \in \mathbb{R}^n_+\}, 1 < p_- \leq p_+ < \infty$.

Corollary 13. *Let* $\omega(x) = \|x\|^{n-\lambda}$ *and* $n > \max\left\{\frac{1}{p_-}, \alpha+\beta+\frac{1}{q_-}\right\}$, $\max\left\{n-\frac{1}{q_+}, n+\alpha+\beta-\frac{1}{p_+}\right\} < \lambda < 1+\alpha+\beta$. *If* $f \in L^{p(\cdot)}(\omega), g \in L^{q(\cdot)}(\omega)$, *then*

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{f(x)g(y)}{\|x\|^{\alpha}\cdot\|y\|^{\beta}|\|x\|-\|y\||^{\lambda-\alpha-\beta}}dxdy \leq (c_1+c_2)\|f\|_{p(\cdot),\omega}\|g\|_{q(\cdot),\omega}, \tag{69}$$

where

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{B\left(1+\alpha+\beta-\lambda, \lambda+\frac{1}{q_+}-n\right)\right.$$
$$\left.+ B\left(1+\alpha+\beta-\lambda, n-\alpha-\beta-\frac{1}{q_-}\right)\right\},$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{B\left(1+\alpha+\beta-\lambda, n-\frac{1}{p_-}\right)\right.$$
$$\left.+ B\left(1+\alpha+\beta-\lambda, \lambda+\frac{1}{p_+}-n-\alpha-\beta\right)\right\}.$$

If $n = 1$ in (69), then

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x^{\alpha}y^{\beta}|x-y|^{\lambda-\alpha-\beta}}dxdy \leq (c_1+c_2)\|f\|_{p(\cdot),\omega}\|g\|_{q(\cdot),\omega}, \tag{70}$$

the corresponding series form is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\alpha}n^{\beta}|m-n|^{\lambda-\alpha-\beta}} \leq (c_1+c_2)\|a\|_{p(\cdot),\omega)\|b\|_{q(\cdot),\omega}}, \tag{71}$$

where

$$c_1 = B\left(1+\alpha+\beta-\lambda, \lambda+\frac{1}{q_+}-1\right) + B\left(1+\alpha+\beta-\lambda, 1-\alpha-\beta-\frac{1}{q_-}\right),$$

$$c_2 = B\left(1+\alpha+\beta-\lambda, 1-\frac{1}{p_-}\right) + B\left(1+\alpha+\beta-\lambda, \lambda-\alpha-\beta+\frac{1}{p_+}-1\right).$$

In particular, if $p(\cdot)$ are $q(\cdot)$ are constants, that is, if $f \in L^p(\omega), g \in L^q(\omega)$, $1 < p < \infty, \frac{1}{p}+\frac{1}{q} = 1$, $\max\{\frac{1}{p}, \alpha+\beta+\frac{1}{q}\} < \lambda < 1+\alpha+\beta < 1+\frac{1}{p}$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha y^\beta |x-y|^{\lambda-\alpha-\beta}} dxdy \le (c_1+c_2)\|f\|_{p,\omega}\|g\|_{q,\omega}, \tag{72}$$

the corresponding series form is

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m^\alpha n^\beta |m-n|^{\lambda-\alpha-\beta}} \le (c_1+c_2)\|a\|_{p,\omega}\|b\|_{q,\omega}, \tag{73}$$

where

$$c_1 = B\left(1+\alpha+\beta-\lambda, \lambda-\frac{1}{p}\right) + B\left(1+\alpha+\beta-\lambda, \frac{1}{p}-\alpha-\beta\right),$$

$$c_2 = B\left(1+\alpha+\beta-\lambda, \frac{1}{q}\right) + B\left(1+\alpha+\beta-\lambda, \lambda-\alpha-\beta-\frac{1}{q}\right).$$

Equations (72) and (73) are the Hardy–Littlewood inequalities (see [19]). Hence, (66) and (69) are new significant extensions of the Hardy–Littlewood inequality on the multiple weighted Orlicz spaces and the variable Lebesgue spaces.

Theorem 4. *Let the conjugate Young's functions* φ, ψ *on* $(0,\infty)$ *be sub-multiplicative. Let* $\omega_1(x) = \|x\|^{-\frac{(n\lambda_1)}{\lambda_2}}$, $\omega_2(y) = \|y\|^{-\frac{(n\lambda_2)}{\lambda_1}}$, *where* λ_1 *and* λ_2 *are real numbers and* $\lambda_1 \times \lambda_2 \ne 0$. *Let* $f \in L_\varphi(\omega_1), g \in L_\psi(\omega_2)$ *and* $\|f\|_{\varphi,\omega_1} > 0, \|g\|_{\psi,\omega_2} > 0$. *If*

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_2}\int_0^\infty \frac{1}{e^u-1}\psi^{-1}(u)u^{(\frac{n}{\lambda_2})-1}du < \infty; \tag{74}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_1}\int_0^\infty \frac{1}{e^u-1}\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)u^{(\frac{n}{\lambda_1})-1}du < \infty, \tag{75}$$

then

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{1}{\exp(\|x\|^{\lambda_1}\cdot\|y\|^{\lambda_2})-1} f(x)g(y)dxdy \le c(\varphi,\psi)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}, \tag{76}$$

where $c(\varphi,\psi) = c_1 + c_2$ *is defined by (74) and (75).*

Corollary 14. *Let* $\lambda_1, \lambda_2, \omega_1$*, and* ω_2 *satisfy the conditions of Theorem 4. If* $f \in L^{p(\cdot)}(\omega_1), g \in L^{q(\cdot)}(\omega_2)$*, then*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{1}{\exp(\|x\|^{\lambda_1}\cdot\|y\|^{\lambda_2})-1} f(x)g(y)dxdy \le c(p,q)\|f\|_{p(\cdot),\omega_1}\|g\|_{q(\cdot),\omega_2}, \tag{77}$$

where

$$c(p,q) = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{\frac{1}{\lambda_2}\left(\int_0^1 \frac{1}{e^u-1}u^{\left(\frac{1}{q+}+\frac{n}{\lambda_2}-1\right)}du + \int_1^\infty \frac{1}{e^u-1}u^{\left(\frac{1}{q-}+\frac{n}{\lambda_2}-1\right)}du\right)\right.$$
$$\left.+\frac{1}{\lambda_1}\left(\int_0^1 \frac{1}{e^u-1}u^{\left(-\frac{1}{p-}+\frac{n}{\lambda_1}-1\right)}du + \int_1^\infty \frac{1}{e^u-1}u^{\left(-\frac{1}{p+}+\frac{n}{\lambda_1}-1\right)}du\right)\right\}. \tag{78}$$

In particular, if $n = 1$, in Corollary 14, then

$$\int_0^\infty\int_0^\infty \frac{1}{\exp(x^{\lambda_1}\cdot y^{\lambda_2})-1} f(x)g(y)dxdy \le c(p,q)\|f\|_{p(\cdot),\omega_1}\|g\|_{q(\cdot),\omega_2}, \tag{79}$$

the corresponding series form is

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{e^{m^{\lambda_1}n^{\lambda_2}}-1}a_m b_n \le c(p,q)\|a\|_{p(\cdot),\omega_1}\|b\|_{q(\cdot),\omega_2}, \tag{80}$$

where $\omega_1(x) = x^{-(\frac{\lambda_1}{\lambda_2})}$, $\omega_2(y) = y^{-(\frac{\lambda_2}{\lambda_1})}$, and

$$c(p,q) = \frac{1}{\lambda_2}\left(\int_0^1 \frac{1}{e^u-1}u^{\left(\frac{1}{q+}+\frac{1}{\lambda_2}-1\right)}du + \int_1^\infty \frac{1}{e^u-1}u^{\left(\frac{1}{q-}+\frac{1}{\lambda_2}-1\right)}du\right)$$
$$+\frac{1}{\lambda_1}\left(\int_0^1 \frac{1}{e^u-1}u^{\left(-\frac{1}{p-}+\frac{1}{\lambda_1}-1\right)}du + \int_1^\infty \frac{1}{e^u-1}u^{\left(-\frac{1}{p+}+\frac{1}{\lambda_1}-1\right)}du\right). \tag{81}$$

Corollary 15. *Let* $\lambda_1, \lambda_2, \omega_1$*, and* ω_2 *satisfy the conditions of Theorem 4. If* $f \in L^p(\omega_1), g \in L^q(\omega_2)$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < \frac{1}{p} < \min\{1, \frac{n}{\lambda_2}, \frac{n}{\lambda_1}-1\}$ *, then*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{1}{\exp(\|x\|^{\lambda_1}\cdot\|y\|^{\lambda_2})-1} f(x)g(y)dxdy \le c(p,q)\|f\|_{p,\omega_1}\|g\|_{q,\omega_2},$$

where

$$c(p,q)=\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\left\{\frac{1}{\lambda_2}\zeta\left(\frac{n}{\lambda_2}+\frac{1}{q}\right)\Gamma\left(\frac{n}{\lambda_2}+\frac{1}{q}\right)\right.$$
$$\left.+\frac{1}{\lambda_1}\zeta\left(\frac{n}{\lambda_1}-\frac{1}{p}\right)\Gamma\left(\frac{n}{\lambda_1}-\frac{1}{p}\right)\right\},$$

and

$$\zeta(\alpha)=\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}}\quad(\alpha>1),\ \Gamma(\beta)=\int_0^{\infty}t^{\beta-1}\mathrm{e}^{-t}dt\quad(\beta>0)$$

are the Riemann–Zeta function and the Gamma function, respectively.

In particular, if $\lambda_1=\lambda_2=1$ and $n=2$ in Corollary 15, then

$$\int_{\mathbb{R}_+^2}\int_{\mathbb{R}_+^2}\frac{1}{\exp(\|x\|\cdot\|y\|)-1}f(x)g(y)dxdy\le c(p,q)\|f\|_{p,\omega_1}\|g\|_{q,\omega_2},$$

where $\omega_1(x)=\|x\|^2,\omega_2(y)=\|y\|^{-2}$ and

$$c(p,q)=\frac{\pi}{2}\times\Gamma\left(1+\frac{1}{q}\right)\left\{\left(1+\frac{1}{q}\right)\zeta\left(2+\frac{1}{q}\right)+\zeta\left(1+\frac{1}{q}\right)\right\}.$$

If $\lambda_1=\lambda_2=1,n=2$ and $p=q=2$ in Corollary 15, then

$$\int_{\mathbb{R}_+^2}\int_{\mathbb{R}_+^2}\frac{1}{\exp(\|x\|\cdot\|y\|)-1}f(x)g(y)dxdy$$
$$\le\frac{\pi^{3/2}}{4}\left\{\frac{3}{2}\zeta\left(\frac{5}{2}\right)+\zeta\left(\frac{3}{2}\right)\right\}\left(\int_{\mathbb{R}_+^2}|f(x)|^2\|x\|^2dx\right)^{1/2}\left(\int_{\mathbb{R}_+^2}|g(y)|^2\|y\|^{-2}dy\right)^{1/2}.$$

Theorem 5. *Let the conjugate Young's functions φ,ψ on $(0,\infty)$ be sub-multiplicative. Let $\omega_1(x)=\|x\|^{-\frac{(n\lambda_1)}{\lambda_2}}$, $\omega_2(y)=\|y\|^{-\frac{(n\lambda_2)}{\lambda_1}}$, where λ_1 and λ_2 are real numbers and $\lambda_1\times\lambda_2\ne0$. Let $f\in L_{\varphi}(\omega_1),g\in L_{\psi}(\omega_2)$ and $\|f\|_{\varphi,\omega_1}>0,\|g\|_{\psi,\omega_2}>0$. If*

$$c_1=\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_2}\int_0^{\infty}\operatorname{csc}h(u)\psi^{-1}(u)u^{\left(\frac{n}{\lambda_2}\right)-1}du<\infty;\tag{82}$$
$$c_2=\frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)\lambda_1}\int_0^{\infty}\operatorname{csc}h(u)\psi\left(\frac{1}{\varphi^{-1}(\psi^{-1}(u))}\right)u^{\left(\frac{n}{\lambda_1}\right)-1}du<\infty,\tag{83}$$

then

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \csc h(\|x\|^{\lambda_1}\cdot\|y\|^{\lambda_2})f(x)g(y)dxdy \le c(\varphi,\psi)\|f\|_{\varphi,\omega_1}\|g\|_{\psi,\omega_2}, \tag{84}$$

where $c(\varphi,\psi) = c_1 + c_2$ *is defined by (82) and (83) , and*

$$\csc h(x) = \frac{2}{\mathrm{e}^x - \mathrm{e}^{-x}}$$

is the hyperbolic cosecant function.

We obtain the following Corollary 16 by taking $\varphi(u) = u^p, \psi(v) = v^q$, $1 < p$, $q < \infty$, $(1/p) + (1/q) = 1$, in Theorem 5.

Corollary 16. *Let* $\lambda_1, \lambda_2, \omega_1$, *and* ω_2 *satisfy the conditions of Theorem 5, and* $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p} < \min\{\frac{n}{\lambda_2}, \frac{n}{\lambda_1} - 1\}$. *If* $f \in L^p(\omega_1), g \in L^q(\omega_2)$, *then*

$$\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \csc h(\|x\|^{\lambda_1}\cdot\|y\|^{\lambda_2})f(x)g(y)dxdy \le c(p,q)\|f\|_{p,\omega_1}\|g\|_{q,\omega_2}, \tag{85}$$

where

$$\begin{aligned} c(p,q) = \frac{\pi^{n/2}}{2^{n-2}\Gamma(n/2)} &\left\{ \frac{1}{\lambda_2}\Gamma\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)\left(1-2^{-\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)}\right)\zeta\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)\right. \\ &\left. + \frac{1}{\lambda_1}\Gamma\left(-\frac{1}{p}+\frac{n}{\lambda_1}\right)\left(1-2^{\left(\frac{1}{p}-\frac{n}{\lambda_1}\right)}\right)\zeta\left(-\frac{1}{p}+\frac{n}{\lambda_1}\right)\right\}. \end{aligned} \tag{86}$$

Proof. By (82) and (83) , we have

$$\begin{aligned} c(p,q) = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} &\left\{ \frac{1}{\lambda_2}\int_0^\infty \csc h(u)u^{\left(\frac{1}{q}+\frac{n}{\lambda_2}-1\right)}du \right. \\ &\left. + \frac{1}{\lambda_1}\int_0^\infty \csc h(u)u^{\left(-\frac{1}{p}+\frac{n}{\lambda_1}-1\right)}du \right\}. \end{aligned} \tag{87}$$

We compute

$$\begin{aligned} \int_0^\infty \csc h(u)u^{\left(\frac{1}{q}+\frac{n}{\lambda_2}-1\right)}du &= \int_0^\infty \frac{2}{e^u - e^{-u}}u^{\left(\frac{1}{q}+\frac{n}{\lambda_2}-1\right)}du \\ &= 2\int_0^\infty \frac{e^{-u}}{1-e^{-2u}}u^{\left(\frac{1}{q}+\frac{n}{\lambda_2}-1\right)}du \\ &= 2\int_0^\infty \left(\sum_{k=0}^\infty e^{-(2k+1)u}\right)u^{\left(\frac{1}{q}+\frac{n}{\lambda_2}-1\right)}du. \end{aligned}$$

By the Lebesgue term-by-term integral theorem, and setting $t = (2k+1)u$, we get

$$\int_0^\infty \csc h(u)u^{\left(\frac{1}{q}+\frac{n}{\lambda_2}-1\right)}du = 2\sum_{k=0}^\infty \left(\frac{1}{2k+1}\right)^{\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)} \int_0^\infty t^{\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)-1}e^{-t}dt$$
$$= 2\Gamma\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)\sum_{k=0}^\infty \left(\frac{1}{2k+1}\right)^{\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)}.$$

We know that

$$\sum_{k=0}^\infty \frac{1}{(2k+1)^\alpha} = \sum_{k=1}^\infty \frac{1}{k^\alpha} - \sum_{k=1}^\infty \frac{1}{(2k)^\alpha} = \left(1-\frac{1}{2^\alpha}\right)\zeta(\alpha),$$

where $\zeta(\alpha)(\alpha > 1)$ is the Riemann–Zeta function. Hence, we have

$$\int_0^\infty \csc h(u)u^{\left(\frac{1}{q}+\frac{n}{\lambda_2}-1\right)}du = 2\Gamma\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)\left(1-2^{-\left(\frac{1}{q}+\frac{n}{\lambda_2}\right)}\right)\zeta\left(\frac{1}{q}+\frac{n}{\lambda_2}\right). \tag{88}$$

By the same way, we get

$$\int_0^\infty \csc h(u)u^{\left(-\frac{1}{p}+\frac{n}{\lambda_1}-1\right)}du = 2\Gamma\left(-\frac{1}{p}+\frac{n}{\lambda_1}\right)\left(1-2^{\left(\frac{1}{p}-\frac{n}{\lambda_1}\right)}\right)\zeta\left(-\frac{1}{p}+\frac{n}{\lambda_1}\right). \tag{89}$$

By (88), (89), and (87), we obtain (86). The proof is complete.

In particular, if $\lambda_1 = \lambda_2 = n = 1$ in Corollary 16, then

$$\int_0^\infty\int_0^\infty \csc h(xy)f(x)g(y)dxdy \le c(p,q)\|f\|_{p,\omega}\|g\|_{q,\omega},$$

The corresponding series form is

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \csc h(mn)a_m b_n \le c(p,q)\|a\|_{p,\omega}\|b\|_{q,\omega},$$

where $\omega(x) = x^{-1}$, and

$$c(p,q) = 2\left\{\Gamma\left(\frac{1}{q}+1\right)\left(1-2^{-\left(\frac{1}{q}+1\right)}\right)\zeta\left(\frac{1}{q}+1\right)+\Gamma\left(\frac{1}{q}\right)\left(1-2^{-\left(\frac{1}{q}\right)}\right)\zeta\left(\frac{1}{q}\right)\right\}.$$

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Hyers–Ulam–Rassias Stability on Amenable Groups

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In Honor of Constantin Carathéodory

Abstract In this chapter, we study the Ulam–Hyers–Rassias stability of the generalized cosine-sine functional equation:

$$\int_K \int_G f(xtk \cdot y) d\mu(t) dk = f(x)g(y) + h(y),\ x, y \in G,$$

where f, g, and h are continuous complex valued functions on a locally compact group G, K is a compact subgroup of morphisms of G, dk is the normalized Haar measure on K, and μ is a complex measure with compact support. Furthermore, we prove a stability theorem in the case where G is amenable, K is a finite subgroup of the automorphisms of G, and μ is a finite K-invariant complex measure, and we obtain also the Hyers–Ulam–Rassias stability of the generalized cosine-sine functional equation:

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y) + 2h(y), x, y \in G,$$

where G is amenable, σ is an involution of G.

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1 Introduction

In 1940 the following stability problem for group homomorphisms was raised by Ulam [34]: Given a group G_1 and a metric group G_2 with metric $d(.,.)$ and a positive number ϵ greater than zero, does there exist a positive number δ greater than zero such that if a function $f : G_1 \longrightarrow G_2$ satisfies the functional inequality $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then there exists a group homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) \leq \epsilon$ for all $x \in G_1$?

The first affirmative answer was given in 1941 by Hyers [19] on Banach spaces.

In 1950 Aoki [1] provided a generalization of the Hyers' theorem for additive mappings, and in 1978 Rassias [26] generalized the Hyers' theorem for linear mappings by considering an unbounded Cauchy difference for sum of powers of norms $\epsilon(\|x\|^p + \|y\|^p)$. Rassias' theorem has been generalized by Găvruta [18] who permitted the Cauchy difference to be bounded by a general control function. Since then, the stability problems for several functional equations have been extensively investigated by a number mathematicians (cf. [8, 9, 12, 17, 21, 22, 25, 27, 28, 30, 31, 33, 35]). The terminology Hyers–Ulam–Rassias stability originates from these historical backgrounds. This terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [16, 20, 23, 24, 29, 32].

The stability of functional equations highlighted a phenomenon which is usually called superstability. Consider the functional equation $E(f) = 0$, and assume we are in a framework where the notion of boundedness of f and of $E(f)$ makes sense. We say that the equation $E(f) = 0$ is superstable if the boundedness of $E(f)$ implies that either f is bounded or f is a solution of $E(f) = 0$. This property was first observed and was proved by Baker et al. [6] in the following theorem

Theorem 1. *Let V be a vector space. If a function $f : V \to R$ satisfies the inequality*

$$|f(x+y) - f(x)f(y)| \leq \epsilon$$

for some $\epsilon > 0$ and for all $x, y \in V$. Then either f is bounded on V or $f(x+y) = f(x)f(y)$ for all $x, y \in V$.

The result was generalized by Baker [5], by replacing V by a semigroup and R by a normed algebra E in which the norm is multiplicative, i.e., $\|uv\| = \|u\|\|v\|$ for all $u, v \in E$.

Badora [2] proved the Hyers–Ulam stability on abelian groups of the functional equation

$$\int_K f(x + k \cdot y)dk = f(x)g(y), \; x, y \in G$$

and the superstability of the spherical function

$$\int_K f(x + k \cdot y)dk = f(x)f(y), \; x, y \in G.$$

In 2012, Badora [3] studied the Hyers–Ulam stability of the generalized cosine-sine functional equation:

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = \int_K f(x+k\cdot y)dk = g(y)f(x) + h(y),\ x,y\in G$$

and

$$\frac{1}{|K|}\sum_{k\in K} f(x+k\cdot y) = g(x)f(y) + h(x),\ x,y\in G,$$

where G is an abelian group and K is a finite subgroup of the automorphism of G with order $|K|$.

Recently, Bouikhalene and Elqorachi [7] obtained the Hyers–Ulam stability of the generalized Wilson's functional equation

$$\int_G\int_K f(xtk\cdot y)d\mu(t)dk = f(x)g(y),\ x,y\in G$$

and the superstability of Badora's functional equation

$$\int_G\int_K f(xtk\cdot y)d\mu(t)dk = f(x)f(y),\ x,y\in G, \tag{1}$$

where K is a compact subgroup of morphisms of G and μ is a K-invariant complex measure with compact support and f,g,h are continuous complex valued functions. We refer also to [10, 11, 13–15].

The main purpose of this chapter is to investigate the Hyers–Ulam–Rassias stability of the generalized cosine-sine functional equation:

$$\int_K\int_G f(xtk\cdot y)d\mu(t)dk = f(x)g(y) + h(y),\ x,y\in G, \tag{2}$$

where K is a compact subgroup of morphisms of G and μ is a K-invariant complex measure with compact support and f,g,h are continuous complex valued functions.

Our results are organized as follows. In Theorem 2 we prove the Hyers–Ulam–Rassias stability of Eq. (2) when f is bounded. In Theorem 3 we prove a superstability result of Eq. (2) in more general case. In Theorems 4 and 5, under the additional conditions that K is a finite group of automorphisms of G and μ is a finite K-invariant measure, we prove the Hyers–Ulam–Rassias stability of Eq. (2) in amenable groups. In Theorem 6 we obtain a stability theorem of a generalized cosine-sine functional equation on amenable groups.

Notations. Throughout this paper G is a Hausdorff locally compact group, e its identity element. K is a compact subgroup of the group $\mathrm{Mor}(G) = \mathrm{Aut}(G)\cup\mathrm{Ant}(G)$: The group of all mappings k of G onto itself are either automorphisms and homeomorphisms ($k \in \mathrm{Aut}(G)$) or anti-automorphisms and homeomorphisms ($k \in \mathrm{Ant}(G)$). The action of $k \in K$ on $x \in G$ will be denoted by $k \cdot x$. So if $k \in \mathrm{Aut}(G)$, we have $k \cdot (xy) = k \cdot xk \cdot y$ for all $x, y \in G$, and if $k \in \mathrm{Ant}(G)$ $k \cdot (xy) = k \cdot yk \cdot x$ for all $x, y \in G$. $M(G)$ is the algebra of all bounded complex measures on G. For $\mu \in M(G)$, $\|\mu\|$ denote the norm of μ. That is, $\|\mu\| = \max_{\|f\|\leq 1, f\in C_b(G)} | < \mu, f > |$, where $C_b(G)$ is the Banach algebra of all bounded and continuous functions on G and $\|f\| = \sup_{x\in G}|f(x)|$. For $\mu \in M(G)$, $k \in K$, and $f \in C_b(G)$, the measure $k \cdot \mu$ is defined by $< k \cdot \mu, f >=< \mu, k \cdot f >$, where $k \cdot f(x) = f(k^{-1} \cdot x)$ for all $x \in G$. A measure μ is said to be a K-invariant measure if $k \cdot \mu = \mu$ for all $k \in K$. A group G is an amenable group if there exists a linear continuous mapping m on $C_b(G)$ which satisfies $\inf f(G) \leq m(G) \leq \sup f(G)$, for all $f \in C_b(G, R)$ and $m(f_a) = m(f)$, $m({}_af) = m(f)$ for all $f \in C_b(G)$ and for all $a \in G$, where $f_a(x) = f(xa)$ and ${}_af(x) = f(ax)$ for all $x \in G$. Finally, $\sigma\colon G \longrightarrow G$ is said to be an involution of G if $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$.

2 Hyers–Ulam–Rassias Stability of a Generalized Cosine-Sine Functional Equation

In this section, we study the Hyers–Ulam–Rassias stability of the functional equation (2). We start by proving some results that we will use later.

Theorem 2. *Let f, g, h: $G \longrightarrow C$ be continuous functions on a group G. φ and ψ are mappings from $G \longrightarrow R$. K is a compact subgroup of the group* Mor(G). *If the functional inequality holds*

$$\left|\int_K \int_G f(xtk \cdot y)d\mu(t)dk - f(x)g(y) - h(y)\right| \leq \min(\varphi(x), \psi(y)) \tag{3}$$

for all $x, y \in G$ and with f bounded on G : $\max_{x\in G} |f(x)| \leq M$ for some $M \in R$. Then there exist F, H: $G \to C$ solutions of equation:

$$\int_K \int_G F(xtk \cdot y)d\mu(t)dk = F(x)g(y) + H(y),\ x, y \in G \tag{4}$$

such that

$$|f(x) - F(x)| \leq M$$

and

$$|h(x) - H(x)| \leq \min(\varphi(x), \psi(x)) + M\|\mu\| + M|g(x)|$$

for all $x \in G$.

Proof. f is assumed to be bounded, so $|f(x)| \leq M$, for all $x \in G$ and for some $M \in R$.

If h is bounded, then we take $F = 0$ and $H = 0$. If h is unbounded, since, the pair (f, g) satisfies (3), so if $f = 0$ then we have $|h(y)| \leq \varphi(x)$, for all $x, y \in G$, which is not true, because h is assumed to be unbounded. Thus, we have $f \neq 0$, so there exists $x_0 \in G$ such that $f(x_0) \neq 0$ and the inequality (3) can be written as follows:

$$\left|\frac{1}{f(x_0)}\int_K\int_G f(x_0tk\cdot y)d\mu(t)dk - g(y) - \frac{h(y)}{f(x_0)}\right| \leq \left|\frac{\varphi(x_0)}{f(x_0)}\right|.$$

which implies that g is unbounded. Let (y_n) be a sequence in G such that $|h(y_n)| \to \infty$ as $n \longrightarrow \infty$. Since $\frac{|h(y_n)|}{|f(x_0)|}$

$$\begin{aligned}
&= \left|\frac{h(y_n)}{f(x_0)} + g(y_n) - \frac{1}{f(x_0)}\int_K\int_G f(x_0tk\cdot y_n)d\mu(t)dk\right.\\
&\quad \left.+\frac{1}{f(x_0)}\int_K\int_G f(x_0tk\cdot y_n)d\mu(t)dk - g(y_n)\right|\\
&\leq \frac{\varphi(x_0)}{|f(x_0)|} + \left|\frac{1}{f(x_0)}\int_K\int_G f(x_0tk\cdot y_n)d\mu(t)dk\right| + |g(y_n)|\\
&\leq \frac{\varphi(x_0)}{|f(x_0)|} + M\|\mu\| + |g(y_n)|.
\end{aligned}$$

Then, we obtain $|g(y_n)| \to \infty$ as $n \longrightarrow \infty$. In the other hand, by using inequality (3) and for $a, b \in G$, we have $|g(y_n)f(a) + h(y_n)| \leq \varphi(a) + M\|\mu\|$ and $|g(y_n)f(b) + h(y_n)| \leq \varphi(b) + M\|\mu\|$. Then, by triangle inequality, we get $|g(y_n)(f(a) - f(b))| \leq \varphi(a) + \varphi(b) + 2M\|\mu\|$. Since $|g(y_n)| \to \infty$ as $n \longrightarrow \infty$, then necessarily we have $f(a) - f(b) = 0$. So, f is constant, i.e., $f = c$, with $c \in C$. Then we can take $F = c$ and $H = -cg + c < \mu, G >$. Finally we obtain $F, H\colon G \longrightarrow C$ solutions of equation:

$$\int_K\int_G F(xtk\cdot y)d\mu(t)dk = F(x)g(y) + H(y),\ x, y \in G$$

and such that

$$\begin{aligned}
|h(x) - H(x)| &= |c < \mu, G > -cg(x) - h(x)|\\
&= \left|\int_K\int_G f(xtk\cdot x)d\mu(t)dk - h(x) - f(x)g(x)\right|\\
&\leq \min(\varphi(x), \psi(x)) + M\|\ \mu\| + M|g(x)|
\end{aligned}$$

for all $x \in G$. This ends the proof.

The following lemma will be helpful in the proof of Theorem 3:

Lemma 1 ([7]). *Let f: $G \longrightarrow C$ be a continuous function. Let μ be a complex measure with compact support and which is K-invariant. Then*

$$\begin{aligned}\int_G\int_K\int_K\int_G & f(zth\cdot(k\cdot ysx))d\mu(t)dhdkd\mu(s)\\ &+\int_G\int_K\int_K\int_G f(zth\cdot(xsk\cdot y))d\mu(t)dhd\mu(s)dk\\ &=\int_G\int_K\int_K\int_G f(ztk\cdot ysh\cdot x)d\mu(t)dhdkd\mu(s)\\ &\quad+\int_G\int_K\int_K\int_G f(zth\cdot xsk\cdot y)d\mu(t)dhdkd\mu(s)\end{aligned}$$

for all $x, y, z \in G$.

Theorem 3. *Let φ be a continuous mapping: $G \to R$, K a compact subgroup of the morphisms of G, $\mu \in M(G)$ is a K-invariant measure with compact support. If the continuous complex valued functions f g, h satisfy*

$$\left|\int_K\int_G f(xtk\cdot y)d\mu(t)dk - f(x)g(y) - h(y)\right| \le \varphi(y) \tag{5}$$

for all $x, y \in G$ with f unbounded on G. Then:

*(**1**) The function g satisfies the functional equation:*

$$\int_K\int_G g(xtk\cdot y)d\mu(t)dk + \int_K\int_G g(k\cdot ytx)d\mu(t)dk = 2g(x)g(y) \tag{6}$$

for all $x, y \in G$.

*(**2**) Furthermore, if $K \subseteq \mathrm{Aut}(G)$ then g satisfies Eq. (1).*

*(**3**) If $K = \{I, \sigma\}$, where I denotes the identity map and σ an involution of G, then g satisfies the d'Alembert's long functional equation:*

$$g(xy) + g(yx) + g(x\sigma(y)) + g(\sigma(y)x) = 4g(x)g(y),\ x, y \in G. \tag{7}$$

Proof. (1) Let f, g, and h satisfy the inequality (5). Let $x, y, z \in G$. By using Lemma 1, we have

$$\begin{aligned}|f(z)| & \left|\int_K\int_G g(xtk\cdot y)d\mu(t)dk + \int_K\int_G g(k\cdot ytx)d\mu(t)dk - 2g(x)g(y)\right|\\ \le & \left|f(z)\int_K\int_G g(xtk\cdot y)d\mu(t)dk + \int_K\int_G h(xtk\cdot y)d\mu(t)dk\right.\\ & \left.-\int_K\int_G\int_K\int_G f(zsk'\cdot(xtk\cdot y))d\mu(t)d\mu(s)dkdk'\right|\end{aligned}$$

$$+ \left| f(z) \int_K \int_G g(k \cdot ytx) d\mu(t) dk + \int_K \int_G h(k \cdot ytx) d\mu(t) dk \right.$$

$$\left. - \int_K \int_G \int_K \int_G f(zsk' \cdot (k \cdot ytx)) d\mu(t) d\mu(s) dk dk' \right|$$

$$+ \left| \int_K \int_G \int_K \int_G f(zsk' \cdot xtk \cdot y) d\mu(t) d\mu(s) dk dk' \right.$$

$$\left. - g(y) \int_K \int_G f(ztk \cdot x) d\mu(t) dk - h(y) \right|$$

$$+ \left| g(y) \left(\int_K \int_G f(ztk \cdot x) d\mu(t) dk - g(x) f(z) - h(x) \right) + h(x) g(y) + h(y) \right|$$

$$+ \left| \int_K \int_G \int_K \int_G f(zsk' \cdot ytk \cdot x) d\mu(t) d\mu(s) dk dk' \right.$$

$$\left. - g(x) \int_K \int_G f(zsk' \cdot y) d\mu(s) dk' - h(x) \right|$$

$$+ \left| g(x) \left(\int_K \int_G f(ztk \cdot y) d\mu(t) dk - g(y) f(z) - h(y) \right) + h(x) + h(y) g(x) \right|$$

$$\leq \int_K \int_G \varphi(xtk \cdot y) d|\mu|(t) dk + \int_K \int_G \varphi(k \cdot ytx) d|\mu|(t) dk$$

$$+ g(y)(\varphi(x) + |h(x)|) + |h(y)| + g(x)(\varphi(y) + |h(y)|) + |h(x)|$$

$$+ \|\mu\|(\varphi(y) + \varphi(x)) + \int_K \int_G |h|(xtk \cdot y) d|\mu|(t) dk$$

$$+ \int_K \int_G |h|(k \cdot ytx) d|\mu|(t) dk$$

Since f is unbounded, then we get the case (1).

(2) Let $K \subseteq \mathrm{Aut}(G)$. For all $x, y, z \in G$, we have

$$|f(z)| \left| \int_K \int_G g(xtk \cdot y) d\mu(t) dk - g(x) g(y) \right|$$

$$= \left| f(z) \int_K \int_G g(xtk \cdot y) d\mu(t) dk - g(x) f(z) g(y) \right|$$

$$\leq \left| f(z) \int_K \int_G g(xsk' \cdot y) d\mu(s) dk' \right.$$

$$\left. - \int_K \int_K \int_G \int_G f(ztk \cdot (xsk' \cdot y)) d\mu(t) d\mu(s) dk dk' \right|$$

$$+ \left| \int_K \int_K \int_G \int_G f(ztk \cdot (xsk' \cdot y)) d\mu(t) d\mu(s) dk dk' - g(x) f(z) g(y) \right|.$$

Since

$$\left|\int_K \int_G f(ztk \cdot y)d\mu(t)dk - f(z)g(y) - h(y)\right| \le \varphi(y)$$

we obtain

$$\begin{aligned}&\left|f(z)\int_K\int_G g(xsk' \cdot y)d\mu(s)dk' - \int_K\int_K\int_G\int_G f(ztk\cdot(xsk'\cdot y))d\mu(t)d\mu(s)dkdk'\right|\\ &\quad\le \int_K\int_G \left(\left|h(xsk'\cdot y)\right| + \left|\varphi(xsk'\cdot y)\right|\right) d\left|\mu\right|(s)dk'\end{aligned}$$

On the other hand, by using $K \subseteq \mathrm{Aut}(G)$, the K-invariance of μ and the invariance of the Haar measure dk, we have

$$\begin{aligned}&\int_K\int_K\int_G\int_G f(ztk\cdot(xsk'\cdot y))d\mu(t)dkd\mu(s)dk'\\ &\quad= \int_K\int_K\int_G\int_G f(ztk\cdot xsk'\cdot y)d\mu(t)dkd\mu(s)dk'.\end{aligned}$$

So, we get

$$\begin{aligned}&\left|\int_K\int_K\int_G\int_G f(ztk\cdot xsk'\cdot y)d\mu(t)d\mu(s)dkdk' - g(x)f(z)g(y)\right|\\ &\quad\le \left|\int_K\int_K\int_G\int_G f(ztk\cdot xsk'\cdot y)d\mu(t)d\mu(s)dkdk'\right.\\ &\qquad\left. - g(y)\int_K\int_G f(ztk\cdot x)d\mu(t)dk - h(y)\right|\\ &\qquad + \left|g(y)\left[\int_K\int_G f(ztk\cdot x)d\mu(t)dk - g(x)f(z) - h(x)\right] + h(y) + g(y)h(x)\right|\\ &\quad\le \varphi(y)\left\|\mu\right\| + \left|g(y)\right|\left(\left|\varphi(x)\right| + \left|h(x)\right|\right) + \left|h(y)\right|.\end{aligned}$$

Since f is assumed to be unbounded, we obtain that g satisfies the functional equation (1). This is case (2).

(3) is a particular case of (1) and this completes the proof of theorem.

Theorem 4. *Let G be an amenable group, K is finite and $K \subseteq \mathrm{Aut}(G)$, $\mu = \Sigma_{i\in I}\alpha_i\delta_{t_i}$ is a discrete normalized and K-invariant measure on G, where $\alpha_i \in C$ and δ_{t_i}, $i \in I$ are Dirac measures. Let φ: $G \to R$ be a function. If f, g, h: $G \to C$ are functions which satisfy*

$$\left|\int_K\int_G f(xtk\cdot y)d\mu(t)dk - f(x)g(y) - h(y)\right| \le \varphi(y) \tag{8}$$

for all $x, y \in G$ *and such that* f *is unbounded on* G*, then, there exists a function* $H: G \to C$ *which satisfies the functional equation:*

$$\int_K \int_G H(xtk \cdot y)d\mu(t)dk = H(x)g(y) + H(y) \tag{9}$$

for all $x, y \in G$ *and*

$$|h(x) - H(x)| \leq \varphi(x) \tag{10}$$

for all $x \in G$.

Proof. Let x *be fixed in* G. The new function

$$z \mapsto \frac{1}{|K|} \sum_{k\in K} \sum_{i\in I} \alpha_i f_{t_i k \cdot x}(z) - f(z)g(x)$$

is bounded by $\varphi(x)+|h(x)|$. *Let* m *be the invariant measure (relating to z) on* $B(G, C)$ and define the mapping $H : G \to C$ *by*

$$H(x) = m\left(\frac{1}{|K|} \sum_{k\in K} \sum_{i} \alpha_i f_{t_i k.x} - fg(x)\right),\ x \in G$$

From inequality (8) and the definition of H, we have

$$\begin{aligned}
|h(y) - H(y)| &= \left| m\left(\frac{1}{|K|} \sum_{k\in K} \sum_{i} \alpha_i f_{t_i k \cdot y} - fg(y)\right) - h(y) \right| \\
&\leq \sup_{x\in G} \left| \frac{1}{|K|} \sum_{k\in K} \sum_{i} \alpha_i f_{t_i k \cdot y}(x) - f(x)g(y) - h(y) \right| \\
&\leq \varphi(y).
\end{aligned}$$

for all $y \in G$. By using the definition of H, we obtain

$$\begin{aligned}
\int_K \int_G H(xs\lambda \cdot y)d\mu(s)d\lambda &= \frac{1}{|K|} \sum_{\lambda\in K} \sum_{j\in I} \alpha_j H(xt_j\lambda \cdot y) \\
&= \frac{1}{|K|} \sum_{\lambda\in K} \sum_{j\in I} \alpha_j m\left[\frac{1}{|K|} \sum_{k\in K} \sum_{i\in I} \alpha_i f_{t_i k \cdot (xt_j \lambda \cdot y)} - fg(xt_j\lambda \cdot y)\right].
\end{aligned}$$

Since $K \subset \text{Aut}(G)$, μ is K-invariant, and dk is a Haar measure, m is a linear map, and from Theorem 3, we get

$$\frac{1}{|K|}\sum_{\lambda\in K}\sum_{j\in I}\alpha_j H(xt_j\lambda .y)$$

$$=m\left[\frac{1}{|K|^2}\sum_{\lambda\in K}\sum_{k\in K}\sum_{i\in I}\sum_{j\in I}\alpha_i\alpha_j f_{t_ik\cdot xt_j\lambda\cdot y}-f\frac{1}{|K|}\sum_{\lambda\in K}\sum_{j\in I}\alpha_j g(xt_j\lambda .y)\right]$$

$$=m\left(\frac{1}{|K|^2}\sum_{\lambda\in K}\sum_{k\in K}\sum_{i\in I}\sum_{j\in I}\alpha_i\alpha_j f_{t_ik.xt_j\lambda .y}-fg(x)g(y)\right)$$

$$=m\left(\frac{1}{|K|}\sum_{k\in K}\sum_{i\in I}\alpha_i\left(\frac{1}{|K|}\sum_{\lambda\in K}\sum_{j\in I}\alpha_j f_{t_ik\cdot xt_j\lambda\cdot y}-f_{t_ik\cdot x}g(y)\right)\right.$$

$$\left.+\left(\frac{1}{|K|}\sum_{k\in K}\sum_{i\in I}\alpha_i f_{t_ik\cdot x}-fg(x)\right)g(y)\right)$$

$$=m\left(\frac{1}{|K|}\sum_{k\in K}\sum_{i\in I}\alpha_i\left(\frac{1}{|K|}\sum_{\lambda\in K}\sum_{j\in I}\alpha_j (f_{t_j\lambda\cdot y})_{t_ik\cdot x}-(fg(y))_{t_ik\cdot x}\right)\right)+H(x)g(y)$$

$$=\frac{1}{|K|}\sum_{k\in K}\sum_{i\in I}\alpha_i m\left(\frac{1}{|K|}\sum_{\lambda\in K}\sum_{j\in I}\alpha_j f_{t_j\lambda\cdot y}-fg(y)\right)_{t_ik\cdot x}+H(x)g(y).$$

Since m is invariant, then we have

$$\frac{1}{|K|}\sum_{\lambda\in K}\sum_{j\in I}\alpha_j H(xt_j\lambda\cdot y)=\left[\frac{1}{|K|}\sum_{k\in K}\sum_{i\in I}\alpha_i\right]H(y)+H(x)g(y).$$

By using $<\mu,G>=\sum_{i\in I}\alpha_i=1$, we get $\frac{1}{|K|}\sum_{k\in K}\sum_{i\in I}\alpha_i=1$ and then

$$\int_K\int_G H(xs\lambda\cdot y)d\mu(s)d\lambda=H(y)+H(x)g(y)$$

for all $x,y\in G$. This ends the proof of theorem.

Theorem 5. *Let G be an amenable group, K is a finite subgroup of* Aut(G)*, and* $\mu=\Sigma_{i\in I}\alpha_i\delta_{t_i}$ *is a discrete normalized and K-invariant measure on G. Let* φ*: G* $\longrightarrow$ *R be a function. If* f*,* g*,* h : $G\longrightarrow C$ *satisfy the inequality*

$$\left|\int_K\int_G f(xtk\cdot y)d\mu(t)dk-f(x)g(y)-h(y)\right|\leq\varphi(y)\qquad(11)$$

for all $x, y \in G$ *and such that* f *is unbounded, then there exist* F, $H : G \to C$ *which satisfy*

$$\int_K \int_G F(xtk \cdot y)d\mu(t)dk = F(x)g(y) + H(y),\ x, y \in G, \tag{12}$$

$$\int_K \int_G H(xtk \cdot y)d\mu(t)dk = H(x)g(y) + H(y),\ x, y \in G, \tag{13}$$

$$\int_K \int_G g(xtk \cdot y)d\mu(t)dk = g(x)g(y),\ x, y \in G, \tag{14}$$

$$\left|\int_G \int_K f(tk \cdot x)dkd\mu(t) - F(x)\right| \leq 2\varphi(x),\ x \in G \tag{15}$$

and

$$|h(x) - H(x)| \leq \varphi(x),\ x \in G \tag{16}$$

Proof. By using Theorem 4, there exists H: $G \to C$ such that

$$\int_K \int_G H(xtk \cdot y)d\mu(t)dk = H(x)g(y) + H(y)$$

for all $x, y \in G$ and

$$|h(x) - H(x)| \leq \varphi(x)$$

for all $x \in G$. Putting $x = e$ in (11), we obtain

$$\left|\int_K \int_G f(tk \cdot y)d\mu(t)dk - f(e)g(y) - h(y)\right| \leq \varphi(y).$$

So, if we take $F = cg + H$, where $c = f(e)$, we get

$$\begin{aligned}
\|F(x) - \int_G \int_K f(tk \cdot x)dkd\mu(t)| &= \left|cg(x) + H(x) - \int_G \int_K f(tk \cdot x)dkd\mu(t)\right| \\
&= \left|cg + h - \int_G \int_K f(tk \cdot x)dkd\mu(t) + H - h\right| \\
&\leq \left|cg + h - \int_G \int_K f(tk \cdot x)dkd\mu(t)| + |H - h\right| \\
&\leq 2\varphi(x)
\end{aligned}$$

for all $x \in G$. From Theorem 3, (2) g satisfies Eq. (1), so for all $x, y \in G$, we have

$$\begin{aligned}\int_K \int_G F(xtk \cdot y)d\mu(t)dk &= c\int_K \int_G g(xtk \cdot y)d\mu(t)dk + \int_K \int_G H(xtk \cdot y)d\mu(t)dk \\ &= cg(x)g(y) + H(x)g(y) + H(y) \\ &= F(x)g(y) + H(y).\end{aligned}$$

This completes the proof of theorem.

In the following corollary, we generalized a result obtained by Badora in [3].

Corollary 1. *Let G be an amenable group, K is a finite subgroup of* Aut(G)*, and $\mu = \delta_e$ is a dirac measure concentrated at the identity element e of G. Let φ: $G \longrightarrow R$ be a function. If f, g, $h : G \longrightarrow C$ satisfy the inequality*

$$\left| \int_K f(xk \cdot y)dk - f(x)g(y) - h(y) \right| \leq \varphi(y) \tag{17}$$

for all $x, y \in G$ and such that f is unbounded, then there exist F, $H : G \to C$ which satisfy

$$\int_K F(xk \cdot y)dk = F(x)g(y) + H(y), \tag{18}$$

$$\int_K H(xtk \cdot y)dk = H(x)g(y) + H(y), \tag{19}$$

$$\int_K g(xk \cdot y)dk = g(x)g(y) \tag{20}$$

for all $x, y \in G$ and

$$\left| \int_K f(k \cdot x)dk - F(x) \right| \leq 2\varphi(x), \tag{21}$$

$$|h(x) - H(x)| \leq \varphi(x) \tag{22}$$

for all $x \in G$.

In the following corollary, we obtain a generalization of a result obtained recently by Badora et al. [4].

Corollary 2. *Let G be an amenable group, $K = \{I\}$. Let φ: $G \longrightarrow R$ be a function. If f, g, $h : G \longrightarrow C$ satisfy the inequality*

$$|f(xy) - f(x)g(y) - h(y)| \leq \varphi(y) \tag{23}$$

for all $x, y \in G$ and such that f is unbounded, then there exist F, $H : G \to C$ which satisfy

$$F(xy) = F(x)g(y) + H(y), \tag{24}$$

$$H(xy) = H(x)g(y) + H(y), \tag{25}$$

$$g(xy) = g(x)g(y), \tag{26}$$

for all $x, y \in G$ and

$$|2f(x) - F(x)| \leq 2\varphi(x), \tag{27}$$

and

$$|h(x) - H(x)| \leq \varphi(x) \tag{28}$$

for all $x \in G$.

Corollary 3. *Let G be an amenable group; let $K = \{I, \sigma\}$, where σ is a homomorphism involutive of G. Let φ: $G \longrightarrow R$ be a function. If f, g, h satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)g(y) - 2h(y)| \leq \varphi(y) \tag{29}$$

for all $x, y \in G$ and such that f is unbounded, then there exist F, $H : G \to C$ which satisfy

$$F(xy) + F(x\sigma(y)) = 2F(x)g(y) + 2H(y), \tag{30}$$

$$H(xy) + H(x\sigma(y)) = 2H(x)g(y) + 2H(y), \tag{31}$$

$$g(xy) + g(x\sigma(y)) = 2g(x)g(y), \tag{32}$$

for all $x, y \in G$ and

$$|f(x) + f(\sigma(x)) - 2F(x)| \leq 4\varphi(x), \tag{33}$$

$$|h(x) - H(x)| \leq \varphi(x) \tag{34}$$

for all $x \in G$.

Theorem 6. *Let G be an amenable group; let $K = \{I, \sigma\}$, where σ is an involution of G. Let φ: $G \longrightarrow R$ be a function. If f, g, h satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)g(y) - 2h(y)| \leq \varphi(y) \tag{35}$$

for all $x, y \in G$ *and such that* f *is unbounded, then there exist* F, $H : G \to C$ *which satisfy*

$$\begin{aligned} &F(xy) + F(yx) + F(x\sigma(y)) + F(\sigma(y)x) \\ &\quad = 2F(x)g(y) + 2F(y)g(x) + 2H(x) + 2H(y),\ x, y \in G, \end{aligned} \tag{36}$$

$$\begin{aligned} &H(xy) + H(yx) + H(x\sigma(y)) + H(\sigma(y)x) \\ &\quad = 2H(x) + 2H(y) + 2g(y)H(x) + 2g(x)H(y), \end{aligned} \tag{37}$$

$$g(xy) + g(yx) + g(x\sigma(y)) + g(\sigma(y)x) = 4g(x)g(y)$$

for all $x, y \in G$ *and*

$$|f(x) + f(\sigma(x)) - 2F(x)| \le 4\varphi(x), \tag{38}$$

$$|h(x) - H(x)| \le \varphi(x) \tag{39}$$

for all $x \in G$.

Proof. For x fixed in G, the function $z \mapsto (f_x + f_{\sigma(x)})(z) - 2f(z)g(x)$ is bounded by $\varphi(x) + |h(x)|$. Let $H(x) = m(f_x + f_{\sigma(x)} - 2fg(x))$, $x \in G$. Since m is additive and g satisfies Eq. (7), we have

$$\begin{aligned} &H(xy) + H(x\sigma(y)) + H(yx) + H(\sigma(y)x) \\ &\quad = m[f_{xy} + f_{\sigma(y)\sigma(x)} - 2fg(xy)] + m[f_{x\sigma(y)} + f_{y\sigma(x)} - 2fg(x\sigma(y))] \\ &\qquad + m[f_{yx} + f_{\sigma(x)\sigma(y)} - 2fg(yx)] + m[f_{\sigma(y)x} + f_{\sigma(x)y} - 2fg(\sigma(y)x)] \\ &\quad = m[f_{xy} + f_{\sigma(y)\sigma(x)} + f_{x\sigma(y)} + f_{y\sigma(x)} + f_{yx} + f_{\sigma(x)\sigma(y)} + f_{\sigma(y)x} + f_{\sigma(x)y} - 8fg(x)g(y)] \\ &\quad = m[(f_y)_x + (f_{\sigma(y)})_x - 2f_x g(y) + (f_{\sigma(x)})_{\sigma(y)} + (f_x)_{\sigma(y)} - 2f_{\sigma(y)}g(x) \\ &\qquad + (f_{\sigma(x)})_y + (f_x)_y - 2f_y g(x) + (f_{\sigma(y)})_{\sigma(x)} + (f_y)_{\sigma(x)} - 2f_{\sigma(x)}g(y) \\ &\qquad + 2[f_x + f_{\sigma(x)} - 2fg(x)]g(y) + 2[f_y + f_{\sigma(y)} - 2fg(y)]g(x) \\ &\quad = m[(f_y)_x + (f_{\sigma(y)})_x - 2f_x g(y)] + m[(f_{\sigma(x)})_{\sigma(y)} + (f_x)_{\sigma(y)} - 2f_{\sigma(y)}g(x)] \\ &\qquad + m[(f_{\sigma(x)})_y + (f_x)_y - 2f_y g(x)] + m[(f_{\sigma(y)})_{\sigma(x)} + (f_y)_{\sigma(x)} - 2f_{\sigma(x)}g(y)] \\ &\qquad + 2g(y)m[f_x + f_{\sigma(x)} - 2fg(x)] + 2g(x)m[f_y + f_{\sigma(y)} - 2fg(y)] \\ &\quad = H(y) + H(x) + H(x) + H(y) + 2g(y)H(x) + 2g(x)H(y) \\ &\quad = 2H(x) + 2H(y) + 2g(y)H(x) + 2g(x)H(y), \end{aligned}$$

where the last identity is due to our assumption that m is linear and invariant.

Now, if we take $F = cg + H$, where $c = f(e)$, then

$$\begin{aligned}\|F(x) - \frac{1}{2}(f(x) + f(\sigma(x)))| &= \left|cg(x) + H(x) - \frac{1}{2}\big(f(x) + f(\sigma(x))\big)\right| \\ &= \left|cg(x) + h(x) - \frac{1}{2}\big(f(x) + f(\sigma(x))\big) + H(x) - h(x)\right| \\ &\leq \left|cg(x) + h(x) - \frac{1}{2}\big(f(x) + f(\sigma(x))\big)\right| + |H(x) - h(x)| \\ &\leq \varphi(x) + \varphi(x)/2\end{aligned}$$

for all $x \in G$. Since g satisfies Eq. (7) and H satisfies Eq. (37), then for all $x, y \in G$, we have

$$\begin{aligned}&F(xy) + F(yx) + F(x\sigma(y)) + F(\sigma(y)x) \\ &\quad = c[g(xy) + g(yx) + g(x\sigma(y)) + g(\sigma(y)x)] \\ &\qquad + H(xy) + H(yx) + H(x\sigma(y)) + H(\sigma(y)x) \\ &\quad = c4g(x)g(y) + 2H(x)g(y) + 2H(y)g(x) + 2H(y) + 2H(x) \\ &\quad = 2F(x)g(y) + 2F(y)g(x) + 2H(y) + 2H(x).\end{aligned}$$

This completes the proof of theorem.

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Closed-Form Solution of a LAN Gateway Queueing Model

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In Honor of Constantin Carathéodory

Abstract In a recent article, an interesting back-to-back queueing model is developed for a gateway linking two LANs. The model ends up with a two-variable functional equation defining the two-dimensional probability generating function (PGF) of the distribution of the gateway occupancy. Unfortunately, however, the article leaves the equation unsolved, citing the traditional difficulty to attack such equations mathematically.

In this chapter, we manage to solve the functional equation for the unknown PGF, utilizing to a great extent the knowledge of the physical properties of the underlying gateway. The closed-form solution obtained for the PGF is validated in several ways, both mathematical and physical. Furthermore, we derive expectations for the gateway occupancy and also validate them both mathematically and physically.

1 Introduction

There are advantages to install several gatewayed local area networks (LANs) instead of installing a single large LAN having the same number of nodes [1]. The gatewaying approach is most fruitful when the nodes can be divided into groups

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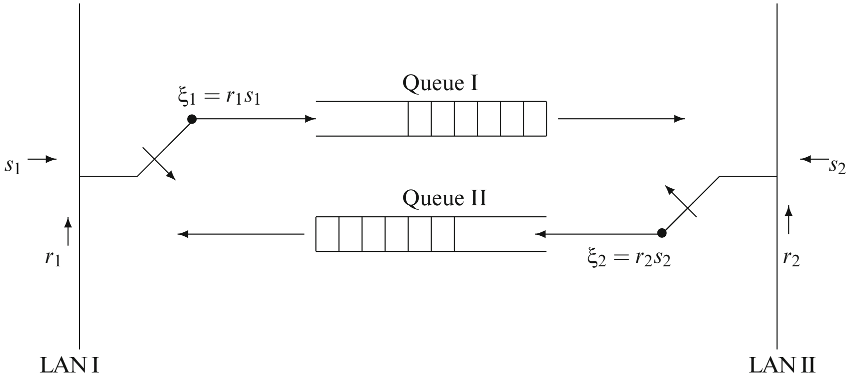

Fig. 1 Gateway modeled as two back-to-back interfering queues

such that a node in a group communicates more often with nodes of the same group than with nodes of another group. The nodes of each group can then be integrated in one LAN, with each two such LANs linked by a gateway. Traffic emanating from a certain node can now be classified as either internal or external, according as the receiving node belongs to the same LAN or a different LAN.

The present gateway model, depicted in Fig. 1 was first introduced in [2]. In this model, the function of the gateway is to handle the external traffic between the two LANs. Specifically, if the gateway links LANs I and II, as shown in Fig. 1, it moves to LAN II the traffic destined to it from LAN I and vice versa. As a consequence, the gateway should be able to distinguish external from internal traffic. Basically, if the gateway recognizes an external packet on either LAN, it will copy it in a buffer and then transmit it to the other LAN when transmission is possible.

The two LANs on the sides of the gateway are assumed to operate as follows:

- Nodes exchange data as packets of fixed size.
- Time is slotted, with the slot length equal to the transmission time of one packet.
- A packet may be transmitted only at the start of a slot.
- Only one packet may be transmitted on a LAN at any given time.

Many LANs satisfy these characteristics such as the slotted CSMA and slotted token ring networks [1].

Clearly, the gateway should have two buffers, one to queue packets going from LAN I to LAN II and one in the other direction, as shown in Fig. 1. It acts as a normal node when seen from any of the two LANs. If it identifies a packet, say, on LAN I destined to LAN II, it will copy it using its LAN I interface and protocol and later sends it onto LAN II using its LAN II interface and protocol. The two LANs as well as the gateway are synchronized, i.e., have the same slot length and boundaries. We will call a LAN *active* in a given slot if it carries in that slot a packet sent by one of its nodes. A LAN that is not active in a slot is called *inactive*. When a LAN is active, the packet it is carrying is either internal or external, according as the destination node is in the same or the other LAN.

Let r_1 denote the probability that LAN I is active in each slot. Thus, $\overline{r_1} = 1 - r_1$ is the probability that LAN I is inactive in each slot. Similarly, let r_2 and $\overline{r_2} = 1 - r_2$ denote the probabilities that LAN II is active and inactive, respectively, in each slot. Furthermore, let s_1 denote the probability that if LAN I is active in a certain slot, then the packet being transmitted is external. Thus, $\overline{s_1} = 1 - s_1$ denotes the probability that if LAN I is active in a certain slot, then the packet being transmitted is internal. And similarly, let s_2 and $\overline{s_2} = 1 - s_2$ denote the probabilities that if LAN II is active in a certain slot, then the packet being transmitted is external and internal, respectively.

External packets of LAN I will be stored in Queue I of the gateway, whereas those of LAN II will be stored in Queue II. The packet at the head of each queue will be transmitted to the destination LAN when the latter is inactive. This means that priority on each LAN is given to its own traffic.

In view of these assumptions, the gateway can be modeled as two back-to-back, coupled (interfering or mutually dependent) queues shown in Fig. 1. Each queue is single server with infinite calling population and infinite waiting room. The coupling of the two queues is due to the fact that one queue cannot have a departure when the other has an arrival. This coupling is signified in the model by the two switch-like mechanisms.

Let ξ_1 denote the probability that LAN I is externally active, i.e., active with a packet destined for LAN II. Note that ξ_1 also represents the packet arrival rate (packets per slot) into Queue I, with mean $1/\xi_1$ slots. This implies that the packet interarrival time into Queue I is geometrically distributed. Using our conventions, then $\overline{\xi_1} = 1 - \xi_1 = 1 - r_1 s_1 = \overline{r_1} + r_1\overline{s_1}$ denotes the probability that LAN I is not externally active, i.e., either inactive or internally active. In a similar manner, let ξ_2 and $\overline{\xi_2} = 1 - \xi_2 = 1 - r_1 s_1 = \overline{r_1} + r_1\overline{s_1}$ denote the probabilities that LAN II is and is not, respectively, externally active. And, again we note that ξ_2 also represents the packet arrival rate into Queue II (packets per slot). This implies that the packet interarrival time into Queue II is geometrically distributed with mean $1/\xi_2$ slots.

Of interest is the steady state distribution of the gateway occupancy given by

$$p_{m,n} = \Pr[X = m, Y = n],$$

where X and Y are the number of packets in Queue I and Queue II, respectively, in steady state. In [2] an attempt was made to find for this distribution its probability generating function (PGF) defined by

$$P(x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n} x^m y^n, \quad |x|,\ |y| \leq 1. \tag{1}$$

The attempt ended with only the following functional equation [3, 4] which defines the required PGF

$$P(x, y) = \frac{1}{xy - M(x, y)}\Big((y-1)(M(x,0) + \overline{r_1}\xi_2 xy)P(x,0) \\ +(x-1)(M(0,y) + \overline{r_2}\xi_1 xy)P(0,y) + (x-1)(y-1)M(0,0)P(0,0)\Big), \tag{2}$$

where

$$M(x, y) = (\overline{r_1} + r_1\overline{s_1}y + \xi_1 xy)(\overline{r_2} + r_2\overline{s_2}x + \xi_2 xy). \tag{3}$$

This equation has not been solved, though, and hence is the present chapter.

The present chapter is organized as follows. In Sect. 2 we introduce some preliminary results which offer a great help in the solution of the functional equation, in Sect. 3 we introduce a solution of the functional equation (2), and in Sect. 4 the solution is validated through several tests, not only to ascertain its correctness but also to gain some insights. In Sect. 5 the expected value of the total occupancy of the gateway is calculated. It is again used to validate the solution. In Sect. 6 we validate the obtained occupancy. Then Sect. 7 focuses on the special case when the operational parameters of the two LANs are identical, resulting in two statistically identical queues in the gateway. The results of Sects. 5 and 7 are used in Sect. 8 to generate numerical values which are plotted and discussed. Finally, Sect. 9 gives the concluding remarks.

2 Preliminaries

In this section we will derive some properties which will help solve the functional equation (2). To that end it is worthwhile to recall the PGF definition (1) in its expanded form, namely,

$$\begin{aligned} P(x, y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n}x^m y^n \\ &= p_{0,0} + p_{1,0}x + p_{2,0}x^2 + p_{3,0}x^3 + \cdots, \\ &\quad + p_{0,1}y + p_{0,2}y^2 + p_{0,3}y^3 + \cdots, \\ &\quad + p_{1,1}xy + p_{1,2}xy^2 + p_{2,1}x^2y + p_{2,2}x^2y^2 + \cdots. \end{aligned} \tag{4}$$

From (4), we see that the partial PGF $P(x, 0)$ is defined by

$$\begin{aligned} P(x, 0) &= \sum_{m=0}^{\infty} p_{m,0}x^m, \quad |x| \le 1 \\ &= p_{0,0} + p_{1,0}x + p_{2,0}x^2 + p_{3,0}x^3 + \cdots, \end{aligned} \tag{5}$$

and that the partial PGF $P(0, y)$ is defined by

$$P(0, y) = \sum_{n=0}^{\infty} p_{0,n} y^n, \quad |y| \le 1$$
$$= p_{0,0} + p_{0,1} y + p_{0,2} y^2 + p_{0,3} y^3 + \cdots . \tag{6}$$

It is also worthwhile to recall from single, discrete queueing theory [5, 6] that for a geo/geo/1 queue with arrival rate λ and service rate μ, the PGF $P(z)$ of its occupancy distribution $p_i, i = 0, 1, 2, \ldots,$ is given by

$$P(z) = \frac{\mu(\overline{\lambda} + \lambda z)}{\mu\overline{\lambda} - \overline{\mu}\lambda z} p_0, \tag{7}$$

where

$$p_0 = \frac{\mu - \lambda}{\mu}. \tag{8}$$

And for such a queue, the expected occupancy $E[P]$ is given by

$$E[P] = \left.\frac{dP(z)}{dz}\right|_{z=1}$$
$$= \frac{\lambda\overline{\lambda}}{\mu - \lambda}. \tag{9}$$

Further, we will call a LAN *inactive* if it is idle all the time. If a queue is not inactive, then it is *active*. For example LAN I is inactive if $r_1 = 0$ and is active otherwise. Also, we will call a LAN *externally inactive* if it is active but with no packets going to the other LAN (through the gateway.) For example, LAN II is externally inactive if $s_2 = 0$ and is externally active otherwise. Finally, we will call a queue in the double-queue gateway system *inactive* if it is empty all the time. If a queue is not inactive, then it is *active*. For example, Queue I is inactive if $\xi_1 = 0$ (i.e., $r_1 = 0$ or $s_1 = 0$).

With the above definitions in mind, and by considering the gateway model depicted in Fig. 1, together with its assumptions, we will derive the following properties.

Property 1. If $s_1 = 0$, then $P(x, y) = P(y)$

If $s_1 = 0$, the arrival rate into Queue I $\xi_1 = r_1 s_1$ is zero, which means that Queue I is inactive. Then the two-queue gateway model of Fig. 1 reduces to a one-queue system, namely, that of Queue II. Let $p_j, j = 0, 1, 2, \ldots$ be the distribution of the occupancy of that queue. As Queue I is inactive, it follows that

$$p_{i,j} = \begin{cases} 0, & i > 0,\ j \ge 0 \\ p_j, & i = 0,\ j \ge 0 \end{cases}. \tag{10}$$

Let $P(z)$ be the PGF of $p_j, j = 0, 1, 2, \ldots$. That is,

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad |z| \leq 1. \tag{11}$$

Then using (4), (10), and (11), we get

$$\begin{aligned} P(x, y)|_{s_1=0} &= \sum_{n=0}^{\infty} p_{0,n} y^n |_{s_1=0} \\ &= \sum_{n=0}^{\infty} p_n y^n \\ &= P(y). \end{aligned} \tag{12}$$

Property 2. If $s_1 = 0$, then $P(x, 0) = p_0$

From (5) and (10), it follows that

$$P(x, 0)|_{s_1=0} = p_{0,0}|_{s_1=0} = p_0.$$

Property 3. If $s_1 = 0$, then $P(0, y) = P(y)$

Using (4), (10), and (11), we get

$$\begin{aligned} P(0, y)|_{s_1=0} &= \sum_{n=0}^{\infty} p_{0,n} y^n |_{s_1=0} \\ &= \sum_{n=0}^{\infty} p_n y^n \\ &= P(y), \end{aligned}$$

Property 4. If $s_2 = 0$, then $P(x, y) = P(x)$

The reasoning here is along the same lines as in Property 2.

Property 5. If $s_2 = 0$, then $P(0, y) = p_0$

The reasoning here is along the same lines as in Property 3.

Property 6. If $s_2 = 0$, then $P(x, 0) = P(x)$

The reasoning here is along the same lines as in Property 4.

Property 7. When LAN I is externally inactive, Queue II behaves as a geo/geo/1 queue with arrival rate ξ_2 and service rate $\overline{r_1}$.

This is evident from the gateway model depicted in Fig. 1 in light of the assumptions of Sect. 1. In each slot, a packet arrives into Queue II with probability ξ_2 and thus does not arrive with probability $\overline{\xi_2}$. This implies that the interarrival time is geometrically distributed with mean $1/\xi_2$. On the output side, if the queue is nonempty, in each slot a packet departs with probability $\overline{r_1}$ (when LAN I is inactive) and thus does not depart with probability r_1. This implies that the service time is geometrically distributed with mean $1/\overline{r_1}$. These are the characteristics of a geo/geo/1 queue. We can also prove the property mathematically, by substituting $s_1 = 0$ which indicates that LAN I is externally inactive and hence that Queue I is inactive. In that case, the two-queue gateway model of Fig. 1 reduces to a one-queue system, namely, Queue II. Let p_j be the probability that Queue II has j packets, $j = 0, 1, 2, \ldots$. and $P(z)$ be its generating function defined by

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad |z| \le 1 \tag{13}$$

By Property 1, we have

$$P(x, y)|_{s_1=0} = P(y). \tag{14}$$

And by Property 2, we have

$$P(x, 0)|_{s_1=0} = p_0. \tag{15}$$

And by Property 4, we have

$$P(0, y)|_{s_1=0} = P(y). \tag{16}$$

Finally, from (3) we have

$$M(x, y)|_{s_1=0} = (\overline{r_2} + r_2 x + r_2\overline{s_2}xy)(\overline{r_1} + r_1 y), \tag{17}$$

$$M(x, 0)|_{s_1=0} = \overline{r_1}(\overline{r_2} + r_2\overline{s_2}x), \tag{18}$$

$$M(0, y)|_{s_1=0} = \overline{r_2}(\overline{r_1} + r_1 y), \tag{19}$$

and

$$M(0, 0)|_{s_1=0} = \overline{r_1 r_2}. \tag{20}$$

Using (14)–(20) into (2) yields

$$P(x, y)|_{s_1=0,x=y} = P(y)$$
$$= \frac{p_0(y-1)(\overline{r_1}(\overline{r_2} + r_2\overline{s_2}y) + \overline{r_1}\xi_2 y^2) + (y-1)\overline{r_1 r_2} + (y-1)\overline{r_2}(\overline{r_1} + r_1 y)P(y)}{y^2 - (\overline{r_1} + r_1 y)(\overline{r_2} + r_2\overline{s_2}y + \xi_2 y^2)}$$

Solving for $P(y)$, performing in the process some straightforward, yet tedious, manipulations, we get

$$P(y) = \frac{(y-1)y\overline{r_1}(\overline{\xi_2} + \xi_2 y)p_0}{y(y + \xi_2 - \xi_2 y - r_1\xi_2 + r_1\xi_2 y + r_1\xi_2 y - \xi_2 r_1 y^2 - 1 - r_1 y + r_1)}$$
$$= \frac{\overline{r_1}(\overline{\xi_2} + \xi_2 y)}{\overline{r_1}\overline{\xi_2} - r_1\xi_2 y}p_0, \tag{21}$$

which is identical to (7), noting that ξ_2 in (21) is λ in (7) and $\overline{r_1}$ in (21) is μ in (7).

Property 8. When Queue II is inactive, Queue I behaves as a geo/geo/1 queue with arrival rate ξ_1 and service rate $\overline{r_2}$.

The reasoning here is along the same lines as in Property 7.

Property 9. If $s_1 = 0$, then

$$P(0,y)|_{s_1=0} = \frac{(\overline{\xi_2} + \xi_2 y)(\overline{r_2} - \xi_1)}{\overline{r_1}\overline{\xi_2} - r_1\xi_2 y} \tag{22}$$

From Property 3 we have

$$P(0,y)|_{s_1=0} = P(y),$$

and using Property 7 we have

$$\begin{aligned} P(0,y)|_{s_1=0} &= P(y) \\ &= \frac{\overline{r_1}(\overline{\xi_2} + \xi_2 y)}{\overline{r_1}\overline{\xi_2} - r_1\xi_2 y} p_0 \end{aligned}$$

substituting for P_0 to get that

$$\begin{aligned} P(0,y)|_{s_1=0} &= \frac{\overline{r_1}(\overline{\xi_2} + \xi_2 y)}{\overline{r_1}\overline{\xi_2} - r_1\xi_2 y} \times \frac{\overline{r_1} - \xi_2}{\overline{r_1}} \\ &= \frac{(\overline{\xi_2} + \xi_2 y)(\overline{r_2} - \xi_1)}{\overline{r_1}\overline{\xi_2} - r_1\xi_2 y} \end{aligned}$$

Property 10. If $x = 1$, $y = 0$, then $P(x,y) = \frac{\overline{r_1} - \xi_2}{\overline{r_1}}$

By Property 9, if $s_1 = 0$, such that $\xi_1 = 0$, then Queue I will be inactive and Queue II will be active, behaving as a geo/geo/1 queueing system with arrival rate ξ_1 and service rate $\overline{r_2}$. Now, if we substitute $x = 1$ in (5), we get

$$\begin{aligned} P(1,0) &= p_{0,0} + p_{1,0} + p_{2,0} + p_{3,0} + \cdots \\ &= \Pr[Y = 0]. \end{aligned}$$

That is, $P(1,0)$ is the marginal probability that Queue II of the gateway is inactive (empty). Since Queue II when isolated from the gateway operates as a geo/geo/1 system with arrival rate ξ_2 and service rate $\overline{r_1}$ and since p_0 then would be as given by (8), then

$$P(1,0) = \frac{\overline{r_1} - \xi_2}{\overline{r_1}}. \tag{23}$$

Property 11. If $x = 0$, $y = 1$, then

$$P(x,y)|_{x=0,y=1} = \frac{\overline{r_2} - \xi_1}{\overline{r_2}}. \tag{24}$$

The reasoning here is along the same lines as in Property 12.

Property 12. If $x = y$, then $P(x, y)$ represents the PGF of the distribution of the total gateway occupancy.

From (4) we get

$$\begin{aligned}
P(x,x) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n}x^{m+n}, \qquad |x| \le 1 \\
&= p_{0,0} + p_{1,0}x + p_{2,0}x^2 + p_{3,0}x^3 + \cdots \\
&\quad + p_{0,1}x + p_{0,2}x^2 + p_{0,3}x^3 + \cdots \\
&\quad + p_{1,1}x^2 + p_{1,2}x^3 + p_{2,1}x^3 + p_{2,2}x^4 + \cdots \\
&= p_{0,0} + (p_{0,1} + p_{1,0})\,x + (p_{0,2} + p_{1,1} + p_{2,0})\,x^2 \\
&\quad + (p_{0,3} + p_{1,2} + p_{2,1} + p_{3,0})\,x^3 + \cdots \\
&= \sum_{m=0}^{\infty} q_m x^m =: Q(x)
\end{aligned}$$

where $q_m = \sum_{i=0}^{m} p_{i,m-i}$, $m = 0, 1, \ldots,$ is the distribution of gateway occupancy (both queues) and $Q(x)$ is its PGF.

3 Solution of Functional Equation

In this section we will solve the two-dimensional functional equation (2) for the generating function $P(x, y)$. It is worth mentioning that such equations arise naturally when trying to analyze two-queue systems using a PGF approach. In these systems, the two queues can be either physical or logical. An example of the latter is a single physical queue that receives customers of two classes, e.g., high and low priority. It should be noted that the literature in the past four decades has witnessed many equations similar to (2).

In [7] an equation arises when considering two queues in a system where the customer on arrival joins the shorter queue. In [8] it arises when considering two interfering queues in a packet radio network consisting of two nodes that transmit their packets over a common channel to a central station. In [9] it arises when analyzing a two-queue system that results from arriving customers simultaneously placing two demands handled independently by two servers. In [10] it arises from a queueing model with applications in databases. In [11] it arises when considering a system of two parallel queues with infinite capacities. In [12] it arises when considering a two-queue system with two types of customers. In [13] it arises when modeling a clocked-buffered 2×2 switch with two parallel servers and two types of customers (jobs). In [14] it arises when modeling a multimedia multiplexer handling traffic of two classes, real-time traffic and non-real time. In [15] it arises when analyzing the performance of a space division output-buffered switch. In [16] it arises when considering a two-stage tandem queue, where customers (jobs) arrive demanding service at both queues before leaving the system. In [17] it arises when analyzing the occupancy of a discrete time priority queue. In [18] it arises when analyzing the delay of an ATM multiplexer (server) handling multimedia traffic of two classes: real time and non-real time. In [19] it arises when analyzing the performance of an output-buffered multichannel switch receiving traffic of two classes.

In all of the above cited works, no closed-form solution has been given for the resulting functional equation. Typically, either an open-form solution, containing summations or products, or a solution for a special case of the equation is given. In the sequel, we will proceed to tackle our two-dimensional functional equation (2), exploiting the knowledge we have about the system it represents, ending with a closed-form solution that is validated in many ways.

By investigating (2), it can be seen that there are three unknowns: the function $P(x, 0)$, the function $P(0, y)$, and the constant $P(0, 0)$. If found they can be plugged into the functional equation (2) and the solution is obtained. It can be seen that the two unknown functions have three requirements:

- The function $P(0, y)$ should satisfy (22) at $s_1 = 0$, hence $\xi_1 = 0$ (i.e., only Queue II is active)

$$
\begin{aligned}
P(0, y)|_{s_1=0} &= p_{0,0} + p_{0,1}y + p_{0,2}y^2 + p_{0,3}y^3 + \cdots |_{s_1=0}, \\
&= \frac{(\overline{\xi_2} + \xi_2 y)(\overline{r_1} - \xi_2)}{\overline{r_1}\,\overline{\xi_2} - r_1 \xi_2 y}
\end{aligned}
\tag{25}
$$

- The function $P(x, 0)$ should satisfy at $s_2 = 0$, hence $\xi_2 = 0$ (i.e., only Queue I is active)

$$
\begin{aligned}
P(x, 0)|_{s_2=0} &= p_{0,0} + p_{1,0}x + p_{2,0}x^2 + p_{3,0}x^3 + \cdots |_{s_2=0} \\
&= \frac{(\overline{\xi_1} + \xi_1 x)(\overline{r_2} - \xi_1)}{\overline{r_2}\,\overline{\xi_1} - r_2 \xi_1 x}
\end{aligned}
\tag{26}
$$

- The function $P(x, 0)$ should satisfy (23) at $x = 1$.

$$
P(1, 0) = \Pr[\, Y = 0] = \frac{\overline{r_1} - \xi_2}{\overline{r_1}}. \tag{27}
$$

- The function $P(0, y)$ should satisfy (24) at $y = 1$.

$$
P(0, 1) = \Pr[X = 0] = \frac{\overline{r_2} - \xi_1}{\overline{r_2}}. \tag{28}
$$

- Both $P(x, 0)$ and $P(0, y)$ should be equal at $x = y = 0$.

$$P(x,0)|_{x=0} = P(0,y)|_{y=0} \tag{29}$$

It can be easily shown that the two functions

$$P(x,0) = \frac{(\overline{\xi_1} + \xi_1 x)(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_1}(\overline{r_2}\overline{\xi_1} - r_2\xi_1 x)} \tag{30}$$

and

$$P(0,y) = \frac{(\overline{\xi_2} + \xi_2 y)(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_2}(\overline{r_1}\overline{\xi_2} - r_1\xi_2 y)} \tag{31}$$

satisfy the five conditions (25)–(29):

- At $s_1 = 0$, hence $\xi_1 = 0$, we have

$$\begin{aligned} P(0,y)|_{\xi_1=0} &= \frac{(\overline{\xi_2} + \xi_2 y)(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_2}(\overline{r_1}\overline{\xi_2} - r_1\xi_2 y)}|_{\xi_1=0} \\ &= \frac{(\overline{\xi_2} + \xi_2 y)(\overline{r_1} - \xi_2)}{\overline{r_1}\overline{\xi_2} - r_1\xi_2 y} \end{aligned}$$

  which satisfies (25).
- At $s_2 = 0$, hence $\xi_2 = 0$, we have

$$\begin{aligned} P(x,0)|_{s_2=0} &= \frac{(\overline{\xi_1} + \xi_1 x)(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_1}(\overline{r_2}\overline{\xi_1} - r_2\xi_1 x)}|_{s_2=0} \\ &= \frac{(\overline{\xi_1} + \xi_1 x)(\overline{r_2} - \xi_1)}{\overline{r_2}\overline{\xi_1} - r_2\xi_1 x} \end{aligned}$$

  which satisfies (26)
- At $x = 1$ we have

$$\begin{aligned} P(x,0)|_{x=1} &= \frac{(\overline{\xi_1} + \xi_1 x)(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_1}(\overline{r_2}\overline{\xi_1} - r_2\xi_1 x)}|_{x=1} \\ &= \frac{(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_1}(1 - r_2 - \xi_1 + r_2\xi_1 - r_2\xi_1)} \\ &= \frac{\overline{r_1} - \xi_2}{\overline{r_1}} \end{aligned}$$

  which satisfies (27)
- At $y = 1$ we have

$$
\begin{aligned}
P(0,y)|_{y=1} &= \frac{(\overline{\xi_2}+\xi_2 y)(\overline{r_1}-\xi_2)(\overline{r_2}-\xi_1)}{\overline{r_2}(\overline{r_1}\overline{\xi_2}-r_1\xi_2 y)}|_{y=1} \\
&= \frac{(\overline{r_1}-\xi_2)(\overline{r_2}-\xi_1)}{\overline{r_2}(1-r_1-\xi_2+r_1\xi_2-r_1\xi_2)} \\
&= \frac{\overline{r_2}-\xi_1}{\overline{r_1}}
\end{aligned}
$$

which satisfies (28).

- At $x = 0$ Eq. (30) and at $y = 0$ Eq. (30), both give the same value, namely,

$$
P(x,0)|_{x=0} = P(0,y)|_{y=0} = \frac{(\overline{r_1}-\xi_2)(\overline{r_2}-\xi_1)}{\overline{r_1 r_2}} \tag{32}
$$

which satisfies (29).

When substituting (30)–(32) in (2), we get

$$
\begin{aligned}
P(x,y) = \frac{1}{xy-M(x,y)} \Bigg\{ & (y-1)(M(x,0)+\overline{r_1}\xi_2 xy)\frac{(\overline{\xi_1}+\xi_1 x)(\overline{r_1}-\xi_2)(\overline{r_2}-\xi_1)}{\overline{r_1}(\overline{r_2}\overline{\xi_1}-r_2\xi_1 x)} \\
&+(x-1)(M(0,y)+\overline{r_2}\xi_1 xy)\frac{(\overline{\xi_2}+\xi_2 y)(\overline{r_1}-\xi_2)(\overline{r_2}-\xi_1)}{\overline{r_2}(\overline{r_1}\overline{\xi_2}-r_1\xi_2 y)} \\
&+(x-1)(y-1)M(0,0)\frac{(\overline{r_1}-\xi_2)(\overline{r_2}-\xi_1)}{\overline{r_1 r_2}} \Bigg\}.
\end{aligned} \tag{33}
$$

From (3), we get

$$
M(x,0) = \overline{r_1}(\overline{r_2}+r_2\overline{s_2}x), \tag{34}
$$

$$
M(0,y) = \overline{r_2}(\overline{r_1}+r_1\overline{s_1}y), \tag{35}
$$

and

$$
M(0,0) = \overline{r_1 r_2}. \tag{36}
$$

Substituting (34)–(36) into (33), we get the final solution of (2) as follows

$$
\begin{aligned}
P(x,y) = {} & \alpha_1\alpha_2\frac{(y-1)(\overline{r_2}+r_2\overline{s_2}x+\xi_2 xy)(\overline{\xi_1}+\xi_1 x)}{(xy-M(x,y))\,(\overline{r_2}\overline{\xi_1}-r_2\xi_1 x)} \\
&+\alpha_1\alpha_2\frac{(x-1)(\overline{r_1}+r_1\overline{s_1}y+\xi_1 xy)(\overline{\xi_2}+\xi_2 y)}{(xy-M(x,y))\,(\overline{r_1}\overline{\xi_2}-r_1\xi_2 y)} \\
&+\frac{\alpha_1\alpha_2(x-1)(y-1)}{(xy-M(x,y))},
\end{aligned} \tag{37}
$$

where

$$\alpha_1 = \overline{r_2} - \xi_1, \alpha_2 = \overline{r_1} - \xi_2. \tag{38}$$

Now that we have obtained $P(x, y)$, we can prove the correctness of the assumption of the two functions (30) and (31) simply by substituting $y = 0$ and $x = 0$, respectively, in (2), to get back (30) and (31). (The details of these substitutions are given in the next section.)

4 Validation of the Solution

In this section, we will validate (37) as a solution of (2). Just as we did with the functional equation (2), we will validate the PGF (37) using two approaches: mathematical and physical.

4.1 Mathematical Validation

Here, we will operate on the PGF (37) as a mathematical entity.

4.1.1 Obtaining $P(x, 0)$

If we put $y = 0$ in (37), we should get (30). Indeed,

$$\begin{aligned} P(x,0) &= \alpha_1\alpha_2 \frac{(-1)(\overline{r_2} + r_2\overline{s_2}x)(\overline{\xi_1} + \xi_1 x)}{(-M(x,0))\,(\overline{r_2}\overline{\xi_1} - r_2\xi_1 x)} + \alpha_1\alpha_2 \frac{(x-1)(\overline{r_1})(\overline{\xi_2})}{(-M(x,0))\,(\overline{r_1}\overline{\xi_2})} \\ &\quad + \frac{\alpha_1\alpha_2(x-1)(-1)}{(-M(x,0))}. \\ &= \alpha_1\alpha_2 \frac{(\overline{r_2} + r_2\overline{s_2}x)(\overline{\xi_1} + \xi_1 x)}{M(x,0)(\overline{r_2}\overline{\xi_1} - r_2\xi_1 x)}, \end{aligned}$$

but since from (3) we get

$$M(x,0) = \overline{r_1}(\overline{r_2} + r_2\overline{s_2}x).$$

also using (38) so that

$$\begin{aligned} P(x,0) &= \alpha_1\alpha_2 \frac{(\overline{\xi_1} + \xi_1 x)}{\overline{r_1}(\overline{r_2}\overline{\xi_1} - r_2\xi_1 x)} \\ &= \frac{(\overline{r_2} - \xi_1)(\overline{r_1} - \xi_2)(\overline{\xi_1} + \xi_1 x)}{\overline{r_1}(\overline{r_2}\overline{\xi_1} - r_2\xi_1 x)} \end{aligned}$$

which is identical to (30).

4.1.2 Obtaining $P(0, y)$

If we put $x = 0$ in (37), we should get (31). Indeed,

$$P(0, y) = \alpha_1\alpha_2 \frac{(y-1)(\overline{r_2})(\overline{\xi_1})}{(-M(0, y))\,(\overline{r_2}\overline{\xi_1})} + \alpha_1\alpha_2 \frac{(-1)(\overline{r_1} + r_1\overline{s_1}y)(\overline{\xi_2} + \xi_2 y)}{(-M(0, y))\,(\overline{r_1}\overline{\xi_2} - r_1\xi_2 y)}$$
$$+ \frac{\alpha_1\alpha_2(-1)(y-1)}{(-M(0, y))},$$

which can be rewritten as follows:

$$P(0, y) = \alpha_1\alpha_2 \frac{(y-1)}{(-M(0, y))} + \alpha_1\alpha_2 \frac{(-1)(\overline{r_1} + r_1\overline{s_1}y)(\overline{\xi_2} + \xi_2 y)}{(-M(0, y))\,(\overline{r_1}\overline{\xi_2} - r_1\xi_2 y)}$$
$$+ \frac{\alpha_1\alpha_2(-1)(y-1)}{(-M(0, y))},$$
$$= \alpha_1\alpha_2 \frac{(\overline{r_1} + r_1\overline{s_1}y)(\overline{\xi_2} + \xi_2 y)}{M(0, y)(\overline{r_1}\overline{\xi_2} - r_1\xi_2 y)}.$$

But from (3) we get

$$M(0, y) = \overline{r_2}(\overline{r_1} + r_1\overline{s_1}y),$$

using (38) to get

$$P(0, y) = \alpha_1\alpha_2 \frac{(\overline{r_1} + r_1\overline{s_1}y)(\overline{\xi_2} + \xi_2 y)}{\overline{r_2}(\overline{r_1} + r_1\overline{s_1}y)(\overline{r_1}\overline{\xi_2} - r_1\xi_2 y)},$$
$$= \frac{(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)(\overline{\xi_2} + \xi_2 y)}{\overline{r_2}(\overline{r_1}\overline{\xi_2} - r_1\xi_2 y)}, \tag{39}$$

which is identical to (31).

4.1.3 Obtaining $P(0, 0)$

If we put $x = y = 0$ in (37), we should get (32). Indeed, using (37), we have

$$P(0, 0) = \alpha_1\alpha_2 \frac{(-1)(\overline{r_2})(\overline{\xi_1})}{(-M(0, 0))\,(\overline{r_2}\overline{\xi_1})} + \alpha_1\alpha_2 \frac{(-1)(\overline{r_1})(\overline{\xi_2})}{(-M(0, 0))\,(\overline{r_1}\overline{\xi_2})}$$
$$+ \frac{\alpha_1\alpha_2(-1)(-1)}{(0 - M(0, 0))}.$$

which can be rewritten as

$$P(0,0) = \frac{\alpha_1 \alpha_2}{M(0,0)}.$$

But from (3) we get

$$M(0,0) = \overline{r_1 r_2},$$

using (38)

$$P(0,0) = \frac{(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_1 r_2}}, \tag{40}$$

which is identical to (32).

4.1.4 Obtaining $P(x,0)$ at $r_1 = 0$

Going back to the definition (5) of $P(x,0)$, we note that when $r_1 = 0$ then $P(x,0) = p_{0,0}$, i.e., the probability that the gateway is empty. Since if $r_1 = 0$, then Queue I is necessarily empty, and the double queue system of the gateway reduces to single queue (Queue II in which case $p_{0,0}$ becomes equivalent to the probability that Queue II is empty, given that its arrival rate is ξ_1 and its service rate is $\overline{r_1} = 1$. For such single queue, using (23), the latter probability is $1 - \xi_2$. Sure enough, from (30) we get

$$P(x,0) = \frac{(\overline{r_2} - \xi_1)(\overline{r_1} - \xi_2)(\overline{\xi_1} + \xi_1 x)}{\overline{r_1}(\overline{r_2}\overline{\xi_1} - r_2 \xi_1 x)},$$

and if we put $r_1 = 0$ in the above equation, we get

$$\begin{aligned} P(x,0)|_{r_1=0} &= \frac{\overline{r_2}(1-\xi_2)}{\overline{r_2}} \\ &= 1 - \xi_2. \end{aligned} \tag{41}$$

which is what we expect.

4.1.5 Obtaining $P(0,y)$ at $r_2 = 0$

In a manner similar to that in the previous section, from (31) we get

$$P(0,y) = \frac{(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)(\overline{\xi_2} + \xi_2 y)}{\overline{r_2}(\overline{r_1}\overline{\xi_2} - r_1 \xi_2 y)},$$

and if we put $r_2 = 0$ in the above equation, we get

$$P(0, y)|_{r_2=0} = \frac{\overline{r_1}(1 - \xi_1)}{\overline{r_1}} = 1 - \xi_1, \tag{42}$$

which is what we expect.

4.1.6 Obtaining $P(1, 0)$

If we put $x = 1,\ y = 0$ in (37), we should get (23). Indeed, using (37), we have

$$P(1, 0) = \alpha_1\alpha_2 \frac{(-1)(\overline{r_2} + r_2\overline{s_2})(\overline{\xi_1} + \xi_1)}{(-M(1, 0))\,(\overline{r_2}\overline{\xi_1} - r_2\xi_1)}.$$

By using (3) we get

$$M(1, 0) = \overline{r_1}(\overline{r_2} + r_2\overline{s_2}),$$

also from (38) so that

$$P(1, 0) = \frac{\alpha_1\alpha_2}{(\overline{r_1})\,(\overline{r_2}\overline{\xi_1} - r_2\xi_1)} = \frac{\overline{r_1} - \xi_2}{\overline{r_1}}, \tag{43}$$

which is identical to (23).

4.1.7 Obtaining $P(0, 1)$

If we put $x = 0,\ y = 1$ in (37), we should get (28). Indeed, using (37), we have

$$P(0, 1) = \alpha_1\alpha_2 \frac{(-1)(\overline{r_1} + r_1\overline{s_1})(\overline{\xi_2} + \xi_2)}{(-M(0, 1))\,(\overline{r_1}\overline{\xi_2} - r_1\xi_2)}$$

and by using (3) we get

$$M(0, 1) = \overline{r_2}(\overline{r_1} + r_1\overline{s_1}),$$

also from (38) so that

$$P(0, 1) = \frac{\alpha_1\alpha_2}{\overline{r_2}(\overline{r_1}\overline{\xi_2} - r_1\xi_2)} = \frac{\overline{r_2} - \xi_1}{\overline{r_2}}, \tag{44}$$

which is identical to (28).

4.1.8 Normalization Condition

If (37) is a proper solution to (2), i.e., if it is a PGF, it must satisfy

$$P(1,1) = 1.$$

From (37) we get

$$\begin{aligned}
P(1,1) &= \lim_{x\to 1} P(x,x) \\
&= \lim_{x\to 1} \alpha_1\alpha_2 \frac{(x-1)(\overline{r_2} + r_2\overline{s_2}x + \xi_2 x^2)(\overline{\xi_1} + \xi_1 x)}{(x^2 - M(x,x))\,(\overline{r_2}\,\overline{\xi_1} - r_2\xi_1 x)} \\
&\quad + \lim_{x\to 1} \alpha_1\alpha_2 \frac{(x-1)(\overline{r_1} + r_1\overline{s_1}x + \xi_1 x^2)(\overline{\xi_2} + \xi_2 x)}{(x^2 - M(x,x))\,(\overline{r_1}\,\overline{\xi_2} - r_1\xi_2 x)} \\
&\quad + \lim_{x\to 1} \frac{\alpha_1\alpha_2(x-1)^2}{(x^2 - M(x,x))}, \\
&= \frac{0}{0}.
\end{aligned}$$

By applying de l'Hospital's rule, we get

$$\begin{aligned}
P(1,1) &= \alpha_1\alpha_2 \lim_{x\to 1} \frac{d/dx\Big((x-1)(\overline{r_2} + r_2\overline{s_2}x + \xi_2 x^2)(\overline{\xi_1} + \xi_1 x)\Big)}{d/dx\Big((x^2 - M(x,x))\,(\overline{r_2}\,\overline{\xi_1} - r_2\xi_1 x)\Big)} \\
&\quad + \alpha_1\alpha_2 \lim_{x\to 1} \frac{d/dx\Big((x-1)(\overline{r_1} + r_1\overline{s_1}x + \xi_1 x^2)(\overline{\xi_2} + \xi_2 x)\Big)}{d/dx\Big((x^2 - M(x,x))\,(\overline{r_1}\,\overline{\xi_2} - r_1\xi_2 x)\Big)} \\
&\quad + \alpha_1\alpha_2 \lim_{x\to 1} \frac{d/dx(x-1)^2}{d/dx\,(x^2 - M(x,x))},
\end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned}
P(1,1) &= \alpha_1\alpha_2 \lim_{x\to 1} \frac{(\overline{r_2} + r_2\overline{s_2}x + \xi_2 x^2)(\overline{\xi_1} + \xi_1 x)}{(2x - M_x(x,x))\,(\overline{r_2}\,\overline{\xi_1} - r_2\xi_1 x)} \\
&\quad + \alpha_1\alpha_2 \lim_{x\to 1} \frac{(\overline{r_1} + r_1\overline{s_1}x + \xi_1 x^2)(\overline{\xi_2} + \xi_2 x)}{(2x - M_x(x,x))\,(\overline{r_1}\,\overline{\xi_2} - r_1\xi_2 x)} \\
&\quad + \alpha_1\alpha_2 \lim_{x\to 1} \frac{2(x-1)}{(2x - M_x(x,x))}.
\end{aligned}$$

So that after applying de l'Hospital's rule we get

$$P(1,1) = \alpha_1\alpha_2 \frac{1}{(2 - M_x(1,1))\,(\overline{r_2}\overline{\xi_1} - r_2\xi_1)} + \alpha_1\alpha_2 \frac{1}{(2 - M_x(1,1))\,(\overline{r_1}\overline{\xi_2} - r_1\xi_2)},$$

which can be rewritten as follows:

$$\begin{aligned} P(1,1) &= \frac{\alpha_1\alpha_2}{(2 - M_x(1,1))} \left(\frac{1}{(\overline{r_2} - \xi_1)} + \frac{1}{(\overline{r_1} - \xi_2)} \right) \\ &= \frac{\alpha_1\alpha_2}{(2 - M_x(1,1))} \left(\frac{(\overline{r_1} - \xi_2) + (\overline{r_2} - \xi_1)}{(\overline{r_2} - \xi_1)(\overline{r_1} - \xi_2)} \right) \\ &= \frac{(\overline{r_1} - \xi_2) + (\overline{r_2} - \xi_1)}{(2 - M_x(1,1))} \end{aligned}$$

then

$$P(1,1) = \frac{2 - r_1 - \xi_1 - r_2 - \xi_2}{2 - M_x(1,1)}. \tag{45}$$

But since from (3) we get

$$M(x,x) = (\overline{r_1} + r_1\overline{s_1}x + \xi_1 x^2)(\overline{r_2} + r_2\overline{s_2}x + \xi_2 x^2),$$

so that

$$\begin{aligned} M_x(x,x) &= \frac{d}{dx} M(x,x) \\ &= (r_1\overline{s_1} + 2\xi_1 x)(\overline{r_2} + r_2\overline{s_2}x + \xi_2 x^2) + (\overline{r_1} + r_1\overline{s_1}x + \xi_1 x^2)(r_2\overline{s_2} + 2\xi_2 x), \end{aligned}$$

then

$$M_x(1,1) = r_1 + \xi_1 + r_2 + \xi_2.$$

Substituting into (45) to get

$$\begin{aligned} P(1,1) &= \frac{2 - r_1 - \xi_1 - r_2 - \xi_2}{2 - (r_1 + \xi_1 + r_2 + \xi_2)} \\ &= 1, \end{aligned} \tag{46}$$

as it should.

4.2 Physical Validation

In this subsection we will exploit our knowledge of the gateway queueing model and try to prove that (37) is a proper solution to (2) and really represents the gateway system of Fig. 1.

4.2.1 Gateway Empty When Both LANs Are Inactive

If both LANs are inactive, no packets will enter the gateway and it will then be inactive with probability 1. Thus, if we substitute $r_1 = r_2 = 0$ into (32), we should get 1. Indeed, if

$$r_1 = r_2 = 0,$$

then

$$\xi_1 = \xi_2 = 0.$$

By substituting these values into (32), we get

$$P(0,0) \mid_{r_1=r_2=0} = \frac{(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_1 r_2}} \mid_{r_1=r_2=0} = 1, \tag{47}$$

as it should.

4.2.2 Gateway Empty When Both LANs Are Externally Inactive

If both LANs are externally inactive, no packets will enter the gateway and it will then be inactive with probability 1. Thus, if we substitute $s_1 = s_2 = 0$ into (32), we should get 1. Indeed, if

$$s_1 = s_2 = 0,$$

then

$$\xi_1 = \xi_2 = 0.$$

By substituting these values into (32), we get

$$P(0,0) \mid_{s_1=s_2=0} = \frac{(\overline{r_1} - \xi_2)(\overline{r_2} - \xi_1)}{\overline{r_1 r_2}} \mid_{s_1=s_2=0} = 1, \tag{48}$$

as it should.

4.2.3 Only Queue II Active When LAN I Is Externally Inactive

If LAN I is externally inactive, no packets will enter Queue I and the gateway will then have only Queue II active. Thus, if we substitute $s_1 = 0$ into (37), we should get the PGF (7) of a geo/geo/1 queue, with arrival rate ξ_2 and service rate $\overline{r_1}$. Indeed, if $s_1 = 0$, then

$$\xi_1 = r_1 s_1 = 0. \tag{49}$$

Let p_j be the probability that Queue II has j packets, $j = 0, 1, 2, \ldots$ and that $P(z)$ is its generating function defined by

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \quad |z| \le 1$$

From the assumption that Queue I is inactive, it follows that

$$p_{i,j} = \begin{cases} 0, & i > 0 \\ p_j, & i = 0 \end{cases}.$$

Using (4) we get

$$\begin{aligned} P(x,y)|_{s_1=0} &= \sum_{n=0}^{\infty} p_{0,n} y^n |_{s_1=0} \\ &= p_{0,0} + p_{0,1} y + p_{0,2} y^2 + p_{0,3} y^3 + \cdots |_{s_1=0} \\ &= p_0 + p_1 y + p_2 y^2 + p_3 y^3 + \cdots \\ &= P(y). \end{aligned}$$

Also in this case we note after using (5) and (6) that

$$P(x,0)|_{s_1=0} = p_{0,0}|_{s_1=0} = \frac{\overline{r_1} - \xi_2}{\overline{r_1}} = \Pr[Y = 0], \tag{50}$$

$$P(0,y)|_{s_1=0} = P(y). \tag{51}$$

By using (3), (49) we note that

$$M(x,y)|_{s_1=0} = (\overline{r_2} + r_2 x + r_2 \overline{s_2} xy)(\overline{r_1} + r_1 y), \tag{52}$$

$$M(x,0)|_{s_1=0} = \overline{r_1}(\overline{r_2} + r_2 \overline{s_2} x), \tag{53}$$

$$M(0,y)|_{s_1=0} = \overline{r_2}(\overline{r_1} + r_1 y), \tag{54}$$

$$M(0,0)|_{s_1=0} = \overline{r_1 r_2}, \tag{55}$$

by using (49), (51), and (55). Equation (37) will take the form

$$P(x,y)|_{s_1=0} = \alpha_1\alpha_2 \frac{(y-1)(\overline{r_2}+r_2\overline{s_2}x+\xi_2 xy)}{(xy-((\overline{r_2}+r_2x+r_2\overline{s_2}xy)(\overline{r_1}+r_1y)))\,(\overline{r_2})}$$
$$+\alpha_1\alpha_2 \frac{(x-1)(\overline{r_1}+r_1y)(\overline{\xi_2}+\xi_2 y)}{(xy-((\overline{r_2}+r_2x+r_2\overline{s_2}xy)(\overline{r_1}+r_1y)))\,(\overline{r_1}\overline{\xi_2}-r_1\xi_2 y)}$$
$$+\frac{\alpha_1\alpha_2(x-1)(y-1)}{(xy-((\overline{r_2}+r_2x+r_2\overline{s_2}xy)(\overline{r_1}+r_1y)))}.$$

which after using (38) can be rewritten as follows:

$$P(x,y)|_{s_1=0} = (\overline{r_2})(\overline{r_1}-\xi_2) \frac{(y-1)(\overline{r_2}+r_2\overline{s_2}x+\xi_2 xy)}{(xy-((\overline{r_2}+r_2x+r_2\overline{s_2}xy)(\overline{r_1}+r_1y)))\,(\overline{r_2})}$$
$$+(\overline{r_2})(\overline{r_1}-\xi_2) \frac{(x-1)(\overline{r_1}+r_1y)(\overline{\xi_2}+\xi_2 y)}{(xy-((\overline{r_2}+r_2x+r_2\overline{s_2}xy)(\overline{r_1}+r_1y)))\,(\overline{r_1}\overline{\xi_2}-r_1\xi_2 y)}$$
$$+\frac{(\overline{r_2})(\overline{r_1}-\xi_2)(x-1)(y-1)}{(xy-((\overline{r_2}+r_2x+r_2\overline{s_2}xy)(\overline{r_1}+r_1y)))}$$

Since

$$P(0,y)|_{s_1=0} = P(y) = \frac{(\overline{r_1}-\xi_2)(\overline{\xi_2}+\xi_2 y)}{(\overline{r_1}\overline{\xi_2}-r_1\xi_2 y)},$$

so that after putting $y=x$ we get

$$P(y) = (\overline{r_1}-\xi_2)\frac{(y-1)(\overline{r_2}+r_2\overline{s_2}y+\xi_2 y^2)}{(y^2-((\overline{r_2}+r_2y+r_2\overline{s_2}y^2)(\overline{r_1}+r_1y)))}$$
$$+\frac{\overline{r_2}(y-1)(\overline{r_1}+r_1y)P(y)}{(y^2-((\overline{r_2}+r_2y+r_2\overline{s_2}y^2)(\overline{r_1}+r_1y)))}$$
$$+\frac{(\overline{r_2})(\overline{r_1}-\xi_2)(y-1)^2}{(y^2-((\overline{r_2}+r_2y+r_2\overline{s_2}y^2)(\overline{r_1}+r_1y)))}.$$

Solving for $P(y)$, we get

$$P(y)\left(1-\frac{\overline{r_2}(y-1)(\overline{r_1}+r_1y)}{(y^2-((\overline{r_2}+r_2y+r_2\overline{s_2}y^2)(\overline{r_1}+r_1y)))}\right)$$
$$=(\overline{r_1}-\xi_2)\frac{(y-1)(\overline{r_2}+r_2\overline{s_2}y+\xi_2 y^2)}{(y^2-((\overline{r_2}+r_2y+r_2\overline{s_2}y^2)(\overline{r_1}+r_1y)))}$$
$$+\frac{\overline{r_2}(\overline{r_1}-\xi_2)(y-1)^2}{(y^2-((\overline{r_2}+r_2y+r_2\overline{s_2}y^2)(\overline{r_1}+r_1y)))}.$$

After some tedious manipulations, we get

$$
\begin{aligned}
P(y) &= \frac{-\xi_2 + r_1\xi_2 + \xi_2^2 + \xi_2 y - r_1\xi_2 y - \xi_2^2 y + 1 - r_1 - \xi_2}{(1 - \xi_2 + r_1\xi_2 - r_1\xi_2 y - r_1)} \\
&= \frac{\overline{r_1}(\overline{\xi_2} + \xi_2 y)}{\overline{r_1}\overline{\xi_2} - r_1\xi_2 y} p_0.
\end{aligned} \tag{56}
$$

which is the single queue equation (7) where $p_0 = \frac{\overline{r_1} - \xi_2}{\overline{r_1}}$.

4.2.4 Only Queue II Active When LAN I Is Inactive

If LAN I is inactive, no packets will enter Queue I and the gateway will then have only Queue II active. Thus, if we substitute $r_1 = 0$ into (37), we should get the PGF of a geo/geo/1 queue with arrival rate ξ_2 and service rate $\overline{r_1} = 1$. Indeed, if $r_1 = 0$, then

$$
\xi_1 = r_1 s_1 = 0. \tag{57}
$$

Let p_j be the probability that Queue II has j packets, $j = 0, 1, 2, \ldots$ and that $P(z)$ is its generating function defined by

$$
P(z) = \sum_{n=0}^{\infty} p_n z^n, \quad |z| \leq 1
$$

From the assumption that Queue I is inactive, it follows that

$$
p_{i,j}|_{r_1=0} = \begin{cases} 0, & i > 0 \\ p_j, & i = 0 \end{cases}
$$

Using this, we get from (4), we get

$$
\begin{aligned}
P(x, y) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} x^m y^n, \quad |x|, \ |y| \leq 1 \\
&= p_{0,0} + p_{1,0}x + p_{2,0}x^2 + p_{3,0}x^3 + \cdots, \\
&\quad + p_{0,1}y + p_{0,2}y^2 + p_{0,3}y^3 + \cdots, \\
&\quad + p_{1,1}xy + p_{1,2}xy^2 + p_{2,1}x^2 y + p_{2,2}x^2 y^2 + \cdots,
\end{aligned} \tag{58}
$$

Substituting into (58) to get

$$
\begin{aligned}
P(x, y)|_{r_1=0} &= p_{0,0} + p_{0,1}y + p_{0,2}y^2 + p_{0,3}y^3 + \cdots, \\
&= \sum_{n=0}^{\infty} p_{0,n} y^n, \quad |y| \leq 1 \\
&= P(y)
\end{aligned}
$$

from (5) we get

$$\begin{aligned} P(x,0)|_{r_1=0} &= p_{0,0}, \\ &= 1-\xi_2, \\ &= \overline{\xi_2}. \end{aligned}$$

From (31) we get

$$P(0,y) = \frac{(\overline{\xi_2}+\xi_2 y)(\overline{r_1}-\xi_2)(\overline{r_2}-\xi_1)}{\overline{r_2}(\overline{r_1}\overline{\xi_2}-r_1\xi_2 y)},$$

from which we get

$$\begin{aligned} P(y) &= P(0,y)|_{r_1=0} \\ &= \frac{(\overline{\xi_2}+\xi_2 y)(1-\xi_2)(\overline{r_2})}{\overline{r_2}(\overline{\xi_2})} \\ &= (\overline{\xi_2}+\xi_2 y). \end{aligned} \tag{59}$$

By using (57), (3) we get

$$M(x,y)|_{r_1=0} = \overline{r_2} + r_2\overline{s_2}x + \xi_2 xy, \tag{60}$$

$$M(x,0)|_{r_1=0} = \overline{r_2} + r_2\overline{s_2}x, \tag{61}$$

$$M(0,y)|_{r_1=0} = M(0,0)|_{r_1=0} = \overline{r_2}, \tag{62}$$

when putting these values into (37) we get

$$\begin{aligned} P(x,y)|_{r_1=0} &= P(y) \\ &= (\overline{r_2})(1-\xi_2)\frac{(y-1)(\overline{r_2}+r_2\overline{s_2}x+\xi_2 xy)}{(xy-(\overline{r_2}+r_2\overline{s_2}x+\xi_2 xy))\,(\overline{r_2})} \\ &\quad +(\overline{r_2})(1-\xi_2)\frac{(x-1)(\overline{\xi_2}+\xi_2 y)}{(xy-(\overline{r_2}+r_2\overline{s_2}x+\xi_2 xy))\,(\overline{\xi_2})} \\ &\quad +\frac{(\overline{r_2})(1-\xi_2)(x-1)(y-1)}{(xy-(\overline{r_2}+r_2\overline{s_2}x+\xi_2 xy)))} \end{aligned}$$

which can be rewritten as

$$\begin{aligned} P(y) &= (1-\xi_2)\frac{(y-1)(\overline{r_2}+r_2\overline{s_2}y+\xi_2 y^2)}{(y^2-(\overline{r_2}+r_2\overline{s_2}y+\xi_2 y^2))} \\ &\quad +(\overline{r_2})\frac{(y-1)(\overline{\xi_2}+\xi_2 y)}{(y^2-(\overline{r_2}+r_2\overline{s_2}y+\xi_2 y^2))} \\ &\quad +\frac{(\overline{r_2})(1-\xi_2)(y-1)^2}{(y^2-(\overline{r_2}+r_2\overline{s_2}y+\xi_2 y^2))}, \end{aligned}$$

also by using (59) the above equation can be rewritten as follows:

$$P(y) = (1-\xi_2)\frac{(y-1)(\overline{r_2}+r_2\overline{s_2}y+\xi_2y^2)}{(y^2-(\overline{r_2}+r_2\overline{s_2}y+\xi_2y^2))} + (\overline{r_2})\frac{(y-1)P(y)}{(y^2-(\overline{r_2}+r_2\overline{s_2}y+\xi_2y^2))} + \frac{(\overline{r_2})(1-\xi_2)(y-1)^2}{(y^2-(\overline{r_2}+r_2\overline{s_2}y+\xi_2y^2))}.$$

Solving for $P(y)$, we get

$$P(y)\left(1-\frac{(y-1)\overline{r_2}}{(y^2-M(y,y))}\right) = \frac{(y-1)(\overline{r_2}+r_2\overline{s_2}y+\xi_2y^2)(1-\xi_2)}{(y^2-M(y,y))} + \frac{(1-\xi_2)\overline{r_2}(y-1)^2}{(y^2-M(y,y))},$$

After some tedious manipulations, we get

$$\begin{aligned} P(y) &= \overline{\xi_2}(y-1)y\frac{\overline{\xi_2}+\xi_2y}{(y-1)(y+1-r_2-\xi_2y)-(y-1)(1-r_2)} \\ &= \overline{\xi_2}+\xi_2y, \end{aligned} \tag{63}$$

which is identical to the single queue equation (7), when $p_0 = \overline{\xi_2} = P_{0,0}$. Note that (63) could have been obtained from (41) by substituting $r_1 = 0$ in the latter.

5 Total System Occupancy

In this section we will find the total system occupancy, i.e., the number of packets in both queues of the gateway. Let us denote this number by the random variable Q and its distribution by q_i, $i = 0, 1, 2, \ldots$. By Property 12, we have

$$Q(x) = P(x,x)$$

Then, from (37) when putting $y = x$, we get

$$\begin{aligned} Q(x) &= P(x,y)|_{x=y} \\ &= \frac{\alpha_1\alpha_2}{x^2-M(x,x)}\left(\frac{(x-1)(\overline{r_2}+r_2\overline{s_2}x+\xi_2x^2)(\overline{\xi_1}+\xi_1x)}{(\overline{r_2}\overline{\xi_1}-r_2\xi_1x)}\right. \\ &\quad \left.+\frac{(x-1)(\overline{r_1}+r_1\overline{s_1}x+\xi_1x^2)(\overline{\xi_2}+\xi_2x)}{(\overline{r_1}\overline{\xi_2}-r_1\xi_2x)}+(x-1)^2\right). \end{aligned}$$

Now, the average gateway occupancy $E[Q]$ will be found using the fact

$$E[Q] = \frac{dQ(x)}{dx}|_{x=1}.$$

After some heavy manipulation, involving applying de l'Hospital's rule three times, we get

$$E[Q] = \frac{r_1\xi_2\alpha_1}{A\alpha_2} + \frac{r_2\xi_1\alpha_2}{A\alpha_1} + \frac{\xi_1(\overline{r_1} + \overline{\xi_1}) + \xi_2(\overline{r_2} + \overline{\xi_2})}{A}, \tag{64}$$

where $\alpha_1 = \overline{r_2} - \xi_1, \alpha_2 = \overline{r_1} - \xi_2$ and $A = \alpha_1 + \alpha_2$.

6 Validation of Total System Occupancy

Here, we will validate the total system occupancy (64).

6.1 Zero Occupancy When Both LANs Are Inactive

If both LANs are externally inactive, no packets will enter the gateway and it will then have no packets. Thus, if we substitute $r_1 = r_2 = 0$ into (64), we should get 0. Indeed, if

$$r_1 = r_2 = 0,$$

then

$$\xi_1 = \xi_2 = 0.$$

Using, (64) to get

$$\begin{aligned} E[Q]|_{r_1=r_2=0} &= \frac{0(1+0) + 0(1+1)}{1+1} + \frac{0(1-0)^2 + 0(1-0)^2}{(1+1)} \\ &= 0 \end{aligned}$$

as it should. so that we can conclude that (64) is a possible expression for the system occupancy.

6.2 Zero Occupancy When Both LANs Are Externally Inactive

If both LANs are externally inactive, no packets will enter the gateway and it will then have no packets. Thus, if we substitute $s_1 = s_2 = 0$ into (64), we should get 0. Indeed, if

$$s_1 = s_2 = 0,$$

then

$$\xi_1 = \xi_2 = 0.$$

Using, (64) to get

$$\begin{aligned} E[Q]|_{s_1=s_2=0} &= \frac{0\alpha_1}{A\alpha_2} + \frac{0\alpha_2}{A\alpha_1} + \frac{0(\overline{r_1} + \overline{\xi_1}) + 0(\overline{r_2} + \overline{\xi_2})}{A} \\ &= 0, \end{aligned}$$

as it should. so that we can conclude that (64) is a possible expression for the system occupancy when $s_1 = s_2 = 0$.

6.3 Total Occupancy When Only LAN I Is Inactive

By putting $r_1 = 0$ so that $\xi_1 = 0$ by putting these values into (64), we get

$$\begin{aligned} E[Q]|_{r_1=0} &= \xi_2 \frac{(\overline{r_2} + \overline{\xi_2})}{1 + \overline{r_2} - \xi_2} \\ &= \xi_2 \frac{(\overline{r_2} + \overline{\xi_2})}{(\overline{r_2} + \overline{\xi_2})} \\ &= \xi_2, \end{aligned} \tag{65}$$

which is the expected number of packets in queue II. This result confirm the well-known occupancy of the single queue system when the service rate $\overline{r_1} = 1$. If we do the opposite, i.e., consider LAN II inactive, we would find

$$E[Q]|_{r_2=0} = \xi_1.$$

6.4 Total Occupancy When LAN I Is Externally Inactive

By putting $s_1 = 0,\ r_1 \neq 0 \rightarrow \xi_1 = 0$ by putting these values into (64), we get $(\overline{r_2} - \xi_1),\ \alpha_2 = (\overline{r_1} - \xi_2)$

$$E[Q]|_{s_1=0} = \frac{r_1\xi_2\alpha_1}{A\alpha_2} + 0 + \frac{0 + \xi_2(\overline{r_2} + \overline{\xi_2})}{A}$$

$$= \frac{\xi_2\overline{\xi_2}}{\overline{r_1} - \xi_2} \tag{66}$$

which is the system occupancy when $s_1 = 0$. This occupancy is correct since it is identical to the occupancy of a geo/geo/1 single queueing system with arrival rate ξ_2 and service rate $\overline{r_1}$.

6.5 Total Occupancy When LAN II Is Externally Inactive

By putting $s_2 = 0, r_2 \neq 0 \rightarrow \xi_2 = 0$ by putting these values into (64), we get

$$E[Q]|_{s_2=0} = \frac{\xi_1(\overline{r_1} + \overline{\xi_1}) + 0(\overline{r_2} + \overline{\xi_2})}{\overline{r_1} - \xi_1 + \overline{r_2} - 0} + \frac{0(\overline{r_2} - \xi_1)^2 + r_2\xi_1(\overline{r_1} - 0)^2}{(\overline{r_1} - \xi_1 + \overline{r_2} - 0)(\overline{r_2} - \xi_1)(\overline{r_1} - 0)}$$

$$= \frac{\xi_1\overline{\xi_1}}{\overline{r_2} - \xi_1} \tag{67}$$

which is the system occupancy when $s_2 = 0$. This occupancy is correct since it is identical to the occupancy of a geo/geo/1 single queueing system with arrival rate ξ_1 and service rate $\overline{r_2}$.

7 Total Occupancy When LANs Are Identical

Here, we assume that the two LANs have the same operational parameters, i.e.,

$$r_1 = r_2 = r,\ s_1 = s_2 = s,$$

in (64) noting that then $\xi_1 = \xi_2 = \xi$. During the next two subsections, we will derive an expression for the total system occupancy when the two LANs are identical. One way directly from the system occupancy equation and the other way from the PGF noting that the two expressions must be identical.

7.1 Derivation from System Occupancy Equation

In this section we will find a special case from (64), and this will be done by substituting

$$r_1 = r_2 = r,\ s_1 = s_2 = s,$$

in (64) noting that $\xi_1 = \xi_2 = \xi$. This gives

$$E[Q]|_{r_1=r_2=r, s_1=s_2=s} = \frac{\xi(\bar{r}+\bar{\xi}) + \xi(\bar{r}+\bar{\xi})}{\bar{r}-\xi+\bar{r}-\xi} + \frac{r\xi(\bar{r}-\xi)^2 + r\xi(\bar{r}-\xi)^2}{(\bar{r}-\xi+\bar{r}-\xi)(\bar{r}-\xi)(\bar{r}-\xi)}$$
$$= \xi\frac{1+\bar{\xi}}{\bar{r}-\xi}, \tag{68}$$

Also, we can arrive at the same result from the other form as follows.

$$E[Q] = \frac{r_1\xi_2\alpha_1^2 + r_2\xi_1\alpha_2^2 + (\xi_1\beta_1 + \xi_2\beta_2)\alpha_1\alpha_2}{\alpha_1\alpha_2^2 + \alpha_1^2\alpha_2}$$
$$= \xi\frac{1+\bar{\xi}}{\bar{r}-\xi} \tag{69}$$

7.2 Derivation Directly from PGF

In Sect. 7.1 above, we derived the expected occupancy of the gateway by substituting

$$r_1 = r_2 = r,\ s_1 = s_2 = s,\ \xi_1 = \xi_2 = \xi,$$

in (64). In this Section, we will derive the same quantity but from the PGF directly. Needless to say, we should end up with the same result. For Identical LANS, we should assume that

$$r_1 = r_2 = r, \tag{70}$$
$$s_1 = s_2 = s, \tag{71}$$
$$\xi_1 = \xi_2 = \xi, \tag{72}$$

when using these assumptions, Eq. (2) will take the form

$$P(x,y) = \frac{1}{xy - M(x,y)}\left((y-1)(M(x,0) + \bar{r}\xi xy)P(x,0)\right.$$
$$\left.+(x-1)(M(0,y) + \bar{r}\xi xy)P(0,y) + (x-1)(y-1)M(0,0)P(0,0)\right),$$

when $y \to x$, the above equation will take the form

$$P(x,x) = \frac{1}{x^2 - M(x,x)} \big((x-1)(M(x,0) + \bar{r}\xi x^2)P(x,0) \\ +(x-1)(M(0,x) + \bar{r}\xi x^2)P(0,x) + (x-1)^2 M(0,0)P(0,0)\big)\,, \quad (73)$$

we note that by using (70)–(72)

$$P(x,0)|_{r_1=r_2, s_1=s_2} = P(0,x),$$

$$M(x,0)|_{r_1=r_2, s_1=s_2} = M(0,x).$$

So that Eq. (73) will be of the form

$$P(x,x) = \frac{1}{x^2 - M(x,x)} \big(2(x-1)(M(x,0) + \bar{r}\xi x^2)P(x,0) + (x-1)^2 M(0,0)P(0,0)\big)\,. \quad (74)$$

Now by using (3), so that

$$M(x,0)|_{r_1=r_2=r, s_1=s_2=s} = \bar{r}(\bar{r} + r\bar{s}x), \quad (75)$$

$$M(0,0)|_{r_1=r_2=r, s_1=s_2=s} = \bar{r}^2 \quad (76)$$

$$M(x,x)|_{r_1=r_2=r, s_1=s_2=s} = (\bar{r} + r\bar{s}x + \xi x^2)^2, \quad (77)$$

so that after using (75)–(77) Eq. (74) will be

$$P(x,x) = \frac{1}{x^2 - M(x,x)} \big(2(x-1)(\bar{r}(\bar{r} + r\bar{s}x) + \bar{r}\xi x^2)P(x,0) + (x-1)^2\bar{r}^2 P(0,0)\big)\,,$$

now by using (30), (32), the above equation will be of the form

$$P(x,x) \\ = \frac{1}{x^2 - M(x,x)} \left(\frac{2(x-1)(\bar{r}(\bar{r} + r\bar{s}x) + \bar{r}\xi x^2)(\bar{r} - \xi)^2(\bar{\xi} + \xi x)}{\bar{r}(\bar{r}\bar{\xi} - r\xi x)} + (x-1)^2(\bar{r} - \xi)^2 \right) \\ = (\bar{r} - \xi)^2 \left(\frac{2(x-1)(\bar{r}(\bar{r} + r\bar{s}x) + \bar{r}\xi x^2)(\bar{\xi} + \xi x)}{\bar{r}(\bar{r}\bar{\xi} - r\xi x)\,(x^2 - M(x,x))} + \frac{(x-1)^2}{(x^2 - M(x,x))} \right).$$

Let us now assume that

$$Q(x) = P(x,x),$$

so that

$$Q(x) = (\bar{r} - \xi)^2 \left(\frac{2(x-1)(\bar{r}(\bar{r} + r\bar{s}x) + \bar{r}\xi x^2)(\bar{\xi} + \xi x)}{\bar{r}(\bar{r}\bar{\xi} - r\xi x)\,(x^2 - M(x,x))} + \frac{(x-1)^2}{(x^2 - M(x,x))} \right),$$

from which we get that

$$\frac{d}{dx}Q(x) = (\bar{r} - \xi)^2 \left(2\frac{d}{dx}\frac{(x-1)(\bar{r}(\bar{r} + r\bar{s}x) + \bar{r}\xi x^2)(\bar{\xi} + \xi x)}{(\bar{r}\bar{\xi} - r\xi x)\,(x^2 - M(x,x))} + \frac{d}{dx}\frac{(x-1)^2}{(x^2 - M(x,x))} \right),$$

so that

$$\begin{aligned} E[Q] &= \frac{d}{dx}Q(x)|_{x=1} \\ &= (\bar{r} - \xi)^2 \left(2\frac{d}{dx}\frac{(x-1)(\bar{r} + r\bar{s}x + \xi x^2)(\bar{\xi} + \xi x)}{(\bar{r}\bar{\xi} - r\xi x)\,(x^2 - M(x,x))}|_{x=1} + \frac{d}{dx}\frac{(x-1)^2}{(x^2 - M(x,x))}|_{x=1} \right). \end{aligned}$$

After some heavy manipulations, we obtain

$$\begin{aligned} E[Q] &= (\bar{r} - \xi)^2 \left(2\frac{[2(\bar{r} - \xi)^2][2(r + \xi) + 2\xi] - [2(\bar{r} - \xi)[1 - (r + \xi)^2 - 2\xi - 2r\xi]]}{2[2(\bar{r} - \xi)^2]^2} \right. \\ &\quad \left. + \frac{[2(\bar{r} - \xi)][2]}{2[2(\bar{r} - \xi)]^2} \right) \\ &= \xi\frac{1 + \bar{\xi}}{\bar{r} - \xi} \end{aligned} \tag{78}$$

which is identical to (68), which is the total system occupancy when assuming full identity on the given system.

7.2.1 Zero Identical System Occupancy When Both LANs Are Inactive

When both LANs are inactive, the gateway should be empty all the time. Let us now put $r = 0$ so that $\xi = 0$ by using these values into (78), we get

$$\begin{aligned} E[Q]|_{r=0} &= 0\frac{1+1}{1-0} \\ &= 0, \end{aligned} \tag{79}$$

7.2.2 Zero Identical System Occupancy When Both LANs Are Externally Inactive

When both LANs are externally inactive, the gateway should be empty all the time. Let us now put $s = 0$ so that $\xi = 0$ into (78) to get

$$E[Q]|_{\xi=0} = 0\frac{1+\overline{\xi}}{\overline{r}-\xi},$$
$$= 0, \tag{80}$$

8 Numerical Results and Discussions

In this section we will generate numerical results for the gateway model. Basically we will calculate the average total gateway occupancy for different operating parameters. It can be easily seen that the gateway operation is controlled by four parameters: r_1, s_1, r_2, s_2. Once values for these parameters are assumed, the gateway occupancy can be calculated using (64), and for the special case when the two LANs are identical, $r_1 = s_1 = s$, $r_2 = s_2 = s$, using (64).

$$E[Q] = \xi_1(\frac{\alpha_2}{\overline{r_1}A} + \frac{1}{\alpha_1}) + \xi_2(\frac{\alpha_1}{\overline{r_2}A} + \frac{1}{\alpha_2}) - (\xi_1 - \xi_2)^2\frac{1}{A}$$

where $\alpha_1 = \overline{r_2} - \xi_1, \alpha_2 = \overline{r_1} - \xi_2$, and $A = \alpha_1 + \alpha_2$.

When identical:

$$E[Q] = \xi\frac{1+\overline{\xi}}{\overline{r}-\xi}. \tag{81}$$

Comparing this occupancy with that of two identical (i.e., arrival rate ξ and service rate $\overline{r}$) but non-interfering queues, we find that the former is greater. Specifically, utilizing (81), we find that

$$E[Q] = \xi\frac{1+\overline{\xi}}{\overline{r}-\xi} > \frac{2\xi\overline{\xi}}{\overline{r}-\xi}$$

This makes perfect sense as the interference between the two queues of the gateway should make departure from them more difficult, hence retaining more packets in the gateway. The extra occupancy Δ added by the interference can be calculated as follows:

$$\Delta = \xi\frac{1+\overline{\xi}}{\overline{r}-\xi} - \frac{2\xi\overline{\xi}}{\overline{r}-\xi}$$
$$= \frac{\xi^2}{\overline{r}-\xi}. \tag{82}$$

For our gateway, if we define the percentage occupancy interference factor, σ, as 100 times the extra occupancy Δ, as given by (9), divided by the occupancy of two identical but non-interfering queues, we get

$$\begin{aligned}\sigma &= 100\frac{\xi^2}{\bar{r}-\xi}\Big/\frac{2\xi\bar{\xi}}{\bar{r}-\xi}\\ &= \frac{50\xi}{\bar{\xi}}.\end{aligned} \tag{83}$$

This factor gives a measure of the impact of the interference between the two queues on the expected gateway occupancy.

Being probabilities, the parameters r_1, s_1, r_2, s_2 can assume values between 0 and 1. However, due to practical considerations, there are some restrictions on those values. We note, for example, that a high value of r_1 will result in a high arrival rate into Queue I, but at the same time a low service rate for Queue II. Thus, if r_2 is also given a high value, increasing the arrival rate of Queue II to the point that it exceeds its service rate, the whole gateway system will be then unstable. Thus, for a stable gateway, the parameters: r_1, s_1, r_2, s_2 should satisfy

$$r_1 s_1 = \xi_1 < 1 - r_2, \tag{84}$$

$$r_2 s_2 = \xi_2 < 1 - r_1, \tag{85}$$

For the general case, we can focus on the gateway from either side, studying only one queue, since it is symmetrical. We arbitrarily focus here on the LAN I side, i.e., the queue to be studied is Queue I. Specifically, we will study the growth of Queue I as its arrival rate from LAN I increases, for different values of r_2 (which immediately determines the service rate of Queue I.) We will keep $s_2 = 1$ throughout this study as this value has no impact on the performance of Queue I.

In the identical case, where $r_1 = r_2 = r$, $s_1 = s_2 = r$, $\xi_1 = \xi_2 = \xi$, we note the following. Once r is chosen, ξ can be changed from 0 upwards, depending on the value assigned to s. However, ξ cannot exceed r, i.e.,

$$\xi \leq r, \tag{86}$$

since $0 \leq s \leq 1$ and by definition we have $\xi = rs$. Furthermore, the arrival rate ξ cannot exceed the service rate $\bar{r}$, i.e.,

$$\xi < 1 - r \tag{87}$$

to ensure stability. Then, to find the maximum value ξ can assume, regardless of the value of r, we substitute in (87) for r by its minimum value which from (86) is ξ, getting

$$\xi < 1 - \xi, \tag{88}$$

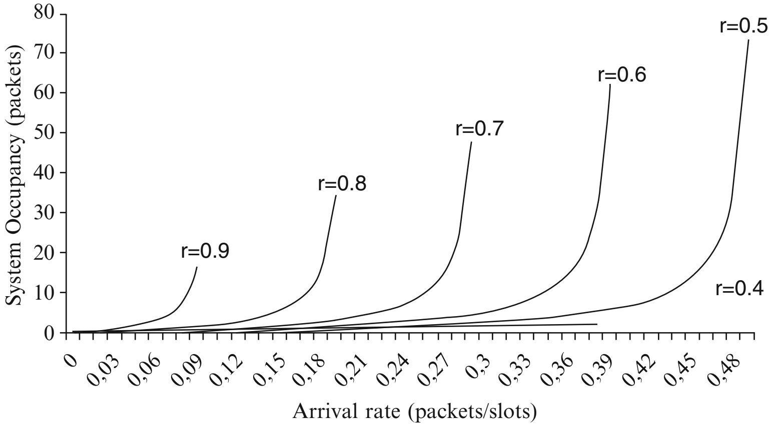

Fig. 2 Gateway occupancy vs. arrival rate for identical LANs

which implies that

$$\xi < 0.5. \tag{89}$$

Thus, in selecting values for r and s in the identical case, they should be such that $rs < 0.5$.

To produce Fig. 2, a 7-column table was constructed, the first for ξ and the remaining six for the expected occupancy at six different activity rates r (from 0.4 to 0.9). To produce the first column, instead of assuming r and then s to get the value of ξ, the column is directly filled up with values for ξ ranging from 0 up to 0.49, regularly spaced at 0.01. Then for each column (corresponding to a given r) in the subsequent six columns, the expected gateway occupancy is calculated using (81). Clearly, the value of s would be different for each entry of the column, depending on the corresponding value of ξ. It should be noted that the six columns are not equal in length. For example, the first of them ($r = 0.4$) stops exactly at the entry $\xi = 0.4$, according to (87). The second column ($r = 0.5$) stops at $\xi = 0.49$, according to (89). As r increases above 0.5, the corresponding columns decrease in length according to (88). For example, the column corresponding to $r = 0.6$ stops at the entry $\xi = 0.39$ and that corresponding to $r = 0.7$ stops at $\xi = 0.29$ and so on. These varying lengths in the table are reflected in the length of the curves of Fig. 2.

We should point out that we focus in Fig. 2 on only LAN activity rates $r = 0.4$, for better visualization. For smaller values of r, e.g., $r = 0.3$ and lower, the occupancy is very small and if plotted would not be noticeable. The reason why the occupancy is small is that the arrival rate $\xi = rs$, being less than or equal to r, is also small, while the service rate, being the complement of r, is large. Actually, the occupancy starts building up at $r = 0.4$ and beyond, where the arrival rate gets higher while the service rate gets lower. We notice from the graph that the total gateway occupancy increases almost linearly with the arrival rate until the difference

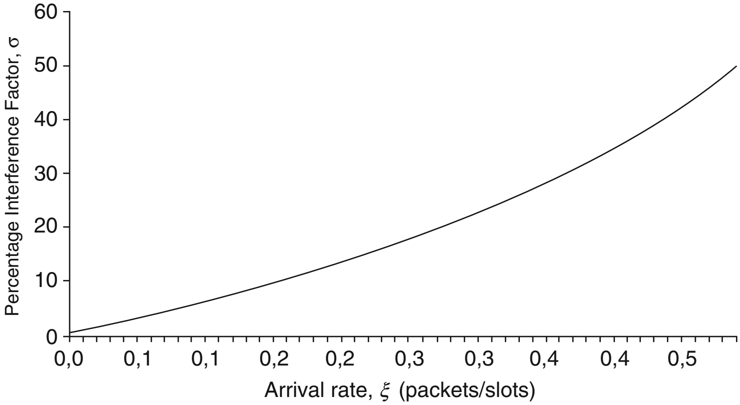

Fig. 3 Identical LANs: percentage occupancy interference factor vs. arrival rate

between the arrival rate ξ and the service rate r becomes small enough at which point the occupancy increases suddenly dramatically. This is the typical behavior of queues in general (see Woodward [5, Chap. 4].)

In Fig. 3, we turn the attention to the impact of the interference between the two queues on the expected gateway occupancy, when the two LANs are still operated with identical parameters. The figure shows the relationship between the percentage interference factor s, calculated using (83), and the arrival rate ξ and reveals two interesting observations. First, the relationship is nonlinear, which is understandable given that the relationship between the expected occupancy of queues and their arrival rates is generally nonlinear. Second, the interference factor has a maximum value of 50. That is, the expected occupancy of the two interfering queues of the gateway is at most 50 higher than the expected occupancy of two identical but non-interfering queues. This limit is actually imposed by the maximum value of the arrival rate ξ, which as explained earlier should be less than 0.5.

9 Conclusions

In this chapter, we have completed the study of a gateway linking two LANs, solving a challenging two-variable functional equation defining the PGF of the joint distribution of the gateway occupancy. The knowledge of the physical properties of the gateway has to a great extent been utilized to obtain the solution, as it is extremely difficult to solve such functional equations using only mathematical tools. The solution is validated using both mathematical and physical techniques. Finally, and for further validation of the solution, expectations are obtained for the gateway occupancy. The results are verified by comparison with anticipated and known results.

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Some Quantum Hermite–Hadamard-Type Inequalities for General Convex Functions

Muhammad Aslam Noor and Themistocles M. Rassias

In Honor of Constantin Carathéodory

Abstract In this chapter, we derive some quantum Hermite–Hadamard type inequalities for general convex functions. A new integral identity for q-differentiable functions is derived, and with the help of this, we obtain some new Hermite–Hadamard-type inequalities for q-differentiable convex functions.

1 Introduction

In recent years, quantum calculus or q-calculus has received special interest by many researchers. However, we would like to point out that the study of quantum calculus was started by Euler (1707–1783), who first introduced the q in tracks of Newton's infinite series. Basically, quantum calculus deals with q-analogues of mathematical objects which can be recaptured as $q \to 1$. For some more details on quantum calculus, see [3–8, 10, 11, 18, 22–24].

Tariboon et al. [23, 24] introduced and investigated the concepts of quantum calculus on finite interval. For further information on quantum calculus on finite intervals, see [17, 23, 24].

It is well known that the function f on the interval $[a, b]$ is a convex function, if and only if, the function f satisfies the inequality

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$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

This inequality is called the Hermite–Hadamard inequality. For the applications and other aspects of Hermite–Hadamard inequalities, see [**?**]. We would like to emphasize that the concept of convex functions has been extended in several directions using some novel and innovative ideas. An important and useful class of convex functions, which was introduced by Noor [12] relative to an arbitrary function. This class of convex functions is nonconvex and is known as general convex functions. Motivated and inspired by the recent activities in this field, we derive some new quantum Hermite–Hadamard inequalities for general convex functions. Our results represent refinement and significant improvement of the known results.

2 Basic Concepts

We now recall some basic concepts of quantum calculus on finite intervals. These results are mainly due to Tariboon et al. [23, 24].

Let $\mathscr{J} = [a,b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. The q-derivative of a function $f : \mathscr{J} \to \mathbb{R}$ at a point $x \in \mathscr{J}$ on $[a,b]$ is defined as follows.

Definition 1. Let $f : \mathscr{J} \to \mathbb{R}$ be a continuous function and let $x \in \mathscr{J}$. Then q-derivative of f on $\mathscr{J}$ at x is defined as

$$\mathscr{D}_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{1}$$

A function f is q-differentiable on $\mathscr{J}$ if $\mathscr{D}_q f(x)$ exists for all $x \in \mathscr{J}$.

Remark 1. Let $f : \mathscr{J} \to \mathbb{R}$ is a continuous function. Let us define the second-order q-derivative on interval $\mathscr{J}$, which is denoted by $\mathscr{D}f_q f$, provided $\mathscr{D}_q f$ is q-differentiable on $\mathscr{J}$ with $\mathscr{D}f_q^2 f = \mathscr{D}_q(\mathscr{D}_q f) : \mathscr{J} \to \mathbb{R}$. Similarly, one can define higher-order q-derivative on $\mathscr{J}$, $\mathscr{D}_q^n : \mathscr{J}_k \to \mathbb{R}$.

Let us elaborate above definitions with the help of an example.

Example 1. Let $x \in [g(a), b]$ and $0 < q < 1$. Then, for $x \neq a$, we have

$$\begin{aligned}\mathscr{D}_q x^2 &= \frac{x^2 - (qx + (1-q)a)^2}{(1-q)(x-a)}\\ &= \frac{(1+q)x^2 - 2qax - (1-q)x^2}{x-a}\\ &= (1+q)x + (1-q)a.\end{aligned}$$

Definition 2. Let $f : \mathscr{J} \to \mathbb{R}$ is a continuous function. A second-order q-derivative on $\mathscr{J}$, which is denoted as $\mathscr{D}_q^2 f$, provided $\mathscr{D}_q f$ is q-differentiable on $\mathscr{J}$ is defined as $\mathscr{D}_q^2 f = \mathscr{D}_q(\mathscr{D}_q f) : \mathscr{J} \to \mathbb{R}$. Similarly higher-order q-derivative on $\mathscr{J}$ is defined by $\mathscr{D}_q^n f =: \mathscr{J}_k \to \mathbb{R}$.

Lemma 1. *Let $\alpha \in \mathbb{R}$. Then*

$$\mathscr{D}_q(x-a)^\alpha = \Big(\frac{1-q^\alpha}{1-q}\Big)(x-a)^{\alpha-1}.$$

Definition 3 ([23, 24]). Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as

$$\int_a^x f(t)_a \mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{2}$$

for $x \in \mathscr{J}$.

Lemma 2. *Let $\alpha \in \mathbb{R} \setminus \{-1\}$. Then*

$$\int_a^x (t-a)^\alpha \mathrm{d}_q t = \Big(\frac{1-q}{1-q^{\alpha+1}}\Big)(x-a)^{\alpha+1}.$$

We now recall the concept of general convex sets and general convex functions, respectively, which are mainly due to Noor [12].

Definition 4 ([12]). The set K_g in a real Hilbert space H is said to be general convex with respect to an arbitrary function $g : H \to H$ such that

$$(1-t)g(u) + tv \in K_g, \quad \forall u, v \in H : g(u), v \in K_g, t \in [0,1].$$

If $g = I$, the identity function, then, we have classical convex set.

Clearly every convex set is a general convex set, but the converse is not true [12].

Definition 5 ([12]). A function $f : K_g \to H$ is said to be general convex, if there exists an arbitrary function $g : H \to H$ such that

$$f((1-t)g(u) + tv) \le (1-t)f(g(u)) + tf(v),$$
$$\forall u, v \in H : g(u), v \in K_g, t \in [0,1]. \tag{3}$$

Definition 6. The function $f : K_g \to H$ is said to be general quasi convex, if there exists an arbitrary function $g : H \to H$ such that

$$f((1-t)g(u) + tv) \le \max\{f(g(u)), f(v)\},$$
$$\forall u, v \in H : g(u), v \in K_g, t \in [0,1]. \tag{4}$$

Definition 7. A function f on the general set K_g is said to be general log-convex function, if there exists an arbitrary function $g : H \to H$, such that

$$f((1-t)g(u) + tv) \leq f^{1-t}(g(u))f^{t}(v),$$
$$\forall u, v \in H : g(u), v \in K_g, t \in [0, 1].$$

Noor [12] has shown that $u \in H : g(u) \in K_g$ is a minimum of the differentiable general convex function f, if and only if $u \in H : g(u) \in K_g$ satisfies

$$\langle f'(g(u)), v - g(u)\rangle \geq 0, \quad \forall v \in K,$$

where f' is the Frechet derivative of f. This above inequality is called the general variational inequality which was introduced and studied by Noor [12] in 1988. For applications, numerical methods, sensitivity analysis, and other aspects of general variational inequalities, see [1, 2, 9, 13, 15, 16, 19–21] and the references therein.

3 Quantum Hermite–Hadamard Inequalities

In this section, we establish some quantum analogues of Hermite–Hadamard-type inequalities for general convexity, essentially using the techniques of Noor et al. [17] and Tariboon et al. [22–24]. We include all the details to convey the main idea and techniques.

Theorem 1. *Let* $f : \mathscr{J} = [g(a), b] \to \mathbb{R}$ *be general convex continuous function on* $\mathscr{J}$ *with respect to an arbitrary function* $g : H \to H$. *Then for* $0 < q < 1$, *we have*

$$f\left(\frac{g(a)+b}{2}\right) \leq \frac{1}{b-g(a)}\int_{g(a)}^{b} f(t)\mathrm{d}_q t \leq \frac{qf(g(a)) + f(b)}{1+q}. \tag{5}$$

Proof. Let f be a general convex function on $[g(a), b]$. Then by taking q-integration with respect to t on $[0, 1]$, we have

$$f\left(\frac{g(a)+b}{2}\right) = \int_0^1 f\left(\frac{(1-t)g(a) + tb + tg(a) + (1-t)b}{2}\right)\mathrm{d}_q t$$
$$\leq \frac{1}{2}\left[\int_0^1 f((1-t)g(a) + tb)\mathrm{d}_q t + \int_0^1 f(tg(a) + (1-t)b)\mathrm{d}_q t\right]$$

$$= \frac{1}{b-g(a)} \int_{g(a)}^{b} f(t)\mathrm{d}_q t = \int_0^1 f((1-t)g(a)+tb)\mathrm{d}_q t$$

$$\leq f(g(a)) \int_0^1 (1-t)\mathrm{d}_q t + f(b) \int_0^1 t\,\mathrm{d}_q t = \frac{qf(g(a)) + f(b)}{1+q}.$$

This completes the proof. □

Note that when $g = I$, the identity function, our result coincides with Theorem 3.2 [24].

Theorem 2. *Let $f, w : I = [g(a), b] \to \mathbb{R}$ be general convex functions, then*

$$\frac{1}{b-g(a)} \int_{g(a)}^{b} f(g(x))w(g(x))\mathrm{d}_q x$$
$$\leq \left(\frac{q(1+q^2)}{(1+q)(1+q+q^2)}\right) f(g(a))w(g(a)) + \left(\frac{q^2}{(1+q)(1+q+q^2)}\right) N(g(a), b)$$
$$+ \left(\frac{1}{1+q+q^2}\right) f(b)w(b),$$

where

$$N(g(a), b) = f(g(a))w(b) + f(b)w(g(a)).$$

Proof. Since f and w are general convex functions, then

$$f((1-t)g(a)+tb) \leq tf(g(a)) + (1-t)f(b),$$
$$w((1-t)g(a)+tb) \leq tw(g(a)) + (1-t)w(b).$$

Multiplying above inequalities and taking q-integral of both sides of above inequality with respect to t on [0, 1], we have

$$\int_0^1 f((1-t)g(a)+tb)w((1-t)g(a)+tb)\mathrm{d}_q t$$

$$\leq f(g(a))w(g(a)) \int_0^1 (1-t)^2 \mathrm{d}_q t + \{f(g(a))w(b) + f(b)w(g(a))\} \int_0^1 t(1-t)\mathrm{d}_q t$$

$$+ f(b)w(b) \int_0^1 t^2 \mathrm{d}_q t.$$

This implies that

$$\begin{aligned}
&\frac{1}{b-g(a)}\int_{g(a)}^{b} f(g(x))w(g(x))\mathrm{d}_q x\\
&\le \left[\frac{q(1+q^2)}{(1+q)(1+q+q^2)}\right] f(g(a))w(g(a))\\
&\quad + \left[\frac{q^2}{(1+q)(1+q+q^2)}\right]\{f(g(a))w(b)+f(b)w(g(a))\}\\
&\quad + \left[\frac{1}{1+q+q^2}\right] f(b)w(b).
\end{aligned}$$

This completes the proof. □

Theorem 3. *Let f and w be general convex functions. Then*

$$\begin{aligned}
&2f\left(\frac{g(a)+b}{2}\right)w\left(\frac{g(a)+b}{2}\right)-\frac{2q^2M(g(a),b)+(1+2q+q^3)N(g(a),b)}{2(1+q)(1+q+q^2)}\\
&\le \frac{1}{(b-g(a))}\int_{g(a)}^{b} f(g(x))w(g(x))\mathrm{d}_q x,
\end{aligned}$$

where $M(g(a),b) = f(g(a))w(g(a)) + f(b)w(b)$ *and* $N(g(a),b) = f(g(a))w(b) + f(b)w(g(a))$.

Proof. Since f and w are general convex function, so

$$\begin{aligned}
&f\left(\frac{g(a)+b}{2}\right)w\left(\frac{g(a)+b}{2}\right)\\
&\le \frac{1}{4}\Bigg[f((1-t)g(a)+tb)w((1-t)g(a)+tb)\\
&\quad + f(tg(a)+(1-t)b)w(tg(a)+(1-t)b)\\
&\quad + [f(g(a))w(g(a))+f(b)w(b)]\{2t(1-t)\}\\
&\quad + [f(g(a))w(b)+f(b)w(g(a))]\{t^2+(1-t)^2\}\Bigg].
\end{aligned}$$

Applying q-integration with respect to t on [0, 1], we have

$$
\begin{aligned}
& f\left(\frac{g(a)+b}{2}\right) w\left(\frac{g(a)+b}{2}\right) \\
& \leq \frac{1}{4}\Bigg[\int_0^1 [f((1-t)g(a)+tb)w((1-t)g(a)+tb) \\
& \quad + f(tg(a)+(1-t)b)w(tg(a)+(1-t)b)]\,\mathrm{d}_q t \\
& \quad + [f(g(a))w(g(a))+f(b)w(b)]\int_0^1 \{2t(1-t)\}\mathrm{d}_q t \\
& \quad + [f(g(a))w(b)+f(b)w(g(a))]\int_0^1 \{t^2+(1-t)^2\}\mathrm{d}_q t\Bigg] \\
& = \frac{1}{2(b-g(a))}\int_{g(a)}^b f(g(x))w(g(x))\mathrm{d}_q x \\
& \quad + \frac{1}{4}\Bigg[\frac{2q^2\{f(g(a))w(g(a))+f(b)w(b)\}}{(1+q)(1+q+q^2)} \\
& \quad + \frac{(1+2q+q^3)[f(g(a))w(b)+f(b)w(g(a))]}{(1+q)(1+q+q^2)}\Bigg].
\end{aligned}
$$

This completes the proof. □

Using the technique of Noor et al. [17], we derive the following auxiliary result which plays an important role in our coming results.

Lemma 3. *Let $f : I = [g(a), b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I with $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$, then*

$$
\begin{aligned}
& \frac{1}{b-g(a)}\int_{g(a)}^b f(g(x))\mathrm{d}_q x - \frac{qf(g(a))+f(b)}{1+q} \\
& = \frac{q(b-g(a))}{1+q}\int_0^1 (1-(1+q)t)\mathscr{D}_q f((1-t)g(a)+t(b))\,\mathrm{d}_q t.
\end{aligned}
$$

Theorem 4. *Let $f : I = [g(a), b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I with $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. If $|\mathscr{D}_q f|^r$, $r \geq 1$ is general convex function, then*

$$\left|\frac{1}{b-g(a)}\int_{g(a)}^{b} f(g(x))\mathrm{d}_q x - \frac{qf(g(a))+f(b)}{1+q}\right|$$

$$\leq \frac{q(b-g(a))}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{1-\frac{1}{r}}$$

$$\times\left[\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(g(a))|^r + \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(b)|^r\right]^{\frac{1}{r}}.$$

Proof. Using Lemma 3 and power mean inequality, we have

$$\left|\frac{1}{b-g(a)}\int_{g(a)}^{b} f(g(x))\mathrm{d}_q x - \frac{qf(g(a))+f(b)}{1+q}\right|$$

$$= \left|\frac{q(b-g(a))}{1+q}\int_0^1 (1-(1+q)t)\mathscr{D}_q f((1-t)g(a)+t(b))\,\mathrm{d}_q t\right|$$

$$\leq \frac{q(b-g(a))}{1+q}\left(\int_0^1 |1-(1+q)t|\,\mathrm{d}_q t\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-(1+q)t||\mathscr{D}_q f((1-t)g(a)+t(b))|^r\,\mathrm{d}_q t\right)^{\frac{1}{r}}$$

$$\leq \frac{q(b-g(a))}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-(1+q)t|[(1-t)|\mathscr{D}_q f(g(a))|^r + t|\mathscr{D}_q f(b)|^r]\,\mathrm{d}_q t\right)^{\frac{1}{r}}$$

$$= \frac{q(b-g(a))}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{1-\frac{1}{r}}$$

$$\times\left[\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(g(a))|^r + \frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(b)|^r\right]^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 5. *Let $f : I = [g(a), b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I with $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. If $|\mathscr{D}_q f|^r$ is quasi general convex function where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then*

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(g(x))\mathrm{d}_q x\right|$$

$$\leq \frac{q(b-a)}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{q(2+q+q^3)}{(1+q)^3}\Big[\sup\{|\mathscr{D}_q f(a)|, |\mathscr{D}_q f(b)|\}\Big]\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 3, Holder's inequality, and the fact that $|\mathscr{D}_q f|^r$ is quasi general convex function, we have

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(g(x))\mathrm{d}_q x\right|$$

$$= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)\,{}_aD_q f((1-t)g(a)+t(b))\mathrm{d}_q t\right|$$

$$\leq \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)^{1-\frac{1}{r}}(1-(1+q)t)^{\frac{1}{r}}\,{}_aD_q f((1-t)g(a)+t(b))\mathrm{d}_q t\right|$$

$$\leq \frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)t|\mathrm{d}_q t\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |1-(1+q)t||{}_aD_q f((1-t)g(a)+t(b))|^r\mathrm{d}_q t\right)^{\frac{1}{r}}$$

$$= \frac{q(b-a)}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{q(2+q+q^3)}{(1+q)^3}\Big[\sup\{|\mathscr{D}_q f(a)|, |\mathscr{D}_q f(b)|\}\Big]\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 6. *Let $f : I = [g(a), b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I with $\mathscr{D}_q$ be continuous and integrable on I where $0 < q < 1$. If If $|\mathscr{D}_q f|^r$ is quasi general convex function where $r > 1$, then*

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(g(x))\mathrm{d}_q x\right|$$
$$\leq \frac{q^2(b-a)(2+q+q^3)}{(1+q)^4}\left(\sup\{|\mathscr{D}_q f(a)|, |\mathscr{D}_q f(b)|\}\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 3, power mean inequality, and the fact that $|\mathscr{D}_q f|^r$ is quasi general convex function, we have

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(g(x))\mathrm{d}_q x\right|$$
$$= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)t)_a D_q f((1-t)g(a)+t(b))\mathrm{d}_q t\right|$$
$$\leq \frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)t|\mathrm{d}_q t\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 |1-(1+q)t||_a D_q f((1-t)g(a)+t(b))|^r \mathrm{d}_q t\right)^{\frac{1}{r}}$$
$$= \frac{q^2(b-a)(2+q+q^3)}{(1+q)^4}\left(\sup\{|\mathscr{D}_q f(a)|, |\mathscr{D}_q f(b)|\}\right)^{\frac{1}{r}}.$$

This completes the proof. □

4 Quantum Ostrowski-Type Inequalities

In this section, we derive some Ostrowski-type inequalities for q-differentiable convex functions. Using the technique of Noor et al. [18], one can prove following auxiliary result. This result plays significant role in our coming result.

Lemma 4. *Let $f : I = [g(a), b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $\mathscr{D}_q$ be continuous and integrable on I where $0 < q < 1$. Then*

$$f(x) - \frac{1}{b-g(a)} \int_{g(a)}^{b} f(u)\mathrm{d}_q u$$

$$= \frac{q(x-g(a))^2}{b-g(a)} \int_0^1 t\mathscr{D}_q f(tx+(1-t)g(a))_0 d_q t + \frac{q(b-x)^2}{b-g(a)} \int_0^1 t\mathscr{D}_q f(tx+(1-t)b)_0 d_q t$$

Theorem 7. *Let $f : I = [g(a), b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $\mathscr{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|\mathscr{D}_q f|$ is general convex function and $|\mathscr{D}_q f(x)| \leq M$, then*

$$\left| f(x) - \frac{1}{b-g(a)} \int_{g(a)}^{b} f(u)\mathrm{d}_q u \right| \leq \frac{qM[(x-g(a))^2 + (b-x)^2]}{(b-g(a))(1+q)}.$$

Proof. Using Lemma 4 and the fact that $|\mathscr{D}_q f|$ is general convex function, we have

$$\left| f(x) - \frac{1}{b-g(a)} \int_{g(a)}^{b} f(u)\mathrm{d}_q u \right|$$

$$= \left| \frac{q(x-g(a))^2}{b-g(a)} \int_0^1 t\mathscr{D}_q f(tx+(1-t)g(a))_0 d_q t + \frac{q(b-x)^2}{b-g(a)} \int_0^1 t\mathscr{D}_q f(tx+(1-t)b)_0 d_q t \right|$$

$$\leq \frac{q(x-g(a))^2}{b-g(a)} \int_0^1 t|\mathscr{D}_q f(tx+(1-t)g(a))|_0 d_q t + \frac{q(b-x)^2}{b-g(a)} \int_0^1 t|_x D_q f(tx+(1-t)b)|_0 d_q t$$

$$\leq \frac{q(x-g(a))^2}{b-g(a)} \int_0^1 t[t|\mathscr{D}_q f(x)| + (1-t)|\mathscr{D}_q f(g(a))|]_0 d_q t$$

$$+ \frac{q(b-x)^2}{b-g(a)} \int_0^1 t[t|_x D_q f(x)| + (1-t)|_x D_q f(b)|]_0 d_q t$$

$$\leq \frac{qM[(x-g(a))^2 + (b-x)^2]}{(b-g(a))(1+q)}.$$

This completes the proof. □

Theorem 8. *Let $f : I = [g(a), b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $\mathscr{D}_q$ be continuous and integrable on I where $0 < q < 1$. If $|\mathscr{D}_q f|^r$ is general convex function and $|\mathscr{D}_q f(x)| \leq M$, then for $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, we have*

$$\left| f(x) - \frac{1}{b-g(a)} \int_{g(a)}^{b} f(u)\mathrm{d}_q u \right| \leq \frac{qM[(x-g(a))^2 + (b-x)^2]}{(b-g(a))} \Big(\frac{1-q}{1-q^{p+1}} \Big)^{\frac{1}{p}}.$$

Proof. Using Lemma 4, Holder's inequality, and the fact that $|\mathscr{D}_q f|^r$ is general convex function, we have

$$\left|f(x) - \frac{1}{b-g(a)}\int_{g(a)}^{b} f(u)\mathrm{d}_q u\right|$$

$$= \left|\frac{q(x-g(a))^2}{b-g(a)}\int_0^1 t\mathscr{D}_q f(tx+(1-t)g(a))_0 d_q t + \frac{q(b-x)^2}{b-g(a)}\int_0^1 t\mathscr{D}_q f(tx+(1-t)b)_0 d_q t\right|$$

$$\leq \frac{q(x-g(a))^2}{b-g(a)}\Big(\int_0^1 t_0^p d_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |\mathscr{D}_q f(tx+(1-t)g(a))|_0^r d_q t\Big)^{\frac{1}{r}}$$

$$+\frac{q(b-x)^2}{b-g(a)}\Big(\int_0^1 t_0^p d_q t\Big)^{\frac{1}{p}}\Big(\int_0^1 |_x D_q f(tx+(1-t)b)|_0^r d_q t\Big)^{\frac{1}{r}}$$

$$\leq \frac{q(x-g(a))^2}{b-g(a)}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [t|\mathscr{D}_q f(x)|^r + (1-t)|\mathscr{D}_q f(g(a))|^r]_0 d_q t\Big)^{\frac{1}{r}}$$

$$+\frac{q(b-x)^2}{b-g(a)}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [t|\mathscr{D}_q f(x)|^r + (1-t)|\mathscr{D}_q f(b)|^r]_0 d_q t\Big)^{\frac{1}{r}}$$

$$\leq \frac{qM[(x-g(a))^2+(b-x)^2]}{(b-g(a))}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}.$$

This completes the proof. □

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General Harmonic Convex Functions and Integral Inequalities

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In Honor of Constantin Carathéodory

Abstract In this chapter, we introduce the notion of general harmonic convex functions using an arbitrary auxiliary function $g : \mathbb{R} \to \mathbb{R}$. We obtain several new integral inequalities for general harmonic convex functions. Special cases which can be derived from our main results are also discussed.

1 Introduction

In recent years, theory of convexity has experienced significant development due to its applications for solving a large number of problems which arise in various branches of pure and applied sciences. Consequently, the classical concepts of convex sets and convex functions have been extended and general in various directions using novel and innovative ideas; see [1, 2, 4, 7, 8, 10**?** –14]. Iscan [4] introduced and investigated the notion of harmonic convex functions. For some recent studies on harmonic convex functions and on its variant forms, see [4–6, 10–12, 14**?**]. An important fact which makes theory of convexity more attractive to researchers is its close relationship with theory of inequalities. Many inequalities known in the literature are proved for convex functions; see [3]. Iscan [4] derived Hermite–Hadamard-type inequality for the class of harmonic convex functions, which reads as:

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Let $f : I_h \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonic convex function with $a < b$ and $a, b \in I_h$, then

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Recently, much attention has been given to derive variant forms of Hermite–Hadamard-type inequalities; see [3].

Motivated by the recent research going on in this field, we introduce and study a new class of harmonic convex functions, which is called general harmonic convex functions. We derive several new Hermite–Hadamard-type integral inequalities for general harmonic convex functions. Several special cases which can be derived from or main results are also discussed.

2 Preliminaries

In this section, we discuss some preliminary concepts which will be helpful in obtaining our main results.

Definition 1. A set $\mathscr{H}_g \subset \mathbb{R} \setminus \{0\}$ is said to be general harmonic convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, such that

$$\frac{g(x)y}{tg(x)+(1-t)y} \in \mathscr{H}_g, \quad \forall x, y \in \mathscr{H}_g, t \in [0, 1]. \tag{2}$$

Note that if $g = I$ where I is the identity function, then we have the definition of classical harmonic convex set; see [13].

Definition 2. A function $f : \mathscr{H}_g \to \mathbb{R}$ is said to be general harmonic convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, such that

$$f\left(\frac{g(x)y}{tg(x)+(1-t)y}\right) \leq (1-t)f(g(x)) + tf(y), \quad \forall g(x), y \in \mathscr{H}_g, t \in [0, 1]. \tag{3}$$

If $g = I$ where I is the identity function, then we have the definition of classical harmonic convex function; see [4].

Definition 3. A function $f : \mathscr{H} \to \mathbb{R}_+$ is said to be general log-harmonic convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, such that

$$f\left(\frac{g(x)y}{tg(x)+(1-t)y}\right) \leq f^{1-t}(g(x))f^t(y), \quad \forall g(x), y \in \mathscr{H}_g, t \in [0, 1]. \tag{4}$$

If $g = I$ where I is the identity function, then we have the definition of classical harmonic log-convex function; see [9].

Definition 4. A function $f : \mathscr{H} \to \mathbb{R}_+$ is said to be general quasi-harmonic convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, such that

$$f\left(\frac{g(x)y}{tg(x) + (1-t)y}\right) \leq \sup\{f(g(x)), f(y)\}, \quad \forall g(x), y \in \mathscr{H}_g, t \in [0,1]. \tag{5}$$

If $g = I$ where I is the identity function, then we have the definition of classical quasi-harmonic convex function; see [14].

3 Integral Inequalities

In this section, we derive Hermite–Hadamard-type inequalities for general harmonic convex functions.

From now onward, we take $I_h = [g(a), b]$, unless otherwise specified.

Theorem 1. *Let* $f : I_h = [g(a), b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be general harmonic convex function. If* $f \in L[g(a), b]$, *then*

$$f\left(\frac{2g(a)b}{g(a)+b}\right) \leq \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \frac{f(x)}{x^2}\,\mathrm{d}x \leq \frac{f(g(a)) + f(b)}{2}. \tag{6}$$

Proof. Let f be general harmonic convex function, we have

$$\begin{aligned}
f\left(\frac{2g(a)b}{g(a)+b}\right) &\leq \frac{1}{2}\left[f\left(\frac{g(a)b}{tg(a)+(1-t)b}\right) + f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right] \\
&= \frac{1}{2}\left[\int_0^1 f\left(\frac{g(a)b}{tg(a)+(1-t)b}\right)\mathrm{d}t + \int_0^1 f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\mathrm{d}t\right] \\
&= \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \frac{f(x)}{x^2}\mathrm{d}x \\
&= \int_0^1 f\left(\frac{g(a)b}{tg(a)+(1-t)b}\right)\mathrm{d}t \\
&\leq \int_0^1 \big[(1-t)f(g(a)) + tf(b)\big]\mathrm{d}t \\
&= \left[f(g(a)) \int_0^1 (1-t)\mathrm{d}t + f(b) \int_0^1 t\mathrm{d}t\right] \\
&= \frac{f(g(a)) + f(b)}{2}.
\end{aligned}$$

This completes the proof. □

Note that if $g = I$ the identity function, then Theorem 1 reduces to (1).

Theorem 2. *Let* $f : I_h = [g(a), b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be general harmonic convex function, then*

$$f\left(\frac{2g(a)b}{g(a)+b}\right) \leq \frac{1}{2}\left[f\left(\frac{4g(a)b}{g(a)+3b}\right)+f\left(\frac{4g(a)b}{3g(a)+b}\right)\right]$$
$$\leq \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x \leq \frac{1}{2}\left[f\left(\frac{2g(a)b}{g(a)+b}\right)+\frac{f(g(a))+f(b)}{2}\right]$$
$$\leq \frac{1}{2}[f(g(a))+f(b)].$$

Proof. By applying the Hermite–Hadamard inequality (6) on each of the interval $\left[g(a), \frac{2g(a)b}{g(a)+b}\right]$ and $\left[\frac{2g(a)b}{g(a)+b}, b\right]$, we get

$$f\left(\frac{4g(a)b}{g(a)+3b}\right) \leq \frac{2g(a)b}{b-g(a)}\int_{g(a)}^{\frac{2g(a)b}{g(a)+b}}\frac{f(x)}{x^2}\mathrm{d}x \leq \frac{1}{2}\left[f(g(a))+f\left(\frac{2g(a)b}{g(a)+b}\right)\right], \quad (7)$$

and

$$f\left(\frac{4g(a)b}{3g(a)+b}\right) \leq \frac{2g(a)b}{b-g(a)}\int_{\frac{2g(a)b}{g(a)+b}}^{b}\frac{f(x)}{x^2}\mathrm{d}x \leq \frac{1}{2}\left[f\left(\frac{2g(a)b}{g(a)+b}\right)+f(b)\right]. \quad (8)$$

Summing up (7) and (8), we have

$$f\left(\frac{2g(a)b}{g(a)+b}\right)$$
$$\leq \frac{1}{2}\left[f\left(\frac{4g(a)b}{g(a)+3b}\right)+f\left(\frac{4g(a)b}{3g(a)+b}\right)\right]$$
$$\leq \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x$$
$$\leq \frac{1}{2}\left[f\left(\frac{2g(a)b}{g(a)+b}\right)+\frac{f(g(a))+f(b)}{2}\right]$$
$$\leq \frac{1}{2}[f(g(a))+f(b)].$$

This completes the proof. □

Now we derive an auxiliary result which will be helpful in our coming result.

Lemma 1. *Let* $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I_h^0 *of* I_h. *If* $f' \in L[g(a), b]$ *and* $\lambda \in [0, 1]$, *then*

$$(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right)+\lambda\left(\frac{f(g(a))+f(b)}{2}\right)-\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x$$
$$=\frac{g(a)b(b-g(a))}{2}\left[\int_0^{\frac{1}{2}}\frac{2t-\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t+\int_{\frac{1}{2}}^{1}\frac{2t-2+\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t\right],$$

where

$$A_t=tg(a)+(1-t)b.$$

Proof. Let

$$I=\frac{g(a)b(b-g(a))}{2}\left[\int_0^{\frac{1}{2}}\frac{2t-\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t+\int_{\frac{1}{2}}^{1}\frac{2t-2+\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t\right]$$
$$=\frac{g(a)b(b-g(a))}{2}\int_0^{\frac{1}{2}}\frac{2t-\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t$$
$$+\frac{g(a)b(b-g(a))}{2}\int_{\frac{1}{2}}^{1}\frac{2t-2+\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t$$
$$=I_1+I_2.$$

Now

$$I_1=\frac{g(a)b(b-g(a))}{2}\int_0^{\frac{1}{2}}\frac{2t-\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t$$
$$=\frac{1}{2}\left|(2t-\lambda)f\left(\frac{g(a)b}{A_t}\right)\right|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}f\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t$$
$$=\frac{(1-\lambda)}{2}f\left(\frac{2g(a)b}{g(a)+b}\right)+\frac{\lambda}{2}f(g(a))-\frac{g(a)b}{b-g(a)}\int_{g(a)}^{\frac{2g(a)b}{g(a)+b}}\frac{f(x)}{x^2}\mathrm{d}x.$$

Similarly, we can show that

$$I_2=\frac{g(a)b(b-g(a))}{2}\int_{\frac{1}{2}}^{1}\frac{2t-2+\lambda}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t$$
$$=\frac{1}{2}\left|(2t-2+\lambda)f\left(\frac{g(a)b}{A_t}\right)\right|_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^{1}f\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t$$
$$=\frac{\lambda}{2}f(b)+\frac{(1-\lambda)}{2}f\left(\frac{2g(a)b}{g(a)+b}\right)-\frac{g(a)b}{b-g(a)}\int_{\frac{2g(a)b}{g(a)+b}}^{b}\frac{f(x)}{x^2}\mathrm{d}x.$$

Thus

$$
\begin{aligned}
&I_1 + I_2\\
&= \frac{(1-\lambda)}{2} f\left(\frac{2g(a)b}{g(a)+b}\right) + \frac{\lambda}{2} f(g(a)) - \frac{g(a)b}{b-g(a)} \int_{g(a)}^{\frac{2g(a)b}{g(a)+b}} \frac{f(x)}{x^2}\mathrm{d}x\\
&\quad + \frac{\lambda}{2} f(b) + \frac{(1-\lambda)}{2} f\left(\frac{2g(a)b}{g(a)+b}\right) - \frac{g(a)b}{b-g(a)} \int_{\frac{2g(a)b}{g(a)+b}}^{b} \frac{f(x)}{x^2}\mathrm{d}x\\
&= (1-\lambda) f\left(\frac{2g(a)b}{g(a)+b}\right) + \lambda\left(\frac{f(g(a))+f(b)}{2}\right) - \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \frac{f(x)}{x^2}\mathrm{d}x,
\end{aligned}
$$

which is the required result. □

We now discuss some special cases which can be obtained from Lemma 1.

If $\lambda = 0, 1, \frac{1}{2}$ and $\frac{1}{3}$, then Lemma 1 reduces to the following results, respectively.

Lemma 2. *Let $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$, then*

$$
f\left(\frac{2g(a)b}{g(a)+b}\right) - \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \frac{f(x)}{x^2}\mathrm{d}x = g(a)b(b-g(a)) \int_0^1 \frac{\nu(t)}{A_t^2} f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t,
$$

where

$$
\nu(t) = \begin{cases} t, & t \in [0, \frac{1}{2}) \\ t-1, & t \in [\frac{1}{2}, 1]. \end{cases}
$$

Lemma 3. *Let $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$, then*

$$
\frac{f(g(a))+f(b)}{2} - \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \frac{f(x)}{x^2}\mathrm{d}x = \frac{g(a)b(b-g(a))}{2} \int_0^1 \frac{2t-1}{A_t^2} f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t.
$$

Lemma 4. *Let $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$, then*

$$
\begin{aligned}
&\frac{1}{4}\left[f(g(a)) + 2f\left(\frac{2g(a)b}{g(a)+b}\right) + f(b)\right] - \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \frac{f(x)}{x^2}\mathrm{d}x\\
&= g(a)b(b-g(a)) \int_0^1 \frac{\nu(t)}{A_t^2} f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t,
\end{aligned}
$$

where

$$
\nu(t) = \begin{cases} t-\frac{1}{4}, & t \in [0, \frac{1}{2}) \\ t-\frac{3}{4}, & t \in [\frac{1}{2}, 1]. \end{cases}
$$

Lemma 5. *Let* $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I_h^0 *of* I_h. *If* $f' \in L[g(a), b]$, *then*

$$\frac{1}{6}\left[f(g(a)) + 4f\left(\frac{2g(a)b}{g(a)+b}\right) + f(b)\right] - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b} \frac{f(x)}{x^2}\mathrm{d}x$$
$$= g(a)b(b-g(a))\int_0^{\frac{1}{2}} \frac{\nu(t)}{A_t^2} f'\left(\frac{g(a)b}{A_t}\right)\mathrm{d}t,$$

where

$$\nu(t) = \begin{cases} t - \frac{1}{6}, \; t \in [0, \frac{1}{2}) \\ t - \frac{5}{6}, \; t \in [\frac{1}{2}, 1]. \end{cases}$$

Using Lemma 1, we now derive some new inequalities for general harmonic convex functions.

Theorem 3. *Let* $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I_h^0 *of* I_h. *If* $f' \in L[g(a), b]$ *and* $|f'|^q$ *is general harmonic convex function on* I_h *for* $q \geq 1$ *and* $\lambda \in [0, 1]$, *then*

$$\left|(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right) + \lambda\left(\frac{f(g(a)) + f(b)}{2}\right) - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq \frac{g(a)b(b-g(a))}{2}\big[(\mu_1(\lambda; g(a), b))^{1-\frac{1}{q}}(\mu_3(\lambda; g(a), b)|f'(g(a))|^q$$
$$+\mu_5(\lambda; g(a), b)|f'(b)|^q)^{\frac{1}{q}}$$
$$+(\mu_2(\lambda; b, g(a)))^{1-\frac{1}{q}}(\mu_6(\lambda; b, g(a))|f'(g(a))|^q + \mu_4(\lambda; b, g(a))|f'(b)|^q)^{\frac{1}{q}}\big],$$

where

$$\mu_1(\lambda; g(a), b) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|}{A_t^2}\mathrm{d}t,$$
$$\mu_2(\lambda; b, g(a)) = \int_{\frac{1}{2}}^{1} \frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t,$$
$$\mu_3(\lambda; g(a), b) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|(1-t)}{A_t^2}\mathrm{d}t,$$
$$\mu_4(\lambda; b, g(a)) = \int_{\frac{1}{2}}^{1} \frac{|2t-2+\lambda|t}{A_t^2}\mathrm{d}t,$$

$$\mu_5(\lambda; g(a), b) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|t}{A_t^2}\mathrm{d}t,$$

$$\mu_6(\lambda; b, g(a)) = \int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|(1-t)}{A_t^2}\mathrm{d}t.$$

Proof. Using Lemma 1 and the power mean inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right)+\lambda\left(\frac{f(g(a))+f(b)}{2}\right)-\frac{g(a)b}{b-g(a)}\int_{g(a)}^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\leq \frac{g(a)b(b-g(a))}{2}\left[\int_0^{\frac{1}{2}} \frac{|2t-\lambda|}{A_t^2}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|\mathrm{d}t+\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|}{A_t^2}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|\mathrm{d}t\right] \\
&\leq \frac{g(a)b(b-g(a))}{2}\left[\left(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|}{A_t^2}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|}{A_t^2}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
&\leq \frac{g(a)b(b-g(a))}{2}\left[\left(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|\big[(1-t)|f'(g(a))|^q+t|f'(b)|^q\big]}{A_t^2}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|\big[(1-t)|f'(g(a))|^q+t|f'(b)|^q\big]}{A_t^2}\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
&= \frac{g(a)b(b-g(a))}{2}\left[\left(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|(1-t)}{A_t^2}|f'(g(a))|^q\mathrm{d}t\right.\right. \\
&\left.+\int_0^{\frac{1}{2}} \frac{|2t-\lambda|t}{A_t^2}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|}{A_t^2}\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\left.\left(\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|(1-t)}{A_t^2}|f'(g(a))|^q\mathrm{d}t+\int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|t}{A_t^2}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
&= \frac{g(a)b(b-g(a))}{2}\left[(\mu_1(\lambda; g(a), b))^{1-\frac{1}{q}}(\mu_3(\lambda; g(a), b)|f'(g(a))|^q+\mu_5(\lambda; g(a), b)|f'(b)|^q)^{\frac{1}{q}}\right. \\
&\left.+(\mu_2(\lambda; b, g(a)))^{1-\frac{1}{q}}(\mu_6(\lambda; b, g(a))|f'(g(a))|^q+\mu_4(\lambda; b, g(a))|f'(b)|^q)^{\frac{1}{q}}\right],
\end{aligned}$$

which is the required result. □

Corollary 1. *Under the conditions of Theorem 3, if $q=1$, then, we have*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right)+\lambda\left(\frac{f(g(a))+f(b)}{2}\right)-\frac{g(a)b}{b-g(a)}\int_{g(a)}^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\leq \frac{g(a)b(b-g(a))}{2}[\mu_3(\lambda; g(a), b)|f'(g(a))|+\mu_5(\lambda; g(a), b)|f'(b)|] \\
&+[\mu_6(\lambda; b, g(a))|f'(g(a))|+\mu_4(\lambda; b, g(a))|f'(b)|],
\end{aligned}$$

where

$$\mu_3(\lambda; g(a), b) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|(1-t)}{A_t^2}\mathrm{d}t,$$

$$\mu_4(\lambda; b, g(a)) = \int_{\frac{1}{2}}^{1} \frac{|2t-2+\lambda|t}{A_t^2}\mathrm{d}t,$$

$$\mu_5(\lambda; g(a), b) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|t}{A_t^2}\mathrm{d}t,$$

$$\mu_6(\lambda; b, g(a)) = \int_{\frac{1}{2}}^{1} \frac{|2t-2+\lambda|(1-t)}{A_t^2}\mathrm{d}t.$$

If $\lambda = 0$, then Theorem 3 reduces to the following result.

Corollary 2. *Let $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $q \geq 1$, then*

$$\left| f\left(\frac{g(a)+b}{2}\right) - \frac{g(a)b}{(b-g(a))}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x \right|$$
$$\leq g(a)b(b-g(a)) \times \big[(\mu_1(0; g(a), b))^{1-\frac{1}{q}}(\mu_3(0; g(a), b)|f'(g(a))|^q$$
$$+\mu_5(0; g(a), b)|f'(b)|^q)^{\frac{1}{q}}$$
$$+(\mu_2(0; b, g(a)))^{1-\frac{1}{q}}(\mu_6(0; b, g(a))|f'(g(a))|^q + \mu_4(0; b, g(a))|f'(b)|^q)^{\frac{1}{q}}\big],$$

where

$$\mu_1(0; g(a), b) = \int_0^{\frac{1}{2}} \frac{t}{A_t^2}\mathrm{d}t$$
$$= \frac{-1}{(g(a)-b)(g(a)+b)} + \frac{1}{(g(a)-b)^2}\ln\left(\frac{g(a)+b}{2b}\right),$$

$$\mu_2(0; b, g(a)) = \int_{\frac{1}{2}}^{1} \frac{|t-1|}{A_t^2}\mathrm{d}t$$
$$= \frac{1}{(g(a)-b)(g(a)+b)} + \frac{1}{(g(a)-b)^2}\ln\left(\frac{g(a)+b}{2g(a)}\right),$$

$$\mu_3(0; g(a), b) = \int_0^{\frac{1}{2}} \frac{t(1-t)}{A_t^2}\mathrm{d}t$$
$$= \frac{-(3g(a)+b)}{2(g(a)+b)(g(a)-b)^2} + \frac{(g(a)+b)}{(g(a)-b)^3}\ln\left(\frac{g(a)+b}{2b}\right),$$

$$\mu_4(0;b,g(a)) = \int_{\frac{1}{2}}^{1} \frac{|t-1|t}{A_t^2}\mathrm{d}t$$
$$= \frac{-(g(a)+3b)}{2(g(a)+b)(g(a)-b)^2} - \frac{(g(a)+b)}{(g(a)-b)^3}\ln\left(\frac{g(a)+b}{2g(a)}\right),$$
$$\mu_5(0;g(a),b) = \int_0^{\frac{1}{2}} \frac{t^2}{A_t^2}\mathrm{d}t$$
$$= \frac{(g(a)+3b)}{2(g(a)+b)(g(a)-b)^2} - \frac{b}{(g(a)-b)^3}\ln\left(\frac{(g(a)+b)^2}{4b^2}\right),$$
$$\mu_6(0;b,g(a)) = \int_{\frac{1}{2}}^{1} \frac{|t-1|(1-t)}{A_t^2}\mathrm{d}t$$
$$= \frac{(3g(a)+b)}{2(g(a)+b)(g(a)-b)^2} + \frac{g(a)}{(g(a)-b)^3}\ln\left(\frac{(g(a)+b)^2}{4g(a)^2}\right).$$

If $\lambda = 1$, then Theorem 3 reduces to the following result.

Corollary 3. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a),b]$ and $|f'|^q$ is general harmonic convex function on I_h for $q \geq 1$, then*

$$\left|\frac{f(g(a))+f(b)}{2} - \frac{g(a)b}{(b-g(a))}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq \frac{g(a)b(b-g(a))}{2} \times \Big[(\mu_1(1;g(a),b))^{1-\frac{1}{q}}(\mu_2(1;g(a),b)|f'(g(a))|^q$$
$$+\mu_3(1;b,g(a))|f'(b)|^q)^{\frac{1}{q}}\Big],$$

where

$$\mu_1(1;g(a),b) = \int_0^1 \frac{|2t-1|}{A_t^2}\mathrm{d}t$$
$$= \frac{1}{g(a)b} - \frac{2}{(b-g(a))^2}\ln\left(\frac{(g(a)+b)^2}{4g(a)b}\right),$$
$$\mu_2(1;g(a),b) = \int_0^1 \frac{|2t-1|(1-t)}{A_t^2}\mathrm{d}t$$
$$= \frac{1}{b(g(a)-b)} - \frac{(3g(a)+b)}{(g(a)-b)^3}\ln\left(\frac{(g(a)+b)^2}{4g(a)b}\right),$$

$$\mu_3(1;b,g(a)) = \int_0^1 \frac{|2t-1|t}{A_t^2}\mathrm{d}t$$
$$= \frac{-1}{g(a)(g(a)-b)} + \frac{(g(a)+3b)}{(g(a)-b)^3}\ln\left(\frac{(g(a)+b)^2}{4g(a)b}\right).$$

If $\lambda = \frac{1}{2}$, then Theorem 3 reduces to the following result, which appears to be new one.

Corollary 4. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a),b]$ and $|f'|^q$ is general harmonic convex function on I_h for $q \geq 1$, then*

$$\left|\frac{1}{4}\left[f(g(a)) + 2f\left(\frac{2g(a)b}{g(a)+b}\right) + f(b)\right] - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq g(a)b(b-g(a)) \times \left[(\mu_1(1/2;g(a),b))^{1-\frac{1}{q}}(\mu_3(1/2;g(a),b)|f'(g(a))|^q\right.$$
$$+\mu_5(1/2;g(a),b)|f'(b)|^q)^{\frac{1}{q}}$$
$$\left.+(\mu_2(1/2;b,g(a)))^{1-\frac{1}{q}}(\mu_6(1/2;b,g(a))|f'(g(a))|^q + \mu_4(1/2;b,g(a))|f'(b)|^q)^{\frac{1}{q}}\right],$$

where

$$\mu_1(1/2;g(a),b) = \int_0^{\frac{1}{2}} \frac{\left|t-\frac{1}{4}\right|}{A_t^2}\mathrm{d}t$$
$$= \frac{1}{4b(g(a)+b)} - \frac{1}{(g(a)-b)^2}\ln\left(\frac{(g(a)+3b)^2}{8b(g(a)+b)}\right),$$
$$\mu_2(1/2;b,g(a)) = \int_{\frac{1}{2}}^{1} \frac{\left|t-\frac{3}{4}\right|}{A_t^2}\mathrm{d}t$$
$$= \frac{1}{4g(a)(g(a)+b)} - \frac{1}{(g(a)-b)^2}\ln\left(\frac{(3g(a)+b)^2}{8g(a)(g(a)+b)}\right),$$

$$\mu_3(1/2;g(a),b) = \int_0^{\frac{1}{2}} \frac{\left|t-\frac{1}{4}\right|(1-t)}{A_t^2}\mathrm{d}t$$
$$= \frac{g(a)}{4b(g(a)-b)} - \frac{(5g(a)+3b)}{4(g(a)-b)^3}\ln\left(\frac{(g(a)+3b)^2}{8b(g(a)+b)}\right),$$
$$\mu_4(1/2;b,g(a)) = \int_{\frac{1}{2}}^{1} \frac{\left|t-\frac{3}{4}\right|t}{A_t^2}\mathrm{d}t$$
$$= \frac{-b}{4g(a)(g(a)-b)} + \frac{(3g(a)+5b)}{4(g(a)-b)^3}\ln\left(\frac{(g(a)+3b)^2}{8g(a)(g(a)+b)}\right),$$

$$\mu_5(1/2;g(a),b) = \int_0^{\frac{1}{2}} \frac{\left|t-\frac{1}{4}\right|t}{A_t^2}\mathrm{d}t$$
$$= \frac{-1}{4(g(a)+b)(g(a)-b)} + \frac{(g(a)+7b)}{4(g(a)-b)^3}\ln\left(\frac{(g(a)+3b)^2}{8b(g(a)+b)}\right),$$
$$\mu_6(1/2;b,g(a)) = \int_{\frac{1}{2}}^1 \frac{\left|t-\frac{3}{4}\right|(1-t)}{A_t^2}\mathrm{d}t$$
$$= \frac{1}{4(g(a)+b)(g(a)-b)} - \frac{(7g(a)+b)}{4(g(a)-b)^3}\ln\left(\frac{(3g(a)+b)^2}{8g(a)(g(a)+b)}\right).$$

If $\lambda = \frac{1}{3}$, then Theorem 3 reduces to the following result.

Corollary 5. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a),b]$ and $|f'|^q$ is general harmonic convex function on I_h for $q \geq 1$, then*

$$\left|\frac{1}{6}\left[f(g(a)) + 4f\left(\frac{2g(a)b}{g(a)+b}\right) + f(b)\right] - \frac{g(a)b}{(b-g(a))}\int_{g(a)}^b \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq g(a)b(b-g(a)) \times \left[(\mu_1(1/3;g(a),b))^{1-\frac{1}{q}}(\mu_3(1/3;g(a),b)|f'(g(a))|^q\right.$$
$$+\mu_5(1/3;g(a),b)|f'(b)|^q)^{\frac{1}{q}}$$
$$\left.+(\mu_2(1/3;b,g(a)))^{1-\frac{1}{q}}(\mu_6(1/3;b,g(a))|f'(g(a))|^q + \mu_4(1/3;b,g(a))|f'(b)|^q)^{\frac{1}{q}}\right],$$

where

$$\mu_1(1/3;g(a),b) = \int_0^{\frac{1}{2}} \frac{\left|t-\frac{1}{6}\right|}{A_t^2}\mathrm{d}t$$
$$= \frac{(g(a)-3b)}{6b(g(a)+b)(g(a)-b)} - \frac{1}{(g(a)-b)^2}\ln\left(\frac{(g(a)+5b)^2}{18b(g(a)+b)}\right),$$
$$\mu_2(1/3;b,g(a)) = \int_{\frac{1}{2}}^1 \frac{\left|t-\frac{5}{6}\right|}{A_t^2}\mathrm{d}t$$
$$= \frac{(3g(a)-b)}{6g(a)(g(a)+b)(g(a)-b)} - \frac{1}{(g(a)-b)^2}\ln\left(\frac{(5g(a)+b)^2}{18g(a)(g(a)+b)}\right),$$
$$\mu_3(1/3;g(a),b) = \int_0^{\frac{1}{2}} \frac{\left|t-\frac{1}{6}\right|(1-t)}{A_t^2}\mathrm{d}t$$
$$= \frac{g(a)^3+b^3+3g(a)b^2-5g(a)^2b}{6b(g(a)+b)(g(a)-b)^3} - \frac{(7g(a)+5b)}{6(g(a)-b)^3}\ln\left(\frac{(g(a)+5b)^2}{18b(g(a)+b)}\right),$$
$$\mu_4(1/3;b,g(a)) = \int_{\frac{1}{2}}^1 \frac{\left|t-\frac{5}{6}\right|t}{A_t^2}\mathrm{d}t$$
$$= -\frac{g(a)^3+b^3+3g(a)^2b-5g(a)b^2}{6g(a)(g(a)+b)(g(a)-b)^3} + \frac{(5g(a)+7b)}{6(g(a)-b)^3}\ln\left(\frac{(5g(a)+b)^2}{18g(a)(g(a)+b)}\right),$$

$$\mu_5(1/3; g(a), b) = \int_0^{\frac{1}{2}} \frac{\left|t-\frac{1}{6}\right|t}{A_t^2}\mathrm{d}t$$
$$= \frac{2b}{3(g(a)+b)(g(a)-b)^2} + \frac{(g(a)+11b)}{6(g(a)-b)^3}\ln\left(\frac{(g(a)+5b)^2}{18b(g(a)+b)}\right),$$
$$\mu_6(1/3; b, g(a)) = \int_{\frac{1}{2}}^{1} \frac{\left|t-\frac{5}{6}\right|(1-t)}{A_t^2}\mathrm{d}t$$
$$= \frac{2g(a)}{3(g(a)+b)(g(a)-b)^2} - \frac{(11g(a)+b)}{6(g(a)-b)^3}\ln\left(\frac{(5g(a)+b)^2}{18g(a)(g(a)+b)}\right).$$

Theorem 4. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\lambda \in [0, 1]$, then*

$$\left|(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right) + \lambda\left(\frac{f(g(a))+f(b)}{2}\right) - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq \frac{g(a)b(b-g(a))}{2}\left[(\mu_7(\lambda, p; g(a), b))^{\frac{1}{p}}\left(\frac{|f'(g(a))|^q + |f'(\frac{2g(a)b}{g(a)+b})|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\mu_8(\lambda, p; b, g(a)))^{\frac{1}{p}}\left(\frac{|f'(\frac{2g(a)b}{g(a)+b})|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right],$$

where

$$\mu_7(\lambda, p; g(a), b) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|^p}{A_t^{2p}}\mathrm{d}t,$$
$$\mu_8(\lambda, p; b, g(a)) = \int_0^{\frac{1}{2}} \frac{|2t-2+\lambda|^p}{A_t^{2p}}\mathrm{d}t.$$

Proof. Using Lemma 1, inequalities (7), (8), and the Holder's integral inequality, we have

$$\left|(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right) + \lambda\left(\frac{f(g(a))+f(b)}{2}\right) - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq \frac{g(a)b(b-g(a))}{2}\left[\int_0^{\frac{1}{2}}\left|\frac{2t-\lambda}{A_t^2}\right|\left|f'\left(\frac{g(a)b}{A_t}\right)\right|\mathrm{d}t + \int_{\frac{1}{2}}^{1}\left|\frac{2t-2+\lambda}{A_t^2}\right|\left|f'\left(\frac{g(a)b}{A_t}\right)\right|\mathrm{d}t\right]$$
$$\leq \frac{g(a)b(b-g(a))}{2}\left[\left(\int_0^{\frac{1}{2}}\left|\frac{2t-\lambda}{A_t^2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|\frac{2t-2+\lambda}{A_t^2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$
\begin{aligned}
&= \frac{g(a)b(b-g(a))}{2}\Bigg[\bigg(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|^p}{A_t^{2p}}\mathrm{d}t\bigg)^{\frac{1}{p}}\bigg(\frac{g(a)b}{b-g(a)}\int_{g(a)}^{\frac{2g(a)b}{g(a)+b}} \frac{|f'(x)|^q}{x^2}\mathrm{d}x\bigg)^{\frac{1}{q}}\\
&+\bigg(\int_{\frac{1}{2}}^{1} \frac{|2t-2+\lambda|^p}{A_t^{2p}}\mathrm{d}t\bigg)^{\frac{1}{p}}\bigg(\frac{g(a)b}{b-g(a)}\int_{\frac{2g(a)b}{g(a)+b}}^{b} \frac{|f'(x)|^q}{x^2}\mathrm{d}x\bigg)^{\frac{1}{q}}\Bigg]\\
&\le \frac{g(a)b(b-g(a))}{2}\Bigg[\bigg(\int_0^{\frac{1}{2}} \frac{|2t-\lambda|^p}{A_t^{2p}}\mathrm{d}t\bigg)^{\frac{1}{p}}\bigg(\frac{|f'(g(a))|^q+|f'(\frac{2g(a)b}{g(a)+b})|^q}{4}\bigg)^{\frac{1}{q}}\\
&+\bigg(\int_{\frac{1}{2}}^{1} \frac{|2t-2+\lambda|^p}{A_t^{2p}}\mathrm{d}t\bigg)^{\frac{1}{p}}\bigg(\frac{|f'(\frac{2g(a)b}{g(a)+b})|^q+|f'(b)|^q}{4}\bigg)^{\frac{1}{q}}\Bigg]\\
&= \frac{g(a)b(b-g(a))}{2}\Bigg[(\mu_7(\lambda,p;g(a),b))^{\frac{1}{p}}\bigg(\frac{|f'(g(a))|^q+|f'(\frac{2g(a)b}{g(a)+b})|^q}{4}\bigg)^{\frac{1}{q}}\\
&+(\mu_8(\lambda,p;b,g(a)))^{\frac{1}{p}}\bigg(\frac{|f'(\frac{2g(a)b}{g(a)+b})|^q+|f'(b)|^q}{4}\bigg)^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

which is the required result. □

If $\lambda = 0$, then Theorem 4 reduces to the following result.

Corollary 6. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$
\begin{aligned}
&\left|f\bigg(\frac{g(a)+b}{2}\bigg)-\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|\\
&\le g(a)b(b-g(a))\Bigg[(\mu_7(0,p;g(a),b))^{\frac{1}{p}}\bigg(\frac{|f'(g(a))|^q+|f'(\frac{2g(a)b}{g(a)+b})|^q}{4}\bigg)^{\frac{1}{q}}\\
&+(\mu_8(0,p;b,g(a)))^{\frac{1}{p}}\bigg(\frac{|f'(\frac{2g(a)b}{g(a)+b})|^q+|f'(b)|^q}{4}\bigg)^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where

$$\mu_7(0,p;g(a),b) = \int_0^{\frac{1}{2}} \frac{|t|^p}{A_t^{2p}}\mathrm{d}t,$$

$$\mu_8(0,p;b,g(a)) = \int_0^{\frac{1}{2}} \frac{|t-1|^p}{A_t^{2p}}\mathrm{d}t.$$

If $\lambda = 1$, then Theorem 4 reduces to the following result.

Corollary 7. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\frac{f(g(a))+f(b)}{2}-\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq \frac{g(a)b(b-g(a))}{2}\Bigg[(\mu_7(1,p;g(a),b))^{\frac{1}{p}}\left(\frac{|f'(g(a))|^q+|f'\big(\frac{2g(a)b}{g(a)+b}\big)|^q}{4}\right)^{\frac{1}{q}}$$
$$+(\mu_8(1,p;b,g(a)))^{\frac{1}{p}}\left(\frac{|f'\big(\frac{2g(a)b}{g(a)+b}\big)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\Bigg],$$

where

$$\mu_7(1,p;g(a),b)=\int_0^{\frac{1}{2}}\frac{|2t-1|^p}{A_t^{2p}}\mathrm{d}t,$$
$$\mu_8(1,p;b,g(a))=\int_0^{\frac{1}{2}}\frac{|2t-1|^p}{A_t^{2p}}\mathrm{d}t.$$

If $\lambda=\frac{1}{2}$, then Theorem 4 reduces to the following result, which appears to be a new one.

Corollary 8. *Let $f: I_h \subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f'\in L[g(a),b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{1}{4}\left[f(g(a))+2f\left(\frac{2g(a)b}{g(a)+b}\right)+f(b)\right]-\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq g(a)b(b-g(a))\Bigg[(\mu_7(1/2,p;g(a),b))^{\frac{1}{p}}\left(\frac{|f'(g(a))|^q+|f'\big(\frac{2g(a)b}{g(a)+b}\big)|^q}{4}\right)^{\frac{1}{q}}$$
$$+(\mu_8(1/2,p;b,g(a)))^{\frac{1}{p}}\left(\frac{|f'\big(\frac{2g(a)b}{g(a)+b}\big)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\Bigg],$$

where

$$\mu_7(1/2,p;g(a),b)=\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{4}|^p}{A_t^{2p}}\mathrm{d}t,$$
$$\mu_8(1/2,p;b,g(a))=\int_{\frac{1}{2}}^{1}\frac{|t-\frac{3}{4}|^p}{A_t^{2p}}\mathrm{d}t.$$

If $\lambda=\frac{1}{3}$, then Theorem 4 reduces to the following result.

Corollary 9. *Let $f: I_h \subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f'\in L[g(a),b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{1}{6}\left[f(g(a))+4f\left(\frac{2g(a)b}{g(a)+b}\right)+f(b)\right]-\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq g(a)b(b-g(a))\left[(\mu_7(1/3,p;g(a),b))^{\frac{1}{p}}\left(\frac{|f'(g(a))|^q+|f'\left(\frac{2g(a)b}{g(a)+b}\right)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\mu_8(1/3,p;b,g(a)))^{\frac{1}{p}}\left(\frac{|f'\left(\frac{2g(a)b}{g(a)+b}\right)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right],$$

where

$$\mu_7(1/3,p;g(a),b)=\int_0^{\frac{1}{2}}\frac{|t-\frac{1}{6}|^p}{A_t^{2p}}\mathrm{d}t,$$
$$\mu_8(1/3,p;b,g(a))=\int_{\frac{1}{2}}^{1}\frac{|t-\frac{5}{6}|^p}{A_t^{2p}}\mathrm{d}t.$$

Theorem 5. *Let* $f: I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I_h^0 *of* I_h. *If* $f' \in L[g(a),b]$ *and* $|f'|^q$ *is general harmonic convex function on* I_h *for* $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\lambda\in[0,1]$, *then*

$$\left|(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right)+\lambda\left(\frac{f(g(a))+f(b)}{2}\right)-\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq\frac{g(a)b(b-g(a))}{4(2(p+1))^{\frac{1}{p}}}\times\frac{(\lambda^{p+1}+(1-\lambda)^{p+1})^{\frac{1}{p}}}{\left[(1-q)(1-2q)(b-g(a))^2\right]^{\frac{1}{q}}}\left[(\mu_9(q;g(a),b)|f'(g(a))|^q\right.$$
$$+\mu_{11}(q;g(a),b)|f'(b)|^q)^{\frac{1}{q}}+(\mu_{12}(q;b,g(a))|f'(g(a))|^q$$
$$\left.+\mu_{10}(q;b,g(a))|f'(b)|^q)^{\frac{1}{q}}\right],$$

where

$$\mu_9(q;g(a),b)=\left[\left(\frac{g(a)+b}{2}\right)^{1-2q}\left[\frac{3g(a)-b}{2}+q(b-g(a))\right]\right.$$
$$\left.+b^{1-2q}[b-2g(a)-2q(b-g(a))]\right],\tag{9}$$
$$\mu_{10}(q;b,g(a))=\left[\left(\frac{g(a)+b}{2}\right)^{1-2q}\left[\frac{3b-g(a)}{2}+q(g(a)-b)\right]\right.$$
$$\left.+g(a)^{1-2q}[g(a)-2b-2q(g(a)-b)]\right],\tag{10}$$

$$\mu_{11}(q; g(a), b) = \left[\left(\frac{g(a)+b}{2}\right)^{1-2q} \times\left[\frac{g(a)-3b}{2} + q(b-g(a))\right] + b^{2-2q}\right], \tag{11}$$

$$\mu_{12}(q; b, g(a)) = \left[\left(\frac{g(a)+b}{2}\right)^{1-2q} \left[\frac{b-3g(a)}{2} + q(g(a)-b)\right] + g(a)^{2-2q}\right]. \tag{12}$$

Proof. Using Lemma 1 and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2g(a)b}{g(a)+b}\right) + \lambda\left(\frac{f(g(a))+f(b)}{2}\right) - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right| \\
&\leq \frac{g(a)b(b-g(a))}{2}\left[\int_0^{\frac{1}{2}} |2t-\lambda|\left|\frac{1}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\right|\mathrm{d}t\right. \\
&\quad\left.+\int_{\frac{1}{2}}^{1} |2t-2+\lambda|\left|\frac{1}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\right|\mathrm{d}t\right] \\
&\leq \frac{g(a)b(b-g(a))}{2}\left[\left(\int_0^{\frac{1}{2}} |2t-\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \left|\frac{1}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1} |2t-2+\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1} \left|\frac{1}{A_t^2}f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
&= \frac{g(a)b(b-g(a))}{2}\left[\left(\int_0^{\frac{1}{2}} |2t-\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \frac{1}{A_t^{2q}}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1} |2t-2+\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1} \frac{1}{A_t^{2q}}\left|f'\left(\frac{g(a)b}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
&\leq \frac{g(a)b(b-g(a))}{2}\left[\left(\int_0^{\frac{1}{2}} |2t-\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\int_0^{\frac{1}{2}} \frac{1}{A_t^{2q}}\left[(1-t)|f'(g(a))|^q + t|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1} |2t-2+\lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1} \frac{1}{A_t^{2q}}\left[(1-t)|f'(g(a))|^q + t|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
&= \frac{g(a)b(b-g(a))}{2}\left[\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\int_0^{\frac{1}{2}} \frac{(1-t)}{A_t^{2q}}|f'(g(a))|^q\mathrm{d}t + \int_0^{\frac{1}{2}} \frac{t}{A_t^{2q}}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1} \frac{(1-t)}{A_t^{2q}}|f'(g(a))|^q\mathrm{d}t + \int_{\frac{1}{2}}^{1} \frac{(1-t)}{A_t^{2q}}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$= \frac{g(a)b(b-g(a))}{2}\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\int_0^{\frac{1}{2}}\frac{(1-t)}{A_t^{2q}}|f'(g(a))|^q+\int_0^{\frac{1}{2}}\frac{t}{A_t^{2q}}|f'(b)|^q\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\frac{(1-t)}{A_t^{2q}}|f'(g(a))|^q+\int_{\frac{1}{2}}^1\frac{t}{A_t^{2q}}|f'(b)|^q\right)^{\frac{1}{q}}\Bigg]$$

$$= \frac{g(a)b(b-g(a))}{4(2(p+1))^{\frac{1}{p}}}\frac{(\lambda^{p+1}+(1-\lambda)^{p+1})^{\frac{1}{p}}}{\left[(1-q)(1-2q)(b-g(a))^2\right]^{\frac{1}{q}}}$$

$$\times\big[(\mu_9(q;g(a),b)|f'(g(a))|^q+\mu_{11}(q;g(a),b)|f'(b)|^q)^{\frac{1}{q}}$$

$$+(\mu_{12}(q;b,g(a))|f'(g(a))|^q+\mu_{10}(q;b,g(a))|f'(b)|^q)^{\frac{1}{q}}\big],$$

which is the required result. □

If $\lambda = 0$, then Theorem 5 reduces to the following result.

Corollary 10. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|f\left(\frac{g(a)+b}{2}\right)-\frac{g(a)b}{(b-g(a))}\int_{g(a)}^b\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq \frac{g(a)b(b-g(a))}{2(2^{p+1}(p+1))^{\frac{1}{p}}}\times\frac{1}{\left[(1-q)(1-2q)(b-g(a))^2\right]^{\frac{1}{q}}}\big[(\mu_9(q;g(a),b)|f'(g(a))|^q$$

$$+\mu_{11}(q;g(a),b)|f'(b)|^q)^{\frac{1}{q}}+(\mu_{12}(q;b,g(a))|f'(g(a))|^q+\mu_{10}(q;b,g(a))|f'(b)|^q)^{\frac{1}{q}}\big],$$

where μ_9, μ_{10}, μ_{11}, and μ_{12} are given by (9)–(12).

If $\lambda = 1$, then Theorem 5 reduces to the following result.

Corollary 11. *Let $f : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\frac{f(g(a))+f(b)}{2}-\frac{g(a)b}{(b-g(a))}\int_{g(a)}^b\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq \frac{g(a)b(b-g(a))}{4(2(p+1))^{\frac{1}{p}}}\times\frac{1}{\left[(1-q)(1-2q)(b-g(a))^2\right]^{\frac{1}{q}}}\big[(\mu_9(q;g(a),b)|f'(g(a))|^q$$

$$+\mu_{11}(q;g(a),b)|f'(b)|^q)^{\frac{1}{q}}+(\mu_{12}(q;b,g(a))|f'(g(a))|^q+\mu_{10}(q;b,g(a))|f'(b)|^q)^{\frac{1}{q}}\big],$$

where μ_9, μ_{10}, μ_{11}, and μ_{12} are given by (9)–(12).

If $\lambda = \frac{1}{2}$, then Theorem 5 reduces to the following result, which appears to be new one.

Corollary 12. *Let $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{1}{4}\left[f(g(a)) + 2f\left(\frac{2g(a)b}{g(a)+b}\right) + f(b)\right] - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq \frac{g(a)b(b-g(a))}{2(4^{p+1}(p+1))^{\frac{1}{p}}} \times \frac{(2)^{\frac{1}{p}}}{\left[(1-q)(1-2q)(b-g(a))^2\right]^{\frac{1}{q}}}\left[(\mu_9(q; g(a), b)|f'(g(a))|^q\right.$$

$$\left.+\mu_{11}(q; g(a), b)|f'(b)|^q)^{\frac{1}{q}} + (\mu_{12}(q; b, g(a))|f'(g(a))|^q + \mu_{10}(q; b, g(a))|f'(b)|^q)^{\frac{1}{q}}\right],$$

where μ_9, μ_{10}, μ_{11}, and μ_{12} are given by (9)–(12).

If $\lambda = \frac{1}{3}$, then Theorem 5 reduces to the following result.

Corollary 13. *Let $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I_h^0 of I_h. If $f' \in L[g(a), b]$ and $|f'|^q$ is general harmonic convex function on I_h for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{1}{6}\left[f(g(a)) + 4f\left(\frac{2g(a)b}{g(a)+b}\right) + f(b)\right] - \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq \frac{g(a)b(b-g(a))}{2(6^{p+1}(p+1))^{\frac{1}{p}}} \times \frac{(1+2^{p+1})^{\frac{1}{p}}}{\left[(1-q)(1-2q)(b-g(a))^2\right]^{\frac{1}{q}}}\left[(\mu_9(q; g(a), b)|f'(g(a))|^q\right.$$

$$+\mu_{11}(q; g(a), b)|f'(b)|^q)^{\frac{1}{q}} + (\mu_{12}(q; b, g(a))|f'(g(a))|^q$$

$$\left.+\mu_{10}(q; b, g(a))|f'(b)|^q)^{\frac{1}{q}}\right],$$

where μ_9, μ_{10}, μ_{11}, and μ_{12} are given by (9)–(12).

4 Inequalities for General Harmonic log-Convex Functions

In this section, we derive Hermite–Hadamard inequalities for general harmonic log-convex functions

Theorem 6. *Let $f : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be an increasing general harmonic* log-*convex function. If $f \in L[g(a), b]$, then*

$$f\left(\frac{2g(a)b}{g(a)+b}\right)L(f(g(a)),f(b))$$
$$\leq \frac{g(a)b}{8(b-g(a))}\int_{g(a)}^{b}\frac{f^4(x)}{x^2}\mathrm{d}x$$
$$+\frac{1}{8}\frac{f(g(a))^2+f(b)^2}{2}\frac{f(g(a))+f(b)}{2}\frac{f(b)-f(g(a))}{\log f(b)-\log f(g(a))}+1.$$

Proof. Let f be general harmonic log-convex functions. Then

$$f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\leq [f(g(a))]^{1-t}[f(b)]^t.$$

Now using the identity

$$8xy\leq x^4+y^4+8, \qquad x,y\in\mathbb{R}.$$

Thus,

$$8f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^t$$
$$\leq f^4\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+[f(g(a))]^{4(1-t)}[f(b)]^{4t}+8.$$

Integrating over [0, 1], we have

$$8\int_0^1 f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^t\mathrm{d}t$$
$$\leq \int_0^1 f^4\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(g(a))]^{4(1-t)}[f(b)]^{4t}\mathrm{d}t+8.$$

Since f is increasing function, we have

$$8\int_0^1 f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t\int_0^1[f(g(a))]^{1-t}[f(b)]^t\mathrm{d}t$$
$$\leq \int_0^1 f^4\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(g(a))]^{4(1-t)}[f(b)]^{4t}\mathrm{d}t+8.$$

From the above inequality, it is easy to observe that

$$f\left(\frac{2g(a)b}{g(a)+b}\right)L(f(g(a)),f(b))$$
$$\leq \frac{g(a)b}{8(b-g(a))}\int_{g(a)}^{b}\frac{f^4(x)}{x^2}\mathrm{d}x$$
$$+\frac{1}{8}K^2(f(g(a)),f(b))A(f(g(a)),f(b))L(f(g(a)),f(b))+1$$

$$= \frac{g(a)b}{8(b-g(a))}\int_{g(a)}^{b}\frac{f^4(x)}{x^2}\mathrm{d}x$$
$$+\frac{1}{8}\frac{f(g(a))^2+f(b)^2}{2}\frac{f(g(a))+f(b)}{2}\frac{f(b)-f(g(a))}{\log f(b)-\log f(g(a))}+1,$$

which is the required result. □

Theorem 7. *Let $f,g: I=[g(a),b]\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be increasing general harmonic* log-*convex functions. If $f\in L[g(a),b]$, then*

$$\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{f^2(x)}{x^2}\mathrm{d}x + A(f(g(a)),f(b))L(f(g(a)),f(b)) + \psi(g(a),b)$$
$$\geq f\left(\frac{2g(a)b}{g(a)+b}\right)L(f(g(a)),f(b)) + 2A(f(g(a)),f(b))L(f(g(a)),f(b))$$
$$-L^2(f(g(a)),f(b))\frac{f(g(a))g(a)^2b}{(b-g(a))^2}\int_{g(a)}^{b}\frac{(b-x)f(x)}{x^3}\mathrm{d}x$$
$$+\frac{f(b)g(a)b^2}{(b-g(a))^2}\int_{g(a)}^{b}\frac{(x-g(a))f(x)}{x^3}\mathrm{d}x,$$

where

$$\psi(g(a),b)=\frac{f^2(g(a))+f(g(a))f(b)+f^2(b)}{3}.$$

Proof. Let f,g be general harmonic log-convex functions on I, we have that

$$f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\leq [f(g(a))]^{1-t}[f(b)]^t \leq (1-t)f(g(a))+tf(b).$$

Using the elementary inequality

$$xy+yz+zx\leq x^2+y^2+z^2,\quad (x,y,z\in\mathbb{R}).$$

We observe that

$$f^2\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+[f(g(a))]^{2(1-t)}[f(b)]^{2t}$$
$$+(1-t)^2f^2(g(a))+t^2f^2(b)+2t(1-t)f(g(a))f(b)$$
$$\geq f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^t$$
$$+(1-t)[f(g(a))]^{2-t}[f(b)]^t+t[f(g(a))]^{1-t}[f(b)]^{1+t}$$
$$+f(g(a))(1-t)f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+f(b)tf\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)$$

Integrating this inequality over t on [0, 1], we deduce that

$$\begin{aligned}&\int_0^1 f^2\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(g(a))]^{2(1-t)}[f(b)]^{2t}\mathrm{d}t\\&+f^2(g(a))\int_0^1(1-t)^2+f^2(b)\int_0^1 t^2\mathrm{d}t+f(g(a))f(b)\int_0^1 2t(1-t)\mathrm{d}t\\&\geq\int_0^1 f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^t\mathrm{d}t\\&+\int_0^1(1-t)[f(g(a))]^{2-t}[f(b)]^t\mathrm{d}t+\int_0^1 t[f(g(a))]^{1-t}[f(b)]^{1+t}\mathrm{d}t\\&+f(g(a))\int_0^1(1-t)f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t+f(b)\int_0^1 tf\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t.\end{aligned}$$

As it is easy to see that

$$\int_0^1 f^2\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t=\frac{g(a)b}{b-g(a)}\int_{g(a)}^b\frac{f^2(x)}{x^2}\mathrm{d}x.$$

By substituting $2t=u$, it is easy to observe that

$$\begin{aligned}\int_0^1[f(g(a))]^{2(1-t)}[f(b)]^{2t}\mathrm{d}t&=f^2(g(a))\int_0^1\left(\frac{f(b)}{f(g(a))}\right)^{2t}\mathrm{d}t\\&=\frac{1}{2}f^2(g(a))\int_0^2\left(\frac{f(b)}{f(g(a))}\right)^{u}\mathrm{d}u\\&=\frac{1}{2}\frac{f^2(b)-f^2(g(a))}{\log f(b)-\log f(g(a))}\\&=\frac{(f(g(a))+f(b))(f(b)-f(g(a)))}{2\log f(b)-\log f(g(a))}\\&=A(f(g(a)),f(b))L(f(g(a)),f(b)).\end{aligned}$$

And using increasing of f and the left half of the Hermite–Hadamard inequality for general harmonic convex function, we get

$$\begin{aligned}&\int_0^1 f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^t\mathrm{d}t\\&\geq\int_0^1 f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t\int_0^1[f(g(a))]^{1-t}[f(b)]^t\mathrm{d}t\\&=\frac{g(a)b}{b-g(a)}\int_{g(a)}^b\frac{f(x)}{x^2}\mathrm{d}x\frac{f(b)-f(g(a))}{\log f(b)-\log f(g(a))}\\&\geq f\left(\frac{2g(a)b}{g(a)+b}\right)L(f(g(a)),f(b)).\end{aligned}$$

By applying simple integration by parts formula, it is easy to observe that

$$\begin{aligned}&\int_0^1 (1-t)[f(g(a))]^{2-t}[f(b)]^t \mathrm{d}t\\&=f^2(g(a))\int_0^1(1-t)\left(\frac{f(b)}{f(g(a))}\right)^{2t}\mathrm{d}t\\&=\frac{-f^2(g(a))}{\log\frac{f(b)}{f(g(a))}}+\frac{f(g(a))f(b)-f^2(g(a))}{\left(\log\frac{f(b)}{f(g(a))}\right)^2},\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 t[f(g(a))]^{1-t}[f(b)]^{1+t}\mathrm{d}t\\&=f(g(a))f(b)\int_0^1 t\left(\frac{f(b)}{f(g(a))}\right)^{t}\mathrm{d}t\\&=\frac{f^2(b)}{\log\frac{f(b)}{f(g(a))}}-\frac{f^2(b)-f(g(a))f(b)}{\left(\log\frac{f(b)}{f(g(a))}\right)^2}.\end{aligned}$$

And by substituting $\frac{g(a)b}{t(g(a))+(1-t)b}=x$, it is easy to observe that

$$f(g(a))\int_0^1(1-t)f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t=\frac{g(a)^2bf(g(a))(b-x)}{(b-g(a))^2}\int_{g(a)}^b\frac{f(x)}{x^3}\mathrm{d}x$$

$$f(b)\int_0^1 tf\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t=\frac{g(a)b^2f(b)(x-g(a))}{(b-g(a))^2}\frac{f(x)}{x^3}\mathrm{d}x.$$

This implies

$$\begin{aligned}&\frac{g(a)b}{b-g(a)}\int_{g(a)}^b\frac{f^2(x)}{x^2}\mathrm{d}x+A(f(g(a)),f(b))L(f(g(a)),f(b))+\psi(g(a),b)\\&\geq f\left(\frac{2g(a)b}{g(a)+b}\right)L(f(g(a)),f(b))+2A(f(g(a)),f(b))L(f(g(a)),f(b))\\&-L^2(f(g(a)),f(b))\frac{f(g(a))g(a)^2b}{(b-g(a))^2}\int_{g(a)}^b\frac{(b-x)f(x)}{x^3}\mathrm{d}x\\&+\frac{f(b)g(a)b^2}{(b-g(a))^2}\int_{g(a)}^b\frac{(x-g(a))f(x)}{x^3}\mathrm{d}x,\end{aligned}$$

which is the required result. □

Theorem 8. *Let $f,g: I_h\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be general harmonic* log*-convex functions. If $fg\in L[g(a),b]$, then*

$$\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{f(x)}{x^2}\right)[g(g(a))]^{\frac{g(a)(x-b)}{x(g(a)-b)}}[g(b)]^{\frac{b(g(a)-x)}{x(g(a)-b)}}\mathrm{d}x$$
$$+\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{g(x)}{x^2}\right)[f(g(a))]^{\frac{g(a)(x-b)}{x(g(a)-b)}}[f(b)]^{\frac{b(g(a)-x)}{x(g(a)-b)}}\mathrm{d}x$$
$$\leq \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x+\frac{1}{2}M(g(a),b).$$

Proof. Let f and g be general harmonic log-convex functions. Then

$$f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\leq [f(g(a))]^{1-t}[f(b)]^{t},$$
$$g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\leq [g(g(a))]^{1-t}[g(b)]^{t}.$$

Now, using $\langle x_1-x_2, x_3-x_4\rangle \geq 0$, $(x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2$, $x_3<x_4$, we have

$$f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[g(g(a))]^{1-t}[g(b)]^{t}+g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^{t}$$
$$\leq f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+[f(g(a))]^{1-t}[f(b)]^{t}[g(g(a))]^{1-t}[g(b)]^{t}.$$

Integrating over [0, 1], we get

$$\int_0^1 f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[g(g(a))]^{\frac{g(a)(x-b)}{x(g(a)-b)}}[g(b)]^{\frac{b(g(a)-x)}{x(g(a)-b)}}\mathrm{d}t$$
$$+\int_0^1 g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[f(g(a))]^{\frac{g(a)(x-b)}{x(g(a)-b)}}[f(b)]^{\frac{b(g(a)-x)}{x(g(a)-b)}}\mathrm{d}t$$
$$\leq \int_0^1 f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t$$
$$+\int_0^1 [f(g(a))]^{1-t}[f(b)]^{t}[g(g(a))]^{1-t}[g(b)]^{t}\mathrm{d}t.$$

From the above inequality, it follows that

$$\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{f(x)}{x^2}\right)[g(g(a))]^{\frac{g(a)(x-b)}{x(g(a)-b)}}[g(b)]^{\frac{b(g(a)-x)}{x(g(a)-b)}}\mathrm{d}x$$
$$+\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{g(x)}{x^2}\right)[f(g(a))]^{\frac{g(a)(x-b)}{x(g(a)-b)}}[f(b)]^{\frac{b(g(a)-x)}{x(g(a)-b)}}\mathrm{d}x$$
$$\leq \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x+L(G^2(f(g(a)),g(g(a))),G^2(f(b),g(b)))$$

$$\leq \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \left(\frac{f(x)g(x)}{x^2}\right) \mathrm{d}x + A(G^2(f(g(a)), g(g(a))), G^2(f(b), g(b)))$$
$$= \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \left(\frac{f(x)g(x)}{x^2}\right) \mathrm{d}x + \frac{1}{2} M(g(a), b),$$

which is the required result.

Theorem 9. *Let $f, g : I_h \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be increasing general harmonic* log-*convex functions. If $fg \in L[g(a), b]$, then*

$$f\left(\frac{2g(a)b}{g(a)+b}\right) L[g(g(a)), g(b)] + g\left(\frac{2g(a)b}{g(a)+b}\right) L[f(g(a)), f(b)]$$
$$\leq \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \left(\frac{f(x)g(x)}{x}\right) \mathrm{d}x + L[f(g(a))g(g(a)), f(b)g(b)].$$
$$\leq \frac{g(a)b}{b-g(a)} \int_{g(a)}^{b} \left(\frac{f(x)g(x)}{x^2}\right) \mathrm{d}x + \frac{f(g(a))g(g(a)) + f(b)g(b)}{2}.$$

Proof. Let f, g be general harmonic log-convex functions. Then

$$f\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right) \leq [f(g(a))]^{1-t}[f(b)]^{t}.$$
$$g\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right) \leq [g(g(a))]^{1-t}[g(b)]^{t}.$$

Now, using $\langle x_1 - x_2, x_3 - x_4 \rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we get

$$f\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)[g(g(a))]^{1-t}[g(b)]^{t} + g\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^{t}$$
$$\leq f\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right) g\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right) + [f(g(a))]^{1-t}[f(b)]^{t}[g(g(a))]^{1-t}[g(b)]^{t}.$$

Integrating over [0, 1] and get

$$\int_0^1 f\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)[g(g(a))]^{1-t}[g(b)]^{t}\mathrm{d}t + \int_0^1 g\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)[f(g(a))]^{1-t}[f(b)]^{t}\mathrm{d}t$$
$$\leq \int_0^1 f\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right) g\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)\mathrm{d}t + \int_0^1 [f(g(a))]^{1-t}[f(b)]^{t}[g(g(a))]^{1-t}[g(b)]^{t}\mathrm{d}t.$$

Now, since f, g are increasing functions, we have

$$\int_0^1 f\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)\mathrm{d}t \int_0^1 [g(g(a))]^{1-t}[g(b)]^{t}\mathrm{d}t + \int_0^1 g\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)\mathrm{d}t \int_0^1 [f(g(a))]^{1-t}[f(b)]^{t}\mathrm{d}t$$
$$\leq \int_0^1 f\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right) g\left(\frac{g(a)b}{t(g(a)) + (1-t)b}\right)\mathrm{d}t + \int_0^1 [f(g(a))]^{1-t}[f(b)]^{t}[g(g(a))]^{1-t}[g(b)]^{t}\mathrm{d}t.$$

From the above inequality, it follows that

$$f\left(\frac{2g(a)b}{g(a)+b}\right)L[g(g(a)),g(b)]+g\left(\frac{2g(a)b}{g(a)+b}\right)L[f(g(a)),f(b)]$$
$$\leq \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x+L[f(g(a))g(g(a)),f(b)g(b)]$$
$$\leq \frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x+\frac{f(g(a))g(g(a))+f(b)g(b)}{2},$$

which is the required result.

Theorem 10. *Let* $f,g : I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be general harmonic* log-*convex functions. If* $fg \in L[g(a),b]$, *then*

$$\frac{g(a)^2b}{(b-g(a))^2}\int_{g(a)}^{b}\frac{(b-x)}{x^3}[\log f(g(a))\log g(x)+\log g(g(a))\log f(x)]\mathrm{d}x$$
$$+\frac{g(a)b^2}{(b-g(a))^2}\int_{g(a)}^{b}\frac{(x-g(a))}{x^3}[\log f(b)\log g(x)+\log g(b)\log f(x)]\mathrm{d}x$$
$$\leq \frac{1}{3}[\log f(g(a))\log g(g(a))+\log f(b)\log g(b)]+\frac{1}{6}[\log f(g(a))g(b)+\log f(b)g(g(a))]$$
$$+\frac{g(a)b}{b-g(a)}\int_{g(a)}^{b}\frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x.$$

Proof. Let f,g be general harmonic log-convex functions. Then

$$\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\leq (1-t)\log f(g(a))+t\log f(b).$$
$$\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\leq (1-t)\log g(g(a))+t\log g(b).$$

Now, using $\langle x_1-x_2,x_3-x_4\rangle \geq 0$, $(x_1,x_2,x_3,x_4 \in \mathbb{R})$ and $x_1<x_2$, $x_3<x_4$, we have

$$\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[(1-t)\log g(g(a))+t\log g(b)]$$
$$+\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)[(1-t)\log f(g(a))+t\log f(b)]$$
$$\leq \log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)$$
$$+[(1-t)\log f(g(a))+t\log f(b)][(1-t)\log g(g(a))+t\log g(b)].$$

we obtain

$$\log g(g(a))(1-t)\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+\log g(b)t\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)$$
$$+\log f(g(a))(1-t)\log g\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)+\log f(b)t\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)$$
$$\leq \log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)$$
$$+[(1-t)\log f(g(a))+t\log f(b)][(1-t)\log g(g(a))+t\log g(b)].$$

Integrating over [0, 1], we get

$$\log g(g(a))\int_0^1(1-t)\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t$$
$$+\log g(b)\int_0^1 t\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t$$
$$+\log f(g(a))\int_0^1(1-t)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t$$
$$+\log f(b)\int_0^1 t\log g\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\mathrm{d}t$$
$$\leq \int_0^1\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\mathrm{d}t$$
$$+\log f(g(a))\log g(g(a))\int_0^1(1-t)^2+\log f(b)\log g(b)\int_0^1 t^2\mathrm{d}t$$
$$+[\log f(g(a))g(b)+\log f(b)g(g(a))]\int_0^1 t(1-t)\mathrm{d}t.$$

Now after simple integration, we have

$$\frac{g(a)^2b}{(b-g(a))^2}\int_{g(a)}^b\frac{(b-x)}{x^3}[\log f(g(a))\log g(x)+\log g(g(a))\log f(x)]\mathrm{d}x$$
$$+\frac{g(a)b^2}{(b-g(a))^2}\int_{g(a)}^b\frac{(x-g(a))}{x^3}[\log f(b)\log g(x)+\log g(b)\log f(x)]\mathrm{d}x$$
$$\leq \frac{1}{3}[\log f(g(a))\log g(g(a))+\log f(b)\log g(b)]+\frac{1}{6}[\log f(g(a))g(b)$$
$$+\log f(b)g(g(a))]$$
$$+\frac{g(a)b}{b-g(a)}\int_{g(a)}^b\frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x,$$

which is the required result. □

Theorem 11. *Let* $f, g : I = [g(a), b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be general harmonic convex functions. If* $fg \in L[g(a), b]$, *then*

$$f\left(\frac{2g(a)b}{g(a)+b}\right)g\left(\frac{2g(a)b}{g(a)+b}\right)\frac{g(a)^2b^2}{(b-g(a))^2}\int_{g(a)}^{b}\frac{\log g(x)}{x^2}\mathrm{d}x\int_{g(a)}^{b}\frac{\log f(x)}{x^2}\mathrm{d}x$$
$$\leq \exp\left[\frac{g(a)b}{2(b-g(a))}\int_{g(a)}^{b}\frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x+\frac{1}{12}M(g(a),b)+\frac{1}{6}N(g(a),b)\right.$$
$$\left.+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)\right].$$

Proof. Let f, g be general harmonic log-convex functions with $t = \frac{1}{2}$, then

$$\log f\left(\frac{2g(a)b}{g(a)+b}\right)\leq\frac{1}{2}\left[\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right].$$
$$\log g\left(\frac{2g(a)b}{g(a)+b}\right)\leq\frac{1}{2}\left[\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right].$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2, x_3 < x_4$, we get

$$\frac{1}{2}\log f\left(\frac{2g(a)b}{g(a)+b}\right)\left[\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+\log g\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right]$$
$$+\frac{1}{2}\log g\left(\frac{2g(a)b}{g(a)+b}\right)\left[\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right]$$
$$\leq\frac{1}{4}\left[\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right]\left[\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\right.$$
$$\left.+\log g\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right]+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)$$
$$=\frac{1}{4}\left[\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\right.$$
$$+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\log g\left(\frac{g(a)b}{(1-t)g(a)+b}\right)$$
$$+\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)$$
$$\left.\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\right]+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)$$
$$\leq\frac{1}{4}\left[\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\right.$$
$$+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\log g\left(\frac{g(a)b}{(1-t)g(a)+b}\right)$$
$$+[(1-t)\log f(g(a))+t\log f(b)][t\log g(g(a))+(1-t)\log g(b)]$$

$$+[t\log f(g(a)) + (1-t)\log f(b)][(1-t)\log g(g(a)) + t\log g(b)]\Bigg]$$

$$+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)$$

$$\leq \frac{1}{4}\Bigg[\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)$$

$$+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\log g\left(\frac{g(a)b}{(1-t)g(a)+b}\right)$$

$$+2t(1-t)[\log f(g(a))\log g(g(a)) + \log f(b)\log g(b)]$$

$$+[\log f(g(a))\log g(b) + \log f(b)\log g(g(a))][t^2+(1-t)^2]\Bigg]$$

$$+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)$$

Integrating over [0, 1] and get

$$\frac{1}{2}\log f\left(\frac{2g(a)b}{g(a)+b}\right)\int_0^1\left[\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+\log g\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right]dt$$

$$+\frac{1}{2}\log g\left(\frac{2g(a)b}{g(a)+b}\right)\int_0^1\left[\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)+\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\right]dt$$

$$\leq \frac{1}{4}\Bigg[\int_0^1\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)dt$$

$$+\int_0^1\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\log g\left(\frac{g(a)b}{(1-t)g(a)+b}\right)dt$$

$$+2[\log f(g(a))\log g(g(a)) + \log f(b)\log g(b)]\int_0^1 t(1-t)dt$$

$$+[\log f(g(a))\log g(b) + \log f(b)\log g(g(a))]\int_0^1[t^2+(1-t)^2]dt\Bigg]$$

$$+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)\int_0^1 dt$$

$$= \frac{1}{4}\Bigg[\int_0^1\log f\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)\log g\left(\frac{g(a)b}{t(g(a))+(1-t)b}\right)dt$$

$$+\int_0^1\log f\left(\frac{g(a)b}{(1-t)g(a)+tb}\right)\log g\left(\frac{g(a)b}{(1-t)g(a)+b}\right)dt$$

$$+\frac{1}{3}[\log f(g(a))\log g(g(a)) + \log f(b)\log g(b)]$$

$$+\frac{2}{3}\big[\log f(g(a))\log g(b) + \log f(b)\log g(g(a))\big]\Bigg]$$

$$+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)$$

From the above inequality, it follows that

$$f\left(\frac{2g(a)b}{g(a)+b}\right)g\left(\frac{2g(a)b}{g(a)+b}\right)\frac{g(a)^2b^2}{(b-g(a))^2}\int_{g(a)}^{b}\frac{\log g(x)}{x^2}\mathrm{d}x\int_{g(a)}^{b}\frac{\log f(x)}{x^2}\mathrm{d}x$$

$$\leq \exp\left[\frac{g(a)b}{2(b-g(a))}\int_{g(a)}^{b}\frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x+\frac{1}{12}M(g(a),b)+\frac{1}{6}N(g(a),b)\right.$$

$$\left.+\log f\left(\frac{2g(a)b}{g(a)+b}\right)\log g\left(\frac{2g(a)b}{g(a)+b}\right)\right],$$

which is the required result. □

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Extension Operator Method for the Exact Solution of Integro-Differential Equations

I.N. Parasidis and E. Providas

In Honor of Constantin Carathéodory

Abstract An exact method for the solution of the linear Fredholm integro-differential equations is proposed. The method is based on the correct extensions of minimal operators in Banach spaces. The integro-differential operator B is formulated as an extension of a minimal operator A_0 and as a perturbation of a correct differential operator $\widehat{A}$. If the operator B is correct, then the unique solution of the integro-differential equation is obtained in closed form. The method can be easily programmed in a computer algebra system. Since there are not any general exact methods for solving integro-differential equations, the present approach can form the base for further study in this direction.

1 Introduction

Integro-differential equations appear in the mathematical modeling and simulation in several areas such as epidemics [24], astrophysics [36], theory of elasticity [9], biomedicine [22], neural networks [16], image processing [32], and finance [31] to mention but a few. For more applications and other details, one can look in the books [17, 25].

Since there is a variety of integro-differential equations, the methods available for determining their solutions vary considerably depending on their origin and

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structure, the limits of integration, the boundedness of the kernel, and the linearity property. In some rather simple situations, it is possible to obtain a solution directly [35], whereas in other cases, for example, in problems involving nonlinear integro-differential equations under the Carathéodory-type conditions (e.g., see [8]), it is very difficult to get a solution. In general, numerical techniques are employed to solve the integro-differential equations. There are many such methods and a plethora of publications describing them. For a survey of the numerical methods available, one can look in the book [1]. Recent advances in numerical techniques for solving Fredholm integro-differential equations can be found in the following papers and the references therein: The Legendre polynomial and variational iteration methods [2], the Galerkin method [5], the Tau method [12], the interpolation collocation method [13], the Taylor expansion method [14], the Chebyshev series method [20], the wavelet Galerkin method [23], the B-spline collocation method [30], and the homotopy perturbation method [37].

The present work is concerned with the exact solution of the linear Fredholm integro-differential equations. Our approach, inspired by the work [27], is based on the correct extensions of minimal operators for solving initial and boundary value problems containing differential or integro-differential equations. The theory of extensions of densely defined minimal operators was initiated by [21, 33, 34] and [3] and developed further by [4, 6, 7, 10, 11, 18, 19, 28, 29] and others.

The paper is organized as follows. In Sect. 2 we quote some definitions and notations. Next we present the extension operator method in a general form, and then in Sect. 4, we focus on its application for solving linear Fredholm integro-differential equations. In Sect. 5 several example problems are solved to demonstrate the efficiency of the method. Finally in Sect. 6, some conclusions and future directions are quoted.

2 Definitions and Notations

Let X be a complex Banach space and X^* its adjoint space, i.e., the set of all complex-valued linear and bounded functionals on X. If $u, v \in X$ and $\psi \in X^*$, then

$$\psi(a_1 u + a_2 v) = a_1 \psi(u) + a_2 \psi(v) \ , \tag{1}$$

where a_1 and a_2 are complex numbers, and there exists $k \in R^+$ such that

$$|\psi(u)| \leq k||u|| \, , \ \text{for all } u \in X \ . \tag{2}$$

The following Banach spaces and notations are used throughout the paper:

$C^0[a, b]$ or $C[a, b]$ or C is the space of all continuous complex-valued functions on $[a, b]$ with the norm

$$||u||_C = \max_{t \in [a,b]} |u(t)|.$$

$C^n[a,b]$ or C^n is the space of all complex-valued functions on $[a,b]$ with continuous derivatives of order n and the norm

$$||u||_{C^n} = ||u||_C + ||u'||_C + \cdots + ||u^{(n)}||_C = \sum_{i=0}^{n} ||u^{(i)}||_C.$$

$L_2(a,b)$ or L_2 is the space of all Lebesgue integrable functions with the norm

$$||u||_{L_2} = \left(\int_a^b |u(t)|^2 dt \right)^{\frac{1}{2}}.$$

$W_2^n(a,b)$ or W_2^n is the Sobolev space, i.e., the space all functions $u \in L_2(a,b)$ with $u^{(i)} \in L_2(a,b), \ \ i = 1,\ldots,n$ and the norm

$$||u||_{W_2^n} = \left(\int_a^b \left[|u(t)|^2 + |u'(t)|^2 + \cdots + |u^{(n)}(t)|^2 \right] dt \right)^{\frac{1}{2}}.$$

Note $L_2 \supset W_2^1 \supset \cdots \supset W_2^n, \ W_2^i \supset C^i, \ i = 1,\ldots,n$ and $L_2 \supset C \supset C^1 \supset \cdots \supset C^n$, for their adjoint spaces hold $[L_2]^* \subset [W_2^1]^* \subset \cdots \subset [W_2^n]^*, \ [W_2^i]^* \subset [C^i]^*, \ i = 1,\ldots,n$ and $[L_2]^* \subset C^* \subset [C^1]^* \subset \cdots \subset [C^n]^*$.

Let X, Y be complex Banach spaces and $A : X \to Y$ an operator with $D(A)$ and $R(A)$ indicating the domain and the range of the operator A, respectively. An operator A is called *closed* if for every sequence x_n in $D(A)$ converging to x_0 with $Ax_n \to y_0, \ y_0 \in Y$, it follows that $x_0 \in D(A)$ and $Ax_0 = y_0$. A closed operator A is called *maximal* if $R(A) = Y$ and $\ker A \neq \{0\}$. A closed operator $A_0 : X \to Y$ is called *minimal* if $R(A_0) \neq Y$ and its inverse A_0^{-1} exists on $R(A_0)$ and is continuous. An operator $\widehat{A} : X \to Y$ is called *correct* if $R(\widehat{A}) = Y$ and its inverse $\widehat{A}^{-1}$ exists and is continuous. Let the operators $A_1, A_2 : X \to Y$; the operator A_2 is said to be an *extension* of A_1, or A_1 is a *restriction* of A_2, in symbol $A_1 \subset A_2$, if $D(A_1) \subseteq D(A_2)$ and $A_1 x = A_2 x$, for all $x \in D(A_1)$. An operator $\widehat{A}$ is called a *correct extension* (resp. *restriction*) of the minimal (resp. maximal) operator A_0 (resp. A) if it is a correct operator and $A_0 \subset \widehat{A}$ (resp. $\widehat{A} \subset A$).

Let $\psi_i \in X^*, \ i = 1,\ldots,m$, Ψ =col$(\psi_1,\ldots,\psi_m)$ denotes the column matrix of $\psi_i, \ i = 1,\ldots,m$ and $\Psi(u)$ =col$(\psi_1(u),\ldots,\psi_m(u))$ denotes the values of Ψ at $u \in X$. In addition, let $q_j \in X, j = 1,\ldots,n$ and $q = (q_1,\ldots,q_n)$ be a vector of X^n. Denote by $\Psi(q)$ the $m \times n$ matrix whose ijth entry is the value of functional ψ_i on element q_j. It is easy to verify that if C is a $n \times k$ constant matrix, then $\Psi(qC) = \Psi(q)C$. Finally, denote by I_m the $m \times m$ identity matrix.

3 Extension Operator Method

Let X and Y be complex Banach spaces and $\widehat{A} : X \to Y$ a linear correct, generally unbounded, operator with $D(\widehat{A}) \subset Z \subseteq X$, where Z is a Banach space, usually a Sobolev space W_2^n or C^n. Consider the perturbed problem:

$$Bu = \widehat{A}u - g\Psi(u) = f, \quad D(B) = D(\widehat{A}), \quad f \in Y, \tag{3}$$

where the inner vector product $g\Psi(u)$ of the vector $g = (g_1, \ldots, g_m) \in Y^m$ and the column vector $\Psi = \text{col}(\psi_1, \ldots, \psi_m) \in (Z^*)^m$ is a perturbation. The operator B is an extension of the minimal operator A_0 defined by

$$A_0 u = \widehat{A}u, \quad D(A_0) = \{u \in D(\widehat{A}) : \Psi(u) = 0\}. \tag{4}$$

We prove that if g and Ψ satisfy certain necessary and sufficient conditions, the operator B is correct. Then we find the solution of the problem (3) by using the solution of the correct problem:

$$\widehat{A}u = f, \quad f \in Y. \tag{5}$$

Theorem 1. *Let X, Y, and Z be complex Banach spaces and $\widehat{A} : X \to Y$ a linear correct operator with $D(\widehat{A}) \subset Z \subseteq X$. Further, let the vector $g = (g_1, \ldots, g_m) \in Y^m$ and the column vector $\Psi = \text{col}(\psi_1, \ldots, \psi_m)$, where $\psi_1, \ldots, \psi_m \in Z^*$ and their restrictions on $D(\widehat{A})$ are linearly independent. Then:*

(i) The operator B defined by

$$Bu = \widehat{A}u - g\Psi(u) = f, \quad D(B) = D(\widehat{A}), \quad f \in Y, \tag{6}$$

is correct if and only if

$$\det W = \det\left[I_m - \Psi(\widehat{A}^{-1}g)\right] \neq 0. \tag{7}$$

(ii) If B is correct, then for any $f \in Y$, the unique solution of (6) is given by

$$u = B^{-1}f = \widehat{A}^{-1}f + (\widehat{A}^{-1}g)\left[I_m - \Psi(\widehat{A}^{-1}g)\right]^{-1}\Psi(\widehat{A}^{-1}f). \tag{8}$$

Proof. (i) and (ii). Suppose B is a correct operator and $\det W = 0$. Since the operator $\widehat{A}$ is correct, then there exists the inverse operator $\widehat{A}^{-1}$. By applying the inverse operator $\widehat{A}^{-1}$ on both sides of (6), we get

$$u - \widehat{A}^{-1}g\Psi(u) = \widehat{A}^{-1}f. \tag{9}$$

Applying next the column vector Ψ on (9), that is, evaluating each of $\psi_1, \ldots, \psi_m$ at u in Eq. (9), we have

$$\begin{aligned}
\Psi(u) - \Psi\left(\widehat{A}^{-1}g\Psi(u)\right) &= \Psi(\widehat{A}^{-1}f), \\
\Psi(u) - \Psi(\widehat{A}^{-1}g)\Psi(u) &= \Psi(\widehat{A}^{-1}f), \\
\left[I_m - \Psi(\widehat{A}^{-1}g)\right]\Psi(u) &= \Psi(\widehat{A}^{-1}f), \\
W\Psi(u) &= \Psi(\widehat{A}^{-1}f) \quad \forall f \in Y,
\end{aligned} \tag{10}$$

where $W = \left[I_m - \Psi(\widehat{A}^{-1}g)\right]$ is a $m \times m$ matrix. Because of the hypothesis $\det W = 0$, we suppose that $rank\, W = k < m$ and that the first k lines of the matrix W are linearly independent. Also, since the components of the vector Ψ are linearly independent on $D(\widehat{A})$, then there exists a biorthogonal set $t_1, \ldots, t_m \in D(\widehat{A})$ such that $\psi_i(t_j) = \delta_{ij},\ i,j = 1, \ldots, m$, where δ_{ij} is the Kronecker symbol. Taking $f = \widehat{A}t_{k+1}$ and substituting into the system (10), we have $W\Psi(u) = \Psi(t_{k+1})$, where because of the orthogonality, all elements of the right hand side are zero except the $k+1$ element which is 1. This means that the rank of the augmented matrix $[W,\ \Psi(t_{k+1})]$ is greater than that of W and therefore the system (10) has no solution (e.g., see [26]). In consequence $Bu = \widehat{A}t_{k+1}$ has no solution and $R(B) \neq Y$. Hence, B is not a correct operator which contradicts our assumption. Therefore (7) is true.

Conversely, suppose now $\det W \neq 0$. Working as above we obtain (9) and (10). Then

$$\Psi(u) = [I_m - \Psi(\widehat{A}^{-1}g)]^{-1}\Psi(\widehat{A}^{-1}f) \tag{11}$$

is uniquely defined for any $f \in Y$. Substituting (11) into (9) and considering (6), we get

$$u = B^{-1}f = \widehat{A}^{-1}f + (\widehat{A}^{-1}g)\left[I_m - \Psi(\widehat{A}^{-1}g)\right]^{-1}\Psi(\widehat{A}^{-1}f) \tag{12}$$

which is the unique solution of (6) for any $f \in Y$. This proves the existence of the inverse B^{-1} and that $R(B) = Y$. It remains to show that the operator B^{-1} is continuous. Let for convenience $W^{-1} = D = (d_{ij}),\ i,j = 1, \ldots, m$. Note that since ψ_j are bounded functionals, there exist constants $k_j > 0$ such that $|\psi_j(\widehat{A}^{-1}f)| \leq k_j||\widehat{A}^{-1}f)||_X,\ j = 1, \ldots, m$. Also, the boundedness of $\widehat{A}^{-1}$ implies the existence of a constant $\delta > 0$ such that $||\widehat{A}^{-1}f||_X \leq \delta||f||_Y$ for all $f \in Y$. Then from (12) we have

$$\begin{aligned}
||B^{-1}f||_X &\leq ||\widehat{A}^{-1}f||_X + ||\sum_{i=1}^{m} d_{i1}\widehat{A}^{-1}g_i||_X \cdot |\psi_1(\widehat{A}^{-1}f)| + \cdots \\
&\quad + ||\sum_{i=1}^{m} d_{im}\widehat{A}^{-1}g_i||_X \cdot |\psi_m(\widehat{A}^{-1}f)| \\
&\leq ||\widehat{A}^{-1}f||_X + \left[\sum_{j=1}^{m}\left(||\sum_{i=1}^{m} d_{ij}\widehat{A}^{-1}g_i||_X\right)k_j\right]||\widehat{A}^{-1}f||_X \\
&\leq \left\{1 + \left[\sum_{j=1}^{m}\left(||\sum_{i=1}^{m} d_{ij}\widehat{A}^{-1}g_i||_X\right)k_j\right]\right\}\delta||f||_Y \\
&\leq c||f||_Y\,,
\end{aligned} \tag{13}$$

where

$$c = \left\{ 1 + \left[\sum_{j=1}^{m} \left(|| \sum_{i=1}^{m} d_{ij} \widehat{A}^{-1} g_i ||_X \right) k_j \right] \right\} \delta > 0.$$

This proves that B^{-1} is bounded and hence continuous. Therefore the operator B is correct. This completes the proof. □

4 Fredholm Integro-Differential Equations

The method presented in the previous section can be applied to several classes of problems. Here, we concentrate on the application of the method to linear Fredholm integro-differential equations.

Consider a standard general form of the linear Fredholm integro-differential equations given by

$$\sum_{k=0}^{n} P_k(x) u^{(k)}(x) = f(x) + \lambda \sum_{j=0}^{l} \int_a^b K_j(x,t) u^{(j)}(t) dt, \quad x \in [a,b], \tag{14}$$

with the initial conditions

$$u^{(k)}(a) = a_k, \quad 0 \leq k \leq n-1, \tag{15}$$

or under the boundary conditions

$$u^{(k)}(a) = a_k, \quad u^{(l)}(b) = b_l, \quad 0 \leq k, l \leq n-1, \quad 0 \leq k+l \leq n-1, \tag{16}$$

where λ is a known parameter, $P_k(x),\ k = 0, \ldots, n,\ f(x)$ and the kernels $K_j(x,t),\ j = 1, \ldots, l \leq n$ are given functions satisfying certain conditions such that Eq. (14) along with (15) or (16) has a unique solution, a_k and b_l are given constants that define the initial or boundary conditions, and $u(x)$ is an unknown function to be determined. Without loss of generality, we consider the case where all conditions are initial conditions. Boundary conditions can be treated in a similar manner. Also, we assume that the initial conditions are homogeneous, i.e., $u^{(k)}(a) = 0$, $k = 0, \ldots, n-1$. Observe that by means of the transformation,

$$v(x) = u(x) - \sum_{k=0}^{n-1} \frac{a_k}{k!} (x-a)^k, \tag{17}$$

Eq. (14) accompanied by the nonhomogeneous conditions (15) can be changed to an integro-differential equation with $v(x)$ being the unknown function and the homogeneous conditions $v^{(k)}(a) = 0, k = 0, \dots, n-1$. In addition, we assume the kernels $K_j(x,t)$ are separable which can be expressed as finite sums of the form

$$K_j(x,t) = \sum_{i=1}^{m} g_{ij}(x)h_{ij}(t), \quad j = 0, \dots, l, \tag{18}$$

or even more interestingly

$$K_j(x,t) = \sum_{i=1}^{m} g_i(x)h_{ij}(t), \quad j = 0, \dots, l, \tag{19}$$

where the functions $g_{ij}(x) = g_i(x), i = 1, \dots, m$ are common for all kernels $K_j(x,t)$, $j = 0, \dots, l$. Then Eq. (14) subjected to initial conditions (15) is written

$$\sum_{k=0}^{n} P_k(x)u^{(k)}(x) = f(x) + \lambda \sum_{i=1}^{m} g_i(x) \int_a^b \sum_{j=0}^{l} h_{ij}(t)u^{(j)}(t)dt,$$
$$u^{(k)}(a) = 0, \quad 0 \le k \le n-1, \quad x \in [a,b]. \tag{20}$$

When $\lambda = 0$, Eq. (14) reduces to an nth order differential equation with varying coefficients:

$$\sum_{k=0}^{n} P_k(x)u^{(k)}(x) = f(x)$$
$$u^{(k)}(a) = 0, \quad 0 \le k \le n-1, \quad x \in [a,b]. \tag{21}$$

If the functions $P_k(x),\ k = 0, \dots, n$ and $f(x)$ are continuous on $[a,b]$, then the problem defined by (21) has a unique solution $u(x)$ on $[a,b]$ (e.g., see [15]). In the following, we assume that $P_k(x),\ k = 0, \dots, n$ and $f(x)$ are continuous on $[a,b]$.

Let now $X = Y = C[a,b]$, $Z = C^n[a,b]$ and define the operator $\widehat{A} : X \to Y$ by

$$\widehat{A}u = \sum_{k=0}^{n} P_k(x)u^{(k)}(x) = f(x),$$
$$D(\widehat{A}) = \{u \in C^n[a,b] : u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0\}. \tag{22}$$

The operator $\widehat{A}$ is a correct operator. In the particular case where $\widehat{A} = u^{(n)}(x)$, the following theorem holds.

Theorem 2. *Let the operator* $\widehat{A} : C[a,b] \to C[a,b]$ *defined by*

$$\widehat{A}u = u^{(n)}(x) = f(x),$$
$$D(\widehat{A}) = \{u \in C^n[a,b] : u(a) = u'(a) = \dots = u^{(n-1)}(a) = 0\} \tag{23}$$

and $f(x) \in C[a,b]$. *Then, the problem (23) is correct and its unique solution is given by*

$$u(x) = \widehat{A}^{-1}f = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f(t)dt. \tag{24}$$

Next, we define the minimal operator A_0 as follows:

$$A_0 u = \widehat{A}u = f(x),$$
$$D(A_0) = \{u \in D(\widehat{A}) : \int_a^b \sum_{j=0}^{l} h_{ij}(t)u^{(j)}(t)dt = 0,\ i = 1,\dots,m\} \tag{25}$$

Finally, we let $\lambda = 1$ and define the operator B by

$$Bu = \sum_{k=0}^{n} P_k(x)u^{(k)}(x) - \sum_{i=1}^{m} g_i(x)\int_a^b \sum_{j=0}^{l} h_{ij}(t)u^{(j)}(t)dt = f(x),$$
$$D(B) = D(\widehat{A}). \tag{26}$$

Note that the operator B is an extension of the minimal operator A_0 and a perturbation of the operator $\widehat{A}$. We write Eq. (20) in the form as in (6) of Theorem 1, namely,

$$Bu = \widehat{A}u - g\Psi(u) = f, \quad D(B) = D(\widehat{A}), \quad f \in Y, \tag{27}$$

where

$$g = (g_1,\dots,g_m) = (g_1(x),\dots,g_m(x)), \quad g_i(x) \in Y,$$
$$\Psi(u) = \mathrm{col}(\psi_1(u),\dots,\psi_m(u))$$
$$= \mathrm{col}\left(\int_a^b \sum_{j=0}^{l} h_{1j}(t)u^{(j)}(t)dt,\dots,\int_a^b \sum_{j=0}^{l} h_{mj}(t)u^{(j)}(t)dt\right). \tag{28}$$

The functionals $\psi_1,\dots,\psi_m$ defined in (28) are linear and bounded on Z. This is proved by the following theorem.

Theorem 3. *Let* $u \in Z = W_2^l(a,b)$, $h_j(t) \in L_2(a,b)$, $j = 0,1,\dots,l$, $b_1 \le b$ *and the functional*

$$\psi(u) = \int_a^{b_1} [\sum_{j=0}^{l} u^{(j)}(t)\overline{h_j(t)}]dt. \tag{29}$$

Then $\psi(u)$ *is linear and bounded on Z, i.e.,* $\psi \in [W_2^l(a,b)]^*$ *and hence* $\psi \in \left[C^l[a,b]\right]^*$.

Proof. It is easy to show that the functional ψ defined by (29) satisfies Eq. (1) and therefore is linear. Next we prove that Eq. (2) is also fulfilled. By using consecutively Hölder inequality, Minkowski inequality, and the inequality

$$\sum_{j=0}^{l} \sqrt{a_j} \le \sqrt{l+1} \sqrt{\sum_{j=0}^{l} a_j}, \quad a_j \ge 0,\ j = 0, \dots, l, \tag{30}$$

we have

$$\begin{aligned} |\psi(u)| &\le \int_a^{b_1} |\sum_{j=0}^{l} u^{(j)}(t)\overline{h_j(t)}|dt \\ &\le \sum_{j=0}^{l} \int_a^b |u^{(j)}(t)\overline{h_j(t)}|dt \\ &\le \sum_{j=0}^{l} \left(\int_a^b |\overline{h_j(t)}|^2 dt\right)^{1/2} \left(\int_a^b |u^{(j)}(t)|^2 dt\right)^{1/2} \\ &\le c \sum_{j=0}^{l} \left(\int_a^b |u^{(j)}(t)|^2 dt\right)^{1/2} \\ &\le c\sqrt{l+1} \left(\sum_{j=0}^{l} \int_a^b |u^{(j)}(t)|^2 dt\right)^{1/2} \\ &= c\sqrt{l+1}||u||_{W_2^l}, \end{aligned} \tag{31}$$

where

$$c = \max_j \left\{ \left(\int_a^b |\overline{h_j(t)}|^2 dt\right)^{1/2} \right\}, \quad j = 0, \dots, l.$$

Equation (31) proves that $\psi \in [W_2^l(a,b)]^*$. Since $[W_2^l(a,b)]^* \subset \left[C^l[a,b]\right]^*$, it follows that $\psi \in \left[C^l[a,b]\right]^*$. This completes the proof. □

It remains to examine if the functionals $\psi_1, \dots, \psi_m$ defined in (28) are linearly independent. This can be done by applying the following proposition.

Proposition 1. *Let the functionals* $\psi_1, \ldots, \psi_n \in Z^*$. *Then* $\psi_1, \ldots, \psi_n$ *are linearly independent elements if there exist a set of linearly independent* $z_1, \ldots, z_n \in Z$ *such that*

$$\det \begin{pmatrix} \psi_1(z_1) & \ldots & \psi_n(z_1) \\ \vdots & \ldots & \vdots \\ \psi_1(z_n) & \ldots & \psi_n(z_n) \end{pmatrix} \neq 0. \tag{32}$$

5 Illustrative Example Problems

Here we implement the extension operator method for solving initial and boundary value problems involving a linear Fredholm integro-differential equation. All examples are taken from the literature on the subject.

Problem 1. Consider first the following integro-differential equation:

$$u'(x) = -10x + \int_{-1}^{1} (x-t)u(t)dt\,, \quad u(0) = 1\,, \quad x \in [-1, 1]. \tag{33}$$

By making the substitution

$$v(x) = u(x) - 1, \tag{34}$$

Eq. (33) is written

$$v'(x) = -8x + x\int_{-1}^{1} v(t)dt - \int_{-1}^{1} tv(t)dt, \quad v(0) = 0, \quad x \in [-1, 1], \tag{35}$$

where now the initial condition is homogeneous. Let $X = Y = C[-1, 1]$, $Z = C^1[-1, 1]$ and the operator $B : X \to Y$ defined by

$$\begin{aligned} Bv &= v'(x) - x\int_{-1}^{1} v(t)dt + \int_{-1}^{1} tv(t)dt = -8x, \\ D(B) &= \{v \in C^1[-1, 1] : v(0) = 0\}. \end{aligned} \tag{36}$$

We express Eq. (36) in the form as in (6), namely,

$$Bv = \widehat{A}v - g\Psi(v) = f\,, \quad D(B) = \{v \in C^1[-1, 1] : v(0) = 0\}\,, \tag{37}$$

where

$$\widehat{A}v = v'(x)\,, \quad D(\widehat{A}) = D(B) \tag{38}$$

and

$$\begin{aligned} g &= (g_1,\, g_2) = (x,\, -1), \\ \Psi(v) &= \text{col}\,(\psi_1(v),\, \psi_2(v)) = \text{col}\left(\int_{-1}^{1} v(t)dt,\, \int_{-1}^{1} tv(t)dt\right), \\ f &= -8x. \end{aligned} \tag{39}$$

By Theorem 3 the functionals $\psi_1, \psi_2 \in \left[C^1[-1,1]\right]^*$, while by Proposition 1, they are linearly independent. From Theorem 2 the differential equation problem

$$\widehat{A}v(x) = v'(x) = f, \quad v(0) = 0 \tag{40}$$

is correct and its solution is given by

$$v(x) = \widehat{A}^{-1}f = \int_0^x f(t)dt. \tag{41}$$

By using the inverse operator $\widehat{A}^{-1}$ and Eq. (39), we have

$$\widehat{A}^{-1}g = \left(\widehat{A}^{-1}g_1,\ \widehat{A}^{-1}g_2\right) = \left(x^2/2,\ -x\right),$$

$$\Psi(\widehat{A}^{-1}g) = \begin{pmatrix} \psi_1(\widehat{A}^{-1}g_1) & \psi_1(\widehat{A}^{-1}g_2) \\ \psi_2(\widehat{A}^{-1}g_1) & \psi_2(\widehat{A}^{-1}g_2) \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & -2/3 \end{pmatrix} \tag{42}$$

and thus

$$\det W = \det\left[I_2 - \Psi(\widehat{A}^{-1}g)\right] = \frac{10}{9} \neq 0\,. \tag{43}$$

In addition we get

$$\widehat{A}^{-1}f = \int_0^x (-8t)dt = -4x^2,$$

$$\Psi(\widehat{A}^{-1}f) = \mathrm{col}\left(\psi_1(\widehat{A}^{-1}f),\ \psi_2(\widehat{A}^{-1}f)\right) = (-8/3,\ 0)\,. \tag{44}$$

Then, from Theorem 1 and Eq. (43) it follows that the operator B is correct and the unique solution of Eq. (37) is given by (8). Substitution of (42) and (44) into (8) yields the exact solution of (37), viz.,

$$v(x) = -6x^2. \tag{45}$$

Consequently, by means of (34), the solution of the integro-differential equation (33) is obtained in closed form:

$$u(x) = 1 - 6x^2. \tag{46}$$

Problem 2. Solve the following third-order integro-differential equation:

$$u'''(x) = e - 2 - x + e^x(3 + x) + \int_0^1 (x-t)u(t)dt,$$

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 2, \quad x \in [0,1]. \tag{47}$$

By using the transformation

$$v(x) = u(x) - x^2 - x, \tag{48}$$

Eq. (47) is written

$$\begin{aligned} &v'''(x) = e - \frac{31}{12} - \frac{x}{6} + e^x(3+x) + x\int_0^1 v(t)dt - \int_0^1 tv(t)dt, \\ &v(0) = v'(0) = v''(0) = 0, \quad x \in [0,1], \end{aligned} \tag{49}$$

where now the initial conditions are homogeneous. Let $X = Y = C[0,1]$, $Z = C^3[0,1]$ and the operator $B : X \to Y$ defined by

$$\begin{aligned} Bv &= v'''(x) - x\int_0^1 v(t)dt + \int_0^1 tv(t)dt = e - \frac{31}{12} - \frac{x}{6} + e^x(3+x), \\ D(B) &= \{v \in C^3[0,1] : v(0) = v'(0) = v''(0) = 0\}, \end{aligned} \tag{50}$$

We write Eq. (50) in the form as in (6), specifically

$$\begin{aligned} Bv &= \widehat{A}v - g\Psi(v) = f, \\ D(B) &= \{v \in C^3[0,1] : v(0) = v'(0) = v''(0) = 0\}, \end{aligned} \tag{51}$$

where

$$\widehat{A}v(x) = v'''(x), \quad D(\widehat{A}) = D(B) \tag{52}$$

and

$$\begin{aligned} g &= (g_1, g_2) = (x, -1), \\ \Psi(v) &= \mathrm{col}(\psi_1(v), \psi_2(v)) = \mathrm{col}\left(\int_0^1 v(t)dt, \int_0^1 tv(t)dt\right), \\ f &= e - \frac{31}{12} - \frac{x}{6} + e^x(3+x). \end{aligned} \tag{53}$$

By Theorem 3 the functionals $\psi_1, \psi_2 \in \left[C^3[0,1]\right]^*$, whereas by Proposition 1, they are linearly independent. From Theorem 2 it follows that the differential equation problem

$$\widehat{A}v(x) = v'''(x) = f, \quad v(0) = v'(0) = v''(0) = 0, \quad f \in C[0,1] \tag{54}$$

is correct and its solution is given by

$$v(x) = \widehat{A}^{-1}f = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt. \tag{55}$$

By using the inverse operator $\widehat{A}^{-1}$ and Eq. (53), we get

$$\widehat{A}^{-1}g_1 = \frac{1}{2}\int_0^x (x-t)^2 t dt = \frac{x^4}{24},$$
$$\widehat{A}^{-1}g_2 = \frac{1}{2}\int_0^x (x-t)^2(-1) dt = -\frac{x^3}{6},$$
$$\widehat{A}^{-1}g = (\widehat{A}^{-1}g_1, \widehat{A}^{-1}g_2) = \left(\frac{x^4}{24}, -\frac{x^3}{6}\right) \tag{56}$$

and

$$\psi_1(\widehat{A}^{-1}g_1) = \frac{1}{24}\int_0^1 t^4 dt = \frac{1}{120}, \quad \psi_1(\widehat{A}^{-1}g_2) = -\frac{1}{6}\int_0^1 t^3 dt = -\frac{1}{24},$$
$$\psi_2(\widehat{A}^{-1}g_1) = \frac{1}{24}\int_0^1 t^5 dt = \frac{1}{144}, \quad \psi_2(\widehat{A}^{-1}g_2) = -\frac{1}{6}\int_0^1 t^4 dt = -\frac{1}{30},$$
$$\Psi(\widehat{A}^{-1}g) = \begin{pmatrix} \psi_1(\widehat{A}^{-1}g_1) & \psi_1(\widehat{A}^{-1}g_2) \\ \psi_2(\widehat{A}^{-1}g_1) & \psi_2(\widehat{A}^{-1}g_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{120} & -\frac{1}{24} \\ \frac{1}{144} & -\frac{1}{30} \end{pmatrix} \tag{57}$$

and hence

$$\det W = \det\left[I_2 - \Psi(\widehat{A}^{-1}g)\right] = \det\begin{pmatrix} \frac{119}{120} & \frac{1}{24} \\ -\frac{1}{144} & \frac{31}{30} \end{pmatrix} = \frac{88561}{86400} \neq 0. \tag{58}$$

Furthermore, we have

$$\widehat{A}^{-1}f = \frac{1}{2}\int_0^x (x-t)^2\left[e - \frac{31}{12} - \frac{t}{6} + e^t(3+t)\right]dt$$
$$= xe^x - \frac{x\left[x^3 + 2x^2(31-12e) + 144x + 144\right]}{144},$$
$$\Psi(\widehat{A}^{-1}f) = \operatorname{col}(\psi_1(\widehat{A}^{-1}f), \psi_2(\widehat{A}^{-1}f))$$
$$= \operatorname{col}\left(\frac{60e+83}{1440}, \frac{4464e - 11537}{4320}\right). \tag{59}$$

Then, from Theorem 1 and Eq. (58), it follows that the operator B is correct and the solution of the problem (51) is given by (8). Substitution of (56), (57), and (59) into (8) yields the exact solution:

$$v(x) = xe^x - x^2 - x. \tag{60}$$

Substituting this into (48), the solution of the integro-differential equation (47) is obtained in closed form:

$$u(x) = xe^x. \tag{61}$$

Problem 3. Find the exact solution of the following third-order integro-differential equation:

$$\begin{aligned} u'''(x) &= e^x + \frac{x(127-69e)}{20} + x\int_0^1 t^2 u(t)dt + 2x\int_0^1 tu'(t)dt + 2x\int_0^1 u''(t)dt, \\ u(0) &= 0, \quad u'(0) = 0, \quad u''(0) = 0, \quad x \in [0,1]. \end{aligned} \tag{62}$$

Let $X = Y = C[0,1]$, $Z = C^3[0,1]$ and the operator $B : X \to Y$ defined by

$$\begin{aligned} Bu &= u'''(x) - x\int_0^1 t^2 u(t)dt - 2x\int_0^1 tu'(t)dt - 2x\int_0^1 u''(t)dt \\ &= e^x + \frac{x(127-69e)}{20}, \\ D(B) &= \{u \in C^3[0,1] : u(0) = 0,\ u'(0) = 0,\ u''(0) = 0\}. \end{aligned} \tag{63}$$

We formulate Eq. (63) as in (6), that is, to say

$$\begin{aligned} Bu &= \widehat{A}u - g\Psi(u) = f, \\ D(B) &= \{u \in C^3[0,1] : u(0) = u'(0) = u''(0) = 0\}, \end{aligned} \tag{64}$$

where

$$\widehat{A}u(x) = u'''(x), \quad D(\widehat{A}) = D(B) \tag{65}$$

and

$$\begin{aligned} g &= x, \\ \Psi(u) &= \int_0^1 [t^2 u(t) + 2tu'(t) + 2u''(t)]dt, \\ f &= e^x + \frac{x(127-69e)}{20}. \end{aligned} \tag{66}$$

By Theorem 3 the functional $\psi \in \left[C^3[0,1]\right]^*$. From Theorem 2 the differential equation problem

$$\widehat{A}u(x) = u'''(x) = f, \quad u(0) = u'(0) = u''(0) = 0, \tag{67}$$

is correct and its solution is given by

$$u(x) = \widehat{A}^{-1}f = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt. \tag{68}$$

By using the inverse operator $\widehat{A}^{-1}$ and Eq. (66), we get

$$\widehat{A}^{-1}g = \frac{1}{2}\int_0^x (x-t)^2 t dt = \frac{x^4}{24},$$
$$\Psi(\widehat{A}^{-1}g) = \int_0^1 \left(\frac{1}{24}t^6 + \frac{1}{3}t^4 + t^2\right) dt = \frac{341}{840} \tag{69}$$

and therefore

$$\det\ W = \det\left[1 - \Psi(\widehat{A}^{-1}g)\right] = \frac{499}{840} \neq 0. \tag{70}$$

Also, we have

$$\widehat{A}^{-1}f = \frac{1}{2}\int_0^x \left[(x-t)^2\left(e^t + \frac{127-69e}{20}t\right)\right] dt$$
$$= e^x - \frac{1}{480}\left[x^4(60e-127) + 240x^2 + 480x + 480\right],$$
$$\Psi(\widehat{A}^{-1}f) = \frac{1100e-2963}{1120} + \frac{227-60e}{300} + \frac{60e-113}{60}$$
$$= \frac{499(60e-127)}{16800}. \tag{71}$$

Then, from Theorem 1 and Eq. (70), it follows that the operator B is correct and the solution of (64) is given by (8). Substituting (69) and (71) into (8), the solution of the integro-differential equation (62) is obtained in closed form:

$$u(x) = e^x - \frac{1}{2}(x^2 + 2x + 2). \tag{72}$$

Problem 4. Consider the following third-order integro-differential equation:

$$u'''(x) = \sin x - x - \int_0^{\pi/2} xtu'(t)dt,$$
$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \quad x \in [0, \pi/2]. \tag{73}$$

By making the substitution

$$v(x) = u(x) + \frac{1}{2}x^2 - 1, \tag{74}$$

the integro-differential equation (73) is written

$$v'''(x) = \sin x - x\left(1 - \frac{\pi^3}{24}\right) - x\int_0^{\pi/2} tv'(t)dt,$$
$$v(0) = v'(0) = v''(0) = 0 \tag{75}$$

where now the initial conditions are homogeneous. Let $X = Y = C[0,\pi]$, $Z = C^3[0,\pi]$ and the operator $B : X \to Y$ defined by

$$Bv = v'''(x) + x\int_0^{\pi/2} tv'(t)dt = \sin x - x\left(1 - \frac{\pi^3}{24}\right),$$
$$D(B) = \{v \in C^3[0,\pi] : \; v(0) = v'(0) = v''(0) = 0\}. \tag{76}$$

We express Eq. (76) in the operator form as in (6), namely,

$$Bv = \widehat{A}v - g\Psi(v) = f,$$
$$D(B) = \{v \in C^3[0,\pi] : \; v(0) = v'(0) = v''(0) = 0\} \tag{77}$$

where

$$\widehat{A}v(x) = v'''(x), \quad D(\widehat{A}) = D(B) \tag{78}$$

and

$$g = -x, \quad \Psi(v) = \int_0^{\pi/2} tv'(t)dt, \quad f = \sin x - x\left(1 - \frac{\pi^3}{24}\right). \tag{79}$$

From Theorem 3 it follows that the functional $\psi \in \left[C^3[0,\pi]\right]^*$. According to Theorem 2, the differential equation problem

$$\widehat{A}v(x) = v'''(x) = f, \quad v(0) = v'(0) = v''(0) = 0 \tag{80}$$

is correct and its solution is given by

$$v(x) = \widehat{A}^{-1}f = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt. \tag{81}$$

Utilizing the inverse $\widehat{A}^{-1}$ and Eq. (79), we get

$$\widehat{A}^{-1}g = \frac{1}{2}\int_0^x (x-t)^2(-t)dt = -\frac{x^4}{24},$$
$$\Psi(\widehat{A}^{-1}g) = \int_0^{\pi/2} t(\widehat{A}^{-1}g)'dt = -\frac{1}{6}\int_0^{\pi/2} t^4 dt = -\frac{\pi^5}{960} \tag{82}$$

and hence

$$\det\, W = \det\left[I_m - \Psi(\widehat{A}^{-1}g)\right] = \frac{960 + \pi^5}{960} \neq 0. \tag{83}$$

Furthermore, we have

$$\begin{aligned}
\widehat{A}^{-1}f &= \frac{1}{2}\int_0^x (x-t)^2 \left[\sin t - t(1 - \frac{\pi^3}{24})\right] dt \\
&= \cos x + \frac{x^4(\pi^3 - 24) + 288x^2 - 576}{576}, \\
\Psi(\widehat{A}^{-1}f) &= \int_0^{\pi/2} t(\widehat{A}^{-1}f)' dt = \frac{\pi^8 - 24\pi^5 + 960\pi^3 - 23040}{23040}.
\end{aligned} \tag{84}$$

Then, from Theorem 1 and Eq. (83), it follows that the operator B is correct and the unique solution of (77) is given by (8). Substitution of (82) and (84) into (8) yields the exact solution of equation (77), viz.,

$$v(x) = \cos x + (x^2 - 2)/2. \tag{85}$$

Substituting this into (74), the solution of the integro-differential equation (73) is obtained in closed form:

$$u(x) = \cos x. \tag{86}$$

Problem 5. Solve the following integro-differential equation:

$$\begin{aligned}
u'(x) &= \cos x + x\int_0^\pi u(t)\sin 2t dt + 2x\int_0^\pi u'(t)\cos 2t dt, \\
u(0) &= 0, \quad x \in [0,\pi].
\end{aligned} \tag{87}$$

We can write (87) conveniently as follows:

$$\begin{aligned}
u'(x) &= \cos x + x\int_0^\pi [u(t)\sin 2t + 2u'(t)\cos 2t]dt, \\
u(0) &= 0, \quad x \in [0,\pi].
\end{aligned} \tag{88}$$

Let $X = Y = C[0,\pi]$, $Z = C^1[0,\pi]$ and the operator $B : X \to Y$ defined by

$$\begin{aligned}
Bu &= u'(x) - x\int_0^\pi \left[u(t)\sin 2t + 2u'(t)\cos 2t\right] dt = \cos x, \\
D(B) &= \{u \in C^1[0,\pi] : \ u(0) = 0\}.
\end{aligned} \tag{89}$$

We express Eq. (89) in the operator form as in (6), viz.,

$$Bu = \widehat{A}u - g\Psi(u) = f, \quad D(B) = \{u \in C^1[0,\pi] : u(0) = 0\}, \tag{90}$$

where

$$\widehat{A}u = u'(x), \quad D(\widehat{A}) = D(B) \tag{91}$$

and

$$g = x, \quad \Psi(u) = \int_0^{\pi} \left[u(t)\sin 2t + 2u'(t)\cos 2t\right] dt, \quad f = cosx. \tag{92}$$

By Theorem 3 the functional $\psi \in \left[C^1[0,\pi]\right]^*$. Theorem 2 states that the differential equation problem

$$\widehat{A}u(x) = u'(x) = f, \quad u(0) = 0 \tag{93}$$

is correct and its solution is given by

$$u(x) = \widehat{A}^{-1}f = \int_0^x f(t)dt. \tag{94}$$

By using the inverse $\widehat{A}^{-1}$ and Eq. (92), we obtain

$$\begin{aligned} \widehat{A}^{-1}g &= \int_0^x t dt = \frac{x^2}{2}, \\ \Psi(\widehat{A}^{-1}g) &= \int_0^{\pi} \left[\frac{t^2}{2}\sin 2t + 2t\cos 2t\right] dt = -\frac{\pi^2}{4} \end{aligned} \tag{95}$$

and thus

$$\det\, W = \det\left[1 - \Psi(\widehat{A}^{-1}g)\right] = 1 + \frac{\pi^2}{4} \neq 0. \tag{96}$$

In addition, we get

$$\begin{aligned} \widehat{A}^{-1}f &= \int_0^x \cos t dt = \sin x, \\ \Psi(\widehat{A}^{-1}f) &= \int_0^{\pi} [\sin t \sin 2t + 2(\sin t)' \cos 2t] dt = 0. \end{aligned} \tag{97}$$

From Theorem 1 and Eq. (96), it follows that the operator B is correct and the unique solution is given by (8). Substitution of (95) and (97) into (8) yields the solution of the integro-differential equation (87) in closed form:

$$u(x) = \sin x. \tag{98}$$

Problem 6. Find the exact solution of the following first-order integro-differential equation:

$$\begin{aligned} u'(x) - u(x) &= -\cos(2\pi x) - 2\pi \sin(2\pi x) - \frac{1}{2}\sin(4\pi x) \\ &+ \int_0^1 \sin(4\pi x + 2\pi t)u(t)dt, \quad u(0) = 1, \quad x \in [0,1]. \end{aligned} \tag{99}$$

By considering the transformation

$$v(x) = u(x) - 1 \tag{100}$$

and since $\int_0^1 \sin(4\pi x + 2\pi t)dt = 0$ Eq. (99) is written

$$\begin{aligned} v'(x) - v(x) &= 1 - \cos(2\pi x) - 2\pi \sin(2\pi x) - \frac{1}{2}\sin(4\pi x) \\ &+ \sin(4\pi x)\int_0^1 \cos(2\pi t)v(t)dt + \cos(4\pi x)\int_0^1 \sin(2\pi t)v(t)dt, \\ v(0) &= 0. \end{aligned} \tag{101}$$

Let $X = Y = C[0,1]$, $Z = C^1[0,1]$ and the operator $B : X \to Y$ defined by

$$\begin{aligned} Bv &= v'(x) - v(x) - \sin(4\pi x)\int_0^1 \cos(2\pi t)v(t)dt - \cos(4\pi x)\int_0^1 \sin(2\pi t)v(t)dt, \\ &= 1 - \cos(2\pi x) - 2\pi \sin(2\pi x) - \frac{1}{2}\sin(4\pi x), \\ D(B) &= \{v \in C^1[0,1] : v(0) = 0\}. \end{aligned} \tag{102}$$

We formulate Eq. (102) as in (6), namely,

$$Bv = \widehat{A}v - g\Psi(v) = f, \quad D(B) = \{v \in C^1[0,1] : v(0) = 0\}, \tag{103}$$

where

$$\widehat{A}v = v'(x) - v(x), \quad D(\widehat{A}) = D(B) \tag{104}$$

and

$$\begin{aligned} g &= (g_1, g_2) = (\sin(4\pi x), \cos(4\pi x)), \\ \Psi(v) &= \mathrm{col}(\psi_1(v), \psi_2(v)) = \mathrm{col}\left(\int_0^1 \cos(2\pi t)v(t)dt, \int_0^1 \sin(2\pi t)v(t)dt\right), \\ f &= 1 - \cos(2\pi x) - 2\pi \sin(2\pi x) - \frac{1}{2}\sin(4\pi x). \end{aligned} \tag{105}$$

According to Theorem 3, the functionals $\psi_1, \psi_2 \in \left[C^1[0,1]\right]^*$, while by Proposition 1, they are linearly independent. The differential equation problem (e.g., see [15])

$$\widehat{A}v = v'(x) - v(x) = f(x), \quad v(0) = 0, \tag{106}$$

is correct and its solution is given by

$$v(x) = \widehat{A}^{-1}f = \int_0^x f(t)e^{x-t}dt. \tag{107}$$

Utilizing the inverse operator $\widehat{A}^{-1}$ and Eq. (105), we get

$$\begin{aligned}
\widehat{A}^{-1}g_1 &= \int_0^x \sin(4\pi t)e^{x-t}dt = \frac{4\pi e^x}{16\pi^2+1} - \frac{4\pi\cos(4\pi x) + \sin(4\pi x)}{16\pi^2+1}, \\
\widehat{A}^{-1}g_2 &= \int_0^x \cos(4\pi t)e^{x-t}dt = \frac{e^x}{16\pi^2+1} - \frac{\cos(4\pi x) - 4\pi\sin(4\pi x)}{16\pi^2+1}, \\
\widehat{A}^{-1}g &= (\widehat{A}^{-1}g_1, \widehat{A}^{-1}g_2)
\end{aligned} \tag{108}$$

and

$$\begin{aligned}
\psi_1(\widehat{A}^{-1}g_1) &= \int_0^1 \cos(2\pi t)\widehat{A}^{-1}g_1 dt = 4\pi k, \\
\psi_1(\widehat{A}^{-1}g_2) &= \int_0^1 \cos(2\pi t)\widehat{A}^{-1}g_2 dt = k, \\
\psi_2(\widehat{A}^{-1}g_1) &= \int_0^1 \sin(2\pi t)\widehat{A}^{-1}g_1 dt = -8\pi^2 k, \\
\psi_2(\widehat{A}^{-1}g_2) &= \int_0^1 \sin(2\pi t)\widehat{A}^{-1}g_2 dt = -2\pi k, \\
\Psi(\widehat{A}^{-1}g) &= \begin{pmatrix} 4\pi k & k \\ -8\pi^2 k & -2\pi k \end{pmatrix},
\end{aligned} \tag{109}$$

where

$$k = \frac{e-1}{(4\pi^2+1)(16\pi^2+1)}.$$

Consequently we compute the determinant

$$\det W = \det\left[I_2 - \Psi(\widehat{A}^{-1}g)\right] = \det\begin{pmatrix} 1-4\pi k & -k \\ 8\pi^2 k & 1+2\pi k \end{pmatrix} = 1 - 2\pi k \neq 0 \tag{110}$$

and the inverse matrix

$$\left[I_2 - \Psi(\widehat{A}^{-1}g)\right]^{-1} = \frac{1}{1-2\pi k}\begin{pmatrix} 1+2\pi k & k \\ -8\pi^2 k & 1-4\pi k \end{pmatrix}. \tag{111}$$

In addition, we have

$$\begin{aligned}
\widehat{A}^{-1}f &= \int_0^x \left[1-\cos(2\pi t) - 2\pi\sin(2\pi t) - \frac{1}{2}\sin(4\pi t)\right] e^{x-t}dt \\
&= \frac{4\pi\cos(4\pi x) + \sin(4\pi x) + 2(16\pi^2+1)(\cos(2\pi x)-1) - 4\pi e^x}{2(16\pi^2+1)}
\end{aligned}$$

$$\psi_1(\widehat{A}^{-1}f) = \int_0^1 \cos(2\pi t)\widehat{A}^{-1}f dt = \frac{1}{2} - 2\pi k,$$

$$\psi_2(\widehat{A}^{-1}f) = \int_0^1 \sin(2\pi t)\widehat{A}^{-1}f dt = 4\pi^2 k,$$

$$\Psi(\widehat{A}^{-1}f) = \mathrm{col}(\frac{1}{2} - 2\pi k,\ 4\pi^2 k). \tag{112}$$

From Theorem 1 and Eq. (110), it follows that the operator B is correct and the solution of (103) is given by (8). Substitution of (108), (111), and (112) into (8) yields the exact solution of (103), viz.,

$$v(x) = \cos(2\pi x) - 1. \tag{113}$$

Substituting this into (100), the solution of the integro-differential equation (99) is obtained in closed form:

$$u(x) = \cos(2\pi x). \tag{114}$$

Problem 7. Solve the following second-order integro-differential boundary value problem:

$$\begin{aligned}
u''(x) &= e^x - (2e-7)x^2 + 6x + 4x^2\int_0^1 u(t)dt, \\
u(0) &= u(1) = 0, \quad x \in [0,1].
\end{aligned} \tag{115}$$

Let $X = Y = C[0,1]$, $Z = C^2[0,1]$ and the operator $B : X \to Y$ defined by

$$\begin{aligned}
Bu &= u''(x) - 4x^2\int_0^1 u(t)dt = e^x - (2e-7)x^2 + 6x, \\
D(B) &= \{u \in C^2[0,1] : u(0) = u(1) = 0\}.
\end{aligned} \tag{116}$$

We express Eq. (116) in the operator form as in (6), specifically

$$Bu = \widehat{A}u - g\Psi(u) = f, \quad D(B) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}, \tag{117}$$

where

$$\widehat{A}u = u''(x), \quad D(\widehat{A}) = D(B) \tag{118}$$

and

$$g = 4x^2, \quad \Psi(u) = \int_0^1 u(t)dt, \quad f(x) = e^x - (2e-7)x^2 + 6x. \tag{119}$$

By Theorem 3 the functional $\psi \in \left[C^2[0,1]\right]^*$. The second-order differential boundary value problem (e.g., see [15])

$$\widehat{A}u = u''(x) = f(x), \quad u(0) = u(1) = 0, \tag{120}$$

is correct and its solution is given by

$$u(x) = \widehat{A}^{-1}f = \int_0^x (x-t)f(t)dt - x\int_0^1 (1-t)f(t)dt. \tag{121}$$

By making use of the inverse operator $\widehat{A}^{-1}$ and Eq. (119), we have

$$\begin{aligned} \widehat{A}^{-1}g &= \int_0^x (x-t)4t^2dt - x\int_0^1 (1-t)4t^2dt = \frac{(x^4-x)}{3}, \\ \Psi(\widehat{A}^{-1}g) &= \frac{1}{3}\int_0^1 (t^4-t)dt = -\frac{1}{10}. \end{aligned} \tag{122}$$

and thus

$$\det W = \det\left[I_m - \Psi(\widehat{A}^{-1}g)\right] = \frac{11}{10} \neq 0. \tag{123}$$

In addition, we get

$$\begin{aligned} \widehat{A}^{-1}f &= \int_0^x (x-t)[e^t - (2e-7)t^2 + 6t]dt - x\int_0^1 (1-t)[e^t - (2e-7)t^2 + 6t]dt \\ &= e^x + \frac{x^4(2e-7) - 12x^3 + x(10e+7) + 12}{12}, \\ \Psi(\widehat{A}^{-1}f) &= \int_0^1 \widehat{A}^{-1}f(t)dt = \frac{11(2e-7)}{40}. \end{aligned} \tag{124}$$

From Theorem 1 and Eq. (123), it follows that the operator B is correct and the solution of the boundary value problem (117) is given by (8). Substituting the values computed in (122) and (124) into (8), the unique solution of the boundary value problem (115) is obtained in closed form:

$$u(x) = e^x + x^3 - xe - 1. \tag{125}$$

6 Conclusions

In this work, we developed and applied an exact extension operator method for solving the Fredholm type linear integro-differential equations. Several example problems from the literature have been solved and the effectiveness of the method in producing the exact solution has been proved. The method is simple and easy to implement. In particular, it can be programmed handily in a modern computer algebra system (CAS) and to become thus a useful tool for scientists and engineers not necessarily familiar with the subject. It is also possible to extend the application of the method presented to singular integro-differential equations, to systems of integro-differential equations, to partial integro-differential equations, and to differential equations with weights.

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Fixed Point Structures, Invariant Operators, Invariant Partitions, and Applications to Carathéodory Integral Equations

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In Honor of Constantin Carathéodory

Abstract The aim of this paper is to present the technique of the fixed point partition with respect to an operator and a fixed point structure, to study the data dependence of the fixed points, Ostrowski property and well posedness of the fixed point problem.

An application to a class of Carthéodory integral equation is given. Some research directions are also presented.

1 Introduction

Let X be a nonempty set and $A : X \to X$ an operator. We denote by F_A the fixed point of A, $F_A := \{x \in X \mid A(x) = x\}$. If $(X, \to)$ is an L-space, then A is weakly Picard operator if $A^n(x) \to x^*(x) \in F_A$ as $n \to \infty$ for all $x \in X$. By $F_A = \{x^*\}$ we understand that A has a unique fixed point and we denote it by x^*. If A is weakly Picard operator and $F_A = \{x^*\}$, then by definition A is Picard operator. The theorem of equivalent statements in the weakly Picard operator theory is well known (see [36, 40, 45]; see also [20, 53]).

Theorem 1. *Let X be a nonempty set and $A : X \to X$ an operator. The following statements are equivalent:*

(i) $F_A = F_{A^n} \neq \emptyset,\ \forall\, n \in \mathbb{N}^*$.

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(ii) *There exists a partition of* X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ *such that:*

(a) $A(X_\lambda) \subset X_\lambda, \ \forall\, \lambda \in \Lambda$

(b) $\forall\, \lambda \in \Lambda$ *there exists* $x^*_\lambda \in X_\lambda$ *such that* $F_{A^n} \cap X_\lambda = \{x^*_\lambda\}$, $\forall\, n \in \mathbb{N}^*$.

(iii) *There exists an L-space structure on* X, $\to$, *such that* $A : (X, \to) \to (X, \to)$ *is a weakly Picard operator.*

(iv) *There exists an L-space structure on* X *and a partition of* X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ *such that:*

(a) $A(X_\lambda) \subset X_\lambda, \ \forall\, \lambda \in \Lambda$

(b) $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is Picard operator for all* $\lambda \in \Lambda$.

(v) *There exist* $l \in]0,1[$, *a complete metric* d *on* X *and a partition of* X, $X = \bigcup X_\lambda$ *such that:*

(a) $A(X_\lambda) \subset X_\lambda$, $F_A \cap X_\lambda = \{x^*_\lambda\}$, $\forall\, \lambda \in \Lambda$.

(b) $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is an* l*-contraction w.r.t. the metric* d, *for all* $\lambda \in \Lambda$.

(vi) *There exist a metric* d *on* X *and a partition of* X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ *such that:*

(a) $A(X_\lambda) \subset X_\lambda$, $F_A \cap X_\lambda = \{x^*_\lambda\}$, $\forall\, \lambda \in \Lambda$.

(b) $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a contraction w.r.t. the metric* d.

It is clear that the above statements are equivalent to the following:

(vii) *There exists an invariant partition of* X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ *such that for each* $\lambda \in \Lambda$, *there exists a complete metric* d_λ *on* X_λ *with respect to which* $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a contraction.*

There are also many situations in which appear invariant partitions w.r.t. an operator (see [2–52, 55, 57–59]). In Theorem 1 these partitions are, in principle, w.r.t. the fixed point structure of contractions (see [41]; see also Sect. 3 of this paper). All these suggest the following notion.

Definition 1. Let X be a nonempty set, $(X, S(X), M)$ be a fixed point structure on X, $Y \subset X$ a nonempty subset of X, and $A : Y \to Y$ an operator. By definition a partition of Y, $Y = \bigcup_{\lambda\in\Lambda} Y_\lambda$ is a fixed point partition of Y w.r.t. the operator A and $(X, S(X), M)$ if:

(a) $A(Y_\lambda) \subset Y_\lambda, \ \forall\, \lambda \in \Lambda$, i.e., $Y = \bigcup_{\lambda\in\Lambda} Y_\lambda$ is an invariant partition w.r.t. the operator A.

(b) $Y_\lambda \in S(X), \ \forall\, \lambda \in \Lambda$.

(c) $A\Big|_{Y_\lambda} \in M(Y_\lambda),\ \forall\, \lambda \in \Lambda.$

The aim of this paper is to present a theory of the fixed point partitions with respect to an operator and a fixed point structure, composed, at least, by the following elements: data dependence of the fixed points, Ostrowski property, and well posedness of the fixed point problem. Some applications to a class of Carathéodory integral equations are also given.

The structure of our paper is the following:

1. Introduction
2. Preliminaries
3. Fixed point structures
4. Invariant operator of an operator
5. Fixed point partitions w.r.t. an operator and a fixed point structure
6. Carathéodory integral equations
7. Some research directions

   References

2 Preliminaries

2.1 Generalized Contractions: Example of Picard Operators

Let $A : X \to X$ be an operator. Then $A^0 := 1_X,\ A^1 := A, \dots, A^{n+1} = A \circ A^n,\ n \in \mathbb{N}$ denote the iterate operators of A. By $I(A)$ we will denote the set of all nonempty invariant subsets of A, i.e., $I(A) := \{Y \subset X | A(Y) \subseteq Y\}$. We also denote by $F_A := \{x \in X |\ x = A(x)\}$ the fixed point set of the operator A. By $F_A = \{x^*\}$, we understand that A has a unique fixed point and we denote it by x^*.

Let X be a nonempty set. Denote by $\Delta(X)$ the diagonal of $X \times X$. Also, let $s(X) := \{(x_n)_{n\in N} | x_n \in X,\ n \in N\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $\mathrm{Lim} : c(X) \to X$ an operator. By definition the triple $(X, c(X), \mathrm{Lim})$ is called an L-space (Fréchet [14]) if the following conditions are satisfied:

(i) If $x_n = x,$ for all $n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$.
(ii) If $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $\mathrm{Lim}(x_{n_i})_{i\in N} = x$.

By definition an element of $c(X)$ is a convergent sequence, $x := \mathrm{Lim}(x_n)_{n\in N}$ is the limit of this sequence, and we also write $x_n \to x$ as $n \to +\infty$.

In what follows we denote an L-space by $(X, \to)$.

Recall now the following important abstract concept.

Definition 2 (Rus [40]). Let $(X, \to)$ be an L-space. An operator $A : X \to X$ is, by definition, a Picard operator if:

(i) $F_A = \{x^*\}$.
(ii) $(A^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 3 (Rus [36]). Let $(X, \to)$ be an L-space. An operator $A : X \to X$ is, by definition, a weakly Picard operator if, for all $x \in X$, we have that $(A^n(x))_{n\in\mathbb{N}} \to x^*(x) \in F_A$ as $n \to \infty$.

If A is weakly Picard operator, then we consider the operator $A^\infty : X \to X$

$$A^\infty(x) := \lim_{n\to+\infty} A^n(x).$$

In particular, since any metric space (X, d) is an L-space, the above concepts can be considered in this context too. In this case, several classical results in fixed point theory can be easily transcribed in terms of (weakly) Picard operators; see [36, 53, 54].

Moreover, a Picard operator for which there exists $\tilde{c} > 0$, such that

$$d(x, x^*) \le \tilde{c}d(x, A(x)), \text{ for all } x \in X,$$

is called a $\tilde{c}$-Picard operator. More generally, a Picard operator for which there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is increasing, continuous in 0, and satisfying $\psi(0) = 0$, such that

$$d(x, x^*) \le \psi(d(x, A(x)), \text{ for all } x \in X,$$

is called a ψ-Picard operator.

For example, Hardy–Rogers' fixed point theorem [17] can be represented, in terms of Picard operators, as follows.

Theorem 2. *Let (X, d) be a complete metric space and $A : X \to X$ be an operator for which there exist $a, b, c, e, f \in \mathbb{R}_+$ with $a + b + c + e + f < 1$ such that, for all $x, y \in X$, we have*

$$d(A(x), A(y)) \le ad(x, y) + bd(x, A(x)) + cd(y, A(y)) + ed(x, A(y)) + fd(y, A(x)).$$

Then, A is a $\tilde{c}$-Picard operator, with $\tilde{c} := \frac{1}{1-\beta}$, where $\beta := \min\{\frac{a+b+e}{1-c-e}, \frac{a+c+f}{1-b-f}\} < 1$.

Another general result was given by Rus (see [38]), as follows.

Theorem 3. *If (X, d) is a complete metric space and $A : X \to X$ is an operator for which there exists a generalized strict comparison function $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ (which means that φ is increasing in each variable and the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\psi(t) := \varphi(t, t, t, t, t)$ satisfies the conditions that $\psi^n(t) \to 0$ as $n \to +\infty$, for all $t > 0$ and $t - \psi(t) \to +\infty$ as $t \to +\infty$) such that, for all $x, y \in X$, we have*

$$d(A(x), A(y)) \le \varphi(d(x, y), d(x, A(x)), d(y, A(y)), d(x, A(y)), d(y, A(x))),$$

then A is a ψ-Picard operator.

For the case of $\mathbb{R}^m_+$-metric spaces (also called generalized metric spaces in the sense of Perov), we have the following fixed point theorem of Perov; see [29].

Theorem 4. *Let (X, d) be a complete $\mathbb{R}_+^m$-metric space and let $A : X \to X$ be an operator with the property that there exists a matrix $S \in M_m(\mathbb{R}_+)$ convergent toward zero such that*

$$d(A(x), A(y)) \leq Sd(x, y), \text{ for all } x, y \in X.$$

Then, A is a Picard operator and

$$d\left(x, x^*\right) \leq S^n (I - S)^{-1} d(x, A(x));$$

For more considerations on ψ-weakly Picard operator theory, see [6, 31, 38, 51, 54].

In Rus [40] the basic theory of Picard operators in the context of L-spaces is presented.

2.2 Applications of the Picard Operator Theory in a Metric Space

We will present in this section some important applications of the Picard operator theory. For this purpose, let (X, d) be a metric space and $A : X \to X$ be an operator. Then:

(i) The fixed point equation

$$x = A(x),\ x \in X$$

is called well posed if $F_A = \{x_A^*\}$ and for any $x_n \in X$, $n \in \mathbb{N}$ a sequence in X such that

$$d(x_n, A(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x_A^* \text{ as } n \to \infty.$$

(ii) The operator A has the Ostrowski property (or the operator A has the limit shadowing property) if $F_A = \{x^*\}$ and for any $x_n \in X$, $n \in \mathbb{N}$ a sequence in X such that

$$(d(x_{n+1}, A(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x^* \text{ as } n \to \infty.$$

(iii) The fixed point equation

$$x = A(x),\ x \in X$$

has the data dependence property if $F_A = \{x_A^*\}$ and for any operator $B : X \to X$ such that there exists $\eta > 0$ with

$$d(A(x), B(x)) \leq \eta, \ \forall\, x \in X,$$

we have that

$$x_B^* \in F_B \text{ implies } d(x_A^*, x_B^*) \leq \psi(\eta) \searrow 0 \text{ as } \eta \searrow 0,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in zero, and $\psi(0) = 0$.

For example, for the data dependence problem, we have the following general abstract result in terms of Picard operators.

Theorem 5. *Let (X, d) be a metric space and $A : X \to X$ be a ψ-Picard operator, such that $F_A = \{x_A^*\}$. Let $B : X \to X$ be an operator such that there exists $\eta > 0$ such that*

$$d(A(x), B(x)) \leq \eta, \ \forall\, x \in X.$$

Then, for any $x_B^* \in F_B$, we have that $d(x_A^*, x_B^*) \leq \psi(\eta)$.

3 Fixed Point Structures

Following [41, 47], we shall present some basic notions and examples for the fixed point structure theory.

Let $\mathscr{C}$ be a class of structured sets and Set^* be the class of nonempty set. We shall use the following notations:

For $X \in Set^*$, $P(X) := \{Y \subset X \mid Y \neq \emptyset\}$; $P(\mathscr{C}) := \{U \in P(X) \mid X \in \mathscr{C}\}$;
$\mathbb{M}(U, V) := \{f : U \to V \mid f \text{ an operator}\}$; $\mathbb{M}(U) := \mathbb{M}(U, U)$;
$S : \mathscr{C} \multimap Set^*$ is a multivalued operator such that $X \multimap S(X) \subset P(X)$;
$M : D_M \subset P(\mathscr{C}) \times P(\mathscr{C}) \multimap \mathbb{M}(P(\mathscr{C}), P(\mathscr{C}))$ is a multivalued operator such that $(U, V) \multimap M(U, V) \subset \mathbb{M}(U, V)$.

The basic notions of this theory is the following.

Definition 4. By a fixed point structure (f.p.s.) on a structured set $X \in \mathscr{C}$, we understand a triple $(X, S(X), M)$ with the following properties:

(a) $U \in S(X)$ implies $(U, U) \in D_M$.
(b) $U \in S(X), f \in M(U)$ implies $F_f \neq \emptyset$.
(c) the operator M is such that:

$(Y, Y) \in D_M, Z \in P(Y)$ with $(Z, Z) \in D_M$ imply
$M(Z) \supset \{f|_Z \mid f \in M(Y)\}$.

Example 1 (The Fixed Point Structure of Contractions). Let $\mathscr{C}$ be the class of complete metric spaces and $X \in \mathscr{C}$. If $S(X) := P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$, $D_M := P(X) \times P(X)$ and $M(Y) := \{f : Y \to Y \mid f \text{ is a contraction}\}$, then $(X, S(X), M)$ is a f.p.s.

Example 2 (The Fixed Point Structure of Nonlinear Contractions). Let $\mathscr{C}$ be the class of complete metric spaces and $X \in \mathscr{C}$. If $S(X) := P_{cl}(X)$, $D_M := P(X) \times P(X)$ and $M(Y) := \{f : Y \to Y \mid$, there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that f is a φ-contraction $\}$, then $(X, S(X), M)$ is a f.p.s.

Example 3 (The Fixed Point Structure of Perov). Let $\mathscr{C}$ be the class of complete $\mathbb{R}^m_+$-metric spaces and $X \in \mathscr{C}$. If $S(X) := P_{cl}(X)$, $D_M := P(X) \times P(X)$ and $M(Y) := \{f : Y \to Y \mid$, there exists a matrix A convergent to zero, such that $d(f(x), f(y)) \leq A \cdot d(x, y)$, $\forall x, y \in Y\}$, then $(X, S(X), M)$ is a f.p.s.

Example 4 (The First Fixed Point Structure of Schauder). Let $\mathscr{C}$ be the class of Banach spaces and $X \in \mathscr{C}$. If $S(X) := P_{cp,cv}(X)$, $D_M := P(X) \times P(X)$ and $M(Y) := C(Y, Y)$, then $(X, S(X), M)$ is a f.p.s.

Example 5 (The Second Fixed Point Structure of Schauder). Let $\mathscr{C}$ be the class of Banach spaces and $X \in \mathscr{C}$. If $S(X) := P_{b,cl,cv}(X)$, $D_M := P(X) \times P(X)$ and $M(Y) := \{f : Y \to Y \mid f$ is a complete continuous $\}$, then $(X, S(X), M)$ is a f.p.s.

4 Invariant Operator of an Operator

Let X and Y be two nonempty sets and $A : X \to X$ an operator. By definition (see [4, 9, 28, 32, 49]), an operator $\Phi : X \to Y$ is invariant with respect to A (or is invariant operator of A) if $\Phi \circ A = \Phi$.

Example 6. Let $X := C[0, 1]$ and $A : C[0, 1] \to C[0, 1]$ be an increasing linear operator such that $A(e_i) = e_i$, $i = 0, 1$, where $e_i(x) = x^i$, $x \in [0, 1]$. Then it is well known that (see [34, 46])

$$A(f)(0) = f(0) \text{ and } A(f)(1) = f(1), \ \forall f \in C[0, 1].$$

Let us take $Y := \mathbb{R}^2$ and $\Phi : C[0, 1] \to \mathbb{R}^2$ be defined by $\Phi(f) = (f(0), f(1))$. It is clear that Φ is an invariant operator of A.

Example 7. Let $X := \mathbb{R}^m$ and $A : \mathbb{R}^m \to \mathbb{R}^m$ be a positive stochastic operator (see [32]) and $Y := \mathbb{R}$. Then the functional $\Phi : \mathbb{R}^m \to \mathbb{R}$ defined by

$$\Phi(x_1, \ldots, x_m) = \sum_{k=1}^{m} x_k$$

is an invariant functional of A.

Example 8. Let $X := C[a,b]$ and $A : X \to X$ be defined by

$$Ax(t) := x(a) + \int_a^t K(t,s,x(s))ds, t \in [a,b],$$

where $K \in C([a,b] \times [a,b] \times \mathbb{R})$. Then, the functional $\Phi : X \to \mathbb{R}$, $\Phi(x) := x(a)$ is an invariant functional of A.

Example 9. Let $X := C[a-h,b]$ (where $a < b$ and $h > 0$) and $A : X \to X$ be defined by

$$Ax(t) := \begin{cases} x(a) + \int_a^t K(t,s,x(s),x(s-h))ds, & t \in [a,b] \\ x(t), & t \in [a-h,a], \end{cases}$$

where $K \in C([a,b]\times[a,b]\times\mathbb{R}\times\mathbb{R})$. Then, the operator $\Phi : C[a-h,b] \to C[a-h,a]$, $\Phi(x) := x|_{[a-h,a]}$ is an invariant operator of A.

Example 10. Let $(X, \to)$ be an L-space and $A : X \to X$ be a weakly Picard operator. Then, the operator $\Phi := A^\infty : X \to F_f$ is an invariant operator of A.

Let X be a nonempty set and $A : X \to X$ an operator. By definition, a partition of X (i.e., $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, $X_\lambda \neq \varnothing$, $\forall \lambda \in \Lambda$ and $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$ implies $X_\lambda \cap X_\mu = \varnothing$) is an invariant partition of X with respect to the operator A if $A(X_\lambda) \subset X_\lambda$, $\forall\, \lambda \in \Lambda$.

Remark 1. Each surjective invariant operator $\Phi : X \to Y$ of an operator $A : X \to X$ generates an invariant partition of X w.r.t. the operator.

Indeed, for $y \in Y$, let $X_y := \{x \in X \mid A(x) = y\}$. Then $X = \bigcup_{y \in Y} X_y$ is an invariant partition of X w.r.t. the operator A. Moreover if in addition $(X, \to)$ and $(Y, \to)$ are L-spaces and Φ is continuous, then $X_y = \overline{X}_y$, $\forall\, y \in Y$.

For example, in the case of Example 6, we consider $X = C[0,1]$, $Y := \mathbb{R}^2$, $X_y := \{f \in C[0,1] \mid f(0) = y_1, f(1) = y_2\}$, $y = (y_1, y_2)$. Then $C[0,1] = \bigcup_{y \in \mathbb{R}^2} X_y$ is an invariant partition of $C[0,1]$ w.r.t. the operator A and $X_y = \overline{X}_y$, $\forall\, y \in Y$.

5 Fixed Point Partitions with Respect to an Operator and a Fixed Point Structure

Following [48, 49], we give the following notion.

Definition 5. Let X be a nonempty set and $A : X \to X$ an operator with $F_A \neq \emptyset$. By definition a partition of X, $X = \bigcup_{x^* \in F_A} X_{x^*}$ is a fixed point partition of X w.r.t. the operator A if:

(i) $A(X_{x^*}) \subset X_{x^*}$, $\forall\, x^* \in F_A$.
(ii) $F_A \cap X_{x^*} = \{x^*\}$, $\forall\, x^* \in F_A$.

By Definition 1 we extended this notion to the notion: fixed point partition with respect to an operator and a f.p.s.

In the case of f.p.s. of contraction, we have the following results.

Theorem 6. *Let (X,d) be a complete metric space and $(X, P_{cl}(X), M)$ be the f.p.s. of contraction. Let $Y \subset X$, $A : Y \to Y$ an operator and $Y = \bigcup_{\lambda\in\Lambda} Y_\lambda$ a fixed point partition of Y w.r.t. A and $(X, P_{cl}(X), M)$. Then we have:*

(1) $Y_\lambda = \overline{Y}_\lambda$, $\forall\, \lambda \in \Lambda$.
(2) $A\big|_{Y_\lambda}$ is an l_λ-contraction.
(3) $F_A \cap Y_\lambda = \{x^*_\lambda\}$, $\forall\, \lambda \in \Lambda$, i.e., $Y = \bigcup Y_\lambda$ is a fixed point partition of Y w.r.t. the operator A.
(4) $A^n(x) \to x^*_\lambda$ as $n \to \infty$, $\forall\, x \in Y_\lambda$, $\forall\, \lambda \in \Lambda$.
(5) $d(x, x^*_\lambda) \le \dfrac{1}{1-l_\lambda} d(x, A(x))$, $\forall\, x \in Y_\lambda$, $\lambda \in \Lambda$.
(6) $\lambda \in \Lambda$, $y_n \in Y_\lambda$, $n \in \mathbb{N}$, $d(y_{n+1}, A(y_n)) \to 0$ as $n \to \infty$ imply that $y_n \to x^*_\lambda$ as $n \to \infty$.
(7) $\lambda \in \Lambda$, $y_n \in Y$, $d(y_n, A(y_n)) \to 0$ as $n \to \infty$ imply that $y_n \to x^*_\lambda$ as $n \to \infty$.

Proof. $Y = \bigcup_{\lambda\in\Lambda} Y_\lambda$ is a fixed point partition of Y w.r.t. A, and $(X, P_{cl}(X), M)$, the f.p.s. of contraction, so $Y_\lambda \in P_{cl}(X)$ and $A\big|_{Y_\lambda}$ is an l_λ-contraction. The conclusions (3)–(7) follow from the properties of contractions (for details see [43, 54]). For (6) see also [51, 53] and for (7): [35, 42, 53]. □

New some remarks on Theorem 6:

(a) Conclusion (6) suggests us the following notion.

Definition 6. Let (X,d) be a metric space, $A : X \to X$ an operator with $F_A \neq \emptyset$ and $X = \bigcup X_\lambda$ a fixed point partition of X w.r.t. the operator A. By definition, the operator A has the Ostrowski property if the following implication holds:

$$\lambda \in \Lambda,\ x_n \in Y_\lambda,\ n \in \mathbb{N},\ d(x_{n+1}, A(x_n)) \to 0 \text{ as } n \to \infty$$

implying that $x_n \to x^*_\lambda$ as $n \to \infty$, where $F_A \cap X_\lambda = \{x^*_\lambda\}$, for each $\lambda \in \Lambda$.

(b) Conclusion (7) suggests us the following notion.

Definition 7. Let (X,d) be a metric space, $A : X \to X$ an operator with $F_A \neq \emptyset$, and $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ a fixed point partition of X w.r.t. the operator A. By definition, the fixed point problem for the operator A is well posed if the following implication holds:

$$\lambda \in \Lambda,\ x_n \in X_\lambda,\ n \in \mathbb{N},\ d(x_n, A(x_n)) \to 0 \text{ as } n \to \infty$$

implying that $x_n \to x^*_\lambda$ as $n \to \infty$, where $F_A \cap X_\lambda = \{x^*_\lambda\}$, for each $\lambda \in \Lambda$.

Theorem 7. *Let* (X,d) *be a complete metric space and* $(X,P_{cl}(X),M)$ *be the f.p.s. of contraction.*

Let $Y\subset X$, $A,B:Y\to Y$ *be two operators and* $Y=\bigcup_{\lambda\in\Lambda}Y_\lambda$ *a fixed point partition of* Y *w.r.t. the operator* A *and* $(X,P_{cl}(X),M)$. *In addition we suppose that:*

(i) There exists on X *a fixed point structure* $(X,S(X),M_1)$ *such that* $Y=\bigcup_{\lambda\in\Lambda}Y_\lambda$ *is a fixed point partition of* Y *w.r.t. the operator* B *and* $(X,S(X),M_1)$.

(ii) $\forall\ \lambda\in\Lambda$ *there exists* $\eta_\lambda>0$ *such that*

$$d(A(x),B(x))\le\eta_\lambda,\ \forall\ x\in Y_\lambda.$$

Let l_λ *be the contraction constant of* $A\Big|_{Y_\lambda}:Y_\lambda\to Y_\lambda$ *and* x^*_λ *the unique fixed point of* A *in* Y_λ.

Then we have

$$d(x^*_\lambda,y^*_\lambda)\le\frac{\eta_\lambda}{1-l_\lambda},\ \forall\ y^*_\lambda\in F_{B\cap Y_\lambda},\ \forall\ \lambda\in\Lambda.$$

Proof. Let $\lambda\in\Lambda$ and $y^*_\lambda\in F_{B\cap Y_\lambda}$. Then

$$\begin{aligned}d\left(x^*_\lambda,y^*_\lambda\right)&\le d\left(A\left(x^*_\lambda\right),A\left(y^*_\lambda\right)\right)+d\left(A\left(y^*_\lambda\right),B\left(y^*_\lambda\right)\right)\\&\le l_\lambda d\left(x^*_\lambda,y^*_\lambda\right)+\eta_\lambda\end{aligned}$$

and we get the conclusion. □

Theorem 8. *Let* (X,d) *be a complete* $\mathbb{R}^m_+$*-metric space and* $(X,P_{cl}(X),M)$ *be the f.p.s. of Perov. Let* $Y\subset X$, $A:Y\to Y$ *an operator and* $Y=\bigcup_{\lambda\in\Lambda}Y_\lambda$ *a fixed point partition of* Y *w.r.t.* A *and* $(X,P_{cl}(X),M)$. *Then we have:*

(1) $Y_\lambda=\overline{Y}_\lambda,\ \forall\ \lambda\in\Lambda$.
(2) $A\Big|_{Y_\lambda}:Y_\lambda\to Y_\lambda$ *is an* S_λ*-contraction.*
(3) $F_A\cap Y_\lambda=\{x^*_\lambda\},\ \forall\ \lambda\in\Lambda$, *i.e.,* $Y=\bigcup_{\lambda\in\Lambda}Y_\lambda$ *is a fixed point partition of* Y *w.r.t. the operator* A.
(4) $A^n(x)\to x^*_\lambda$ *as* $n\to\infty$, $\forall\ x\in Y_\lambda$, $\forall\ \lambda\in\Lambda$.
(5) $d(x,x^*_\lambda)\le(I-S_\lambda)^{-1}d(x,A(x)),\ \forall\ x\in Y_\lambda,\ \forall\ \lambda\in\Lambda$.
(6) The operator A *has the Ostrowski property.*
(7) The fixed point problem for the operator A *is well posed.*

Proof. The proof is similar with the proof of Theorem 6; $Y=\bigcup_{\lambda\in\Lambda}Y_\lambda$ is a fixed point partition of Y w.r.t. A and $(X,P_{cl}(X),M)$, the f.p.s. of Perov, so $Y_\lambda\in P_{cl}(X)$ and $A\Big|_{Y_\lambda}$ is an S_λ-contraction. The conclusions (3)–(7) follow from the properties of Perov contractions (for details see [54]). □

Theorem 9. *Let (X, d) be a complete $\mathbb{R}^m_+$-metric space and $(X, P_{cl}(X), M)$ be the f.p.s. of Perov. Let $Y \subset X$, $A, B : Y \to Y$ be two operators and $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$ a fixed point partition of Y w.r.t. the operator A and $(X, P_{cl}(X), M)$. In addition we suppose that:*

(i) There exists on X a fixed point structure $(X, S(X), M_1)$ such that $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$ is a fixed point partition of Y w.r.t. the operator B and $(X, S(X), M_1)$.
(ii) $\forall\, \lambda \in \Lambda$ there exists $\eta_\lambda \in \mathbb{R}^m_+$ such that

$$d(A(x), B(x)) \leq \eta_\lambda,\ \forall\, x \in Y_\lambda.$$

Let S_λ be the contraction matrix of $A\Big|_{Y_\lambda} : Y_\lambda \to Y_\lambda$ and x^*_λ the unique fixed point of A in Y_λ.

Then we have

$$d(x^*_\lambda, y^*_\lambda) \leq (I - S_\lambda)^{-1}\eta_\lambda,\ \forall\, y^*_\lambda \in F_{B\cap Y_\lambda},\ \forall\, \lambda \in \Lambda.$$

Proof. Let $\lambda \in \Lambda$ and $y^*_\lambda \in F_{B\cap Y_\lambda}$. Then

$$\begin{aligned} d\left(x^*_\lambda, y^*_\lambda\right) &\leq d\left(A\left(x^*_\lambda\right), A\left(y^*_\lambda\right)\right) + d\left(A\left(y^*_\lambda\right), B\left(y^*_\lambda\right)\right) \\ &\leq S_\lambda d\left(x^*_\lambda, y^*_\lambda\right) + \eta_\lambda \end{aligned}$$

and we get the conclusion. □

Theorem 10. *Let (X, d) be a complete metric space and $(X, P_{cl}(X), M)$ be the f.p.s. of nonlinear contraction. Let $Y \subset X$, $A : Y \to Y$ be an operator and $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$ a fixed point partition of Y w.r.t. A and $(X, P_{cl}(X), M)$. Then we have:*

(1) $Y_\lambda = \overline{Y}_\lambda,\ \forall\, \lambda \in \Lambda$.
(2) $A\Big|_{Y_\lambda}$ is an φ_λ-contraction.
(3) $F_A \cap Y_\lambda = \{x^*_\lambda\},\ \forall\, \lambda \in \Lambda$, i.e., $Y = \bigcup Y_\lambda$ is a fixed point partition of Y w.r.t. the operator A.
(4) $A^n(x) \to x^*_\lambda$ as $n \to \infty$, $\forall\, x \in X_\lambda$, $\forall\, \lambda \in \Lambda$.
(5) If $(X, P_{cl}(X), M)$ is the f.p.s. of nonlinear contraction with the property that

$$\mathrm{M(Y)} := \{\mathrm{f} : \mathrm{Y} \to \mathrm{Y} \mid \textit{there exists a strict comparison function } \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \textit{ such that } f \textit{ is a } \varphi\textit{-contraction}\}$$

then

$$d(x, x^*_\lambda) \leq \psi_{\varphi_\lambda}\left(d(x, A(x))\right),\ \forall\, x \in Y_\lambda, \lambda \in \Lambda,$$

where $\psi_{\varphi_\lambda} : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi_{\varphi_\lambda}(t) := \sup\{s \mid s - \varphi_\lambda(s) \le t\}$.

(6) If $(X, P_{cl}(X), M)$ *is the f.p.s. of nonlinear contraction with the property that*

$$\mathrm{M(Y)} := \{\mathrm{f : Y \to Y} \mid \text{ there exists a strong subadditive comparison function } \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } f \text{ is a } \varphi\text{-contraction }\}$$

then the operator A has the Ostrowski property.

(7) If $(X, P_{cl}(X), M)$ *is the f.p.s. of nonlinear contraction as in* (5), *then the fixed point problem for the operator A is well posed.*

Proof. The conclusions (3)–(7) follow from the properties of φ-contractions (for details see [43, 51]). □

Theorem 11. *Let* (X, d) *be a complete metric space and* $(X, P_{cl}(X), M)$ *be the f.p.s. of nonlinear contraction with the property that*

$$\mathrm{M(Y)} := \{\mathrm{f : Y \to Y} \mid \text{ there exists a strict comparison function } \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } f \text{ is a } \varphi - \text{contraction }\}.$$

Let $Y \subset X$, $A, B : Y \to Y$ *be two operators and* $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$ *a fixed point partition of Y w.r.t. the operator A and* $(X, P_{cl}(X), M)$. *In addition we suppose that:*

(i) There exists on X a fixed point structure $(X, S(X), M_1)$ *such that* $Y = \bigcup_{\lambda \in \Lambda} Y_\lambda$ *is a fixed point partition of Y w.r.t. the operator B and* $(X, S(X), M_1)$.

(ii) $\forall\ \lambda \in \Lambda$ *there exists* $\eta_\lambda > 0$ *such that*

$$d(A(x), B(x)) \le \eta_\lambda, \ \forall\ x \in Y_\lambda.$$

Let φ_λ *be the strict comparison function such that* $A\Big|_{Y_\lambda} : Y_\lambda \to Y_\lambda$ *is a* φ_λ*-contraction and* x^*_λ *the unique fixed point of A in* Y_λ.

Then we have

$$d(x^*_\lambda, y^*_\lambda) \le \psi_{\varphi_\lambda}(\eta_\lambda), \ \forall\ y^*_\lambda \in F_{B \cap Y_\lambda}, \ \forall\ \lambda \in \Lambda.$$

Proof. Let $\lambda \in \Lambda$ and $y^*_\lambda \in F_{B \cap Y_\lambda}$. Then

$$\begin{aligned} d\left(x^*_\lambda, y^*_\lambda\right) &\le d\left(A\left(x^*_\lambda\right), A\left(y^*_\lambda\right)\right) + d\left(A\left(y^*_\lambda\right), y^*_\lambda\right) \\ &\le \varphi_\lambda\left(d\left(x^*_\lambda, y^*_\lambda\right)\right) + d\left(A\left(y^*_\lambda\right), B\left(y^*_\lambda\right)\right) \\ &\le \varphi_\lambda\left(d\left(x^*_\lambda, y^*_\lambda\right)\right) + \eta_\lambda \end{aligned}$$

so

$$d\left(x^*_\lambda, y^*_\lambda\right) \le \psi_{\varphi_\lambda}(\eta_\lambda).$$

□

6 Carathéodory Integral Equations

Let $(\mathbb{B}, +, \mathbb{R}, |\cdot|)$ be a Banach space. By definition (see [1, 11, 16, 33, 56]) and operator, $f : [a; b] \times \mathbb{B} \to \mathbb{B}$ is a Carathéodory operator if:

(1) $f(t, \cdot) : \mathbb{B} \to \mathbb{B}$ is continuous for a.e. $t \in [a; b]$.
(2) $f(\cdot, u) : [a; b] \to \mathbb{B}$ is strongly measurable for all $u \in \mathbb{B}$.
(3) For every $r > 0$, there exists $h_r \in L^1[a; b]$ such that

$$u \in \mathbb{B},\ |u| \leq r \Longrightarrow |f(t, u)| \leq h_r(t),\ \text{for a.e. } t \in [a; b].$$

Let us consider the following integral equations.

$$x(t) = g(t, x(t), x(a)) + \int_a^t f(s, x(s))ds,\ t \in [a; b]. \tag{1}$$

In what follows we suppose that:

(i) $f : [a; b] \times \mathbb{B} \to \mathbb{B}$ is a Carathéodory operator.
(ii) There exists $l_f \in L^1[a; b]$ such that

$$|f(t, u) - f(t, v)| \leq l_f(t)\,|u - v|,\ \forall u, v \in \mathbb{B},\ \text{and a.e. } t \in [a; b].$$

(iii) $g \in C([a; b] \times \mathbb{B} \times \mathbb{B}, \mathbb{B})$.
(iv) There exists $l_g \in]0; 1[$ such that

$$|g(t, u, \lambda) - g(t, v, \lambda)| \leq l_g\,|u - v|,\ \forall t \in [a; b],\ u, v, \lambda \in \mathbb{B}.$$

Let $X = (C([a; b], \mathbb{B}), \|\cdot\|_\tau)$ the Banach space where

$$\|x\|_\tau := \max_{t \in [a;b]} \left(|x(t)|\, e^{-\tau \int_a^t l_f(s)ds} \right)$$

and

$$A(x)(t) = g(t, x(t), x(a)) + \int_a^t f(s, x(s))ds,\ t \in [a; b].$$

Let

$$Y_\lambda = \{x \in X \mid x(a) = \lambda\}.$$

It is clear that $X = \bigcup_{\lambda \in \mathbb{B}} Y_\lambda$ is a partition of X.

Corresponding to Eq. (1), we consider the equation in $\lambda \in \mathbb{B}$:

$$\lambda = g(a, \lambda, \lambda) \tag{2}$$

and we denote by S_g the solution set of (2).

Theorem 12. *We suppose that the conditions* (i)–(iv) *are satisfied. Then we have:*

(1) $A\big|_{Y_\lambda}$ *is an* l_λ*-contraction for all* $\lambda \in S_g$.
(2) $F_A \cap Y_\lambda = \{x_\lambda^*\}$, $\forall\, \lambda \in S_g$, *i.e.,* $Y = \bigcup_{\lambda \in S_g} Y_\lambda$ *is a fixed point partition of* Y *w.r.t. the operator* A.
(3) $A^n(x) \to x_\lambda^*$ *as* $n \to \infty$, $\forall\, x \in Y_\lambda$, $\forall\, \lambda \in S_g$.
(4) $d(x, x_\lambda^*) \le \dfrac{1}{1 - l_\lambda} d(x, A(x))$, $\forall\, x \in Y_\lambda$, $\lambda \in S_g$.
(5) $\lambda \in S_g$, $y_n \in Y_\lambda$, $n \in \mathbb{N}$, $d(y_{n+1}, A(y_n)) \to 0$ *as* $n \to \infty$ *imply that* $y_n \to x_\lambda^*$ *as* $n \to \infty$.
(6) $\lambda \in S_g$, $y_n \in Y$, $d(y_n, A(y_n)) \to 0$ *as* $n \to \infty$ *imply that* $y_n \to x_\lambda^*$ *as* $n \to \infty$.

Proof. If f is a Carathéodory operator, then $A : X \to X$.

It is easy to see that if x is a solution of (1), then $x(a) \in S_g$, so $F_A \cap Y_\lambda \neq \varnothing \iff \lambda \in S_g$. This implies that *card* F_A = *card* S_g. Also, it is clear that if $\lambda \in S_g$, then $Y_\lambda \in I(A)$ and $Y_\lambda \in P_{cl}(X)$.

Let $\lambda \in S_g$ and $x, y \in Y_\lambda$ and then

$$|A(x)(t) - A(y)(t)| \le \left(l_g + \frac{1}{\tau}\right) \|x - y\|_\tau \, e^{\tau \int_a^t l_f(s)ds},$$

so

$$\|A(x) - A(y)\|_\tau \le \left(l_g + \frac{1}{\tau}\right) \|x - y\|_\tau$$

Since $l_g < 1$ then we can choose τ big enough such that $l_g + \frac{1}{\tau} < 1$; therefore $A\big|_{Y_\lambda}$ is an l_λ-contraction for $\lambda \in S_g$, where

$$l_\lambda = l_g + \frac{1}{\tau}.$$

This proves that $A\big|_{Y_\lambda}$ is an l_λ-contraction for all $\lambda \in S_g$, so $Y = \bigcup_{\lambda \in S_g} Y_\lambda$ is a fixed point partition of Y w.r.t. the operator A and $(X, P_{cl}(X), M)$ the f.p.s. of contractions. The conclusion is obtained from Theorem 6. □

7 Some Research Directions

7.1 A Technique for a New Fixed Point Theorem

Let $(X, S(X), M)$ be a f.p.s., $Y \subset X$ a nonempty subset of X, $A : Y \to Y$ an operator, and $\Phi : X \to \Lambda$ an surjective invariant operator of A. Let

$$X_\lambda = \{x \in X \mid \Phi(x) = \lambda\}, \quad \lambda \in \Lambda.$$

Let

$$\Lambda_1 = \{\lambda \in \Lambda \mid X_\lambda \cap Y \neq \varnothing\}$$

and $Y_\lambda = X_\lambda \cap Y$. Then:

(a) $Y = \bigcup_{\lambda \in \Lambda_1} Y_\lambda$ is an invariant partition of Y w.r.t. the operator A.

(b) If $Y_\lambda \in S(X)$ and $A\big|_{Y_\lambda} \in M(Y_\lambda)$, then $Y = \bigcup_{\lambda \in \Lambda_1} Y_\lambda$ is a fixed point partition of Y w.r.t. the operator A and the f.p.s. $(X, S(X), M)$, i.e., $F_A \cap Y_\lambda \neq \varnothing$.

The problem is to give new fixed point theorems using the above scheme.

For example, let X be a Banach space, $(X, P_{b,cl,cv}(X), M)$ the Schauder f.p.s., $Y \in P_{b,cl}(X)$, $A : Y \to Y$ a complete continuous operator, and $\Phi : X \to \mathbb{R}$ a nontrivial continuous linear functional which is invariant for A. Let $X = \bigcup_{\lambda \in \mathbb{R}} X_\lambda$ be the partition of X generated by Φ. If $Y \cap X_\lambda$ is convex, then $F_A \neq \varnothing$. Moreover, if $X_\lambda \cap Y \neq \varnothing$, then $F_A \cap (X_\lambda \cap Y) \neq \varnothing$.

References: [26, 36, 39].

Example 11. Let $X = \mathbb{R}^2$ and $Y \subset \mathbb{R}^2$ be defined by the polygon $P_1P_2P_3P_4P_5P_6$ where $P_1(-1, 3)$, $P_2(3, -1)$, $P_3(1, -1)$, $P_4(1, -4)$, $P_5(-2, -1)$, and $P_6(-1, -1)$. Let $A : Y \to Y$ be defined by

$$A(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_1 + 2x_2}{2}\right).$$

The operator $\Phi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Phi(x_1, x_2) = x_1 + x_2$$

is a linear functional which is invariant for A.

$X = \mathbb{R}^2 = \bigcup_{\lambda \in \mathbb{R}} X_\lambda$ is the partition of X generated by Φ.

Then $F_A \cap (X_\lambda \cap Y) \neq \emptyset$ for all $\lambda \in [-3;2]$.

Proof. Notice that $Y \in P_{b,cl}\left(\mathbb{R}^2\right)$ and Y is not a convex set.

For $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\Phi\left(A\left(x_1, x_2\right)\right) = \frac{x_1}{2} + \frac{x_1 + 2x_2}{2} = x_1 + x_2 = \Phi\left(x_1, x_2\right),$$

so $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is a linear functional invariant for A.

$Y_\lambda = X_\lambda \cap Y \in P_{b,cl,cv}(X)$ and we have that

$$\Lambda_1 = \{\lambda \in \mathbb{R} \mid X_\lambda \cap Y \neq \emptyset\} = [-3;2],$$

so $F_A \cap Y_\lambda \neq \emptyset$ for all $\lambda \in [-3;2]$. By computation it is easy to see that $F_A \cap Y_\lambda = \{(0, \lambda)\}$ for $\lambda \in [-3;2]$ (Fig. 1) □

7.2 The Case of the Banach Space $C\left(\overline{\Omega}\right)$

Let $\Omega \subset \mathbb{R}^m$ be an open convex subset, $X := C\left(\overline{\Omega}\right)$ the Banach space with max norm, and $A : C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}\right)$ a continuous linear operator. The problem is to give examples of invariant operators $\Phi : C\left(\overline{\Omega}\right) \to \Lambda$ invariant w.r.t. the operator A. References: [28, 32, 40].

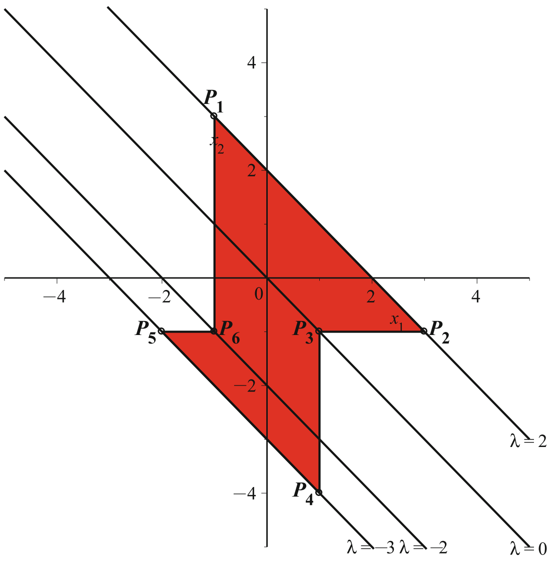

Fig. 1 The set Y from Example 11

7.3 The Case of the Ordered Sets

Let $(X, \leq)$ be an ordered set, in which conditions there exists a surjective operator $\Phi : X \to \Lambda$ such that $(X_\lambda, \leq)$ is a complete lattice.

Another problem is to give new fixed point theorems on the way as in Sect. 7.1.

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On the Best Hyers–Ulam Stability Constants for Some Equations and Operators

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In Honor of Constantin Carathéodory

Abstract In this paper we review some existing results on the best constant in Hyers–Ulam stability of some classical functional equations and some linear operators in approximation theory. We also present some new proofs of these results and some remarks on this topic.

1 Introduction

In 1940, S.M. Ulam posed the following problem [36]: *Let* $(G, \cdot, d)$ *be a metric group and* ε *a positive number. Does there exist a positive constant* k *such that for every* $f : G \to G$ *satisfying*

$$d(f(xy), f(x)f(y)) \le \varepsilon, \ \forall\, x, y \in G \tag{1}$$

there exists a homomorphism $g : G \to G$ *of the group* G *with the property*

$$d(f(x), g(x)) \le k \cdot \varepsilon, \ \forall\, x \in G? \tag{2}$$

If such a constant k exists, we say that the equation of the homomorphism

$$f(xy) = f(x) \cdot f(y) \tag{H}$$

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is stable. The first answer to Ulam's question was given a year later, for the Cauchy functional equation, by Hyers [8]. Generally we say that a functional equation is stable in Hyers–Ulam sense if for every solution of the perturbed equation (called **approximate solution**), there exists a solution of the equation (**exact solution**) near it. For more details, approaches, and results on Hyers–Ulam stability, we refer the reader to [2–5, 9, 13–15, 21–23, 28–33].

The number k from Ulam's problem is called a Hyers–Ulam constant of the equation (H) (briefly, HUS-constant), and the infimum of all HUS-constants of the equation (H) is denoted by K_H; generally K_H is not a HUS-constant of the equation (H) (see [6]). In case when K_H is a HUS-constant for the equation (H), it is called the best HUS-constant of (H). The problem of studying the best HUS-constant was first posed in [27].

In the literature there are just a few results on the best constant in Hyers–Ulam stability of equations and operators, and we mention here the characterization of the stability of linear operators and the representation of their best constants obtained by Miura, Takahashi et al. [6, 7]. In [10, 11] the stability of operators is studied with applications to nonlinear analysis; some open problems are also posed there.

In this paper we review some existing results (see [23–26]) and present some new facts concerning the best HUS-constant.

2 Best Constants of Cauchy, Jensen, and Quadratic Equations

Let X be a normed space and Y a Banach space over $\mathbb{R}$. We obtain the best HUS-constant for some classical functional equations: Cauchy, Jensen, and Quadratic equations, i.e.,

$$f(x+y) = f(x) + f(y) \tag{C}$$

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \tag{J}$$

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{Q}$$

with the unknown $f : X \to Y$.

The first result on the stability of Cauchy's equation, given in the next theorem, was obtained by Hyers in 1941 [8].

Theorem 1. *Let ε be a positive number. Then for every function $f : X \to Y$ satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \quad x, y \in X \tag{3}$$

there exists a unique additive function $a : X \to Y$ with the property

$$\|f(x) - a(x)\| \leq \varepsilon, \quad x \in X. \tag{4}$$

The first result on Hyers–Ulam stability of the Jensen equation was given by Kominek [16], but our approach will be based on a result given by Lee and Jun on generalized stability of equation (J) (see Theorem 1.2 in [17] for $p = 0$).

Theorem 2. *Let $\varepsilon > 0$. Then for every $f : X \to Y$ satisfying the relation*

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \right\| \leq \varepsilon, \quad x, y \in X \tag{5}$$

there exists a unique additive mapping $a : X \to Y$ such that

$$\|f(x) - a(x) - f(0)\| \leq 2\varepsilon, \quad x \in X. \tag{6}$$

For more results on the stability of the equation (J), we refer the reader to [12].

The solutions of the equation (Q) are called quadratic functions. A function $f : X \to Y$ is quadratic if and only if there exists a unique symmetric and biadditive function $B : X \times X \to Y$ such that

$$f(x) = B(x,x), \quad x \in X.$$

A first result on Hyers–Ulam stability of the equation (Q) was obtained by Skof and generalized later by Cholewa; for more details, see [9, p. 45]. The result of Skof and Cholewa is contained in the next theorem.

Theorem 3. *Let $\varepsilon > 0$ and $f : X \to Y$ be a function satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon, \ \forall\, x, y \in X. \tag{7}$$

Then there exists a unique quadratic function $q : X \to Y$ with the property

$$\|f(x) - q(x)\| \leq \frac{\varepsilon}{2}, \ \forall\, x \in X. \tag{8}$$

Let us denote in what follows by K_C, K_J, and K_Q the best constants for the equations (C), (J), and (Q), respectively. For the next theorem contained in [26], we will give a new proof without using the unicity of the additive/quadratic function in Theorems 1–3.

Theorem 4. *The following relations hold:*

1) $K_C = 1$.
2) $K_J = 2$.
3) $K_Q = \frac{1}{2}$.

Proof. 1) Suppose that Cauchy's equation (C) has a HUS-constant $k < 1$. Let $u \in Y$, $\|u\| = 1$. Consider the function $f : X \to Y, f(x) = u$, $x \in X$. The function f satisfies (3) with $\varepsilon = 1$, and consequently there exists an additive function $a : X \to Y$ such that

$$\|f(x) - a(x)\| \le k, \ \ x \in X.$$

This entails $\|u - a(nx)| \le k$ for every $x \in X$ and for every $n \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of all positive integers.

We get

$$\left\| \frac{u}{n} - a(x) \right\| \le \frac{k}{n}, \ \ x \in X, \ \ n \in \mathbb{N}.$$

Letting $n \to \infty$ in the previous relation, we get $a(x) = 0$, $x \in X$, i.e.,

$$1 = \|u\| \le k < 1,$$

a contradiction. We conclude that Cauchy's equation cannot have a HUS-constant smaller than 1, thus $K_C = 1$.

2) Suppose that Jensen's equation has a HUS-constant $k < 2$. Let H be a closed hyperplane in X containing the origin and H_+, H_- the two open half-spaces determined by H. Let $u \in Y$, $\|u\| = 2$.

Consider the function $f : X \to Y$,

$$f(x) = \begin{cases} u, & x \in H_+, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that

$$\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x) + f(y)}{2} \right\| \le 1, \quad \forall x, y \in X.$$

According to Theorem 2 with $\varepsilon = 1$, there exists an additive function $a : X \to Y$ such that $\|f(x) - a(x) - f(0)\| \le k$, $\forall x \in X$. Let $x \in H_+$ and $n \in \mathbb{N}$. Then $\|f(nx) - a(nx)\| \le k$, which entails $\left\| \frac{f(nx)}{n} - a(x) \right\| \le \frac{k}{n}$, $\forall n \in \mathbb{N}$.

Letting $n \to \infty$, we get $a(x) = 0$, and so $2 = \|u\| = \|f(x)\| \le k < 2$, a contradiction.

3) Suppose that the quadratic equation has a HUS-constant $k < \frac{1}{2}$. Let $u \in Y$, $\|u\| = \frac{1}{2}$, and consider the function $f : X \to Y, f(x) = u$, $x \in X$. Then (7) is satisfied for $\varepsilon = 1$; therefore there exists a quadratic function $q : X \to Y$ such that

$$\|u - q(x)\| \le k, x \in X.$$

On the other hand, there exists a symmetric and biadditive function $B : X \times X \to Y$ such that

$$q(x) = B(x,x), \quad x \in X.$$

It follows

$$\|u - B(nx, nx)\| \le k, \quad x \in X, \quad n \in \mathbb{N},$$

which is equivalent to

$$\left\| \frac{u}{n^2} - B(x,x) \right\| \le \frac{k}{n^2}, \quad x \in X, \quad n \in \mathbb{N}.$$

Letting $n \to \infty$, we get $B(x,x) = 0$, $x \in X$, which entails

$$\frac{1}{2} = \|u\| \le k < \frac{1}{2},$$

contradiction. We conclude that $K_Q = \frac{1}{2}$. □

Remark 1. The above result concerning Jensen's equation corrects the earlier version from [26].

3 Hyers–Ulam Stability of Linear Operators

Let A and B be normed spaces and consider a mapping $T : A \to B$. The following definition can be found in [35].

Definition 1. We say that T has the **Hyers–Ulam stability property** (briefly, T is **HU-stable**) if there exists a constant $K > 0$ such that:

(i) For any $g \in T(A)$, $\varepsilon > 0$ and $f \in A$ with $\|Tf - g\| \le \varepsilon$, there exists an $f_0 \in A$ with $Tf_0 = g$ and $\|f - f_0\| \le K \cdot \varepsilon$.

The number K is called a Hyers–Ulam stability constant (briefly HUS-constant) and the infimum of all HUS-constants of T is denoted by K_T; generally, K_T is not a HUS-constant of T (see [6, 7]).

Suppose now that T is a bounded linear operator and let $N(T)$ and $R(T)$ be its kernel and range, respectively.

Define the one-to-one operator $\widetilde{T} : A/N(T) \to R(T)$ by

$$\widetilde{T}(f + N(T)) = Tf, \; f \in A \tag{9}$$

and let $\widetilde{T}^{-1} : R(T) \to A/N(T)$ be the inverse of $\widetilde{T}$.

Theorem 5. *([35]) Let A and B be Banach spaces and $T : A \to B$ a bounded linear operator. The following statements are equivalent:*

(a) *T is HU-stable.*
(b) *$R(T)$ is closed.*
(c) *$\widetilde{T}^{-1}$ is bounded.*

Moreover, if one of the conditions (a)–(c) is satisfied, then

$$K_T = \|\widetilde{T}^{-1}\|. \tag{10}$$

A result on Hyers–Ulam stability of the product of two linear operators is given in the next theorem.

Theorem 6. *[24] Let X, Y, and Z be normed spaces and $A : Y \to Z$ and $B : X \to Y$ two linear operators which are HU-stable with HUS-constants K_1, respectively K_2. If $N(A) \subset R(B)$, then $AB : X \to Z$ is HU-stable with HUS-constant K_1K_2.*

Proof. Let $x \in X$, $\|ABx\| \le 1$. Then there exists $a \in N(A)$ such that $\|Bx - a\| \le K_1$. According to the hypothesis, $a = Bc$ for some $c \in X$. Thus $\|B(x - c)\| \le K_1$, which entails the existence of $b \in N(B)$ such that $\|x - c - b\| \le K_1K_2$.

Since $AB(b + c) = ABc = Aa = 0$, we conclude that $b + c \in N(AB)$, and so AB is HU-stable with HUS-constant K_1K_2. □

A question arises naturally. If A and B have the best constants K_A and K_B, is K_AK_B the best constant for $A \cdot B$? The following example shows that the answer to this question can be negative.

Example 1. Let $a, b \in \mathbb{R}$, $a < b$, and $p \in C[a, b]$ such that $m = min\, p$, $M = max\, p$, $0 < m < M$. Consider the linear operators $A, B : C[a, b] \to C[a, b]$ given by

$$Af = p \cdot f, \quad Bf = \frac{f}{p}, \quad f \in C[a, b].$$

Then $K_{AB} = 1$, $K_A = \frac{1}{m}$, $K_B = M$.

Indeed, we have $AB = BA = I$; therefore $K_{AB} = 1$.

On the other hand A and B are bijective operators, so according to Theorem 5,

$$K_A = \left\|A^{-1}\right\| = \frac{1}{m}$$

$$K_B = \left\|B^{-1}\right\| = M.$$

It follows $K_AK_B = \frac{M}{m} > 1 = K_{AB}$. It is easy to verify that $\frac{1}{m}$ and M are HUS-constants; therefore K_A and K_B are the best HUS-constants.

4 Hyers–Ulam Stability of Some Classical Operators from Approximation Theory

The main results used in our approach for obtaining, in some concrete cases, the explicit value of K_T are the formula (10) and a result by Lubinsky and Ziegler [18] concerning coefficient bounds in the Lorentz representation of a polynomial.

Let $P \in \Pi_n$, where Π_n is the set of all polynomials of degree at most n with real coefficients. Then P has a unique representation of the form

$$P(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k} \tag{11}$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$. Let T_n denote the Chebyshev polynomial of the first kind. The following representation holds

$$T_n(2x-1) = \sum_{k=0}^{n} d_{n,k} (-1)^{n-k} x^k (1-x)^{n-k} \tag{12}$$

where

$$d_{n,k} := \sum_{j=0}^{\min\{k,n-k\}} \binom{n}{2j} \binom{n-2j}{k-j} 4^j, \; k = 0, 1, \ldots, n. \tag{13}$$

The authors of the present article proved in [24] that

$$d_{n,k} = \binom{2n}{2k}, \; k = 0, 1, \ldots, n, \tag{14}$$

therefore

$$T_n(2x-1) = \sum_{k=0}^{n} \binom{2n}{2k} (-1)^{n-k} x^k (1-x)^{n-k}. \tag{15}$$

Theorem 7 (Lubinsky and Ziegler [18]). *Let P have the representation (11) and let $0 \le k \le n$. Then*

$$|c_k| \le d_{n,k} \cdot \| P \|_\infty$$

with equality if and only if P is a constant multiple of $T_n(2x-1)$.

(i) Stancu Operators

Let $C[0,1]$ be the linear space of all continuous functions $f:[0,1]\to\mathbb{R}$, endowed with the supremum norm denoted by $\|\cdot\|$, and a and b real numbers, $0\le a\le b$. The Stancu operator [34] $S_n : C[0,1]\to\Pi_n$ is defined by

$$S_nf(x)=\sum_{k=0}^{n}f\left(\frac{k+a}{n+b}\right)\binom{n}{k}x^k(1-x)^{n-k},\ f\in C[0,1].$$

We have

$$N(S_n)=\left\{f\in C[0,1]:f\left(\frac{k+a}{n+b}\right)=0,\ 0\le k\le n\right\},$$

which is a closed subspace of $C[0,1]$, and $R(S_n)=\Pi_n$. The operator $\widetilde{S}_n : C[0,1]/N(S_n)\to\Pi_n$ is bijective and $\widetilde{S}_n^{-1}:\Pi_n\to C[0,1]/N(S_n)$ is bounded since $\dim\Pi_n=n+1$, so according to Theorem 5, the operator S_n is HU-stable (see also [24]).

Theorem 8 ([25]). *For $n\ge 1$, we have*

$$K_{S_n}=\binom{2n}{2\left[\frac{n}{2}\right]}\Big/\binom{n}{\left[\frac{n}{2}\right]}.$$

Proof. Let $p\in\Pi_n$, $\|p\|\le 1$, $p(x)=\sum_{k=0}^{n}c_k(p)x^k(1-x)^{n-k}$.

Consider the piecewise affine function $f_p\in C[0,1]$ defined by

$$\begin{aligned}
&f_p(t)=c_0(p), && t\in\left[0,\frac{a}{n+b}\right]\\
&f_p(t)=c_n(p), && t\in\left[\frac{n+a}{n+b},1\right]\\
&f_p\left(\frac{k+a}{n+b}\right)=c_k(p)\Big/\binom{n}{k},\ 0\le k\le n.
\end{aligned}$$

Then $S_nf_p=p$ and $\widetilde{S}_n^{-1}(p)=f_p+N(S_n)$.

As usual, the norm of $\widetilde{S}_n^{-1}:\Pi_n\to C[0,1]/N(S_n)$ is defined by

$$\|\widetilde{S}_n^{-1}\|=\sup_{\|p\|\le 1}\|\widetilde{S}_n^{-1}(p)\|=\sup_{\|p\|\le 1}\inf_{h\in N(S_n)}\|f_p+h\|.$$

Clearly

$$\inf_{h\in N(S_n)} \|f_p + h\| = \|f_p\| = \max_{0\le k\le n} |c_k(p)|/\binom{n}{k}.$$

Therefore

$$\begin{aligned}\|\widetilde{S}^{-1}\| &= \sup_{\|p\|\le 1} \max_{0\le k\le n} |c_k(p)|/\binom{n}{k} \\ &\le \sup_{\|p\|\le 1} \max_{0\le k\le n} d_{n,k}\cdot \|p\|/\binom{n}{k} = \max_{0\le k\le n} d_{n,k}/\binom{n}{k}.\end{aligned}$$

On the other hand, let $q(x) = T_n(2x-1)$, $x \in [0,1]$.

Then $\|q\| = 1$ and $|c_k(q)| = d_{n,k}$, $0 \le k \le n$, according to Theorem 7. Consequently

$$\|\widetilde{S}_n^{-1}\| \ge \max_{0\le k\le n} |c_k(q)|/\binom{n}{k} = \max_{0\le k\le n} d_{n,k}/\binom{n}{k}$$

and so

$$\|\widetilde{S}_n^{-1}\| = \max_{0\le k\le n} \frac{d_{n,k}}{\binom{n}{k}} = \max_{0\le k\le n} \frac{\binom{2n}{2k}}{\binom{n}{k}}.$$

Let

$$a_k = \binom{2n}{2k}/\binom{n}{k},\ 0 \le k \le n.$$

Then

$$\frac{a_{k+1}}{a_k} = \frac{2n-2k-1}{2k+1},\ 0 \le k \le n.$$

The inequality $\frac{a_{k+1}}{a_k} \ge 1$ is satisfied if and only if $k \le \left[\frac{n-1}{2}\right]$; therefore

$$\max_{0\le k\le n} a_k = a_{\left[\frac{n-1}{2}\right]+1} = \begin{cases} a_{\left[\frac{n}{2}\right]}, & n \text{ even} \\ a_{\left[\frac{n}{2}\right]+1}, & n \text{ odd.}\end{cases}$$

Since $a_{\left[\frac{n}{2}\right]+1} = a_{\left[\frac{n}{2}\right]}$ if n is an odd number, we conclude that

$$K_{S_n} = \|\widetilde{S}_n^{-1}\| = \binom{2n}{2\left[\frac{n}{2}\right]} \Big/ \binom{n}{\left[\frac{n}{2}\right]}.$$

The theorem is proved. □

Remark 2. K_{S_n} does not depend on a and b. For $a = b = 0$, the Stancu operator reduces to the classical Bernstein operator. Therefore the infimum of the HUS-constants of the Bernstein operator is

$$K_{B_n} = \binom{2n}{2\left[\frac{n}{2}\right]} \Big/ \binom{n}{\left[\frac{n}{2}\right]}.$$

(ii) Kantorovich Operators

Let $X = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is bounded and Riemann integrable}\}$ be endowed with the supremum norm denoted by $\|\cdot\|$.

The Kantorovich operators [1] are defined by

$$K_n f(x) = (n+1) \sum_{k=0}^{n} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right) \binom{n}{k} x^k (1-x)^{n-k}$$

for $f \in X$ and $x \in [0, 1]$. The kernel of K_n is given by

$$N(k_n) = \left\{ f \in X : \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt = 0,\ 0 \le k \le n \right\}$$

and $N(K_n)$ is a closed subspace of X.

The operators K_n are HU-stable since their ranges are finite dimensional spaces (see also [24]).

Theorem 9 ([25]). *The following relation holds*

$$K_{K_n} = \binom{2n}{2\left[\frac{n}{2}\right]} \Big/ \binom{n}{\left[\frac{n}{2}\right]}.$$

(iii) An Extremal Property of K_{B_n}

We consider a class of generalized positive linear operators defined on $C[0, 1]$ endowed with the supremum norm $\|\cdot\|$. Let $L_n : C[0, 1] \to \Pi_n$ be defined by

$$L_n f(x) := \sum_{k=0}^{n} A_{n,k}(f)\binom{n}{k}x^k(1-x)^{n-k},\ f \in C[0,1]$$

where $A_{n,k} : C[0,1] \to \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $0 \le k \le n$ are positive linear functionals satisfying $A_{n,k}(\mathbf{1}) = 1$, $0 \le k \le n$. Suppose that the range of L_n is Π_n. Then it follows easily that

$$N(L_n) = \{f \in C[0,1] : A_{n,k}(f) = 0,\ 0 \le k \le n\}$$

and $\widetilde{L}_n : C[0,1]/N(L_n) \to \Pi_n$ is a bijective operator. The operator L_n is HU-stable since the operator $\widetilde{L}_n^{-1}$ is bounded. The following result is proved in [25].

Theorem 10 ([25]). *The following inequality holds*

$$K_{L_n} \ge K_{B_n}.$$

Related results can be found in [19, 20].

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More on the Metric Projection onto a Closed Convex Set in a Hilbert Space

Biagio Ricceri

In Honor of Constantin Carathéodory

Abstract Let H be a real Hilbert space and X a nonempty compact convex subset of H, with $0 \notin X$. For each $x \in H$, denote by $P(x)$ the unique point of X such that $\|x - P(x)\| = \text{dist}(x, X)$. For each $r > 0$, set $\gamma(r) = \inf_{\|x\|^2=r} \|x - P(x)\|^2$. Moreover, for each $\lambda > 1$, denote by $\hat{u}_\lambda$ the unique fixed point of the map $\frac{1}{\lambda}P$. In this paper, in particular, we highlight the following facts: the function $\lambda \to h(\lambda) := \|\hat{u}_\lambda\|^2$ is decreasing in $]1, +\infty[$ and its range is $]0, \|P(0)\|^2[$; the function γ is C^1, decreasing and strictly convex in $]0, \|P(0)\|^2[$, and one has $\gamma'(r) = -h^{-1}(r)$ for all $r \in]0, \|P(0)\|^2[$.

Here and in what follows, $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert space and X is a nonempty closed convex subset of H. For each $x \in H$, we denote by $P(x)$ the metric projection of x on X, that is, the unique global minimum of the restriction of the functional $y \to \|x-y\|$ to X. There is no doubt that the map P is among the most important and studied ones within convex analysis, functional analysis, and optimization theory. For the above reason, we think that it is of interest to highlight some properties of P which do not appear in the wide literature concerning P. We collect such properties in Theorems 1, 2, and 3 below. First, we fix some notations. For each $r > 0$, we put

$$B_r = \{x \in H : \|x\|^2 \leq r\}$$

and

$$S_r = \{x \in H : \|x\|^2 = r\} .$$

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Moreover, for each $x \in X$, we set

$$J(x) = \frac{1}{2}(\|x\|^2 - \|x - P(x)\|^2 + \|P(0)\|^2) .$$

Furthermore, for each $r > 0$, we put

$$\gamma(r) = \inf_{x \in S_r} \|x - P(x)\|^2 .$$

Finally, since P is nonexpansive in H, for each $\lambda \in]-1, 1[$, the map λP is a contraction and hence has a unique fixed point that we denote by $\hat{y}_\lambda$.

Theorem 1. *Assume that* $0 \notin X$.

Then, the following assertions hold:

(c_1) *The function* $\lambda \to g(\lambda) := J(\hat{y}_\lambda)$ *is increasing in* $]-1, 1[$ *and its range is* $]-\|P(0)\|^2, \|P(0)\|^2[$.

(c_2) *For each* $r \in]-\|P(0)\|^2, \|P(0)\|^2[$, *the point* $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ *is the unique point of minimal norm of* $J^{-1}(r)$ *toward which every minimizing sequence in* $J^{-1}(r)$, *for the norm, converges .*

(c_3) *The function* $r \to \hat{x}_r$ *is continuous in* $]-\|P(0)\|^2, \|P(0)\|^2[$.

(c_4) *The function* $\lambda \to h(\lambda) := \|\hat{y}_{\frac{1}{\lambda}}\|^2$ *is decreasing in* $]1, +\infty[$ *and its range is* $]0, \|P(0)\|^2[$.

(c_5) *For each* $r \in]0, \|P(0)\|^2[$, *the point* $\hat{v}_r := \hat{y}_{\frac{1}{h^{-1}(r)}}$ *is the unique global maximum of* $J_{|S_r}$ *toward which every maximizing sequence for* $J_{|S_r}$ *converges .*

(c_6) *The function* $r \to \hat{v}_r$ *is continuous in* $]0, \|P(0)\|^2[$.

Assuming, in addition, that X is compact, the following assertions hold:

(c_7) *The function* γ *is* C^1, *decreasing and strictly convex in* $]0, \|P(0)\|^2[$.

(c_8) *One has*

$$P(\hat{v}_r) = -\gamma'(r)\hat{v}_r$$

for all $r \in]0, \|P(0)\|^2[$.

(c_9) *One has*

$$\gamma'(r) = -h^{-1}(r)$$

for all $r \in]0, \|P(0)\|^2[$.

Proof. Clearly, the set of all fixed points of P agrees with X. Now, fix $u \in H$ and $\lambda < 1$. We show that

$$P(u + \lambda(P(u) - u)) = P(u) . \tag{1}$$

If $u \in X$, this is clear. Thus, assume $u \notin X$ and hence $P(u) \neq u$. Let $\varphi : H \to \mathbf{R}$ be the continuous linear functional defined by

$$\varphi(x) = \langle P(u) - u, x \rangle$$

for all $x \in H$. Clearly, $\|\varphi\|_{H^*} = \|P(u) - u\|$. We have

$$\operatorname{dist}(u + \lambda(P(u) - u), \varphi^{-1}(\varphi(P(u))) = \frac{|\varphi(u + \lambda(P(u) - u)) - \varphi(P(u))|}{\|\varphi\|_{H^*}} = (1 - \lambda)\|P(u) - u\| \,. \tag{2}$$

Moreover, by a classical result [6, Corollary 25.23], we have

$$\langle P(u) - u, P(u) - x\rangle \leq 0$$

for all $x \in X$, that is,

$$X \subseteq \varphi^{-1}([\varphi(P(u)), +\infty[) \,. \tag{3}$$

Also, notice that

$$\operatorname{dist}(u + \lambda(P(u) - u), \varphi^{-1}(\varphi(P(u)))) = \operatorname{dist}(u + \lambda(P(u) - u), \varphi^{-1}([\varphi(P(u)), +\infty[)). \tag{4}$$

Indeed, otherwise, it would exist $w \in H$, with $\varphi(w) > \varphi(P(u))$, such that

$$\|u + \lambda(P(u) - u) - w\| < \operatorname{dist}(u + \lambda(P(u) - u), \varphi^{-1}(\varphi(P(u))) \,.$$

Then, since $\varphi(u+\lambda(P(u)-u)) < \varphi(P(u))$ (indeed $\varphi(u+\lambda(P(u)-u))-\varphi(P(u)) = (\lambda - 1)\|P(u) - u\|^2$), by connectedness and continuity, in the open ball centered at $u+\lambda(P(u)-u)$, of radius $\operatorname{dist}(u+\lambda(P(u)-u), \varphi^{-1}(\varphi(P(u)))$, it would exist a point at which φ takes the value $\varphi(P(u))$, which is absurd. So, (4) holds. Now, from (2), (3), and (4), it follows that

$$\begin{aligned}(1 - \lambda)\|P(u) - u\| &\leq \operatorname{dist}(u + \lambda(P(u) - u), X) \leq \|u + \lambda(P(u) - u) - P(u)\| \\ &= (1 - \lambda)\|P(u) - u\|\end{aligned}$$

which yields (1). From (1), in particular, we infer that $P(0) = P(-P(0))$. On the other hand, if $\tilde{x} \in H$ is such that $\tilde{x} = -P(\tilde{x})$, then, applying (1) with $u = \tilde{x}$ and $\lambda = \frac{1}{2}$, we get $P(0) = P(\tilde{x})$ and so $\tilde{x} = -P(0)$. Therefore, $-P(0)$ is the unique fixed point of $-P$. Now, let us recall that J is a Fréchet differentiable convex functional whose derivative is equal to P ([1], Proposition 2.2). This allows us to use the results of Ricceri [3]. Therefore, (c_1), (c_2), and (c_3) follow, respectively, from (a_1), (a_2), and (a_3) of Theorem 3.2 of [3], since (with the notation of that result) we have $\eta_1 = J(-P(0)) = -\|P(0)\|^2$ and $\theta_1 = \inf_X J = \|P(0)\|^2$, while (c_4), (c_5), and (c_6) follow, respectively, from (b_1), (b_2), and (b_3) of Theorem 3.3 of [3], since $\theta_2 = \|P(0)\|^2$. Now, assume that X is also compact. Then, J turns out to be sequentially weakly continuous ([5], Corollary 41.9). Moreover, J has no local maxima since P has no zeros. At this point, (c_7), (c_8), and (c_9) follow, respectively, from (b_4), (b_5), and (b_6) of Theorem 3.3 of [3], since, for a constant k_0, we have

$$\sup_{S_r} J = -\frac{1}{2}\gamma(r) + k_0$$

for all $r > 0$. The proof is complete. △

Theorem 2. *Let $Q : H \to H$ be a continuous and monotone potential operator such that*

$$\lim_{\|x\|\to+\infty} I(x) := \int_0^1 \langle Q(sx), x\rangle ds = +\infty\ .$$

Set

$$\lambda^* = \inf_{r>\inf_H I}\ \inf_{x\in I^{-1}(]-\infty,r[)} \frac{J(x) - \inf_{y\in I^{-1}(]-\infty,r])} J(y)}{r - I(x)}\ .$$

Then, the equation

$$P(x) + \lambda Q(x) = 0$$

has a solution in H for every $\lambda > \lambda^*$. Moreover, when $\lambda^* > 0$, the same equation has no solution in H for every $\lambda < \lambda^*$.

Proof. Since Q is a monotone potential operator, the functional I turns out to be convex of class C^1 and its derivative agrees with Q. Now, the conclusion follows from Theorem 2.4 of [2], since, by convexity, the solutions of the equation $P(x) + \lambda Q(x) = 0$ are exactly the global minima in H of the functional $J + \lambda I$. △

Theorem 3. *Let $(T, \mathscr{F}, \mu)$ be a measure space, with $0 < \mu(T) < +\infty$, and assume that $0 \notin X$.*

Then, for every $\eta \in L^\infty(T)$, with $\eta \geq 0$, for every $r \in]0, \|P(0)\|^2[$ and for every $p \geq 2$, if we put

$$U_{\eta,r} = \left\{u \in L^p(T, H) : \int_T \eta(t)\|u(t)\|^2 d\mu = r\int_T \eta(t)d\mu\right\}\ ,$$

we have

$$\inf_{u\in U_{\eta,r}} \int_T \eta(t)\|u(t) - P(u(t))\|^2 d\mu = \inf_{x\in S_r} \|x - P(x)\|^2 \int_T \eta(t)d\mu \tag{5}$$

and

$$\sup_{u\in U_{\eta,r}} \int_T \eta(t)\|u(t) - P(u(t))\|^2 d\mu = \sup_{x\in S_r} \|x - P(x)\|^2 \int_T \eta(t)d\mu\ . \tag{6}$$

Proof. Applying Theorem 5 of [4] to J and $-J$, respectively, we obtain

$$\inf_{u\in V_{\eta,r}} \int_T \eta(t)J(u(t))d\mu = \inf_{S_r} J \int_T \eta(t)d\mu \tag{7}$$

and

$$\sup_{u\in V_{\eta,r}} \int_T \eta(t)J(u(t))d\mu = \sup_{S_r} J \int_T \eta(t)d\mu \ , \tag{8}$$

where

$$V_{\eta,r} = \left\{ u \in L^p(T,H) : \int_T \eta(t)\|u(t)\|^2 d\mu \leq r \int_T \eta(t)d\mu \right\} \ .$$

Now, observe that $J_{|S_r}$ has a global minimum. Indeed, since J is weakly lower semicontinuous and B_r is weakly compact, $J_{|B_r}$ has a global minimum, say $\hat{w}_r$. Notice that $\hat{w}_r \in S_r$, since, otherwise, $P(\hat{w}_r) = 0$ which is impossible since $0 \notin X$. So, $\hat{w}_r$ is a global minimum of $J_{|S_r}$. Furthermore, from Theorem 1, we know that $J_{|S_r}$ has a global maximum, say $\hat{v}_r$. Denote by the same symbols the constant functions (from T into Y) taking, respectively, the values $\hat{w}_r$ and $\hat{v}_r$. Since $\mu(T) < +\infty$, we have $\hat{w}_r, \hat{v}_r \in U_{\eta,r}$. So, from (7) and (8), it follows, respectively,

$$\inf_{u\in V_{\eta,r}} \int_T \eta(t)J(u(t))d\mu = \int_T J(\hat{w}_r)\eta(t)d\mu \geq \inf_{u\in U_{\eta,r}} \int_T \eta(t)J(u(t))d\mu$$

and

$$\sup_{u\in V_{\eta,r}} \int_T \eta(t)J(u(t))d\mu = \int_T J(\hat{v}_r)\eta(t)d\mu \leq \sup_{u\in U_{\eta,r}} \int_T \eta(t)J(u(t))d\mu \ .$$

Therefore

$$\begin{aligned}
\inf_{S_r} J \int_T \eta(t)d\mu &= \frac{1}{2}(r + \|P(0)\|^2 - \sup_{x\in S_r} \|x - P(x)\|^2) \int_T \eta(t)d\mu \\
&= \frac{1}{2} \inf_{u\in U_{\eta,r}} \int_T \eta(t)(\|u(t)\|^2 - \|u(t) - P(u(t))\|^2 + \|P(0)\|^2)d\mu \\
&= \frac{1}{2}(r + \|P(0)\|^2) \int_T \eta(t)d\mu - \frac{1}{2} \sup_{u\in U_{\eta,r}} \int_T \eta(t)\|u(t) - P(u(t))\|^2 d\mu
\end{aligned}$$

which yields (6). Likewise

$$\begin{aligned}
\sup_{S_r} J \int_T \eta(t)d\mu &= \frac{1}{2}(r + \|P(0)\|^2 - \inf_{x\in S_r} \|x - P(x)\|^2) \int_T \eta(t)d\mu \\
&= \frac{1}{2} \sup_{u\in U_{\eta,r}} \int_T \eta(t)(\|u(t)\|^2 - \|u(t) - P(u(t))\|^2 + \|P(0)\|^2)d\mu
\end{aligned}$$

$$= \frac{1}{2}(r + \|P(0)\|^2) \int_T \eta(t) d\mu - \frac{1}{2} \inf_{u \in U_{\eta,r}} \int_T \eta(t) \|u(t) - P(u(t))\|^2 d\mu$$

which yields (5). △

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Carathéodory Functions in Partial Differential Equations

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In Honor of Constantin Carathéodory

Abstract We show how Carathéodory functions can be used in solving problems in partial differential equations.

1 Carathéodory's Theorem

1.1 Carathéodory Functions

In extending Peano's theorem for ordinary differential equations, Carathéodory [1] proved the following:

Theorem 1. *Let f(t,x) be a function defined on*

$$\mathbf{R} : |t-\tau| \leq a, \quad |x-\xi| \leq b, \tag{1}$$

where (τ, ξ) is a fixed point in the (t,x) plane, and a and b are positive real numbers. Assume that $f(x,t)$ is continuous in t for a.e. x and measurable in x for every t. Assume that there is an integrable function m(t) in $|t-\tau| \leq a$ such that

$$|f(t,x)| \leq m(t), \quad (t,x) \in \mathbf{R}. \tag{2}$$

Then there exists an absolutely continuous function u(t) on some interval $|t-\tau| \leq \alpha$ such that

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$$u'(t) = f(t, u(t)) \text{ a.e. in } |t - \tau| \leq \alpha, \quad u(\tau) = \xi. \tag{3}$$

In Peano's theorem, $f(t, x)$ was assumed continuous in both variables, and $m(t)$ was assumed bounded. Then $u(t)$ is continuously differential and satisfies (3) everywhere. In order to allow solutions that are only absolutely continuous, Carathéodory needed functions $f(t, x)$ such that $f(t, u(t))$ is measurable for all continuous $u(t)$. Functions described in the theorem, i.e., functions $f(x, t)$ continuous in t for a.e. x and measurable in x for every t, fit the bill. They are now known as Carathéodory functions. In the study of partial differential equations, it is required that $f(t, u(t))$ be measurable for all measurable $u(t)$. Carathéodory functions accomplish this as well.

2 Semilinear Boundary Value Problems

2.1 Introduction

Many elliptic semilinear problems can be described in the following way. Let Ω be a domain in $\mathbb{R}^n$, and let A be a self-adjoint operator on $L^2(\Omega)$. We assume that $A \geq \lambda_0 > 0$ and that

$$C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^{m,2}(\Omega) \tag{4}$$

for some $m > 0$, where $C_0^\infty(\Omega)$ denotes the set of test functions in Ω (i.e., infinitely differentiable functions with compact supports in Ω) and $H^{m,2}(\Omega)$ denotes the Sobolev space. If m is an integer, the norm in $H^{m,2}(\Omega)$ is given by

$$\|u\|_{m,2} := \left(\sum_{|\mu| \leq m} \|D^\mu u\|^2 \right)^{1/2}. \tag{5}$$

Here D^μ represents the generic derivative of order $|\mu|$ and the norm on the right-hand side of (5) is that of $L^2(\Omega)$. We shall not assume that m is an integer.

Let q be any number satisfying

$$\begin{aligned} 2 \leq q &\leq 2n/(n - 2m), \quad && 2m < n \\ 2 \leq q &< \infty, && n \leq 2m \end{aligned}$$

and let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. This means that $f(x, t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$. We make the following assumptions:

(A) The function $f(x, t)$ satisfies

$$|f(x, t)| \leq V(x)^q (|t|^{q-1} + W(x)) \tag{6}$$

and

$$f(x,t)/V(x)^q = o(|t|^{q-1}) \text{ as } |t| \to \infty, \tag{7}$$

where $V(x) > 0$ is a function in $L^q(\Omega)$ such that

$$\|Vu\|_q \leq C\|u\|_D, \quad u \in D \tag{8}$$

and W is a function in $L^\infty(\Omega)$. Here

$$\|u\|_q := \left(\int_\Omega |u(x)|^q dx\right)^{1/q}, \tag{9}$$

$$\|u\|_D := \|A^{1/2}u\| \tag{10}$$

and $q' = q/(q-1)$. If Ω and $V(x)$ are bounded, then (8) will hold automatically by the Sobolev inequality. However, there are functions $V(x)$ which are unbounded and such that (8) holds even on unbounded regions Ω. With the norm (10), D becomes a Hilbert space. Define

$$F(x,t) := \int_0^t f(x,s)ds \tag{11}$$

and

$$G(u) := \|u\|_D^2 - 2\int_\Omega F(x,u)\,dx. \tag{12}$$

We want $G(u)$ to be defined for all $u \in D$. For this purpose we need $F(x,u(x))$ and $f(x,u(x))$ to be measurable for all measurable $u(x)$. This follows from the fact that $f(x,t)$ is a Carathéodory function. Once we know that such functions are measurable, we can obtain integrability by means of estimates. We shall show that G is a continuously differentiable functional on the whole of D. First we note

Lemma 1. *Under hypothesis (A), $F(x,u(x))$ and $v(x)f(x,u(x))$ are in $L^1(\Omega)$ whenever $u,\ v \in D$.*

Proof. By (6) and (11), we have

$$|F(x,u)| \leq C(|Vu|^q + |V^q u|W). \tag{13}$$

Since $Vu \in L^q(\Omega)$ and $V^{q-1} \in L^{q'}(\Omega)$, the right-hand side is in $L^1(\Omega)$. Similarly

$$|vf(x,u)| \leq |Vv|(|Vu|^{q-1} + V^{q-1}W). \tag{14}$$

Since Vu and Vv are in $L^q(\Omega)$, the same reasoning applies.

Next we have

Lemma 2. *$G(u)$ has a Fréchet derivative $G'(u)$ on D given by*

$$(G'(u), v)_D = 2(u, v)_D - 2(f(\cdot, u), v). \tag{15}$$

Proof. We have by (12)

$$\begin{aligned} &G(u+v) - G(u) - 2(u, v)_D + 2(f(u), v) \\ &\quad = \|v\|_D^2 - 2\int_\Omega [F(x, u+v) - F(x, u) - vf(x, u)]dx. \end{aligned} \tag{16}$$

The first term on the right-hand side of (16) is clearly $o(\|v\|_D)$ as $\|v\|_D \to 0$. Since

$$\begin{aligned} F(x, u+v) - F(x, u) &= \int_0^1 [dF(x, u+\theta v)/d\theta]d\theta \\ &= \int_0^1 f(x, u+\theta v)v d\theta, \end{aligned}$$

the integral in (16) equals $\int_\Omega \int_0^1 [f(x, u+\theta v) - f(x, u)]v d\theta dx$. By Hölder's inequality, this is bounded by

$$\left(\int_\Omega \int_0^1 |V^{-1}[f(x, u+\theta v) - f(x, u)]|^{q'} d\theta dx\right)^{1/q'} \|Vv\|_q. \tag{17}$$

In view of (16), the lemma will be proved if we can show that the expression (17) is $o(\|v\|_D)$. By (8) the second factor is $O(\|v\|_D)$. Hence it suffices to show that the first factor in (17) is $o(1)$. The integrand is bounded by

$$(|V(u+\theta v)|^{q-1} + |Vu|^{q-1} + 2V^{q-1}W)^{q'} \le C(|Vu|^q + |Vv|^q + V^q W). \tag{18}$$

If the first factor in (17) did not converge to 0 with $\|v\|_D$, then there would be a sequence $\{v_k\} \subset D$ such that $\|v_k\|_D \to 0$, while

$$\int_\Omega \int_0^1 |V^{-1}[f(x, u+\theta v_k) - f(x, u)]|^{q'} d\theta dx \ge \epsilon > 0. \tag{19}$$

In view of (8), $\|Vv_k\|_q \to 0$. Thus there is a renamed subsequence such that $Vv_k \to 0$ a.e. But by (18), the integrand of (19) is majorized by

$$C(|Vu|^q + |Vv_k|^q + V^q W)$$

which converges in $L^1(\Omega)$ to

$$C(|Vu|^q + V^q W).$$

Moreover, the integrand converges to 0 a.e. Hence the left-hand side of (19) converges to 0, contradicting (19). This proves the lemma.

Lemma 3. *The derivative $G'(u)$ given by (15) is continuous in u.*

Proof. By (15), we have

$$\begin{aligned}(G'(u_1) - G'(u_2), v)_D &= 2(u_1 - u_2, v)_D - 2\int_\Omega v[f(x,u_1) - f(x,u_2)]dx \\ &\leq 2\|u_1 - u_2\|_D \|v\|_D \\ &\quad + 2\|Vv\|_q \left(\int_\Omega |V^{-1}[f(x,u_1) - f(x,u_2)]|^{q'} dx\right)^{1/q'}.\end{aligned}$$

Thus

$$\begin{aligned}\|G'(u_1) - G'(u_2)\|_D &\leq 2\|u_1 - u_2\|_D \\ &+ C\left(\int_\Omega |V^{-1}[f(x,u_1) - f(x,u_2)]|^{q'} dx\right)^{1/q'}. \end{aligned} \tag{20}$$

Reasoning as in the proof of Lemma 2, we show that the right-hand side of (20) converges to 0 as $u_1 \to u_2$ in D.

2.2 Mountain Pass Geometry

We now add hypotheses to obtain the simplest configuration leading to a solution of

$$Au = f(x,u), \ u \in D. \tag{21}$$

By a solution of (21), we shall mean a function $u \in D$ such that

$$(u,v)_D = (f(\cdot,u), v), \quad v \in D. \tag{22}$$

If $f(x,u)$ is in $L^2(\Omega)$, then a solution of (22) is in $D(A)$ and solves (21) in the classical sense. Otherwise we call it a weak (or semistrong) solution.

We add the following to the hypotheses of Sect. 1:

(B) If λ_0 is in the spectrum of A, then it is an isolated eigenvalue, with its eigenspace finite dimensional and contained in $L^\infty(\Omega)$. If $q = 2$ in (6), then multiplication by $V(x)$ is a compact operator from D to $L^2(\Omega)$.

(C) There is a $\delta > 0$ such that

$$2F(x,t) \leq \lambda_0 t^2, \quad |t| \leq \delta.$$

We have the following.

Lemma 4. *Under hypotheses (A)–(C), the following alternative holds. Either:*

(a) there is an infinite number of $y(x) \in D(A) \setminus \{0\}$ *such that*

$$Ay = f(x, y) = \lambda_0 y \tag{23}$$

or

(b) for each $\rho > 0$ *sufficiently small, there is an* $\epsilon > 0$ *such that*

$$G(u) \geq \epsilon, \ \ \|u\|_D = \rho. \tag{24}$$

Proof. Let $\lambda_1 > \lambda_0$ be the next point in the spectrum of A, and let N_0 denote the eigenspace of λ_0. We take $M = N_0^{\perp} \cap D$. By hypothesis (B), there is a $\rho > 0$ such that

$$\|y\|_D \leq \rho \Rightarrow |y(x)| \leq \delta/2, \quad y \in N_0.$$

Now suppose $u \in D$ satisfies

$$\|u\|_D \leq \rho \text{ and } |u(x)| \geq \delta \tag{25}$$

for some $x \in \Omega$. We write

$$u = w + y, \ \ w \in M, \ \ y \in N_0. \tag{26}$$

Then for those $x \in \Omega$ satisfying (25), we have

$$\delta \leq |u(x)| \leq |w(x)| + |y(x)| \leq |w(x)| + (\delta/2).$$

Hence

$$|y(x)| \leq \delta/2 \leq |w(x)|, \tag{27}$$

and consequently,

$$|u(x)| \leq 2|w(x)| \tag{28}$$

for all such x. Now we have by (6) and (21)

$$\begin{aligned} G(u) &\geq \|u\|_D^2 - \lambda_0 \int_{|u|<\delta} u^2 dx - C \int_{|u|>\delta} (|Vu|^q + |V^q u|W)\, dx \\ &\geq \|u\|_D^2 - \lambda_0 \|u\|^2 - C' \int_{|u|>\delta} |Vu|^q dx \\ &\geq \|w\|_D^2 - \lambda_0 \|w\|^2 - C'' \int_{2|w|>\delta} |Vw|^q dx \end{aligned}$$

in view of the fact that $\|y\|_D^2 = \lambda_0\|y\|^2$ and (28) holds. We shall show that

$$\int_{2|w|>\delta} |Vw|^q dx/\|w\|_D^2 \to 0 \text{ as } \|w\|_D \to 0. \tag{29}$$

Assuming this for the moment, we see that

$$G(u) \geq \left(1 - \frac{\lambda_0}{\lambda_1} - o(1)\right)\|w\|_D^2, \quad \|u\|_D \leq \rho. \tag{30}$$

Now suppose alternative (b) of the lemma did not hold. Then there would be a sequence such that

$$G(u_k) \to 0, \ \|u_k\|_D = \rho. \tag{31}$$

If ρ is taken sufficiently small, (30) implies that $\|w_k\|_D \to 0$. Consequently, $\|y_k\|_D \to \rho$. Since N_0 is finite dimensional, there is a renamed subsequence such that $y_k \to y_0$ in N_0. Thus we have

$$\|y_0\| = \rho, \ G(y_0) = 0, \ |y_0(x)| \leq \delta/2, \quad x \in \Omega.$$

Consequently, hypothesis (C) implies

$$2F(x, y_0(x)) \leq \lambda_0 y_0(x)^2, \quad x \in \Omega. \tag{32}$$

Since

$$\int_\Omega \{\lambda_0 y_0(x)^2 - 2F(x, y_0(x))\}\, dx = G(y_0) = 0$$

and the integrand is ≥ 0 a.e. by (32), we see that

$$2F(x, y_0(x)) \equiv \lambda_0 y_0(x)^2, \quad x \in \Omega.$$

Let $\varphi(x)$ be any function in $C_0^\infty(\Omega)$. Then for $t > 0$ sufficiently small

$$t^{-1}[2F(x, y_0 + t\varphi) - \lambda_0(y_0 + t\varphi)^2 - 2F(x, y_0) + \lambda_0 y_0^2] \leq 0.$$

Taking the limit as $t \to 0$, we have

$$(f(x, y_0) - \lambda_0 y_0)\varphi(x) \leq 0, \quad x \in \Omega.$$

Since this is true for every $\varphi \in C_0^\infty(\Omega)$, we see that

$$f(x, y_0(x)) \equiv \lambda_0 y_0(x), \quad x \in \Omega.$$

Since $y_0 \in N_0$, it follows that (23) holds. Thus the lemma will be proved once we have established (29). First assume that $q > 2$. Then we have

$$\int |Vw|^q dx \le C\|w\|_D^q = o(\|w\|_D^2)$$

by (8). Next assume that $q = 2$ and there is a sequence $\{w_k\}$ such that

$$\int_{2|w_k|>\delta} |Vw_k|^2 dx/\|w_k\|_D^2 \ge \epsilon > 0 \tag{33}$$

while $\rho_k = \|w_k\|_D \to 0$. Let $\tilde{w}_k = w_k/\rho_k$. Then $\|\tilde{w}_k\|_D = 1$. Let $\mu_k(x)$ be the characteristic function of the set of those $x \in \Omega$ such that $2|w_k(x)| \ge \delta$. Then (33) becomes

$$\int_\Omega |V\tilde{w}_k|^2 \mu_k(x)\, dx \ge \epsilon. \tag{34}$$

But

$$\begin{aligned} \mu_k(x) &= 1 \text{ when } 2|\tilde{w}_k(x)| \ge \delta/\rho_k \to \infty \\ &= 0 \text{ when } 2|\tilde{w}_k(x)| < \delta/\rho_k. \end{aligned}$$

Hence $\mu_k(x) \to 0$ a.e. Since $q = 2$, we know that there is a renamed subsequence such that $V\tilde{w}_k$ converges in $L^2(\Omega)$. But

$$V(x)^2\tilde{w}_k(x)^2\mu_k(x) \le V(x)^2\tilde{w}_k(x)^2,$$

and the right-hand side converges in $L^1(\Omega)$. Since the left-hand side converges a.e. to 0, we see that

$$\int |V(x)\tilde{w}_k(x)|^2\mu_k(x)\, dx \to 0.$$

Hence there cannot exist an $\epsilon > 0$ such that

$$\int_{2|w_k|>\delta} |V\tilde{w}_k|^2 dx \ge \epsilon.$$

This completes the proof of (29) and of the lemma.

2.3 *Finding a Critical Sequence*

Under the hypotheses of Lemma 4, we see that if (22) does not have a solution, then (23) holds. We want to find a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c,\ \epsilon \le c \le \infty,\ \|G'(u_k)\|/(1 + \|u_k\|_D) \to 0. \tag{35}$$

For this purpose we shall make use of

Lemma 5. *Assume that there is a* $\delta > 0$ *such that*

$$G(0) < \alpha \leq G(u), \quad u \in \partial B_\delta \tag{36}$$

and that there is a $\varphi_0 \in \partial B_1$ *such that*

$$G(R\varphi_0) \leq m_R, \quad R > R_0, \tag{37}$$

where

$$\partial B_\delta = \{u \in H : \|u\|_H = \delta\}.$$

If

$$m_R/R^{\beta+1} \to 0 \text{ as } R \to \infty \tag{38}$$

for some $\beta \geq 0$*, then there is a sequence* $\{u_k\} \subset E$ *such that*

$$G(u_k) \to c, \; \alpha \leq c \leq \infty, \; G'(u_k)/(\|u_k\| + 1)^\beta \to 0. \tag{39}$$

The simplest hypotheses that will guarantee this are:

(D) The point λ_0 is an eigenvalue of A with a corresponding eigenfunction $\varphi_0 \geq 0$.
(E) There are functions $W_0(x) \in L^1(\Omega),\; h(t)$ locally bounded such that

$$2F(x,t) \geq \lambda_0 t^2 - W_0(x)h(t), \quad x \in \Omega,\; t > 0 \tag{40}$$

and

$$h(t)/t^2 \to 0 \text{ as } t \to \infty.. \tag{41}$$

We have

Lemma 6. *Under hypotheses* (A)–(E)*, there is a sequence satisfying (35).*

Proof. Under hypotheses (A)–(C), Lemma 4 shows us that (36) holds. We must show that (37) and (38) hold as well under the additional assumptions $(D) and (E)$. By (40) we have

$$G(R\varphi_0) \leq R^2(\|\varphi_0\|_D^2 - \lambda_0\|\varphi_0\|^2) + \int_\Omega W_0(x)h(R\varphi_0(x))\,dx$$

since $\varphi_0 \geq 0$. Hence

$$G(R\varphi_0)/R^2 \leq \int_\Omega W_0(x)[h(R\varphi_0)/R^2\varphi_0^2]\varphi_0^2 dx.. \tag{42}$$

Now for any $\eta > 0$, there is a t_0 such that

$$|h(t)|/t^2 \leq \eta \text{ for } t > t_0$$

and since $h(t)$ is locally bounded,

$$|h(t)| \leq K, \quad 0 \leq t \leq t_0.$$

Thus the integral on the right-hand side of (42) is bounded by

$$\eta \int_{R\varphi_0 > t_0} W_0(x)\varphi_0(x)^2 dx + K \int_{R\varphi_0 \leq t_0} W_0(x)\, dx/R^2.$$

This shows that

$$G(R\varphi_0)/R^2 \to 0 \text{ as } R \to \infty. \tag{43}$$

Hence the hypotheses of Lemma 5 are satisfied and our lemma follows.

2.4 Obtaining a Solution

Now that we have a critical sequence, i.e., a sequence satisfying

$$G(u_k) \to c,\ G'(u_k)/(1 + \|u_k\|_D)^\beta \to 0, \tag{44}$$

we would like to know when such a sequence leads to a solution of (21). For this purpose we assume only the assumptions made in Sect. 1, i.e., that A is a self-adjoint operator on $L^2(\Omega), A \geq \lambda_0 > 0$ and (4) holds for some $m > 0$. Moreover, $f(x, t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying hypothesis (A). This is all we assume concerning A and $f(x, t)$ in the present section. We have

Lemma 7. *Let A and $f(x, t)$ satisfy the hypotheses stated above, and assume that there is a sequence $\{u_k\} \subset D$ satisfying (44) with*

$$-\infty \leq c \leq \infty, \quad -\infty < \beta < \infty. \tag{45}$$

If

$$\|u_k\|_D \leq C, \tag{46}$$

then c is finite and there is a $u \in D$ such that

$$G(u) = c,\ G'(u) = 0. \tag{47}$$

Proof. It follows from (46) that there is a subsequence which converges weakly in D to a limit u. For any compact subset $K \subset \Omega$, the imbedding of $H_0^m(\Omega)$ in $L^2(K)$ is compact. Thus we can find a subsequence which converges to u in $L^2(K)$. We can then find a subsequence of this subsequence which converges to u a.e. in K. By taking a set of compact subsets of Ω which exhaust Ω, we can find a renamed subsequence which not only converges to u weakly in D, but strongly in $L^2(K)$ for each compact subset K of Ω and also a.e. in Ω. For such a subsequence I claim that

$$\int_\Omega F(x, u_k(x))\,dx \to \int_\Omega F(x, u(x))\,dx \tag{48}$$

$$\int_\Omega f(x, u_k(x))v(x)\,dx \to \int_\Omega f(x, u(x))v(x)\,dx, \quad v \in D \tag{49}$$

and

$$\int_\Omega f(x, u_k(x))u_k(x)\,dx \to \int_\Omega f(x, u(x))u(x)\,dx. \tag{50}$$

To see this, let $w_r(t)$ be the continuous function defined by

$$\begin{aligned} w_r(t) &= t, \quad |t| \le r \\ &= r, \quad t > r \\ &= -r, \quad t < -r. \end{aligned}$$

Let $\epsilon > 0$ be given and pick r so large that

$$|f(x,t)| \le \epsilon V(x)^q |t|^{q-1}, \quad |t| \ge r \tag{51}$$

and

$$|F(x,t) - F(x, w_r(t))| \le \epsilon V(x)^q |t|^q, \quad |t| \ge r. \tag{52}$$

This can be done by (7). Now

$$\begin{aligned} \int_\Omega [F(x,u_k) - F(x,u)]dx = &\int_{|u_k|>r} [F(x,u_k) - F(x, w_r(u_k))]dx \\ &+ \int_\Omega [F(x, w_r(u_k)) - F(x, w_r(u))]dx \\ &+ \int_{|u|>r} [F(x, w_r(u)) - F(x,u)]dx. \end{aligned} \tag{53}$$

In view of (52), the first integral on the right-hand side can be estimated by

$$\epsilon \int |Vu_k|^q dx \le \epsilon C \|u_k\|_D^q \le \epsilon C_1$$

and by (8) and (46). A similar estimate holds for the third integral. On the other hand, the integrand in the middle integral converges to 0 a.e. in Ω. Moreover, it is majorized by

$$C(|Vw_r(u_k)|^q + |Vw_r(u_k)|W + |Vw_r(u)|^q + |Vw_r(u)|W) \leq 2C(V^q r^q + VrW)$$

which is a function in $L^1(\Omega)$. Thus the middle integral converges to 0 as $k \to \infty$. Hence the left-hand side of (53) can be made as small as desired by taking k sufficiently large. This shows that (48) holds. One proves (49) and (50) in a similar way using (51). Once we have (49), we see that

$$\begin{aligned} (G'(u_k), v) &= 2(u_k, v)_D - 2\int_\Omega f(x, u_k)v\,dx \\ &\to 2(u, v)_D - 2\int f(x, u)v\,dx \\ &= (G'(u), v). \end{aligned}$$

Hence u is a solution of the second equation in (47). Since

$$\|u_k\|_D^2 - \int_\Omega f(x, u_k)u_k dx = (G'(u_k), u_k)/2 \to 0,$$

we see by (50) that

$$\|u_k\|_D^2 \to \int_\Omega f(x, u)u dx = \|u\|_D^2.$$

This implies that $u_k \to u$ strongly in D. Consequently the first equation in (47) holds as well, showing in particular that c is finite. This proves the lemma.

2.5 Solving the Problem

Lemma 6 gives us sufficient conditions for the functional G to have a critical sequence. Lemma 7 tells us that a bounded critical sequence produces a solution. In this section we give additional conditions which will imply that (46) holds. We will then be assured that we indeed have a solution of (21). We assume:

(F) There are functions $V_1, W_1 \in L^2(\Omega)$ such that multiplication by V_1 is compact from D to $L^2(\Omega)$, and

$$|f(x,t)| \leq V_1^2|t| + V_1 W_1, \quad x \in \Omega,\ t \in \mathbf{R} \tag{54}$$

and

$$f(x,t)/t \to \alpha_{\pm}(x) \text{ as } \to \pm\infty \text{ a.e.} \tag{55}$$

(G) The only solution of

$$Au = \alpha_+ u^+ - \alpha_- u^-, \ u \in D \tag{56}$$

is $u \equiv 0$, where $u^{\pm} = \max\{\pm u, 0\}$. We have

Lemma 8. *Under hypotheses* (*B*), (*F*)*, and* (*G*)*, every sequence satisfying (44) with* $\beta \leq 1$ *is bounded in D.*

Proof. Suppose there were a renamed subsequence such that

$$\rho_k = \|u_k\|_D \to \infty. \tag{57}$$

Define

$$\tilde{u}_k = u_k/\rho_k.$$

Then

$$\|\tilde{u}_k\|_D = 1. \tag{58}$$

It therefore follows that there is a renamed subsequence converging weakly to a function $\tilde{u} \in D$ and such that $V_1\tilde{u}_k$ converges strongly to $V_1\tilde{u}$ in $L^2(\Omega)$ and a.e. in Ω. Then by (54)

$$|f(x,u_k)\tilde{u}_k|/\rho_k \leq |V_1\tilde{u}_k|^2 + |V_1\tilde{u}_k|W_1/\rho_k \to |V_1\tilde{u}|^2 \text{ in } L^1(\Omega). \tag{59}$$

If $\tilde{u}(x) \neq 0$, then

$$f(x,u_k)\tilde{u}_k/\rho_k = [f(x,u_k)/u_k]\tilde{u}_k^2 \to \alpha_{\pm}(x)[\tilde{u}(x)^{\pm}]^2 \ a.e. \tag{60}$$

by (55). If $\tilde{u}(x) = 0$, then (60) holds by (59). Hence (59) and (60) imply

$$\rho_k^{-1}\int_\Omega f(x,u_k)\tilde{u}_k dx \to \int_\Omega \{\alpha_+(\tilde{u}^+)^2 + \alpha_-(\tilde{u}^-)^2\}\, dx. \tag{61}$$

By (44)

$$(G'(u_k),\tilde{u}_k)/2\rho_k = \|\tilde{u}_k\|_D^2 - \rho_k^{-1}(f(u_k),\tilde{u}_k) \to 0 \tag{62}$$

since $\beta \leq 1$. By (58) and (61), this implies

$$\int_{\Omega} \{\alpha_+(\tilde{u}^+)^2 + \alpha_-(\tilde{u}^-)^2\}\, dx = 1. \tag{63}$$

In particular, we see that $\tilde{u} \not\equiv 0$. Moreover, for any $v \in D$, we have by (44)

$$(G'(u_k), v)/2\rho_k = (\tilde{u}_k, v)_D - \rho_k^{-1}(f(u_k), v) \to 0. \tag{64}$$

But

$$|f(x, u_k)v|/\rho_k \le |V\tilde{u}_k||Vv| + |Vv|W/\rho_k \to |V\tilde{u}||Vv| \text{ in } L^1(\Omega) \tag{65}$$

and

$$f(x, u_k)v/\rho_k = [f(x, \rho_k\tilde{u}_k)/\rho_k\tilde{u}_k]\tilde{u}_k v \to [\alpha_+\tilde{u}^+ - \alpha_-\tilde{u}^-]v \; a.e. \tag{66}$$

if $\tilde{u} \neq 0$. If $\tilde{u}(x) = 0$, (66) follows from (65). Thus (64)–(66) imply

$$(\tilde{u}, v)_D = (\alpha_+\tilde{u}^+ - \alpha_-\tilde{u}^-, v), \quad v \in D.$$

Since A is self-adjoint and $\tilde{u} \in L^2(\Omega)$, we see that $\tilde{u}$ is a solution of (56). Hypothesis (G) says that $\tilde{u}$ must therefore be 0. But this contradicts (63). Hence (57) cannot hold, and the lemma is proved.

We can now summarize the conclusions of Lemmas 6–8.

Theorem 2. *Let A be a self-adjoint operator in $L^2(\Omega)$ such that $A \ge \lambda_0 > 0$ and (4) holds for some $m > 0$. Assume that λ_0 is an eigenvalue of A with eigenfunction $\varphi_0 \ge 0$ in $L^\infty(\Omega)$. Assume also*

$$2F(x, t) \le \lambda_0 t^2, \quad |t| \le \delta \text{ for some } \delta > 0 \tag{67}$$

$$2F(x, t) \ge \lambda_0 t^2 - W_0(x)h(t), \quad t > 0,\; x \in \Omega, \tag{68}$$

where $W_0 \in L^1(\Omega)$ and $h(t)$ is a locally bounded function satisfying

$$h(t)/t^2 \to 0 \text{ as } t \to \infty. \tag{69}$$

Assume that $f(x, t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying hypotheses (F) and (G). Then (21) has a solution $u \neq 0$.

Proof. Hypotheses (A)–(G) are satisfied. If (23) has a solution, then (21) does indeed have a solution $y \neq 0$. Otherwise (24) holds for some $\epsilon > 0$ and $\rho > 0$ by Lemma 4. By Lemma 6 there is a critical sequence satisfying (35) with $\beta = 1$. Lemma 8 guarantees that this sequence is bounded, and Lemma 7 shows that this leads to a solution of (47). But $G'(u) = 0$ is equivalent to (22). Hence u is a solution of (21). Moreover, since $c \ge \epsilon > 0$ by (35) and $G(0) = 0$, we see that $u \neq 0$, and the proof is complete.

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Basic Tools, Increasing Functions, and Closure Operations in Generalized Ordered Sets

Árpád Száz

In Honor of Constantin Carathéodory

Abstract Having in mind Galois connections, we establish several consequences of the following definitions.

An ordered pair $X(\leq) = (X, \leq)$ consisting of a set X and a relation $\leq$ on X is called a goset (generalized ordered set).

For any $x \in X$ and $A \subseteq X$, we write $x \in \text{ub}_X(A)$ if $a \leq x$ for all $a \in A$, and $x \in \text{int}_X(A)$ if $\text{ub}_X(x) \subseteq A$, where $\text{ub}_X(x) = \text{ub}_X(\{x\})$.

Moreover, for any $A \subseteq X$, we also write $A \in \mathscr{U}_X$ if $A \subseteq \text{ub}_X(A)$, and $A \in \mathscr{T}_X$ if $A \subseteq \text{int}_X(A)$. And in particular, $A \in \mathscr{E}_X$ if $\text{int}_X(A) \neq \emptyset$.

A function f of one goset X to another Y is called increasing if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$.

In particular, an increasing function φ of X to itself is called a closure operation if $x \leq \varphi(x)$ and $\varphi(\varphi(x)) \leq \varphi(x)$ for all $x \in X$.

The results obtained extend and supplement some former results on increasing functions and can be generalized to relator spaces.

1 Introduction

Ordered sets and *Galois connections* occur almost everywhere in mathematics [12]. They allow of transposing problems and results from one world of our imagination to another one.

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In [48], having in mind a terminology of Birkhoff [2, p. 1], an ordered pair $X(\leq) = (X, \leq)$ consisting of a set X and a relation $\leq$ on X is called a *goset* (generalized ordered set).

In particular, a goset $X(\leq)$ is called a *proset* (preordered set) if the relation $\leq$ is reflexive and transitive. And, a proset $X(\leq)$ is called a *poset* (partially ordered set) if the relation $\leq$ is in addition antisymmetric.

In a goset X, we may define several algebraic and topological basic tools. For instance, for any $x \in X$ and $A \subseteq X$, we write $x \in \mathrm{ub}_X(A)$ if $a \leq x$ for all $a \in A$, and $x \in \mathrm{int}_X(A)$ if $\mathrm{ub}_X(x) \subseteq A$, where $\mathrm{ub}_X(x) = \mathrm{ub}_X(\{x\})$.

Moreover, we write $A \in \mathscr{U}_X$ if $A \subseteq \mathrm{ub}_X(A)$, $A \in \mathscr{T}_X$ if $A \subseteq \mathrm{int}_X(A)$, and $A \in \mathscr{E}_X$ if $\mathrm{int}_X(A) \neq \emptyset$. However, these families are in general much weaker tools than the relations ub_X and int_X which are actually equivalent tools.

In [58], in accordance with [11, Definition 7.23], an ordered pair (f, g) of functions f of one goset X to another Y and g of Y to X is called a *Galois connection* if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$.

In this case, by taking $\varphi = g \circ f$, we can at once see that $f(u) \leq f(v) \iff u \leq g\big(f(v)\big) \iff u \leq (g \circ f)(v) \iff u \leq \varphi(v)$ for all $u, v \in X$. Therefore, the ordered pair (f, φ) is a *Pataki connection* by a terminology of Száz [58].

A function f of one goset X to another Y is called *increasing* if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$. And, an increasing function φ of X to itself is called a *closure operation* on X if $x \leq \varphi(x)$ and $\varphi\big(\varphi(x)\big) \leq \varphi(x)$ for all $x \in X$.

In [53], we have proved that if (f, φ) is a Pataki connection between the prosets X and Y, then f is increasing and φ is a closure operation such that $f \leq f \circ \varphi$ and $f \circ \varphi \leq f$. Thus, $f = f \circ \varphi$ if in particular Y is a poset.

Moreover, we have also proved that a function φ of a proset X to itself is a closure operation if and only if (φ, φ) is a Pataki connection or equivalently (f, φ) is a Pataki connection for some function f of X to another proset Y.

Thus, increasing functions are, in a certain sense, natural generalizations of not only closure operations but also Pataki and Galois connections. Therefore, it seems plausible to extend some results on these connections to increasing functions.

For instance, having in mind a supremum property of Galois connections [51], we shall show that a function f of one goset X to another Y is increasing if and only if $f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$ for all $A \subseteq X$.

If X is *reflexive* in the sense that the inequality relation in it is reflexive, then we may write max instead of ub. While, if X and Y are sup-*complete* and *antisymmetric* and f is increasing, then we can state that $\sup_Y\big(f[A]\big) \leq f\big(\sup_X(A)\big)$.

Here, the relations $\max_X$ and $\sup_X$ are defined by $\max_X(A) = A \cap \mathrm{ub}_X(A)$ and $\sup_X(A) = \min_X(\mathrm{ub}_X(A)) = \mathrm{ub}_X(A) \cap \mathrm{lb}_X(\mathrm{ub}_X(A))$ for all $A \subseteq X$. Moreover, the goset X is called sup-complete if $\sup_X(A) \neq \emptyset$ for all $A \subseteq X$.

In particular, we shall show that if φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X, then $\varphi\big(\sup_X(A)\big) = \varphi\big(\sup_X\big(\varphi[A]\big)\big)$ for all $A \subseteq X$. Moreover, if $Y = \varphi[X]$ and $A \subseteq Y$, then $\sup_Y(A) = \varphi\big(\sup_X(A)\big)$.

In addition to the above results, we shall also show that a function f of one goset X to another Y is increasing if and only if $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big)$ for all $A \subseteq X$, or equivalently $f^{-1}[B] \in \mathscr{T}_X$ for all $B \in \mathscr{T}_Y$ if in particular Y is a proset.

Finally, by writing R and S in place of the inequalities in the gosets X and Y, we shall show that a function f of one *simple relator space* $X(R)$ to another $Y(S)$ is increasing if and only if $f \circ R \subseteq S \circ f$, or equivalently $R \subseteq f^{-1} \circ S \circ f$.

The latter fact, together with some basic operations for relators [56], allows of several natural generalizations of the notion of increasingness of functions to pairs $(\mathscr{F}, \mathscr{G})$ of relators on one relator space $(X, Y)(\mathscr{R})$ to another $(Z, W)(\mathscr{S})$.

Here, a family $\mathscr{R}$ of relations on X to Y is called a *relator*, and the ordered pair $(X, Y)(\mathscr{R}) = \big((X, Y), \mathscr{R}\big)$ is called a *relator space*. Thus, relator spaces are substantial generalizations of not only *ordered sets* but also *uniform spaces*.

Moreover, analogously to Galois and Pataki connections [55, 60], increasing functions are also very particular cases of *upper, lower, and mildly semicontinuous pairs of relators*. Unfortunately, these were not considered in [35, 46, 56].

2 Binary Relations and Ordered Sets

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. If in particular $F \subseteq X^2$, with $X^2 = X \times X$, then we may simply say that F is a relation on X. In particular, $\Delta_X = \{(x, x) : x \in X\}$ is called the *identity relation* on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images* of x and A under F, respectively. If $(x, y) \in F$, then we may also write $x F y$.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain* and *range* of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a *total relation* on X to Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. Thus, it can be shown $F^c = X \times Y \setminus F$ and $F^c[A]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$. (See [57].)

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$. Thus, the operations c and -1 are compatible in the sense $(F^c)^{-1} = (F^{-1})^c$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$.

While, if G is a relation on Z to W, then the *box product relation* $F \boxtimes G$ can be naturally defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$. (See [57].)

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for an arbitrary family of relations too.

If F is a relation on X to Y, then a subset Φ of F is called a *partial selection relation* of F. Thus, we also have $D_\Phi \subseteq D_F$. Therefore, a partial selection relation Φ of F may be called *total* if $D_\Phi = D_F$.

The total selection relations of a relation F will usually be simply called the selection relations of F. Thus, the axiom of choice can be briefly expressed by saying that every relation F has a selection function.

If F is a relation on X to Y and $U \subseteq D_F$, then the relation $F \,|\, U = F \cap (U \times Y)$ is called the *restriction* of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subseteq D_G$ and $F = G \,|\, D_F$, then G is called an *extension* of F.

For any relation F on X to Y, we may naturally define two *set-valued functions*, $F^\diamond$ of X to $\mathscr{P}(Y)$ and $F^\Diamond$ of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$, such that $F^\diamond(x) = F(x)$ for all $x \in X$ and $F^\Diamond(A) = F[A]$ for all $A \subset X$.

Functions of X to $\mathscr{P}(Y)$ can be identified with relations on X to Y. While, functions of $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ are more general objects than relations on X to Y. They were called *corelations* on X to Y in [59].

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R^{-1} \subseteq R$, and *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$.

Thus, a reflexive and transitive (symmetric) relation may be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence (partial order) relation*.

For instance, for $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup A^c \times X$ is a preorder relation on X. (See [24, 52].) While, for a *pseudo-metric* d on X and $r > 0$, the *surrounding* $B_r^d = \big\{(x, y) \in X^2 : \ d(x, y) < r\big\}$ is a tolerance relation on X.

Moreover, we may recall that if $\mathscr{A}$ is a *partition* of X, i. e., a family of pairwise disjoint, nonvoid subsets of X such that $X = \bigcup \mathscr{A}$, then $S_{\mathscr{A}} = \bigcup_{A \in \mathscr{A}} A^2$ is an equivalence relation on X, which can, to some extent, be identified with $\mathscr{A}$.

According to algebra, for any relation R on X, we may naturally define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also naturally define $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, R^∞ is the smallest preorder relation containing R [16].

Note that R is a preorder on X if and only if $R = R^\infty$. Moreover, $R^\infty = R^{\infty\infty}$ and $(R^\infty)^{-1} = (R^{-1})^\infty$. Therefore, R^{-1} is also a preorder on X if R is a preorder on X. Moreover, R^∞ is already an equivalence on X if R is symmetric.

According to [48], an ordered pair $X(\leq) = \big(X, \leq\big)$, consisting of a set X and a relation $\leq$ on X, will be called a *generalized ordered set* or an *ordered set without axioms*. And, we shall usually write X in place of $X(\leq)$.

In the sequel, a generalized ordered set $X(\leq)$ will, for instance, be called *reflexive* if the relation $\leq$ is reflexive on X. Moreover, it is called a preordered (partially ordered) set if $\leq$ is a preorder (partial order) on X.

Having in mind a widely used terminology of Birkhoff [2, p. 1], a generalized ordered set will be briefly called a *goset*. Moreover, a preordered (partially ordered) set will be call a *proset (poset)*.

Thus, every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the *universal relation* X^2. And, the *power set* $\mathscr{P}(X)$ of X is a poset with the ordinary set inclusion $\subseteq$.

In this respect, it is also worth mentioning that if in particular X a goset, then for any $A, B \subseteq X$ we may also naturally write $A \leq B$ if $a \leq b$ for all $a \in A$ and $b \in B$. Thus, $\mathscr{P}(X)$ is also a goset with this extended inequality.

Moreover, if $X(\leq)$ is a goset and $Y \subseteq X$, then by taking $\leq_Y = \leq \cap Y^2$, we can also get a goset $Y(\leq_Y)$. This *subgoset* inherits several properties of the original goset. Thus, for instance, every family of sets is a poset with set inclusion.

In the sequel, trusting to the reader's good sense to avoid confusions, for any goset $X(\leq)$ and operation $\star$ on relations on X, we shall use the notation $X^\star$ for the goset $X(\leq^\star)$. Thus, for instance, X^{-1} will be called the *dual* of the goset X.

Several definitions on posets can be naturally extended to gosets [48]. And, even to arbitrary *relator spaces* [47] which include *ordered sets* [11], *context spaces* [15], and *uniform spaces* [14] as the most important particular cases.

Moreover, most of the definitions can also be naturally extended to *corelator spaces* $(X, Y)(\mathscr{U}) = \big((X, Y), \mathscr{U}\big)$ consisting of two sets X and Y and a family $\mathscr{U}$ of corelations on X to Y. However, it is convenient to investigate first gosets.

3 Upper and Lower Bounds

According to [48], for instance, we may naturally introduce the following

Definition 1. For any subset A of a goset X, the elements of the sets

$$\operatorname{ub}_X(A) = \big\{x \in X: \quad A \leq \{x\}\big\} \qquad \text{and} \qquad \operatorname{lb}_X(A) = \big\{x \in X: \quad \{x\} \leq A\big\}$$

will be called the *upper and lower bounds* of the set A in X, respectively.

Remark 1. Thus, for any $x \in X$ and $A \subseteq X$, we have

(1) $x \in \operatorname{ub}_X(A)$ if and only if $a \leq x$ for all $a \in A$,
(2) $x \in \operatorname{lb}_X(A)$ if and only if $x \leq a$ for all $a \in A$.

Remark 2. Hence, by identifying singletons with their elements, we can see that

(1) $\operatorname{ub}_X(x) = \leq(x) = [\,x, +\infty\,[\, = \big\{y \in X: \quad x \leq y\big\}$,
(2) $\operatorname{lb}_X(x) = \geq(x) = \,]-\infty, x\,] = \big\{y \in X: \quad x \geq y\big\}$.

This shows that the relation ub_X is somewhat more natural tool than lb_X.

By using Remark 1, we can easily establish the following

Theorem 1. *For any subset A of a goset X, we have*

(1) $\operatorname{ub}_X(A) = \operatorname{lb}_{X^{-1}}(A)$,
(2) $\operatorname{lb}_X(A) = \operatorname{ub}_{X^{-1}}(A)$.

Proof. If $x \in \mathrm{ub}_X(A)$, then by Remark 1 we have $a \leq x$ for all $a \in A$. This implies that $x \leq^{-1} a$ for all $a \in A$. Hence, since $X^{-1} = X(\leq^{-1})$, we can already see that $x \in \mathrm{lb}_{X^{-1}}(A)$. Therefore, $\mathrm{ub}_X(A) \subseteq \mathrm{lb}_{X^{-1}}(A)$.

The converse inclusion can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by taking X^{-1} in place of X.

Remark 3. This theorem shows that the relations ub_X and lb_X are equivalent tools in the goset X.

By using Remark 1, we can also easily establish the following theorem.

Theorem 2. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\mathrm{ub}_Y(A) = \mathrm{ub}_X(A) \cap Y$,
(2) $\mathrm{lb}_Y(A) = \mathrm{lb}_X(A) \cap Y$.

Concerning the relations ub_X and lb_X, we can also easily prove the following theorem.

Theorem 3. *For any family $(A_i)_{i \in I}$ subsets of a goset X, we have*

(1) $\mathrm{ub}_X\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \mathrm{ub}_X\big(A_i\big)$,
(2) $\mathrm{lb}_X\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \mathrm{lb}_X\big(A_i\big)$.

Proof. If $x \in \mathrm{ub}_X\big(\bigcup_{i \in I} A_i\big)$, then by Remark 1 we have $a \leq x$ for all $a \in \bigcup_{i \in I} A_i$. Hence, it is clear that we also have $a \leq x$ for all $a \in A_i$ with $i \in I$. Therefore, $x \in \mathrm{ub}_X(A_i)$ for all $i \in I$, and thus $x \in \bigcap_{i \in I} \mathrm{ub}_X(A_i)$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 1.

From the above theorem, by identifying singletons with their elements, we can immediately derive the following corollary.

Corollary 1. *For any subset A of a goset X, we have*

(1) $\mathrm{ub}_X(A) = \bigcap_{a \in A} \mathrm{ub}_X(a)$,
(2) $\mathrm{lb}_X(A) = \bigcap_{a \in A} \mathrm{lb}_X(a)$.

Remark 4. Hence, by using Remark 2 and a basic fact on complement relations mentioned in Sect. 2, we can immediately derive that

(1) $\mathrm{ub}_X(A) = \leq^c [A]^c$. (2) $\mathrm{lb}_X(A) = \geq^c [A]^c$.

From Corollary 1, we can also immediately derive the first two assertions of

Theorem 4. *If X is a goset, then*

(1) $\mathrm{ub}_X(\emptyset) = X$ *and* $\mathrm{lb}_X(\emptyset) = X$,
(2) $\mathrm{ub}_X(B) \subseteq \mathrm{ub}_X(A)$ *and* $\mathrm{lb}_X(B) \subseteq \mathrm{lb}_X(A)$ *if* $A \subseteq B \subseteq X$,
(3) $\bigcup_{i \in I} \mathrm{ub}_X\big(A_i\big) \subseteq \mathrm{ub}_X\Big(\bigcap_{i \in I} A_i\Big)$ *and* $\bigcup_{i \in I} \mathrm{lb}_X\big(A_i\big) \subseteq \mathrm{lb}_X\Big(\bigcap_{i \in I} A_i\Big)$ *if* $A_i \subseteq X$ *for all* $i \in I$.

Proof. To prove the first part of (3), we can note that if $A_i \subseteq X$ for all $i \in I$, then $\bigcap_{i \in I} A_i \subseteq A_i$ for all $i \in I$. Hence, by using (2), we can already infer that $\mathrm{ub}_X(A_i) \subseteq \mathrm{ub}_X\big(\bigcap_{i \in I} A_i\big)$ for all $i \in I$, and thus the required inclusion is also true.

However, it is now more important to note that, as an immediate consequence of the corresponding definitions, we can also state the following theorem which actually implies most of the properties of the relations ub_X and lb_X.

Theorem 5. *For any two subsets A and B of a goset X, we have*

$$B \subseteq \mathrm{ub}_X(A) \iff A \subseteq \mathrm{lb}_X(B)\,.$$

Proof. By Remark 1, it is clear that each of the above inclusions is equivalent to the property that $a \leq b$ for all $a \in A$ and $b \in B$.

Remark 5. This property can be briefly expressed by writing that $A \leq B$, or equivalently $A \times B \subseteq \leq$, that is, $B \in \mathrm{Ub}_X(A)$, or equivalently $A \in \mathrm{Lb}_X(B)$ by the notations of our former paper [47].

From Theorem 5, it is clear that in particular we have

Corollary 2. *For any subset A of a goset X, we have*

(1) $\mathrm{ub}_X(A) = \big\{x \in X: \ A \subseteq \mathrm{lb}_X(x)\big\}$,
(2) $\mathrm{lb}_X(A) = \big\{x \in X: \ A \subseteq \mathrm{ub}_X(x)\big\}$.

Remark 6. Moreover, from Theorem 5, we can see that, for any $A, B \subseteq X$, we have

$$\mathrm{lb}_X(A) \subseteq^{-1} B \iff A \subseteq \mathrm{ub}_X(B)\,.$$

This shows that the set-valued functions lb_X and ub_X form a *Galois connection* between the poset $\mathscr{P}(X)$ and its dual in the sense of [11, Definition 7.23], suggested by Schmidt's reformulation [36, p. 209] of Ore's definition of Galois connexions [30].

Remark 7. Hence, by taking $\Phi_X = \mathrm{ub}_X \circ \mathrm{lb}_X$, for any $A, B \subseteq X$, we can infer that

$$\mathrm{lb}_X(A) \subseteq^{-1} \mathrm{lb}_X(B) \iff A \subseteq \Phi_X(B)\,.$$

This shows that the set-valued functions lb_X and Φ_X form a *Pataki connection* between the poset $\mathscr{P}(X)$ and its dual in the sense of [51, Remark 3.8] suggested by a fundamental unifying work of Pataki [32] on the basic refinements of relators studied each separately by the present author in [42].

Remark 8. By [53, Theorem 4.7], this fact implies that $\mathrm{lb}_X = \mathrm{lb}_X \circ \Phi_X$, and Φ_X is a *closure operation* on the poset $\mathscr{P}(X)$ in the sense of [2, p. 111].

By an observation, attributed to Richard Dedekind by Erné [12, p. 50], this is equivalent to the requirement that the set function Φ_X with itself forms a Pataki connection between the poset $\mathscr{P}(X)$ and itself.

4 Interiors and Closures

Because of Remark 2, we may also naturally introduce the following

Definition 2. For any subset A of a goset X, the sets

$$\operatorname{int}_X(A) = \big\{ x \in X : \ \operatorname{ub}_X(x) \subseteq A \big\} \quad \text{and} \quad \operatorname{cl}_X(A) = \big\{ x \in X : \ \operatorname{ub}_X(x) \cap A \neq \emptyset \big\}$$

will be called the *interior and closure* of the set A in X, respectively.

Remark 9. Recall that, by Remark 2, we have $\operatorname{ub}_X(x) = \leq (x) = [\, x, \ +\infty \, [$ for all $x \in X$.

Therefore, the present one-sided interiors and closures, when applied to subsets of the real line $\mathbb{R}$, greatly differ from the usual ones.

The latter ones can only be derived from a *relator* (family of relations) which has to consist of at least countable many tolerance or preorder relations.

By using Definition 2, we can easily prove the following theorem.

Theorem 6. *For any subset A of a goset X, we have*

(1) $\operatorname{int}_X(A) = X \setminus \operatorname{cl}_X(X \setminus A)$,
(2) $\operatorname{cl}_X(A) = X \setminus \operatorname{int}_X(X \setminus A)$.

Proof. If $x \in \operatorname{int}_X(A)$, then by Definition 2 we have $\operatorname{ub}(x) \subseteq A$. Hence, we can infer that $\operatorname{ub}(x) \cap (X \setminus A) = \emptyset$. Therefore, by Definition 2, we have $x \notin \operatorname{cl}_X(X \setminus A)$, and thus $x \in X \setminus \operatorname{cl}_X(X \setminus A)$. This shows that $\operatorname{int}_X(A) \subseteq X \setminus \operatorname{cl}_X(X \setminus A)$.

The converse inclusion can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by writing $X \setminus A$ in place of A, and applying complementation.

Remark 10. This theorem shows that the relations int_X and cl_X are also equivalent tools in the goset X.

By using the complement operation $\mathscr{C}$, defined by $\mathscr{C}(A) = A^c = X \setminus A$ for all $A \subseteq X$, the above theorem can be reformulated in a more concise form.

Corollary 3. *For any goset X, we have*

(1) $\operatorname{int}_X = \big(\operatorname{cl}_X \circ \mathscr{C}\big)^c = \operatorname{cl}_X^c \circ \mathscr{C}$,
(2) $\operatorname{cl}_X = \big(\operatorname{int}_X \circ \mathscr{C}\big)^c = \operatorname{int}_X^c \circ \mathscr{C}$.

Proof. To prove the second part of (1), note that by the corresponding definitions, for any $A \subseteq X$, we have

$$\big(\operatorname{cl}_X \circ \mathscr{C}\big)^c(A) = \big(\operatorname{cl}_X \circ \mathscr{C}\big)(A)^c = \operatorname{cl}_X\big(\mathscr{C}(A)\big)^c = \operatorname{cl}_X^c\big(\mathscr{C}(A)\big) = \big(\operatorname{cl}_X^c \circ \mathscr{C}\big)(A).$$

Now, in contrast to Theorems 1 and 2, we can only state the following two theorems.

Theorem 7. *For any subset A of a goset X, we have*

(1) $\operatorname{int}_{X^{-1}}(A) = \{x \in X : \operatorname{lb}_X(x) \subseteq A\}$,
(2) $\operatorname{cl}_{X^{-1}}(A) = \{x \in X : \operatorname{lb}_X(x) \cap A \neq \emptyset\}$.

Theorem 8. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\operatorname{int}_X(A) \cap Y \subseteq \operatorname{int}_Y(A)$,
(2) $\operatorname{cl}_Y(A) \subseteq \operatorname{cl}_X(A) \cap Y$.

However, concerning the relations int_X and cl_X, we can also easily prove

Theorem 9. *For any family $(A_i)_{i \in I}$ subsets of a goset X, we have*

(1) $\operatorname{int}_X\Big(\bigcap_{i \in I} A_i\Big) = \bigcap_{i \in I} \operatorname{int}_X(A_i)$,
(2) $\operatorname{cl}_X\Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} \operatorname{cl}_X(A_i)$.

Proof. If $x \in \operatorname{int}_X\big(\bigcap_{i \in I} A_i\big)$, then by Definition 2 we have $\operatorname{ub}_X(x) \subseteq \bigcap_{i \in I} A_i$. Therefore, $\operatorname{ub}_X(x) \subseteq A_i$, and thus $x \in \operatorname{int}_X(A_i)$ for all $i \in I$. Therefore, $x \in \bigcap_{i \in I} \operatorname{int}_X(A_i)$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 6.

Remark 11. This theorem shows that, despite Remark 10, there are cases when the relation cl_X is a more convenient tool than int_X.

Namely, from assertion (2), by identifying singletons with their elements, we can immediately derive the following corollary.

Corollary 4. *For any subset A of a goset X, we have*

$$\operatorname{cl}_X(A) = \bigcup_{a \in A} \operatorname{cl}_X(a).$$

Remark 12. Note that, for any $x, y \in X$, we have

$$y \in \operatorname{cl}_X(x) \iff \operatorname{ub}_X(y) \cap \{x\} \neq \emptyset \iff x \in \operatorname{ub}_X(y) \iff y \in \operatorname{lb}_X(x),$$

and thus also $\operatorname{cl}_X(x) = \operatorname{lb}_X(x)$. Hence, by using Theorem 1, we can immediately infer that $\operatorname{cl}_X(x) = \operatorname{ub}_{X^{-1}}(x)$.

Therefore, as an immediate consequence of the above results, we can also state

Theorem 10. *For any subset A of a goset X, we have*

$$\operatorname{cl}_X(A) = \bigcup_{a \in A} \operatorname{lb}_X(a) = \bigcup_{a \in A} \operatorname{ub}_{X^{-1}}(a).$$

Remark 13. Hence, by using Remark 2 and Theorem 1, we can at once see that

$$\mathrm{cl}_X(A) = \bigcup_{a \in A} \geq (a) = \geq [A] \qquad \text{and} \qquad \mathrm{cl}_{X^{-1}}(A) = \bigcup_{a \in A} \leq (a) = \leq [A].$$

And thus, by Theorem 6, also $\mathrm{int}_X(A) = \geq [A^c]^c$ and $\mathrm{int}_{X^{-1}}(A) = \leq [A^c]^c$.

Now, analogously to Theorem 4, we can also easily establish the following

Theorem 11. *If X is a goset, then*

(1) $\mathrm{cl}_X(\emptyset) = \emptyset$ *and* $\mathrm{int}_X(X) = X$,
(2) $\mathrm{cl}_X(A) \subseteq \mathrm{cl}_X(B)$ *and* $\mathrm{int}_X(A) \subseteq \mathrm{int}_X(B)$ *if* $A \subseteq B \subseteq X$,
(3) $\mathrm{cl}_X\Big(\bigcap_{i \in I} A_i\Big) \subseteq \bigcap_{i \in I} \mathrm{cl}_X(A_i)$ *and* $\bigcup_{i \in I} \mathrm{int}_X(A_i) \subseteq \mathrm{int}_X\Big(\bigcup_{i \in I} A_i\Big)$ *if* $A_i \subseteq X$ *for all* $i \in I$.

However, it is now more important to note that, analogously to Theorem 5, we also have the following theorem which actually implies most of the properties of the relations int_X and cl_X.

Theorem 12. *For any two subsets A and B of a goset X, we have*

$$B \subseteq \mathrm{int}_X(A) \iff \mathrm{cl}_{X^{-1}}(B) \subseteq A.$$

Proof. If $B \subseteq \mathrm{int}_X(A)$, then by Definition 2, we have $\mathrm{ub}_X(b) \subseteq A$ for all $b \in B$. Hence, by Theorem 10, we can already see that $\mathrm{cl}_{X^{-1}}(B) = \bigcup_{b \in B} \mathrm{ub}_X(b) \subseteq A$.

The converse implication can be proved quite similarly by reversing the above argument.

Remark 14. Recall that, by Remark 13, we have $\mathrm{cl}_{X^{-1}}(B) = \leq [B]$. Therefore, by Theorem 12, the inclusion $B \subseteq \mathrm{int}_X(A)$ can also be reformulated by stating that $\leq [B] \subseteq A$, or equivalently $\leq [B] \cap A^c = \emptyset$. That is, $B \in \mathrm{Int}_X(A)$, or equivalently $B \notin \mathrm{Cl}_X(A^c)$ by the notations of Száz [47].

From Theorem 12, it is clear that in particular we have

Corollary 5. *For any subset A of a goset X, we have*

$$\mathrm{int}_X(A) = \big\{ x \in X : \quad \mathrm{cl}_{X^{-1}}(x) \subseteq A \big\}.$$

Remark 15. From Theorem 12, we can also see that, for any $A, B \subseteq X$, we have

$$\mathrm{cl}_{X^{-1}}(A) \subseteq B \iff A \subseteq \mathrm{int}_X(B).$$

This shows that the set-valued functions $\mathrm{cl}_{X^{-1}}$ and int_X form a Galois connection between the poset $\mathscr{P}(X)$ and itself.

Remark 16. Thus, by taking $\Phi_X = \text{int}_X \circ \text{cl}_{X^{-1}}$, for any $A, B \subseteq X$ we can state that

$$\text{cl}_{X^{-1}}(A) \subseteq \text{cl}_{X^{-1}}(B) \quad \Longleftrightarrow \quad A \subseteq \Phi_X(B)\,.$$

This shows that the set-valued functions $\text{cl}_{X^{-1}}$ and Φ_X form a Pataki connection between the poset $\mathscr{P}(X)$ and itself. Thus, $\text{cl}_{X^{-1}} = \text{cl}_{X^{-1}} \circ \Phi_X$, and Φ_X is closure operation on the poset $\mathscr{P}(X)$.

Remark 17. The upper- and lower-bound Galois connection, described in Remark 6, was first studied by Birkhoff [2, p. 122] under the name *polarities*.

While, the closure–interior Galois connection, described in Remark 15, has been only considered in [61] with reference to Davey and Priestly [11, Exercise 7.18].

5 Open and Closed Sets

Definition 3. For any goset X, the members of the families

$$\mathscr{T}_X = \{A \subseteq X : \quad A \subseteq \text{int}_X(A)\} \qquad \text{and} \qquad \mathscr{F}_X = \{A \subseteq X : \quad \text{cl}_X(A) \subseteq A\}$$

are called the *open and closed subsets* of X, respectively.

Remark 18. Thus, by Definition 2 and Theorem 10, for any $A \subseteq X$, we have

(1) $A \in \mathscr{T}_X$ if and only if $\text{ub}_X(a) \subseteq A$ for all $a \in A$.
(2) $A \in \mathscr{F}_X$ if and only if $\text{lb}_X(a) \subseteq A$ for all $a \in A$.

Namely, by Definition 2, for any $a \in A$ we have $a \in \text{int}_X(A)$ if and only if $\text{ub}_X(a) \subseteq A$. Moreover, by Theorem 10, we have $\text{cl}_X(A) = \bigcup_{a \in A} \text{lb}_X(a)$.

Remark 19. Because of Remarks 2 and 18, the members of the families $\mathscr{T}_X$ and $\mathscr{F}_X$ may also be called the *ascending and descending subsets* of X.

Namely, for instance, by the above mentioned remarks, for any $A \subseteq X$ we have $A \in \mathscr{T}_X$ if and only if for any $a \in A$ and $x \in X$, with $a \leq x$, we also have $x \in A$.

Remark 20. Moreover, from Remarks 2 and 18, we can also see that

(1) $\mathscr{T}_X = \{A \subseteq X : \ \leq [A] \subseteq A\}$. (2) $\mathscr{F}_X = \{A \subseteq X : \ \geq [A] \subseteq A\}$.

Namely, for instance, by a basic definition on relations and Remark 2, for any $A \subseteq X$ we have $\leq [A] = \bigcup_{a \in A} \leq (a) = \bigcup_{a \in A} \text{ub}_X(a)$.

By using Definition 3 and Theorem 6, we can also easily prove the following theorem.

Theorem 13. *For any goset X, we have*

(1) $\mathscr{T}_X = \{A \subseteq X : \quad A^c \in \mathscr{F}_X\}$,
(2) $\mathscr{F}_X = \{A \subseteq X : \quad A^c \in \mathscr{T}_X\}$.

Proof. If $A \in \mathscr{T}_X$, then by Definition 3 we have we have $A \subseteq \operatorname{int}_X(A)$. Hence, by using Theorem 6, we can infer that $\operatorname{cl}_X(A^c) = \operatorname{int}_X(A)^c \subseteq A^c$. Therefore, by Definition 3, the inclusion $A^c \in \mathscr{F}_X$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 6.

Remark 21. This theorem shows that the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are also equivalent tools in the goset X.

By using the element-wise complementation, defined by $\mathscr{A}^c = \{A^c : A \in \mathscr{A}\}$ for all $\mathscr{A} \subseteq \mathscr{P}(X)$, Theorem 13 can also be reformulated in a more concise form.

Corollary 6. *For any goset X, we have*

(1) $\mathscr{T}_X = \mathscr{F}_X{}^c$,
(2) $\mathscr{F}_X = \mathscr{T}_X{}^c$.

Now, as an immediate consequence of Remark 20, we can also state the following theorem which can also be easily proved with the help of Definition 3 and Theorem 12.

Theorem 14. *For any goset X, we have*

(1) $\mathscr{T}_X = \mathscr{F}_{X^{-1}}$,
(2) $\mathscr{F}_X = \mathscr{T}_{X^{-1}}$.

Proof. If $A \in \mathscr{T}_X$, then by Definition 3, we have $A \subseteq \operatorname{int}_X(A)$. Hence, by using Theorem 12, we can infer that $\operatorname{cl}_{X^{-1}}(A) \subseteq A$. Therefore, $A \in \mathscr{F}_{X^{-1}}$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by writing X^{-1} in place of X.

Remark 22. Moreover, because of Remark 14 and Theorem 13, for any $A \subseteq X$ we can also state that $A \in \mathscr{T}_X$ if and only if $A \in \operatorname{Int}_X(A)$, and $A \in \mathscr{F}_X$ if and only if $A^c \notin \operatorname{Cl}_X(A)$.

By using Definition 3 and Theorem 8, we can easily establish the following theorem.

Theorem 15. *For any subset Y of a goset X, we have*

(1) $\mathscr{T}_X \cap \mathscr{P}(Y) \subseteq \mathscr{T}_Y$,
(2) $\mathscr{F}_X \cap \mathscr{P}(Y) \subseteq \mathscr{F}_Y$.

Proof. Namely, if, for instance, $A \in \mathscr{T}_X \cap \mathscr{P}(Y)$, then $A \in \mathscr{T}_X$ and $A \in \mathscr{P}(Y)$. Therefore, $A \subseteq \operatorname{int}_X(A)$ and $A \subseteq Y$. Hence, by Theorem 8, we can already see that $A \subset \operatorname{int}_X(A) \cap Y \subseteq \operatorname{int}_Y(A)$, and thus $A \in \mathscr{T}_Y$ also holds.

Moreover, by using Definition 3 and Theorems 9 and 11, we can also easily prove the following.

Theorem 16. *For any goset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are ultratopologies [10] (complete rings [1]) in the sense that they are closed under arbitrary unions and intersections.*

Proof. Namely, if, for instance, $A_i \in \mathscr{T}_X$ for all $i \in I$, then $A_i \subseteq \operatorname{int}_X(A_i)$ for all $i \in I$. Hence, by using Theorems 9 and 11, we can already infer that

$$\bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} \operatorname{int}_X(A_i) = \operatorname{int}_X\Big(\bigcap_{i\in I} A_i\Big) \quad \text{and} \quad \bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} \operatorname{int}_X(A_i) \subseteq \operatorname{int}_X\Big(\bigcup_{i\in I} A_i\Big).$$

Therefore, the sets $\bigcap_{i\in I} A_i$ and $\bigcup_{i\in I} A_i$ are also in $\mathscr{T}_X$.

Remark 23. From the above theorem, by taking the empty subfamily of $\mathscr{T}_X$ and $\mathscr{F}_X$, we can immediately infer that $\{\emptyset, X\} \subseteq \mathscr{T}_X \cap \mathscr{F}_X$.

Finally, we note that the following theorem is also true

Theorem 17. *For any subset A of a goset X, we have*

(1) $\bigcup \mathscr{T}_X \cap \mathscr{P}(A) \subseteq \operatorname{int}_X(A)$,
(2) $\operatorname{cl}_X(A) \subseteq \bigcap \mathscr{F}_X \cap \mathscr{P}^{-1}(A)$.

Proof. Define $B = \bigcup \mathscr{T}_X \cap \mathscr{P}(A)$. Then, we evidently have $B \subseteq A$. Moreover, by Theorem 16, we can see that $B \in \mathscr{T}_X$. Hence, by using Definition 3 and Theorem 11, we can already infer that $B \subseteq \operatorname{int}_X(B) \subseteq \operatorname{int}_X(A)$. Therefore, (1) is true.

Moreover, from (1), by using Theorem 12 and the fact that $U \in \mathscr{P}^{-1}(V)$ if and only if $V \subseteq U$, we can easily see that (2) is also true.

Example 1. If, for instance, $X = \mathbb{R}$ and $\preceq$ is a relation on X such that

$$\preceq(x) = \{x-1\} \cup [x, +\infty[$$

for all $x \in X$, then by using Remarks 2 and 18 we can easily see that $\mathscr{T}_X = \{\emptyset, X\}$, and thus by Corollary 6 also $\mathscr{F}_X = \{\emptyset, X\}$.

Namely, if $A \in \mathscr{T}_X$ such that $A \neq \emptyset$, then there exists $x \in X$ such that $x \in A$, and thus by the abovementioned remarks $\preceq(x) = \operatorname{ub}_X(x) \subseteq A$. Therefore,

$$\{x-1\} \cup [x, +\infty[\subseteq A.$$

Hence, we can see that $x - 1 \in A$. Therefore, $\preceq(x-1) \subseteq A$, and thus

$$\{x-2\} \cup [x-1, +\infty[\subseteq A.$$

Hence, by induction, it is clear that for any $n \in \mathbb{N}$ we also have

$$\{x-n-1\} \cup [x-n, +\infty[\subseteq A.$$

Thus, by the Archimedean property of $\mathbb{N}$ in $\mathbb{R}$, we necessarily have $A = X$.

Now, by using that $\mathscr{F}_X = \{\emptyset, X\}$, we can easily see that

$$\bigcup \mathscr{F}_X \cap \mathscr{P}^{-1}(A) = \emptyset \quad \text{if} \quad A = \emptyset \qquad \text{and} \qquad \bigcup \mathscr{F}_X \cap \mathscr{P}^{-1}(A) = X \quad \text{if} \quad A \neq \emptyset\,.$$

Moreover, we can also easily see that, for any $x, y \in X$,

$$y \in \operatorname{lb}_X(x) \iff x \in \operatorname{ub}_X(y) \iff x \in \preceq (y) \iff x \in \{y-1\} \cup [y, +\infty[$$
$$\iff x = y-1 \text{ or } y \leq x \iff y \leq x \text{ or } y = x+1 \iff y \in\,]-\infty, x] \cup \{x+1\}\,.$$

Therefore,

$$\operatorname{lb}_X(x) = \,]-\infty, x] \cup \{x+1\}\,.$$

Thus, by Theorem 10,

$$\operatorname{cl}_X(A) = \bigcup_{a \in A} \operatorname{lb}_X(a) = \bigcup_{a \in A} \big(\,]-\infty, a] \cup \{a+1\}\big)\,.$$

for all $A \subseteq X$. Hence, it is clear that equality in the assertion (2) of Theorem 17 need not be true.

Remark 24. This shows that the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are, in general, much weaker tools in the goset X than the relations int_X and cl_X. However, later we see that this is not the case if X is in particular a proset.

6 Fat and Dense Sets

Note that a subset A of a goset X may be called *upper bounded* if $\operatorname{ub}_X(A) \neq \emptyset$. Therefore, in addition to Definition 3, we may also naturally introduce the following.

Definition 4. For any goset X, the members of the families

$$\mathscr{E}_X = \{A \subseteq X : \operatorname{int}_X(A) \neq \emptyset\} \qquad \text{and} \qquad \mathscr{D}_X = \{A \subseteq X : \operatorname{cl}_X(A) = X\}$$

are called the *fat and dense subsets* of X, respectively.

Remark 25. Thus, by Definition 2, for any $A \subseteq X$, we have

(1) $A \in \mathscr{E}_X$ if and only if $\operatorname{ub}_X(x) \subseteq A$ for some $x \in X$.
(2) $A \in \mathscr{D}_X$ if and only if $\operatorname{ub}_X(x) \cap A \neq \emptyset$ for all $x \in X$.

Remark 26. Moreover, by Remark 13 and Theorem 10, we can also see that

$$\mathscr{D}_X = \{ A \subseteq X : \quad X = \geq [A] \} = \{ A \subseteq X : \quad X = \bigcup_{a \in A} \mathrm{lb}_X(a) \}.$$

Therefore, for any $A \subseteq X$, we have $A \in \mathscr{D}_X$ if and only if for any $x \in X$ there exists $a \in A$ such that $x \in \mathrm{lb}_X(a)$, i. e., $x \leq a$.

Remark 27. Because of the above two remarks, the members of the families $\mathscr{E}_X$ and $\mathscr{D}_X$ may also be called the *residual and cofinal subsets* of X.

Namely, for instance, by Remarks 2 and 25, for any $A \subseteq X$, we have $A \in \mathscr{E}_X$ if and only if there exists $x \in X$ such that for any $y \in X$, with $x \leq y$, we have $y \in A$.

By using Definition 4 and Theorem 6, we can easily prove the following.

Theorem 18. *For any goset X, we have*

(1) $\mathscr{E}_X = \{ A \subseteq X : \quad A^c \notin \mathscr{D}_X \}$,
(2) $\mathscr{D}_X = \{ A \subseteq X : \quad A^c \notin \mathscr{E}_X \}$.

Proof. If $A \in \mathscr{E}_X$, then by Definition 4 we have $\mathrm{int}_X(A) \neq \emptyset$. Hence, by Theorem 6, we can infer that $\mathrm{cl}_X(A^c) = X \setminus \mathrm{int}_X(A) \neq X$. Therefore, $A^c \notin \mathscr{D}_X$ also holds.

The converse implication can be proved quite similarly by reversing the above argument. Moreover, (2) can be derived from (1) by using Theorem 6.

Remark 28. This theorem shows that the families $\mathscr{E}_X$ and $\mathscr{D}_X$ are also equivalent tools in the goset X.

By using element-wise complementation, Theorem 18 can also be written in a more concise form.

Corollary 7. *For any goset X, we have*

(1) $\mathscr{E}_X = \big(\mathscr{P}(X) \setminus \mathscr{D}_X \big)^c$,
(2) $\mathscr{D}_X = \big(\mathscr{P}(X) \setminus \mathscr{E}_X \big)^c$.

Moreover, concerning the families $\mathscr{E}_X$ and $\mathscr{D}_X$, we can also prove the following.

Theorem 19. *For any goset X, we have*

(1) $\mathscr{E}_X = \{ E \subseteq X : \quad \forall\ D \in \mathscr{D}_X : \quad E \cap D \neq \emptyset \}$,
(2) $\mathscr{D}_X = \{ D \subseteq X : \quad \forall\ E \in \mathscr{E}_X : \quad E \cap D \neq \emptyset \}$.

Proof. If $E \in \mathscr{E}_X$, then by Remark 25, there exists $x \in X$ such that $\mathrm{ub}_X(x) \subseteq E$. Moreover, if $D \in \mathscr{D}_X$, then by Remark 25, we have $\mathrm{ub}_X(x) \cap D \neq \emptyset$. Therefore, $E \cap D \neq \emptyset$ also holds.

Conversely, if $E \subseteq X$ such that $E \cap D \neq \emptyset$ for all $D \in \mathscr{D}_X$, then we can also easily see that $E \in \mathscr{E}_X$. Namely, if $E \notin \mathscr{E}_X$, then by Theorem 18 we necessarily have $E^c \in \mathscr{D}_X$. Therefore, $E \cap E^c \neq \emptyset$ which is a contradiction.

Hence, it is clear that (1) is true. Assertion (2) can be proved quite similarly.

Now, a counterpart of Theorem 14 is not true. However, analogously to Theorems 15 and 16, we can also state the following two theorems.

Theorem 20. *For any subset Y of a goset X, we have*

(1) $\mathscr{E}_X \cap \mathscr{P}(Y) \subseteq \mathscr{E}_Y$,
(2) $\mathscr{D}_X \cap \mathscr{P}(Y) \subseteq \mathscr{D}_Y$.

Theorem 21. *For any goset X, the families $\mathscr{E}_X$ and $\mathscr{D}_X$ are ascending subfamilies of the poset $\mathscr{P}(X)$ such that*

(1) $\mathscr{T}_X \setminus \{\emptyset\} \subseteq \mathscr{E}_X$,
(2) $\mathscr{F}_X \cap \mathscr{D}_X \subseteq \{X\}$.

From this theorem, we can immediately derive the following

Corollary 8. *For any subset A of a goset X, the following assertions are true:*

(1) If $B \subseteq A$ for some $B \in \mathscr{T}_X \setminus \{\emptyset\}$, then $A \in \mathscr{E}_X$.
(2) If $A \in \mathscr{D}_X$, then $A \setminus B \neq \emptyset$ for all $B \in \mathscr{F}_X \setminus \{X\}$.

Proof. To check (2), note that if the conclusion of (2) does not hold, then there exists $B \in \mathscr{F}_X \setminus \{X\}$ such that $A \setminus B = \emptyset$, and thus $A \cap B^c = \emptyset$. Hence, by defining $C = B^c$ and using Theorem 13, we can already see that $C \in \mathscr{T}_X \setminus \{\emptyset\}$ such that $A \cap C = \emptyset$, and thus $C \subseteq A^c$. Therefore, by (1), $A^c \in \mathscr{E}_X$, and thus by Theorem 18, we have $A \notin \mathscr{D}_X$.

Remark 29. The converses of the above assertions need not be true. Namely, if X is as in Example 1, then $\mathscr{T}_X = \{\emptyset, X\}$, but $\mathscr{E}_X$ is quite a large subfamily of $\mathscr{P}(X)$.

This shows that there are cases when even the families $\mathscr{E}_X$ and $\mathscr{D}_X$ are better tools in a goset X than $\mathscr{T}_X$ and $\mathscr{F}_X$. However, later we shall see that this is not the case if X is in particular a proset.

The duality and several advantages of fat and dense sets in relator spaces, over the open and closed ones, were first revealed by the present author at a Prague Topological Symposium in 1991 [40]. However, nobody was willing to accept this.

Remark 30. An ascending subfamily $\mathscr{A}$ of the poset $\mathscr{P}(X)$ is usually called a *stack* in X. It is called proper if $\emptyset \notin \mathscr{A}$ or equivalently $\mathscr{A} \neq \mathscr{P}(X)$.

In particular, a stack $\mathscr{A}$ in X is called a *filter* if $A, B \in \mathscr{A}$ implies $A \cap B \in \mathscr{A}$. And, $\mathscr{A}$ is called a *grill* if $A \cup B \in \mathscr{A}$ implies $A \in \mathscr{A}$ or $B \in \mathscr{A}$. These are usually assumed to be nonempty and proper.

Several interesting historical facts on stacks, lters, grills and nets can be found in the works [62, 63] of Thron

Concerning the families $\mathscr{E}_X$ and $\mathscr{D}_X$, we can also easily establish the following two theorems.

Theorem 22. *For any poset X, the following assertions are equivalent :*

(1) $\mathscr{E}_X \neq \emptyset$,
(2) $X \in \mathscr{E}_X$,

(3) $\emptyset \notin \mathscr{D}_X$,
(4) $X \neq \emptyset$.

Proof. To prove the equivalence of (1) and (4), note that, by Theorem 21, assertions (1) and (2) are equivalent. Moreover, by Remark 25, assertion (2) holds if and only there exists $x \in X$ such that $\mathrm{ub}_X(x) \subseteq X$. That is, (4) holds.

Theorem 23. *For any poset X, the following assertions are equivalent:*

(1) $\emptyset \notin \mathscr{E}_X$,
(2) $\mathscr{D}_X \neq \emptyset$,
(3) $X \in \mathscr{D}_X$,
(4) $X = \geq [X]$.

Proof. To prove the equivalence of (1) and (4), note that by Remark 25 assertion (1) holds if and only if, for any $x \in X$, we have $\mathrm{ub}_X(x) \not\subseteq \emptyset$. That is, $\mathrm{ub}_X(x) \neq \emptyset$, or equivalently $\leq (x) \neq \emptyset$. That is, the relation $\leq$ is total in the sense that its domain is the whole X.

Remark 31. A subset $\mathscr{B}$ of a stack $\mathscr{A}$ in X is called a *base* of $\mathscr{A}$ if for each $A \in \mathscr{A}$ there exists $B \in \mathscr{B}$ such that $B \subseteq A$. That is, $\mathscr{B}$ is a cofinal subset of the poset $\mathscr{A}^{-1} = \mathscr{A}(\subseteq^{-1}) = \mathscr{A}(\supseteq)$.

Note that if $\mathscr{B} \subseteq \mathscr{P}(X)$, then the family

$$\mathscr{B}^* = \mathrm{cl}_{\mathscr{P}^{-1}}(\mathscr{B}) = \left\{ A \subseteq X : \quad \exists\, B \in \mathscr{B} : \quad B \subseteq A \right\}$$

is already a stack in X such that $\mathscr{B}$ is a base of $\mathscr{B}^*$.

Now, as a more important addition to Theorem 21, we can also easily prove

Theorem 24. *For any goset X, the stack $\mathscr{E}_X$ has a base $\mathscr{B}$ with* $\mathrm{card}(\mathscr{B}) \leq \mathrm{card}(X)$.

Proof. By Remarks 25 and 31, it is clear that the family $\mathscr{B}_X = \left\{ \mathrm{ub}_X(x) : \quad x \in X \right\}$ is a base of $\mathscr{E}_X$.

Moreover, we can note that the function f, defined by $f(x) = \mathrm{ub}_X(x)$ for $x \in X$, is onto $\mathscr{B}_X$. Hence, by the axiom of choice, the cardinality condition follows.

Namely, now f^{-1} is a relation of $\mathscr{B}_X$ to X. Hence, by choosing a selection function φ of f^{-1}, we can see that φ is an injection of $\mathscr{B}$ to X.

Remark 32. Now, a corresponding property of the family $\mathscr{D}_X$ should, in principle, be derived from the above theorem by using either Theorem 18 or 19.

Remark 33. The importance of the study of the cardinalities of the bases of the stack of all fat sets in a relator space, concerning a problem of mine on paratopologically simple relators, was first recognized by J. Deák (1994) and G. Pataki (1998). (For the corresponding results, see Pataki [31].)

7 Maximum, Minimum, Supremum, and Infimum

According to [48], we may also naturally introduce the following.

Definition 5. For any subset A of a goset X, the members of the sets

$$\max_X(A) = A \cap \mathrm{ub}_X(A) \qquad \text{and} \qquad \min_X(A) = A \cap \mathrm{lb}_X(A)$$

are called the *maxima* and *minima* of the set A in X, respectively.

Remark 34. Thus, for any subset A of a goset X, we have

(1) $\mathrm{ub}_X(A) = \max_X(A)$ if and only if $\mathrm{ub}_X(A) \subseteq A$.
(2) $\mathrm{lb}_X(A) = \min_X(A)$ if and only if $\mathrm{lb}_X(A) \subseteq A$.

Moreover, from Definition 5, we can see that the properties of the relations $\max_X$ and $\min_X$ can be immediately derived from the results of Sect. 3.

For instance, from Theorems 1 and 2 and Corollaries 1 and 2, by using Definition 5, we can immediately derive the following four theorems.

Theorem 25. *For any subset A of a goset X, we have*

(1) $\max_X(A) = \min_{X^{-1}}(A)$,
(2) $\max_X(A) = \min_{X^{-1}}(A)$.

Remark 35. This theorem shows that the relations $\max_X$ and $\min_X$ are also equivalent tools in the goset X.

Theorem 26. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\max_Y(A) = \max_X(A)$,
(2) $\min_Y(A) = \min_X(A)$.

Theorem 27. *For any subset A of a goset X, we have*

(1) $\max_X(A) = \bigcap_{a \in A} A \cap \mathrm{ub}_X(a)$,
(2) $\min_X(A) = \bigcap_{a \in A} A \cap \mathrm{lb}_X(a)$.

Theorem 28. *For any subset A of a goset X, we have*

(1) $\max_X(A) = \{x \in A : \ A \subseteq \mathrm{lb}_X(x)\}$,
(2) $\min_X(A) = \{x \in A : \ A \subseteq \mathrm{ub}_X(x)\}$.

Remark 36. By Corollary 2, for instance, we may also naturally define

$$\mathrm{ub}_X^*(A) = \{x \in X : \ A \cap \mathrm{ub}_X(x) \subseteq \mathrm{lb}_X(x)\},$$

and also $\max_X^*(A) = A \cap \mathrm{ub}_X^*(A)$ for all $A \subseteq X$.

Thus, for any $x \in X$ and $A \subseteq X$, we have $x \in \mathrm{ub}_X^*(A)$ if and only if $x \leq a$ implies $a \leq x$ for all $a \in A$. Therefore, $\max_X^*(A)$ is just the family of all *maximal elements* of A.

The most important theorems on a poset X give some sufficient conditions in order that the set $\max^*(X)$ be nonempty. (See, for instance, [18, p. 33] and the references of [54].)

Now, by using Definition 5, we may also naturally introduce

Definition 6. For any subset A of a goset X, the members of the sets

$$\sup_X(A) = \min_X\big(\mathrm{ub}_X(A)\big) \qquad \text{and} \qquad \inf_X(A) = \max_X\big(\mathrm{lb}_X(A)\big)$$

are called the *suprema* and *infima* of the set A in X, respectively.

Thus, by Definition 5, we evidently have the following

Theorem 29. *For any subset A of a goset X, we have*

(1) $\sup_X(A) = \mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$,
(2) $\inf_X(A) = \mathrm{lb}_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$.

Hence, by Theorem 1, it is clear that we also have the following.

Theorem 30. *For any subset A of a goset X, we have*

(1) $\sup_X(A) = \inf_{X^{-1}}(A)$,
(2) $\inf_X(A) = \sup_{X^{-1}}(A)$.

Remark 37. This theorem shows that the relations $\sup_X$ and $\inf_X$ are also equivalent tools in the goset X.

However, instead of an analogue of Theorem 2, we can only prove

Theorem 31. *If X is a goset and $Y \subseteq X$, then for any $A \subseteq Y$ we have*

(1) $\sup_X(A) \cap Y \subseteq \sup_Y(A)$,
(2) $\inf_X(A) \cap Y \subseteq \inf_Y(A)$.

Proof. To prove (1), by using Theorems 2, 4, and, 29 we can see that

$$\begin{aligned}\sup_Y(A) &= \mathrm{ub}_Y(A) \cap \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big)\\ &= \mathrm{ub}_X(A) \cap Y \cap \ \mathrm{lb}_X\big(\mathrm{ub}_X(A) \cap Y\big) \cap \mathrm{ub}_X(A) \cap Y\\ &= \mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A) \cap Y\big) \cap Y \supseteq \mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big) \cap Y\\ &= \sup_X(A) \cap Y.\end{aligned}$$

Remark 38. In connection with inclusion (2), Tamás Glavosits, my PhD student, showed that the corresponding equality need not be true even if X is a finite poset.

For this, he took $X = \{a, b, c, d\}$, $Y = X \setminus \{b\}$ and $A = Y \setminus \{a\}$, and considered the preorder $\leq$ on X generated by the relation $R = \{(a,b), (b,c), (b, d)\}$.

Thus, he could at once see that $\inf_Y(A) = \max_Y\big(\mathrm{lb}_Y(A)\big) = \max_Y\big(\{a\}\big) = \{a\}$, but $\inf_X(A) = \max_X\big(\mathrm{lb}_X(A)\big) = \max_X\big(\{a,\, b\}\big) = \{b\}$, and thus $\inf_X(A) \cap Y = \emptyset$.

Now, by using Theorem 29, we can also easily prove the following theorem which shows that the relations $\sup_X$ and $\inf_X$ are, in a certain sense, better tools in the goset X than $\max_X$ and $\min_X$.

Theorem 32. *For any subset A of a goset X, we have*

(1) $\max_X(A) = A \cap \sup_X(A)$,
(2) $\min_X(A) = A \cap \inf_X(A)$.

Proof. To prove (2), note that by Theorem 29 and Definition 5, we have

$$A \cap \inf_X(A) = A \cap \mathrm{lb}_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) = \min_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big).$$

Moreover, by Definition 5 and Remark 8, we have

$$\min_X(A) \subseteq A \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big), \quad \text{and so} \quad \min_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) = \min_X(A).$$

Remark 39. By the above theorem, for any subset A of a goset X, we have

(1) $\max_X(A) = \sup_X(A)$ if and only if $\sup_X(A) \subseteq A$.
(2) $\min_X(A) = \inf_X(A)$ if and only if $\inf_X(A) \subseteq A$.

Moreover, by using Theorem 29, we can also easily prove the following theorem which will make a basic theorem on supremum and infimum completeness properties to be completely obvious.

Theorem 33. *For any subset A of a goset X, we have*

(1) $\sup_X(A) = \inf_X\big(\mathrm{ub}_X(A)\big)$,
(2) $\inf_X(A) = \sup_X\big(\mathrm{lb}_X(A)\big)$.

Proof. To prove (2), note that by Theorem 29 and Remark 8, we have

$$\begin{aligned}\inf_X(A) &= \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \cap \mathrm{lb}_X(A)\\ &= \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \cap \mathrm{lb}_X\Big(\mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)\Big) = \sup_X\big(\mathrm{lb}_X(A)\big).\end{aligned}$$

Remark 40. Concerning our references to Remark 8 in the proofs of Theorems 32 and 33, note that the assertions

$$A \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \qquad \text{and} \qquad \mathrm{lb}_X(A) = \mathrm{lb}_X\Big(\mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)\Big)$$

can also be easily proved directly, by using Definition 1, without using the corresponding theorems on Pataki connections.

Definition 7. A goset X is called *inf-complete* (*sup-complete*) if $\inf_X(A) \neq \emptyset$ $(\sup_X(A) \neq \emptyset)$ for all $A \subseteq X$.

Remark 41. Quite similarly, a goset X may, for instance, be also naturally called *min-complete* if $\min_X(A) \neq \emptyset$ for all nonvoid subset A of X.

Thus, the set $\mathbb{Z}$ of all integers is min-, but not inf-complete. While, the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is inf-, but not min-complete.

Now, as an immediate consequence of Theorem 33, we can state the following straightforward extension of [2, Theorem 3, p. 112].

Theorem 34. *For a goset X, the following assertions are equivalent:*

(1) X is inf-complete,
(2) X is sup-complete.

Remark 42. Similar equivalences of several modified inf- and sup-completeness properties of gosets have been established in [3, 4].

Finally, we note that, by Definition 5 and Theorem 27, we evidently have

Theorem 35. *For any subset A of a goset X, we have*

(1) $\inf_X(A) = \big\{ x \in \mathrm{lb}_X(A) : \quad \mathrm{lb}_X(A) \subseteq \mathrm{lb}_X(x) \big\}$,
(2) $\sup_X(A) = \big\{ x \in \mathrm{ub}_X(A) : \quad \mathrm{ub}_X(A) \subseteq \mathrm{ub}_X(x) \big\}$.

Moreover, by using this theorem, we can also easily prove the following.

Theorem 36. *For any subset A of a proset X, we have*

(1) $\inf_X(A) = \big\{ x \in X : \quad \mathrm{lb}_X(x) = \mathrm{lb}_X(A) \big\}$,
(2) $\sup_X(A) = \big\{ x \in X : \quad \mathrm{ub}_X(x) = \mathrm{ub}_X(A) \big\}$.

Proof. Define

$$\Phi(A) = \big\{ x \in X : \quad \mathrm{lb}_X(x) = \mathrm{lb}_X(A) \big\}.$$

Now, if $x \in \Phi(A)$, we can see that

(a) $\mathrm{lb}_X(x) \subseteq \mathrm{lb}_X(A)$,
(b) $\mathrm{lb}_X(A) \subseteq \mathrm{lb}_X(x)$.

From (a), since X is reflexive, and thus $x \leq x$, i. e., $x \in \mathrm{lb}_X(x)$, we can infer that $x \in \mathrm{lb}_X(A)$. Hence, by (b) and Theorem 35, we can already see that $x \in \inf_X(A)$. Therefore, $\Phi(A) \subseteq \inf_X(A)$ even if X is assumed to be only a reflexive goset.

Conversely, if $x \in \inf_X(A)$, then by Theorem 35 we also have
(c) $x \in \mathrm{lb}_X(A)$, (d) $\mathrm{lb}_X(A) \subseteq \mathrm{lb}_X(x)$.

From (c), we can infer that $x \leq a$ for all $a \in A$. Hence, by using the transitivity of X we can easily see that if $y \in \mathrm{lb}_X(x)$, and thus $y \leq x$, then $y \leq a$ also holds for all $a \in A$, and thus $y \in \mathrm{lb}_X(A)$. Therefore, $\mathrm{lb}_X(x) \subseteq \mathrm{lb}_X(A)$ even if X is assumed to be only a transitive goset. Hence, by using (d), we can already see that $\mathrm{lb}_X(x) = \mathrm{lb}_X(A)$, and thus $x \in \Phi(x)$. Therefore, $\inf_X(A) \subseteq \Phi(A)$ even if X is assumed to be only a transitive goset.

The above arguments show that (1) is true. Moreover, from (1) by using Theorems 1 and 30, we can at once see that (2) is also true.

8 Self-bounded Sets

Analogously to Definition 3, for instance, we may also naturally introduce

Definition 8. For any goset X, the members of the family

$$\mathcal{U}_X = \{ A \subseteq X : \quad A \subset \text{ub}_X(A) \}$$

are called the *self-upper-bounded subsets* of X.

Remark 43. Thus, by the corresponding definitions, for any $A \subseteq X$, we have $A \in \mathcal{U}_X$ if and only if $x \leq y$ for all $x, y \in A$.

Therefore, $A \in \mathcal{U}_X$ if and only if $A \leq A$ or equivalently $A^2 \subseteq \leq$. That is, by the notations of Száz [47], we have $A \in \text{Ub}_X(A)$ or equivalently $A \in \text{Lb}_X(A)$.

Because of the above remark, we evidently have the following three theorems.

Theorem 37. *For any goset X, we have $\mathcal{U}_X = \mathcal{U}_{X^{-1}}$.*

Theorem 38. *For any subset Y of goset X, we have $\mathcal{U}_Y = \mathcal{U}_X \cap \mathcal{P}(Y)$.*

Theorem 39. *For any goset X, we have*

$$\mathcal{U}_X = \{ A \subseteq X : \quad \forall\ x, y \in A : \quad \{x, y\} \in \mathcal{U}_X \}.$$

Hence, it is clear that, in particular, we also have the following corollary.

Corollary 9. *For any goset X, the family $\mathcal{U}_X$ is a descending subset of the poset $\mathcal{P}(X)$ such that $\bigcup \mathcal{V} \in \mathcal{U}_X$ for any chain $\mathcal{V}$ in $\mathcal{U}_X$.*

However, it is now more important to note that, by using the corresponding definitions, we can also prove the following

Theorem 40. *For any subset A of a goset X, the following assertions are equivalent:*

(1) $A \in \mathcal{U}_X$,
(2) $A = \max_X(A)$,
(3) $A \subseteq \sup_X(A)$,
(4) $A \subseteq \text{lb}_X(A)$,
(5) $A = \min_X(A)$,
(6) $A \subseteq \inf_X(A)$.

Proof. By Definitions 5 and 8, we evidently have

$$A \in \mathcal{U}_X \iff A \subseteq \text{ub}_X(A) \iff A \subseteq A \cap \text{ub}_X(A) \iff A \subseteq \max_X(A).$$

Hence, since $\max_X(A) \subseteq A$, it is clear that (1) and (2) are equivalent.

Moreover, by using Definition 8 and Theorem 5, we can at once see that (1) and (4) are also equivalent. Hence, by using the inclusion $A \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$ and Theorem 29, we can also easily see that

$$A \in \mathscr{U}_X \iff A \subseteq \mathrm{lb}_X(A) \iff A \subseteq \mathrm{lb}_X(A) \cap \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big) \iff A \subseteq \inf_X(A)\,.$$

Therefore, (1) and (6) are also equivalent. The proofs of the remaining implications are quite similar.

Remark 44. This theorem shows that, in a goset X, the family $\mathscr{U}_X$ is just the collection of all fixed elements of the set-valued functions $\max_X$ and $\min_X$.

Now, as some immediate consequences of Theorem 40 and Definition 6, we can also state

Corollary 10. *For any subset A of a goset X, the following assertions are equivalent:*

(1) $\mathrm{ub}_X(A) \in \mathscr{U}_X$;
(2) $\mathrm{ub}_X(A) = \sup_X(A)$;
(3) $\mathrm{ub}_X(A) \subseteq \mathrm{ub}_X\big(\mathrm{ub}_X(A)\big)$;
(4) $\mathrm{ub}_X(A) \subseteq \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$.

Corollary 11. *For any subset A of a goset X, the following assertions are equivalent:*

(1) $\mathrm{lb}_X(A) \in \mathscr{U}_X$;
(2) $\mathrm{lb}_X(A) = \inf_X(A)$;
(3) $\mathrm{lb}_X(A) \subseteq \mathrm{lb}_X\big(\mathrm{lb}_X(A)\big)$;
(4) $\mathrm{lb}_X(A) \subseteq \mathrm{ub}_X\big(\mathrm{lb}_X(A)\big)$.

However, it is now more important to note that, by using Theorem 40, we can also easily prove the following theorem.

Theorem 41. *For any goset X, we have*

(1) $\mathscr{U}_X = \big\{ \max_X(A) : \quad A \subseteq X \big\}$,
(2) $\mathscr{U}_X = \big\{ \min_X(A) : \quad A \subseteq X \big\}$.

Proof. If $V \in \mathscr{U}_X$, then by Theorem 40, we have $V = \max_X(V)$. Therefore, V is in the family $\mathscr{A} = \{ \max_X(A) : \ A \subseteq X \}$.

Conversely, if $V \in \mathscr{A}$, then there exists $A \in \mathscr{A}$ such that $V = \max_X(A)$. Hence, by Definition 5, it follows that $V \subseteq A$ and $V \subseteq \mathrm{ub}_X(A)$. Now, by Theorem 4, we can also see that $\mathrm{ub}_X(A) \subseteq \mathrm{ub}_X(V)$. Therefore, $V \subseteq \mathrm{ub}_X(V)$, and thus $V \in \mathscr{U}_X$ also holds.

This proves (1). Moreover, (2) can be derived from (1) by using Theorems 25 and 37.

Remark 45. This theorem shows that, in a goset X, the family $\mathscr{U}_X$ is just the range of the set-valued functions $\max_X$ and $\min_X$.

By using Remark 43, we can also easily prove the following three theorems.

Theorem 42. *For any goset X, the following assertions are equivalent:*

(1) X *is reflexive,*
(2) $\{x\} \in \mathscr{U}_X$ *for all* $x \in X$.

Theorem 43. *If X is an antisymmetric goset, then for any $A \in \mathscr{U}_X$ we have* $\operatorname{card}(A) \leq 1$.

Proof. If $A \in \mathscr{U}_X$ and $x, y \in A$, then by Remark 43 we have $x \leq y$ and $y \leq x$. Hence, by the assumed antisymmetry of $\leq$, it follows that $x = y$.

Theorem 44. *If X is reflexive goset such that* $\operatorname{card}(A) \leq 1$ *for all $A \in \mathscr{U}_X$, then X is antisymmetric.*

Proof. If $x, y \in X$ such that $x \leq y$ and $y \leq x$, then by taking $A = \{x, y\}$ we can see that $A \leq A$, and thus $A \in \mathscr{U}_X$. Hence, by the assumption, it follows that $\operatorname{card}(A) \leq 1$. Therefore, we necessarily have $x = y$.

From the latter two theorems, by using Theorem 41, Definition 6 and Theorem 32, we can immediately derive the following two theorems.

Theorem 45. *If X is an antisymmetric goset, then under the notation $\Phi = \max_X$, $\min_X$, $\sup_X$, or $\inf_X$, for any $A \subseteq X$ we have* $\operatorname{card}\big(\Phi(A)\big) \leq 1$.

Theorem 46. *If X is a reflexive goset such that, under the notation $\Phi = \max_X$, $\min_X$, $\sup_X$, or $\inf_X$, for any $A \subseteq X$ we have* $\operatorname{card}\big(\Phi(A)\big) \leq 1$, *then X is antisymmetric.*

Proof. Note that, if, for instance, $\operatorname{card}\big(\sup_X(A)\big) \leq 1$ for all $A \subseteq X$, then by Theorem 32, we also have $\operatorname{card}\big(\max_X(A)\big)$ for all $A \subseteq X$. Hence, by using Theorem 41, we can infer that $\operatorname{card}(A) \leq 1$ for all $A \in \mathscr{U}_X$. Therefore, by Theorem 44, we can state that X is antisymmetric.

Remark 46. In connection with the above results, it is worth noticing that the goset X considered in Example 1 is reflexive, but not antisymmetric.

Namely, concerning the relation $\preceq$, we can easily see that, for any $x, y \in X$, we have both $x \preceq y$ and $y \preceq x$ if and only if $x = y$ or $x = y - 1$ or $y = x - 1$.

Therefore, for any $A \subseteq X$, we have $A \in \mathscr{U}_X$ if and only if $A = \emptyset$ or $A = \{x\}$ or $A = \{x, x - 1\}$ for some $x \in X$.

This fact, together with $\mathscr{T}_X = \{\emptyset, X\}$, shows that there are cases when even the family $\mathscr{U}_X$ is also a better tool than the family $\mathscr{T}_X$.

In the sequel, beside reflexivity and antisymmetry, we shall also need a further, similarly simple and important, property of gosets.

Definition 9. A goset X will be called *linear* if for any $x, y \in X$, with $x \neq y$, we have either $x \leq y$ or $y \leq x$.

Remark 47. If X is a goset, then for any $x, y \in X$,we may also write $x < y$ if both $x \leq y$ and $x \neq y$.

Therefore, if the goset X is linear, then for any $x, y \in X$, with $x \neq y$, we actually have either $x < y$ or $y < x$.

Moreover, as an immediate consequence of the corresponding definitions, we can also state the following.

Theorem 47. *For a goset* X, *the following assertions are equivalent :*

(1) X *is reflexive and linear,*
(2) *For any* $x, y \in X$, *we have either* $x \leq y$ *or* $y \leq x$,
(3) $\max_X(A) \neq \emptyset$ $(\min_X(A) \neq \emptyset)$ *for all* $A \subseteq X$ *with* $1 \leq \operatorname{card}(A) \leq 2$.

Proof. To check the implication (3) $\Longrightarrow$ (2), note that if $x, y \in X$, then $A = \{x, y\}$ is a subset of X such that $1 \leq \operatorname{card}(A) \leq 2$. Therefore, if (3) holds, then there exists $z \in X$ such that $z \in \max_X(A)$. Hence, by Definition 5, it follows that $z \in A$ and $z \in \mathrm{ub}_X(A)$. Therefore, we have either $z = x$ or $z = y$. Moreover, we have $x \leq z$ and $y \leq z$. Hence, if $z = x$, we can see that $y \leq x$. While, if $z = y$, we can see that $x \leq y$. Therefore, (2) also holds.

From this theorem, it is clear that in particular we have

Corollary 12. *If* X *is a min-complete (max-complete) goset, then* X *is reflexive and linear.*

The importance of reflexive, linear, and antisymmetric gosets is also apparent from the next two simple theorems.

Theorem 48. *If* X *is an antisymmetric goset, then* $x < y$ *implies* $y \not\leq x$ *for all* $x, y \in X$.

Theorem 49. *If* X *is a reflexive and linear goset, then* $x \not\leq y$ *implies* $y < x$ *for all* $x, y \in X$.

Proof. If $x, y \in X$ such that $x \not\leq y$, then by Theorem 47 we have $y \leq x$. Moreover, by the reflexivity of X, we also have $x \neq y$, and hence $y \neq x$. Therefore, $y < x$ also holds.

Remark 48. Therefore, if X is a reflexive, linear, and antisymmetric goset, then for any $x, y \in X$,we have

$$x \not\leq y \quad \Longleftrightarrow \quad x <^{-1} y \, .$$

Note that, analogously to the equivalences in Remarks 6 and 15, this is again a Galois connection property.

9 The Importance of Reflexivity and Transitivity

Several simple characterizations of reflexivity and transitivity of a goset X, in terms of the relations ub_X and lb_X, and their compositions considered in Sect. 7, have been given in [49].

Now, by using the techniques of the theory of relator spaces, we shall give some more delicate characterizations of these properties in terms of the relations int_X and cl_X and the families $\mathscr{T}_X$ and $\mathscr{F}_X$.

Theorem 50. *For any goset X, the following assertions are equivalent :*

(1) X is reflexive,
(2) $x \in \mathrm{ub}_X(x)$ for all $x \in X$,
(3) $\mathrm{int}_X(A) \subseteq A$ for all $A \subseteq X$,
(4) $\mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \subseteq \mathrm{ub}_X(x)$ for all $x \in X$.

Proof. By Remark 2, it is clear that (1) and (2) are equivalent. Moreover, if $A \subseteq X$ and $x \in \mathrm{int}_X(A)$, then by Definition 2 we have $\mathrm{ub}_X(x) \subseteq A$. Hence, if (2) holds, we can infer that $x \in A$, and thus (3) also holds.

Now, since (3) trivially implies (4), it remains to show only that (4) also implies (2). However, for this, it is enough to note only that, for any $x \in X$, we have $\mathrm{ub}_X(x) \subseteq \mathrm{ub}_X(x)$, and hence $x \in \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$ by Definition 2.

From this theorem, by using Theorem 6, we can immediately derive

Corollary 13. *For any goset X, the following assertions are equivalent :*

(1) X is reflexive,
(3) $A \subseteq \mathrm{cl}_X(A)$ for all $A \subseteq X$.

Proof. For instance, if (1) holds, then by Theorem 50, for any $A \subseteq X$, we have $\mathrm{int}_X(A^c) \subseteq A^c$. Hence, by using Theorem 6, we can already infer that $A \subseteq \mathrm{int}_X(A^c)^c = \mathrm{cl}_X(A)$. Therefore, (2) also holds.

From the above results, by Definition 3, it is clear that we also have

Theorem 51. *If X is a reflexive goset, then*

(1) $\mathscr{T}_X = \big\{A \subseteq X : \ A = \mathrm{int}_X(A)\big\}$,
(2) $\mathscr{F}_X = \big\{A \subseteq X : \ A = \mathrm{cl}_X(A)\big\}$.

Remark 49. This theorem shows that, in a reflexive goset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are just the collections of all fixed elements of the set-valued functions int_X and cl_X, respectively.

However, it is now more important to note that, in addition to Theorem 50, we can also prove the following.

Theorem 52. *For any goset X, the following assertions are equivalent :*

(1) X *is transitive,*
(2) $\mathrm{ub}_X(x) \in \mathscr{T}_X$ *for all* $x \in X$,
(3) $\mathrm{int}_X(A) \in \mathscr{T}_X$ *for all* $A \subseteq X$,
(4) $\mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \in \mathscr{T}_X$ *for all* $x \in X$,
(5) $x \in \mathrm{int}_X\Big(\mathrm{int}_X\big(\mathrm{ub}_X(x)\big)\Big)$ *for all* $x \in X$.

Proof. If (1) holds, then the inequality relation $\leq$ in X is transitive. Therefore, if $x \in X$ and $y \in \mathrm{ub}_X(x)$, then by Remark 2, for any $z \in \mathrm{ub}_X(y)$ we also have $z \in \mathrm{ub}_X(x)$. Hence, we can see that $\mathrm{ub}_X(y) \subseteq \mathrm{ub}_X(x)$, and thus by Definition 2 we have $y \in \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. This shows that $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$, and thus by Definition 3 we have $\mathrm{ub}_X(x) \in \mathscr{T}_X$. Therefore, (2) also holds.

Conversely, if (2) holds, then by Definition 3, for any $x \in X$, we have $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. Therefore, by Definition 2, for any $y \in \mathrm{ub}_X(x)$ we have $\mathrm{ub}_X(y) \subseteq \mathrm{ub}_X(x)$. Therefore, $z \in \mathrm{ub}_X(y)$ implies $z \in \mathrm{ub}_X(x)$. Hence, by Remark 2, it is clear that the inequality relation $\leq$ in X is transitive, and (1) also holds.

Next, we show that (2) also implies (3). For this, note that if $A \subseteq X$ and $x \in \mathrm{int}_X(A)$, then by Definition 2 we have $\mathrm{ub}_X(x) \subseteq A$. Hence, by using Theorem 11, we can infer that $\mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \subseteq \mathrm{int}_X(A)$. Moreover, if (2) holds, then by Definition 3 we also have $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. Thus, $\mathrm{ub}_X(x) \subseteq \mathrm{int}_X(A)$ is also true. Hence, by Definition 2, it follows that $x \in \mathrm{int}_X\big(\mathrm{int}_X(A)\big)$. This shows that $\mathrm{int}_X(A) \subseteq \mathrm{int}_X\big(\mathrm{int}_X(A)\big)$, and thus by Definition 3 we also have $\mathrm{int}_X(A) \in \mathscr{T}_X$. Therefore, (3) also holds.

Now, since (3) trivially implies (4), it remains only to show only that (4) implies (5), and (5) implies (2). For this, note that if (4) holds, then by Definitions 2 and 3, for any $x \in X$, we have $x \in \mathrm{int}_X\big(\mathrm{ub}_X(x)\big) \subseteq \mathrm{int}_X\Big(\mathrm{int}_X\big(\mathrm{ub}_X(x)\big)\Big)$. Therefore, (5) also holds. Moreover, if (5) holds, then by Definition 2, for any $x \in X$, we have $\mathrm{ub}_X(x) \subset \mathrm{int}_X\big(\mathrm{ub}_X(x)\big)$. Therefore, $\mathrm{ub}_X(x) \in \mathscr{T}_X$, and thus (2) also holds.

From this theorem, by using Theorems 6 and 13, we can immediately derive

Corollary 14. *For any goset X, the following assertions are equivalent :*

(1) X *is transitive,*
(2) $\mathrm{cl}_X(A) \in \mathscr{F}_X$ *for all* $A \subseteq X$.

Now, as an immediate consequence of the above results, we can also state

Theorem 53. *For a proset X, we have*

(1) $\mathscr{T}_X = \big\{\mathrm{int}_X(A) : \quad A \subseteq X\big\}$,
(2) $\mathscr{F}_X = \big\{\mathrm{cl}_X(A) : \quad A \subseteq X\big\}$.

Remark 50. This theorem shows that in a proset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are just the ranges of the set-valued functions int_X and cl_X, respectively.

However, it is now more important to note that, by using Theorems 50 and 52, we can also easily prove the following.

Theorem 54. *For any goset X, the following assertions are equivalent:*

(1) X *is reflexive and transitive,*
(2) $\text{int}_X(A) = \bigcup \mathscr{T}_X \cap \mathscr{P}(A)$ *for all* $A \subseteq X$,
(3) $\text{cl}_X(A) = \bigcap \mathscr{F}_X \cap \mathscr{P}^{-1}(A)$ *for all* $A \subseteq X$.

Proof. Suppose that (1) holds and $A \subseteq X$. Define

$$B = \text{int}_X(A) \qquad \text{and} \qquad C = \bigcup \mathscr{T}_X \cap \mathscr{P}(A).$$

Then, by Theorems 50 and 52, we can see that $B \subseteq A$ and $B \in \mathscr{T}_X$, and thus $B \in \mathscr{T}_X \cap \mathscr{P}(A)$. Therefore, $B \subseteq \bigcup \mathscr{T}_X \cap \mathscr{P}(A) = C$. Moreover, from Theorem 17, we can see that $C \subseteq B$ is always true. Therefore, (2) also holds.

Conversely, if (2) holds, then for any $A \subseteq X$ we evidently have $\text{int}_X(A) \subseteq A$. Thus, by Theorem 50, X is reflexive. Moreover, by Theorem 16, we can see that $\text{int}_X(A) \in \mathscr{T}_X$. Therefore, by Theorem 52, X is also transitive. Thus, (1) also holds.

Now, to complete the proof, it remains to note only that the equivalence of (2) and (3) is an immediate consequence of Theorems 6 and 13.

Remark 51. This theorem shows that in a proset X the relation int_X or cl_X and the family $\mathscr{T}_X$ or $\mathscr{F}_X$ are also equivalent tools.

Now, by using Theorems 50 and 52, we can also easily prove the following.

Theorem 55. *For any subset A of a proset X, we have*

(1) $A \in \mathscr{E}_X$ *if and only if* $B \subseteq A$ *for some* $B \in \mathscr{T}_X \setminus \{\emptyset\}$,
(2) $A \in \mathscr{D}_X$ *if and only if* $A \setminus B \neq \emptyset$ *for all* $B \in \mathscr{F}_X \setminus \{X\}$.

Proof. According to Remark 31, define $\mathscr{B} = \mathscr{T}_X \setminus \{\emptyset\}$ and $\mathscr{A} = \mathscr{B}^*$. Then, for any $A \subseteq X$, we have $A \in \mathscr{A}$ if and only if $B \subseteq A$ for some $B \in \mathscr{B}$.

Now, if $A \in \mathscr{E}_X$, then by Remark 25, there exists $x \in X$ such that $\text{ub}_X(x) \subseteq A$. Moreover, by Theorems 50 and 52, we have $x \in \text{ub}_X(x)$ and $\text{ub}_X(x) \in \mathscr{T}_X$, and hence $\text{ub}_X(x) \in \mathscr{B}$. Therefore, $A \in \mathscr{A}$ also holds. This shows that $\mathscr{E}_X \subseteq \mathscr{A}$.

Moreover, from Corollary 8, we can see that $\mathscr{A} \subseteq \mathscr{E}_X$ is always true. Therefore, (1) also holds. Now, (2) can be easily derived from (1) by using Theorems 13 and 18.

Remark 52. By Remark 31, assertion (1) means only that, in a proset X, the family $\mathscr{T}_X \setminus \{\emptyset\}$ is also a base for the stack $\mathscr{E}_X$.

Beside Remark 51, this also shows that, in a proset X, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ are better tools than the families $\mathscr{E}_X$ and $\mathscr{D}_X$.

10 An Interior Operation and the Preorder Closure

Because of Theorems 50 and 52, in addition to the operations c, -1, and ∞ mentioned in Sect. 2, we may also naturally introduce some further unary operations on relations and thus also on gosets.

For instance, in accordance with [44, Definition 3.1], we may naturally introduce

Definition 10. For any goset X, we define a relation $\leq^{\circ}$ on X such that

$$\leq^{\circ}(x) = \operatorname{int}_X\big(\operatorname{ub}_X(x)\big)$$

for all $x \in X$. Moreover, according to a notation of Sect. 2, we write $X^{\circ} = X(\leq^{\circ})$.

Remark 53. Thus, by the corresponding definitions, for any $x, y \in X$, we have

$$x \leq^{\circ} y \iff y \in \leq^{\circ}(x) \iff y \in \operatorname{int}_X\big(\operatorname{ub}_X(x)\big) \iff \operatorname{ub}_X(y) \subseteq \operatorname{ub}_X(x).$$

Therefore, $\leq^{\circ}$ is already a preorder relation on X, and thus X° is a proset.

Moreover, as an immediate consequence of Theorems 50 and 52, we can state

Theorem 56. *For any goset X, we have*

(1) $\leq^{\circ} \subseteq \leq$ *if X is reflexive,*
(2) $\leq \subseteq \leq^{\circ}$ *if and only if X is transitive.*

Proof. To derive (2) from Theorem 52, note that for any $x \in X$ we have

$$\leq(x) \subseteq \leq^{\circ}(x) \iff \operatorname{ub}_X(x) \subseteq \operatorname{int}_X\big(\operatorname{ub}_X(x)\big) \iff x \in \operatorname{int}_X\Big(\operatorname{int}_X\big(\operatorname{ub}_X(x)\big)\Big).$$

From this theorem, by Remark 53, it is clear that in particular we also have

Corollary 15. *For any goset X, the following assertions are equivalent:*

(1) $\leq = \leq^{\circ}$,
(2) *X is a proset,*
(3) $y \in \operatorname{ub}_X(x) \iff \operatorname{ub}_X(y) \subseteq \operatorname{ub}_X(x)$ *for all* $x, y \in X$.

Remark 54. Note that, analogously to the statements of Remarks 7 and 16, assertion (3) is again a Pataki connection property.

Concerning assertion (3), it is also worth mentioning that $\leq$ is an equivalence relation on X if and only if it is total and, under the notation $X = X(\leq)$, for any $x, y \in X$ we have $y \in \operatorname{ub}_X(x)$ if and only if $\operatorname{ub}_X(x) \cap \operatorname{ub}_X(y) \neq \emptyset$.

Moreover, from Theorem 56, by using Remark 53 and a basic property of the relation $\leq^{\infty}$, we can also immediately derive the following.

Theorem 57. *For any goset X, we have*

(1) $\leq^{\circ} \subseteq \leq^{\infty}$ *if X is reflexive,*
(2) $\leq^{\infty} \subseteq \leq^{\circ}$ *if and only if X is transitive.*

Hence, it is clear that in particular we also have the following.

Corollary 16. *For a reflexive goset X, the following assertions are equivalent:*

(1) $\leq^{\circ} = \leq^{\infty}$,

(2) *X is transitive.*

Remark 55. Now, analogously to Definition 10, for any goset X, we may also naturally define a relation $\leq^{-}$ on X such that

$$\leq^{-}(x) = \mathrm{cl}_X\big(\mathrm{ub}_X(x)\big)$$

for all $x \in X$. Moreover, now we may also naturally write $X^{-} = X(\leq^{-})$.

Thus, in addition to the inclusions $\leq \subseteq \leq^{-}$ and $\leq^{-} \subseteq \leq$, we may also naturally investigate the inclusions $\leq^{\circ} \subseteq \leq^{-}$ and $\leq^{-} \subseteq \leq^{\circ}$. (See [44].)

However, it now is more important to note that the generated preorder relations can always be expressed in terms of the Pervin relations of the open sets defined by the original relations [26, 27].

Theorem 58. *If X is a goset, then for any $x \in X$, we have*

$$\leq^{\infty}(x) = \bigcap_{A \in \mathscr{T}_X} R_A = \bigcap \big\{ A \in \mathscr{T}_X : \quad x \in A \big\}.$$

Proof. Recall that, for any $A \subseteq X$, we have $R_A = A^2 \cup A^c \times X$. Therefore,

$$R_A(x) = A \quad \text{if} \quad x \in A \qquad \text{and} \qquad R_A(x) = X \quad \text{if} \quad x \in A^c.$$

Hence, we can easily see that $x \in R_A(x)$ and

$$\big(R_A \circ R_A\big)(x) = R_A\,[R_A(x)] = \bigcup_{x \in A} R_A(x) \subseteq R_A(x)$$

for all $x \in X$. Therefore, $\Delta_X \subseteq R_A$ and $R_A \circ R_A \subseteq R_A$, and thus R_A is a preorder relation on X.

Hence, by a basic theorem on preorder relations, it is clear that $S = \bigcap_{A \in \mathscr{T}_X} R_A$ is also a preorder relation on X. Moreover, we can note that, for any $x \in X$, we have

$$S(x) = \Big(\bigcap_{A \in \mathscr{T}_X} R_A\Big)(x) = \bigcap_{A \in \mathscr{T}_X} R_A(x) = \bigcap \big\{ A \in \mathscr{T}_X : \quad x \in A \big\}.$$

Furthermore, if $x \in X$ and $y \in \leq^{\infty}(x)$, then by using the inclusion $\leq \subseteq \leq^{\infty}$ and the transitivity of $\leq^{\infty}$, we can also easily see that

$$\mathrm{ub}_X(y) = \leq_X(y) \subseteq \leq [\leq^{\infty}(x)] \subseteq \leq^{\infty}[\leq^{\infty}(x)] = \big(\leq^{\infty} \circ \leq^{\infty}\big)(x) \subseteq \leq^{\infty}(x).$$

Therefore, $y \in \text{int}_X\big(\leq^\infty (x)\big)$. This shows that $\leq^\infty (x) \subseteq \text{int}_X\big(\leq^\infty (x)\big)$ and thus $\leq^\infty (x) \in \mathscr{T}_X$. Hence, since $x \in \leq^\infty (x)$ also holds, we can already infer that $S(x) \subseteq \leq^\infty (x)$. Therefore, $S \subseteq \leq^\infty$ is also true.

On the other hand, if $A \in \mathscr{T}_X$, then by Remark 18, for any $x \in A$, we have $\leq (x) = \text{ub}_X(x) \subseteq A = R_A(x)$. Therefore, $\leq \subseteq R_A$. Hence, since R_A is a preorder relation on X, we can already infer that $\leq^\infty \subseteq R_A^\infty = R_A$. Therefore, $\leq^\infty \subseteq S$, and thus the required assertion is also true.

Remark 56. Note that if X is a goset, then by using Theorem 16 from the above theorem, we can also see that $\leq^\infty (x) \in \mathscr{T}_X$ for all $x \in X$.

From Theorem 58, we can also immediately derive the following

Corollary 17. *For any goset X, the following assertions are equivalent:*

(1) X is proset,

(2) $\leq = \bigcap_{A \in \mathscr{T}_X} R_A$.

Now, according to the definitions of [21, 33], we may also have

Definition 11. A goset X is called *well-chained* if the inequality relation $\leq$ in it is well-chained in the sense that $\leq^\infty = X^2$.

Remark 57. By using the definition of $\leq^\infty$, the above property can be reformulated in a detailed form that for any $x, y \in X$, with $x \neq y$, there exists a finite sequence $(x_i)_{i=0}^n$ in X, with $x_0 = x$ and $x_n = y$, such that $x_{i-1} \leq x_i$ for all $i = 1, 2, \ldots, n$.

Remark 58. During the long evolution of the concept of “connected”, the denition of “chain connectedness”, and also that of “archwise connectedness”, has been replaced by the present “modern denition of connectedness”. (See Thron [62, p. 29] and Wilder [66].)

However, in the theory relator spaces, it has turned out that the latter, celebrated connectedness is a particular case of well-chainedness, and well-chainedness is a particular case of *simplicity*. Unfortunately, our fundamental works [20, 21, 31, 33] on on these subjects were also strongly rejected by the leading topologists working in the editorial boards of various mathematical journals.

In this respect, it is also worth mentioning that Császár [9] also observed that “the concept of a connected set belongs rather to the theory of generalized topological spaces instead of topology in the strict sense.” However, he has not quoted our former paper [33], despite that he knew that each increasing operation γ on $\mathscr{P}(X)$, with $\gamma(X) = X$, can be written in the form $\gamma = \text{int}_{\mathscr{R}}$ with some nonvoid relator $\mathscr{R}$ on X. (For the proof of this and some more general results, see [41] and the references therein.)

By using Definition 11, from Theorem 58, we can easily derive the following.

Theorem 59. *For a goset X, the following assertions are equivalent:*

(1) X *is well-chained,*
(2) $\mathscr{T}_X = \{\emptyset, X\}$,
(2) $\mathscr{F}_X = \{\emptyset, X\}$.

Proof. To see that (1) implies (2), note that, by Theorem 58, for any $x \in X$, we have

$$\leq^{\infty}(x) = \bigcap \{A \in \mathscr{T}_X : \quad x \in A\}.$$

Therefore, for any $A \in \mathscr{T}_X$ and $x \in A$, we have $\leq^{\infty}(x) \subseteq A$. Moreover, if (1) holds, then $\leq^{\infty} = X^2$, and thus $\leq^{\infty}(x) = X$ for all $x \in X$. Therefore, if $A \neq \emptyset$, then $A = X$, and thus (2) also holds.

Remark 59. This theorem shows that, analogously to Example 1, the families $\mathscr{T}_X$ and $\mathscr{F}_X$ in a well-chained goset X are also quite useless tools.

Now, in addition to Theorem 59, we can also easily prove the following.

Theorem 60. *For a proset X, the following assertions are equivalent:*

(1) X *is well-chained,*
(2) $\mathscr{E}_X = \{X\}$,
(3) $\mathscr{D}_X = \mathscr{P}(X) \setminus \{\emptyset\}$.

Proof. If (1) holds, then by Theorems 55 and 59, it is clear that (2) also holds. (Note that this implication can also be easily proved by using the corresponding definitions.)

On the other hand, if (2) holds, then by Remark 25, for any $x \in X$, we necessarily have $\operatorname{ub}_X(x) = X$, and thus $\leq(x) = X$. Therefore, $\leq = X^2$, and thus (1) also holds.

This shows that (1) and (2) are equivalent. Moreover, by Theorem 19, it is clear that (2) and (3) are always equivalent.

Remark 60. In [33], as a consequence of some other results, we have proved that if $X = X(\mathscr{R})$ is a relator space with $\mathscr{R} \neq \emptyset$ and $\operatorname{card}(X) > 1$, then X is paratopologically well-chained if and only if $\mathscr{E}_X = \{X\}$.

Moreover, X is paratopologically connected if and only if $\mathscr{E}_X \subseteq \mathscr{D}_X$. Therefore, the "hyperconnectedness," introduced by Levine [22] and studied by several further authors, is a particular case of our paratopological connectedness.

11 Comparisons of Inequalities

Because of the inclusion $\leq \subseteq \leq^{\infty}$, it is also of some interest to prove the following.

Theorem 61. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$, *the following assertions are equivalent :*

(1) $\leq_1 \subseteq \leq_2$,
(2) $\text{ub}_{X_1} \subseteq \text{ub}_{X_2}$,
(3) $\text{lb}_{X_1} \subseteq \text{lb}_{X_2}$.

Proof. If (1) holds, then by Remark 2, we have $\text{ub}_{X_1}(x) = \leq_1(x) \subseteq \leq_2(x) = \text{ub}_{X_2}(x)$ for all $x \in X$. Hence, by using Corollary 1, we can already infer that

$$\text{ub}_{X_1}(A) = \bigcap_{a \in A} \text{ub}_{X_1}(a) \subseteq \bigcap_{a \in A} \text{ub}_{X_2}(a) = \text{ub}_{X_2}(A)$$

for all $A \subseteq X$. Therefore, (2) also holds.

Conversely, if (2) holds, then in particular, we have

$$\text{ub}_{X_1}(x) = \text{ub}_{X_1}\big(\{x\}\big) \subseteq \text{ub}_{X_2}\big(\{x\}\big) = \text{ub}_{X_2}(x),$$

and hence $\leq_1(x) \subseteq \leq_2(x)$ for all $x \in X$. Therefore, (1) also holds.

This shows that (1) and (2) are equivalent. Hence, by using Theorem 1, we can easily see that (1) and (3) are also equivalent.

From this theorem, by Definition 8, it is clear that in particular we also have

Corollary 18. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$, *with* $\leq_1 \subseteq \leq_2$, *we have* $\mathscr{U}_{X_1} \subseteq \mathscr{U}_{X_2}$.

Proof. Namely, if $A \in \mathscr{U}_{X_1}$, then by Definition 8 we have $A \subseteq \text{ub}_{X_1}(A)$. Moreover, by Theorem 61, now we also have $\text{ub}_{X_1}(A) \subseteq \text{ub}_{X_2}(A)$. Therefore, $A \subseteq \text{ub}_{X_2}(A)$, and thus $A \in \mathscr{U}_{X_2}$ also holds.

Remark 61. Note if X is a reflexive and antisymmetric goset, then by Theorems 42 and 43 we have $\mathscr{U}_X = \{\{\emptyset\}\} \cup \{\{x\}\}_{x \in X}$.

Therefore, the converse of the above corollary need not be true even if in particular $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$ are posets.

However, by using Theorem 61, we can also easily prove the following.

Theorem 62. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$, *the following assertions are equivalent :*

(1) $\leq_1 \subseteq \leq_2$,
(2) $\text{int}_{X_2} \subseteq \text{int}_{X_1}$,
(3) $\text{cl}_{X_1} \subseteq \text{cl}_{X_2}$.

Proof. If $A \subseteq X$ and $x \in \operatorname{int}_{X_2}(A)$, then by Definition 2 we have $\operatorname{ub}_{X_2}(x) \subseteq A$. Moreover, if (1) holds, then by Theorem 61 we also have $\operatorname{ub}_{X_1}(x) \subseteq \operatorname{ub}_{X_2}(x)$. Therefore, $\operatorname{ub}_{X_1}(x) \subseteq A$, and thus $x \in \operatorname{int}_{X_1}(A)$ is also true. This, shows that $\operatorname{int}_{X_2}(A) \subseteq \operatorname{int}_{X_1}(A)$ for all $A \subseteq X$. Therefore, (2) also holds.

Moreover, if (2) holds, then by using Theorem 6 we can easily see that (3) also holds. Therefore, we need only show that (3) also implies (1). For this, note that if (3) holds, then in particular by Remark 12 we have

$$\operatorname{lb}_{X_1}(x) = \operatorname{cl}_{X_1}\big(\{x\}\big) \subseteq \operatorname{cl}_{X_2}\big(\{x\}\big) = \operatorname{lb}_{X_2}(x)$$

for all $x \in X$. Hence, by using Corollary 1, we can see that $\operatorname{lb}_{X_1}(A) \subseteq \operatorname{lb}_{X_2}(A)$ for all $A \subseteq X$. Therefore, $\operatorname{lb}_{X_1} \subseteq \operatorname{lb}_{X_2}$, and thus by Theorem 61 assertion (1) also holds.

From this theorem, by Definitions 3 and 4, it is clear that we also have

Corollary 19. *For any two gosets $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$, with $\leq_1 \subseteq \leq_2$, we have*

(1) $\mathscr{T}_{X_2} \subseteq \mathscr{T}_{X_1}$,
(2) $\mathscr{F}_{X_2} \subseteq \mathscr{F}_{X_1}$,
(3) $\mathscr{E}_{X_2} \subseteq \mathscr{E}_{X_1}$,
(4) $\mathscr{D}_{X_1} \subseteq \mathscr{D}_{X_2}$.

Proof. For instance, if $A \in \mathscr{D}_{X_1}$, then by Definition 4 we have $X = \operatorname{cl}_{X_1}(A)$. Moreover, by Theorem 62, now we also have $\operatorname{cl}_{X_1}(A) \subseteq \operatorname{cl}_{X_2}(A)$. Therefore, $X = \operatorname{cl}_{X_2}(A)$, and thus $A \in \mathscr{D}_{X_2}$ also holds. Therefore, (4) is true.

Now, by using the above results and Theorems 17 and 54, we can also prove

Theorem 63. *For any goset $X_1 = X(\leq_1)$ and proset $X_2 = X(\leq_2)$, the following assertions are equivalent:*

(1) $\leq_1 \subseteq \leq_2$,
(2) $\mathscr{T}_{X_2} \subseteq \mathscr{T}_{X_1}$,
(3) $\mathscr{F}_{X_2} \subseteq \mathscr{F}_{X_1}$.

Proof. If (1) holds, then by Corollary 19 assertion (2) also holds. Conversely, if (2) holds, then by Theorems 17 and 54 we have

$$\operatorname{int}_{X_2}(A) = \bigcup \mathscr{T}_{X_2} \cap \mathscr{P}(A) \subseteq \bigcup \mathscr{T}_{X_1} \cap \mathscr{P}(A) \subseteq \operatorname{int}_{X_1}(A)$$

for all $A \subseteq X$. Therefore, $\operatorname{int}_{X_2} \subseteq \operatorname{int}_{X_2}$, and thus by Theorem 62 assertion (1) also holds.

This shows that (1) and (2) are equivalent. Moreover, by Theorem 13, it is clear that (2) and (3) are always equivalent.

However, concerning fat and dense sets, we can only prove the following.

Theorem 64. *For any two gosets* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$*, the following assertions are equivalent :*

(1) $\mathscr{E}_{X_1} \subseteq \mathscr{E}_{X_2}$,

(2) $\mathscr{D}_{X_2} \subseteq \mathscr{D}_{X_1}$,

(3) There exists a function φ *of* X *to itself such that* $\leq_2 \circ \varphi \subseteq \leq_1$,

(4) There exists a relation R *of* X *to itself such that* $\leq_2 \circ R \subseteq \leq_1$.

Proof. By Remarks 2 and 25, for any $x \in X$, we have $\leq_1 (x) \in \mathscr{E}_{X_1}$. Therefore, if (1) holds, then we also have $\leq_1 (x) \in \mathscr{E}_{X_2}$. Hence, by using Remarks 2 and 25, we can infer that there exists $y \in X$ such that $\leq_2 (y) \subseteq \leq_1 (x)$.

Hence, by the axiom of choice, it is clear that there exists a function φ of X to itself such that $\leq_2 \big(\varphi(x)\big) \subseteq \leq_1 (x)$, and thus $(\leq_2 \circ \varphi)(x) \subseteq \leq_1 (x)$ for all $x \in X$. Therefore, (3) also holds.

On the other hand, if (3) holds, then by Remark 2 for any $x \in X$, we have $\text{ub}_{X_2}\big(\varphi(x)\big) = \leq_2 \big(\varphi(x)\big) \subseteq \leq_1 (x) = \text{ub}_{X_1}(x)$. Hence, by Remark 25, it is clear that (1) also holds.

Now, since (3) trivially implies (4), and (3) follows from (4) by choosing a selection function φ of R, it remains only to note that, by Theorem 18, assertions (1) and (2) are also equivalent.

Finally, we note that, by using the above theorem, we can also easily prove the following theorem whose converse seems not to be true.

Theorem 65. *If* $X_1 = X(\leq_1)$ *and* $X_2 = X(\leq_2)$ *are gosets, with* $\leq_1 \subseteq \leq_2$*, such that either* $\mathscr{E}_{X_1} \subseteq \mathscr{E}_{X_2}$ *or* $\mathscr{D}_{X_2} \subseteq \mathscr{D}_{X_1}$*, then there exists a function* φ *of* X *to itself such that* $\leq_1 = \leq_1 \circ \varphi^{\infty}$.

Proof. Now, by Theorem 64, there exists a function φ of X to itself such that $\leq_2 \circ \varphi \subseteq \leq_1$. Hence, by using that $\leq_1 \subseteq \leq_2$, we can already infer that

$$\leq_1 \circ \varphi \subseteq \leq_2 \circ \varphi \subseteq \leq_1 \subseteq \leq_2, \qquad \text{and thus} \qquad \leq_1 \circ \varphi^2 \subseteq \leq_2 \circ \varphi \subseteq \leq_1 .$$

Hence, by induction, it is clear that we actually have $\leq_1 \circ \varphi^n \subseteq \leq_1$ for all $n \in \mathbb{N}$. Moreover, we can also note that $\leq_1 \circ \varphi^0 = \leq_1 \circ \Delta_X = \leq_1$.

Hence, by using a basic theorem on relations, we can infer that

$$\leq_1 \circ \varphi^{\infty} = \leq_1 \circ \bigcup_{n=0}^{\infty} \varphi^n = \bigcup_{n=0}^{\infty} \leq_1 \circ \varphi^n \subseteq \bigcup_{n=0}^{\infty} \leq_1 = \leq_1 .$$

Thus, since $\leq_1 = \leq_1 \circ \varphi^0 \subseteq \leq_1 \circ \varphi^{\infty}$, the required equality is also true.

12 The Importance of the Preorder Closure and Complementation

From the inclusion $\leq\,\subseteq\,\leq^{\infty}$, by using Theorems 61 and 62 and the notation $X^{\infty} = X(\leq^{\infty})$, we can immediately derive the following.

Theorem 66. *For any goset X, we have*

(1) $\mathrm{ub}_X \subseteq \mathrm{ub}_{X^{\infty}}$,
(2) $\mathrm{lb}_X \subseteq \mathrm{lb}_{X^{\infty}}$,
(3) $\mathrm{int}_{X^{\infty}} \subseteq \mathrm{int}_X$,
(4) $\mathrm{cl}_X \subseteq \mathrm{cl}_{X^{\infty}}$.

Moreover, by using Corollary 19, Remark 56, and Theorem 13, we can also prove the following.

Theorem 67. *For any goset X, we have*

(1) $\mathscr{T}_X = \mathscr{T}_{X^{\infty}}$,
(2) $\mathscr{F}_X = \mathscr{F}_{X^{\infty}}$,
(3) $\mathscr{E}_{X^{\infty}} \subseteq \mathscr{E}_X$,
(4) $\mathscr{D}_X \subset \mathscr{D}_{X^{\infty}}$.

Proof. From Corollary 19, we can at once see that the inclusions (3), (4), and $\mathscr{T}_{X^{\infty}} \subseteq \mathscr{T}_X$ are true.

On the other hand, if $A \in \mathscr{T}_X$, then by Theorem 58 we have $\leq^{\infty}(x) \subseteq A$ for all $x \in A$. Hence, by Remark 18, we can see that $A \in \mathscr{T}_{X^{\infty}}$. Therefore, $\mathscr{T}_{X^{\infty}} \subseteq \mathscr{T}_X$, and thus (1) is also true. Hence, by Theorem 13, it is clear that (2) is also true.

Remark 62. Note that if X is as in Example 1, then $\mathscr{T}_X = \{\emptyset, X\}$, and thus by Theorems 59 and 60, we have $\mathscr{E}_{X^{\infty}} = \{X\}$. However, because of Remark 25, $\mathscr{E}_X$ is quite a large subfamily of $\mathscr{P}(X)$. Therefore, the equalities in (3) and (4) need not be true.

Now, by using Theorems 63 and 67 and Corollary 18, we can also prove

Theorem 68. *For any two gosets $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$, the following assertions are equivalent:*

(1) $\mathscr{T}_{X_2} \subseteq \mathscr{T}_{X_1}$,
(2) $\mathscr{F}_{X_2} \subseteq \mathscr{F}_{X_1}$,
(3) $\leq_1\,\subseteq\,\leq_2^{\infty}$,
(4) $\leq_1^{\infty}\,\subseteq\,\leq_2^{\infty}$.

Proof. If (1) holds, then by Theorem 67 we can see that $\mathscr{T}_{X_2^{\infty}} \subseteq \mathscr{T}_{X_1}$ also holds. Hence, by using Theorem 63, we can already infer that (3) also holds.

Moreover, if (3) holds, then by using the corresponding properties of the operation ∞, we can also easily see that $\leq_1^{\infty} \subseteq \leq_2^{\infty\,\infty} = \leq_2^{\infty}$, and thus (4) also holds.

On the other hand, if (4) holds, then because of $\leq_1 \subseteq \leq_1^{\infty}$ it is clear that (3) also holds. Moreover, if (3) holds, then by using Theorem 67 and Corollary 19, we can see that $\mathscr{T}_{X_2} = \mathscr{T}_{X_2^{\infty}} \subseteq \mathscr{T}_{X_1}$, and thus (1) also holds. Now, to complete the proof, it remains only to note that, by Theorem 13, assertions (1) and (2) are also equivalent.

Remark 63. From this theorem, we can at once see that, for any two gosets $X_1 = X(\leq_1)$ and $X_2 = X(\leq_2)$, we have

$$\mathscr{T}_{X_1} \subseteq^{-1} \mathscr{T}_{X_2} \iff X_1 \leq X_2^{\infty},$$

in the sense that $\leq_1 \subseteq \leq_2^{\infty}$.

This shows that, analogously to Remarks 7 and 16, the set-valued functions $\mathscr{T}$ and ∞ also form a Pataki connection.

Thus, the counterparts of the corresponding parts of Remarks 8 and 16 can also be stated. However, it would be more interesting to look for a generating Galois connection.

Now, by Theorems 64 and 65, we can also state the following two theorems.

Theorem 69. *For any goset X, the following assertions are equivalent:*

(1) $\mathscr{E}_X \subseteq \mathscr{E}_{X^{\infty}}$,
(2) $\mathscr{D}_{X^{\infty}} \subseteq \mathscr{D}_X$,
(3) there exists a function φ of X to itself such that $\leq^{\infty} \circ \varphi \subseteq \leq$,
(4) there exists a relation R of X to itself such that $\leq^{\infty} \circ R \subseteq \leq$.

Remark 64. Note that, by Theorem 67, we may write equality in the assertions (1) and (2) of the above theorem and also in the conditions of the following.

Theorem 70. *If X is a goset, such that $\mathscr{E}_X \subseteq \mathscr{E}_{X^{\infty}}$, or equivalently $\mathscr{D}_{X^{\infty}} \subseteq \mathscr{D}_X$, then there exists function φ of X to itself such that $\leq = \leq \circ \varphi^{\infty}$.*

Finally, we note that, by using the notation $X^c = X(\leq^c)$, we can also prove the following particular case of [47, Theorem 4.11], which in addition to the results of [17, 57] also shows the importance of complement relations.

Theorem 71. *For any goset X, we have*

(1) $\mathrm{lb}_X = \big(\mathrm{cl}_{X^c}\big)^c$,
(2) $\mathrm{cl}_X = \big(\mathrm{lb}_{X^c}\big)^c$.

Proof. By using Remarks 4 and 13, instead of Corollary 1 and Theorem 10, we can at once see that

$$\mathrm{lb}_X^c(A) = \mathrm{lb}_X(A)^c = \geq^c [A] = \mathrm{cl}_{X^c}(A)$$

for all $A \subseteq X$. Therefore, $\mathrm{lb}_X^c = \mathrm{cl}_{X^c}$, and thus (1) is also true.

Now, (2) can be immediately derived from (1) by writing X^c in place of X and applying complementation.

Remark 65. This theorem shows that the relations lb_X and cl_X are also equivalent tools in the goset X.

Hence, by Remarks 3 and 10, it is clear that the relations ub_X and int_X are also equivalent tools in the goset X.

Remark 66. By using Theorem 1 and Corollary 3, and the corresponding properties of inversion and complementations, the assertions (1) and (2) of Theorem 71 can be reformulated in several different forms.

For instance, as an immediate consequence of Theorem 71 and Corollary 3, we can at once state the following.

Corollary 20. *For any goset X, we have*

(1) $\mathrm{lb}_X = \mathrm{int}_{X^c} \circ \mathscr{C}$,

(2) $\mathrm{int}_X = \mathrm{lb}_{X^c} \circ \mathscr{C}$.

Remark 67. Analogously to Theorem 10, the above results also show that, despite Remark 2, there are cases when the relation lb_X is a more convenient tool in the goset X than ub_X.

13 Some Further Results on the Basic Tools

As some converses to Theorems 3, 9, 16, and 24, we can also easily prove the following theorems.

Theorem 72. *If Φ is a relation on $\mathscr{P}(X)$ to X, for some set X, such that*

$$\Phi\Big(\bigcup_{i\in I} A_i\Big) = \bigcap_{i\in I} \Phi\left(A_i\right)$$

for any family $(A_i)_{i\in I}$ subsets of X, then there exists a relation $\leq$ on X such that, under the notation $X = X(\leq)$, we have $\Phi = \mathrm{ub}_X$ $(\Phi = \mathrm{lb}_X)$.

Proof. For any $x, y \in X$, define $x \leq y$ if $y \in \Phi(x)$, where $\Phi(x) = \Phi\big(\{x\}\big)$. Then, by Remark 2, we have $\Phi(x) = \mathrm{ub}_X(x)$ for all $x \in X$. Hence, by using the assumed union-reversingness of Φ and Corollary 1, we can already see that

$$\Phi(A) = \bigcap_{a\in A} \Phi(a) = \bigcap_{a\in A} \mathrm{ub}_X(a) = \mathrm{ub}_X(A)$$

for all $A \subseteq X$. Therefore, $\Phi = \mathrm{ub}_X$ is also true.

This proves the first statement of the theorem. The second statement can be derived from the first one by using Theorem 1.

Theorem 73. *If Ψ is a relation on $\mathscr{P}(X)$ to X, for some set X, such that*

$$\Psi\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} \Psi\left(A_i\right)$$

for any family $(A_i)_{i\in I}$ *subsets of* X, *then there exists a relation* $\leq$ *on* X *such that, under the notation* $X = X(\leq)$, *we have* $\Psi = \mathrm{cl}_X$.

Proof. For any $x, y \in X$, define $x \leq y$ if $x \in \Psi(y)$, where $\Psi(y) = \Psi(\{y\})$. Then, by Remark 2, we have $\mathrm{lb}_X(y) = \Psi(y)$ for all $y \in X$. Hence, by using the assumed union preservingness of Ψ and Theorem 10, we can already see that

$$\Psi(A) = \bigcup_{a\in A} \Psi(a) = \bigcup_{a\in A} \mathrm{lb}_X(a) = \mathrm{cl}_X(A)$$

for all $A \subseteq X$. Therefore, the required equality is also true.

From this theorem, by using Corollary 3, we can easily derive the following.

Corollary 21. *If* Φ *is a relation on* $\mathscr{P}(X)$ *to* X, *for some set* X, *such that*

$$\Phi\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} \Phi\big(A_i\big)$$

for any family $\big(A_i\big)_{i\in I}$ *of subsets of* X, *then there exists a relation* $\leq$ *on* X *such that, under the notation* $X = X(\leq)$, *we have* $\Phi = \mathrm{int}_X$.

Proof. Define $\Psi = (\Phi \circ \mathscr{C})^c$. Then, by using the assumed intersection-preservingness of Φ and De Morgan's law, we can see that Ψ is an union-preserving relation on $\mathscr{P}(X)$ to X. Therefore, by Theorem 73, there exists a relation $\leq$ on X such that in the goset $X = X(\leq)$ we have $\Psi = \mathrm{cl}_X$. Hence, by using the definition of Ψ and Corollary 3, we can see that $\Phi = \big(\Psi \circ \mathscr{C}\big)^c = \big(\mathrm{cl}_X \circ \mathscr{C}\big)^c = \mathrm{int}_X$ also holds.

Theorem 74. *If* $\mathscr{A}$ *is a family of subsets of a set* X *such that* $\mathscr{A}$ *is closed under arbitrary unions and intersections, then there exists a preorder relation* $\leq$ *on* X *such that, under the notation* $X = X(\leq)$, *we have* $\mathscr{A} = \mathscr{T}_X$ $(\mathscr{A} = \mathscr{F}_X)$.

Proof. Define

$$\leq\, = \bigcap_{A\in A} R_A \qquad \text{where} \qquad R_A = A^2 \cup A^c \times X.$$

Then, from the proof of Theorem 58, we know that $\leq$ is a preorder relation on X such that, under the notation $X = X(\leq)$, for any $x \in X$ we have

$$\mathrm{ub}_X(x) = \leq(x) = \bigcap \big\{A \in \mathscr{A} : \quad x \in A\big\}.$$

Hence, since $\mathscr{A}$ is closed under arbitrary intersections, it is clear that $\mathrm{ub}_X(x) \in \mathscr{A}$ for all $x \in X$. Moreover, we can also note that $x \in \mathrm{ub}_X(x)$ for all $x \in X$.

Therefore, if $V \in \mathscr{T}_X$, that is, by Remark 18 we have $\mathrm{ub}_X(x) \subseteq V$ for all $x \in V$, then we necessarily have $V = \bigcup_{x\in V} \mathrm{ub}_X(x)$. Hence, since $\mathscr{A}$ is also closed under arbitrary unions, it is clear that $V \in \mathscr{A}$. Therefore, $\mathscr{T}_X \subseteq \mathscr{A}$.

Conversely, if $V \in \mathscr{A}$, then for any $x \in V$ we have

$$\text{ub}_X(x) = \bigcap \big\{ A \in \mathscr{A} : \quad x \in A \big\} \subseteq V .$$

Therefore, by Remark 18, we have $V \in \mathscr{T}_X$. Thus, $\mathscr{A} \subseteq \mathscr{T}_X$ also holds.

This proves that $\mathscr{A} = \mathscr{T}_X$, and thus the first statement of the theorem is true. The second statement of the theorem can be derived from the first one by using Theorem 14.

Remark 68. In principle, the first statement of the above theorem can also be proved with the help of Corollary 21 . However, this proof requires an intimate connection between interior operations and families of sets.

For this, one can note that if Φ is a relation on $\mathscr{P}(X)$ to X such that

$$\Phi(B) = \bigcup \big(\mathscr{A} \cap \mathscr{P}(B) \big)$$

for all $B \subseteq X$, then by this definition and the assumed union property of $\mathscr{A}$, we have

(a) $\mathscr{A} = \big\{ B \subseteq X : \quad B = \Phi(B) \big\}$, (b) $\Phi(B) \in \mathscr{A} \cap \mathscr{P}(B)$ for all $B \subseteq X$.

Moreover, by using (b), the assumed intersection property of $\mathscr{A}$ and the definition of Φ, we can see that Φ is union preserving.

However, it is now more important to note that, analogously to Theorem 74, we also have the following.

Theorem 75. *If $\mathscr{A}$ is a nonvoid stack in X, for some set X, having a base $\mathscr{B}$ with* $\text{card}(\mathscr{B}) \leq \text{card}(X)$, *then there exists a relation $\leq$ on X such that, under the notation $X = X(\leq)$, we have $\mathscr{A} = \mathscr{E}_X$.*

Proof. Since $\text{card}(\mathscr{B}) \leq \text{card}(X)$, there exists an injective function φ of $\mathscr{B}$ onto a subset Y of X. Choose $B \in \mathscr{B}$ and define a relation $\leq$ on X such that

$$\leq (x) = \varphi^{-1}(x) \quad \text{if} \quad x \in Y \qquad \text{and} \qquad \leq (x) = B \quad \text{if} \quad x \in Y^c .$$

Then, under the notation $X = X(\leq)$, we evidently have

$$\mathscr{B} = \big\{ \text{ub}_X(x) : \quad x \in X \big\} .$$

Hence, since $\mathscr{B}$ is a base of $\mathscr{A}$, we can already infer that

$$\mathscr{A} = \big\{ A \subseteq X : \quad \exists\, x \in X : \quad \text{ub}_X(x) \subseteq A \big\} = \mathscr{E}_X .$$

Remark 69. Now, a corresponding theorem for the family $\mathscr{D}_X$ should, in principle, be derived from the above theorem by using either Theorem 18 or 19 .

However, it would now be even more interesting to prove a counterpart of Theorems 74 and 75 for the family $\mathscr{U}_X$.

14 Increasing Functions

Increasing functions are usually called isotone, monotone, or order-preserving in algebra. Moreover, in [11, p. 186] even the extensive maps are called increasing. However, we prefer to use the following terminology of analysis [38, p. 128].

Definition 12. If f is a function of one goset X to another Y, then we say that:

(1) f is *increasing* if $u \leq v$ implies $f(u) \leq f(v)$ for all $u, v \in X$.
(2) f is *strictly increasing* if $u < v$ implies $f(u) < f(v)$ for all $u, v \in X$.

Remark 70. Quite similarly, the function f may, for instance, be called *decreasing* if $u \leq v$ implies $f(v) \leq f(u)$ for all $u, v \in X$.

Thus, we can note that f is a decreasing function of X to Y if and only if it is an increasing function of X to the dual Y^{-1} of Y.

Therefore, the study of decreasing functions can be traced back to that of the increasing ones. The following two obvious theorems show that almost the same is true in connection with the strictly increasing ones.

Theorem 76. *If f is an injective, increasing function of one goset X to another Y, then f is strictly increasing.*

Remark 71. Conversely, we can at once see that if f is a strictly increasing function of an arbitrary goset X to a reflexive one Y, then f is increasing.

Moreover, we can also easily prove the following

Theorem 77. *If f is a strictly increasing function of a linear goset X to an arbitrary one Y, then f is injective.*

Proof. If $u, v \in X$ such that $u \neq v$, then by Remark 47 we have either $u < v$ or $v < u$. Hence, by using the strict increasingness of f, we can already infer that either $f(u) < f(v)$ or $f(v) < f(u)$, and thus $f(u) \neq f(v)$.

Now, as an immediate consequence of the above results, we can also state

Corollary 22. *For a function f of a linear goset X to a reflexive one Y, the following assertions are equivalent:*

(1) f is strictly increasing,
(2) f is injective and increasing.

In this respect, the following is also worth proving.

Theorem 78. *If f is a strictly increasing function of a linear goset X onto an antisymmetric one Y, then f^{-1} is a strictly increasing function of Y onto X.*

Proof. From Theorem 77, we know that f is injective. Hence, since $f[X] = Y$, we can see that f^{-1} is a function of Y onto X. Therefore, we need only show that f^{-1} is also strictly increasing.

For this, suppose that $z, w \in Y$ such that $z < w$. Define $u = f^{-1}(z)$ and $v = f^{-1}(w)$. Then, $u, v \in X$ such that $z = f(u)$ and $w = f(v)$. Hence, since $z \neq w$, we can also see that $u \neq v$. Moreover, by Remark 47, we have either $u < v$ or $v < u$. However, if $v < u$, then by the strict increasingness of f we also have $f(v) < f(u)$, and thus $w < z$. Hence, by using the inequality $z < w$ and the antisymmetry of Y, we can already infer that $z = w$. This contradiction proves that $u < v$, and thus $f^{-1}(z) < f^{-1}(w)$.

Hence, by using Theorem 76 and Remark 71, we can immediately derive

Corollary 23. *If f is an injective, increasing function of a reflexive, linear goset X onto an antisymmetric one Y, then f^{-1} is an injective, increasing function of Y onto X.*

Analogously to [58], we shall now also use the following.

Definition 13. If φ is an unary operation on a goset X, then we say that:

(1) φ is *extensive (intensive)* if $\Delta_X \leq \varphi$ $\big(\varphi \leq \Delta_X\big)$.
(2) φ is *upper (lower) semi-idempotent* if $\varphi \leq \varphi^2$ $\big(\varphi^2 \leq \varphi\big)$.

Remark 72. Moreover, φ may be naturally called *upper (lower) semi-involutive* if φ^2 is extensive (intensive). That is, $\Delta_X \leq \varphi^2$ $(\varphi^2 \leq \Delta_X)$.

Remark 73. In this respect, it is also worth noticing that φ is upper (lower) semi-idempotent if and only if its restriction to its range is extensive (intensive). Therefore, if φ is extensive (intensive), then φ is upper (lower) semi-idempotent.

The importance of extensive operations is also apparent from the following.

Theorem 79. *If φ is a strictly increasing operation on a min-complete, antisymmetric goset X, then φ is extensive.*

Proof. If φ is not extensive, then the set $A = \{x \in X : \ x \not\leq \varphi(x)\}$ is not void. Thus, by the min-completeness of X, there exists $a \in \min_X(A)$. Hence, by the definition of $\min_X$, we can see that $a \in A$ and $a \in \mathrm{lb}_X(A)$. Thus, in particular, by the definition of A, we have $a \not\leq \varphi(a)$. Hence, by using Corollary 12 and Theorem 49, we can infer that $\varphi(a) < a$. Thus, since φ is strictly increasing, we also have $\varphi\big(\varphi(a)\big) < \varphi(a)$. Hence, by using Theorem 48, we can infer that $\varphi(a) \not\leq \varphi\big(\varphi(a)\big)$. Thus, by the definition of A, we also have $\varphi(a) \in A$. Hence, by using that $a \in \mathrm{lb}_X(A)$, we can infer that $a \leq \varphi(a)$. This contradiction shows that φ is extensive.

Remark 74. To feel the importance of extensive operations, it is also worth noticing that if φ is an extensive operation on an antisymmetric goset, then each maximal element x of X is already a fixed point of φ in the sense that $\varphi(x) = x$.

This fact has also been strongly emphasized by Brøndsted [6]. Moreover, fixed point theorems for extensive maps (which are sometimes called expansive, progressive, increasing, or inflationary) were also proved in [19], [11, p. 188], and [29].

The following theorem shows that, in contrast to the injective, increasing functions, the inverse of an injective, extensive operation need not be extensive.

Theorem 80. *If φ is an injective, extensive operation on an antisymmetric goset X such that $X = \varphi[X]$ and φ^{-1} is also extensive, then $\varphi = \Delta_X$.*

Proof. By the extensivity of φ and φ^{-1}, for every $x \in X$, we have $x \leq \varphi(x)$ and $\varphi(x) \leq \varphi^{-1}\big(\varphi(x)\big)$. Hence, by noticing that $\varphi^{-1}\big(\varphi(x)\big) = x$ and using the antisymmetry of X, we can already infer that $\varphi(x) = x$, and thus $\varphi(x) = \Delta_X(x)$. Therefore, the required equality is also true.

From this theorem, by using Theorems 78 and 79, we can immediately derive

Corollary 24. *If φ is a strictly increasing operation on a min-complete, antisymmetric goset X such that $X = \varphi[X]$, then $\varphi = \Delta_X$.*

Proof. Now, from Corollary 12 and Theorem 78, we can see that φ^{-1} is also strictly increasing. Thus, by Theorem 79, both φ and φ^{-1} are extensive. Therefore, by Theorem 80, the required equality is also true.

In general, the idempotent operations are quite different from both upper and lower semi-idempotent ones. However, we may still naturally have the following.

Definition 14. An increasing, extensive (intensive) operation is called a *preclosure (preinterior) operation*. And, a lower semi-idempotent (upper semi-idempotent) preclosure (preinterior) operation is called a *closure (interior) operation*.

Moreover, an extensive (intensive) lower semi-idempotent (upper semi-idempotent) operation is called a *semiclosure (semi-interior) operation*. While, an increasing and upper (lower) semi-idempotent operation is called an *upper (lower) semimodification operation*.

Remark 75. Thus, φ is, for instance, an interior operation on a goset X if and only if it is a closure operation on the dual X^{-1} of X.

15 Algebraic Properties of Increasing Functions

Concerning increasing functions, we can also prove the following.

Theorem 81. *For a function f of one goset X to another Y, the following assertions are equivalent:*

(1) *f is increasing,*
(2) *$f[\mathrm{ub}_X(x)] \subseteq \mathrm{ub}_Y\big(f(x)\big)$ for all $x \in X$,*
(3) *$f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$ for all $A \subseteq X$.*

Proof. If $A \subseteq X$ and $y \in f[\mathrm{ub}_X(A)]$, then there exists $x \in \mathrm{ub}_X(A)$ such that $y = f(x)$. Thus, for any $a \in A$, we have $a \leq x$. Hence, if (1) holds, we can infer that $f(a) \leq f(x)$, and thus $f(a) \leq y$. Therefore, $y \in \mathrm{ub}_Y\big(f[A]\big)$, and thus (3) also holds.

The remaining implications $(3) \Longrightarrow (2) \Longrightarrow (1)$ are even more obvious.

From this theorem, by using Definition 8, we can immediately derive

Corollary 25. *If f is an increasing function of one goset X to another Y, then for any $A \in \mathscr{U}_X$ we have $f[A] \in \mathscr{U}_Y$.*

Proof. Namely, if $A \in \mathscr{U}_X$, then by Definition 8, we have $A \subseteq \mathrm{ub}_X(A)$. Hence, by using Theorem 81, we can infer that $f[A] \subseteq f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$. Thus, by Definition 8, we also have $f[A] \in \mathscr{U}_Y$.

Moreover, by using Theorem 81, we can also prove the following.

Theorem 82. *If f is an increasing function of one goset X onto another Y, then for any $B \subseteq Y$ we have*

$$\mathrm{ub}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{ub}_Y(B)].$$

Proof. Now, by Theorem 81 and a basic theorem on relations, we have

$$f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] \subseteq \mathrm{ub}_Y\left(f\left[f^{-1}[B]\right]\right) = \mathrm{ub}_Y\left(\big(f \circ f^{-1}\big)[B]\right).$$

Moreover, by using that Y is the range of f, we can easily see that $\Delta_Y \subseteq f \circ f^{-1}$. Hence, we can immediately infer that $B \subseteq \big(f \circ f^{-1}\big)[B]$, and thus also

$$\mathrm{ub}_Y\left(\big(f \circ f^{-1}\big)[B]\right) \subseteq \mathrm{ub}_Y(B).$$

Therefore, we actually have $f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] \subseteq \mathrm{ub}_Y(B)$, and thus also

$$\big(f^{-1} \circ f\big)\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] = f^{-1}\left[f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right]\right] \subseteq f^{-1}[\mathrm{ub}_Y(B)].$$

Moreover, since X is the domain of f, we can note that $\Delta_X \subseteq f^{-1} \circ f$, and thus

$$\mathrm{ub}_X\big(f^{-1}[B]\big) \subseteq \big(f^{-1} \circ f\big)\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right].$$

Therefore, the required inclusion is also true.

Now, as a partial converse to this theorem, we can also prove the following.

Theorem 83. *If f is an injective function of one goset X to another Y such that*

$$\mathrm{ub}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{ub}_Y(B)]$$

for all $B \subseteq X$, then f is increasing.

Proof. Now, by some basic theorems on relations, for any $B \subseteq Y$, we also have

$$f\left[\mathrm{ub}_X\big(f^{-1}[B]\big)\right] \subseteq f\left[f^{-1}[\mathrm{ub}_Y(B)]\right] = \big(f \circ f^{-1}\big)[\mathrm{ub}_Y(B)].$$

Moreover, since f is a function, we also have $f \circ f^{-1} \subseteq \Delta_X$, and thus also $\big(f \circ f^{-1}\big)\,[\,\mathrm{ub}_Y(B)\,] \subseteq \mathrm{ub}_Y(B)$. Therefore, we actually have

$$f\big[\,\mathrm{ub}_X\big(f^{-1}[\,B\,]\big)\,\big] \subseteq \mathrm{ub}_Y(B)\,.$$

Hence, it is clear that, for any $A \subseteq X$, we have

$$f\big[\,\mathrm{ub}_X\big(\big(f^{-1}\circ f\big)\,[\,A\,]\big)\,\big] = f\big[\,\mathrm{ub}_X\big(f^{-1}\big[\,f\,[\,A\,]\,\big]\big)\,\big] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)\,.$$

Moreover, by using that f is injective, we can note that $f^{-1} \circ f \subseteq \Delta_X$, and thus also $\big(f^{-1}\circ f\big)\,[\,A\,] \subseteq A$. Hence, we can infer that $\mathrm{ub}_x(A) \subseteq \mathrm{ub}_X\big(\big(f^{-1}\circ f\big)\,[\,A\,]\big)$, and thus also

$$f\,[\,\mathrm{ub}_X(A)\,] \subseteq f\big[\,\mathrm{ub}_X\big(\big(f^{-1}\circ f\big)\,[\,A\,]\big)\,\big]\,.$$

Therefore, we actually have

$$f\,[\,\mathrm{ub}_X(A)\,] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)\,.$$

Hence, by Theorem 81, we can already see that f is increasing.

Remark 76. Note that f is an increasing function of X to Y if and only if it is an increasing function of X^{-1} to Y^{-1}.

Therefore, in the above theorems, we may write lb in place of ub . However, because of Theorems 29 and 4, we cannot write sup instead of ub .

Despite this, by using Theorem 81, we can also prove the following.

Theorem 84. *For a function f of a reflexive goset X to an arbitrary one Y, the following assertions are equivalent :*

(1) f *is increasing,*
(2) $f\,[\,\max_X(A)\,] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$,
(3) $f\,[\,\max_X(A)\,] \subseteq \max_Y\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$,
(4) $f\,[\,\max_X(A)\,] \subseteq \mathrm{ub}_Y\big(f\,[\,A\,]\big)$ *for all* $A \subseteq X$ *with* $\mathrm{card}(A) \leq 2$.

Proof. If (1) holds, then by Theorem 81 and a basic theorem on relations, for any $A \subseteq X$, we have

$$\begin{aligned} f\,[\,\max_X(A)\,] &= f\,[\,A \cap \mathrm{ub}_X(A)\,] \subseteq f\,[\,A\,] \cap f\,[\,\mathrm{ub}_X(A)\,] \\ &\subseteq f\,[\,A\,] \cap \mathrm{ub}_Y\big(f\,[\,A\,]\big) = \max_Y\big(f\,[\,A\,]\big). \end{aligned}$$

Therefore, (3) also holds even if X is not assumed to be reflexive.

Thus, since the implication (3) $\Longrightarrow$ (2) $\Longrightarrow$ (4) trivially hold, we need only show that (4) also implies (1). For this, note that if $u,\ v \in X$ such that $u \leq v$, then by taking $A = \{u, v\}$ and using the reflexivity of X we can see that $v \in \mathrm{ub}_X(A)$, and thus

$$v \in A \cap \mathrm{ub}_X(A) = \max_X(A).$$

Hence, if (4) holds, we can infer that

$$f(v) \in f[\max_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big) = \mathrm{ub}_Y\big(\{f(u), f(v)\}\big).$$

Thus, in particular $f(u) \leq f(v)$, and thus (1) also holds.

Now, as a useful consequence of this theorem, we can also easily prove

Corollary 26. *If f is a function on a reflexive goset X to an arbitrary one Y such that*

$$f[\sup_X(A)] \subseteq \sup_Y\big(f[A]\big)$$

for all $A \subseteq X$ with $\mathrm{card}(A) \leq 2$, then f is already increasing.

Proof. If A is as above, then by Theorems 29 and 32 we have

$$f[\max_X(A)] \subseteq f[\sup_X(A)] \subseteq \sup_Y\big(f[A]\big) \subseteq \mathrm{ub}_Y\big(f[A]\big).$$

Therefore, by Theorem 84, the required assertion is also true.

Because of Theorems 29 and 4, a converse of this corollary is certainly not true. However, by using Theorem 81, we can also prove the following two theorems.

Theorem 85. *If f is an increasing function of one goset X to another Y, then for any $A \subseteq X$ we have*

$$\mathrm{lb}_Y\big(\mathrm{ub}_Y\big(f[A]\big)\big) \subseteq \mathrm{lb}_Y\big(f[\mathrm{ub}_X(A)]\big).$$

Proof. Now, by Theorem 81, we have $f[\mathrm{ub}_X(A)] \subseteq \mathrm{ub}_Y\big(f[A]\big)$. Hence, by using Theorem 4, we can immediately derive the required inclusion.

Theorem 86. *If f is an increasing function of one sup-complete, antisymmetric goset X to another Y, then for any $A \subseteq X$ we have*

$$\sup_Y\big(f[A]\big) \leq f\big(\sup_X(A)\big).$$

Proof. If $\alpha = \sup_X(A)$, then by Theorems 29 and 45 and, and the usual identification of singletons with their elements, we also have $\alpha \in \mathrm{ub}_X(A)$, and thus $f(\alpha) \in f[\mathrm{ub}_X(A)]$. Hence, by using Theorem 81, we can already infer that $f(\alpha) \in \mathrm{ub}_Y\big(f[A]\big)$.

While, if $\beta = \sup_Y\big(f[A]\big)$, then by Theorems 29 and 45, and the usual identification of singletons with their elements, we also have $\beta \in \mathrm{lb}_Y\big(\mathrm{ub}_Y\big(f[A]\big)\big)$. Hence, by using that $f(\alpha) \in \mathrm{ub}_Y\big(f[A]\big)$, we can already infer that $\beta \leq f(\alpha)$, and thus the required equality is also true.

By using the dual of Theorem 81 mentioned in Remark 76, we can quite similarly prove the following theorem which can also be derived from Theorem 86 by dualization.

Theorem 87. *If f is an increasing function of one inf-complete, antisymmetric goset X to another Y, then for any $A \subseteq X$ we have*

$$f\big(\inf_X(A)\big) \leq \inf_Y\big(f[A]\big).$$

Remark 77. Note that, by Theorem 34, in the latter theorem we may also write sup-complete instead of inf-complete.

Therefore, as an immediate consequence of Theorems 86 and 87, we can state

Corollary 27. *If f is an increasing function of a sup-complete, antisymmetric goset X to a sup-complete, transitive and antisymmetric goset Y, and A is a nonvoid subset of X such that $f\big(\inf_X(A)\big) = f\big(\sup_X(A)\big)$, then*

$$\inf_Y\big(f[A]\big) = f\big(\inf_X(A)\big) \qquad \textit{and} \qquad \sup_Y\big(f[A]\big) = f\big(\sup_X(A)\big).$$

16 Topological Properties of Increasing Functions

In principle, the following theorem can be derived from the dual Theorem 81 by using Theorem 71. However, it is now more convenient to give a direct proof.

Theorem 88. *For a function f of one goset X to another Y, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big)$ *for all* $A \subseteq X$,
(3) $\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B)]$ *for all* $B \subseteq B \subseteq Y$,
(4) $f^{-1}[\mathrm{int}_Y(B)] \subseteq \mathrm{int}_X\big(f^{-1}[B]\big)$ *for all* $B \subseteq Y$.

Proof. If $A \subseteq X$ and $y \in f[\mathrm{cl}_X(A)]$, then there exists $x \in \mathrm{cl}_X(A)$ such that $y = f(x)$. Thus, by Definition 2, we have $\mathrm{ub}_X(x) \cap A \neq \emptyset$. Therefore, there exists $a \in A$ such that $a \in \mathrm{ub}_X(x)$, and thus $x \leq a$. Hence, if (1) holds, we can infer that $f(x) \leq f(a)$, and thus $f(a) \in \mathrm{ub}_Y\big(f(x)\big) = \mathrm{ub}_Y(y)$. Now, since $f(a) \in f[A]$ also holds, we can already see that $f(a) \in \mathrm{ub}_Y(y) \cap f[A]$, and thus $\mathrm{ub}_Y(y) \cap f[A] \neq \emptyset$. Therefore, by Definition 2, we also have $y \in \mathrm{cl}_Y\big(f[A]\big)$. This shows that $f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big)$, and thus (2) also holds.

While, if $B \subseteq Y$, then $f^{-1}[B] \subseteq X$. Therefore, if (2) holds, then we have

$$f\big[\mathrm{cl}_X(f^{-1}[B])\big] \subseteq \mathrm{cl}_Y\big(f\big[f^{-1}[B]\big]\big) = \mathrm{cl}_Y\big(\big(f \circ f^{-1}\big)[B]\big).$$

Moreover, since f is a function, we can easily see that $f \circ f^{-1} \subseteq \Delta_Y$, and thus $\big(f \circ f^{-1}\big)[B] \subseteq B$. Hence, by using Theorem 11, we can infer that

$$\mathrm{cl}_Y\big(\big(f \circ f^{-1}\big)[B]\big) \subseteq \mathrm{cl}_Y(B).$$

Therefore, we actually have $f\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big] \subseteq \mathrm{cl}_Y(B)$, and thus also

$$\big(f^{-1} \circ f\big)\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big] = f^{-1}\big[f\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big]\big] \subseteq f^{-1}\big[\mathrm{cl}_Y(B)\big].$$

Moreover, since X is the domain of f, we can note that $\Delta_X \subseteq f^{-1} \circ f$, and thus

$$\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq \big(f^{-1} \circ f\big)\big[\mathrm{cl}_X\big(f^{-1}[B]\big)\big].$$

Therefore, we actually have $\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B)]$, and thus (3) also holds.

On the other hand, if $B \subseteq Y$, then by using Theorem 6 and a basic fact on inverse images, we can also see that

$$f^{-1}[\mathrm{int}_Y(B)] = f^{-1}\big[\mathrm{cl}_Y(B^c)^c\big] = f^{-1}\big[\mathrm{cl}_Y(B^c)\big]^c.$$

Moreover, if (3) holds, then we can also see that $\mathrm{cl}_X\big(f^{-1}[B^c]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B^c)]$, and thus

$$f^{-1}[\mathrm{cl}_Y(B^c)]^c \subseteq \mathrm{cl}_X\big(f^{-1}[B^c]\big)^c = \mathrm{cl}_X\big(f^{-1}[B]^c\big)^c = \mathrm{int}_X\big(f^{-1}[B]\big).$$

This shows that $f^{-1}[\mathrm{int}_Y(B)] \subseteq \mathrm{int}_X\big(f^{-1}[B]\big)$, and thus (4) also holds.

Now, it remains to show that (4) also implies (1). For this, note that, by Definition 2, for any $x \in X$ we have $f(x) \in \mathrm{int}_Y\big(\mathrm{ub}_Y\big(f(x)\big)\big)$, and thus

$$x \in f^{-1}\big(f(x)\big) \subseteq f^{-1}\big[\mathrm{int}_Y\big(\mathrm{ub}_Y\big(f(x)\big)\big)\big].$$

Moreover, if (4) holds, then we also have

$$f^{-1}\big[\mathrm{int}_Y\big(\mathrm{ub}_Y\big(f(x)\big)\big)\big] \subseteq \mathrm{int}_X\big(f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]\big).$$

This shows that $x \in \mathrm{int}_X\big(f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]\big)$, and thus by Definition 2 we have $\mathrm{ub}_X(x) \subseteq f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]$. Hence, we can already infer that

$$f[\mathrm{ub}_X(x)] \subseteq f\big[f^{-1}\big[\mathrm{ub}_Y\big(f(x)\big)\big]\big] = \big(f \circ f^{-1}\big)\big[\mathrm{ub}_Y\big(f(x)\big)\big] \subseteq \mathrm{ub}_Y\big(f(x)\big).$$

Therefore, by Theorem 81, assertion (1) also holds.

From this theorem, by using Definition 3, we can immetiately derive

Corollary 28. *If f is an increasing function of one goset X to another Y, then*

(1) $B \in \mathscr{T}_Y$ *implies* $f^{-1}[B] \in \mathscr{T}_X$,
(2) $B \in \mathscr{F}_Y$ *implies* $f^{-1}[B] \in \mathscr{F}_X$.

Proof. If $B \in \mathscr{T}_Y$, then by Definition 3 we have $B \subseteq \operatorname{int}_Y(B)$. Hence, by using Theorem 88 and the increasingness of f, we can already infer that

$$f^{-1}[B] \subseteq f^{-1}[\operatorname{int}_Y(B)] \subseteq \operatorname{int}_X\big(f^{-1}[B]\big).$$

Therefore, by Definition 3, we also have $f^{-1}[B] \in \mathscr{T}_X$.

This shows that (1) is true. Moreover, by using Theorem 13, we can easily see that (1) and (2) are equivalent even if f is not assumed to be increasing.

For instance, if $B \in \mathscr{F}_Y$, then by Theorem 13, we have $B^c \in \mathscr{T}_X$. Hence, if (1) holds, we can infer $f^{-1}[B^c] \in \mathscr{T}_X$. Now, by using that $f^{-1}[B^c] = f^{-1}[B]^c$, we can already see that $f^{-1}[B]^c \in \mathscr{T}_X$, and thus by Theorem 13 we also have $f^{-1}[B] \in \mathscr{F}_X$. Therefore, (2) also holds.

Remark 78. Moreover, if f is as in the above corollary, then by using the assertion (2) of Theorem 88 we can immediately see that if $A \subseteq X$ such that $f[A] \in \mathscr{F}_Y$, then $f[\operatorname{cl}_X(A)] \subseteq f[A]$. Note that this fact can also be derived from Corollary 28.

However, it is now more important to note that, in addition to the Corollary 28, we can also prove the following.

Theorem 89. *For a function f of a goset X to a proset Y, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $B \in \mathscr{T}_Y$ *implies* $f^{-1}[B] \in \mathscr{T}_X$,
(3) $B \in \mathscr{F}_Y$ *implies* $f^{-1}[B] \in \mathscr{F}_X$.

Proof. Now, by Corollary 28 and its proof, we need actually show only that (3) also implies (1). For this, note that if $B \subseteq Y$, then by Corollary 14 we have $\operatorname{cl}_Y(B) \in \mathscr{F}_Y$. Hence, if (3) holds, we can infer that $f^{-1}[\operatorname{cl}_Y(B)] \in \mathscr{F}_X$. Therefore, by Definition 3, we have

$$\operatorname{cl}_X\big(f^{-1}[\operatorname{cl}_Y(B)]\big) \subseteq f^{-1}[\operatorname{cl}_Y(B)].$$

Moreover, by Corollary 13, now we also have $B \subseteq \operatorname{cl}_Y(B)$, and thus also $f^{-1}[B] \subseteq f^{-1}[\operatorname{cl}_Y(B)]$. Hence, by using Theorem 11, we can infer that

$$\operatorname{cl}_X\big(f^{-1}[B]\big) \subseteq \operatorname{cl}_X\big(f^{-1}[\operatorname{cl}_Y(B)]\big).$$

This shows that

$$\mathrm{cl}_X\big(f^{-1}[B]\big) \subseteq f^{-1}[\mathrm{cl}_Y(B)]\,.$$

Therefore, by Theorem 88, assertion (1) also holds.

Remark 79. Note that the assertion (2) of Theorem 88, and the assertions (3) of Theorems 81 and 84, are more natural than the assertions (3) and (4) of Theorem 88 and the assertions (2) and (3) of Theorem 89.

Namely, the assertion (2) of Theorem 88, in a detailed form, means only that, for any $A \subseteq X$, the inclusion $x \in \mathrm{cl}_X((A)$ implies $f(x) \in \mathrm{cl}_Y\big(f[A]\big)$. That is, if x is "near" to A in X, then $f(x)$ is also "near" to $f[A]$ in Y.

Actually, the nearness of one set to another is an even more natural concept than that of a point to a set. Note that, according to a general definition of Száz [47], for any two subsets A and B of a goset X, we have $B \in \mathrm{Cl}_X(A)$ if and only if $\mathrm{cl}_X(A) \cap B \neq \emptyset$.

Now, by using Theorem 88, we can also prove the following.

Theorem 90. *If f is an increasing function of one goset X onto another Y, then*

(1) $A \in \mathscr{D}_X$ *implies* $f[A] \in \mathscr{D}_Y$,
(2) $B \in \mathscr{E}_Y$ *implies* $f^{-1}[B] \in \mathscr{E}_X$.

Proof. If $A \in \mathscr{D}_X$, then by Definition 4 we have $X = \mathrm{cl}_X(A)$. Hence, by using Theorem 88 and our assumptions on f, we can already infer that

$$Y = f[X] = f[\mathrm{cl}_X(A)] \subseteq \mathrm{cl}_Y\big(f[A]\big),$$

and thus $Y = \mathrm{cl}_Y\big(f[A]\big)$. Therefore, by Definition 4, we also have $f[A] \in \mathscr{D}_Y$.

This shows that (1) is true. Moreover, by using Theorem 19, we can easily see that (1) and (2) are equivalent even if f is not assumed to be increasing and onto Y.

For instance, if $A \in \mathscr{D}_X$ and (1) holds, then $f[A] \in \mathscr{D}_Y$. Therefore, if $B \in \mathscr{E}_Y$, then by Theorem 19 we have $f[A] \cap B \neq \emptyset$. Hence, it follows that $A \cap f^{-1}[B] \neq \emptyset$. Therefore, by Theorem 19, we have $f^{-1}[B] \in \mathscr{E}_X$, and thus (2) also holds.

Remark 80. Moreover, if f is as in the above theorem, then by using the assertion (3) of Theorem 88 we can also easily see that if $B \subseteq Y$ such that $f^{-1}[B] \in \mathscr{D}_X$, then $B \in \mathscr{D}_Y$. However, this fact can be more easily derived from Theorem 90.

17 Algebraic Properties of Closure Operations

Theorem 91. *If φ is a closure operation on an inf-complete, antisymmetric goset X, then for any $A \subseteq X$ we have*

$$\inf_X\big(\varphi[A]\big) = \varphi\left(\inf_X\big(\varphi[A]\big)\right).$$

Proof. Now, by Theorem 87, we have $\varphi\big(\inf_X(A)\big) \leq \inf_X\big(\varphi[A]\big)$. Hence, by writing $\varphi[A]$ in place of A, we can see that

$$\varphi\,\big(\inf_X\big(\varphi[A]\big)\big) \leq \inf_X\,(\varphi[\varphi[A]])\;.$$

Moreover, by using the antisymmetry of X, we can see that φ is now idempotent. Therefore, $\varphi[\varphi[A]] = (\varphi\circ\varphi)[A] = \varphi^2[A] = \varphi[A]$. Thus, we actually have

$$\varphi\,\big(\inf_X\big(\varphi[A]\big)\big) \leq \inf_X\,(\varphi[A])\;.$$

Moreover, by extensivity of φ, the converse inequality is also true. Hence, by using the antisymmetry of X, we can see that the required equality is also true.

Remark 81. It can be easily seen that an operation φ on a set X is idempotent if and only if $\varphi[X]$ is the family of all fixed points of φ.

Namely, $\varphi^2 = \varphi$ if and only if $\varphi^2(x) = \varphi(x)$, i.e., $\varphi\big(\varphi(x)\big) = \varphi(x)$ for all $x \in X$. That is, $\varphi(x) \in \mathrm{Fix}(\varphi)$ for all $x \in X$, or equivalently $\varphi[X] \subseteq \mathrm{Fix}(\varphi)$. Thus, since the converse inclusion always holds, the required assertion is also true.

Therefore, by using Theorem 91, we can also prove the following.

Corollary 29. *Under the conditions of Theorem 91, for any $A \subseteq \varphi[X]$, we have*

$$\inf_X\big(A\big) = \varphi\big(\inf_X(A)\big)\,.$$

Proof. Now, because of the antisymmetry of X, the operation φ is idempotent. Thus, by Remark 81, we have $\varphi(y) = y$ for all $y \in \varphi[X]$. Hence, by using the assumption $A \subseteq \varphi[X]$, we can see that $\varphi[A] = A$. Thus, Theorem 91 gives the required equality.

Remark 82. Note that if φ is an extensive, idempotent operation on a reflexive, antisymmetric goset X, then $\varphi[X]$ is also the family of all elements x of X which are φ-closed in the sense that $\varphi(x) \leq x$.

Therefore, if in addition to the conditions of Theorem 91, X is reflexive, then the assertion of Corollary 29 can also be expressed by stating that the infimum of any family of φ-closed elements of X is also φ-closed.

Now, instead of an analogue of Theorem 91 for supremum, we can only prove

Theorem 92. *If φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X, then for any $A \subseteq X$ we have*

$$\varphi\big(\sup_X(A)\big) = \varphi\,\big(\sup_X\big(\varphi[A]\big)\big)\;.$$

Proof. Define $\alpha = \sup_X(A)$ and $\beta = \sup_X\big(\varphi[A]\big)$. Then, by Theorem 86, we have $\beta \leq \varphi(\alpha)$. Hence, since φ is increasing, we can infer that $\varphi(\beta) \leq \varphi\big(\varphi(\alpha)\big)$. Moreover, since φ is now idempotent, we also have $\varphi\big(\varphi(\alpha)\big) = \varphi(\alpha)$. Therefore, $\varphi(\beta) \leq \varphi(\alpha)$.

On the other hand, since φ is extensive, for any $x \in A$ we have $x \leq \varphi(x)$. Moreover, since $\beta \in \mathrm{ub}_X\big(\varphi[A]\big)$, we also have $\varphi(x) \leq \beta$. Hence, by using the transitivity of X, we can infer that $x \leq \beta$. Therefore, $\beta \in \mathrm{ub}_X(A)$. Now, by using that $\alpha \in \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$, we can see that $\alpha \leq \beta$. Hence, by using the increasingness of φ, we can infer that $\varphi(\alpha) \leq \varphi(\beta)$. Therefore, by the antisymmetry of X, we actually have $\varphi(\alpha) = \varphi(\beta)$, and thus the required equality is also true.

From this theorem, we only get the following counterpart of Theorem 91.

Corollary 30. *Under the conditions of Theorem 92, for any $A \subseteq X$, the following assertions are equivalent:*

(1) $\sup_X\big(\varphi[A]\big) = \varphi\big(\sup_X(A)\big)$,
(2) $\sup_X\big(\varphi[A]\big) = \varphi\big(\sup_X\big(\varphi[A]\big)\big)$.

Now, in addition to Theorems 26 and 31, we can also prove

Theorem 93. *If φ is a closure operation on an inf-complete, antisymmetric goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\inf_Y(A) = \inf_X(A).$$

Proof. If $\alpha = \inf_X(A)$, then by Corollary 29 we have $\alpha = \varphi(\alpha)$, and hence $\alpha \in Y$. Therefore, under the usual identification of singletons with their elements, $\alpha = \inf_X(A) \cap Y$ also holds.

On the other hand, by Theorem 31, we always have $\inf_X(A) \cap Y \subseteq \inf_Y(A)$. Therefore, $\alpha \in \inf_Y(A)$ also holds. Hence, by using Theorem 45, we can already see that $\alpha = \inf_Y(A)$ is also true.

From this theorem, it is clear that in particular we also have

Corollary 31. *Under the conditions of Theorem 93, the subgoset Y is also inf-complete.*

Remark 83. Hence, by Theorem 34, we can see that the subgoset Y is also sup-complete.

Now, instead of establishing an analogue of Theorem 93 for supremum, it is convenient to prove first some more general theorems.

Theorem 94. *If φ is an idempotent operation on a goset X and $Y = \varphi[X]$, then for any $A \subseteq Y$ we have*

$$\mathrm{ub}_Y(A) \subseteq \varphi[\mathrm{ub}_X(A)].$$

Proof. If $\beta \in \mathrm{ub}_Y(A)$, then by Theorem 2 we have $\beta \in Y$ and $\beta \in \mathrm{ub}_X(A)$. Hence, by Remark 81, we can see that $\beta = \varphi(\beta)$, and thus $\beta \in \varphi[\mathrm{ub}_X(A)]$. Therefore, the required inclusion is also true.

Remark 84. By dualization, it is clear that in the above theorem we may also write lb in place of ub.

However, it is now more important to note that we also have the following.

Theorem 95. *If* φ *is an extensive operation on a transitive goset* X *and* $Y = \varphi\,[X]$, *then for any* $A \subseteq Y$ *we have*

$$\varphi\,[\,\mathrm{ub}_X(A)\,] \subseteq \mathrm{ub}_Y(A)\,.$$

Proof. If $\beta \in \mathrm{ub}_X(A)$, then because of $\beta \leq \varphi(\beta)$ and the transitivity of X, we also have $\varphi(\beta) \in \mathrm{ub}_X(A)$. Hence, since $\varphi(\beta) \in Y$, we can already see that $\varphi(\beta) \in \mathrm{ub}_X(A) \cap Y = \mathrm{ub}_Y(A)$, and thus the required inclusion is also true.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 32. *If* φ *is a semiclosure operation on a transitive, antisymmetric goset* X *and* $Y = \varphi\,[X]$, *then for any* $A \subseteq Y$ *we have*

$$\mathrm{ub}_Y(A) = \varphi\,[\,\mathrm{ub}_X(A)\,]\,.$$

However, it is now more important to note that, in addition to Theorem 95, we can also prove the following.

Theorem 96. *If* φ *is a lower semimodification operation on a transitive goset* X *and* $Y = \varphi\,[X]$, *then for any* $A \subseteq Y$ *we have*

$$\varphi\left[\,\mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)\,\right] \subseteq \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big)\,.$$

Proof. Suppose that $\beta \in \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)$. If $v \in \mathrm{ub}_Y(A)$, then by Theorem 2 we have $v \in Y$ and $v \in \mathrm{ub}_X(A)$. Hence, by using the assumed property of β, we can infer that $\beta \leq v$. Now, since φ is increasing, we can also state that $\varphi(\beta) \leq \varphi(v)$.

Moreover, since $v \in Y$, we can see that there exists $u \in X$ such that $v = \varphi(u)$. Hence, by using that φ is lower semi-idempotent, we can infer that

$$\varphi(v) = \varphi\big(\varphi(u)\big) = \varphi^2(u) \leq \varphi(u) = v\,.$$

Now, by using the transitivity of X, we can also see that $\varphi(\beta) \leq v$. Therefore, $\varphi(\beta) \in \mathrm{lb}_X\big(\mathrm{ub}_Y(A)\big)$. Hence, since $\varphi(\beta) \in Y$ also holds, we can already infer that $\varphi(\beta) \in \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big)$. Therefore, the required inclusion is also true.

Now, by using Theorems 95 and 96, we can also prove the following.

Theorem 97. *If* φ *is a closure operation on a transitive goset* X *and* $Y = \varphi\,[A]$, *then for any* $A \subseteq Y$ *we have*

$$\varphi\,[\,\sup_X(A)\,] \subseteq \sup_Y(A)\,.$$

Proof. By Theorems 29, 95, and 96, and a basic fact on relations, we have

$$\varphi\,[\,\sup_X(A)\,] = \varphi\,[\,\mathrm{ub}_X(A) \cap \mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)\,]$$
$$\subseteq \varphi\,[\,\mathrm{ub}_X(A)\,] \cap \varphi\,\big[\,\mathrm{lb}_X\big(\mathrm{ub}_X(A)\big)\,\big] \subseteq \mathrm{ub}_Y(A) \cap \mathrm{lb}_Y\big(\mathrm{ub}_Y(A)\big) = \sup_Y(A)\,.$$

Hence, it is clear that, analogously to Corollary 31, we can also state

Corollary 33. *If in addition to the conditions of Theorem 97, the goset X sup-complete, then the subgoset Y is also sup-complete.*

From Theorem 97, by using Theorem 45, we can also immediately derive the following counterpart of Theorem 93 and Corollary 29.

Theorem 98. *If φ is a closure operation on a sup-complete, transitive, and antisymmetric goset X and $Y = \varphi\,[A\,]$, then for any $A \subseteq Y$ we have*

$$\sup_Y(A) = \varphi\big(\sup_X(A)\big)\,.$$

18 Generalizations of Increasingness to Relator Spaces

A family $\mathscr{R}$ of relations on one set X to another Y is called a *relator* on X to Y. And, the ordered pair $(X,\,Y)(\mathscr{R}) = \big((X,\,Y),\,\mathscr{R}\big)$ is called a *relator space*. (For the origins, see [65], [28], [14], [39], and the references therein.)

If in particular $\mathscr{R}$ is a relator on X to itself, then we may simply say that $\mathscr{R}$ is a relator on X. And, by identifying singletons with their elements, we may naturally write $X(\mathscr{R})$ in place of $(X,X)(\mathscr{R})$, since $(X,X) = \{\{X\},\,\{X,\,X\}\} = \{\{X\}\}$.

Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [11] and *uniform spaces* [14]. However, they are insufficient for several important purposes. (See, for instance, [15, 46].)

A relator $\mathscr{R}$ on X to Y, or a relator space $(X,\,Y)(\mathscr{R})$ is called *simple* if there exists a relation R on X to Y such that $\mathscr{R} = \{R\}$. In this case, by identifying singletons with their elements, we may write $(X,\,Y)(R)$ in place of $(X,\,Y)(\{R\})$.

According to our former definition, a simple relator space $X(R)$ may be called a *goset* (generalized ordered set). Moreover, by Ganter and Wille [15, p. 17], a simple relator space $\big(X,Y\big)(R)$ may be called called a *formal context* or *context space*.

A relator $\mathscr{R}$ on X, or a relator space $X(\mathscr{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathscr{R}$ is a reflexive relation on X. Thus, we may also naturally speak of *preorder, tolerance*, and *equivalence relators*.

For any family $\mathscr{A}$ of subsets of X, the family $\mathscr{R}_{\mathscr{A}} = \{R_A : \ A \in \mathscr{A}\}$ is a preorder relator on X. While, for any family $\mathscr{D}$ of *pseudo-metrics* on X, the family $\mathscr{R}_{\mathscr{D}} = \{B_r^d : \ r > 0,\ d \in \mathscr{D}\}$ is a tolerance relator on X.

Moreover, if $\mathfrak{S}$ is a family of partitions of X, then $\mathscr{R}_{\mathfrak{S}} = \{ S_{\mathscr{A}} : \mathscr{A} \in \mathfrak{S} \}$ is an equivalence relator on X. Uniformities generated by such practically important relators seem to have been investigated only by Levine [23].

Now, according to Definition 12, a function f of one simple relator space $X(R)$ to another $Y(S)$ may be naturally called *increasing* if for any $u, v \in X$

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v)\,.$$

Hence, by noticing that

$$u\,R\,v \quad \Longleftrightarrow \quad v \in R(u) \quad \Longleftrightarrow \quad (u, v) \in R,$$

and

$$f(u)\,S\,f(v) \quad \Longleftrightarrow \quad f(v) \in S\big(f(u)\big) \quad \Longleftrightarrow \quad \big(f(u), f(v)\big) \in S,$$

that is,

$$f(u) S f(v) \quad \Longleftrightarrow \quad f(v) \in (S \circ f)(u) \quad \Longleftrightarrow \quad (f \boxtimes f)(u, v) \in S,$$

we can easily establish the following.

Theorem 99. *For a function f of one simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $f \circ R \subseteq S \circ f$,
(3) $(f \boxtimes f)[R] \subseteq S$,
(4) $f \circ R \circ f^{-1} \subseteq S$,
(5) $R \subseteq (f \boxtimes f)^{-1}[S]$,
(6) $R \subseteq f^{-1} \circ S \circ f$.

Proof. By the above argument and the corresponding definitions, it is clear that

$$(1) \quad \Longleftrightarrow \quad \forall\ (u, v) \in R: \quad (f \boxtimes f)(u, v) \in S \quad \Longleftrightarrow \quad (3)$$

and

$$\begin{aligned} (1) \quad &\Longleftrightarrow \quad \forall\ u \in X: \ \ \forall\ v \in R(u): \ \ f(v) \in (S \circ f)(u) \\ &\Longleftrightarrow \quad \forall\ u \in X: \ \ f[R(u)] \subseteq (S \circ f)(u) \\ &\Longleftrightarrow \quad \forall\ u \in X: \ \ \big(f \circ R\big)(u) \subseteq (S \circ f)(u) \quad \Longleftrightarrow \quad (2)\,. \end{aligned}$$

Moreover, if (2) holds, then by using that $f \circ f^{-1} \subseteq \Delta_Y$ we can see that

$$f \circ R \circ f^{-1} \subseteq S \circ f \circ f^{-1} \subseteq S \circ \Delta_Y = S,$$

and thus (4) also holds.

Conversely, if (4) holds, then by using that $\Delta_X \subseteq f^{-1} \circ f$ we can similarly see that

$$f \circ R = f \circ R \circ \Delta_X \subseteq f \circ R \circ f^{-1} \circ f \subseteq S \circ f,$$

and thus (2) also holds. Therefore, (2) and (4) are also equivalent.

Now, it is enough to prove only that (3) and (2) are also equivalent to (5) and (6), respectively.

For this, it is convenient to note that if φ is a function of one set U to another V, then because of the inclusions $\Delta_U \subseteq \varphi^{-1} \circ \varphi$ and $\varphi \circ \varphi^{-1} \subseteq \Delta_V$, for any $A \subseteq U$ and $B \subseteq Y$, we have

$$\varphi[A] \subseteq B \quad \Longleftrightarrow \quad A \subseteq \varphi^{-1}[B].$$

That is, the set functions φ and φ^{-1} also form a Galois connection.

Namely, if, for instance, (2) holds, then for any $x \in X$ we have

$$f[R(x)] = (f \circ R)(x) \subseteq (S \circ f)(x).$$

Hence, by using the abovementioned fact, we can already infer that

$$R(x) \subseteq f^{-1}[(S \circ f)(x)] = \big(f^{-1} \circ S \circ f\big)(x).$$

Therefore, (6) also holds. While, if (6) holds, then by using a reverse argument, we can quite similarly see that (2) also holds.

From Theorem 99, by using the uniform closure operation $*$ defined by

$$\mathscr{R}^* = \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathscr{R} : \quad R \subseteq S \big\}$$

for any relator $\mathscr{R}$ on X to Y, we can immediately derive the following.

Corollary 34. *For a function f of one simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing,*
(2) $S \circ f \in \big\{ f \circ R \big\}^*$,
(3) $S \in \big\{ (f \boxtimes f)[R] \big\}^*$,
(4) $S \in \big\{ f \circ R \circ f^{-1} \big\}^*$,
(5) $(f \boxtimes f)^{-1}[S] \in \{R\}^*$,
(6) $f^{-1} \circ S \circ f \in \{R\}^*$.

Remark 85. Now, by using the notations $\mathscr{F} = \{f\}$, $\mathscr{R} = \{R\}$ and $\mathscr{S} = \{S\}$, instead of (2) we may also write the more instructive inclusions

$$\mathscr{S} \circ \mathscr{F} \subseteq (\mathscr{F} \circ \mathscr{R})^*, \quad (\mathscr{S}^* \circ \mathscr{F})^* \subseteq (\mathscr{F} \circ \mathscr{R}^*)^*, \quad (\mathscr{S}^* \circ \mathscr{F}^*)^* \subseteq (\mathscr{F}^* \circ \mathscr{R}^*)^*.$$

The second one, whenever we think arbitrary relators in place of $\mathscr{R}$ and $\mathscr{S}$, already shows the $*$-invariance of the increasingness of $\mathscr{F}$ with respect to those relators.

From Corollary 34, by using the following obvious extensions of the operations -1 and $\circ$ from relations to relators, defined by

$$\mathscr{R}^{-1} = \{R^{-1} : \quad R \in \mathscr{R}\} \qquad \text{and} \qquad \mathscr{S} \circ \mathscr{R} = \{S \circ R : \quad R \in \mathscr{R}, \ S \in \mathscr{S}\}$$

for any relator $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, we can easily derive the following generalization of [46, Definition 4.1], which is also closely related to [60, Definition 15.1] .

Definition 15. Let $(X, Y)(\mathscr{R})$ and $(Z, W)(\mathscr{S})$ be relator spaces, and suppose that $\square$ is a direct unary operation for relators. Then, for any two relators $\mathscr{F}$ on X to Z and $\mathscr{G}$ on Y to W, we say that the pair

(1) $(\mathscr{F}, \mathscr{G})$ is *mildly* $\square$*-increasing* if

$$\Big(\big(\mathscr{G}^{\square}\big)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square} \Big)^{\square} \subseteq \mathscr{R}^{\square}.$$

(2) $(\mathscr{F}, \mathscr{G})$ is *upper* $\square$*-semi-increasing* if

$$\big(\mathscr{S}^{\square} \circ \mathscr{F}^{\square}\big)^{\square} \subseteq \big(\mathscr{G}^{\square} \circ \mathscr{R}^{\square}\big)^{\square}.$$

(3) $(\mathscr{F}, \mathscr{G})$ is *lower* $\square$*-semi-increasing* if

$$\Big(\big(\mathscr{G}^{\square}\big)^{-1} \circ \mathscr{S}^{\square} \Big)^{\square} \subseteq \Big(\mathscr{R}^{\square} \circ \big(\mathscr{F}^{\square}\big)^{-1} \Big)^{\square}.$$

Remark 86. A function $\square$ of the class of all relator spaces to that of all relators is called a *direct unary operation for relators* if, for any relator space $(X, Y)(\mathscr{R})$, the value $\square\big((X, Y)(\mathscr{R})\big)$ is a relator on X to Y.

In this case, trusting to the reader's good sense to avoid confusions, we shall simply write $\mathscr{R}^{\square}$ instead of $\mathscr{R}^{\square_{XY}} = \square\big((X, Y)(\mathscr{R})\big)$. Thus, $*$ is a direct, while -1 is a non-direct unary operation for relators.

19 Some Useful Simplifications of Definition 15

The rather difficult increasingness properties given in Definition 15 can be greatly simplified whenever the operation $\square$ has some useful additional properties.

For instance, by using an analogue of Definition 14, we can easily establish

Theorem 100. *If in addition to the assumptions of Definition 15, $\square$ is a closure operation for relators, then*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*-increasing if and only if*

$$\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \circ \mathscr{F}^{\square} \subseteq \mathscr{R}^{\square}.$$

(2) $(\mathscr{F}, \mathscr{G})$ *is upper* $\square$*-semi-increasing if and only if*

$$\mathscr{S}^{\square} \circ \mathscr{F}^{\square} \subseteq \left(\mathscr{G}^{\square} \circ \mathscr{R}^{\square}\right)^{\square}.$$

(3) $(\mathscr{F}, \mathscr{G})$ *is lower* $\square$*-semi-increasing if and only if*

$$\left(\mathscr{G}^{\square}\right)^{-1} \circ \mathscr{S}^{\square} \subseteq \left(\mathscr{R}^{\square} \circ \left(\mathscr{F}^{\square}\right)^{-1}\right)^{\square}.$$

Remark 87. To check this, note that an operation $\square$ for relators is a closure operation if and only if, for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X to Y, we have

$$\mathscr{U}^{\square} \subseteq \mathscr{V}^{\square} \iff \mathscr{U} \subseteq \mathscr{V}^{\square}.$$

That is, the set functions $\square$ and $\square$ form a Pataki connection.

Now, by calling an operation $\square$ for relators to be *inversion and composition compatible* if

$$\left(\mathscr{R}^{\square}\right)^{-1} = \left(\mathscr{R}^{-1}\right)^{\square} \qquad \text{and} \qquad \left(\mathscr{S} \circ \mathscr{R}\right)^{\square} = \left(\mathscr{S}^{\square} \circ \mathscr{R}\right)^{\square} = \left(\mathscr{S} \circ \mathscr{R}^{\square}\right)^{\square}$$

for any relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z, we can also easily establish

Theorem 101. *If in addition to the assumptions of Definition 15, $\square$ is an inversion and composition compatible operation for relators, then*

(1) $(\mathscr{F}, \mathscr{G})$ *is mildly* $\square$*-increasing if and only if*

$$\left(\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F}\right)^{\square} \subseteq \mathscr{R}^{\square}.$$

(2) $\big(\mathscr{F}, \mathscr{G}\big)$ *is upper $\Box$-semi-increasing if and only if*

$$\big(\mathscr{S} \circ \mathscr{F}\big)^{\Box} \subseteq \big(\mathscr{G} \circ \mathscr{R}\big)^{\Box}.$$

(3) $\big(\mathscr{F}, \mathscr{G}\big)$ *is lower $\Box$-semi-increasing if and only if*

$$\big(\mathscr{G}^{-1} \circ \mathscr{S}\big)^{\Box} \subseteq \big(\mathscr{R} \circ \mathscr{F}^{-1}\big)^{\Box}.$$

Remark 88. To check this, note that if $\Box$ is a composition compatible operation for relators, then for any three relators $\mathscr{R}$ on X to Y, $\mathscr{S}$ on Y to Z, and $\mathscr{T}$ on Z to W, we have

$$(\mathscr{S} \circ \mathscr{R})^{\Box} = (\mathscr{S}^{\Box} \circ \mathscr{R}^{\Box})^{\Box} \qquad \text{and} \qquad (\mathscr{T} \circ \mathscr{S} \circ \mathscr{R})^{\Box} = (\mathscr{T}^{\Box} \circ \mathscr{S}^{\Box} \circ \mathscr{R}^{\Box})^{\Box}.$$

From the above theorem, it is clear that in particular we also have

Corollary 35. *If in addition to the assumptions of Definition 15, $\Box$ is an inversion and composition compatible closure operation for relators, then*

(1) $\big(\mathscr{F}, \mathscr{G}\big)$ *is mildly $\Box$-increasing if and only if*

$$\mathscr{G}^{-1} \circ \mathscr{S} \circ \mathscr{F} \subseteq \mathscr{R}^{\Box}.$$

(2) $\big(\mathscr{F}, \mathscr{G}\big)$ *is upper $\Box$-semi-increasing if and only if*

$$\mathscr{S} \circ \mathscr{F} \subseteq \big(\mathscr{G} \circ \mathscr{R}\big)^{\Box}.$$

(3) $\big(\mathscr{F}, \mathscr{G}\big)$ *is lower $\Box$-semi-increasing if and only if*

$$\mathscr{G}^{-1} \circ \mathscr{S} \subseteq \big(\mathscr{R} \circ \mathscr{F}^{-1}\big)^{\Box}.$$

Concerning inversion compatible operations, we can also prove the following.

Theorem 102. *If in addition to the assumptions of Definition 15, $\Box$ is an inversion compatible operation for relators, then*

(1) $\big(\mathscr{F}, \mathscr{G}\big)$ *is mildly $\Box$-increasing with respect to the relators $\mathscr{R}$ and $\mathscr{S}$ if and only if $\big(\mathscr{G}, \mathscr{F}\big)$ is mildly $\Box$-increasing with respect to the relators $\mathscr{R}^{-1}$ and $\mathscr{S}^{-1}$.*

(2) $\big(\mathscr{F}, \mathscr{G}\big)$ *is upper $\Box$-semi-increasing with respect to the relators $\mathscr{R}$ and $\mathscr{S}$ if and only if $\big(\mathscr{G}, \mathscr{F}\big)$ is lower $\Box$-semi-increasing with respect to the relators $\mathscr{R}^{-1}$ and $\mathscr{S}^{-1}$.*

Proof. To prove the "only if part" of (2), note that by the assumed inversion compatibility of $\square$ and a basic inversion property of the element-wise composition of relators, we have

$$\Big(\big(\mathscr{S}^{\square}\circ\mathscr{F}^{\square}\big)^{\square}\Big)^{-1}=\Big(\big(\mathscr{S}^{\square}\circ\mathscr{F}^{\square}\big)^{-1}\Big)^{\square}$$
$$=\Big(\big(\mathscr{F}^{\square}\big)^{-1}\circ\big(\mathscr{S}^{\square}\big)^{-1}\Big)^{\square}$$
$$=\Big(\big(\mathscr{F}^{\square}\big)^{-1}\circ\big(\mathscr{S}^{-1}\big)^{\square}\Big)^{\square},$$

and quite similarly

$$\Big(\big(\mathscr{G}^{\square}\circ\mathscr{R}^{\square}\big)^{\square}\Big)^{-1}=\Big(\big(\mathscr{R}^{-1}\big)^{\square}\circ\big(\mathscr{G}^{\square}\big)^{-1}\Big)^{\square}.$$

Therefore, if $\big(\mathscr{S}^{\square}\circ\mathscr{F}^{\square}\big)^{\square}\subseteq\big(\mathscr{G}^{\square}\circ\mathscr{R}^{\square}\big)$ holds, then we also have

$$\Big(\big(\mathscr{F}^{\square}\big)^{-1}\circ\big(\mathscr{S}^{-1}\big)^{\square}\Big)^{\square}\subseteq\Big(\big(\mathscr{R}^{-1}\big)^{\square}\circ\big(\mathscr{G}^{\square}\big)^{-1}\Big)^{\square}.$$

Remark 89. Such types of arguments indicate that we actually have to keep in mind only the definition of upper $\square$-semi-increasingness, since the other two ones can be easily derived from this one under some simplifying assumptions.

Remark 90. Unfortunately, Theorems 102 and 101 have only a limited range of applicability since several important closure operations on relators are not inversion or composition compatible.

Remark 91. However, it can be easily seen that a union-preserving operation $\square$ for relators is inversion compatible if and only if $\{R^{-1}\}^{\square}\subseteq\big(\{R\}^{\square}\big)^{-1}$ for any relation R on X to Y.

Moreover, a closure operation $\square$ for relators is composition compatible if and only if

$$\mathscr{S}\circ\mathscr{R}^{\square}\subseteq\big(\mathscr{S}\circ\mathscr{R}\big)^{\square}\qquad\text{and}\qquad\mathscr{S}^{\square}\circ\mathscr{R}\subseteq\big(\mathscr{S}\circ\mathscr{R}\big)^{\square}$$

for any two relators $\mathscr{R}$ on X to Y and $\mathscr{S}$ on Y to Z.

Remark 92. By using the latter facts, one can more easily see that, for instance, the uniform closure operation $*$ is inversion and composition compatible.

20 Some Further Important Unary Operations for Relators

In addition to the operation $*$, the functions $\#$, $\wedge$, and Δ, defined by

$$\mathscr{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \ \exists\ R \in \mathscr{R} : \quad R[A] \subseteq S[A] \big\},$$
$$\mathscr{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \ \exists\ R \in \mathscr{R} : \quad R(x) \subseteq S(x) \big\},$$

and

$$\mathscr{R}^{\Delta} = \big\{ S \subset X \times Y : \quad \forall\ x \in X : \ \exists\ u \in X : \ \exists\ R \in \mathscr{R} : \quad R(u) \subseteq S(x) \big\}$$

for any relator $\mathscr{R}$ on X to Y, are also important closure operations for relators.

Thus, we evidently have $\mathscr{R} \subseteq \mathscr{R}^{*} \subseteq \mathscr{R}^{\#} \subseteq \mathscr{R}^{\wedge} \subseteq \mathscr{R}^{\Delta}$ for any relator $\mathscr{R}$ on X to Y. Moreover, if in particular $X = Y$, then in addition to the above inclusions we can also easily prove that $\mathscr{R}^{\infty} \subseteq \mathscr{R}^{*\infty} \subseteq \mathscr{R}^{\infty *} \subseteq \mathscr{R}^{*}$, where

$$\mathscr{R}^{\infty} = \big\{ R^{\infty} : \quad R \in \mathscr{R} \big\}.$$

In addition to ∞, it is also worth considering the operation ∂, defined by

$$\mathscr{R}^{\partial} = \big\{ S \subseteq X^{2} : \quad S^{\infty} \in \mathscr{R} \big\}$$

for any relator $\mathscr{R}$ on X. Namely, for any two relators $\mathscr{R}$ and $\mathscr{S}$ on X, we have

$$\mathscr{R}^{\infty} \subseteq \mathscr{S} \quad \Longleftrightarrow \quad \mathscr{R} \subseteq \mathscr{S}^{\partial}.$$

This shows that the set functions ∞ and ∂ also form a Galois connection. Therefore, $\infty = \infty\,\partial\,\infty$, and $\infty\,\partial$ is also closure operation for relators.

Moreover, for any relator $\mathscr{R}$ on X to Y, we may also naturally define

$$\mathscr{R}^{c} = \big\{ R^{c} : \quad R \in \mathscr{R} \big\},$$

where $R^{c} = X \times Y \setminus R$. Thus, for instance, we may also naturally consider the operation $\circledast = c * c$ which seems to play the same role in order theory as the operation $*$ does in topology.

Unfortunately, the operations $\wedge$ and Δ are not inversion Compatible; therefore, in addition to these operations we have also to consider the operations $\vee = \wedge - 1$ and $\nabla = \Delta - 1$, which already have very curious properties.

For instance, the operations $\vee\vee$ and $\nabla\nabla$ coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined by

$$\mathscr{R}^{\bullet} = \big\{ \delta_{\mathscr{R}} \big\}^{*}, \qquad \text{where} \qquad \delta_{\mathscr{R}} = \bigcap \mathscr{R},$$

and

$$\mathscr{R}^{\blacklozenge} = \mathscr{R} \quad \text{if} \quad \mathscr{R} = \big\{ X \times Y \big\} \qquad \text{and} \qquad \mathscr{R}^{\blacklozenge} = \mathscr{P}(X \times Y) \quad \text{if} \quad \mathscr{R} \neq \big\{ X \times Y \big\}.$$

Because of the above important operations for relators, Definition 15 offers an abundance of natural increasingness properties for relations. Moreover, from the results of Sects. 15 and 16, one can also immediately derive several reasonable definitions for the increasingness of relations.

However, in [58], a relation F on a goset X to a set Y has been called increasing if the induced set-valued function $F^{\diamond}$ is increasing. That is, $u \leq v$ implies $F(u) \subseteq F(v)$ for all $u, v \in X$. Thus, it can be easily seen that F is increasing if and only if F^{-1} is *ascending valued* in the sense that $F^{-1}(y)$ is an ascending subset of X for all $y \in Y$.

If $\mathscr{R}$ is a relator on X to Y, then by extending the corresponding parts of Definitions 1 and 2 ,we may also naturally define

$$\operatorname{Lb}_{\mathscr{R}}(B) = \big\{ A \subseteq X : \quad \exists\, R \in \mathscr{R} : \ A \times B \subseteq R \big\} \quad \text{and} \quad \operatorname{lb}_{\mathscr{R}}(B) = X \cap \operatorname{Lb}_{\mathscr{R}}(B),$$

and

$$\operatorname{Int}_{\mathscr{R}}(B) = \big\{ A \subseteq X : \quad \exists\, R \in \mathscr{R} : \ R\,[A\,] \subseteq B \big\} \quad \text{and} \quad \operatorname{int}_{\mathscr{R}}(B) = X \cap \operatorname{Int}_{\mathscr{R}}(B)$$

for all $B \subseteq Y$. However, these relations are again not independent of each other.

Namely, by the corresponding definitions, it is clear that

$$\begin{aligned} A \times B \subseteq R \quad &\Longleftrightarrow \quad \forall\, a \in A : \ B \subseteq R(a) \quad \Longleftrightarrow \quad \forall\, a \in A : \ R(a)^{c} \subseteq B^{c} \\ &\Longleftrightarrow \quad \forall\, a \in A : \ R^{c}(a) \subseteq B^{c} \quad \Longleftrightarrow \quad R^{c}\,[A\,] \subseteq B^{c}. \end{aligned}$$

Therefore, we have

$$A \in \operatorname{Lb}_{\mathscr{R}}(B) \quad \Longleftrightarrow \quad A \in \operatorname{Int}_{\mathscr{R}^{c}}(B^{c}) \quad \Longleftrightarrow \quad A \in \big(\operatorname{Int}_{\mathscr{R}^{c}} \circ \mathscr{C}\big)(B)\,.$$

Hence, we can already see that

$$\operatorname{Lb}_{\mathscr{R}} = \operatorname{Int}_{\mathscr{R}^{c}} \circ \mathscr{C}, \qquad \text{and thus also} \qquad \operatorname{lb}_{\mathscr{R}} = \operatorname{int}_{\mathscr{R}^{c}} \circ \mathscr{C}\,.$$

These formulas, proved first in [47], establish at least as important relationship between order and topological theories as the famous Euler formulas do between exponential and trigonometric functions [38, p. 227] .

To see the importance of the operations $\#$ and $\textcircled{\#} = c\,\#\,c$, by using Pataki connections on power sets [50], it can be shown that, for any relator $\mathscr{R}$ on X to Y, $\mathscr{S} = \mathscr{R}^{\#}$ $\big(\mathscr{S} = \mathscr{R}^{\textcircled{\#}}\big)$ is the largest relator on X to Y such that $\operatorname{Int}_{\mathscr{S}} = \operatorname{Int}_{\mathscr{R}}$ $\big(\operatorname{Lb}_{\mathscr{S}} = \operatorname{Lb}_{\mathscr{R}}\big)$.

Concerning the operations $\wedge$ and $\textcircled{\wedge} = c \wedge c$, we can quite similarly see that $\mathscr{S} = \mathscr{R}^{\wedge}$ $\big(\mathscr{S} = \mathscr{R}^{\textcircled{\wedge}}\big)$ is the largest relator on X to Y such that $\operatorname{int}_{\mathscr{S}} = \operatorname{int}_{\mathscr{R}}$ $\big(\operatorname{lb}_{\mathscr{S}} = \operatorname{lb}_{\mathscr{R}}\big)$. Moreover, if in particular $\mathscr{R}$ is a relator on X, then some similar assertions holds for the families

$$\tau_{\mathscr{R}} = \big\{ A \subseteq X : \quad A \in \operatorname{Int}_{\mathscr{R}}(A) \big\} \qquad \text{and} \qquad \ell_{\mathscr{R}} = \big\{ A \subseteq X : \quad A \in \operatorname{Lb}_{\mathscr{R}}(A) \big\}\,.$$

However, if $\mathscr{R}$ is a relator on X, then for the families

$$\mathscr{T}_{\mathscr{R}} = \{ A \subseteq X : \quad A \subseteq \operatorname{int}_{\mathscr{R}}(A) \} \qquad \text{and} \qquad \mathscr{L}_{\mathscr{R}} = \{ A \subseteq X : \quad A \subseteq \operatorname{lb}_{\mathscr{R}}(A) \}$$

there does not exist a largest relator $\mathscr{S}$ on X such that $\mathscr{T}_{\mathscr{S}} = \mathscr{T}_{\mathscr{R}}$ $\left(\mathscr{L}_{\mathscr{S}} = \mathscr{L}_{\mathscr{R}} \right)$.

In the light of this and some other disadvantages of the family $\mathscr{T}_{\mathscr{R}}$, it is rather curious that most of the works in topology and analysis have been based on open sets suggested by Tietze [64] and standardized by Bourbaki [5] and Kelley [18].

Moreover, it also a striking fact that, despite the results of Pervin [34], Fletcher and Lindgren [14], and the present author [52], topologies and their generalizations are still intensively investigated, without generalized uniformities, by a great number of mathematicians.

The study of the various generalized topologies is mainly motivated by some recent papers of Á. Császár. For instance, the authors of [7, 25] write that: "The theory of generalized topological spaces, which was founded by Á. Császár, is one of the most important developments of general topology in recent years."

For any relator $\mathscr{R}$ on X to Y, we may also naturally define

$$\mathscr{E}_{\mathscr{R}} = \{ B \subseteq Y : \quad \operatorname{int}_{\mathscr{R}}(B) \neq \emptyset \} \qquad \text{and} \qquad \mathfrak{E}_{\mathscr{R}} = \{ B \subseteq Y : \quad \operatorname{lb}_{\mathscr{R}}(B) \neq \emptyset \}.$$

In a relator space $X(\mathscr{R})$, the family $\mathscr{E}_{\mathscr{R}}$ of all *fat sets* is frequently a more important tool than the family $\mathscr{T}_{\mathscr{R}}$ of all *topologically open sets*. Namely, if $\mathscr{R}$ is a relator on X to Y, then it can be shown that $\mathscr{S} = \mathscr{R}^{\Delta}$ is the largest relator on X to Y such that $\mathscr{E}_{\mathscr{S}} = \mathscr{E}_{\mathscr{R}}$.

Moreover, if $\mathscr{R}$ is a relator on X to Y, then for any goset Γ, and *nets* $x \in X^{\Gamma}$ and $y \in Y^{\Gamma}$,we may naturally define $x \in \operatorname{Lim}_{\mathscr{R}}(y)$ if the net (x, y) is eventually in each $R \in \mathscr{R}$ in the sense that $(x, y)^{-1}[R] \in \mathscr{E}_{\Gamma}$. Now, for any $a \in X$, we may also naturally write $a \in \lim_{\mathscr{R}}(y)$ if $(a) \in \operatorname{Lim}_{\mathscr{R}}(y)$, where (a) is an abbreviation for the constant net $(a)_{\gamma \in \Gamma} = \Gamma \times \{a\}$.

In a relator space $(X, Y)(\mathscr{R})$, the *convergence relation* $\operatorname{Lim}_{\mathscr{R}}$, suggested by Efremović and Švarc [13], is a much stronger tool than the *proximal interior relation* $\operatorname{Int}_{\mathscr{R}}$ suggested by Smirnov [37]. If $\mathscr{R}$ is a relator on X to Y, then it can be shown that $\mathscr{S} = \mathscr{R}^{*}$ is the largest relator on X to Y such that $\operatorname{Lim}_{\mathscr{S}} = \operatorname{Lim}_{\mathscr{R}}$.

Now, following the ideas of Császár [8], for any relator $\mathscr{R}$ on X to Y, we may also naturally consider the *hyperrelators*

$$\mathfrak{H}_{\mathscr{R}} = \{ \operatorname{Int}_R : \quad R \in \mathscr{R} \} \qquad \text{and} \qquad \mathfrak{K}_{\mathscr{R}} = \{ \operatorname{Lim}_R : \quad R \in \mathscr{R} \}.$$

By the corresponding definitions, it is clear that

$$\operatorname{Int}_{\mathscr{R}} = \bigcup_{R \in \mathscr{R}} \operatorname{Int}_R \qquad \text{and} \qquad \operatorname{Lim}_{\mathscr{R}} = \bigcap_{R \in \mathscr{R}} \operatorname{Lim}_R .$$

Therefore, the above hyperrelators are much stronger tools in the relator space $(X, Y)(\mathscr{R})$ than the relations $\operatorname{Int}_{\mathscr{R}}$ and $\operatorname{Lim}_{\mathscr{R}}$.

For instance, a net $y \in Y^{\Gamma}$ may be naturally called *convergence Cauchy* with respect to the relator $\mathscr{R}$ if $\lim_R(y) \neq \emptyset$ for all $R \in \mathscr{R}$. Hence, since

$$\lim_{\mathscr{R}}(y) = \bigcap_{R \in \mathscr{R}} \lim_R(y),$$

we can at once see that a convergent net is convergence Cauchy, but the converse statement need not be true.

However, it can be shown that the net y is convergent with respect to the relator $\mathscr{R}$ if and only if it convergence Cauchy with respect to the *topological closure* $\mathscr{R}^{\wedge}$ of $\mathscr{R}$. (See [43].) Therefore, the two notions are in a certain sense equivalent.

The same is true in connection with the notions *adherent* and *adherence Cauchy*, which are defined by using $\mathscr{D}_{\Gamma}$ instead of $\mathscr{E}_{\Gamma}$. Moreover, it is also noteworthy that a similar situation holds in connection with the concepts *compact* and *precompact*. (See [45].)

Now, according to the ideas of Száz [59], we may also naturally consider corelator spaces, mentioned in Sect. 2, instead of relator spaces. However, the increasingness properties (1) and (3) considered in Definition 15 cannot be immediately generalized to such spaces. Namely, in contrast to relations, the ordinary inverse of a correlation is usually not a correlation.

Finally, we note that, in addition to the results of Sect.17, it would also be desirable to to establish some topological properties of closure operations by supplementing the results of Sect. 16. Moreover, it would be desirable to extend the notion of closure operations to arbitrary relator spaces.

However, in this direction, we could only observe that a unary operation φ on a simple relator space $X(R)$ is extensive if and only if $\varphi \subseteq R$. Moreover, φ is lower semi-idempotent if and only if $\varphi \,|\, \varphi[X] \subseteq R^{-1}$.

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Addition-Like Variational Principles in Asymmetric Spaces

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In Honor of Constantin Carathéodory

Abstract A class of addition-like smooth variational principles in asymmetric spaces is established, under the general directions in Yongxin and Shuzhong (J Math Anal Appl 246:308–319, 2000). The obtained results lie in the logical segment between Dependent Choice principle (DC) and Ekeland's variational principle (EVP); so, they are equivalent with both (DC) and (EVP).

1 Introduction

Let X be a nonempty set; and $d : X \times X \to R_+ := [0, \infty[$ be a *metric* over it; the couple (X, d) will be referred to as a *metric space*. Further, let $\varphi \in \mathscr{F}(X, R \cup \{\infty\})$ be a function with

(ip-lsc-1) φ is inf-proper: $\inf[\varphi(X)] > -\infty$ and $\mathrm{Dom}(\varphi) := \{x \in X; \varphi(x) < \infty\} \neq \emptyset$ (hence, $-\infty < \inf[\varphi(X)] < \infty$)

(ip-lsc-2) φ is d-lsc: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \xrightarrow{d} x$.

(Here, for each couple A, B of nonempty sets, $\mathscr{F}(A, B)$ stands for the class of all functions from A to B; when $A = B$, we write $\mathscr{F}(A, A)$ as $\mathscr{F}(A)$). The following statement in Ekeland [13] (referred to as Ekeland's variational principle; in short: EVP) is our starting point.

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Theorem 1. *Let these conditions hold; as well as*

(d-com) X is d-complete: each d-Cauchy sequence is d-convergent.

Then, for each (starting point) $u \in \mathrm{Dom}(\varphi)$*, there exists (another point)* $v \in \mathrm{Dom}(\varphi)$*, such that*

(11-a) $d(u,v) \leq \varphi(u) - \varphi(v)$ *(hence* $\varphi(u) \geq \varphi(v)$*)*
(11-b) $d(v,x) > \varphi(v) - \varphi(x)$*, for all* $x \in X \setminus \{v\}$*.*

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory, and global analysis; see, in this direction, the 1997 monograph by Hyers et al. [19, Chap. 5]. So, it cannot be surprising that, soon after, many extensions of (EVP) were proposed. For example, the (abstract) *order* one starts from the fact that, with respect to the *(quasi-) order* (i.e., reflexive and transitive relation)

(q-ord) $(x, y \in X)$: $x \leq y$ iff $d(x,y) + \varphi(y) \leq \varphi(x)$

the point $v \in X$ appearing in the second conclusion above is *maximal*; so that, (EVP) is nothing but a Zorn–Bourbaki maximal statement, in the denumerable variant of it expressed by the 1976 Brezis–Browder ordering principle [6] (in short: BB); see also Turinici [28]. The *dimensional* way of extension refers to the ambient space (R) of $\varphi(X)$ being substituted by a (topological or not) *vector space*. An account of the results in this area is to be found in the 2003 monograph by Goepfert et al. [18, Chap. 3]; however, as shown in Turinici [27], the sequential statements of this type are all reducible to (BB) above. Further, the *(pseudo) metrical* one consists in conditions imposed to the ambient metric over X being relaxed. Some basic results in this direction were obtained in the 1996 paper by Kada et al. [20]; but, as established in Turinici [29], all these are again deductible from (BB). Finally, we must add to this list (cf. Turinici [30]) the 1987 *smooth* variational principle in Borwein and Preiss [3] (in short: BP).

Having these precise, note that an interesting extension of (BP)—hence, of (EVP) as well—was obtained in Yongxin and Shuzhong [33] along the *asymmetric spaces* [generated by standard metrics]. A functional extension of this result to the same setting—and practically, with the same argument—was obtained in the paper by Farkas [15]; further enlargements of this research line (to the framework of asymmetric spaces generated by Bakhtin metrics) have been carried out—again with the same reasoning—by Farkas et al. [16]. Taking the preceding observations into account, it is legitimate to ask whether the obtained results are effective (logical) extensions of (EVP). As we shall see (in Sect. 7), the answer to this is *negative*; moreover (cf. Sect. 8), all results of the quoted authors do not seem to be valid in the genuine asymmetric setting. A way of correcting this drawback is proposed in Sects. 4–6. The basic tools for establishing these results are the *Dependent Choice principle* (discussed in Sect. 2) and some auxiliary facts involving *r-asymmetric spaces* (cf. Sect. 3). Further aspects will be delineated in a separate paper.

2 Dependent Choice Principles

Throughout this exposition, the axiomatic system in use is Zermelo–Fraenkel's (abbreviated: ZF), as described by Cohen [9, Chap. 2]. The notations and basic facts to be considered in this system are more or less standard. Some important ones are discussed below.

(A) Let X be a nonempty set. By a *relation* over X, we mean any nonempty part $\mathscr{R} \subseteq X \times X$. For simplicity, we sometimes write $(x, y) \in \mathscr{R}$ as $x\mathscr{R}y$. Note that $\mathscr{R}$ may be regarded as a mapping between X and 2^X (=the class of all subsets in X). In fact, denote for $x \in X$:

$X(x, \mathscr{R}) = \{y \in X; x\mathscr{R}y\}$ (the section of $\mathscr{R}$ through x);

then, the desired mapping representation is $(\mathscr{R}(x) = X(x, \mathscr{R}); x \in X)$. A basic example of such object is

$\mathscr{I} = \{(x, x); x \in X\}$ (the identical relation over X).

Let X be a nonempty set. By a *sequence* in X, we mean any mapping $x \in \mathscr{F}(N, X)$, where $N = \{0, 1, \ldots\}$ is the class of *natural numbers*. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$ or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with

$(i(n); n \geq 0)$ is divergent (in the sense: $i(n) \to \infty$ as $n \to \infty$)

will be referred to as a *subsequence* of $(x_n; n \geq 0)$.

(B) Remember that, an outstanding part of (ZF) is the *Axiom of Choice* (abbreviated: AC); which, in a convenient manner, may be written as

(AC) For each nonempty set X, there exists a (selective) function $f : (2)^X \to X$, with $f(Y) \in Y$, for each $Y \in (2)^X$.

(Here, $(2)^X$ stands for the class of all nonempty elements in 2^X). Sometimes, when the ambient set X is endowed with denumerable-type structures, the use of selective functions like before may be substituted by a weaker form of (AC), called *Dependent Choice principle* (in short: DC). Some preliminaries are needed. Let X be a nonempty set. For each natural number $k \geq 1$, call the map $F : N(k, >) \to X$, a *k-sequence*; if $k \geq 1$ is generic, we talk about a *finite* sequence. The following result, referred to as the *Finite Dependent Choice property* (in short: (DC-fin)) is available in the *strongly reduced* Zermelo–Fraenkel system (ZF-AC). Call the relation $\mathscr{R}$ over X, *proper* when

$(X(x, \mathscr{R}) =)\mathscr{R}(x)$ is nonempty, for each $x \in X$.

Note that, in this case, $\mathscr{R}$ is to be viewed as a mapping between X and $(2)^X$; the couple $(X, \mathscr{R})$ will be then referred to as a *proper relational structure*. Given $a \in X$, let us say that the k-sequence $F : N(k, >) \to X$ (where $k \geq 2$) is $(a, \mathscr{R})$*-iterative* provided the following holds:

$F(0) = a; F(i)\mathscr{R}F(i+1)$ [i.e., $F(i+1) \in \mathscr{R}(F(i))$], $\forall i \in N(k-1, >)$.

Lemma 1. *Let the relational structure* $(X, \mathscr{R})$ *be proper. Then, for each* $k \geq 2$*, the following property holds (in (ZF-AC)):*

($\Pi(k)$) for each $a \in X$*, there exists an* $(a, \mathscr{R})$*-iterative* k*-sequence.*

Proof. Clearly, $\Pi(2)$ is true; just take $b \in \mathscr{R}(a)$ and define $F : N(2, >) \to X$ as $F(0) = a$, $F(1) = b$. Assume that $\Pi(k)$ is true, for some $k \geq 2$; we claim that $\Pi(k+1)$ is true as well. In fact, let $F : N(k, >) \to X$ be an $(a, \mathscr{R})$-iterative k-sequence, assured by hypothesis. As $\mathscr{R}$ is proper, $\mathscr{R}(F(k-1))$ is nonempty; let u be some element of it. The map $G : N(k+1, >) \to X$ introduced as

$G(i) = F(i), i \in N(k, >); G(k) = u$

is an $(a, \mathscr{R})$-iterative $(k+1)$-sequence; and then, we are done.

Now, it is natural to see what happens when k "tends to infinity." At a first glance, the following *Dependent Choice principle* (in short: DC) is obtainable in (ZF-AC) from this "limit" process. Given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; \mathscr{R})$*-iterative* provided

$x_0 = a; x_n \mathscr{R} x_{n+1}$ (i.e., $x_{n+1} \in \mathscr{R}(x_n)$), $\forall n$.

Proposition 1. *Let the relational structure* $(X, \mathscr{R})$ *be proper. Then, for each* $a \in X$*, there is an* $(a, \mathscr{R})$*-iterative sequence in* X*.*

Concerning this aspect, we stress that—from a technical perspective—the limit process in question does not work in the strongly reduced system (ZF-AC), because it involves an infinite choice process; whence, (DC) is not obtainable from the axioms of the underlying system. On the other hand, this principle—proposed, independently, by Bernays [2] and Tarski [26]—is deductible from (AC), but not conversely (cf. Wolk [32]). Moreover, by the developments in Moskhovakis [24, Chap. 8], and Schechter [25, Chap. 6], the *reduced system* (ZF-AC+DC) is comprehensive enough so as to cover the "usual" mathematics; see also Moore [23, Appendix 2, Table 4].

(C) A "diagonal" version of this principle may be stated as follows. Let $(\mathscr{R}_n; n \geq 0)$ be a sequence of relations on X. Given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; (\mathscr{R}_n; n \geq 0))$*-iterative* provided

$x_0 = a; x_n \mathscr{R}_n x_{n+1}$ (i.e., $x_{n+1} \in \mathscr{R}_n(x_n)$), $\forall n$.

The following *Diagonal Dependent Choice principle* (in short: DDC) is effectively needed in our future developments.

Proposition 2. *Let* $(\mathscr{R}_n; n \geq 0)$ *be a sequence of proper relations on* X*. Then, for each* $a \in X$*, there exists at least one* $(a; (\mathscr{R}_n; n \geq 0))$*-iterative sequence in* X*.*

Clearly, (DDC) includes (DC); to which it reduces when $(\mathscr{R}_n; n \geq 0)$ is constant. The reciprocal of this is also true. In fact, letting the premises of (DDC) hold, put $P = N \times X$; and let $\mathscr{S}$ be the (proper) relation over P introduced as

$$\mathscr{S}(i,x) = \{i+1\} \times \mathscr{R}_i(x), \ (i,x) \in P.$$

It will suffice applying (DC) to $(P, \mathscr{S})$ and $b := (0,a) \in P$ to get the conclusion in the statement; we do not give details.

Summing up, (DDC) is provable in (ZF-AC+DC). This is valid as well for its variant, referred to as the *Selected Dependent Choice principle* (in short: SDC). Given the map $F : N \to (2)^X$ and the relation $\mathscr{R}$ over X, let us say that F is *$\mathscr{R}$-chainable*, provided

$$(\forall n \geq 0): \quad \mathscr{R}(x) \cap F(n+1) \neq \emptyset, \ \forall x \in F(n).$$

Proposition 3. *Let the map $F : N \to (2)^X$ and the relation $\mathscr{R}$ over X be such that F is $\mathscr{R}$-chainable. Then, for each $a \in F(0)$ there exists a sequence $(x(n); n \geq 0)$ in X with the properties*

$$x(0) = a; \ x(n) \in F(n), \ \forall n; \ x(n)\mathscr{R}x(n+1), \ \forall n.$$

As before, (SDC) $\Longrightarrow$ (DC) ($\Longleftrightarrow$ (DDC)); just take $(F(n) = X; n \geq 0)$. But the reciprocal is also true, in the sense (DDC) $\Longrightarrow$ (SDC). This follows from

Proof (Proposition 3). Let the premises of (SDC) be admitted. Define a sequence of relations $(\mathscr{R}_n; n \geq 0)$ over X as: for each $n \geq 0$,

$$\mathscr{R}_n(x) = \mathscr{R}(x) \cap F(n+1), \ \text{if } x \in F(n); \mathscr{R}_n(x) = \{x\}, \text{if } x \in X \setminus F(n).$$

Clearly, $\mathscr{R}_n$ is proper, for all $n \geq 0$. So, by (DDC), it follows that, for the starting $a \in F(0)$, there exists an $(a; (\mathscr{R}_n; n \geq 0))$-iterative sequence $(x(n); n \geq 0)$ in X. Combining with the very definition of $(\mathscr{R}_n; n \geq 0)$ yields the desired conclusion.

In particular, when $\mathscr{R} = X \times X$, F is $\mathscr{R}$-chainable. The corresponding variant of (SDC) is just the *Denumerable Axiom of Choice* (in short (AC-N)):

Proposition 4. *Let $F : N \to (2)^X$ be a function. Then, for each $a \in F(0)$, there exists a function $f : N \to X$ with $f(0) = a$ and $f(n) \in F(n)$, $\forall n \geq 0$.*

As a consequence of the above facts,

(DC) $\Longrightarrow$ (AC-N) in the strongly reduced system (ZF-AC); i.e.,
(AC-N) is deductible in the reduced system (ZF-AC+DC).

A direct verification of this is obtainable by taking $P = N \times X$ and introducing the relation over it:

$$\mathscr{R}(n,x) = \{n+1\} \times F(n+1), (n,x) \in P;$$

we do not give details. The reciprocal of the written inclusion is not true; see, for instance, Moskhovakis [24, Chap. 8, Sect. 8.25].

3 Asymmetric Spaces

In the following, some auxiliary facts involving extended asymmetrics and related functional spaces will be discussed.

(A) Remember that $R_+ = [0, \infty[$ denotes the positive (real) half-axis. Here, the symbol ∞ has no concrete existence; because, ultimately, $R_+ = \{x \in R; x \geq 0\}$.

Now, let ∞ be an element not belonging to R; and put

$R_+ \cup \{\infty\} = [0, \infty]$ (the extended positive (real) half-axis).

Before developing the specific facts to be used, we must introduce an ordering and algebraic/topological structure over this generalized positive interval.

(A-1) Let us extend the (strict) order ($<$) on R_+ to a similar object on $R_+ \cup \{\infty\}$. Precisely, define the relation $\mathscr{S}$ on $R_+ \cup \{\infty\}$ as

$$\mathscr{S} = \{(a, \infty); a \in R_+\} \cup \{(t, s) \in R_+ \times R_+; t < s\}.$$

It is not hard to see that

(so-1) $\mathscr{S}$ is irreflexive:
$t\mathscr{S}t$ is false, for each $t \in R_+ \cup \{\infty\}$

(so-2) $\mathscr{S}$ is transitive:
$t_1, t_2, t_3 \in R_+ \cup \{\infty\}$, $t_1\mathscr{S}t_2$, $t_2\mathscr{S}t_3 \Longrightarrow t_1\mathscr{S}t_3$.

For simplicity, we again denote by ($<$) this relation $\mathscr{S}$; i.e.,

($t, s \in R_+ \cup \{\infty\}$): $t\mathscr{S}s$ iff (by definition) $t < s$;

and call it the *strict order* of $R_+ \cup \{\infty\}$. The associated relation ($\leq$) on $R_+ \cup \{\infty\}$ introduced as

$$t \leq s \text{ iff either } t < s \text{ or } t = s$$

is therefore reflexive, transitive, and antisymmetric; hence an *order* on $R_+ \cup \{\infty\}$. Moreover, ($\leq$) is *total*; i.e.,

(R-tot) $\forall t, s \in R_+ \cup \{\infty\}$: $t \leq s$ or $t > s$.

Finally, $R_+ \cup \{\infty\}$ is ($\leq$)-*complete*; i.e.,

(R-com) each subset A of $R_+ \cup \{\infty\}$ admits a supremum, $\sup(A)$.

(A-2) Further, we introduce a convergence structure on $R_+ \cup \{\infty\}$. Given a sequence $(\lambda_n; n \geq 0)$ in $R_+ \cup \{\infty\}$ and an element $\lambda \in R_+ \cup \{\infty\}$, let us say that $\lambda_n \to \lambda$ (and read: (λ_n) *converges* to λ), provided

(Case $\lambda < \infty$): for each $\varepsilon > 0$, there exists $n(\varepsilon) \geq 0$ such that $n \geq n(\varepsilon)$ implies $\max\{0, \lambda - \varepsilon\} \leq \lambda_n \leq \lambda + \varepsilon$

(Case $\lambda = \infty$): for each $\delta > 0$, there exists $n(\delta) \geq 0$ such that $n \geq n(\delta)$ implies $\lambda_n \geq \delta$.

An equivalent way of expressing this is the following. Denote, for each sequence $(\lambda_n; n \geq 0)$ in $R_+ \cup \{\infty\}$

$$\liminf_n(\lambda_n) = \sup_k \inf\{\lambda_k, \lambda_{k+1}, \ldots\},$$
$$\limsup_n(\lambda_n) = \inf_k \sup\{\lambda_k, \lambda_{k+1}, \ldots\}$$

(the *inferior* and *superior* limit of our sequence). Then, for any such sequence (λ_n) in $R_+ \cup \{\infty\}$ and any element λ in $R_+ \cup \{\infty\}$, we have

$$\lambda_n \to \lambda \text{ iff } \liminf_n(\lambda_n) = \limsup_n(\lambda_n) = \lambda;$$

since the verification is immediate, we do not give details. The convergence structure $(\to)$ we just defined has the properties

(conv-1) $((\to)$ is reflexive)
$(\forall u \in R_+ \cup \{\infty\})$: $(t_n = u; n \geq 0)$ fulfills $t_n \to u$

(conv-2) $((\to)$ is hereditary)
if $t_n \to t$, then $r_n \to t$, for each subsequence (r_n) of (t_n);

so, it fulfills the general requirements in Kasahara [21]. This allows to give a (sequential) characterization of continuity for functions in the classes $\mathscr{F}(R_+ \cup \{\infty\})$ and/or $\mathscr{F}(R_+ \cup \{\infty\} \times R_+ \cup \{\infty\}, R_+ \cup \{\infty\})$. For example, we say that the function $\Gamma : R_+ \cup \{\infty\} \to R_+ \cup \{\infty\}$ is *continuous* at $u \in R_+ \cup \{\infty\}$, provided

for each sequence (t_n) in $R_+ \cup \{\infty\}$ with $t_n \to u$, we have $\Gamma(t_n) \to \Gamma(u)$.

Likewise, we say that the function $\Delta : R_+ \cup \{\infty\} \times R_+ \cup \{\infty\} \to R_+ \cup \{\infty\}$ is *continuous* at $(u, v) \in R_+ \cup \{\infty\} \times R_+ \cup \{\infty\}$, if

for each couple of sequences (t_n) and (s_n) in $R_+ \cup \{\infty\}$ with $t_n \to u$, $s_n \to v$, we have $\Delta(t_n, s_n) \to \Delta(u, v)$.

These will be useful in the sequel.

(A-3) Under the above preliminaries, we may now introduce a basic convention. Let $(\oplus)$ be a binary operation on R_+, endowed with the properties

(bo-1) $(\oplus)$ is associative:
$(t_1 \oplus t_2) \oplus t_3 = t_1 \oplus (t_2 \oplus t_3)$, for all $t_1, t_2, t_3 \in R_+$

(bo-2) $(\oplus)$ has $0 \in R_+$ as a null element:
$t \oplus 0 = 0 \oplus t = t$, $\forall t \in R_+$

(bo-3) $(\oplus)$ is first variable continuous increasing:
$\forall s \in R_+$, the map $t \mapsto t \oplus s$ is continuous and increasing over R_+

(bo-4) $(\oplus)$ is second variable continuous strictly increasing:
$\forall t \in R_+$, the map $s \mapsto t \oplus s$ is continuous and strictly increasing over R_+;
hence, in particular, $t \oplus s_1 \leq t \oplus s_2$ implies $s_1 \leq s_2$;

we then say that $(\oplus)$ is an *addition-like operation* on R_+. For an easy reference, we also list the following extra conditions to be considered:

(boe-1) $(\oplus)$ is commutative: $t \oplus s = s \oplus t$, for each $t, s \in R_+$

(boe-2) $(\oplus)$ is continuous (on $R_+ \times R_+$):
$t_n \to t$ and $s_n \to s$ imply $t_n \oplus s_n \to t \oplus s$.

The usefulness of these conventions will become clear a bit further.

Suppose that we defined such an object. Then, if we take (by definition)

$$a \oplus \infty = \infty \oplus a = \infty, \forall a \in R_+ \cup \{\infty\}$$

we get an extended operation over $R_+ \cup \{\infty\}$; for each $t, s \in R_+ \cup \{\infty\}$, $t \oplus s$ will be referred to as the $(\oplus)$-*sum* between t and s. Moreover, one may define the *repeated* $(\oplus)$-sum on $R_+ \cup \{\infty\}$, according to the iterative procedure

$$\oplus\{t_1, t_2\} = t_1 \oplus t_2; \oplus\{t_1, t_2, t_3\} = [\oplus\{t_1, t_2\}] \oplus t_3; \dots$$
$$\oplus\{t_i; 1 \leq i \leq n+1\} = [\oplus\{t_i; 1 \leq i \leq n\}] \oplus t_{n+1}, \forall n \geq 3.$$

This extended operation $(\oplus)$ has all properties of the initial one, with $R_+ \cup \{\infty\}$ in place of R_+. The assertion is immediate in case of associative and null element property. It is also clear in case of (first and second variable) increasing properties; we do not give details. To complete the claim, it remains to verify that this extended operation is continuous at infinity with respect to the first and second variable. A positive answer to this is contained in

Proposition 5. *Let* $(\oplus)$ *be the extended operation we just defined. Then,*

(31-1) $t \oplus s \geq \max\{t, s\}$, *for all* $t, s \in R_+ \cup \{\infty\}$

(31-2) $(\oplus)$ *is first variable continuous at infinity: for each* $s \in R_+ \cup \{\infty\}$*, and each sequence* (t_n) *in* $R_+ \cup \{\infty\}$ *with* $t_n \to \infty$*, we have* $t_n \oplus s \to \infty = \infty \oplus s$

(31-3) $(\oplus)$ *is second variable continuous at infinity: for each* $t \in R_+ \cup \{\infty\}$*, and each sequence* (s_n) *in* $R_+ \cup \{\infty\}$ *with* $s_n \to \infty$*, we have* $t \oplus s_n \to \infty = t \oplus \infty$

(31-4) $(\oplus)$ *is (globally) continuous at* (∞, ∞)*: for each couple of sequences* (t_n) *and* (s_n) *in* $R_+ \cup \{\infty\}$ *with* $t_n \to \infty$*,* $s_n \to \infty$*, we have* $t_n \oplus s_n \to \infty = \infty \oplus \infty$.

Proof. (i): For each $t, s \in R_+ \cup \{\infty\}$, we have (by the properties of our operation)

$$t \oplus s \geq t \oplus 0 = t, \; t \oplus s \geq 0 \oplus s = s;$$

wherefrom, the first conclusion follows.

(ii), (iii), (iv): Evident, by the above relation.

In the following, some important examples of such operations will be given.

Example 1. Let $(+)$ denote the *(standard) addition* of R_+. Clearly, $(+)$ is associative and commutative. On the other hand, $(+)$ has $0 \in R$ as null element and is continuous; hence, all the more, first and second variable continuous. In addition, $(+)$ is first and second variable strictly increasing; hence, it is an addition-like on R_+. Finally, if we take

$$a + \infty = \infty + a = \infty, \forall a \in R_+ \cup \{\infty\}.$$

the resulting operation $(+)$ is an addition-like on $R_+ \cup \{\infty\}$, endowed with the commutative property. This, in particular, allows us to define the *repeated* addition on $R_+ \cup \{\infty\}$, according to

$\sum\{t_1, t_2\} = t_1 + t_2$; $\sum\{t_1, t_2, t_3\} = \sum\{t_1, t_2\} + t_3$; ...
$\sum\{t_i; 1 \leq i \leq n+1\} = \sum\{t_i; n \leq i \leq n\} + t_{n+1}$, $\forall n \geq 3$.

In fact, some other properties of this operation are available; we do not give details.

Example 2. Let $(\oplus)$ stand for the binary operation over R_+:

$(t, s \in R_+)$: $t \oplus s = t + s + ts$;

referred to as *multi-addition*. Clearly, $(\oplus)$ is associative and commutative. On the other hand, $(+)$ has $0 \in R$ as null element and is continuous (hence, all the more, first and second variable continuous). In addition, $(+)$ is first and second variable strictly increasing; hence, it is an addition-like on R_+. To verify this, it will suffice to consider the second variable case; i.e., the property

$$(\forall t \in R_+, \forall s_1, s_2 \in R_+): s_1 < s_2 \text{ implies } t \oplus s_1 < t \oplus s_2.$$

Suppose that $s_1 < s_2$; then, by definition,

$t + s_1 < t + s_2$, $ts_1 \leq ts_2$; wherefrom $t \oplus s_1 < t \oplus s_2$;

and we are done. Finally, under the notation

$R_+^0 =]0, \infty[$ (i.e., $R_+^0 = \{x \in R; x > 0\}$) (the strict positive half-axis),

let us accept the conventions

$a + \infty = \infty + a = \infty$, $\forall a \in R_+ \cup \{\infty\}$;
$a \cdot \infty = \infty \cdot a = \infty$, $\forall a \in R_+^0 \cup \{\infty\}$; $0 \cdot \infty = \infty \cdot 0 = 0$.

We then get the relations

$a \oplus \infty = \infty \oplus a = \infty$, $\forall a \in R_+ \cup \{\infty\}$;

and this yields an extended multi-addition $(\oplus)$ on $R_+ \cup \{\infty\}$, which is an addition-like on $R_+ \cup \{\infty\}$. In particular, we may define the *repeated* multi-addition on $R_+ \cup \{\infty\}$, according to the iterative procedure (see above)

$\oplus\{t_1, t_2\} = t_1 \oplus t_2$; $\oplus\{t_1, t_2, t_3\} = [\oplus\{t_1, t_2\}] \oplus t_3$; ...
$\oplus\{t_i; 1 \leq i \leq n+1\} = [\oplus\{t_i; n \leq i \leq n\}] \oplus t_{n+1}$, $\forall n \geq 3$.

As before, some other properties of this operation are valid; but these will be not effectively needed later.

Example 3. Let $(*)$ be an addition-like operation over R_+; and $\beta \in \mathscr{F}(R_+)$ be a function endowed with the properties

β is a strictly increasing continuous bijection from R_+ to R_+.

Note that, by these conditions, one gets

(sic-bij) β^{-1} is a strictly increasing continuous bijection from R_+ to R_+.

In fact, the strictly increasing bijection property for β^{-1} is clear; so, it remains to establish that β^{-1} is continuous, i.e., right and left continuous:

(right-con) $(\forall s \geq 0)$: $\beta^{-1}(s) = \beta^{-1}(s+) := \inf\{\beta^{-1}(t); t > s\}$
(left-con) $(\forall s > 0)$: $\beta^{-1}(s) = \beta^{-1}(s-) := \sup\{\beta^{-1}(t); t < s\}$.

We shall discuss the right continuous property; the left continuous one will be handled in a similar way. So, let $s \geq 0$ be arbitrary fixed. Assume by contradiction that the written relation is not true:

$$\inf\{\beta^{-1}(t); t > s\} > \beta^{-1}(s).$$

Combining with the bijection property of β^{-1}, there exists some $r \in R_+$ such that

$$\beta^{-1}(t) > \beta^{-1}(r) > \beta^{-1}(s), \text{ for all } t > s.$$

This, along with the strictly increasing property of β, gives

$$t > r > s, \text{ for all } t > s, \text{ a contradiction}$$

(just take any $t \in]s, r[$ to verify this). Hence, the working assumption above is not acceptable; and our assertion follows.

Having these precise, define a new operation $(\oplus)$ over R_+, according to

$$t \oplus s = \beta^{-1}(\beta(t) * \beta(s)),\ t, s \in R_+.$$

Clearly, $(\oplus)$ is associative but not in general commutative. On the other hand, as

$$\beta(0) = 0 \text{ (hence, } \beta^{-1}(0) = 0),$$

$(\oplus)$ has $0 \in R$ as null element. Moreover, since both β and β^{-1} are continuous and strictly increasing, $(\oplus)$ is first variable continuous increasing and second variable continuous strictly increasing; hence, it is an addition-like on R_+. Further, if we take

$$a \oplus \infty = \infty \oplus a = \infty,\ \forall a \in R_+ \cup \{\infty\},$$

the resulting operation $(\oplus)$ on $R_+ \cup \{\infty\}$ is an addition-like over $R_+ \cup \{\infty\}$. [The claim follows at once by taking

$$\beta(\infty) = \beta^{-1}(\infty) = \infty$$

and noting that (the extended functions) β and β^{-1} are continuous at infinity:

$$\beta(t_n) \to \infty = \beta(\infty) \text{ and } \beta^{-1}(t_n) \to \infty = \beta^{-1}(\infty) \text{ as } t_n \to \infty;$$

we do not give details.] This, in particular, allows us to define the extended operation $(\oplus)$ on $R_+ \cup \{\infty\}$, according to

$$\oplus\{t_1, t_2\} = t_1 \oplus t_2;\ \oplus\{t_1, t_2, t_3\} = [\oplus\{t_1, t_2\}] \oplus t_3;\ \ldots$$
$$\oplus\{t_i; 1 \leq i \leq n+1\} = [\oplus\{t_i; n \leq i \leq n\}] \oplus t_{n+1},\ \forall n \geq 3.$$

Note finally that, if β is the identity function of $\mathscr{F}(R_+)$, the associated operation $(\oplus)$ is just the (standard) addition.

(B) Let X be a nonempty set. Call the subset Y of X, *almost-singleton* (in short *asingleton*) provided [$y_1, y_2 \in Y$ implies $y_1 = y_2$] and *singleton* if, in addition, Y is nonempty; note that, in this case, $Y = \{y\}$, for some $y \in X$.

Any map $g : X \times X \to R_+ \cup \{\infty\}$ with the property

(refle) g is reflexive ($g(x, x) = 0$, $\forall x \in X$)

will be referred to as a *(generalized) reflexive asymmetric* (in short *r-asymmetric*) of X; the associated structure (X, g) will be referred to as a *r-asymmetric space*.

Suppose that we introduced such an object. A natural convergence and Cauchy structure over X may be defined as below.

(B-1) We say that the sequence (x_n) in X, *g-converges* to $x \in X$ (and write: $x_n \xrightarrow{g} x$) iff $g(x_n, x) \to 0$ as $n \to \infty$; that is

$$\forall \delta > 0, \exists p = p(\delta), \forall n\colon\ (p \leq n \Longrightarrow g(x_n, x) < \delta).$$

Then x is called a *g-limit* of (x_n); the set of all these will be denoted as $g - \lim_n(x_n)$ [or, simply, $\lim_n(x_n)$ when g is understood]; if such elements exist, we say that (x_n) is *g-convergent*.

Clearly, the introduced convergence $(\xrightarrow{g})$ has the properties

(conve-1) $((\xrightarrow{g})$ is reflexive)
$(\forall u \in X)$: $(x_n = u; n \geq 0)$ fulfills $x_n \xrightarrow{g} u$

(conve-2) $((\xrightarrow{g})$ is hereditary)
if $x_n \xrightarrow{g} x$, then $y_n \xrightarrow{g} x$, for each subsequence (y_n) of (x_n);

so, it fulfills the general requirements in Kasahara [21]. Concerning some other convergence properties to be used, the following one is of interest:

(sepa) $(\xrightarrow{g})$ is separated (referred to as g is separated):
$\lim_n(x_n)$ is an asingleton, for each sequence $(x_n; n \geq 0)$ in X.

In this case, note that—given the g-convergent sequence (x_n)—we must have

$\lim_n(x_n) = \{z\}$ (for some $z \in X$);

this will be written as $\lim_n(x_n) = z$. Moreover (under the same framework)

(suf) g is sufficient: $g(x, y) = 0$ implies $x = y$.

In fact, let $u, v \in X$ be such that $g(u, v) = 0$. The constant sequence $(x_n = u; n \geq 0)$ fulfills $x_n \xrightarrow{g} u$, $x_n \xrightarrow{g} v$; so that (by the separated property), $u = v$.

(B-2) Further, let us say that the sequence $(x_n; \geq 0)$ in X is *g-Cauchy*, when $g(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; i.e.,

$$\forall \delta > 0, \exists q = q(\delta), \forall (m, n)\colon\ (q \leq m < n \Longrightarrow g(x_m, x_n) < \delta).$$

The class of all these will be indicated as Cauchy(X, g); some basic properties of it are described below:

(Cauchy-1) (inclusion of constant sequences):
$\forall u \in X$, the constant sequence $(x_n = u; n \geq 0)$ is g-Cauchy

(Cauchy-2) (the hereditary property):
$(x_n; n \geq 0)$ is g-Cauchy implies $(y_n; n \geq 0)$ is g-Cauchy,
for each subsequence $(y_n; n \geq 0)$ of $(x_n; n \geq 0)$.

In the following, a useful property is established for the sequences in X that are not endowed with the g-Cauchy property.

Proposition 6. *Suppose that* $(x_n; n \geq 0)$ *is a sequence in* X*, with*

$(x_n; n \geq 0)$ *is not g-Cauchy.*

There exist then a number $\eta > 0$ *and a couple of rank-sequences* $(m(j); j \geq 0)$, $(n(j); j \geq 0)$*, with*

(32-1) $j \leq m(j) < n(j)$, $\forall j \geq 0$*; hence,* $(m(j) \to \infty$, $n(j) \to \infty)$ *as* $j \to \infty$
(32-2) $d(x_{m(j)}, x_{n(j)}) \geq \eta$, $\forall j \geq 0$.

Proof. By the very definition of this concept, the negation of g-Cauchy property for $(x_n; n \geq 0)$ means: there exists $\eta > 0$, such that

$$\mathscr{R}_j := \{(m,n) \in N \times N; j \leq m < n, d(x_m, x_n) \geq \eta\} \neq \emptyset, \ \forall j \geq 0.$$

In this case, denote for each $j \geq 0$,

$m(j) = \min \mathrm{Dom}(\mathscr{R}_j)$, $n(j) = \min \mathscr{R}_j(m(j))$.

The couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$ fulfills the desired properties; and then, conclusion follows.

Finally, note that (by the properties of our asymmetric) a g-convergent sequence need not be g-Cauchy; however, X is called *g-complete*, when each g-Cauchy sequence is g-convergent.

(C) Now, according to our previous conventions, we may introduce a *g-closure* operator on X as follows. Let Y be a subset of X. We say that $u \in X$ is *g-adherent* to Y, when

(adh-df) $u \in \lim_n(x_n)$, for some sequence $(x_n; n \geq 0)$ in Y.

The set of all such points will be called the *g-closure* of Y and denoted as $\mathrm{cl}_g(Y)$.

The basic properties of the mapping $Y \mapsto \mathrm{cl}_g(Y)$ are contained in the following

Lemma 2. *Under the above conventions, we have*

(adh-1) *(progressiveness)* $Y \subseteq \mathrm{cl}_g(Y)$, $\forall Y \in 2^X$,
(adh-2) *(identity)* $\emptyset = \mathrm{cl}_g(\emptyset)$, $X = \mathrm{cl}_g(X)$,
(adh-3) *(monotonicity)* $Y_1 \subseteq Y_2$ *implies* $\mathrm{cl}_g(Y_1) \subseteq \mathrm{cl}_g(Y_2)$,
(adh-4) *(additivity)* $\mathrm{cl}_g(U \cup V) = \mathrm{cl}_g(U) \cup \mathrm{cl}_g(V)$, $\forall U, V \in 2^X$.

Proof. (adh-1), (adh-2), (adh-3): Evident.

(adh-4): The right to left inclusion is clear, by the monotone property; so, it remains to verify the left to right inclusion. Let $w \in \mathrm{cl}_g(U \cup V)$ be arbitrary fixed:

$$w \in \lim_n(z_n), \text{ for some sequence } (z_n; n \geq 0) \text{ in } U \cup V.$$

If the alternative below holds

for each (index) h, there exists (another index) $k > h$, with $x_k \in U$,

then a strictly ascending rank-sequence $(i(n); n \geq 0)$ may be found, such that

$$w \in \lim_n(z_{i(n)}), \text{ where } (z_{i(n)}; n \geq 0) = \text{sequence in } U;$$

so, $w \in \mathrm{cl}_g(U)$. Otherwise (if the opposite alternative holds), we must have (by the choice of our sequence)

there exists an index h, fulfilling: $z_k \in V$, for all $k > h$;

and this tells us that

$$w \in \lim_n(z_{h+1+n}), \text{ where } (z_{h+1+n}; n \geq 0) = \text{sequence in } V;$$

so, necessarily, $w \in \mathrm{cl}_g(V)$.

When $Y = \mathrm{cl}_g(Y)$, we say that Y is *g-closed*; this, according to the above, may be characterized as

if $(x_n; n \geq 0)$ is a sequence in Y and $x_n \stackrel{g}{\longrightarrow} x$, then $x \in Y$.

Denote, for simplicity

$\mathscr{K}(g) = \{Y \in 2^X; Y = \mathrm{cl}_g(Y)\}$ (the class of all g-closed subsets in X).

Some basic properties of this class are listed below. [Since the verification is immediate, we do not give details.]

Lemma 3. *Under the introduced conventions:*

(clo-1) $\emptyset, X \in \mathscr{K}(g)$
(clo-2) $U, V \in \mathscr{K}(g) \Longrightarrow U \cup V \in \mathscr{K}(g)$
(clo-3) $\mathscr{A} \subseteq \mathscr{K}(g) \Longrightarrow \cap\mathscr{A} \in \mathscr{K}(g)$.

A useful completion of these facts is the following. Denote

$\mathscr{K}(g, \mathrm{non})$ = the class of all nonempty g-closed parts of X.

This class is nonvoid, because $X \in \mathscr{K}(g, \mathrm{non})$. Moreover, under

g is sufficient (see above),

all singletons of X belong to $\mathscr{K}(g, \mathrm{non})$. In fact, let $u \in X$ be arbitrary fixed; and put $U = \{u\}$. If $v \in \mathrm{cl}_g(U)$, then, by definition, $v \in \lim_n(u_n)$, for some sequence $(u_n; n \geq 0)$ of U. This yields $d(u, v) = 0$, wherefrom (as g is sufficient) $u = v$, i.e., $v \in U$. Summing up, we have $U = \mathrm{cl}_g(U)$; and the claim follows.

Technically speaking, the ambient r-asymmetric structure (X, g) must be viewed as a (sequential) convergence space—endowed with a closure operator—and not as a topological space. To motivate our assertion, define

$\mathscr{T}(g) = \{Z \in 2^X; X \setminus Z \in \mathscr{K}(g)\}$ (the class of all *g-open* subsets in X).

By the result above, the following properties of this class are available:

(open-1) $\emptyset, X \in \mathscr{T}(g)$
(open-2) $U, V \in \mathscr{T}(g) \Longrightarrow U \cap V \in \mathscr{T}(g)$
(open-3) $\mathscr{B} \subseteq \mathscr{T}(g) \Longrightarrow \cup\mathscr{B} \in \mathscr{T}(g)$.

This class is therefore a *topology* on X, according to Bourbaki [5, Chap. I, Sect. 1]. Hence, it has an associated closure operator, introduced as: for each $Y \in 2^X$,

$x \in \text{cl}_{(g)}(Y)$ iff $U \cap Y \neq \emptyset$, whenever $U \in \mathscr{T}(g)$ fulfills $x \in U$.

However, this closure operator $Y \mapsto \text{cl}_{(g)}(Y)$ does not coincide with the initial closure operator $Y \mapsto \text{cl}_g(Y)$. An explanation of this bad property is due to the fact that—by the very choice of $g(.,.)$—the closure operator $Y \mapsto \text{cl}_g(Y)$ does not satisfy a condition like

(adh-5) (idempotence) $\text{cl}_g(\text{cl}_g(Y)) = \text{cl}_g(Y)$, $\forall Y \in 2^X$;

so that, it cannot be viewed as a (genuine) Kuratowski closure operator [22, Chap. I, Sect. 4]. Further aspects may be found in Engelking [14, Chap. 1, Sect. 1.2].

(D) We are now introducing a basic notion. Let $\varphi : X \to R_+ \cup \{\infty\}$ be a function; we call it *g-lsc* on X, provided

$\varphi(x) \leq \liminf_n \varphi(x_n)$, whenever $x_n \xrightarrow{g} x$.

A useful characterization of this concept is offered by

Lemma 4. *The function $\varphi : X \to R_+ \cup \{\infty\}$ is g-lsc on X iff*

$[\varphi \leq t] := \{x \in X; \varphi(x) \leq t\}$ is g-closed, for each $t \in R_+$.

Proof. (i) Suppose that φ is g-lsc on X; and (given $t \in R_+$), let $(x_n; n \geq 0)$ be a sequence in $[\varphi \leq t]$ (hence, $\varphi(x_n) \leq t$, for all n), with $x_n \xrightarrow{g} x$. Then,

$$\varphi(x) \leq \liminf_n \varphi(x_n) \leq t.$$

(ii) Suppose that the property in this statement is true; and assume that $x_n \xrightarrow{g} x$. If $\liminf_n \varphi(x_n) = \infty$, we are done; so, it remains to discuss the case of

$\liminf_n \varphi(x_n)(= \sup_k \inf\{\varphi(x_k), \varphi(x_{k+1}), \ldots\}) < \infty$.

Assume by contradiction that $\liminf_n \varphi(x_n) < \varphi(x)$; and pick some $\lambda \in R_+$ with $\liminf_n \varphi(x_n) < \lambda < \varphi(x)$. By the previous relation, we must have

$$\inf\{\varphi(x_k), \varphi(x_{k+1}), \ldots\} < \lambda, \ \forall k \geq 0.$$

This, by the very definition of the involved concept, tells us that there exists a subsequence $(y_n := x_{i(n)}; n \geq 0)$ of $(x_n; n \geq 0)$, with

$$(y_n \xrightarrow{g} x \text{ and}) \ \varphi(y_n) < \lambda, \text{ for all } n \geq 0.$$

By the quoted property in the statement, we then have $\varphi(x) \leq \lambda < \varphi(x)$, contradiction. Hence, our conclusion must be true.

Denote for simplicity

$$\mathscr{F}(g - lsc)(X, R_+ \cup \{\infty\}) = \{\varphi \in \mathscr{F}(X, R_+ \cup \{\infty\});\ \varphi \text{ is } g\text{-lsc}\}.$$

A natural question is to determine whether this class is invariant to algebraic and/or topological operations upon its functions.

(D-1) Concerning the algebraic properties of our class, let $(\oplus)$ be an addition-like over $R_+ \cup \{\infty\}$. The corresponding functional operation over $\mathscr{F}(X, R_+ \cup \{\infty\})$ will be denoted in the same way; i.e., for $\psi, \chi \in \mathscr{F}(X, R_+ \cup \{\infty\})$, $\psi \oplus \chi$ denotes the function in $\mathscr{F}(X, R_+ \cup \{\infty\})$ introduced as

$$(\psi \oplus \chi)(x) = \psi(x) \oplus \chi(x),\ x \in X.$$

Then, call the addition-like $(\oplus)$, *g-lsc compatible* when

$$\psi, \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\}) \Longrightarrow \psi \oplus \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\}).$$

As we shall see, a property of this type is possible under the regularity conditions imposed upon our addition-like $(\oplus)$. Precisely, we have

Proposition 7. *Under these conventions,*

(33-1) *For each* $\lambda > 0$ *and* $\psi \in \mathscr{F}(g - lsc)(X, R_+ \cup \{\infty\})$*, we necessarily have* $\lambda\psi \in \mathscr{F}(g - lsc)(X, R_+ \cup \{\infty\})$
(33-2) *The addition-like* $(\oplus)$ *is g-lsc compatible (see above)*
(33-3) *The (standard) addition is g-lsc compatible*
(33-4) *The multi-addition is g-lsc compatible.*

Proof. (i) Evident.

(ii) Letting $\psi, \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$, assume that the sequence $(x_n; n \geq 0)$ in X fulfills

$$\psi(x_n) \oplus \chi(x_n) \leq t,\ \forall n \text{ (for some } t \in R_+)$$
$$x_n \xrightarrow{g} x, \text{ for some } x \in X;$$

note that, by the former of these (and definition of extended operation)

$$(\psi(x_n); n \geq 0) \text{ and } (\chi(x_n); n \geq 0) \text{ are sequences in } R_+.$$

The alternative

$$\sup\{\psi(x_n); n \geq 0\} = \infty \text{ or } \sup\{\chi(x_n); n \geq 0\} = \infty$$

is impossible; for, e.g., if the former of these holds, then (by the above remark involving finite values of our sequences)

$$\sup\{\psi(x_n); n \geq k\} = \infty,\ \forall k \geq 0;\ \text{ whence } \limsup_n \psi(x_n) = \infty;$$

and this yields

$$t \geq \limsup_n(\psi(x_n) \oplus \chi(x_n)) \geq \limsup_n(\psi(x_n)) = \infty > t,$$

contradiction. Hence, necessarily

$$\sup\{\psi(x_n); n \geq 0\} < \gamma < \infty,\ \sup\{\chi(x_n); n \geq 0\} < \delta < \infty,$$

for some $\gamma, \delta \in R_+$; or, equivalently,

both sequences $(\psi(x_n); n \geq 0)$ and $(\chi(x_n); n \geq 0)$ are bounded (in R_+).

Denote, for $k \geq 0$

$\alpha_k = \inf\{\psi(x_k), \psi(x_{k+1}), \ldots\}$, $\beta_k = \inf\{\chi(x_k), \chi(x_{k+1}), \ldots\}$.

Clearly, $(\alpha_k; k \geq 0)$ and $(\beta_k; k \geq 0)$ are ascending and

$\alpha := \lim_n(\alpha_n) = \liminf_n(\psi(x_n))$, $\beta := \lim_n(\beta_n) = \liminf_n(\chi(x_n))$;

moreover, by the increasing properties of $(\oplus)$,

$$\alpha_k \oplus \beta_k \leq \psi(x_k) \oplus \chi(x_k) \leq t, \text{ for all } k \geq 0.$$

Let $m, n \geq 0$ be a couple of ranks with $m \leq n$. From the first variable increasing property of $(\oplus)$, we have (by the above)

$$\alpha_m \oplus \beta_n \leq \alpha_n \oplus \beta_n \leq t, \text{ for all these } (m, n).$$

Passing to limit as $n \to \infty$, we get (by the second variable continuous property of our addition-like $(\oplus)$)

$$\alpha_m \oplus \beta \leq t, \text{ for all } m;$$

and this, combined with the first variable continuous property of $(\oplus)$, yields

$\alpha \oplus \beta \leq t$, where α, β have the precise meaning.

On the other hand, by the g-lsc property of our functions,

$$\psi(x) \leq \liminf_n \psi(x_n) = \alpha,\ \chi(x) \leq \liminf_n \chi(x_n) = \beta.$$

Combining with the above gives (by the increasing properties of $(\oplus)$)

$$\psi(x) \oplus \chi(x) \leq \alpha \oplus \beta \leq t;$$

and conclusion follows.

(iii) The (standard) addition is an addition-like; hence, the preceding part is applicable here.

(iv) The multi-addition is an addition-like; hence, the addition-like-type conclusion is applicable here.

The particular compatible properties above are obtained by means of a general statement involving addition-like operations. For practical reasons, it would be useful having a direct verification of them. This is provided by the following

Proposition 8. *Under the precise conventions, we have:*

(34-1) *the addition* $(+)$ *is g-lsc compatible:*
$\psi, \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$ *imply* $\psi + \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$

(34-2) *the multiplication* $(.)$ *is g-lsc compatible:*
$\psi, \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$ *imply*
$\psi\chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$

(34-3) *the multi-addition is g-lsc compatible:*
$\psi, \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$ *imply*
$\psi + \chi + \psi\chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$.

Proof. (i) Letting $\psi, \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$, assume that the sequence $(x_n; n \geq 0)$ in $[\psi + \chi \leq t]$ (where $t \in R_+$) fulfills $x_n \xrightarrow{g} x$, for some $x \in X$; note that, necessarily,

$(\psi(x_n); n \geq 0)$ and $(\chi(x_n); n \geq 0)$ are sequences in R_+.

The alternative

$\sup\{\psi(x_n); n \geq 0\} = \infty$ or $\sup\{(\chi(x_n); n \geq 0\} = \infty$

is impossible, for, e.g., if the former of these holds, then (by the above remark involving finite values of our sequences)

$\sup\{\psi(x_n); n \geq k\} = \infty, \forall k \geq 0;$ whence $\limsup_n \psi(x_n) = \infty$;

and this yields

$$t \geq \limsup_n(\psi(x_n) + \chi(x_n)) \geq \limsup_n(\psi(x_n)) = \infty > t;$$

contradiction. Hence, necessarily

$$\sup\{\psi(x_n); n \geq 0\} < \alpha^* < \infty, \ \sup\{\chi(x_n); n \geq 0\} < \beta^* < \infty,$$

for some $\alpha^*, \beta^* \in R_+$; or, equivalently,

both sequences $(\psi(x_n); n \geq 0)$ and $(\chi(x_n); n \geq 0)$ are bounded (in R_+).

But then, by a two-step construction, there must be a subsequence $(y_n; n \geq 0)$ of $(x_n; n \geq 0)$, such that

$$\lim_n(\psi(y_n)) = \alpha, \lim_n(\chi(y_n)) = \beta, \text{ for some } \alpha \in [0, \alpha^*], \beta \in [0, \beta^*];$$

moreover, by the very choice of our sequence, $\alpha + \beta \leq t$. On the other hand, by the g-lsc property of our functions,

$$\psi(x) \leq \lim_n \psi(y_n) = \alpha, \ \chi(x) \leq \lim_n \chi(y_n) = \beta.$$

Combining with the above gives $\psi(x) + \chi(x) \leq t$; and conclusion follows.

(ii) Letting $\psi, \chi \in \mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$, assume that the sequence $(x_n; n \geq 0)$ in $[\psi\chi \leq t]$ (where $t \in R_+$) fulfills $x_n \overset{g}{\longrightarrow} x$, for some $x \in X$. If the alternative below holds (effectively)

$$\{n \in N; \psi(x_n) = 0\} \text{ is infinite or } \{n \in N; \chi(x_n) = 0\} \text{ is infinite,}$$

then we are done; for, e.g., if the former of these is true, the g-lsc property of ψ (at $x \in X$) gives

$$\psi(x) = 0, \text{ wherefrom } \psi(x)\chi(x) = 0 \leq t.$$

It remains now to discuss the opposite alternative; clearly, without loss, it may be written as

$$\psi(x_n) > 0 \text{ and } \chi(x_n) > 0 \text{ (hence, } \psi(x_n) < \infty \text{ and } \chi(x_n) < \infty\text{), for all } n.$$

Now, as before, if the alternative below holds (effectively)

$$\inf\{\psi(x_n); n \geq 0\} = 0 \text{ or } \inf\{\chi(x_n); n \geq 0\} = 0,$$

then, we are again done; for, e.g., if the former of these is true, then (by the preceding remark about values of our sequences)

$$\inf\{\psi(x_n); n \geq k\} = 0, \forall k \geq 0; \text{ whence } \liminf_n(\psi(x_n)) = 0;$$

and then, by the g-lsc property of ψ, one gets

$$\psi(x) = 0, \text{ wherefrom } \psi(x)\chi(x) = 0 \leq t.$$

In this case, we have to discuss the alternative

$$\inf\{\psi(x_n); n \geq 0\} > \alpha_*, \ \inf\{\chi(x_n); n \geq 0\} > \beta_*,$$

for some $\alpha_*, \beta_* \in R_+^0$. In this case, the alternative

$$\sup\{\psi(x_n); n \geq 0\} = \infty \text{ or } \sup\{(\chi(x_n); n \geq 0\} = \infty$$

is impossible; for, e.g., if the former of these holds, then (by the above remark involving finite values of our sequences)

$$\sup\{\psi(x_n); n \geq k\} = \infty, \forall k \geq 0; \text{ whence } \limsup_n(\psi(x_n)) = 0;$$

so that, by the choice of our sequence, one gets (cf. the preceding conclusion)

$$t \geq \limsup_n(\psi(x_n)\chi(x_n)) \geq \limsup_n(\psi(x_n)\beta_*) = \infty > t,$$

contradiction; hence, necessarily

$$\sup\{\psi(x_n); n \geq 0\} < \alpha^*, \ \sup\{\chi(x_n); n \geq 0\} < \beta^*,$$

for some $\alpha^*, \beta^* \in R_+^0$. Summing up,

both sequences $(\psi(x_n); n \geq 0)$ and $(\chi(x_n); n \geq 0)$ are bounded in R_+^0.

But then, by a two-step construction, there must be a subsequence $(y_n; n \geq 0)$ of $(x_n; n \geq 0)$, such that

$$\lim_n(\psi(y_n)) = \alpha, \lim_n(\chi(y_n)) = \beta, \text{ for some } \alpha \in [\alpha_*, \alpha^*], \beta \in [\beta_*, \beta^*];$$

moreover, by the very choice of our sequence, $\alpha\beta \leq t$. On the other hand, by the g-lsc property of our functions,

$$\psi(x) \leq \lim_n \psi(y_n) = \alpha, \ \chi(x) \leq \lim_n \chi(y_n) = \beta.$$

Combining with the above gives $\psi(x)\chi(x) \leq t$; and conclusion follows.

(iii) Evident.

(D-2) We are now passing to the (order and) topological properties of functions in $\mathscr{F}(X, R_+ \cup \{\infty\})$. Define a (pointwise) order on $\mathscr{F}(X, R_+ \cup \{\infty\})$ as

$$\psi \leq \chi \text{ iff } \psi(t) \leq \chi(t), \forall t \in R_+ \cup \{\infty\}.$$

Also, for each ascending sequence $(\psi_n; n \geq 0)$ in $\mathscr{F}(X, R_+ \cup \{\infty\})$, the (pointwise) limit function $\psi \in \mathscr{F}(X, R_+ \cup \{\infty\})$, introduced as

$$\psi(x) = \lim_n(\psi_n(x)) = \sup_n(\psi_n(x)), x \in X$$

is well defined; for simplicity, we denote it as $\psi = \lim_n(\psi_n)$. Call the function $\omega \in \mathscr{F}(X, R_+)$, *g-continuous*, provided

$$x_n \xrightarrow{g} x \text{ implies } \omega(x_n) \to \omega(x).$$

Proposition 9. *Under the precise conventions,*

(35-1) *If* $(\psi_n; n \geq 0)$ *is an ascending sequence in* $\mathscr{F}(g - lsc)(X, R_+ \cup \{\infty\})$*, then* $\psi = \lim_n(\psi_n)$ *belongs to* $\mathscr{F}(g - lsc)(X, R \cup \{\infty\})$

(35-2) *If* $\Lambda \in \mathscr{F}(R_+)$ *is increasing lsc and* $\omega \in \mathscr{F}(X, R_+)$ *is g-lsc, then the composed function* $\psi = \Lambda \circ \omega$ *(i.e.,* $(\psi(x) = \Lambda(\omega(x)); x \in X)$*) belongs to the class* $\mathscr{F}(g - lsc)(X, R_+)$

(35-3) *If* $\Lambda \in \mathscr{F}(R_+)$ *is lsc and* $\omega \in \mathscr{F}(X, R_+)$ *is g-continuous, then the composed function* $\psi = \Lambda \circ \omega$ *(i.e.,* $(\psi(x) = \Lambda(\omega(x)); x \in X)$*) belongs to the class* $\mathscr{F}(g - lsc)(X, R_+)$.

Proof. (i) Let $(\psi_n; n \geq 0)$ be an ascending sequence in $\mathscr{F}(g-lsc)(X, R_+ \cup \{\infty\})$; and $\psi = \lim_n(\psi_n)$ stand for its (pointwise) limit (see above). By the very definition of this function, we have

$$[\psi \leq t] = \cap\{[\psi_n \leq t]; n \geq 0\}, \ t \in R_+;$$

and this, by a previous auxiliary statement (relative to g-closed sets), yields the desired conclusion.

(ii) Suppose that $\Lambda \in \mathscr{F}(R_+)$ is increasing lsc and $\omega \in \mathscr{F}(X, R_+)$ is g-lsc. Letting $t \in R_+$ be arbitrary fixed, denote $M(t) = \{r \in R_+; \Lambda(r) \leq t\}$; clearly, by the increasing property of Λ, we derive

$$M(t) \text{ is left hereditary: } r \in M(t) \Longrightarrow [0, r] \subseteq M(t).$$

If $M(t) = \emptyset$, then

$$\{x \in X; \Lambda(\omega(x)) \leq t\} = \emptyset;$$

and we are done. Suppose now that $M(t) \neq \emptyset$; and put $\sigma = \sup(M)$. If $\sigma = \infty$, then (by the above left hereditary property)

$$M(t) = R_+, \text{ whence, } \{x \in X; \Lambda(\omega(x)) \leq t\} = X;$$

and conclusion follows. Assume now that $\sigma < \infty$. As $\Lambda(.)$ is lsc, we must have $\sigma \in M(t)$; so, again by the left hereditary property, $M(t) = [0, \sigma]$. As a direct consequence, we have the representation

$$\{x \in X; \Lambda(\omega(x)) \leq t\} = \{x \in X; \omega(x) \leq \sigma\};$$

and this (via $\omega(.)$ being g-lsc) appears as g-closed.

(iii) Let (x_n) be a sequence in X and x be some element of X in such a way that $x_n \stackrel{g}{\longrightarrow} x$. By the g-continuous hypothesis about ω, we have

$$t_n \to t, \text{ where } (t_n := \omega(x_n); n \geq 0) \text{ and } t := \omega(x).$$

On the other hand, by the lsc condition upon Λ, we have

$$\Lambda(t) \leq \liminf_n \Lambda(t_n).$$

This, by the introduced notations, may be written as

$$\Lambda(\omega(x)) \leq \liminf_n \Lambda(\omega(x_n));$$

and conclusion follows. The proof is complete.

Concerning the last property, note that the imposed hypotheses are not minimal so as to get the written conclusion; but, for the concrete applications to be considered, this will suffice.

(E) Let X be a nonempty set; and $(e(.,.), d(.,.))$ be a couple of (generalized) r-asymmetrics over it; the triple $(X; e, d)$ will be then referred to as a *double r-asymmetric space*. Call the couple (e, d), *admissible* provided

(adm-1) (e, d) is right semi-lsc: $x \mapsto e(w, x)$ is d-lsc, for each $w \in X$
(adm-2) X is (e, d)-complete: each e-Cauchy sequence is d-convergent
(adm-3) (e, d) is Cauchy separated:

(x_n) is e-Cauchy and $(x_n \stackrel{e}{\longrightarrow} u, x_n \stackrel{d}{\longrightarrow} u, x_n \stackrel{e}{\longrightarrow} v)$ imply $u = v$.

As we shall see, this is the most important condition required by our main variational results. So, it is natural to look for concrete circumstances under which it holds. Two basic situations of this type are described below.

(E-1) Let us say that the couple (e, d) is *KST-admissible*, when

(kst-1) (e, d) is right semi-lsc: $x \mapsto e(w, x)$ is d-lsc, for each $w \in X$
(kst-2) (e, d) is Cauchy transitive: each e-Cauchy sequence is d-Cauchy.

(This is related to the developments in Kada et al. [20]; we do not give details.) For the moment, (e, d) fulfills one of the three conditions appearing in the admissible property. Concerning the remaining ones, the possibility of reaching them is certified by the following auxiliary facts below.

Proposition 10. *Let the couple of r-asymmetrics (e, d) be KST-admissible (see above); with, in addition*

(d-com) X is d-complete: each d-Cauchy sequence in X is d-convergent
(C-sep) e is Cauchy separated:
(x_n) *is e-Cauchy and* $(x_n \xrightarrow{e} u,\ x_n \xrightarrow{e} v)$ *imply* $u = v$.

Then,

(36-1) (x_n) *is e-Cauchy and* $x_n \xrightarrow{d} x$ *imply* $x_n \xrightarrow{e} x$
(36-2) *X is (e, d)-complete: each e-Cauchy sequence in X is d-convergent*
(36-3) *(e, d) is Cauchy separated:*

(x_n) *is e-Cauchy and* $(x_n \xrightarrow{e} u,\ x_n \xrightarrow{d} u,\ x_n \xrightarrow{e} v)$ *imply* $u = v$.
Hence, summing up,

(36-4) *(e, d) is KST-admissible, X is d-complete, and e is Cauchy separated imply (e, d) is admissible.*

Proof. (i) Suppose that the e-Cauchy sequence (x_n) in X fulfills $x_n \xrightarrow{d} x$, for some $x \in X$. Let $\eta > 0$ be arbitrary fixed. By the imposed e-Cauchy property, there exists $k = k(\eta) \geq 0$, such that

$$k \leq m \leq n \text{ implies } e(x_m, x_n) \leq \eta.$$

For the (arbitrary) fixed $m \geq k$, the map $y \mapsto e(x_m, y)$ is d-lsc. So, passing to limit as $n \to \infty$ in the above relation yields

$$e(x_m, x) \leq \eta, \text{ for all ranks } m \geq k.$$

This (by the arbitrariness of $\eta > 0$) tells us that $x_n \xrightarrow{e} x$.

(ii) Let (x_n) be an e-Cauchy sequence in X. As (e, d) is Cauchy transitive, (x_n) is d-Cauchy too; and this, along with X being d-complete, assures us that (x_n) is d-convergent.

(iii), (iv) Evident, by the involved definitions.

(E-2) Clearly, the key condition in the preceding statement is the Cauchy transitive one; so, it is natural asking to what extent is this available under concrete cases. An appropriate answer to this may be given along the lines below. Let us say that (e,d) is *YS-admissible*, when

(ys-1) (e,d) is right semi-lsc: $x \mapsto e(w,x)$ is d-lsc, for each $w \in X$
(ys-2) (e,d) is Cauchy convergence transitive:
(y_n) and (z_n) are e-Cauchy and $e(y_n,z_n) \to 0$ imply $d(y_n,z_n) \to 0$.

(This is related to the developments in Yongxin and Shuzhong [33]; we do not give details.) As before, (e,d) fulfills a condition required by the admissible property. The possibility of reaching the remaining ones is certified by the following (relative) auxiliary facts below.

Proposition 11. *Let the couple of r-asymmetrics (e,d) be YS-admissible (see above); with, in addition*

(d-com) *X is d-complete: each d-Cauchy sequence in X is d-convergent*
(eC-sep) *d is e-Cauchy separated:*
(x_n) is e-Cauchy and ($x_n \xrightarrow{d} u$, $x_n \xrightarrow{d} v$) imply $u = v$.

Then,

(37-1) *(e,d) is Cauchy transitive [each e-Cauchy sequence in X is d-Cauchy]; hence, (e,d) is KST-admissible*
(37-2) *e is Cauchy separated:*
(x_n) is e-Cauchy and ($x_n \xrightarrow{e} u$, $x_n \xrightarrow{e} v$) imply $u = v$.
Summing up, we derived that
(37-3) *(e,d) is YS-admissible, X is d-complete, and d is e-Cauchy separated imply (e,d) is admissible.*

Proof. (i) Let $(x_n; n \geq 0)$ be an e-Cauchy sequence in X; and suppose by contradiction that $(x_n; n \geq 0)$ is not d-Cauchy. By an auxiliary fact above, there must be an $\eta > 0$ and a couple of rank-sequences $(m(j); j \geq 0)$ and $n(j); j \geq 0)$, such that

$$j \leq m(j) < n(j),\ \forall j;\ \text{hence, } (m(j) \to \infty, n(j) \to \infty) \text{ as } j \to \infty$$
$$d(x_{m(j)}, x_{n(j)}) \geq \eta, \text{ for all } j \geq 0.$$

Define the subsequences of $(x_n; n \geq 0)$

$$(y_j = x_{m(j)}; j \geq 0),\ (z_j = x_{n(j)}; j \geq 0);$$

clearly, both these are e-Cauchy sequences—because, so is $(x_n; n \geq 0)$. On the other hand, by the underlying e-Cauchy property of our sequence $(x_n; n \geq 0)$,

$$e(y_j, z_j) = e(x_{m(j)}, x_{n(j)}) \to 0 \text{ as } j \to \infty;$$

and this, combined with (e,d) being Cauchy convergence transitive, gives

$$d(y_j, z_j) = d(x_{m(j)}, x_{n(j)}) \to 0 \text{ as } j \to \infty;$$

in contradiction with a previous relation. Hence, $(x_n; n \geq 0)$ is d-Cauchy; and the claim follows.

(ii) Suppose that the e-Cauchy sequence (x_n) in X fulfills $x_n \xrightarrow{e} u$, $x_n \xrightarrow{e} v$. As (e, d) is Cauchy convergence transitive, we have $x_n \xrightarrow{d} u$, $x_n \xrightarrow{d} v$; and this, along with d being e-Cauchy separated, yields $u = v$, hence the assertion.

(iii) Evident, by the above.

Concerning the last property, note that the imposed hypotheses are not minimal so as to get the written conclusion; but, for the concrete applications to be considered, this will suffice.

4 Main Results

Under these preliminaries, we may now pass to the questions addressed in our introductory part.

(A) Let X be a nonempty set and $(e(.,.), d(.,.))$ be a couple of (generalized) r-asymmetrics over it; the triple $(X; e, d)$ will be then referred to as a *double r-asymmetric space*. Suppose for the moment that

(r-s-lsc) (e, d) is right semi-lsc: $x \mapsto e(w, x)$ is d-lsc, for each $w \in X$

Further, let $(\oplus)$ be an addition-like over R_+; i.e.,

(adli-1) $(\oplus)$ is associative and has $0 \in R_+$ as its null element

(adli-2) $(\oplus)$ is first variable continuously increasing and second variable continuously strictly increasing.

Extend this operation to $R_+ \cup \{\infty\}$, by means of

$$a \oplus \infty = \infty \oplus a = \infty, \forall a \in R_+ \cup \{\infty\}.$$

As precise, the new object $(\oplus)$ is an addition-like on $R_+ \cup \{\infty\}$; so, in particular

for each $t \in R_+$, and each sequence (s_n) in $R_+ \cup \{\infty\}$ with $s_n \to s \in R_+ \cup \{\infty\}$, we have $t \oplus s_n \to t \oplus s$.

Finally, let the corresponding functional operation over $\mathscr{F}(X, R_+ \cup \{\infty\})$ be denoted in the same way; i.e., for $\psi, \chi \in \mathscr{F}(X, R_+ \cup \{\infty\})$, let $\psi \oplus \chi$ denote the function in $\mathscr{F}(X, R_+ \cup \{\infty\})$, introduced as

$$(\psi \oplus \chi)(x) = \psi(x) \oplus \chi(x), x \in X.$$

Remember that, under these conditions (cf. a previous fact)

$(\oplus)$ is d-lsc compatible:
$\psi, \chi \in \mathscr{F}(d{-}lsc)(X, R_+ \cup \{\infty\}) \Longrightarrow \psi \oplus \chi \in \mathscr{F}(d{-}lsc)(X, R_+ \cup \{\infty\})$.

Given a sequence $(\mu_n; n \geq 0)$ in R_+^0 and a sequence $(u_n; n \geq 0)$ in X, let us attach them a sequence $(\zeta_n; n \geq 0)$ of functions in $\mathscr{F}(X, R_+ \cup \{\infty\})$, according to

$$\zeta_n(x) = \mu_n e(u_n, x), x \in X, n \geq 0.$$

By the right semi-lsc property of (e, d), any function of this type is d-lsc. Then—with the aid of this starting sequence—let us construct another functional sequence $(\psi_n; n \geq 0)$ in $\mathscr{F}(X, R_+ \cup \{\infty\})$, by means of iterative procedure

$$\psi_0 = \zeta_0, \psi_1 = \psi_0 \oplus \zeta_1; \text{ and, in general, } \psi_{n+1} = \psi_n \oplus \zeta_{n+1}, n \geq 0.$$

Note that, by the same property of (e, d), any such function is d-lsc. Moreover, from the second variable (strict) increasing property of $(\oplus)$,

$$\psi_n(x) = \psi_{n-1}(x) \oplus \zeta_n(x) \geq \psi_{n-1}(x) \oplus 0 = \psi_{n-1}(x), \ \forall x \in X, \forall n \geq 1.$$

As a consequence, $(\psi_n; n \geq 0)$ is (pointwise) ascending; so that, its (pointwise) limit $\psi_\infty = \lim_n(\psi_n)$, introduced as

$$\psi_\infty(x) = \lim_n(\psi_n(x)) = \sup\{\psi_n(x); n \geq 0\}, x \in X$$

is well defined as an element of $\mathscr{F}(X, R_+ \cup \{\infty\})$. In addition, ψ_∞ is d-lsc; just take an auxiliary fact into account, combined with

$$[\psi_\infty \leq t] = \cap\{[\psi_n \leq t]; n \geq 0\}, \ \forall t \in R_+.$$

Having these precise, fix some function $\varphi \in \mathscr{F}(X, R_+ \cup \{\infty\})$, with:

(p-lsc-1) φ is proper: $\mathrm{Dom}(\varphi) := \{x \in X; \varphi(x) < \infty\} \neq \emptyset$;
whence $0 \leq \inf[\varphi(X)] < \infty$

(p-lsc-2) φ is d-lsc; or, equivalently (see above):
$[\varphi \leq t]$ is d-closed, for each $t \in R_+$.

Denote, for $\varepsilon > 0$, $U \in 2^X$,

$$U(\varphi, \varepsilon) = \{x \in U; \varphi(x) \leq \inf[\varphi(X)] \oplus \varepsilon\} \text{ (hence, } U(\varphi, \varepsilon) \subseteq U \cap \mathrm{Dom}(\varphi)).$$

Our first main result in this exposition is as follows.

Theorem 2. *Let the couple (e, d) of (generalized) r-asymmetrics over X be such that one of the (admissible-type) conditions below holds:*

(ad-1) (e, d) *is admissible*

(ad-2) (e, d) *is KST-admissible, X is d-complete, and e is Cauchy separated*

(ad-3) (e, d) *is YS-admissible, X is d-complete, and d is e-Cauchy separated.*

In addition, let $(\oplus)$ be an addition-like on R_+ (hence, so is its extension $(\oplus)$ over $R_+ \cup \{\infty\}$). Finally, let $(\mu_n; n \geq 0)$ be a sequence in R_+^0; and $u_0 \in X$, $\varepsilon > 0$, $U_0 \in \mathscr{K}(d)$ be taken according to

(reg) $u_0 \in U_0(\varphi, \mu_0\varepsilon)$ *(whence, $u_0 \in U_0 \cap \mathrm{Dom}(\varphi)$).*

Then, a sequence $(u_n; n \geq 0)$ *in* $U_0(\varphi, \mu_0\varepsilon)$ *and an element* $v \in U_0(\varphi, \mu_0\varepsilon)$ *may be determined, in such a way that*

(41-a) (u_n) *is e-Cauchy and* $(u_n \stackrel{e}{\longrightarrow} v,\ u_n \stackrel{d}{\longrightarrow} v)$
(41-b) $\varphi(v) \oplus \psi_\infty(v) \leq \varphi(u_0)$*; hence,* $\varphi(v) \leq \varphi(u_0)$ *and* $\psi_\infty(v) < \infty$
(41-c) $\varphi(x) \oplus \psi_\infty(x) > \varphi(v) \oplus \psi_\infty(v)$*, for each* $x \in U_0 \setminus \{v\}$.

Proof. By a couple of auxiliary results exposed in a previous place, all admissible-type conditions imposed upon (e, d) are finally reducible to the (general) admissible one; so, we may assume in the following that (e, d) is admissible. There are several steps to be passed.

Step 1. Denote for simplicity:

Θ = the class of all triples (t, θ, T) in $X \times \mathscr{F}(d-lsc)(X, R_+ \cup \{\infty\}) \times \mathscr{K}(d)$, fulfilling $t \in T$, $\varphi(t) \oplus \theta(t) < \infty$, $T \subseteq U_0(\varphi, \mu_0\varepsilon)$.

For each $n \geq 0$, let $\mathscr{R}_n$ stand for the relation over Θ, introduced as for each (a, α, A) and (b, β, B) in Θ,

$(a, \alpha, A)\mathscr{R}_n(b, \beta, B)$ iff
$b \in A$, $\varphi(b) \oplus \alpha(b) \leq \Lambda \oplus \mu_{n+1}\varepsilon/2^{n+1}$
(where $\Lambda := \inf\{\varphi(x) \oplus \alpha(x); x \in A\}$),
$(\beta(x) = \alpha(x) \oplus \mu_{n+1}e(b, x); x \in X)$,
$B = \{x \in A; \varphi(x) \oplus \beta(x) \leq \varphi(b) \oplus \beta(b)\}$.

The definition is consistent and $\mathscr{R}_n$ is proper on Θ (for each $n \geq 0$); because, given any triple (a, α, A) in Θ,

(copr-1) $b \in A$ exists, via
$0 \leq \inf[\varphi(X)] \leq \Lambda \leq \varphi(a) \oplus \alpha(a) < \infty$, $\Lambda < \Lambda \oplus \mu_{n+1}\varepsilon/2^{n+1}$
(copr-2) $\beta(.)$ is d-lsc, as a $(\oplus)$-sum of two d-lsc functions (see above)
(copr-3) B is d-closed (as $\varphi \oplus \beta$ is d-lsc and $A \in \mathscr{K}(d)$), with $b \in B$
(copr-4) $\varphi(b) \oplus \beta(b) = \varphi(b) \oplus \alpha(b) \leq \Lambda \oplus \mu_{n+1}\varepsilon/2^{n+1} < \infty$
(copr-5) $B \subseteq U_0(\varphi, \mu_0\varepsilon)$, because $B \subseteq A \subseteq U_0(\varphi, \mu_0\varepsilon)$;

hence, summing up, $(b, \beta, B) \in \Theta$.

Having these precise, take the triple (u_0, ε, U_0) like in the statement; and put

$$(\gamma_0(x) = \mu_0 e(u_0, x); x \in X) \text{ (hence, } \gamma_0 = \psi_0),$$
$$T_0 = \{x \in U_0; \varphi(x) \oplus \gamma_0(x) \leq \varphi(u_0)\}.$$

Clearly, γ_0 is d-lsc, and T_0 is d-closed; moreover,

(start-1) $\varphi(u_0) \oplus \gamma_0(u_0) = \varphi(u_0) < \infty$ (whence, $u_0 \in T_0$)
(start-2) $(\forall x \in T_0)$: $\varphi(x) \leq \varphi(x) \oplus \gamma_0(x) \leq \varphi(u_0) \leq \inf[\varphi(X)] \oplus \mu_0\varepsilon$
(which gives $T_0 \subseteq U_0(\varphi, \mu_0\varepsilon)$);

hence, necessarily, $(u_0, \gamma_0, T_0) \in \Theta$.

By the (Diagonal) Dependent Choice principle, there exists

$((u_n, \gamma_n, T_n); n \geq 0)$ = sequence in Θ, such that

$$(u_n, \gamma_n, T_n)\mathscr{R}_n(u_{n+1}, \gamma_{n+1}, T_{n+1}), \text{ for all } n.$$

The basic properties of this sequence are concentrated in

Proposition 12. *Under the above constructions, we have:*

(41-1) $(\gamma_n(x) = \oplus\{\mu_i e(u_i, x); i \leq n\}; x \in X)$ *[i.e., $\gamma_n = \psi_n$], for all $n \geq 0$ (where the sequence $(\psi_n; n \geq 0)$ was already introduced)*

(41-2) *$(T_n; n \geq 0)$ is a descending sequence of nonempty d-closed sets in X fulfilling $u_n \in T_{n-1} \cap T_n$, for all $n \geq 0$ (where, by definition, $T_{-1} = T_0$)*

(41-3) *the sequence $(\varphi(u_n) \oplus \psi_n(u_n); n \geq 0)$ is descending in R_+; i.e.,*

$\varphi(u_n) \oplus \psi_n(u_n) \leq \varphi(u_{n-1}) \oplus \psi_{n-1}(u_{n-1}), \forall n \geq 0$
(where, by convention, $u_{-1} = u_0$, $\psi_{-1} = \psi_0$)

(41-4) *$e(u_n, y) \leq \varepsilon/2^n$, for each $y \in T_n$ and each $n \geq 0$.*

Proof (Proposition 12). We shall use an inductive argument upon our (already constructed) iterative sequence in Θ.

(iter-0) Let the triple $(u_0, \gamma_0, T_0) \in \Theta$ be introduced as before. By this very definition, we get

$\gamma_0 = \psi_0, u_0 \in T_0 = T_{-1} \cap T_0;$

hence, (41-1)+(41-2) hold for $n = 0$; further, it is clear that (41-3) holds for $n = 0$. Finally, letting $y \in T_0$ be arbitrary fixed, we have (via $\gamma_0 = \psi_0$)

$$\varphi(y) \oplus \psi_0(y) \leq \varphi(u_0);$$

This, along with $T_0 \subseteq \text{Dom}(\varphi)$ and the choice of u_0, gives (by the first variable increasing property of $(\oplus)$)

$$\inf[\varphi(X)] \oplus \mu_0 e(u_0, y) \leq \inf[\varphi(X)] \oplus \mu_0 \varepsilon,$$

whence (by the second variable strictly increasing property of $(\oplus)$), (41-4) holds too for $n = 0$.

(iter-1) For the moment, (41-1)–(41-4) above hold for $n = 0$. Starting from the triple $(u_0, \gamma_0 = \psi_0, T_0) \in \Theta$, the next one $(u_1, \gamma_1, T_1) \in \Theta$ (according to the precise relations) fulfills (via $\gamma_0 = \psi_0$)

$u_1 \in T_0, \varphi(u_1) \oplus \psi_0(u_1) \leq L_0 \oplus \mu_1 \varepsilon/2^1,$
where $L_0 = \inf\{\varphi(x) \oplus \psi_0(x); x \in T_0\},$
$(\gamma_1(x) = \psi_0(x) \oplus \mu_1 e(u_1, x); x \in X),$
$T_1 = \{x \in T_0; \varphi(x) \oplus \gamma_1(x) \leq \varphi(u_1) \oplus \gamma_1(u_1)\}.$

This firstly means that

$\gamma_1 = \psi_1, u_1 \in T_0 \cap T_1;$

hence, (41-1)+(41-2) hold for $n = 1$. Secondly, by the very definition of T_1 (and $\gamma_1 = \psi_1$)

$$(\forall x \in T_1): \ \varphi(x) \oplus \psi_1(x) \leq \varphi(u_1) \oplus \psi_1(u_1) = \varphi(u_1) \oplus \psi_0(u_1).$$

Since $u_1 \in T_0$, we must have

$$\varphi(u_1) \oplus \psi_1(u_1) = \varphi(u_1) \oplus \psi_0(u_1) \leq \varphi(u_0) = \varphi(u_0) \oplus \psi_0(u_0);$$

i.e., (41-3) is holding for $n = 1$. Finally, let $y \in T_1$ be arbitrary fixed; hence, by definition, $y \in T_0$ and (via $\gamma_1 = \psi_1$)

$$\varphi(y) \oplus \psi_1(y) \leq \varphi(u_1) \oplus \psi_1(u_1);$$

or equivalently (by the associative property of $(\oplus)$),

$$(\varphi(y) \oplus \psi_0(y)) \oplus \mu_1 e(u_1, y) \leq \varphi(u_1) \oplus \psi_0(u_1).$$

Combining with $y \in T_0$ (and construction of u_1) gives (by the first variable increasing property of $(\oplus)$)

$$L_0 \oplus \mu_1 e(u_1, y) \leq \varphi(u_1) \oplus \psi_0(u_1) \leq L_0 \oplus \mu_1 \varepsilon/2^1;$$

whence (from the second variable strictly increasing property), (41-4) holds for $n = 1$.

(iter-2) For the moment, (41-1)–(41-4) above hold for $n \in \{0, 1\}$. Starting from the triple $(u_1, \gamma_1 = \psi_1, T_1) \in \Theta$ as before, the next one $(u_2, \gamma_2, T_2) \in \Theta$ (according to the relations above) fulfills

$u_2 \in T_1, \varphi(u_2) \oplus \psi_1(u_2) \leq L_1 \oplus \mu_2 \varepsilon/2^2$,
where $L_1 = \inf\{\varphi(x) \oplus \psi_1(x); x \in T_1\}$,
$(\gamma_2(x) = \psi_1(x) \oplus \mu_2 e(u_2, x); x \in X)$,
$T_2 = \{x \in T_1; \varphi(x) \oplus \gamma_2(x) \leq \varphi(u_2) \oplus \gamma_2(u_2)\}$.

This firstly means that

$$\gamma_2 = \psi_2 \text{ and } u_2 \in T_1 \cap T_2;$$

hence, (41-1)+(41-2) hold for $n = 2$. Secondly, by the very definition of T_2 (and $\gamma_2 = \psi_2$)

$$(\forall x \in T_2): \ \varphi(x) \oplus \psi_2(x) \leq \varphi(u_2) \oplus \psi_2(u_2) = \varphi(u_2) \oplus \psi_1(u_2).$$

Since $u_2 \in T_1$, we must have

$$\varphi(u_2) \oplus \psi_2(u_2) = \varphi(u_2) \oplus \psi_1(u_2) \leq \varphi(u_1) \oplus \psi_1(u_1);$$

i.e., (41-3) is holding for $n = 2$. Finally, let $y \in T_2$ be arbitrary fixed; hence, by definition, $y \in T_1$ and (via $\gamma_2 = \psi_2$)

$$\varphi(y) \oplus \psi_2(y) \leq \varphi(u_2) \oplus \psi_2(u_2);$$

or, equivalently (by the associative property of $(\oplus)$)

$$(\varphi(y) \oplus \psi_1(y)) \oplus \mu_2 e(u_2, y) \leq \varphi(u_2) \oplus \psi_1(u_2).$$

Combining with $y \in T_1$ (and construction of u_2), gives (by the first variable increasing property of $(\oplus)$)

$$L_1 \oplus \mu_2 e(u_2, y) \leq \varphi(u_2) \oplus \psi_1(u_2) \leq L_1 \oplus \mu_2 \varepsilon/2^2;$$

whence (from the second variable strictly increasing property), (41-4) holds for $n = 2$.

...

(iter-h) Assume that (41-1)–(41-4) above hold for $n \in \{0, \ldots, h-1\}$ (where $h \geq 1$). Starting from the triple $(u_{h-1}, \gamma_{h-1} = \psi_{h-1}, T_{h-1}) \in \Theta$ as before, the next one $(u_h, \gamma_h, T_h) \in \Theta$ (according to the relations above) fulfills

$u_h \in T_{h-1}, \varphi(u_h) \oplus \psi_{h-1}(u_h) \leq L_{h-1} \oplus \mu_h \varepsilon/2^h$,
where $L_{h-1} = \inf\{\varphi(x) \oplus \psi_{h-1}(x); x \in T_{h-1}\}$,
$(\gamma_h(x) = \psi_{h-1}(x) \oplus \mu_h e(u_h, x); x \in X)$,
$T_h = \{x \in T_{h-1}; \varphi(x) \oplus \gamma_h(x) \leq \varphi(u_h) \oplus \gamma_h(u_h)\}$.

This firstly means that

$$\gamma_h = \psi_h \text{ and } u_h \in T_{h-1} \cap T_h;$$

hence, (41-1)+(41-2) hold for $n = h$. Secondly, by the very definition of T_h (and $\gamma_h = \psi_h$)

$$(\forall x \in T_h): \ \varphi(x) \oplus \psi_h(x) \leq \varphi(u_h) \oplus \psi_h(u_h) = \varphi(u_h) \oplus \psi_{h-1}(u_h).$$

Since $u_h \in T_{h-1}$, we must have

$$\varphi(u_h) \oplus \psi_h(u_h) = \varphi(u_h) \oplus \psi_{h-1}(u_h) \leq \varphi(u_{h-1}) \oplus \psi_{h-1}(u_{h-1});$$

i.e., (41-3) is holding for $n = h$. Finally, let $y \in T_h$ be arbitrary fixed; hence, by definition, $y \in T_{h-1}$ and (via $\gamma_h = \psi_h$)

$$\varphi(y) \oplus \psi_h(y) \leq \varphi(u_h) \oplus \psi_h(u_h);$$

or equivalently (by the associative property of $(\oplus)$)

$$(\varphi(y) \oplus \psi_{h-1}(y)) \oplus \mu_h e(u_h, y) \leq \varphi(u_h) \oplus \psi_{h-1}(u_h).$$

Combining with $y \in T_{h-1}$ (and construction of u_h) gives (by the first variable increasing property of $(\oplus)$)

$$L_{h-1} \oplus \mu_h e(u_h, y) \leq \varphi(u_h) \oplus \psi_{h-1}(u_h) \leq L_{h-1} \oplus \mu_h \varepsilon/2^h;$$

whence (from the second variable strictly increasing property), (41-4) holds for $n = h$. This concludes our argument.

Having these preliminary facts obtained, let us go further with our proof, by passing to the following:

Step 2. From the above properties of $(T_n; n \geq 0)$, we get

$$e(u_n, u_m) \leq \varepsilon/2^n, \text{ whenever } n \leq m;$$

which tells us that $(u_n; n \geq 0)$ is an e-Cauchy sequence. As X is (e, d)-complete, $u_n \xrightarrow{d} v$ as $n \to \infty$, for some $v \in X$. In addition, by the d-closeness of sets in $(T_n; n \geq 0)$, we derive $v \in \cap\{T_n; n \geq 0\}$; so that

$$e(u_n, v) \leq \varepsilon/2^n, \text{ for all } n \text{ (whence, } u_n \xrightarrow{e} v);$$

and the first conclusion in our statement holds.

Step 3. Let $k \geq 0$ be arbitrary fixed. By the descending in R_+ property of the sequence $(\varphi(u_n) \oplus \psi_n(u_n); n \geq 0)$, we get

$$\varphi(u_k) \oplus \psi_k(u_k) \geq \varphi(u_n) \oplus \psi_n(u_n), \ \forall n \geq k.$$

Let $m \geq k$ be arbitrary fixed. The ascending property of our functional sequence $(\psi_n; n \geq 0)$ gives by the above

$$\varphi(u_k) \oplus \psi_k(u_k) \geq \varphi(u_n) \oplus \psi_m(u_n), \text{ whenever } n \geq m.$$

Combining with $\varphi \oplus \psi_m$ being d-lsc gives (passing to limit as $n \to \infty$ and noting that $\varphi(u_k) \oplus \psi_k(u_k) < \infty$)

$$\varphi(u_k) \oplus \psi_k(u_k) \geq \varphi(v) \oplus \varphi_m(v) \text{ (for all } m \geq k).$$

By the second variable continuity of (extended operation) $(\oplus)$, we get

$$\lim_m (\varphi(v) \oplus \varphi_m(v)) = \varphi(v) \oplus \varphi_\infty(v).$$

This yields (passing to limit as $m \to \infty$ in the preceding relation)

$$\varphi(u_k) \oplus \psi_k(u_k) \geq \varphi(v) \oplus \varphi_\infty(v), \text{ for all } k \geq 0.$$

Note that, as a direct consequence,

$$\varphi_\infty(v) < \infty; \text{ hence, } \varphi(v) \oplus \psi_\infty(v) < \infty.$$

Step 4. In particular, when $k = 0$, we get the second conclusion in the statement. On the other hand, let $x \in U_0 \setminus \{v\}$ be arbitrary fixed. If $\psi_\infty(x) = \infty$, we are done because (see above)

$$\varphi(x) \oplus \psi_\infty(x) = \infty > \varphi(v) \oplus \psi_\infty(v).$$

Suppose now that

$\psi_\infty(x) < \infty$, whence, $\psi_n(x) < \infty$, $\forall n$.

If $x \in U_0 \setminus T_0$, we are again done (with $k = 0$), in view of

$$\varphi(x) \oplus \psi_\infty(x) \geq \varphi(x) \oplus \psi_0(x) > \varphi(u_0) \geq \varphi(v) \oplus \psi_\infty(v).$$

It remains then to discuss the alternative $x \in T_0$. Suppose by contradiction that

$x \in \cap\{T_n; n \geq 0\}$; i.e., $x \in T_n$, for each $n \geq 0$.

By a previous auxiliary fact involving the sequence $(T_n; n \geq 0)$, we have

$$e(u_n, x) \leq \varepsilon/2^n, \text{ for all } n \geq 0.$$

As a consequence, $u_n \xrightarrow{e} x$ as $n \to \infty$; and this, along with ($u_n \xrightarrow{e} v$, $u_n \xrightarrow{d} v$), gives (as (e, d) is Cauchy separated), $x = v$; contradiction. Hence, necessarily,

$$x \in T_0 \setminus \cap\{T_n; n \geq 0\} = \cup\{T_{i-1} \setminus T_i; i \geq 1\},$$

wherefrom, there must be some uniquely determined index $k = k(x) \geq 1$, such that

$$x \in T_{k-1} \setminus T_k; \text{ i.e., } x \in T_{k-1} \text{ and } x \notin T_k.$$

By the very definition of T_k, we therefore have

$$\varphi(x) \oplus \psi_k(x) > \varphi(u_k) \oplus \psi_k(u_k);$$

and this, combined with a previous relation, gives the desired fact. The proof is thereby complete.

Concerning the argument proposed here, we must stress that, essentially, it is the one in Yongxin and Shuzhong [33]; so, it is natural that our main result above be referred to as *addition-like Yongxin–Shuzhong variational principle on asymmetric spaces*; in short: (YS-adli). Note that, further extensions of these facts are possible—within the class of quasi-ordered r-asymmetric spaces—under the lines in Turinici [30]; we shall discuss these in a separate paper.

(B) Let again X be a nonempty set and (e, d) be a couple of (generalized) r-asymmetrics over it; the triple $(X; e, d)$ will be then referred to as a *double r-asymmetric space*. Suppose for the moment that

(r-s-lsc) (e, d) is *right semi-lsc*: $x \mapsto e(w, x)$ is d-lsc, for each $w \in X$.

Further, let $(\oplus)$ be an addition-like over R_+; i.e.,

(adli-1) $(\oplus)$ is associative and has $0 \in R_+$ as its null element
(adli-2) $(\oplus)$ is first variable continuously increasing and second variable continuously strictly increasing.

Extend this operation to $R_+ \cup \{\infty\}$, by means of

$$a \oplus \infty = \infty \oplus a = \infty, \forall a \in R_+ \cup \{\infty\};$$

the new object $(\oplus)$ is an addition-like over $R_+ \cup \{\infty\}$ (see above). Finally, let the corresponding addition-like over $\mathscr{F}(X, R_+ \cup \{\infty\})$ be denoted in the same way; i.e., for $\psi, \chi \in \mathscr{F}(X, R_+ \cup \{\infty\})$, let $\psi \oplus \chi$ denote the function in $\mathscr{F}(X, R_+ \cup \{\infty\})$

$$(\psi \oplus \chi)(x) = \psi(x) \oplus \chi(x), x \in X.$$

Remember that, under these conditions (cf. a previous fact)

$(\oplus)$ is d-lsc compatible:
$\psi, \chi \in \mathscr{F}(d-lsc)(X, R_+ \cup \{\infty\}) \Longrightarrow \psi \oplus \chi \in \mathscr{F}(d-lsc)(X, R_+ \cup \{\infty\})$.

Given $g \in \{e, d\}$, call it $(\oplus)$-*triangular*, provided

(g-tri) $g(x, z) \leq g(x, y) \oplus g(y, z), \forall x, y, z \in X.$

Technically speaking, the $(\oplus)$-triangular condition upon $e(.,.)$ and/or $d(.,.)$ was not effectively needed for the arguments we just described to work; so, the variational statement above is in particular valid for such triangular r-asymmetrics. However, when this condition is imposed upon $e(.,.)$, the variational statement in question may be written in a very simple and useful way. It is our aim in the sequel to give its corresponding form; further particular aspects will be discussed a bit further.

To begin with, assume that

(e-tri) e is $(\oplus)$-triangular (see above).

Further, concerning the additive-like operation $(\oplus)$, we must accept the following (extra) regularity condition:

(adli-3) $(\oplus)$ is commutative: $t \oplus s = s \oplus t, \forall t, s \in R_+$.

A useful consequence of these facts (and the previous ones) is to be stated as follows. For each sequence $(t_n; n \geq 0)$ in R_+, denote

$$\oplus\{t_i; i \geq 0\} = \lim_n t_{[n]}, \text{ where } (t_{[n]} = \oplus\{t_i; i \leq n\}; n \geq 0).$$

Lemma 5. *Let the triple of sequences* $(t_n; n \geq 0)$, $(a_n; n \geq 0)$, *and* $(b_n; n \geq 0)$ *in* R_+ *be such that*

$t_i \leq a_i \oplus b_i$, *for all* $i \geq 0$.

Then, necessarily,

$$\oplus\{t_n \geq 0\} \leq [\oplus\{a_n; n \geq 0\}] \oplus [\oplus\{b_n; n \geq 0\}].$$

Proof. There are two parts to be passed.

(i) Let the triples $(\tau_1, \alpha_1, \beta_1)$ and $(\tau_2, \alpha_2, \beta_2)$ over R_+ be such that

$\tau_1 \le \alpha_1 \oplus \beta_1, \tau_2 \le \alpha_2 \oplus \beta_2.$

Then, necessarily,

$\tau_1 \oplus \tau_2 \le (\alpha_1 \oplus \alpha_2) \oplus (\beta_1 \oplus \beta_2).$

In fact, by the first and second variable increasing properties,

$$\tau_1 \oplus \tau_2 \le (\alpha_1 \oplus \beta_1) \oplus (\alpha_2 \oplus \beta_2).$$

This, along with the associative and commutative properties of $(\oplus)$, gives us the desired conclusion.

(ii) The case of

either $\oplus\{a_n; n \ge 0\} = \infty$ or $\oplus\{b_n; n \ge 0\} = \infty$

is clear; so, without loss, one may assume that

$\oplus\{a_n; n \ge 0\} < \infty$ and $\oplus\{b_n; n \ge 0\} < \infty$.

By the first part, one has (according to the admitted notations)

$$t_{[n]} \le [\oplus\{a_i; i \le n\}] \oplus [\oplus\{b_i; i \le n\}], \ \forall n.$$

From the first and second variable increasing property, we have

$$t_{[n]} \le [\oplus\{a_n; n \ge 0\}] \oplus [\oplus\{b_n; n \ge 0\}], \ \forall n.$$

Passing to limit as $n \to \infty$ yields

$$\lim_n (t_{[n]}) \le [\oplus\{a_n; n \ge 0\}] \oplus [\oplus\{b_n; n \ge 0\}];$$

and conclusion follows.

Finally, denote, for each $\mu \in R_+$:

$H(\mu) = \inf \Xi(\mu)$, where $\Xi(\mu) = \{\nu \in R_+; \mu(t \oplus s) \le \mu t \oplus \nu s, \forall t, s \in R_+\}$.

[Here, by convention, $\inf(\emptyset) = \infty$.] The definition is therefore meaningful and yields a function $H : R_+ \to R_+ \cup \{\infty\}$; its basic properties are concentrated in the following

Proposition 13. *Under these conventions:*

(42-1) $\mu(t \oplus s) \le \mu t \oplus H(\mu)s$, *for each* $t, s \in R_+$, *and each* $\mu \in R_+$
(42-2) $H(0) = 0$ *and* $\mu \le H(\mu)$, *for each* $\mu \in R_+$.

Proof. (i) Let $\mu \in R_+$ be arbitrary fixed. If $H(\mu) = \infty$, we are done; so, without loss, assume that $H(\mu) < \infty$. By the second variable continuity condition imposed upon $(\oplus)$, the following properties hold, for each $\mu \in R_+$

$\Xi(\mu)$ is a (nonempty) closed subset of R_+
$\Xi(\mu)$ is *right-hereditary*: $\nu \in \Xi(\mu) \Longrightarrow [\nu, \infty[\subseteq \Xi(\mu)$;

so that, $\Xi(\mu) = [H(\mu), \infty[$. This gives the desired conclusion.

(ii) As $\Xi(0) = R_+$, we must have $H(0) = 0$. On the other hand, taking $t = 0$ in the preceding relation gives

$$\mu s \leq H(\mu)s, \text{ for each } s \in R_+;$$

and, from this, all is clear.

Now, for each sequence $(\rho_n; n \geq 0)$ in $R_+ \cup \{\infty\}$, denote

$\Gamma((\rho_n); t) = \oplus\{\rho_i t; i \geq 0\}, t \in R_+$.

Note that, by this very definition,

$\Gamma((\rho_n); 0) = 0$; and (if $t > 0$)
$\Gamma((\rho_n); t) = \infty$, whenever $\rho_i = \infty$, for some $i \geq 0$.

We are now in position to get an appropriate answer to the posed question. Fix some function $\varphi \in \mathscr{F}(X, R_+ \cup \{\infty\})$, with

(p-d-lsc) φ is proper, d-lsc (see above).

Remember that, for $\varepsilon > 0$, $U \in 2^X$, we denoted

$U(\varphi, \varepsilon) = \{x \in U; \varphi(x) \leq \inf[\varphi(X)] \oplus \varepsilon\}$ (hence, $U(\varphi, \varepsilon) \subseteq U \cap \mathrm{Dom}(\varphi)$).

Our second main result in this exposition is as follows:

Theorem 3. *Let the couple (e, d) of (generalized) r-asymmetrics over X be such that one of the (admissible-type) conditions below holds:*

(ad-1) (e, d) *is admissible*
(ad-2) (e, d) *is KST-admissible, X is d-complete, and e is Cauchy separated*
(ad-3) (e, d) *is YS-admissible, X is d-complete, and d is e-Cauchy separated.*

In addition, let the commutative addition-like $(\oplus)$ be such that e is $(\oplus)$-triangular. Finally, let $(\mu_n; n \geq 0)$ be a sequence in R^0_+ and $u_0 \in X$, $\varepsilon > 0$, $U_0 \in \mathscr{K}(d)$ be taken according to

(reg) $u_0 \in U_0(\varphi, \mu_0\varepsilon)$ *(whence, $u_0 \in U_0 \cap \mathrm{Dom}(\varphi)$).*

Then, a sequence $(u_n; n \geq 0)$ in $U_0(\varphi, \mu_0\varepsilon)$ and an element $v \in U_0(\varphi, \mu_0\varepsilon)$ may be determined, in such a way that

(42-a) (u_n) *is e-Cauchy and* $(u_n \xrightarrow{e} v,\ u_n \xrightarrow{d} v)$
(42-b) $\varphi(v) \oplus \psi_\infty(v) \leq \varphi(u_0)$*; hence,* $\varphi(v) \leq \varphi(u_0)$ *and* $\psi_\infty(v) < \infty$
(42-c) $\varphi(x) \oplus \Gamma((H(\mu_n)); e(v, x)) > \varphi(v)$*, for each* $x \in U_0 \setminus \{v\}$.

Proof. By the first main result, it follows that for $u_0 \in X$, $\varepsilon > 0$, $U_0 \in \mathscr{K}(d)$ taken as before, there exists a sequence $(u_n; n \geq 0)$ in $U_0(\varphi, \mu_0\varepsilon)$ and an element $v \in U_0(\varphi, \mu_0\varepsilon)$, with the properties

(42-aa) (u_n) is e-Cauchy and $(u_n \xrightarrow{e} v,\ u_n \xrightarrow{d} v)$
(42-bb) $\varphi(v) \oplus \psi_\infty(v) \leq \varphi(u_0)$; hence, $\varphi(v) \leq \varphi(u_0)$ and $\psi_\infty(v) < \infty$
(42-cc) $\varphi(x) \oplus \psi_\infty(x) > \varphi(v) \oplus \psi_\infty(v)$, for each $x \in U_0 \setminus \{v\}$.

As a consequence, the first and second part in the statement are fulfilled. It remains now to establish that, from the last conclusion above, the final part in our statement is fulfilled too. So, let $x \in U_0 \setminus \{v\}$ be arbitrary fixed. If

either $\varphi(x) = \infty$ or $\Gamma((H(\mu_n)); e(v,x)) = \infty$,

then, we are done, in view of

$$\varphi(x) \oplus \Gamma((H(\mu_n)); e(v,x)) = \infty > \varphi(v).$$

Assume now that

$\varphi(x) < \infty$ and $\Gamma((H(\mu_n)); e(v,x)) < \infty$.

From the $(\oplus)$-triangular property and the very definition of $H(.)$,

$$\mu_i e(u_i, x) \leq \mu_i e(u_i, v) \oplus H(\mu_i) e(v,x),\ \ \forall i \geq 0.$$

This, along with an auxiliary fact above, gives

$$\psi_\infty(x) \leq \psi_\infty(v) \oplus \Gamma((H(\mu_n)); e(v,x));$$

or equivalently (as $(\oplus)$ is commutative)

$$\psi_\infty(x) \leq \Gamma((H(\mu_n)); e(v,x)) \oplus \psi_\infty(v).$$

Replacing in the last conclusion above gives (by the associative and second variable strictly increasing property of $(\oplus)$)

$$\begin{aligned}
&[\varphi(x) \oplus \Gamma((H(\mu_n)); e(v,x))] \oplus \psi_\infty(v) \\
&\quad = \varphi(x) \oplus [\Gamma((H(\mu_n)); e(v,x)) \oplus \psi_\infty(v)] \\
&\quad \geq \varphi(x) \oplus \psi_\infty(x) > \varphi(v) \oplus \psi_\infty(v).
\end{aligned}$$

This yields (by the first variable increasing property of $(\oplus)$)

$$\varphi(x) \oplus \Gamma((H(\mu_n)); e(v,x)) > \varphi(v);$$

and the last part of our statement follows as well. The proof is thereby complete.

By definition, this result will be referred to as *addition-like triangular Yongxin–Shuzhong variational principle on asymmetric spaces*, in short (YS-adli-tri). As before, further extensions of these facts are possible, over the class of quasi-ordered r-asymmetric spaces; we do not give details.

5 Standard Addition Case

As already precise in a previous place, one basic choice for the addition-like $(\oplus)$ used in our main results is $(+)$ (standard addition); i.e.,

$(\forall t, s \in R_+)$: $t \oplus s$ is identical with $t + s$.

It is our aim in the following to discuss the corresponding versions of underlying statements with respect to such a choice. This, aside from providing us a motivational base for the developed methods, has some important practical applications.

(A) Let X be a nonempty set and $(e(.,.), d(.,.))$ be a couple of (generalized) r-asymmetrics over it; the triple $(X; e, d)$ will be then referred to as a *double r-asymmetric space*. Suppose for the moment that

(r-s-lsc) (e, d) is *right semi-lsc*: $x \mapsto e(w, x)$ is d-lsc, for each $w \in X$.

Let $(\mu_n; n \geq 0)$ be a sequence in R^0_+. (Sometimes, we may also impose to this sequence a regularity condition like

$\lim_n(\sum_{i\leq n} \mu_i) = 1$; i.e., $\sum_n \mu_n = 1$;

we then say that $(\mu_n; n \geq 0)$ is a *unitary sequence* in R^0_+. But, for the moment, this is not the case.) Further, taking a sequence $(u_n; n \geq 0)$ in X, let us construct a sequence $(\psi_n; n \geq 0)$ of functions in $\mathscr{F}(X, R_+ \cup \{\infty\})$, according to the convention

$\psi_n(x) = \sum_{i\leq n} \mu_i e(u_i, x), x \in X, n \geq 0$;

note that, as $(+)$ is d-lsc compatible (see above), any function of this type is d-lsc. Clearly, $(\psi_n; n \geq 0)$ is pointwise ascending; hence, the (pointwise) limit $\psi_\infty = \lim_n(\psi_n)$, of it, introduced as

$\psi_\infty(x) = \lim_n(\psi_n(x)) = \sum_{i\geq 0} \mu_i e(u_i, x), x \in X$

is well defined as an element of $\mathscr{F}(X, R_+ \cup \{\infty\})$. Moreover [by the remarks above], ψ_∞ is d-lsc; just take an auxiliary fact into account, combined with

$$[\psi_\infty \leq t] = \cap\{[\psi_n \leq t]; n \geq 0\}, \ \forall t \in R_+.$$

Having these precise, fix some function $\varphi \in \mathscr{F}(X, R_+ \cup \{\infty\})$, with

(p-lsc-1) φ is proper: $\text{Dom}(\varphi) := \{x \in X; \varphi(x) < \infty\} \neq \emptyset$;
whence $0 \leq \inf[\varphi(X)] < \infty$

(p-lsc-2) φ is d-lsc; or, equivalently (see above):
$[\varphi \leq t]$ is d-closed, for each $t \in R_+$.

Denote, for $\varepsilon > 0$, $U \in 2^X$,

$$U(\varphi, \varepsilon) = \{x \in U; \varphi(x) \leq \inf[\varphi(X)] + \varepsilon\} \text{ (hence, } U(\varphi, \varepsilon) \subseteq U \cap \mathrm{Dom}(\varphi)).$$

The following (standard) addition-type variational result involving our data is our starting point:

Theorem 4. *Let the couple (e, d) of (generalized) r-asymmetrics over X be such that one of the (admissible-type) conditions below holds:*

(ad-1) (e, d) *is admissible*
(ad-2) (e, d) *is KST-admissible, X is d-complete, and e is Cauchy separated*
(ad-3) (e, d) *is YS-admissible, X is d-complete, and d is e-Cauchy separated.*

In addition, let $(\mu_n; n \geq 0)$ be a sequence in R^0_+; and $u_0 \in X$, $\varepsilon > 0$, $U_0 \in \mathscr{K}(d)$ be taken according to

(reg) $u_0 \in U_0(\varphi, \mu_0\varepsilon)$ *(whence,* $u_0 \in U_0 \cap \mathrm{Dom}(\varphi)$*).*

Then, a sequence $(u_n; n \geq 0)$ in $U_0(\varphi, \mu_0\varepsilon)$ and an element $v \in U_0(\varphi, \mu_0\varepsilon)$ may be determined, in such a way that

(51-a) (u_n) *is e-Cauchy and* $(u_n \xrightarrow{e} v,\ u_n \xrightarrow{d} v)$
(51-b) $\varphi(v) + \psi_\infty(v) \leq \varphi(u_0)$*; hence,* $\varphi(v) \leq \varphi(u_0)$ *and* $\psi_\infty(v) < \infty$
(51-c) $\varphi(x) + \psi_\infty(x) > \varphi(v) + \psi_\infty(v)$*, for each* $x \in U_0 \setminus \{v\}$.

Proof. (Sketch) It is immediately seen that, under $(\oplus)$ being identical with $(+)$, the first main result applies.

As already precise, the argument proposed here is, essentially, the one in Yongxin and Shuzhong [33]; so, it is natural that our statement above be referred to as *(standard) addition Yongxin–Shuzhong variational principle on asymmetric spaces*, in short (YS-ad). Note that, further extensions of these facts are possible—within the class of quasi-ordered r-asymmetric spaces—under the lines in Turinici [30]; we shall discuss these in a separate paper.

(B) Let again X be a nonempty set and (e, d) be a couple of (generalized) r-asymmetrics over it; the triple $(X; e, d)$ will be then referred to as a *double r-asymmetric space*. Given $g \in \{e, d\}$, call it *triangular*, provided

(ad-g-tri) $g(x, z) \leq g(x, y) + g(y, z)$, $\forall x, y, z \in X$.

Technically speaking, such triangular conditions were not effectively needed for the arguments of the (standard) additive-type result above to work; so, the variational statement in question is in particular valid for such triangular r-asymmetrics. However, when this condition is imposed upon e, our variational result may be written in a way that allows us direct comparisons with Ekeland's variational principle [13] (in short EVP). It is our aim in the sequel to give its corresponding form; some other aspects will be discussed a bit further.

To begin with, assume that

(ad-e-tri) e is triangular (see above).

Let the function $\varphi : X \to R_+ \cup \{\infty\}$ be taken as

(p-lsc) φ is proper, d-lsc.

Remember that, for each $\varepsilon > 0$, $U \in 2^X$, we denoted

$$U(\varphi, \varepsilon) = \{x \in U; \varphi(x) \leq \inf[\varphi(X)] + \varepsilon\} \text{ (hence, } U(\varphi, \varepsilon) \subseteq U \cap \mathrm{Dom}(\varphi)).$$

The following variational result (referred to as *(standard) addition triangular Yongxin–Shuzhong variational principle on asymmetric spaces*, in short (YS-ad-tri)) is available:

Theorem 5. *Let the couple* (e, d) *of (generalized) r-asymmetrics over X be such that one of the (admissible-type) conditions below holds:*

(ad-1) (e, d) *is admissible*
(ad-2) (e, d) *is KST-admissible, X is d-complete, and e is Cauchy separated*
(ad-3) (e, d) *is YS-admissible, X is d-complete, and d is e-Cauchy separated.*

Suppose in addition that e is triangular; and let (the starting point) $u \in \mathrm{Dom}(\varphi)$ *be arbitrary fixed. There exists then (another point)* $v \in \mathrm{Dom}(\varphi)$*, such that*

(52-a) $\varphi(v) + e(u, v) \leq \varphi(u)$ *(hence,* $\varphi(v) \leq \varphi(u)$*)*
(52-b) $\varphi(x) + e(v, x) > \varphi(v)$, $\forall x \in X \setminus \{v\}$.

Note that a direct argument for proving these conclusions is available, via second main result (YS-adli-tri). However, for simplicity reasons, it will be more convenient for us to deduce the conclusions in question from the (standard) additive result above (YS-ad).

Proof (Theorem 5). There are three parts to be passed.

Part 1. Denote, for simplicity $u_0 := u$; as well as

$$U_0 = \{x \in X; \varphi(x) + e(u_0, x) \leq \varphi(u_0)\}.$$

Clearly, U_0 is nonempty; because u_0 is an element of it. Moreover, we claim that

U_0 is d-closed (i.e., $U_0 \in \mathscr{K}(d)$).

In fact, by this very definition,

$$U_0 = \{x \in X; \omega(x) \leq \varphi(u_0)\}, \text{ where } (\omega(x) = \varphi(x) + e(u_0, x); x \in X).$$

Now, by the imposed conditions upon (e, d) and φ (and a previous auxiliary fact), $\omega(.)$ is d-lsc; and this, along with a level set characterization of this concept (see above), gives the desired fact.

Part 2. Let $(\mu_n; n \geq 0)$ be a sequence in R^0_+ with

$\sum_n \mu_n = \lim_n(\sum_{i \leq n} \mu_i) = 1$; i.e., $(\mu_n; n \geq 0)$ is unitary;

and take the strictly positive number $\varepsilon > 0$ according to

$\varphi(u_0) - \inf[\varphi(X)] \leq \mu_0\varepsilon$; or, equivalently, $u_0 \in X(\varphi, \mu_0\varepsilon)$.

As $u_0 \in U_0$, we must have

$u_0 \in U_0(\varphi, \mu_0\varepsilon)$ (whence, $u_0 \in U_0 \cap \mathrm{Dom}(\varphi)$).

By the (standard) additive variational principle (YS-ad), there exists a sequence $(u_n; n \geq 0)$ in $U_0(\varphi, \mu_0\varepsilon)$ and an element $v \in U_0(\varphi, \mu_0\varepsilon)$, with

(52-aa) (u_n) is e-Cauchy and $(u_n \xrightarrow{e} v,\ u_n \xrightarrow{d} v)$
(52-bb) $\varphi(v) + \psi_\infty(v) \leq \varphi(u_0)$; hence, $\varphi(v) \leq \varphi(u_0)$ and $\psi_\infty(v) < \infty$
(52-cc) $\varphi(x) + \psi_\infty(x) > \varphi(v) + \psi_\infty(v)$, for each $x \in U_0 \setminus \{v\}$.

In particular, as $v \in U_0$, one gets

$$e(u_0, v) \leq \varphi(u_0) - \varphi(v);$$

and the first half of our statement is holding.

Part 3. Let $x \in X \setminus \{v\}$ be arbitrary fixed. If one has that

$\varphi(x) = \infty$ or $e(v, x) = \infty$,

we are done; so, without loss, assume in the following that

$\varphi(x) < \infty$ and $e(v, x) < \infty$.

Two alternatives occur.

Alter 3-1. Suppose that $x \in U_0 \setminus \{v\}$. From the second conclusion above,

$$e(u_n, v) < \infty, \text{ for all } n.$$

This, along with the triangular condition, yields

$$e(u_n, x) - e(u_n, v) \leq e(v, x),\ \forall n;$$

wherefrom (by our notations)

$$\psi_m(x) - \psi_m(v) \leq \left(\sum_{i \leq m} \mu_i\right) e(v, x),\ \forall m.$$

Passing to limit as $m \to \infty$, we therefore get

$$\psi_\infty(x) - \psi_\infty(v) \leq e(v, x);$$

and this, in combination with the last conclusion above, gives us the second half of our statement.

Alter 3-2. Suppose that $x \in X \setminus U_0$; i.e., (by the above definition)

$$e(u_0, x) > \varphi(u_0) - \varphi(x) \text{ (in view of } \varphi(x), e(v, x), \varphi(v) < \infty).$$

If, by absurd, the second half of our statement is false, we must have

$$e(v, x) \leq \varphi(v) - \varphi(x) \text{ (cf. the finiteness properties above).}$$

Taking the first conclusion of the statement into account gives (by our triangular property of $e(.,.)$)

$$e(u_0, x) \leq e(u_0, v) + e(v, x) \leq$$
$$\varphi(u_0) - \varphi(v) + \varphi(v) - \varphi(x) = \varphi(u_0) - \varphi(x);$$

in contradiction with a previous relation involving the same data. The proof is thereby complete.

(C) According to a previous remark, these results may be given as extensions of related ones in Yongxin and Shuzhong [33]. However, in the quoted paper, the objective function φ is taken as an element of $\mathscr{F}(X, R \cup \{\infty\})$, fulfilling:

(ge-p-lsc-1) φ is inf-proper: $\inf[\varphi(X)] > -\infty$ and $\mathrm{Dom}(\varphi) := \{x \in X; \varphi(x) < \infty\} \neq \emptyset$ (hence, $-\infty < \inf[\varphi(X)] < \infty$)

(ge-p-lsc-2) φ is d-lsc on X: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \xrightarrow{d} x$.

But these results may be reduced to the ones we already stated. In fact, let us consider the *translated* function $\Phi \in \mathscr{F}(X, R_+ \cup \{\infty\})$ introduced as

$$\Phi(x) = \varphi(x) - \inf[\varphi(X)], x \in X.$$

By the conditions above, it follows that

(I) Φ is proper: $\mathrm{Dom}(\Phi) := \{x \in X; \Phi(x) < \infty\} \neq \emptyset$
(II) $\inf[\Phi(X)] = 0$ and Φ is d-lsc.

Summing up, the results above are applicable to Φ; and from their corresponding conclusions (in terms of Φ), we get the ones in terms of φ; hence the assertion. In particular, when

(st-m) $e = d =$ (standard) metric on X,

the triangular statement above yields, in a direct way, Ekeland's variational principle [13] (in short EVP). For a different line of reasoning in establishing this fact, see Turinici [29] and the references therein.

6 Functional Case (Bakhtin Metrics)

In the following, a functional extension of the above result is given, within the class of Bakhtin metrical structures.

Let X be a nonempty set. Any map $g : X \times X \to R_+ \cup \{\infty\}$ with

(ref-su) g is reflexive sufficient ($x = y$ iff $g(x, y) = 0$)
(sym) g is symmetric ($g(x, y) = g(y, x)$, $\forall x, y \in X$)

will be referred to as a *reflexive sufficient symmetric* (in short *rs-symmetric*) of X. Given $s \geq 1$, let us say that g is *s-triangular*, provided

(s-tri) $g(x, z) \leq s(g(x, y) + g(y, z))$, $\forall x, y, z \in X$.

Let $B(g)$ denote the class of all these; i.e.,

$B(g) = \{s \in [1, \infty[; g \text{ is } s\text{-triangular}\}$;

note at this moment that

$$B(g) \text{ is right-hereditary: } s \in B(g) \text{ implies } [s, \infty[\subseteq B(g).$$

If $B(g)$ is nonempty, we say that $g(.,.)$ is a *(generalized) Bakhtin metric*; and the couple (X, g) will be called a *(generalized) Bakhtin metric space*. (This convention is motivated by the fact that the study of such structures was initiated by Bakhtin [1]; see also Czerwik [10]). Likewise, any number $s \in B(g)$ will be referred to as a *Bakhtin characteristic* of $g(.,.)$; in this case, g is also referred to as a *(generalized) s-metric*.

Fix in the following a Bakhtin metric $g(.,.)$; and let $s \in B(g)$ be some Bakhtin characteristic of it. As $g(.,.)$ is in particular r-asymmetric, we may introduce a g-convergence and g-Cauchy structure over X, under our general model. Note at this moment that any g-convergent sequence in X is g-Cauchy too; when the reciprocal holds too, we say that X is *g-complete*. Moreover,

$g(.,.)$ is separated: $x_n \xrightarrow{g} u$ and $x_n \xrightarrow{g} v$ imply $u = v$.

In fact, given a sequence $(x_n; n \geq 0)$ in X like in the premise above, one has (by symmetry and s-triangular inequality)

$$g(u, v) \leq s(g(x_n, u) + g(x_n, v)), \ \forall n;$$

so that, passing to limit as $n \to \infty$, yields $g(u, v) = 0$, wherefrom (by sufficiency) $u = v$. [Note that—as already mentioned—the sufficiency of $g(.,.)$ follows from such a property; but this is not essential to us.]

Under these preliminaries, we may now pass to the effective part of our developments. Let in the following $d : X \times X \to R_+ \cup \{\infty\}$ be a (generalized) Bakhtin metric over X. (Remember that by the above developments, d is separated.) Let also $f : X \times X \to R_+$ be a (standard) r-asymmetric over X and $\Delta \in \mathscr{F}(R_+)$ be a function with the ad hoc property

$\Delta(0) = 0$; hence, $\Delta(f(x, x)) = 0$, $\forall x \in X$.

The mapping $e : X \times X \to R_+$ introduced as

$$e(x,y) = \Delta(f(x,y)),\ x,y \in X$$

is a (standard) r-asymmetric over X, as it can be directly seen. A natural question to be posed is that of determining sufficient conditions under which the KST-admissible and YS-admissible properties be transferable from (f,d) to (e,d). For technical reasons, it would be useful working with the strong version of the latter property. Namely, given $h \in \{f,e\}$, let us consider the properties:

(sp-1) (h,d) is right semi-lsc:
$x \mapsto h(w,x)$ is d-lsc, for each $w \in X$

(sp-2) (h,d) is Cauchy transitive:
(x_n) is h-Cauchy implies (x_n) is d-Cauchy

(sp-3) (h,d) is convergence transitive:
$h(x_n,y_n) \to 0$ implies $d(x_n,y_n) \to 0$.

When (sp-1)+(sp-2) hold, then (h,d) is called (as before) *KST-admissible*; and, if (sp-1)+(sp-3) hold, then (h,d) is called *strongly YS-admissible*.

For an appropriate answer to this, we need to assume that $\Delta \in \mathscr{F}(R_+)$ is a *Feng–Liu function*; i.e.,:

(f-liu) Δ is increasing, lsc, and $\Delta^{-1}(0) = \{0\}$.

(This convention is related to the developments in Feng and Liu [17]; we do not give details). The following properties of this functional class will be useful for us.

Proposition 14. *Let $\Delta \in \mathscr{F}(R_+)$ be a Feng–Liu function. Then:*

(61-1) $\Delta(.)$ *is a small function, in the sense:*
for each $\varepsilon > 0$, there exists $\delta > 0$, such that
$\forall (t \geq 0)$*: ($\Delta(t) < \delta$ implies $t < \varepsilon$)*

(61-2) $\Delta(.)$ *is zero-convergence transitive: for each sequence (t_n) in R_+, we have*
$\Delta(t_n) \to 0$ *implies* $t_n \to 0$.

Proof. (i) Assume that Δ is not a small function; whence, for some $\varepsilon > 0$,

to each $\delta > 0$, there corresponds some $t_\delta \geq \varepsilon$, such that $\Delta(t_\delta) < \delta$.

As Δ is increasing, this yields

$$\Delta(\varepsilon) < \delta, \text{ for all } \delta > 0;$$

and, therefore, $\Delta(\varepsilon) = 0$, contradiction; hence the claim.

(ii) Evident, by the small property of $\Delta(.)$ we just established.

Concerning the transfer problem above, a useful answer to it is contained in the following:

Proposition 15. *Under the above conventions, we have:*

(62-1) *(f,d) is KST-admissible $\Longrightarrow$ (e,d) is KST-admissible*

(62-2) *(f, d) is strongly YS-admissible $\Longrightarrow$ (e, d) is strongly YS-admissible*
(62-3) *f is separated $\Longrightarrow$ e is separated.*

Proof. (i) Suppose that (f, d) is right semi-lsc; and let $w \in X$ be arbitrary fixed. By definition, the partial map $x \mapsto f(w, x)$ is d-lsc. This, along with the hypotheses about Δ, tells us that the partial map

$$x \mapsto e(w, x) = \Delta(f(w, x))$$

is d-lsc, if one takes a previous fact into account. As $w \in X$ was arbitrarily chosen, one gets that (e, d) is right semi-lsc.

(ii) Assume that (f, d) is Cauchy transitive; and let (x_n) be an e-Cauchy sequence in X. For the arbitrary fixed $\varepsilon > 0$, let $\delta > 0$ be the number attached by the small property of Δ. Given this $\delta > 0$, there exists some rank $n(\delta) \geq 0$, with

$$n(\delta) \leq i \leq j \text{ implies } e(x_i, x_j) = \Delta(f(x_i, x_j)) < \delta.$$

But then, from the quoted property,

$$n(\delta) \leq i \leq j \text{ implies } f(x_i, x_j) < \varepsilon;$$

which (by the arbitrariness of $\varepsilon > 0$), means that (x_n) is f-Cauchy. By hypothesis, (x_n) is then d-Cauchy; and this tells us that (e, d) is Cauchy transitive.

(iii) Suppose that (f, d) is convergence transitive; and let (x_n), (y_n) be a couple of sequences in X with

$$e(x_n, y_n) = \Delta(f(x_n, y_n)) \to 0 \text{ as } n \to \infty.$$

By the zero-convergence transitivity of Δ (see above), we must have $f(x_n, y_n) \to 0$ as $n \to \infty$. Combining with the working hypothesis gives $d(x_n, y_n) \to 0$. As the sequences (x_n) and (y_n) were arbitrarily chosen with the imposed property, we therefore derive that (e, d) is convergence transitive.

(iv) Suppose that f is separated and let the sequence (x_n) in X and the points $u, v \in X$ be such that

$$x_n \xrightarrow{e} u, x_n \xrightarrow{e} v;$$

or, equivalently (by definition),

$$e(x_n, u) = \Delta(f(x_n, u)) \to 0 \text{ and } e(x_n, v) = \Delta(f(x_n, v)) \to 0, \text{ as } n \to \infty.$$

Combining with the zero-convergence transitivity of Δ, we then have

$$x_n \xrightarrow{f} u, x_n \xrightarrow{f} v;$$

wherefrom (by the imposed hypothesis) $u = v$; which tells us that the associated map e is necessarily separated.

Now, by simply combining this with our (standard) addition Yongxin–Shuzhong variational principle (YS-ad), one gets a useful functional version of it, expressed as below. Let $d(.,.)$ be a (generalized) Bakhtin metric over X and $f(.,.)$ be a (standard) r-asymmetric on X. Further, letting $\Delta : R_+ \to R_+$ be a Feng–Liu function, define the associated to (f, Δ) (standard) r-asymmetric

$$e(x, y) = \Delta(f(x, y)), x, y \in X.$$

Given the sequence $(\mu_n; n \geq 0)$ in R^0_+ and the sequence $(u_n; n \geq 0)$ in X, let us attach them a sequence $(\psi_n; n \geq 0)$ of functions in $\mathscr{F}(X, R_+)$, as

$$\psi_n(x) = \sum_{i \leq n} \mu_i e(u_i, x), x \in X, n \geq 0.$$

Let also $\psi_\infty := \lim_n(\psi_n)$ stand for their pointwise limit

$$\psi_\infty(x) = \lim_n \psi_n(x) = \sum_{i \geq 0} \mu_i e(u_i, x), x \in X;$$

note that this function may belong to $\mathscr{F}(X, R_+ \cup \{\infty\})$. Finally, take some inf-proper d-lsc function $\varphi \in \mathscr{F}(X, R_+ \cup \{\infty\})$ with

(p-lsc) $\quad \varphi$ is proper, d-lsc.

Remember that, for each $\varepsilon > 0$, $U \in 2^X$, we introduced the notation

$$U(\varphi, \varepsilon) = \{x \in U; \varphi(x) \leq \inf[\varphi(X)] + \varepsilon\} \text{ (hence, } U(\varphi, \varepsilon) \subseteq U \cap \mathrm{Dom}(\varphi)).$$

The following statement (called the *(standard) addition functional Yongxin–Shuzhong variational principle* on Bakhtin structures; in short: (YS-ad-f-Bakhtin)) is now available.

Theorem 6. *Let the (generalized) Bakhtin metric $d(.,.)$ on X and the (standard) r-asymmetric $f(.,.)$ on X be such that X is d-complete, and one of the extra conditions below are holding*

(KST-se) $\quad$ *(f, d) is KST-admissible and f is separated*
(s-YS) $\quad$ *(f, d) is strongly YS-admissible.*

In addition, let $(\mu_n; n \geq 0)$ be a sequence in R^0_+ and $u_0 \in X$, $\varepsilon > 0$, $U_0 \in \mathscr{K}(d)$ be taken according to

(reg) $\quad u_0 \in U_0(\varphi, \mu_0\varepsilon)$ *(whence, $u_0 \in U_0 \cap \mathrm{Dom}(\varphi)$).*

Finally, let $\Delta : R_+ \to R_+$ be a Feng–Liu function; and let $e(.,.)$ stand for the associated to (f, Δ) (standard) r-asymmetric on X. Then, a sequence $(u_n; n \geq 0)$ in $U_0(\varphi, \mu_0\varepsilon)$ and an element $v \in U_0(\varphi, \mu_0\varepsilon)$ may be determined, such that:

(61-a) $\quad$ *(u_n) is e-Cauchy and $(u_n \xrightarrow{e} v$, $u_n \xrightarrow{d} v)$*
(61-b) $\quad \varphi(v) + \psi_\infty(v) \leq \varphi(u_0)$*; hence,* $\varphi(v) \leq \varphi(u_0)$ *and* $\psi_\infty(v) < \infty$
(61-c) $\quad \varphi(x) + \psi_\infty(x) > \varphi(v) + \psi_\infty(v)$*, for each* $x \in U_0 \setminus \{v\}$.

Note that further extensions of this result are possible when $f(.,.)$ is a generalized r-asymmetric on X and $\Delta \in \mathscr{F}(R_+ \cup \{\infty\})$ is an (extended) positive function; we do not give details.

A basic particular case of these developments is $f = d$. Precisely, let X be a nonempty set and $d : X \times X \to R_+$ be a (standard) Bakhtin metric over it, fulfilling (in addition)

(se-lsc) d is semi-lsc: $x \mapsto d(w, x)$ is d-lsc, for each $w \in X$.

Note that, in this case, the couple (d, d) is right semi-lsc. On the other hand, (d, d) is trivially endowed with the Cauchy and convergence transitive properties; whence, (d, d) is both KST-admissible and strongly YS-admissible. Moreover, as established in a previous place, d is separated. Further, letting $\Delta : R_+ \to R_+$ be a Feng–Liu function, define the associated to (d, Δ) (standard) *rs-symmetric*

$$e(x, y) = \Delta(d(x, y)), x, y \in X.$$

Given the sequence $(\mu_n; n \geq 0)$ in R^0_+ and the sequence $(u_n; n \geq 0)$ in X, let us attach them a sequence $(\psi_n; n \geq 0)$ of functions in $\mathscr{F}(X, R_+)$, as

$$\psi_n(x) = \sum_{i \leq n} \mu_i e(u_i, x), x \in X, n \geq 0;$$

and let $\psi_\infty := \lim_n(\psi_n)$ stand for their (pointwise) limit

$$\psi_\infty(x) = \lim_n \psi_n(x) = \sum_{i \geq 0} \mu_i e(u_i, x), x \in X;$$

clearly, this last function may belong to $\mathscr{F}(X, R_+ \cup \{\infty\})$. Finally, take some function $\varphi \in \mathscr{F}(X, R_+ \cup \{\infty\})$ with

(p-lsc) φ is proper, d-lsc (see above).

The following result (referred to as *(standard) addition Yongxin–Shuzhong variational principle in Bakhtin metric spaces* (in short: (YS-ad-Bakhtin)) is now holding.

Theorem 7. *Let the (standard) Bakhtin metric $d(.,.)$ on X be such that d is semi-lsc and X is d-complete. In addition, let $(\mu_n; n \geq 0)$ be a sequence in R^0_+ and $u_0 \in X$, $\varepsilon > 0$, $U_0 \in \mathscr{K}(d)$ be taken according to*

$$u_0 \in U_0(\varphi, \mu_0\varepsilon) \text{ (whence, } u_0 \in U_0 \cap \mathrm{Dom}(\varphi)).$$

Finally, let $\Delta : R_+ \to R_+$ be a Feng–Liu function; and let $e(.,.)$ stand for the associated to (d, Δ) (standard) rs-symmetric on X. Then, a sequence $(u_n; n \geq 0)$ in $U_0(\varphi, \mu_0\varepsilon)$ and an element $v \in U_0(\varphi, \mu_0\varepsilon)$ may be determined, in such a way that

(62-a) *(u_n) is e-Cauchy and $(u_n \xrightarrow{e} v$, $u_n \xrightarrow{d} v)$*
(62-b) *$\varphi(v) + \psi_\infty(v) \leq \varphi(u_0)$; hence, $\varphi(v) \leq \varphi(u_0)$ and $\psi_\infty(v) < \infty$*
(62-c) *$\varphi(x) + \psi_\infty(x) > \varphi(v) + \psi_\infty(v)$, for each $x \in U_0 \setminus \{v\}$.*

The following particular cases of this statement are to be noted.

(PC-1) Let $d : X \times X \to R_+$ be a (standard) Bakhtin metric, with Bakhtin characteristic $s = s(d) > 1$; remember that d is separated. Further, let the sequence $(\mu_n; n \geq 0)$ of strictly positive numbers be defined as

$$\mu_n = 1/s^{n+1}, n \geq 0; \text{ hence } \sum_n \mu_n = s/(s-1).$$

Finally, take

$\Delta(t) = t, t \in R_+$ (the identical function).

The corresponding $U_0 = X$ version of (YS-ad-Bakhtin) extends a related 2011 statement in Bota et al. [4] based on the semi-lsc condition upon $d(.,.)$ being substituted by the bilateral condition

(bi-cont) $d(.,.)$ is *continuous*:
$x_n \xrightarrow{d} x, y_n \xrightarrow{d} y$ imply $d(x_n, y_n) \to d(x, y)$.

Note that an equivalence between these regularity conditions upon d cannot be reached, in general; we do not give details.

(PC-2) Suppose that $d : X \times X \to R_+$ is a *(standard) metric* on X; clearly, d is continuous and separated. The corresponding version of (YS-ad-Bakhtin) extends a related 1987 statement in Borwein and Preiss [3] (in short (BP)), to which it reduces when the Feng–Liu function Δ is taken as

$\Delta(t) = t^p, t \in R_+$, for some $p \geq 1$.

(PC-3) A limit version of the preceding one is $p = 1$. So, let X be a nonempty set and $d : X \times X \to R_+$ be a (standard) metric over it. Hence, by the previous conventions, $d(.,.)$ is a Bakhtin metric on X, with Bakhtin characteristic $s = 1$. In particular, $d(.,.)$ is triangular and separated (hence, sufficient); moreover, in view of the Lipschitz-type relations (deductible from the triangle inequality)

$$|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v), \ \forall x, y, u, v \in X,$$

it follows that

$$d(.,.) \text{ is continuous; hence, } d \text{ is semi-lsc.}$$

An application of (YS-ad-Bakhtin) to these data is possible, under the lines of (YS-ad-tri); and the obtained particular version of it is just Ekeland variational principle [13] (in short EVP).

Finally, note that further extensions of our developments to quasi-ordered structures of this type are possible, under the lines in Turinici [30]. These will be discussed elsewhere.

7 Converse Question

By the developments above, we have the inclusions

$$((\text{DDC}) \iff) \ (\text{DC}) \implies (\text{YS-adli}) \implies (\text{YD-adli-tri})$$

$$\text{(YS-adli)} \Longrightarrow \text{(YS-ad)} \Longrightarrow \text{(YS-ad-tri)} \Longrightarrow \text{(EVP)}$$

$$\text{(YS-adli)} \Longrightarrow \text{(YS-adli-tri)} \Longrightarrow \text{(YS-ad-tri)} \Longrightarrow \text{(EVP)}$$

$$\text{(YS-ad)} \Longrightarrow \text{(YS-ad-f-Bakhtin)} \Longrightarrow \text{(YS-ad-Bakhtin)} \Longrightarrow \text{(EVP)}.$$

So, it is natural to ask whether these inclusion chains may be reversed. At a first glance, a negative answer is highly expectable, because (DC) is "too general" with respect to (EVP). However, the situation is exactly opposite, in the sense: (EVP) includes (DC); and then, we closed the circle between all such principles. [Clearly, the natural setting for solving this problem is (ZF-AC), referred to (see above) as the *strongly reduced* Zermelo–Fraenkel system.] An early result of this type was provided in 1987 by Brunner [8]; for a different answer to the same, we refer to the 1999 paper by Dodu and Morillon [12]. It is our aim in the following to show that a further extension of this last result is possible, in the sense: (DC) is deductible from a certain Lipschitz-bounded countable version of (EVP).

Let $(X, \leq)$ be a partially ordered structure. Remember that $z \in X$ is $(\leq)$-*maximal* if $z \leq w \in X$ implies $z = w$; the class of all these will be denoted as $\max(X, \leq)$. In this case, we say that $(\leq)$ is a *Zorn order* when

$\max(X, \leq)$ is (nonempty and) cofinal in X
(for each $u \in X$, there exists $v \in \max(X, \leq)$ with $u \leq v$).

In particular, when $d(.,.)$ is a *(standard) metric* on X and $\varphi : X \to R_+$ is some function, a good example of partial order on X is that introduced by the convention

$x \leq_{(d,\varphi)} y$ iff $d(x, y) \leq \varphi(x) - \varphi(y)$;

referred to as the *Brøndsted order [7]* attached to the couple (d, φ). Further, let us say that φ is *d-Lipschitz*, provided

$|\varphi(x) - \varphi(y)| \leq Ld(x, y), \forall x, y \in X$, for some $L > 0$;

note that, any such function is uniformly continuous on X.

The following stronger variant of (EVP) enters in our discussion.

Theorem 8. *Let the metric space* (X, d) *and the function* $\varphi : X \to R_+$ *satisfy*

(bd-com) *X is d-bounded and d-complete*
(d-Lip) *φ is d-Lipschitz (hence, bounded)*
(count) *$\varphi(X)$ is (at most) countable.*

Then, $(\leq_{(d,\varphi)})$ *is a Zorn order.*

We call this, the Lipschitz-bounded countable version of (EVP) (in short (EVP-Lbc)). By the above developments, we thus have

$$\text{(DC)} \Longrightarrow \text{(EVP)} \Longrightarrow \text{(EVP-Lbc)}.$$

The remarkable fact to be added is that this last principle yields (DC); so, it completes the circle between all these.

Proposition 16. *We have (in the strongly reduced Zermelo–Fraenkel system)*

(EVP-Lbc) $\Longrightarrow$ *(DC).*

As a consequence of this:

(71-1) *the variational principles (YS-adli), (YS-adli-tri), (YS-ad), (YS-ad-tri), (YS-ad-f-Bakhtin), (YS-ad-Bakhtin) are all equivalent with both (DC) and (EVP); hence, mutually equivalent.*

(71-2) *any maximal/variational principle (VP) with (DC)* $\Longrightarrow$ *(VP)* $\Longrightarrow$ *(EVP) is equivalent with both (DC) and (EVP).*

The proof of this result may be found in Turinici [31]. However, for completeness reasons, we shall provide the argument, with certain modifications.

Proof. There are several steps to be followed.

Part 0. Let M be a nonempty set and $\mathscr{R}$ be a proper relation over M. Fix $a \in M$; and take some other point α, that does not belong to M. Put $P = M \cup \{\alpha\}$; and let $d(.,.)$ stand for the discrete metric on P:

$$d(s,t) = 0, \text{ if } s = t; \;\; d(s,t) = 1, \text{ if } s \neq t.$$

(In fact, $d(.,.)$ is even an *ultrametric* on P; but, this is not essential for us.)

Part 1. Let $\mathscr{S}(P)$ stand for the class of all sequences $x = (x(n); n \geq 0)$ with elements in P. Denote $X = \{x \in \mathscr{S}(P); x(0) = a\}$; and let us introduce the map

$$d_\infty(x,y) = \textstyle\sum_n 2^{-n} d(x(n), y(n)), \text{ for } x = (x(n)) \text{ and } y = (y(n)) \text{ in } X.$$

It is not hard to see that d_∞ is a (standard) metric on X; moreover,

$$d_\infty(x,y) \leq \textstyle\sum_{n \geq 1} 2^{-n} = 1, \forall x,y \in X; \text{ whence, } X \text{ is } d_\infty\text{-bounded.}$$

A natural question to be discussed here is the completeness property. In this direction, we have the following:

Lemma 6. *Under the above conventions,*

X is d_∞-complete: each d_∞-Cauchy sequence in X is d_∞-convergent.

Proof. Let $(x^n; n \geq 0)$ be a sequence in X; it may be written as

$$(x^n = (x^n(0), x^n(1), \ldots) = (a, x^n(1), \ldots); n \geq 0).$$

Assume that

$(x^n; n \geq 0)$ is d_∞-Cauchy or, equivalently,
$\forall \varepsilon > 0$: $C(\varepsilon) := \{n \in N; n \leq p \leq q \Longrightarrow d_\infty(x^p, x^q) < \varepsilon\} \neq \emptyset$.

As a consequence, the map $\varepsilon \mapsto C(\varepsilon)$ is increasing on R^0_+, in the sense:

$\varepsilon_* < \varepsilon^*$ implies $C(\varepsilon_*) \subseteq C(\varepsilon^*)$;

so that, the map

$(\Gamma : R^0_+ \to N)$: $\quad \Gamma(\varepsilon) := \min[C(\varepsilon)], \varepsilon > 0$

is decreasing on R^0_+:

$\varepsilon_* < \varepsilon^*$ implies $\Gamma(\varepsilon_*) \geq \Gamma(\varepsilon^*)$.

Let $(\varepsilon_n; n \geq 0)$ be a strictly descending sequence in R^0_+ with

$\varepsilon_n < 2^{-n}$, for all n (hence, $\varepsilon_n \to 0$).

Denote for simplicity

$m(k) = \Gamma(\varepsilon_k), n(k) = m(k) + k, \ k \geq 0.$

By the properties above, the map $k \mapsto m(k)$ is increasing; hence, the map $k \mapsto n(k)$ is strictly increasing. For the moment, it is clear that

$$x^{n(0)}(0) = x^p(0) = a, \ \forall p \geq n(0).$$

Further, by the very definition of these maps,

$$n(1) \leq p \leq q \Longrightarrow d_\infty(x^p, x^q) < \varepsilon_1.$$

Combining with the definition of d_∞ gives

$$2^{-1} d(x^p(1), x^q(1)) < \varepsilon_1, \text{ if } n(1) \leq p \leq q;$$

so that (as $\varepsilon_1 < 2^{-1}$),

$$x^{n(1)}(1) = x^p(1), \text{ for all } p \geq n(1).$$

The procedure may continue indefinitely; it gives us, for the strictly ascending sequence of ranks $(n(i); i \geq 0)$, an evaluation like

$$x^{n(i)}(i) = x^p(i), \ \text{ for all } p \geq n(i) \text{ and all } i \geq 0.$$

Let $y = (y(i); i \geq 0)$ be the "diagonal" sequence $(y(i) = x^{n(i)}(i); i \geq 0)$; clearly, it is an element of X. We claim that our initial sequence $(x^n; n \geq 0)$ is convergent (modulo d_∞) to y. In fact, let $\varepsilon > 0$ be arbitrary fixed; and $h = h(\varepsilon)$ be such that

$2^{-j} < \varepsilon$, for all $j \geq h$ (possibly, since $\lim_n(2^{-n}) = 0$).

For each $n \geq n(h)$ (hence $n \geq n(i)$, whenever $i \leq h$), we have (by the above)

$$\begin{aligned} d(x^n, y) &= \textstyle\sum_{i \leq h} 2^{-i} d(x^n(i), x^{n(i)}(i)) + \sum_{i > h} 2^{-i} d(x^n(i), x^{n(i)}(i)) \\ &= \textstyle\sum_{i > h} 2^{-i} d(x^n(i), x^{n(i)}(i)) \leq \sum_{i > h} 2^{-i} = 2^{-h} < \varepsilon; \end{aligned}$$

and, from this, we are done.

Part 2. Let Y stand for the class of all sequences $x = (x(n); n \geq 0)$ in X with

$$(\forall n): \quad x(n), x(n+1) \in M \Longrightarrow x(n)\mathscr{R}x(n+1).$$

Note that $Y \neq \emptyset$; for, given $b \in \mathscr{R}(a)$, the sequence $y = (y(n); n \geq 0)$ in X introduced as below

$$(y(0) = a, y(1) \in b; y(n) = \alpha, n \geq 2)$$

is an element of it.

Lemma 7. *The subset Y is d_∞-closed, hence, d_∞-complete as well.*

Proof. Let $(x^n := (x^n(0) = a, x^n(1), \ldots); n \geq 0)$ be a sequence in Y and $y = (y(n); n \geq 0)$ be an element of X with

$x^n \to y$ (modulo d_∞); that is,
$d_\infty(x^n, y) := \sum_i 2^{-i} d(x^n(i), y(i)) \to 0$, as $n \to \infty$.

Note that, as a direct consequence of this,

$$x^n(i) \xrightarrow{d} y(i) \text{ as } n \to \infty, \ \forall i \geq 0.$$

Further, as d_∞ is metric, $(x^n; n \geq 0)$ appears as d_∞-Cauchy; so, by a preceding statement, there exists a strictly ascending sequence of ranks $(n(i); i \geq 0)$, with

$$(\forall i \geq 0): \quad x^{n(i)}(i) = x^p(i), \quad \forall p \geq n(i).$$

In this case, the d_∞-limit $y = (y(n); n \geq 0)$ of our sequence must have the form

$y(i) = x^{n(i)}(i)$, for all $i \geq 0$.

We now claim that the representation of Y gives us the desired conclusion: $y \in Y$. In fact, let $i \geq 0$ be such that $y(i), y(i+1) \in M$. By the previous relations,

$$y(i) = x^{n(i)}(i) = x^{n(i+1)}(i) \in M; \ \ y(i+1) = x^{n(i+1)}(i+1) \in M.$$

This, along with $x^{n(i+1)} \in Y$ yields

$$x^{n(i+1)}(i)\mathscr{R}x^{n(i+1)}(i+1); \text{ that is, } y(i)\mathscr{R}y(i+1).$$

The argument is thereby complete.

Part 3. Now, let us note that conclusion of our statement is equivalent with $Y \cap \mathscr{S}(M) \neq \emptyset$. For, taking some sequence $y = (y(n); n \geq 0)$ in this intersection, we have $y(n), y(n+1) \in M$, $\forall n$; so that, by definition, $y(n)\mathscr{R}y(n+1)$, $\forall n$, whence, $(y(n); n \geq 0)$ is $(a, \mathscr{R})$-iterative. Assume by contradiction that this is not true:

$Y \cap \mathscr{S}(M) = \emptyset$; i.e., for each $y = (y(n); n \geq 0) \in Y$,
there exists some $k = k(y) \geq 1$, such that $y(k) = \alpha$.

As a consequence, the functions below are well defined:

$$g(y) = \min\{k \geq 1; y(k) = \alpha\}, \varphi(y) = 2^{2-g(y)}, y \in Y.$$

Some basic properties of these are described in

Proposition 17. *The following are valid:*

(72-1) *the functions g, φ are continuous on Y; precisely,*

$$\forall y \in Y, \exists \beta = \beta(y) > 0 : z \in Y, d_\infty(z, y) < \beta \Longrightarrow g(z) = g(y), \varphi(z) = \varphi(y)$$

(72-2) *the function φ is d_∞-Lipschitz, in the sense:*

$$|\varphi(x) - \varphi(y)| \leq 4d_\infty(x, y), \quad \forall x, y \in Y$$

(72-3) *$g(Y)$ is countable; hence, so is $\varphi(Y)$.*

Proof. (i) Fix $y = (y(n); n \geq 0) \in Y$, and put $r = g(y)$; we therefore have

$$r \geq 1, y(r) = \alpha, y(k) \in M, \forall k \in N(r, >).$$

Take some $\beta \in]0, 2^{-r}[$; and let $z = (z(n); n \geq 0) \in Y$ be such that $d_\infty(y, z) < \beta$. By the definition of our metric,

$$2^{-k} d(y(k), z(k)) < \beta < 2^{-r}, \quad \forall k \in N(r, \geq);$$

and this yields

$$z(k) = y(k), \ \forall k \in N(r, \geq).$$

In particular, we must have

$$(z(k) \in M, \quad \forall k \in N(r, >)) \text{ and } z(r) = \alpha;$$

so that $g(z) = r = g(y)$ (whence, $\varphi(z) = \varphi(y)$).

(ii) Let $x = (x(n); n \geq 0)$ and $y = (y(n); n \geq 0)$ be two points in Y. Denote for simplicity $r = g(x)$, $s = g(y)$. If $r = s$, all is clear; so, it remains the opposite case $r \neq s$; without loss, one may assume that $r < s$. As a consequence,

$$x = (x(0) = a, \ldots, x(r-1), \alpha, \ldots, x(s-1), x(s), \ldots),$$
$$y = (y(0) = a, \ldots, y(r-1), y(r), \ldots y(s-1), \alpha, \ldots).$$

In particular, $y(r) \in M$; hence $y(r) \neq \alpha$; and then,

$$d_\infty(x, y) \geq 2^{-r} \geq 2^{-r} - 2^{-s} = |2^{-r} - 2^{-s}|.$$

This gives the conclusion we need.

(iii) Evident.

Part 4. We show that, under the introduced conventions,

$$\text{for each } v \in Y, \text{ there exists } y \in Y \setminus \{v\} \text{ such that } d_\infty(v, y) \le \varphi(v) - \varphi(y);$$

or, in other words, each element of Y is non-maximal with respect to the Brøndsted ordering attached to d_∞ and φ:

$(z, w \in Y)$: $\quad z \le w$ iff $d_\infty(z, w) \le \varphi(z) - \varphi(w)$.

In fact, let $v = (v(n); n \ge 0)$ be its representation. Put $g(v) = r$; hence,

$$r \ge 1;\; v(0), \ldots, v(r-1) \in M;\; v(r) = \alpha.$$

Note that, by the definition of Y, one gets the relations

$$v(i)\mathscr{R}v(i+1), \text{ whenever } i \le r-2.$$

Take $y = (y(n); n \ge 0)$ in $Y \setminus \{v\}$ according to

$y(k) = v(k)$, $\forall k \in N(r, >)$; $y(h) = \alpha$, $\forall h \in N(r+1, <)$;
$y(r), y(r+1) \in M$; $y(i)\mathscr{R}y(i+1)$, $\forall i \in \{r-1, r\}$.

(The last relation is possible, by the Finite Dependent Choice property). As a consequence of this, $g(y) = r + 2$. Now, the desired relation above becomes

$$d_\infty(v, y) \le 2^{2-r} - 2^{-r} = 3 \cdot 2^{-r}.$$

According to the representation of $y \in Y \setminus \{v\}$, this means

$$\sum_{i \ge r} 2^{-i} d(v(i), y(i)) \le 3 \cdot 2^{-r}.$$

But then, the last relation is clear, in view of

$$\sum_{i \ge r} 2^{-i} d(v(i), y(i)) \le \sum_{i \ge r} 2^{-i} = 2^{1-r} < 3 \cdot 2^{-r}.$$

Part 5. We may now pass to the final part of our argument. By the above facts, (EVP-Lbc) is applicable to the metric space (Y, d_∞) and the function $\varphi : Y \to R_+$ (introduced as before). Hence, the associated Brøndsted order $(\le)$ (see above) is a Zorn one. As a consequence, there exists, for the starting point (in Y)

$$u = (u(n); n \ge 0):\; u(0) = a, u(n) = \alpha, \forall n \ge 1,$$

some other point $v = (v(n); n \ge 0)$ in Y with

(zorn-1) $u \le v$: $d_\infty(u,v) \le \varphi(u) - \varphi(v)$
(zorn-2) v is $(\le)$-maximal: $d_\infty(v,y) > \varphi(v) - \varphi(y)$, $\forall y \in Y \setminus \{v\}$.

This, however, contradicts the preceding step and shows that $Y \cap \mathscr{S}(M) \neq \emptyset$. But then (by the very definition of Y), there must be some sequence $y = (y(n); n \ge 0)$ in M, with

$$y(0) = a \text{ and } y(n)\mathscr{R}y(n+1),\ \forall n.$$

The proof is complete.

In particular, when the boundedness and Lipschitz properties are ignored, this result is just the one in Dodu and Morillon [12]. Further aspects may be found in Turinici [30].

Summing up, all variational principles in this exposition (derived from (DC)) are nothing but logical equivalents of (EVP). So, it is natural to ask whether the remaining (sequential) ones—including the smooth variational principle in Deville et al. [11]—are endowed as well with such a property. The answer to this is affirmative; further aspects will be delineated elsewhere.

8 Yongxin–Shuzhong Approaches

As already precise, all smooth variational results—which the present exposition is based on—emerge from a 2000 contribution in the area due to Yongxin and Shuzhong [33]. So, any discussion of such statements must begin with the line of argument provided by these authors. In the following, we shall however follow the developments in Farkas et al. [16]; because, apart from giving the most general result in the area, their exposition is better structured than the previous ones. [The only intervention in authors' text refers to numbering of certain relations; this will help us to formulate our comments about their proof.]

(A) The concept of *Bakhtin metric* was already introduced; and some basic properties of it were listed in a previous place. An interesting completion of these refers to the so-called Cantor's intersection theorem for the subsequent Bakhtin metrical structure; see Bota et al. [4] for details. Let (X,d) be a Bakhtin metric space. For each nonempty subset Y of X, denote (as in the standard metrical case)

$$\operatorname{diam}(Y) = \sup\{d(u,v); u, v \in Y\} \text{ (the diameter of } Y).$$

Lemma 8. *Let (X,d) be a complete Bakhtin metric space and $(F_n; n \ge 0)$ be a nonincreasing sequence*

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$$

of nonempty closed subsets of X, with

$$\operatorname{diam}(F_n) \to 0 \text{ as } n \to \infty.$$

Then, necessarily,

$\cap\{F_n; n \geq 0\}$ *contains one and only one point.*

(B) Having these precise, we may now proceed to a discussion of the quoted result. Let X be a nonempty set; and $d : X \times X \to R_+$ be a Bakhtin metric over it, endowed with the properties

(FMN-1) d is continuous and (X, d) is complete.

Further, let $f : X \to R \cup \{\infty\}$ be an (extended) function with

(FMN-2) f is proper, lsc, bounded from below;

and $\rho : X \times X \to R_+ \cup \{\infty\}$ be an (extended) positive function with

(FMN-3) $\rho(x, x) = 0$, $\forall x \in X$
(FMN-4) for each couple of sequences (y_n) and (z_n) in X, we have $\rho(y_n, z_n) \to 0$ implies $d(y_n, z_n) \to 0$
(FMN-5) for each $z \in X$, the function $y \mapsto \rho(y, z)$ is lsc.

Let also $h : R_+ \to R_+$ be a function with

(FMN-6) h is continuous and nonincreasing;

and take a sequence (δ_n) in R^0_+. For any $x_0 \in X$ and any sequence (x_n) in X (to be constructed further), let us introduce the notation

$$\mathscr{A}[x; m] = f(x) + h(d(x_0, x)) \sum_{n=0}^{m} \delta_n \rho(x, x_n),\ x \in X,\ m \in N \cup \{\infty\}.$$

The main authors' result is

Theorem 9. *Let the Bakhtin metric $d(.,.)$, the (extended) function f, and the (extended) positive function ρ be taken as in (FMN-1) – (FMN-5). Further, let $h : R_+ \to R_+$ be a positive function as in (FMN-6) and (δ_n) be a sequence in R^0_+. Finally, take $x_0 \in X$ and $\varepsilon > 0$ according to*

(FMN-7) $f(x_0) \leq \inf[f(X)] + \varepsilon$.

There exists then a sequence (x_n) in X and an element x_ε in X such that

(I) $x_n \to x_\varepsilon$ *as* $n \to \infty$
(II) $h(d(x_0, x_\varepsilon))\rho(x_\varepsilon, x_n) \leq \varepsilon/2^n\delta_0$, *for all* $n \in N$
(III) $A[x_\varepsilon; \infty] \leq f(x_0)$
(IV) $\mathscr{A}[x; \infty] > \mathscr{A}[x_\varepsilon; \infty]$, *for each* $x \neq x_\varepsilon$.

[In fact, this result has also a variant where the case of

$\{n \in N; \delta_n > 0\}$ = finite, in the sense: there exists $k \in N$, with $(\delta_i > 0$, for $0 \leq i \leq k)$ and $(\delta_j = 0$, for $j > k)$

is being considered; but, this is not important for us.]

The proof proposed by the authors is that given below:

Proof. Define the following set:

$$\mathscr{W}(x_0) = \{x \in X; \mathscr{A}[x;0] \le f(x_0)\}.$$

By the assumption (FMN-3), we have $\rho(x_0, x_0) = 0$; so, $x_0 \in \mathscr{W}(x_0)$. Therefore, the set $\mathscr{W}(x_0)$ is nonempty. From the lower semicontinuity of the functions f and $\rho(., x_0)$ and the continuity of the function h, we deduce that $\mathscr{W}(x_0)$ is a closed subset of X. We can choose the element $x_1 \in \mathscr{W}(x_0)$ such that

$$\mathscr{A}[x_1;0] \le \inf\{\mathscr{A}[x;0]; x \in \mathscr{W}(x_0)\} + \varepsilon\delta_1/2\delta_0;$$

and consider the set

$$\mathscr{W}(x_1) = \{x \in \mathscr{W}(x_0); \mathscr{A}[x;1] \le \mathscr{A}[x_1;0]\}.$$

Similarly as above, we obtain that $\mathscr{W}(x_1) \neq \emptyset$ (since $x_1 \in \mathscr{W}(x_1)$) and $\mathscr{W}(x_1)$ is a nonempty closed subset of $\mathscr{W}(x_0)$, which means that $\mathscr{W}(x_1)$ is a nonempty closed subset of X as well.

Using the method of mathematical induction, we can define a point $x_{n-1} \in \mathscr{W}(x_{n-2})$ and a set $\mathscr{W}(x_{n-1})$, such that

$$\mathscr{W}(x_{n-1}) = \{x \in \mathscr{W}(x_{n-2}); \mathscr{A}[x;n-1] \le \mathscr{A}[x_{n-1};n-2]\}.$$

It is easy to see that $\mathscr{W}(x_{n-1}) \neq \emptyset$ and $\mathscr{W}(x_{n-1})$ is a closed subset of X. We can choose $x_n \in \mathscr{W}(x_{n-1})$ such that

$$\mathscr{A}[x_n;n-1] \le \inf\{\mathscr{A}[x;n-1]; x \in \mathscr{W}(x_{n-1})\} + \varepsilon\delta_n/2^n\delta_0$$

and consider the set

$$\mathscr{W}(x_n) = \{x \in \mathscr{W}(x_{n-1}); \mathscr{A}[x;n] \le \mathscr{A}[x_n;n-1]\},$$

which is also a closed subset of X.

Let

(rela-1) z be an arbitrary element of $\mathscr{W}(x_n)$.

Then, from the definition of $\mathscr{W}(x_n)$, we have the inequality

$$\mathscr{A}[z;n] \le \mathscr{A}[x_n;n-1],$$

which means that

$$\begin{aligned} f(z) &+ h(d(x_0,z))\delta_n\rho(z,x_n) + h(d(x_0,z))\textstyle\sum_{i=0}^{n-1}\delta_i\rho(z,x_i) \\ &\le f(x_n) + h(d(x_0,x_n))\textstyle\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i). \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} &h(d(x_0,z))\delta_n\rho(z,x_n) \\ &\quad \le (f(x_n) + h(d(x_0,x_n))\textstyle\sum_{i=0}^{n-1}\delta_i\rho(x_n,x_i)) - (f(z) + h(d(x_0,z))\textstyle\sum_{i=0}^{n-1}\delta_i\rho(z,x_i)) \\ &\quad \le \mathscr{A}[x_n;n-1] - \inf\{\mathscr{A}[x;n-1]; x \in \mathscr{W}(x_{n-1})\} \le \varepsilon\delta_n/2^n\delta_0; \end{aligned}$$

therefore

(rela-2) $\quad h(d(x_0, z))\rho(z, x_n) \leq \varepsilon/2^n\delta_0.$

So, if $n \to \infty$, then

(rela-3) $\quad \rho(z, x_n) \to 0.$

Then, from (FMN-4), it follows that

(rela-4) $\quad d(z, x_n) \to 0$ as $n \to \infty$.

Therefore,

(rela-5) $\quad \mathrm{diam}(\mathscr{W}(x_n)) \to 0$, whenever $n \to \infty$

and we obtain a descending sequence $(\mathscr{W}(x_n); n \geq 0)$ of nonempty closed subsets of X, i.e.,

(rela-6) $\quad \mathscr{W}(x_0) \supseteq \mathscr{W}(x_1) \supseteq \ldots \supseteq \mathscr{W}(x_n) \supseteq \ldots$

such that

$$\mathrm{diam}(\mathscr{W}(x_n)) \to 0, \text{ whenever } n \to \infty.$$

Applying the Cantor's intersection theorem for the set sequence $(\mathscr{W}(x_n); n \in N)$, we conclude that there exists an $x_\varepsilon \in X$ such that (see Lemma 8)

$$\cap\{\mathscr{W}(x_n); n \in N\} = \{x_\varepsilon\}.$$

We can observe that

(rela-7) $\quad z = x_\varepsilon$ satisfies (rela-2);

therefore

$$x_n \to x_\varepsilon \text{ as } n \to \infty.$$

If $x \neq x_\varepsilon$, then there exists $m \in N$ such that

(rela-8) $\quad \mathscr{A}[x; m] > \mathscr{A}[x_m; m-1].$

It is clear that, if $q \geq m$, then

$$\mathscr{A}[x_m; m-1] \geq \mathscr{A}[x_q; q-1] \geq \mathscr{A}[x_\varepsilon; q-1].$$

Combining this relation with inequality (rela-8), we get the following estimation:

$$\mathscr{A}[x; m] \geq \mathscr{A}[x_\varepsilon; q-1].$$

Hence, if $q, m \to \infty$, we obtain the desired relation (**IV**).

Comments About the Proof:

(Com-1) The evaluation (rela-2) is a **local** one; because, for the (arbitrary but) **fixed** element $z \in \mathscr{W}(x_n)$, an (inequality) connection like

(Rela-k) $\quad h(d(x_0, z))\rho(z, x_k) \leq \varepsilon/2^k\delta_0$

between z and some term x_k (of that sequence) is possible whenever $z \in \mathscr{W}(x_k)$. This, however, cannot hold for all $k \geq n$; i.e., a global version of (Rela-k) like

(Rela-glo) $h(d(x_0, z))\rho(z, x_k) \leq \varepsilon/2^k\delta_0$, for all $k \geq n$

cannot hold (in general). To motivate such an assertion, it will suffice noting that as long as $z \in \mathscr{W}(x_n)$ is an (arbitrary but) **fixed** element and $(\mathscr{W}(x_i); i \geq 0)$ is a descending sequence, it is possible that

$z \notin \mathscr{W}(x_k)$, for some $k > n$; and then $z \notin \mathscr{W}(x_h)$ for all $h \geq k$.

Hence, for a **fixed** $z \in \mathscr{W}(x_n)$, (Rela-glo) may be false; and, in this case, evaluation (rela-3) is unacceptable.

(Com-2) If $z \in \mathscr{W}(x_n)$ satisfies

$h(d(x_0, z)) = 0$ (not impossible, by the choice of $h(.)$),

then (rela-2) holds in a trivial way, because, it is retainable with

$\rho(z, x_n)$= arbitrary in $R_+ \cup \{\infty\}$.

But then, evidently,

(rela-2) $\Longrightarrow$ (rela-3) may be not valid.

(Com-3) As a consequence, (rela-4) is unacceptable as well. This conclusion may be also confirmed as follows. Assume, by contradiction that (rela-4) is true. Then, we must have (as $z \in \mathscr{W}(x_n)$ is arbitrary)

$\lim_n(x_n) = z$, for each $z \in \mathscr{W}(x_n)$;

and this, by the properties of the Bakhtin metric d, gives

$\mathrm{diam}(\mathscr{W}(x_n)) = 0$, wherefrom, $\mathrm{diam}(\mathscr{W}(x_k)) = 0$, for all $k \geq n$.

Hence, all reasoning involving Cantor's intersection theorem is, from now on, absolutely useless. In fact, the whole iterative procedure becomes trivial; for, in view of preceding relations,

$\mathscr{W}(x_k) = \{x_n\}$, for all $k \geq n$; whence $x_\varepsilon = x_n$;

i.e., the *final* point x_ε of this process is always identical with the nth point x_n of the same; impossible, as simple examples show.

(Com-4) It follows from the preceding discussion that the main result in Farkas et al. [16] is not in general true; this affects in a direct way all statements of the quoted paper.

(Com-5) On the other hand, the main result in Farkas [15] is just a particular case—obtained with the same argument—of the one due to Farkas et al. [16], when the ambient Bakhtin metric $d(.,.)$ is a (standard) metric (on X). Hence, the underlying result is also unacceptable; this affects all statements of the quoted paper.

(Com-6) All these proofs have as common origin the one proposed by Yongxin and Shuzhong [33] in establishing their main result. As a consequence, the result in question is (like before) not valid, in general.

(Com-7) The final part of argument in Bota et al. [4] is identical with the one discussed in **(Com-1)**; hence, it is not acceptable. However, the result established there is ultimately correct, if one uses a direct approach, based on the properties of Bakhtin metric d; we do not give details.

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Compositional Yang-Hilbert-Type Integral Inequalities and Operators

Bicheng Yang

In Honor of Constantin Carathéodory

Abstract By using the Real and Functional Analysis and estimating the weight functions, we build two kinds of compositional Yang-Hilbert-type integral inequalities with the best possible constant factors. The equivalent forms and the reverses are also considered. Four kinds of compositional Yang-Hilbert-type integral operators are defined and the related composition formulas are given.

1 Introduction

If $f(x), g(y) \geq 0, f, g \in L^2(\mathbf{R}_+),\ ||f||_2, ||g||_2 > 0$, then we have the following well-known Hilbert's integral inequality and the equivalent form, which published in 1911 (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi ||f||_2 ||g||_2, \tag{1}$$

$$\left[\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^2 dy\right]^{\frac{1}{2}} < \pi ||f||_2, \tag{2}$$

where the constant factor π is the best possible.

In 1925, by introducing a pair of conjugate exponents (p, q) $(\frac{1}{p} + \frac{1}{q} = 1)$, Hardy et al. [2] gave extensions of (1) and (2) as follows:

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For $p > 1$, $f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$, $||f||_p, ||g||_q > 0$, we have the following Hardy-Hilbert's integral inequality and the equivalent form

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{3}$$

$$\left[\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx\right)^p dy\right]^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/p)} ||f||_p, \tag{4}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$.

Definition 1. If $\lambda \in \mathbf{R} = (-\infty, \infty), \mathbf{R}_+ = (0, \infty), k_\lambda(x, y)$ is a measurable function in $\mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+$, satisfying for any $t, x, y \in \mathbf{R}_+$,

$$k_\lambda(tx, ty) = t^{-\lambda} k_\lambda(x, y), \tag{5}$$

then we call $k_\lambda(x, y)$ as homogeneous function of degree $-\lambda$ in $\mathbf{R}_+^2$.

In 1934, by introducing a general nonnegative homogeneous function $k_1(x, y)$ of degree -1, Hardy et al. [3] gave extensions of (3) and (4) as follows:

For $p > 1, \frac{1}{p} + \frac{1}{q} = 1$,

$$k_p = \int_0^\infty k_1(u, 1) u^{\frac{-1}{p}} du \in \mathbf{R}_+,$$

$f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$, $||f||_p, ||g||_q > 0$, we have the following Hardy-Hilbert-type integral inequality and the equivalent form

$$\int_0^\infty \int_0^\infty k_1(x, y) f(x) g(y) dxdy < k_p ||f||_p ||g||_q, \tag{6}$$

$$\left[\int_0^\infty \left(\int_0^\infty k_1(x, y) f(x) dx\right)^p dy\right]^{\frac{1}{p}} < k_p ||f||_p, \tag{7}$$

where the constant factor k_p is the best possible.

Some applications of Hardy-Hilbert-type inequalities are provided in [3, 4].

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [5] gave an extension of (3) with the homogeneous kernel of degree $-\lambda$ as $\frac{1}{(x+y)^\lambda}$. In 2009, by introducing a general nonnegative homogeneous function $k_\lambda(x, y)$ of degree $-\lambda$ and adding another pair of conjugate exponents (r, s) $(\frac{1}{r} + \frac{1}{s} = 1)$, Yang [6] gave extensions of (6) and (7) as follows:

For $p, r > 1, \lambda \in \mathbf{R}_+, \phi(x) = x^{p(1-\frac{\lambda}{r})-1}, \psi(y) = y^{q(1-\frac{\lambda}{s})-1} (x, y \in \mathbf{R}_+)$,

$$k_\lambda(r) = \int_0^\infty k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du \in \mathbf{R}_+,$$

$f(x),\ g(y) \geq 0,$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} = \left(\int_0^\infty \phi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, we have the following Yang-Hilbert-type integral inequality and the equivalent form

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y) dx dy < k_\lambda(r) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{8}$$

$$\left[\int_0^\infty y^{\frac{p\lambda}{s}-1} \left(\int_0^\infty k_\lambda(x,y) f(x) dx \right)^p dy \right]^{\frac{1}{p}} < k_\lambda(r) ||f||_{p,\phi}, \tag{9}$$

where the constant factor $k_\lambda(r)$ is the best possible.

Remark 1. When $\lambda = 1, r = q, s = p$, (8) and (9) reduce respectively to (6) and (7). Hence, Yang-Hilbert-type integral inequalities are extensions of Hardy-Hilbert-type integral inequalities with multiparameters and a best possible constant factor.

Using (2), we may define Hilbert's integral operator $T : L^2(\mathbf{R}_+) \to L^2(\mathbf{R}_+)$ as follows (cf. [7]):

For any $f \in L^2(\mathbf{R}_+)$, there exists a unique $Tf \in L^2(\mathbf{R}_+)$, satisfying

$$Tf(y) = \int_0^\infty \frac{f(x)}{x+y} dx (y \in \mathbf{R}_+).$$

Then by (2), we have $||Tf||_2 \le \pi ||f||_2$, namely, T is a bounded linear operator satisfying $||T|| \le \pi$. Since the constant factor in (2) is the best possible, it follows that $||T|| = \pi$.

About the discrete analogues of (1) and (2), in 1950, Wilhelm [8] gave a similar operator expression. In 2002, by using the operator theory, Zhang [9] gave some improvements of (2) and the discrete analogues. In 2006–2009, [10] considered a new Hilbert-type operator and its applications, and [11, 12] gave some multiple Hilbert-type operator expressions.

By using (9), we can define Yang-Hilbert-type integral operator $T : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\varphi}(\mathbf{R}_+)$ as follows (cf. [6]):

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unique $Tf \in L_{p,\phi}(\mathbf{R}_+)$, satisfying

$$Tf(y) = y^{\lambda-1} \int_0^\infty \frac{f(x)}{x+y} dx (y \in \mathbf{R}_+). \tag{10}$$

Then by (9), we have $||Tf||_{p,\phi} \le k_\lambda(r) ||f||_{p,\phi}$, namely, T is a bounded linear operator satisfying $||T|| \le k_\lambda(r)$. Since the constant factor in (9) is the best possible, we have $||T|| = k_\lambda(r)$.

Some other kinds of Yang-Hilbert-type inequalities are provided by [13–19]

On the composition of two Hilbert-type operators, the main objective is to build a few expression formulas as

$$||T_1 \cdot T_2|| = ||T_1|| \cdot ||T_2||. \tag{11}$$

Recently, [20] published a composition of two discrete Hilbert-Hardy-type operators with the particular kernels; [21] published a composition of two half-discrete Hilbert-Hardy-type operators with the particular kernels, and [22, 23] published some compositions of two Hardy-type integral operators with the particular kernels. [24] provided a composition of two general Hilbert-Hardy-type integral operators. These works are hard and interested.

By using the way of Real and Functional Analysis and estimating the weight functions, we build two kinds of compositional Yang-Hilbert-type integral inequalities with the best possible constant factors. The equivalent forms and the reverses are also considered. Four kinds of compositional Yang-Hilbert-type integral operators are defined and the related composition formulas such as (11) are given, which are some extensions of the results in [22, 23] and [24].

2 General Yang-Hilbert-Type Integral Inequalities and the Operator Expressions

First, for the needing of Sects. 3 and 5, we give a weight function and study some general Yang-Hilbert-type integral inequalities with multiparameters and the best constant factors, which were partly mentioned in [6]. The equivalent forms, the reverse, the operator expressions, and a large number of particular examples are also discussed.

2.1 A Weight Function and a Lemma

Definition 2. If $\sigma \in \mathbf{R}$, $h(t)$ is a nonnegative measurable function in $\mathbf{R}_+$, define the following weight function:

$$\omega(\sigma, y) := y^{\sigma} \int_0^{\infty} h(xy) x^{\sigma-1} dx (y \in \mathbf{R}_+). \tag{12}$$

Setting $t = xy$ in (12), we find

$$\omega(\sigma, y) = k(\sigma) := \int_0^{\infty} h(t) t^{\sigma-1} dt. \tag{13}$$

Lemma 1. *If* $p > 0 (p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, \sigma \in \mathbf{R}$, $k(\sigma)$ *is defined by (13), both* $h(t)$ *and* $f(t)$ *are nonnegative measurable functions in* $\mathbf{R}_+$, *then, (i) for* $p > 1$, *we have the following inequality:*

$$J := \left[\int_0^{\infty} y^{p\sigma-1} \left(\int_0^{\infty} h(xy) f(x) dx \right)^p dy \right]^{\frac{1}{p}}$$
$$\leq k(\sigma) \left(\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right)^{\frac{1}{p}}; \tag{14}$$

(ii) for $0 < p < 1$, *we have the reverse of (14).*

Proof. (i) By Hölder's inequality with weight (cf. [25]) and (12), it follows that

$$
\begin{aligned}
&\int_0^\infty h(xy)f(x)dx \\
&= \int_0^\infty h(xy)\left[\frac{x^{(1-\sigma)/q}}{y^{(1-\sigma)/p}}f(x)\right]\left[\frac{y^{(1-\sigma)/p}}{x^{(1-\sigma)/q}}\right]dx \\
&\leq \left[\int_0^\infty h(xy)\frac{x^{(1-\sigma)p/q}}{y^{1-\sigma}}f^p(x)dx\right]^{\frac{1}{p}} \\
&\quad \times \left[\int_0^\infty h(xy)\frac{y^{(1-\sigma)q/p}}{x^{1-\sigma}}dx\right]^{\frac{1}{q}} \\
&= (\omega(\sigma,y))^{\frac{1}{q}}y^{\frac{1}{p}-\sigma}\left[\int_0^\infty h(xy)\frac{x^{(1-\sigma)(p-1)}}{y^{1-\sigma}}f^p(x)dx\right]^{\frac{1}{p}}. \qquad (15)
\end{aligned}
$$

Then by (13), (12), and Fubini theorem (cf. [26]), we have

$$
\begin{aligned}
J^p &\leq k^{p-1}(\sigma)\int_0^\infty \int_0^\infty h(xy)\frac{x^{(1-\sigma)(p-1)}}{y^{1-\sigma}}f^p(x)dxdy \\
&= k^{p-1}(\sigma)\int_0^\infty \left[\int_0^\infty h(xy)\frac{x^{(1-\sigma)(p-1)}}{y^{1-\sigma}}dy\right]f^p(x)dx \\
&= k^{p-1}(\sigma)\int_0^\infty \omega(\sigma,x)x^{p(1-\sigma)-1}f^p(x)dx. \qquad (16)
\end{aligned}
$$

Still by (13), we obtain (14).

(ii) For $0 < p < 1$, by the reverse Hölder's inequality with weight (cf. [25]), (12) and (13), we can obtain the reverse of (15) and (16). Then we find the reverse of (14) by using (13).

The lemma is proved. □

2.2 *Two Equivalent Inequalities with the Nonhomogeneous Kernel*

Theorem 1. *Suppose that* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, h(t) \geq 0,$

$$k(\sigma) = \int_0^\infty h(t)t^{\sigma-1}dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty, 0 < \int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy < \infty,$$

then we have the following equivalent inequalities:

$$I := \int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy$$
$$< k(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (17)$$

$$J = \left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}, \quad (18)$$

where the constant factor $k(\sigma)$ is the best possible.

Proof. We first proved that (15) keeps the form of strict inequality for any $y \in \mathbf{R}_+$. Otherwise, there exists a $y > 0$, such that (15) keeps the form of equality. Then, there exist two constants A and B, such that they are not all zero, and (cf. [25])

$$A\frac{x^{(1-\sigma)p/q}}{y^{1-\sigma}}f^p(x) = B\frac{y^{(1-\sigma)q/p}}{x^{1-\sigma}} \text{ a.e. in } \mathbf{R}_+.$$

If $A = 0$, then $B = 0$, which is impossible. Assuming that $A \neq 0$, then it follows that

$$x^{p(1-\sigma)-1}f^p(x) = y^{q(1-\sigma)}\frac{B}{Ax} \text{ a.e. in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty,$$

in virtue of $\int_0^\infty \frac{1}{x}dx = \infty$. Hence, both (15) and (16) keep the form of strict inequalities, and then we have (18).

By Hölder's inequality (cf. [25]), we find

$$I = \int_0^\infty \left(y^{\sigma-\frac{1}{p}}\int_0^\infty h(xy)f(x)dx\right)(y^{\frac{1}{p}-\sigma}g(y))dy$$
$$\le J\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}. \quad (19)$$

Then by (18), we have (17).

On the other hand, assuming that (17) is valid, we set

$$g(y) := y^{p\sigma-1}\left(\int_0^\infty h(xy)f(x)dx\right)^{p-1}, y \in \mathbf{R}_+.$$

Then we find

$$J^p = \int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy.$$

By (16), in view of

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

it follows that $J < \infty$. If $J = 0$, then, (18) is trivially valid; if $J > 0$, then by (17), we have

$$\begin{aligned}
0 &< \int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy = J^p = I \\
&< k(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}} < \infty, \\
J &= \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{p}} < k(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}},
\end{aligned}$$

and then (18) follows, which is equivalent to (17).

For any $n \in \mathbf{N} = \{1, 2, \cdots\}$, we set functions $f_n(x)$ and $g_n(y)$ as follows:

$$f_n(x) := \begin{cases} 0, x \in (0,1), \\ x^{\sigma - \frac{1}{np} - 1}, x \in [1, \infty), \end{cases}$$

$$g_n(y) := \begin{cases} y^{\sigma + \frac{1}{nq} - 1}, y \in (0,1], \\ 0, y \in (1, \infty). \end{cases}$$

Then we find

$$\begin{aligned}
L_n &:= \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g_n^q(y)dy\right]^{\frac{1}{q}} \\
&= \left(\int_1^\infty x^{-\frac{1}{n}-1} dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1} dy\right)^{\frac{1}{q}} = n.
\end{aligned}$$

In view of Fubini theorem (cf. [26]), it follows that

$$\begin{aligned}
I_n &:= \int_0^\infty \int_0^\infty h(xy) f_n(x) g_n(y) dxdy \\
&= \int_1^\infty x^{\sigma - \frac{1}{np} - 1} \left(\int_0^1 h(xy) y^{\sigma + \frac{1}{nq} - 1} dy\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty x^{-\frac{1}{n}-1}\left(\int_0^x h(t)t^{\sigma+\frac{1}{nq}-1}dt\right)dx \\
&= \int_1^\infty x^{-\frac{1}{n}-1}\left(\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^x h(t)t^{\sigma+\frac{1}{nq}-1}dt\right)dx \\
&= n\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^\infty x^{-\frac{1}{n}-1}\left(\int_1^x h(t)t^{\sigma+\frac{1}{nq}-1}dt\right)dx \\
&= n\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^\infty \left(\int_t^\infty x^{-\frac{1}{n}-1}dx\right)h(t)t^{\sigma+\frac{1}{nq}-1}dt \\
&= n\left(\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^\infty h(t)t^{\sigma-\frac{1}{np}-1}dt\right).
\end{aligned}
$$

If there exists a positive number $k \le k(\sigma)$, such that (17) is still valid when replacing $k(\sigma)$ to k, then in particular, it follows that $\frac{1}{n}I_n < k\frac{1}{n}L_n$, and

$$
\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^\infty h(t)t^{\sigma-\frac{1}{np}-1}dt < k.
$$

Since both $\{h(t)t^{\sigma+\frac{1}{nq}-1}\}_{n=1}^\infty (t \in (0,1])$ and $\{h(t)t^{\sigma-\frac{1}{np}-1}\}_{n=1}^\infty (t \in (1,\infty))$ are nonnegative and increasing, then by Levi theorem (cf. [26]), it follows that

$$
\begin{aligned}
k(\sigma) &= \int_0^1 h(t)t^{\sigma-1}dt + \int_1^\infty h(t)t^{\sigma-1}dt \\
&= \lim_{n\to\infty}\left(\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^\infty h(t)t^{\sigma-\frac{1}{np}-1}dt\right) \le k,
\end{aligned}
$$

and then $k = k(\sigma)$ is the best possible constant factor of (17).

The constant factor in (18) is still the best possible. Otherwise, we would reach a contradiction by (19) that the constant factor in (17) is not the best possible.

The theorem is proved. □

Theorem 2. *Replacing $p > 1$ to $0 < p < 1$ in Theorem 1, we have the equivalent reverses of (17) and (18). If there exists a constant $\delta_0 > 0$, such that for any $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma]$, $k(\widetilde{\sigma}) \in \mathbf{R}_+$, then the constant factor in the reverses of (17) and (18) is the best possible.*

Proof. By Lemma 1 and the reverse Hölder's inequality, we have the reverses of (17)–(19). By the same way, we can set $g(y)$ as Theorem 1 and prove that the reverses of (17) and (18) are equivalent.

For any $n > \frac{2}{\delta_0|q|} (n \in \mathbf{N})$, we set $f_n(x)$ and $g_n(y)$ as Theorem 1. If there exists a positive number $k \ge k(\sigma)$, such that the reverse of (17) is valid when replacing $k(\sigma)$ to k, then in particular, it follows that $\frac{1}{n}I_n > k\frac{1}{n}L_n$, and

$$
\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^\infty h(t)t^{\sigma-\frac{1}{np}-1}dt > k. \tag{20}
$$

Since $\{h(t)t^{\sigma-\frac{1}{np}-1}\}_{n=1}^{\infty}(t \in (1,\infty))$ is still nonnegative and increasing, then by Levi theorem, it follows that

$$\lim_{n\to\infty}\int_1^{\infty} h(t)t^{\sigma-\frac{1}{np}-1}dt = \int_1^{\infty} h(t)t^{\sigma-1}dt.$$

Since for $n > \frac{2}{\delta_0|q|}$,

$$0 \le h(t)t^{\sigma+\frac{1}{nq}-1} \le h(t)t^{(\sigma-\frac{\delta_0}{2})-1}(t \in (0,1]),$$

and

$$0 \le \int_0^1 h(t)t^{(\sigma-\frac{\delta_0}{2})-1}dt \le k(\sigma-\frac{\delta_0}{2}) < \infty,$$

then by Lebesgue dominated convergence theorem (cf. [26]), it follows that

$$\lim_{n\to\infty}\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt = \int_0^1 h(t)t^{\sigma-1}dt.$$

In view of the above results and (20), we have

$$k(\sigma) = \lim_{n\to\infty}\left(\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1}dt + \int_1^{\infty} h(t)t^{\sigma-\frac{1}{np}-1}dt\right) \ge k,$$

and then $k = k(\sigma)$ is the best possible constant factor in the reverse of (17).

By the same way, we can prove that the constant factor in the reverse of (18) is the best possible by using the reverse of (19).

The theorem is proved. □

Assuming that $h(xy) = 0(0 < \frac{1}{y} \le x)$, we find $h(t) = 0(t \ge 1)$ and

$$k(\sigma) = k_1(\sigma) := \int_0^1 h(t)t^{\sigma-1}dt.$$

In view of Theorems 1 and 2, we have

Corollary 1. *Suppose that* $p > 0(p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, h(t) \ge 0, k_1(\sigma) \in \mathbf{R}_+$, $f(x), g(y) \ge 0$, *satisfying*

$$0 < \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x)dx < \infty, 0 < \int_0^{\infty} y^{q(1-\sigma)-1} g^q(y)dy < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^{\frac{1}{y}} h(xy)f(x)g(y)dxdy$$
$$< k_1(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (21)$$

$$\left[\int_0^\infty y^{p\sigma-1}\left(\int_0^{\frac{1}{y}} h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k_1(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}, \quad (22)$$

where the constant factor $k_1(\sigma)$ is the best possible.

(ii) If $0 < p < 1$, then we have the equivalent reverses of (21) and (22). Assuming that there exists a constant $\delta_0 > 0$, such that for any $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma]$, $k_1(\widetilde{\sigma}) \in \mathbf{R}_+$, then the constant factor in the reverses of (21) and (22) is the best possible.

Assuming that $h(xy) = 0(0 < x \le \frac{1}{y})$, we find $h(t) = 0(0 < t \le 1)$ and

$$k(\sigma) = k_2(\sigma) := \int_1^\infty h(t)t^{\sigma-1}dt.$$

In view of Theorems 1 and 2, we have

Corollary 2. *Suppose that* $p > 0(p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, h(t) \ge 0, k_2(\sigma) \in \mathbf{R}_+$, $f(x), g(y) \ge 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty, 0 < \int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities:

$$\int_0^\infty \int_{\frac{1}{y}}^\infty h(xy)f(x)g(y)dxdy$$
$$< k_2(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (23)$$

$$\left[\int_0^\infty y^{p\sigma-1}\left(\int_{\frac{1}{y}}^\infty h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k_2(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}, \quad (24)$$

where the constant factor $k_2(\sigma)$ is the best possible.

(ii) If $0 < p < 1$, then we have the equivalent reverses of (23) and (24). Assuming that there exists a constant $\delta_0 > 0$, such that for any $\widetilde{\sigma} \in (\sigma - \delta_0, \sigma]$, $k_2(\widetilde{\sigma}) \in \mathbf{R}_+$, then the constant factor in the reverses of (23) and (24) is the best possible.

Remark 2. Setting $B_{y,0} := (0, \infty), B_{y,1} := (0, \frac{1}{y}), B_{y,2} := (\frac{1}{y}, \infty)(y > 0)$, and $k_0(\sigma) = k(\sigma)$, then we can reform (17), (21), and (23) as follows:

$$\int_0^\infty \int_{B_{y,j}} h(xy)f(x)g(y)dxdy$$
$$< k_j(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}, \qquad (25)$$

where the constant factors $k_j(\sigma)(j = 0, 1, 2)$ are the best possible. Also we can reform (18), (22), and (24) as follows:

$$\left[\int_0^\infty y^{p\sigma-1}\left(\int_{B_{y,j}} h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k_j(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}, \qquad (26)$$

where the constant factors $k_j(\sigma)(j = 0, 1, 2)$ are the best possible.

2.3 Two Equivalent Inequalities with the Homogeneous Kernel

Replacing x to $\frac{1}{x}$ in the inequalities of Theorems 1 and 2, setting $h(t)$ as $k_\lambda(1, t)$, since

$$h\left(\frac{y}{x}\right) = k_\lambda\left(1, \frac{y}{x}\right) = x^\lambda k_\lambda(x, y),$$

also replacing $f(\frac{1}{x})$ to $x^{2-\lambda} f(x)$, by simplification, we have

Theorem 3. *Suppose that* $p > 0 (p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda,\ k_\lambda(x, y)$ *is a nonnegative homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2$,

$$k_\lambda(\mu) := \int_0^\infty k_\lambda(t, 1)t^{\mu-1}dt \in \mathbf{R}_+,$$

$f(x), g(y) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\mu)-1} f^p(x)dx < \infty, 0 < \int_0^\infty g^{q(1-\sigma)-1}(y)dy < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k_\lambda(\mu)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (27)$$

$$\left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty k_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} < k_\lambda(\mu)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}, \quad (28)$$

where the constant factor $k_\lambda(\mu)$ is the best possible.

(ii) If $0 < p < 1$, then we have the equivalent reverses of (27) and (28). Assuming that there exists a constant $\delta_0 > 0$, such that for any $\widetilde{\mu} \in [\mu, \mu + \delta_0)$, $k_\lambda(\widetilde{\mu}) \in \mathbf{R}_+$, then the constant factor in the reverses of (27) and (28) is the best possible.

Proof. For $p > 1$, it is evident that (27) and (28) are value and equivalent. If the constant factor $k_\lambda(\mu)$ in (27) is not the best possible, then replacing x to $\frac{1}{x}$ and $f(\frac{1}{x})$ to $x^{2-\lambda}f(x)$ in (27), setting $k_\lambda(1,t) = h(t)$, we would reach a contradiction that the constant factor $k(\sigma)(= k_\lambda(\mu))$ in (17) is not the best possible. By the same way, we can prove that the other parts of this theorem are valid. □

Assuming that $k_\lambda(x,y) = 0(x \geq y > 0)$, we find $k_\lambda(t,1) = 0(t \geq 1)$ and

$$k_\lambda(\mu) = k_{\lambda,1}(\mu) := \int_0^1 k_\lambda(t,1)t^{\mu-1}dt.$$

In view of Theorem 3, we have

Corollary 3. *Suppose that* $p > 0(p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda$, $k_\lambda(x,y)$ *is a nonnegative homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2, k_{\lambda,1}(\mu) \in \mathbf{R}_+, f(x), g(y) \geq 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\mu)-1} f^p(x)dx < \infty, 0 < \int_0^\infty g^{q(1-\sigma)-1}(y)dy < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities:

$$\int_0^\infty \int_0^y k_\lambda(x,y)f(x)g(y)dxdy$$
$$< k_{\lambda,1}(\mu)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (29)$$

$$\left[\int_0^\infty y^{p\sigma-1}\left(\int_0^y k_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k_{\lambda,1}(\mu)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}, \quad (30)$$

where the constant factor $k_{\lambda,1}(\mu)$ is the best possible.

(ii) If $0 < p < 1$, then we have the equivalent reverses of (29) and (30). Assuming that there exists a constant $\delta_0 > 0$, such that for any $\widetilde{\mu} \in [\mu, \mu + \delta_0)$, $k_{\lambda,1}(\widetilde{\mu}) \in \mathbf{R}_+$, then the constant factor in the reverses of (29) and (30) is the best possible.

Assuming that $k_\lambda(x,y) = 0(0 < x \le y)$, we find $k_\lambda(t,1) = 0(0 < t \le 1)$ and

$$k_\lambda(\mu) = k_{\lambda,2}(\mu) := \int_1^\infty k_\lambda(t,1)t^{\mu-1}dt.$$

In view of Theorem 3, we have

Corollary 4. *Suppose that* $p > 0(p \ne 1), \frac{1}{p} + \frac{1}{q} = 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda$, $k_\lambda(x,y)$ *is a nonnegative homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2, k_{\lambda,2}(\mu) \in \mathbf{R}_+, f(x), g(y) \ge 0$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\mu)-1}f^p(x)dx < \infty, 0 < \int_0^\infty g^{q(1-\sigma)-1}(y)dy < \infty.$$

(i) If $p > 1$, then we have the following equivalent inequalities:

$$\int_0^\infty \int_y^\infty k_\lambda(x,y)f(x)g(y)dxdy$$
$$< k_{\lambda,2}(\mu)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \quad (31)$$

$$\left[\int_0^\infty y^{p\sigma-1}\left(\int_y^\infty k_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k_{\lambda,2}(\mu)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}, \quad (32)$$

where the constant factor $k_{\lambda,2}(\mu)$ is the best possible.

(ii) If $0 < p < 1$, then we have the equivalent reverses of (31) and (32). Assuming that there exists a constant $\delta_0 > 0$, such that for any $\widetilde{\mu} \in [\mu, \mu + \delta_0)$, $k_{\lambda,2}(\widetilde{\mu}) \in \mathbf{R}_+$, then the constant factor in the reverses of (31) and (32) is the best possible.

Remark 3. Setting $A_{y,0} := (0, \infty), A_{y,1} := (0, y), A_{y,2} := (y, \infty)(y > 0)$, and $k_{\lambda,0}(\mu) = k_\lambda(\mu)$, then we can reform (27), (29), and (31) as follows:

$$\int_0^\infty \int_{A_{y,i}} k_\lambda(x, y) f(x) g(y) dx dy$$
$$< k_{\lambda,i}(\mu) \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \quad (33)$$

where the constant factors $k_{\lambda,i}(\mu)(i = 0, 1, 2)$ are the best possible. We can also reform (28), (30), and (32) as follows:

$$\left[\int_0^\infty y^{p\sigma-1} \left(\int_{A_{y,i}} k_\lambda(x, y) f(x) dx \right)^p dy \right]^{\frac{1}{p}}$$
$$< k_{\lambda,i}(\mu) \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}}, \quad (34)$$

where the constant factors $k_{\lambda,i}(\mu)(i = 0, 1, 2)$ are the best possible.

2.4 Operator Expressions and Some Examples on the Norms

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda$. We set the following functions:

$$\varphi(x) := x^{p(1-\sigma)-1}, \psi(y) := y^{q(1-\sigma)-1}, \phi(x) := x^{p(1-\mu)-1} (x, y \in \mathbf{R}_+),$$

wherefrom $\psi^{1-p}(y) = y^{p\sigma-1}$. Define the following real normed linear space:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{ f; ||f||_{p,\varphi} := \left(\int_0^\infty \varphi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \left(\int_0^\infty \phi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\psi}(\mathbf{R}_+) = \left\{ g; ||g||_{q,\psi} := \left(\int_0^\infty \psi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}_+) = \left\{ h; ||h||_{p,\psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}.$$

Note. For $0 < p < 1$, we still use the above formal symbols in the following.

(a) In view of Remark 2, for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_j(y) := \int_{B_{y,j}} h(xy)f(x)dx (y \in \mathbf{R}_+; j = 0, 1, 2),$$

by (26), we have $H_j \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$ and

$$||H_j||_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y)H_j^p(y)dy \right)^{\frac{1}{p}} \le k_j(\sigma)||f||_{p,\varphi} < \infty. \quad (35)$$

Definition 3. For any $j = 0, 1, 2$, define a general Yang-Hilbert-type integral operators with the nonhomogeneous kernel $T_j : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows:

For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique $T_jf = H_j \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+,\ T_jf(y) = H_j(y)$.

In view of (35), it follows that

$$||T_jf||_{p,\psi^{1-p}} = ||H_j||_{p,\psi^{1-p}} \le k_j(\sigma)||f||_{p,\varphi},$$

and then the operator T_j is bounded satisfying

$$||T_j|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_jf||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \le k_j(\sigma).$$

Since the constant factor $k_j(\sigma)$ in (35) is the best possible, we have

$$||T_j|| = k_j(\sigma). \quad (36)$$

If we define the formal inner product of T_jf and $g(\in L_{q,\psi}(\mathbf{R}_+))$ as

$$(T_jf, g) := \int_0^\infty \left(\int_{B_{y,j}} h(xy)f(x)dx \right) g(y)dy$$
$$= \int_0^\infty \int_{B_{y,j}} h(xy)f(x)g(y)dxdy,$$

then we can rewrite (25) and (26) as follows:

For any $j = 0, 1, 2$, we have the following equivalent inequalities:

$$(T_j f, g) < ||T_j|| \cdot ||f||_{p,\varphi} ||g||_{q,\psi}, \tag{37}$$

$$||T_j f||_{p,\psi^{1-p}} < ||T_j|| \cdot ||f||_{p,\varphi}, \tag{38}$$

where the constant factor $||T_j||$ is the best possible.

(b) In view of Remark 3, for $f \in L_{p,\phi}(\mathbf{R}_+)$, setting

$$K_i(y) := \int_{A_{y,i}} k_\lambda(x,y) f(x) dx (y \in \mathbf{R}_+; i = 0, 1, 2),$$

by (34), we have

$$||K_i||_{p,\psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y) K_i^p(y) dy \right)^{\frac{1}{p}} \le k_{\lambda,i}(\mu) ||f||_{p,\phi} < \infty. \tag{39}$$

Definition 4. For any $i = 0, 1, 2$, define a general Yang-Hilbert-type integral operators with the homogeneous kernel $\widetilde{T}_i : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows,

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unique $\widetilde{T}_i f = K_i \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+, \widetilde{T}_i f(y) = K_i(y)$.

In view of (39), it follows that

$$||\widetilde{T}_i f||_{p,\psi^{1-p}} = ||K_i||_{p,\psi^{1-p}} \le k_{\lambda,i}(\mu) ||f||_{p,\phi},$$

and then the operator $\widetilde{T}_i$ is bounded satisfying

$$||\widetilde{T}_i|| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R}_+)} \frac{||\widetilde{T}_i f||_{p,\psi^{1-p}}}{||f||_{p,\phi}} \le k_{\lambda,i}(\mu).$$

Since the constant factor $k_{\lambda,i}(\mu)$ in (39) is the best possible, we have

$$||\widetilde{T}_i|| = k_{\lambda,i}(\mu). \tag{40}$$

If we define the formal inner product of $\widetilde{T}_i f$ and $g (\in L_{q,\psi}(\mathbf{R}_+))$ as follows:

$$(\widetilde{T}_i f, g) := \int_0^\infty \left(\int_{A_{y,i}} k_\lambda(x,y) f(x) dx \right) g(y) dy$$
$$= \int_0^\infty \int_{A_{y,i}} k_\lambda(x,y) f(x) g(y) dx dy,$$

then we can rewrite (33) and (34) as follows.

For any $i = 0, 1, 2$, we have the following equivalent inequalities:

$$(\widetilde{T}_i f, g) < ||\widetilde{T}_i|| \cdot ||f||_{p,\phi} ||g||_{q,\psi}, \tag{41}$$

$$||\widetilde{T}_i f||_{p,\psi^{1-p}} < ||\widetilde{T}_i|| \cdot ||f||_{p,\phi}, \tag{42}$$

where the constant factor $||\widetilde{T}_i||$ is the best possible.

Example 1. (a) We set

$$h(t) = k_\lambda(1,t) = \frac{1}{(1+t)^\lambda} (\mu, \sigma > 0, \mu + \sigma = \lambda).$$

Then we have

$$h(xy) = \frac{1}{(1+xy)^\lambda}, k_\lambda(x,y) = \frac{1}{(x+y)^\lambda}$$

and obtain

$$k(\sigma) = k_\lambda(\mu) = \int_0^\infty \frac{t^{\sigma-1}}{(1+t)^\lambda} dt = B(\mu, \sigma) \in \mathbf{R}_+.$$

In view of (36) and (40), we have

$$||T_0|| = ||\widetilde{T}_0|| = B(\mu, \sigma).$$

(b) We set

$$h(t) = k_\lambda(1,t) = \frac{\ln t}{t^\lambda - 1} (\mu, \sigma > 0, \mu + \sigma = \lambda).$$

Then we have

$$h(xy) = \frac{\ln(xy)}{(xy)^\lambda - 1}, k_\lambda(x,y) = \frac{\ln(y/x)}{y^\lambda - x^\lambda}$$

and obtain

$$\begin{aligned} k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{(\ln t) t^{\sigma-1}}{t^\lambda - 1} dt \\ &= \frac{1}{\lambda^2} \int_0^\infty \frac{(\ln u) u^{(\sigma/\lambda)-1}}{u-1} du \\ &= \left[\frac{\pi}{\lambda \sin \pi(\sigma/\lambda)} \right]^2 \in \mathbf{R}_+. \end{aligned}$$

In view of (36) and (40), we have

$$||T_0|| = ||\widetilde{T}_0|| = \left[\frac{\pi}{\lambda \sin \pi(\sigma/\lambda)}\right]^2.$$

(c) We set

$$h(t) = k_\lambda(1, t) = \frac{|\ln t|^\beta}{(\max\{1, t\})^\lambda}$$

$(\beta > -1, \mu, \sigma > 0, \mu + \sigma = \lambda)$. Then we have

$$h(xy) = \frac{|\ln(xy)|^\beta}{(\max\{1, xy\})^\lambda}, k_\lambda(x, y) = \frac{|\ln(y/x)|^\beta}{(\max\{x, y\})^\lambda}$$

and obtain

$$\begin{aligned}
k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{|\ln t|^\beta t^{\sigma-1}}{(\max\{1, t\})^\lambda} dt \\
&= \int_0^1 (-\ln t)^\beta t^{\sigma-1} dt + \int_1^\infty \frac{(\ln t)^\beta t^{\sigma-1}}{t^\lambda} dt \\
&= \int_0^1 (-\ln t)^\beta (t^{\sigma-1} + t^{\mu-1}) dt = \left(\frac{1}{\sigma^{\beta+1}} \frac{1}{\mu^{\beta+1}}\right) \int_0^\infty v^\beta e^{-v} dv \\
&= \left(\frac{1}{\sigma^{\beta+1}} + \frac{1}{\mu^{\beta+1}}\right) \Gamma(\beta + 1) \in \mathbf{R}_+.
\end{aligned}$$

In view of (36) and (40), we have

$$||T_0|| = ||\widetilde{T}_0|| = \left(\frac{1}{\sigma^{\beta+1}} + \frac{1}{\mu^{\beta+1}}\right) \Gamma(\beta + 1).$$

We still can find that

$$\begin{aligned}
||T_1|| = ||\widetilde{T}_2|| &= \frac{1}{\sigma^{\beta+1}} \Gamma(\beta + 1), \\
||T_2|| = ||\widetilde{T}_1|| &= \frac{1}{\mu^{\beta+1}} \Gamma(\beta + 1).
\end{aligned}$$

(d) We set

$$h(t) = k_\lambda(1, t) = \frac{|\ln t|^\beta}{(\min\{1, t\})^\lambda}$$

$(\beta > -1, \mu, \sigma < 0, \mu + \sigma = \lambda)$. Then we have

$$h(xy) = \frac{|\ln(xy)|^{\beta}}{(\min\{1, xy\})^{\lambda}}, k_{\lambda}(x, y) = \frac{|\ln(y/x)|^{\beta}}{(\min\{x, y\})^{\lambda}}$$

and obtain

$$\begin{aligned} k(\sigma) = k_{\lambda}(\mu) &= \int_0^{\infty} \frac{|\ln t|^{\beta} t^{\sigma-1}}{(\min\{1, t\})^{\lambda}} dt \\ &= \int_0^1 \frac{(-\ln t)^{\beta} t^{\sigma-1}}{t^{\lambda}} dt + \int_1^{\infty} (\ln t)^{\beta} t^{\sigma-1} dt \\ &= \int_0^1 (-\ln t)^{\beta} (t^{-\mu-1} + t^{-\sigma-1}) dt \\ &= \left[\frac{1}{(-\mu)^{\beta+1}} \frac{1}{(-\sigma)^{\beta+1}} \right] \int_0^{\infty} v^{\beta} e^{-v} dv \\ &= \left[\frac{1}{(-\mu)^{\beta+1}} + \frac{1}{(-\sigma)^{\beta+1}} \right] \Gamma(\beta + 1) \in \mathbf{R}_{+}. \end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \left[\frac{1}{(-\mu)^{\beta+1}} + \frac{1}{(-\sigma)^{\beta+1}} \right] \Gamma(\beta + 1).$$

We still can find that

$$||T_1|| = ||\widetilde{T}_2|| = \frac{1}{(-\mu)^{\beta+1}} \Gamma(\beta + 1),$$

$$||T_2|| = ||\widetilde{T}_1|| = \frac{1}{(-\sigma)^{\beta+1}} \Gamma(\beta + 1).$$

(e) We set

$$h(t) = k_{\lambda}(1, t) = \frac{|\ln t|^{\beta}}{1 + t^{\lambda}}$$

$(\beta > -1, \mu, \sigma > 0, \mu + \sigma = \lambda)$. Then we have

$$h(xy) = \frac{|\ln(xy)|^{\beta}}{1 + (xy)^{\lambda}}, k_{\lambda}(x, y) = \frac{|\ln(y/x)|^{\beta}}{x^{\lambda} + y^{\lambda}}$$

and obtain by Lebesgue term by term theorem that

$$
\begin{aligned}
k(\sigma) &= k_\lambda(\mu) = \int_0^\infty \frac{|\ln t|^\beta t^{\sigma-1}}{1+t^\lambda}dt \\
&= \int_0^1 \frac{(-\ln t)^\beta t^{\sigma-1}}{t^\lambda+1}dt + \int_1^\infty \frac{(\ln t)^\beta t^{\sigma-1}}{t^\lambda+1}dt \\
&= \int_0^1 \frac{(-\ln t)^\beta (t^{\sigma-1}+t^{\mu-1})}{t^\lambda+1}dt \\
&= \int_0^1 (-\ln t)^\beta \sum_{k=0}^\infty (-1)^k t^{k\lambda}(t^{\sigma-1}+t^{\mu-1})dt \\
&= \int_0^1 (-\ln t)^\beta \sum_{k=0}^\infty [t^{2k\lambda} - t^{(2k+1)\lambda}](t^{\sigma-1}+t^{\mu-1})dt \\
&= \int_0^1 (-\ln t)^\beta \sum_{k=0}^\infty t^{2k\lambda}(t^{\sigma-1}+t^{\mu-1})(1-t^\lambda)dt \\
&= \sum_{k=0}^\infty \int_0^1 (-\ln t)^\beta t^{2k\lambda}(t^{\sigma-1}+t^{\mu-1})(1-t^\lambda)dt \\
&= \sum_{k=0}^\infty (-1)^k \int_0^1 (-\ln t)^\beta (t^{k\lambda+\sigma-1}+t^{k\lambda+\mu-1})dt \\
&= \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(k\lambda+\sigma)^{\beta+1}} + \frac{1}{(k\lambda+\mu)^{\beta+1}}\right]\Gamma(\beta+1) \in \mathbf{R}_+.
\end{aligned}
$$

In view of (36) and (37), we have

$$
||T_0|| = ||\widetilde{T}_0|| = \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(k\lambda+\sigma)^{\beta+1}} + \frac{1}{(k\lambda+\mu)^{\beta+1}}\right]\Gamma(\beta+1).
$$

We still can find that

$$
||T_1|| = ||\widetilde{T}_2|| = \Gamma(\beta+1)\sum_{k=0}^\infty (-1)^k \frac{1}{(k\lambda+\sigma)^{\beta+1}},
$$

$$
||T_2|| = ||\widetilde{T}_1|| = \Gamma(\beta+1)\sum_{k=0}^\infty (-1)^k \frac{1}{(k\lambda+\mu)^{\beta+1}}.
$$

(f) We set

$$
h(t) = k_\lambda(t,1) = \frac{(\min\{1,t\})^\eta}{|1-t|^{\lambda+\eta}}
$$

$(\eta > -\min\{\mu, \sigma\}, \mu + \sigma = \lambda < 1 - \eta)$. Then we have

$$h(xy) = \frac{(\min\{1, xy\})^\eta}{|1 - xy|^{\lambda+\eta}}, k_\lambda(x, y) = \frac{(\min\{x, y\})^\eta}{|x - y|^{\lambda+\eta}}$$

and obtain

$$\begin{aligned} k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{(\min\{1, t\})^\eta}{|1 - t|^{\lambda+\eta}} t^{\sigma-1} dt \\ &= \int_0^1 \frac{t^{\eta+\sigma-1}}{(1 - t)^{\lambda+\eta}} dt + \int_1^\infty \frac{t^{\sigma-1}}{(t - 1)^{\lambda+\eta}} dt \\ &= \int_0^1 \frac{t^{\eta+\sigma-1} + t^{\eta+\mu-1}}{(1 - t)^{\lambda+\eta}} dt \\ &= B(1 - \lambda - \eta, \eta + \sigma) + B(1 - \lambda - \eta, \eta + \mu) \in \mathbf{R}_+. \end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = B(1 - \lambda - \eta, \eta + \sigma) + B(1 - \lambda - \eta, \eta + \mu).$$

We still can find that

$$\begin{aligned} ||T_1|| &= ||\widetilde{T}_2|| = B(1 - \lambda - \eta, \eta + \sigma), \\ ||T_2|| &= ||\widetilde{T}_1|| = B(1 - \lambda - \eta, \eta + \mu). \end{aligned}$$

(g) We set

$$h(t) = k_\lambda(1, t) = \frac{(\min\{t, 1\})^\eta |\ln t|^\beta}{(\max\{t, 1\})^{\lambda+\eta}}$$

$(\eta > -\min\{\mu, \sigma\}, \mu + \sigma = \lambda, \beta > -1)$. Then we have

$$\begin{aligned} h(xy) &= \frac{(\min\{1, xy\})^\eta |\ln xy|^\beta}{(\max\{1, xy\})^{\lambda+\eta}}, \\ k_\lambda(x, y) &= \frac{(\min\{x, y\})^\eta |\ln y/x|^\beta}{(\max\{x, y\})^{\lambda+\eta}} \end{aligned}$$

and obtain

$$\begin{aligned} k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{(\min\{1, t\})^\eta |\ln t|^\beta}{(\max\{1, t\})^{\lambda+\eta}} t^{\sigma-1} dt \\ &= \int_0^1 (-\ln t)^\beta t^{\sigma+\eta-1} dt + \int_1^\infty (\ln t)^\beta \frac{t^{\sigma-1}}{t^{\lambda+\eta}} dt \end{aligned}$$

$$= \int_0^1 (-\ln t)^\beta (t^{\sigma+\eta-1} + t^{\mu+\eta-1}) dt$$

$$= \left[\frac{1}{(\sigma+\eta)^{\beta+1}} + \frac{1}{(\mu+\eta)^{\beta+1}}\right] \Gamma(\beta+1) \in \mathbf{R}_+.$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \left[\frac{1}{(\sigma+\eta)^{\beta+1}} + \frac{1}{(\mu+\eta)^{\beta+1}}\right] \Gamma(\beta+1).$$

We still can find that

$$||T_1|| = ||\widetilde{T}_2|| = \frac{1}{(\sigma+\eta)^{\beta+1}} \Gamma(\beta+1),$$

$$||T_2|| = ||\widetilde{T}_1|| = \frac{1}{(\mu+\eta)^{\beta+1}} \Gamma(\beta+1).$$

Example 2. (a) We set

$$h(t) = k_\lambda(1,t) = \frac{1}{\prod_{k=1}^s (\max\{a_k, t\})^{\lambda/s}}$$

$(s \in \mathbf{N}, 0 = a_0 < a_1 \leq \cdots \leq a_s < \infty,\ \mu, \sigma > 0,\ \mu + \sigma = \lambda)$. Then we have

$$h(xy) = \frac{1}{\prod_{k=1}^s (\max\{a_k, xy\})^{\lambda/s}},$$

$$k_\lambda(x,y) = \frac{1}{\prod_{k=1}^s (\max\{a_k x, y\})^{\lambda/s}}.$$

If for any $i \in \{1, \cdots, s-1\}$, $\sigma - \frac{i\lambda}{s} \neq 0$, then we obtain

$$k(\sigma) = k_\lambda(\mu) = \int_0^\infty \frac{1}{\prod_{k=1}^s (\max\{a_k, t\})^{\lambda/s}} t^{\sigma-1} dt$$

$$= \sum_{i=0}^{s-1} \int_{a_i}^{a_{i+1}} \frac{t^{\sigma-1} dt}{\prod_{k=1}^s (\max\{a_k, t\})^{\lambda/s}} + \int_{a_s}^\infty \frac{t^{\sigma-1} dt}{\prod_{k=1}^s (\max\{a_k, t\})^{\lambda/s}}$$

$$= \sum_{i=0}^{s-1} \int_{a_i}^{a_{i+1}} \frac{t^{\sigma-\frac{i\lambda}{s}-1} dt}{\prod_{k=i+1}^s a_k^{\lambda/s}} + \int_{a_s}^\infty t^{-\mu-1} dt$$

$$= \frac{a_1^\sigma}{\sigma \prod_{k=1}^s a_k^{\lambda/s}} + \sum_{i=1}^{s-1} \frac{a_{i+1}^{\sigma-\frac{i\lambda}{s}} - a_i^{\sigma-\frac{i\lambda}{s}}}{(\sigma - \frac{i\lambda}{s}) \prod_{k=i+1}^s a_k^{\lambda/s}} + \frac{a_s^{-\mu}}{\mu} \in \mathbf{R}_+.$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{a_1^{\sigma}}{\sigma \prod_{k=1}^{s} a_k^{\lambda/s}} + \sum_{i=1}^{s-1} \frac{a_{i+1}^{\sigma-\frac{i\lambda}{s}} - a_i^{\sigma-\frac{i\lambda}{s}}}{(\sigma - \frac{i\lambda}{s}) \prod_{k=i+1}^{s} a_k^{\lambda/s}} + \frac{a_s^{-\mu}}{\mu}.$$

If there exists a $i_0 \in \{1, \cdots, s-1\}$, such that $\sigma - \frac{i_0\lambda}{s} = 0$, then it means that the above corresponding term

$$\frac{a_{i_0+1}^{\sigma-\frac{i_0\lambda}{s}} - a_{i_0}^{\sigma-\frac{i\lambda}{s}}}{(\sigma - \frac{i_0\lambda}{s}) \prod_{k=i_0+1}^{s} a_k^{\lambda/s}}$$

equals to $\left(\prod_{k=i_0+1}^{s} a_k^{\lambda/s}\right)^{-1} \ln(\frac{a_{i_0+1}}{a_{i_0}})$.

In particular, (i) if $s = 1$ (or $a_s = \cdots = a_1$), then we find

$$h(t) = k_\lambda(1, t) = \frac{1}{(\max\{a_1, t\})^\lambda},$$
$$h(xy) = \frac{1}{(\max\{a_1, xy\})^\lambda}, k_\lambda(x, y) = \frac{1}{(\max\{a_1 x, y\})^\lambda},$$

and

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{a_1^{\mu}} \frac{\lambda}{\sigma\mu};$$

(ii) if $s = 2$, then it follows that

$$h(t) = k_\lambda(1, t) = \frac{1}{(\max\{a_1, t\} \max\{a_2, t\})^{\lambda/2}}$$

and

$$h(xy) = \frac{1}{(\max\{a_1, xy\} \max\{a_2, xy\})^{\lambda/2}},$$
$$k_\lambda(x, y) = \frac{1}{(\max\{a_1 x, y\} \max\{a_2 x, y\})^{\lambda/2}}.$$

We obtain for $\sigma \neq \frac{\lambda}{2}$ that

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{a_2^{\lambda/2}} \left(\frac{a_1^{\sigma-\frac{\lambda}{2}}}{\sigma} + \frac{a_2^{\sigma-\frac{\lambda}{2}} - a_1^{\sigma-\frac{\lambda}{2}}}{\sigma - \frac{\lambda}{2}} + \frac{a_2^{\sigma-\frac{\lambda}{2}}}{\mu} \right);$$

for $\sigma = \frac{\lambda}{2}$,

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{a_2^{\lambda/2}}\left[\frac{\lambda}{\sigma\mu} + \ln(\frac{a_2}{a_1})\right].$$

(b) We set

$$h(t) = k_\lambda(1,t) = \frac{1}{\prod_{k=1}^{s}(\min\{a_k, t\})^{\lambda/s}}$$

$(s \in \mathbf{N}, 0 = a_0 < a_1 \le \cdots \le a_s < \infty,\ \mu, \sigma < 0,\ \mu + \sigma = \lambda)$. Then we have

$$h(xy) = \frac{1}{\prod_{k=1}^{s}(\min\{a_k, xy\})^{\lambda/s}},$$

$$k_\lambda(x,y) = \frac{1}{\prod_{k=1}^{s}(\min\{a_k x, y\})^{\lambda/s}}.$$

If for any $i \in \{1, \cdots, s-1\}$, $-\mu + \frac{i\lambda}{s} \neq 0$, then we obtain

$$\begin{aligned} k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{1}{\prod_{k=1}^{s}(\min\{a_k, t\})^{\lambda/s}} t^{\sigma-1} dt \\ &= \sum_{i=0}^{s-1} \int_{a_i}^{a_{i+1}} \frac{t^{\sigma-1}}{\prod_{k=1}^{s}(\min\{a_k, t\})^{\lambda/s}} dt \\ &\quad + \int_{a_s}^{\infty} \frac{t^{\sigma-1}}{\prod_{k=1}^{s}(\min\{a_k, t\})^{\lambda/s}} dt \\ &= \sum_{i=0}^{s-1} \int_{a_i}^{a_{i+1}} \frac{t^{-\mu+\frac{i\lambda}{s}-1}}{\prod_{k=1}^{i} a_k^{\lambda/s}} dt + \int_{a_s}^{\infty} \frac{t^{\sigma-1}}{\prod_{k=1}^{s} a_k^{\lambda/s}} dt \\ &= \frac{a_1^{-\mu}}{-\mu} + \sum_{i=1}^{s-1} \frac{a_{i+1}^{-\mu+\frac{i\lambda}{s}} - a_i^{-\mu+\frac{i\lambda}{s}}}{(-\mu + \frac{i\lambda}{s}) \prod_{k=1}^{i} a_k^{\lambda/s}} + \frac{a_s^{\sigma}}{-\sigma \prod_{k=1}^{s} a_k^{\lambda/s}} \in \mathbf{R}_+. \end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{a_1^{-\mu}}{-\mu} + \sum_{i=1}^{s-1} \frac{a_{i+1}^{-\mu+\frac{i\lambda}{s}} - a_i^{-\mu+\frac{i\lambda}{s}}}{(-\mu + \frac{i\lambda}{s}) \prod_{k=1}^{i} a_k^{\lambda/s}} + \frac{a_s^{\sigma}}{-\sigma \prod_{k=1}^{s} a_k^{\lambda/s}}.$$

If there exists a $i_0 \in \{1, \cdots, s-1\}$, such that $-\mu + \frac{i_0\lambda}{s} = 0$, then it means that the above corresponding term

$$\frac{a_{i_0+1}^{-\mu+\frac{i_0\lambda}{s}} - a_{i_0}^{-\mu+\frac{i_0\lambda}{s}}}{(-\mu + \frac{i_0\lambda}{s}) \prod_{k=1}^{i_0} a_k}$$

equals to $(\prod_{k=1}^{i_0} a_k)^{-1} \ln(\frac{a_{i_0+1}}{a_{i_0}})$.

In particular, (i) for $s = 1$ (or $a_s = \cdots = a_1$), we find

$$h(t) = k_\lambda(1,t) = \frac{1}{(\min\{a_1, t\})^\lambda},$$
$$h(xy) = \frac{1}{(\min\{a_1, xy\})^\lambda}, k_\lambda(x,y) = \frac{1}{(\min\{a_1 x, y\})^\lambda},$$

and

$$||T_0|| = ||\widetilde{T}_0|| = a_1^\mu \frac{(-\lambda)}{\sigma\mu};$$

(ii) for $s = 2$, it follows that

$$h(t) = k_\lambda(1,t) = \frac{1}{(\min\{a_1, t\} \min\{a_2, t\})^{\lambda/2}}$$

and

$$h(xy) = \frac{1}{(\min\{a_1, xy\} \min\{a_2, xy\})^{\lambda/2}},$$
$$k_\lambda(x,y) = \frac{1}{(\min\{a_1 x, y\} \min\{a_2 x, y\})^{\lambda/2}}.$$

We obtain for $\mu \neq \frac{\lambda}{2}$ that

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{a_1^{\lambda/2}} \left(\frac{a_1^{-\mu+\frac{\lambda}{2}}}{-\mu} + \frac{a_2^{-\mu+\frac{\lambda}{2}} - a_1^{-\mu+\frac{\lambda}{2}}}{-\mu + \frac{\lambda}{2}} + \frac{a_2^{-\mu+\frac{\lambda}{2}}}{-\sigma} \right);$$

for $\mu = \frac{\lambda}{2}$,

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{a_1^{\lambda/2}} \left[\frac{(-\lambda)}{\sigma\mu} + \ln(\frac{a_2}{a_1}) \right].$$

Example 3. (a) We set

$$h(t) = k_0(1,t) = \ln(1 + \frac{\rho}{t^\eta})(\rho > 0, 0 < \sigma < \eta).$$

Then we have

$$h(xy) = \ln[1 + \frac{\rho}{(xy)^\eta}], k_0(x,y) = \ln[1 + \rho(\frac{x}{y})^\eta]$$

and obtain

$$\begin{aligned}
k(\sigma) = k_0(\mu) &= \int_0^\infty t^{\sigma-1}\ln(1+\frac{\rho}{t^\eta})dt \\
&= \frac{1}{\sigma}\int_0^\infty \ln(1+\frac{\rho}{t^\eta})dt^\sigma \\
&= \frac{1}{\sigma}\left[t^\sigma \ln(1+\frac{\rho}{t^\eta})|_0^\infty - \int_0^\infty t^\sigma d\ln(1+\frac{\rho}{t^\eta})\right] \\
&= \frac{\eta}{\sigma}\int_0^\infty \frac{t^{\sigma-1}}{(t^\eta/\rho)+1}dt = \frac{\rho^{\sigma/\eta}}{\sigma}\int_0^\infty \frac{u^{(\sigma/\eta)-1}}{u+1}du \\
&= \frac{\rho^{\sigma/\eta}\pi}{\sigma\sin\pi(\sigma/\eta)} \in \mathbf{R}_+.
\end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{\rho^{\sigma/\eta}\pi}{\sigma\sin\pi(\sigma/\eta)}.$$

(b) We set

$$h(t) = k_0(1,t) = \arctan(\frac{\rho}{t^\eta})(\rho > 0, 0 < \sigma < \eta).$$

Then we have

$$h(xy) = \arctan(\frac{\rho}{(xy)^\eta}), k_0(x,y) = \arctan\rho(\frac{x}{y})^\eta$$

and obtain

$$\begin{aligned}
k(\sigma) = k_0(\mu) &= \int_0^\infty t^{\sigma-1}\arctan(\frac{\rho}{t^\eta})dt \\
&= \frac{1}{\sigma}\int_0^\infty \arctan(\frac{\rho}{t^\eta})dt^\sigma \\
&= \frac{1}{\sigma}\left[t^\sigma \arctan(\frac{\rho}{t^\eta})|_0^\infty - \int_0^\infty t^\sigma d\arctan(\frac{\rho}{t^\eta})\right] \\
&= \frac{\eta}{\sigma\rho}\int_0^\infty \frac{t^{\eta+\sigma-1}dt}{(t^{2\eta}/\rho^2)+1} = \frac{\rho^{\sigma/\eta}}{2\sigma}\int_0^\infty \frac{u^{[(\eta+\sigma)/(2\eta)]-1}}{u+1}du \\
&= \frac{\rho^{\sigma/\eta}\pi}{2\sigma\sin\pi[(\eta+\sigma)/(2\eta)]} = \frac{\rho^{\sigma/\eta}\pi}{2\sigma\cos\pi[\sigma/(2\eta)]} \in \mathbf{R}_+.
\end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{\rho^{\sigma/\eta}\pi}{2\sigma\cos\pi[\sigma/(2\eta)]}.$$

(c) We set

$$h(t) = k_0(1,t) = e^{-\rho t^\eta}(\rho, \sigma, \eta > 0).$$

Then we have

$$h(xy) = e^{-\rho(xy)^\eta}, k_0(x,y) = e^{-\rho(\frac{y}{x})^\eta}$$

and obtain

$$k(\sigma) = k_0(\mu) = \int_0^\infty t^{\sigma-1}e^{-\rho t^\eta}dt$$
$$= \frac{1}{\eta\rho^{\sigma/\eta}}\int_0^\infty e^{-u}u^{\frac{\sigma}{\eta}-1}du = \frac{1}{\eta\rho^{\sigma/\eta}}\Gamma(\frac{\sigma}{\eta}) \in \mathbf{R}_+.$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{\eta\rho^{\sigma/\eta}}\Gamma(\frac{\sigma}{\eta}).$$

Example 4. (a) We set

$$h(t) = k_0(1,t) = \csc h(\rho t^\eta) = \frac{2}{e^{\rho t^\eta} - e^{-\rho t^\eta}}(\rho > 0, 0 < \eta < \sigma).$$

We call $\csc h(\cdot)$ as hyperbolic cosecant function (cf. [27]). Then we have

$$h(xy) = \frac{2}{e^{\rho(xy)^\eta} - e^{-\rho(xy)^\eta}}, k_0(x,y) = \frac{2}{e^{\rho(\frac{y}{x})^\eta} - e^{-\rho(\frac{y}{x})^\eta}}.$$

By Lebesgue term-by-term integration theorem, we obtain

$$k(\sigma) = k_0(\mu) = \int_0^\infty \frac{2t^{\sigma-1}dt}{e^{\rho t^\eta} - e^{-\rho t^\eta}} = \int_0^\infty \frac{2t^{\sigma-1}dt}{e^{\rho t^\eta}(1-e^{-2\rho t^\eta})}$$
$$= 2\int_0^\infty t^{\sigma-1}\sum_{k=0}^\infty e^{-(2k+1)\rho t^\eta}dt = 2\sum_{k=0}^\infty \int_0^\infty t^{\sigma-1}e^{-(2k+1)\rho t^\eta}dt$$
$$= \frac{2}{\eta\rho^{\sigma/\eta}}\sum_{k=0}^\infty \frac{1}{(2k+1)^{\sigma/\eta}}\int_0^\infty e^{-u}u^{\frac{\sigma}{\eta}-1}du$$
$$= \frac{2}{\eta\rho^{\sigma/\eta}}\Gamma(\frac{\sigma}{\eta})\sum_{k=0}^\infty \frac{1}{(2k+1)^{\sigma/\eta}}$$

$$= \frac{2}{\eta\rho^{\sigma/\eta}}\Gamma(\frac{\sigma}{\eta})\left[\sum_{k=1}^{\infty}\frac{1}{k^{\sigma/\eta}} - \sum_{k=1}^{\infty}\frac{1}{(2k)^{\sigma/\eta}}\right]$$

$$= \frac{2}{\eta\rho^{\sigma/\eta}}(1 - \frac{1}{2^{\sigma/\eta}})\Gamma(\frac{\sigma}{\eta})\zeta(\frac{\sigma}{\eta}) \in \mathbf{R}_+,$$

where $\zeta(\frac{\sigma}{\eta}) = \sum_{k=1}^{\infty}\frac{1}{k^{\sigma/\eta}}$ ($\zeta(\cdot)$ is Riemann zeta function). In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{2}{\eta\rho^{\sigma/\eta}}(1 - \frac{1}{2^{\sigma/\eta}})\Gamma(\frac{\sigma}{\eta})\zeta(\frac{\sigma}{\eta}).$$

(b) We set

$$h(t) = k_0(1,t) = e^{-\rho t^\eta}\cot h(\rho t^\eta) = e^{-\rho t^\eta}\frac{e^{\rho t^\eta} + e^{-\rho t^\eta}}{e^{\rho t^\eta} - e^{-\rho t^\eta}}$$

$$= \frac{1 + e^{-2\rho t^\eta}}{e^{\rho t^\eta} - e^{-\rho t^\eta}} = \frac{e^{-\rho t^\eta} + e^{-3\rho t^\eta}}{1 - e^{-2\rho t^\eta}}(\rho > 0, 0 < \eta < \sigma).$$

We call $\cot h(\cdot)$ as hyperbolic cotangent function (cf. [27]). Then we have

$$h(xy) = \frac{1 + e^{-2\rho(xy)^\eta}}{e^{\rho(xy)^\eta} - e^{-\rho(xy)^\eta}}, k_0(x,y) = \frac{1 + e^{-2\rho(\frac{y}{x})^\eta}}{e^{\rho(\frac{y}{x})^\eta} - e^{-\rho(\frac{y}{x})^\eta}}.$$

By Lebesgue term-by-term integration theorem, we obtain

$$k(\sigma) = k_0(\mu) = \int_0^\infty \frac{(e^{-\rho t^\eta} + e^{-3\rho t^\eta})t^{\sigma-1}}{1 - e^{-2\rho t^\eta}}dt$$

$$= \int_0^\infty t^{\sigma-1}\sum_{k=0}^{\infty}(e^{-(2k+1)\rho t^\eta} + e^{-(2k+3)\rho t^\eta})dt$$

$$= \frac{1}{\eta\rho^{\sigma/\eta}}\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)^{\sigma/\eta}} + \frac{1}{(2k+3)^{\sigma/\eta}}\right]\int_0^\infty e^{-u}u^{\frac{\sigma}{\eta}-1}du$$

$$= \frac{1}{\eta\rho^{\sigma/\eta}}\Gamma(\frac{\sigma}{\eta})\left[2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\sigma/\eta}} - 1\right]$$

$$= \frac{1}{\eta\rho^{\sigma/\eta}}\Gamma(\frac{\sigma}{\eta})\left[(2 - \frac{1}{2^{(\sigma/\eta)-1}})\zeta(\frac{\sigma}{\eta}) - 1\right] \in \mathbf{R}_+.$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{\eta\rho^{\sigma/\eta}}\Gamma(\frac{\sigma}{\eta})\left[(2 - \frac{1}{2^{(\sigma/\eta)-1}})\zeta(\frac{\sigma}{\eta}) - 1\right].$$

(c) We set

$$h(t) = k_0(1, t) = \sec h(\rho t^\eta) = \frac{2}{e^{\rho t^\eta} + e^{-\rho t^\eta}} (\rho, \eta, \sigma > 0).$$

We call $\sec h(\cdot)$ as hyperbolic secant function (cf. [27]). Then we have

$$h(xy) = \frac{2}{e^{\rho (xy)^\eta} + e^{-\rho (xy)^\eta}}, k_0(x, y) = \frac{2}{e^{\rho (\frac{y}{x})^\eta} + e^{-\rho (\frac{y}{x})^\eta}}.$$

By Lebesgue term-by-term integration theorem, we obtain

$$\begin{aligned}
k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{2t^{\sigma-1}dt}{e^{\rho t^\eta} + e^{-\rho t^\eta}} = \int_0^\infty \frac{2t^{\sigma-1}dt}{e^{\rho t^\eta}(1 + e^{-2\rho t^\eta})} \\
&= 2\int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\rho t^\eta} dt \\
&= 2\int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty [e^{-(4k+1)\rho t^\eta} - e^{-(4k+3)\rho t^\eta}] dt \\
&= 2\sum_{k=0}^\infty \int_0^\infty t^{\sigma-1} [e^{-(4k+1)\rho t^\eta} - e^{-(4k+3)\rho t^\eta}] dt \\
&= 2\sum_{k=0}^\infty (-1)^k \int_0^\infty t^{\sigma-1} e^{-(2k+1)\rho t^\eta} dt \\
&= \frac{2}{\eta \rho^{\sigma/\eta}} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\sigma/\eta}} \int_0^\infty e^{-u} u^{\frac{\sigma}{\eta}-1} du \\
&= \frac{2}{\eta \rho^{\sigma/\eta}} \Gamma(\frac{\sigma}{\eta}) \xi(\frac{\sigma}{\eta}) \in \mathbf{R}_+,
\end{aligned}$$

where $\xi(\frac{\sigma}{\eta}) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\sigma/\eta}}$. In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{2}{\eta \rho^{\sigma/\eta}} \Gamma(\frac{\sigma}{\eta}) \xi(\frac{\sigma}{\eta}).$$

(d) We set

$$\begin{aligned}
h(t) = k_0(1, t) = e^{-\rho t^\eta} \tan h(\rho t^\eta) &= e^{-\rho t^\eta} \frac{e^{\rho t^\eta} - e^{-\rho t^\eta}}{e^{\rho t^\eta} + e^{-\rho t^\eta}} \\
&= \frac{1 - e^{-2\rho t^\eta}}{e^{\rho t^\eta} + e^{-\rho t^\eta}} = \frac{e^{-\rho t^\eta} - e^{-3\rho t^\eta}}{1 + e^{-2\rho t^\eta}} (\rho, \eta, \sigma > 0).
\end{aligned}$$

We call $\tan h(\cdot)$ as hyperbolic tangent function (cf. [27]). Then we have

$$h(xy) = \frac{1 - e^{-2\rho (xy)^\eta}}{e^{\rho (xy)^\eta} + e^{-\rho (xy)^\eta}}, k_0(x, y) = \frac{1 - e^{-2\rho (\frac{y}{x})^\eta}}{e^{\rho (\frac{y}{x})^\eta} + e^{-\rho (\frac{y}{x})^\eta}}.$$

By Lebesgue term-by-term integration theorem, we obtain

$$\begin{aligned}
k(\sigma) = k_0(\mu) &= \int_0^\infty \frac{(e^{-\rho t^\eta} - e^{-3\rho t^\eta})t^{\sigma-1}}{1+e^{-2\rho t^\eta}} dt \\
&= \int_0^\infty (e^{-\rho t^\eta} - e^{-3\rho t^\eta})t^{\sigma-1} \sum_{k=0}^\infty (-1)^k e^{-2k\rho t^\eta} dt \\
&= \int_0^\infty (e^{-\rho t^\eta} - e^{-3\rho t^\eta})t^{\sigma-1} \sum_{k=0}^\infty e^{-4k\rho t^\eta}(1-e^{-2\rho t^\eta}) dt \\
&= \sum_{k=0}^\infty \int_0^\infty (e^{-\rho t^\eta} - e^{-3\rho t^\eta})t^{\sigma-1} e^{-4k\rho t^\eta}(1-e^{-2\rho t^\eta}) dt \\
&= \sum_{k=0}^\infty (-1)^k \int_0^\infty (e^{-(2k+1)\rho t^\eta} - e^{-(2k+3)\rho t^\eta})t^{\sigma-1} dt \\
&= \frac{1}{\eta\rho^{\sigma/\eta}} \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(2k+1)^{\sigma/\eta}} - \frac{1}{(2k+3)^{\sigma/\eta}}\right] \int_0^\infty e^{-u} u^{\frac{\sigma}{\eta}-1} du \\
&= \frac{1}{\eta\rho^{\sigma/\eta}} \Gamma(\frac{\sigma}{\eta}) \left[2\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\sigma/\eta}} - 1\right] \\
&= \frac{1}{\eta\rho^{\sigma/\eta}} \Gamma(\frac{\sigma}{\eta})(2\xi(\frac{\sigma}{\eta}) - 1) \in \mathbf{R}_+.
\end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{1}{\eta\rho^{\sigma/\eta}} \Gamma(\frac{\sigma}{\eta})(2\xi(\frac{\sigma}{\eta}) - 1).$$

Lemma 2. *If* $\mathbf{C}$ *is the set of complex numbers and* $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, $z_k \in \mathbf{C}\backslash\{z|\mathrm{Re}z \geq 0,\ \mathrm{Im}z = 0\}$ $(k = 1, 2, \cdots, n)$ *are different points, the function* $f(z)$ *is analytic in* $\mathbf{C}_\infty$ *except for* $z_i (i = 1, 2, \cdots, n)$*, and* $z = \infty$ *is a zero point of* $f(z)$ *whose order is not less than 1, and then for* $\alpha \in \mathbf{R}$, *we have*

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{2\pi i}{1-e^{2\pi\alpha i}} \sum_{k=1}^n \mathrm{Res}[f(z)z^{\alpha-1}, z_k], \tag{43}$$

where $0 < \mathrm{Im}\ln z = \arg z < 2\pi$. *In particular, if* $z_k (k = 1, \cdots, n)$ *are all poles of order 1, setting* $\varphi_k(z) = (z - z_k)f(z)(\varphi_k(z_k) \neq 0)$, *then*

$$\int_0^\infty f(x)x^{\alpha-1}dx = \frac{\pi}{\sin\pi\alpha} \sum_{k=1}^n (-z_k)^{\alpha-1}\varphi_k(z_k). \tag{44}$$

Proof. In view of the theorem (cf. [28], P. 118), we have (43). We find

$$\begin{aligned} 1 - e^{2\pi\alpha i} &= 1 - \cos 2\pi\alpha - i\sin 2\pi\alpha \\ &= -2i\sin\pi\alpha(\cos\pi\alpha + i\sin\pi\alpha) \\ &= -2ie^{i\pi\alpha}\sin\pi\alpha. \end{aligned}$$

In particular, since $f(z)z^{\alpha-1} = \frac{1}{z-z_k}(\varphi_k(z)z^{\alpha-1})$, it is obvious that

$$\mathrm{Re}s[f(z)z^{\alpha-1}, -a_k] = z_k{}^{\alpha-1}\varphi_k(z_k) = -e^{i\pi\alpha}(-z_k)^{\alpha-1}\varphi_k(z_k).$$

Then by (43), we obtain (44).

The lemma is proved. □

Example 5. (a) We set

$$h(t) = k_\lambda(1,t) = \frac{1}{\prod_{k=1}^{s}(a_k + t^{\lambda/s})}$$

($s \in \mathbf{N}, 0 < a_1 < \cdots < a_s < \infty,\ \mu,\sigma > 0,\ \mu + \sigma = \lambda$). Then we have the kernels

$$h(xy) = \frac{1}{\prod_{k=1}^{s}[a_k + (xy)^{\lambda/s}]},$$

$$k_\lambda(x,y) = \frac{1}{\prod_{k=1}^{s}(a_k x^{\lambda/s} + y^{\lambda/s})}.$$

For $f(z) = \frac{1}{\prod_{k=1}^{s}(z+a_k)}$, $z_k = -a_k$, by (44), we can find

$$\varphi_k(z_k) = (z + a_k)\frac{1}{\prod_{i=1}^{s}(z + a_i)}|_{z=-a_k} = \prod_{j=1(j\neq k)}^{s}\frac{1}{a_j - a_k}$$

and obtain

$$\begin{aligned} k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{t^{\sigma-1}}{\prod_{k=1}^{s}(a_k + t^{\lambda/s})}dt \\ &= \frac{s}{\lambda}\int_0^\infty \frac{u^{(s\sigma/\lambda)-1}}{\prod_{k=1}^{s}(u + a_k)}du \\ &= \frac{\pi s}{\lambda\sin\pi(s\sigma/\lambda)}\sum_{k=1}^{s}a_k^{\frac{s\sigma}{\lambda}}\prod_{j=1(j\neq k)}^{s}\frac{1}{a_j - a_k} \in \mathbf{R}_+. \end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{\pi s}{\lambda \sin \pi (s\sigma/\lambda)} \sum_{k=1}^{s} a_k^{\frac{s\sigma}{\lambda}} \prod_{j=1(j\neq k)}^{s} \frac{1}{a_j - a_k}.$$

In particular, (i) for $s = 1, h(t) = k_\lambda(t, 1) = \frac{1}{a_1+t^\lambda}$, we have

$$h(xy) = \frac{1}{a_1 + (xy)^\lambda}, k_\lambda(x, y) = \frac{1}{a_1 x^\lambda + y^\lambda},$$

and

$$||T_0|| = ||\widetilde{T}_0|| = \frac{\pi}{\lambda \sin \pi (\sigma/\lambda)} a_1^{\frac{\sigma}{\lambda}-1};$$

(ii) for $s = 2$,

$$h(t) = k_\lambda(1, t) = \frac{1}{(a_1 + t^{\lambda/2})(a_2 + t^{\lambda/2})},$$

we have

$$h(xy) = \frac{1}{[a_1 + (xy)^{\lambda/2}][a_2 + (xy)^{\lambda/2}]},$$

$$k_\lambda(x, y) = \frac{1}{(a_1 x^{\lambda/2} + y^{\lambda/2})(a_2 x^{\lambda/2} + y^{\lambda/2})},$$

and

$$||T_0|| = ||\widetilde{T}_0|| = \frac{2\pi}{\lambda \sin \pi (2\sigma/\lambda)} \frac{1}{(a_2 - a_1)} (a_1^{\frac{2\sigma}{\lambda}-1} - a_2^{\frac{2\sigma}{\lambda}-1}).$$

(b) We set

$$h(t) = k_\lambda(1, t) = \frac{1}{t^\lambda + 2ct^{\lambda/2} \cos \gamma + c^2}$$

$(c > 0, |\gamma| < \frac{\pi}{2}, \mu, \sigma > 0,\ \mu + \sigma = \lambda)$. Then we have

$$h(xy) = \frac{1}{(xy)^\lambda + 2c(xy)^{\lambda/2} \cos \gamma + c^2},$$

$$k_\lambda(x, y) = \frac{1}{y^\lambda + 2c(xy)^{\lambda/2} \cos \gamma + c^2 x^\lambda}.$$

By (44), we can find

$$\begin{aligned}
k(\sigma) = k_\lambda(\mu) &= \int_0^\infty \frac{t^{\sigma-1}}{t^\lambda + 2ct^{\lambda/2}\cos\gamma + c^2} dt \\
&= \frac{2}{\lambda}\int_0^\infty \frac{u^{(2\sigma/\lambda)-1}}{u^2 + 2cu\cos\gamma + c^2} du \\
&= \frac{2}{\lambda}\int_0^\infty \frac{u^{(2\sigma/\lambda)-1}}{(u + ce^{i\gamma})(u + ce^{-i\gamma})} du \\
&= \frac{2\pi}{\lambda \sin \pi(2\sigma/\lambda)}\left[\frac{(ce^{i\gamma})^{(2\sigma/\lambda)-1}}{c(e^{-i\gamma} - e^{i\gamma})} + \frac{(ce^{-i\gamma})^{(2\sigma/\lambda)-1}}{c(e^{i\gamma} - e^{-i\gamma})}\right] \\
&= \frac{2\pi \sin\gamma(1 - 2\sigma/\lambda)}{\lambda \sin\pi(2\sigma/\lambda)\sin\gamma} c^{\frac{2\sigma}{\lambda}-2} \in \mathbf{R}_+.
\end{aligned}$$

In view of (36) and (37), we have

$$||T_0|| = ||\widetilde{T}_0|| = \frac{2\pi \sin\gamma(1 - 2\sigma/\lambda)}{\lambda \sin\pi(2\sigma/\lambda)\sin\gamma} c^{\frac{2\sigma}{\lambda}-2}.$$

3 First Kind of Compositional Yang-Hilbert-Type Integral Inequalities

In this section, by using a few lemmas, we obtain two equivalent first kind of compositional Yang-Hilbert-type integral inequalities and the reverses with the best possible constant factors. Some corollaries are deduced. We agree on that $p > 0 (p \neq 1), \frac{1}{p} + \frac{1}{q} = 1,\ \mu, \sigma \in \mathbf{R},\ \mu + \sigma = \lambda$ in the following.

3.1 Some Lemmas

Lemma 3 (cf. [29], Lemma 2.2.5). *Suppose that* $\lambda \in A_i (\neq \Phi) \subset \mathbf{R},\ k_\lambda^{(i)}(x, y)$ *are homogeneous functions of degree* $-\lambda$ *in* $\mathbf{R}_+^2$,

$$k_\lambda^{(i)}(\mu) := \int_0^\infty k_\lambda^{(i)}(u, 1) u^{\mu-1} du, \tag{45}$$

there exists a constant $\delta_0 > 0$, *such that* $k_\lambda^{(i)}(\mu \pm \delta_0) \in \mathbf{R}_+ (i = 1, 2)$. *Then for any* $\delta \in [0, \delta_0)$, *we have* $k_\lambda^{(i)}(\mu \pm \delta) \in \mathbf{R}_+$, *and*

$$\lim_{\delta \to 0+} k_\lambda^{(i)}(\mu \pm \delta) = k_\lambda^{(i)}(\mu) (i = 1, 2) \tag{46}$$

Proof. We find

$$0 \le k_\lambda^{(i)}(\mu \pm \delta) = \int_0^1 k_\lambda^{(i)}(u,1)u^{\mu\pm\delta-1}du + \int_1^\infty k_\lambda^{(i)}(u,1)u^{\mu\pm\delta-1}du$$
$$\le \int_0^1 k_\lambda^{(i)}(u,1)u^{(\mu-\delta_0)-1}du + \int_1^\infty k_\lambda^{(i)}(u,1)u^{(\mu+\delta_0)-1}du$$
$$\le k_\lambda^{(i)}(\mu-\delta_0) + k_\lambda^{(i)}(\mu+\delta_0) < \infty.$$

If there exists a constant $\eta \in (-\delta_0, \delta_0)$, such that $k_\lambda^{(i)}(\mu+\eta) = 0$, then

$$k_\lambda^{(i)}(u,1)u^{\mu+\eta-1} = 0 \ a.e. \ in \ (0,\infty),$$

namely, $k_\lambda^{(i)}(u,1) = 0$ a.e. in $(0,\infty)$, and it follows that

$$k_\lambda^{(i)}(\mu \pm \delta_0) = \int_0^\infty k_\lambda^{(i)}(u,1)u^{\mu\pm\delta_0-1}du = 0,$$

which contradicts the fact that $k_\lambda^{(i)}(\mu \pm \delta_0) \in \mathbf{R}_+$. Hence, for any $\delta \in [0,\delta_0)$, $k_\lambda^{(i)}(\mu \pm \delta) \in \mathbf{R}_+$.

Since we find

$$k_\lambda^{(i)}(u,1)u^{\mu\pm\delta-1} \le g(u) := \begin{cases} k_\lambda^{(i)}(u,1)u^{(\mu-\delta_0)-1}, u \in (0,1], \\ k_\lambda^{(i)}(u,1)u^{(\mu+\delta_0)-1}, u \in (1,\infty), \end{cases}$$

and

$$0 \le \int_0^\infty g(u)du = \int_0^1 g(u)du + \int_1^\infty g(u)du$$
$$\le k_\lambda^{(i)}(\mu-\delta_0) + k_\lambda^{(i)}(\mu+\delta_0) < \infty,$$

then by Lebesgue dominated convergence theorem (cf. [26]), it follows that (46) follows.

The lemma is proved. □

With the assumptions of Lemma 3, we set the following conditions:

Condition (i). For any $\lambda \in A_1 \cap A_2 (\neq \Phi)$, there exist constants $\delta_1 \in (0,\delta_0)$ and $L_1 > 0$, such that

$$k_\lambda^{(2)}(u,1)u^{\mu-\delta_1} \le L_1(u \in (0,1)) \tag{47}$$

Condition (ii). For any $\lambda \in (0,1) \cap A_1 \cap A_2 (\neq \Phi)$, there exists a constant $L_2 > 0$, such that

$$k_\lambda^{(2)}(u,1)(1-u)^\lambda \le L_2(u \in (0,1)). \tag{48}$$

Example 6. (a) For $\lambda \in A_1 = A_2 = \mathbf{R}_+,\ \delta_0 > 0$, the functions

$$k_\lambda^{(2)}(u,1) = \frac{1}{(u+1)^\lambda}, \frac{1}{u^\lambda + 1}, \frac{\ln u}{u^\lambda - 1}, \frac{|\ln u|^\beta}{(\max\{u,1\})^\lambda} (\beta \geq 0)$$

satisfy for using Condition (i); for $\lambda = 0, 0 < \delta_0 < \mu + \eta, \beta \geq 0$, the function

$$k_0^{(2)}(u,1) = \left(\frac{\min\{u,1\}}{\max\{u,1\}}\right)^\eta |\ln u|^\beta$$

satisfies for using Condition (i); for $\lambda = 0, \rho, \eta, \mu > 0, \delta_0 \in (0,\mu)$, the functions

$$k_0^{(2)}(u,1) = \ln\left[1 + \rho(\frac{1}{u})^\eta\right], \arctan \rho(\frac{1}{u})^\eta,$$

and $e^{-\rho u^\eta}$ satisfy for using Condition (i). In fact, for $\delta_1 \in (0,\delta_0)$, we obtain

$$\lim_{u\to 0+} k_\lambda^{(2)}(u,1)u^{\mu-\delta_1} = 0.$$

In view of the continuity, $k_\lambda^{(2)}(u,1)u^{\mu-\delta_1}$ is bounded in $(0,1)$.

(b) For $\lambda \in A_1 \cap A_2 = (0,1)$, the function

$$k_\lambda^{(2)}(u,1) = \frac{1}{|u-1|^\lambda}$$

satisfy for using Condition (ii).

Definition 5. On the assumptions of Lemma 3, define the following real function:

$$\widetilde{F}_k(y) := \begin{cases} y^{\lambda-1}\int_1^\infty k_\lambda^{(2)}(x,y)x^{\mu-\frac{1}{pk}-1}dx, y \in (1,\infty), \\ 0, y \in (0,1], \end{cases}$$

where $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\}(k \in \mathbf{N})$.

Setting $u = x/y (y > 1)$, we find

$$\begin{aligned} \widetilde{F}_k(y) &= y^{\mu-\frac{1}{pk}-1}\int_{\frac{1}{y}}^\infty k_\lambda^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du \\ &= y^{\mu-\frac{1}{pk}-1}\left[\int_0^\infty k_\lambda^{(2)}(u,1)u^{(\mu-\frac{1}{pk})-1}du - \int_0^{\frac{1}{y}} k_\lambda^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right] \\ &= y^{\mu-\frac{1}{pk}-1}k_\lambda^{(2)}(\mu - \frac{1}{pk}) - F_k(y), \end{aligned} \tag{49}$$

$$F_k(y) := y^{\mu-\frac{1}{pk}-1}\int_0^{\frac{1}{y}} k_\lambda^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du (y \in (1,\infty)).$$

(a) If $k_\lambda^{(2)}(u,1)$ satisfies Condition (i) (for $\lambda \in A_1 \cap A_2$), then by (47), we have

$$0 \le F_k(y) \le L_1 y^{\mu-\frac{1}{pk}-1} \int_0^{\frac{1}{y}} u^{-\mu+\delta_1} u^{\mu-\frac{1}{pk}-1} du$$
$$= \frac{L_1 y^{\mu-\delta_1-1}}{\delta_1 - \frac{1}{pk}} (y \in (1,\infty));$$

(b) if $k_\lambda^{(2)}(u,1)$ satisfies Condition (ii) (for $\lambda \in (0,1) \cap A_1 \cap A_2$), then by (48), we have

$$0 \le F_k(y) \le L_2 y^{\mu-\frac{1}{pk}-1} \int_0^{\frac{1}{y}} \frac{u^{\mu-\frac{1}{pk}-1}}{(1-u)^\lambda} du$$
$$= L_2 y^{\lambda-1} \int_0^1 \frac{v^{\mu-\frac{1}{pk}-1}}{(y-v)^\lambda} dv \le \frac{L_2 y^{\lambda-1}}{(y-1)^\lambda} \int_0^1 v^{\mu-\frac{1}{pk}-1} dv$$
$$= \frac{L_2}{\mu - \frac{1}{pk}} \frac{y^{\lambda-1}}{(y-1)^\lambda} (y \in (1,\infty)).$$

Remark 4. In view of the cases (a) and (b), there exists a large constant $L > 0$, such that

(a) $F_k(y) \le L y^{\mu-\delta_1-1} (y \in (1,\infty); \lambda \in A_1 \cap A_2)$;
(b) $F_k(y) \le L \frac{y^{\lambda-1}}{(y-1)^\lambda} (y \in (1,\infty); \lambda \in (0,1) \cap A_1 \cap A_2)$.

Lemma 4. *On the assumptions of Lemma 3, if $k_\lambda^{(1)}(x,y)$ is a symmetric function such that for any $x, y \in \mathbf{R}_+$, $k_\lambda^{(1)}(y,x) = k_\lambda^{(1)}(x,y)$, and $k_\lambda^{(2)}(u,1)$ satisfies Condition (i) for $\lambda \in A_1 \cap A_2$ or Condition (ii) for $\lambda \in (0,1) \cap A_1 \cap A_2$, then we have*

$$\widetilde{L}_k := \frac{1}{k} \int_1^\infty \int_1^\infty k_\lambda^{(1)}(x,y) \widetilde{F}_k(y) x^{\sigma-\frac{1}{qk}-1} dy dx$$
$$= \prod_{i=1}^{2} k_\lambda^{(i)}(\mu) + o(1) (k \to \infty). \tag{50}$$

Proof. In view of (49), we have

$$\widetilde{L}_k = \frac{1}{k} \int_1^\infty \int_1^\infty k_\lambda^{(1)}(x,y) \left(y^{\mu-\frac{1}{pk}-1} k_\lambda^{(2)} \left(\mu - \frac{1}{pk} \right) - F_k(y) \right)$$
$$\times x^{\sigma-\frac{1}{qk}-1} dy dx = I_1 - I_2 \le I_1, \tag{51}$$

where we define

$$I_1 := \frac{1}{k}k_\lambda^{(2)}(\mu - \frac{1}{pk})\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right)x^{\sigma-\frac{1}{qk}-1}dx,$$
$$I_2 := \frac{1}{k}\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)F_k(y)dy\right)x^{\sigma-\frac{1}{qk}-1}dx.$$

Since $k_\lambda^{(1)}(y,x) = k_\lambda^{(1)}(x,y)$, we obtain by Fubini theorem that (cf. [26])

$$\begin{aligned}
&\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right)x^{\sigma-\frac{1}{qk}-1}dx \\
&= \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\frac{1}{pk}-1}dy\right)x^{\sigma-\frac{1}{qk}-1}dx \\
&= \int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)x^{-\frac{1}{k}-1}dx \\
&= \int_1^\infty \left(\int_{\frac{1}{x}}^1 k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)x^{-\frac{1}{k}-1}dx \\
&\quad + \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right)x^{-\frac{1}{k}-1}dx \\
&= \int_0^1 \left(\int_{\frac{1}{u}}^\infty x^{-\frac{1}{k}-1}dx\right)k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du \\
&\quad + k\int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du \\
&= k\left(\int_0^1 k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du + \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right).
\end{aligned}$$

Since $\{k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}\}_{k=1}^\infty$ $(u \in (1,\infty))$ is increasing, by Levi theorem (cf. [26]), it follows that

$$\int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du \to \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-1}du(k\to\infty).$$

Since $\frac{1}{|q|k} < \delta_1$, then we find $\frac{1}{qk} > -\delta_1 > -\delta_0$ and

$$k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1} \le k_\lambda^{(1)}(u,1)u^{\mu-\delta_0-1}(u \in (0,1)),$$

and

$$0 \le \int_0^1 k_\lambda^{(1)}(u,1)u^{\mu-\delta_0-1}du$$
$$\le \int_0^\infty k_\lambda^{(1)}(u,1)u^{\mu-\delta_0-1}du = k_\lambda^{(1)}(\mu-\delta_0) < \infty,$$

and then by Lebesgue dominated convergence theorem (cf. [26]), we have

$$\int_0^1 k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du \to \int_0^1 k_\lambda^{(1)}(u,1)u^{\mu-1}du(k\to\infty).$$

Hence, by Lemma 3, we find

$$\begin{aligned} I_1 &= k_\lambda^{(2)}(\mu-\frac{1}{pk}) \\ &\quad\times\left(\int_0^1 k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du + \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right) \\ &\quad\to \prod_{i=1}^2 k_\lambda^{(i)}(\mu)(k\to\infty). \end{aligned} \tag{52}$$

We estimate I_2.

(a) If $k_\lambda^{(2)}(u,1)$ satisfies Condition (i) for $\lambda \in A_1 \cap A_2$, then by Remark 4(a), we have

$$\begin{aligned} 0 \le J_2 &:= \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)F_k(y)dy\right) x^{\sigma-\frac{1}{qk}-1}dx \\ &\le L\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\delta_1-1}dy\right) x^{\sigma-\frac{1}{qk}-1}dx \\ &= L\int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\delta_1-1}du\right) x^{-\delta_1-\frac{1}{qk}-1}dx \\ &\le L\int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1)u^{\mu-\delta_1-1}du\right) x^{-\delta_1-\frac{1}{qk}-1}dx \\ &= \frac{L}{\delta_1+\frac{1}{qk}} k_\lambda^{(1)}(\mu-\delta_1) < \infty; \end{aligned}$$

(b) if $k_\lambda^{(2)}(u,1)$ satisfies Condition (ii) for $\lambda \in (0,1)\cap A_1 \cap A_2$, then by Remark 4(b), we have

$$
\begin{aligned}
0 \le J_2 &\le L\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x)x^{\sigma-\frac{1}{qk}-1}dx\right)\frac{y^{\lambda-1}}{(y-1)^\lambda}dy \\
&= L\int_1^\infty \left(\int_0^y k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du\right)\frac{y^{\sigma-\frac{1}{qk}-1}}{(y-1)^\lambda}dy \\
&\le L\int_0^1 \left(\int_0^\infty k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du\right)\frac{v^{\mu+\frac{1}{qk}-1}}{(1-v)^\lambda}dv \\
&= Lk_\lambda^{(1)}(\mu+\frac{1}{qk})B(1-\lambda,\mu+\frac{1}{qk}) < \infty.
\end{aligned}
$$

Therefore, in view of (a) and (b), we have $I_2 \to 0 (k\to\infty)$.
By (51), (52), and the above results, we have (50).
The lemma is proved. □

3.2 Main Results

We set functions $\phi(x) := x^{p(1-\mu)-1}$, $\psi(y) := y^{q(1-\sigma)-1} (x, y \in \mathbf{R}_+)$ in the following.

Theorem 4. *Suppose that (a) $\lambda \in A_i (\neq \Phi) \subset \mathbf{R}$, $k_\lambda^{(i)}(x,y)$ are homogeneous functions of degree $-\lambda$ in $\mathbf{R}_+^2$,*

$$k_\lambda^{(i)}(\mu) = \int_0^\infty k_\lambda^{(i)}(u,1)u^{\mu-1}du,$$

there exists a constant $\delta_0 > 0$, such that $k_\lambda^{(i)}(\mu\pm\delta_0) \in \mathbf{R}_+ (i = 1, 2)$; (b) $k_\lambda^{(1)}(x,y)$ is a symmetric function; (c) if we use Condition (i), then $\lambda \in A_1 \cap A_2$; if we use Condition (ii), then $\lambda \in (0,1)\cap A_1 \cap A_2$. Then for $p > 1, f(x), g(y) \ge 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, and

$$F_\lambda(y) := \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx, y \in \{y\in\mathbf{R}_+; f(y) > 0\}, \\ 0, y \in \{y \in \mathbf{R}_+; f(y) = 0\}, \end{cases}$$

we have the following equivalent inequalities:

$$I := \int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)g(x)dydx < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi}||g||_{q,\psi} \tag{53}$$

$$J := \left[\int_0^\infty x^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi}, \tag{54}$$

where the constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ is the best possible.

Proof. Since $k_\lambda^{(1)}(y,x) = k_\lambda^{(1)}(x,y)$, by (28), we have

$$J = \left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(1)}(y,x)F_\lambda(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$= \left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$\leq k_\lambda^{(1)}(\mu)||F_\lambda||_{p,\phi}, \tag{55}$$

$$||F_\lambda||_{p,\phi} = \left[\int_0^\infty y^{p(1-\mu)-1}\left(y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$= \left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(2)}(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< k_\lambda^{(2)}(\mu)||f||_{p,\phi}. \tag{56}$$

Then (54) follows.

By Hölder's inequality (cf. [25]), we have

$$I = \int_0^\infty \left(x^{\sigma-\frac{1}{p}}\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)(x^{-\sigma+\frac{1}{p}}g(x))dx$$
$$\leq J||g||_{q,\psi}. \tag{57}$$

Then by (54), we have (53).

On the other hand, suppose that (53) is valid. We set

$$g(x) := x^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)^{p-1} (x \in \mathbf{R}_+),$$

and find $||g||_{q,\psi}^q = J^p$. If $J = 0$, then (54) is trivially valid; if $J = \infty$, then by (57), we have $||F_\lambda||_{p,\varphi} = \infty$, which contradicts inequality (56). Assuming that $0 < J < \infty$, then by (53), we have

$$||g||_{q,\psi}^q = J^p = I < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi}||g||_{q,\psi},$$

$$||g||_{q,\psi}^{q-1} = J < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi},$$

and we have (54), which is equivalent to (53).

For any $k > \max\{\frac{1}{q\delta_1}, \frac{1}{p\delta_1}\}(k \in \mathbf{N})$, we set $\tilde{f}(x) = \tilde{g}(y) = 0 (x, y \in (0, 1])$;

$$\tilde{f}(x) = x^{\mu - \frac{1}{pk} - 1}, \tilde{g}(y) = y^{\sigma - \frac{1}{qk} - 1} (x, y \in (1, \infty)).$$

Then we have $\widetilde{F}_k(y) = 0 (y \in (0, 1])$;

$$\widetilde{F}_k(y) = y^{\lambda - 1} \int_1^\infty k_\lambda^{(2)}(x, y) x^{\mu - \frac{1}{pk} - 1} dx$$
$$= y^{\lambda - 1} \int_0^\infty k_\lambda^{(2)}(x, y) \tilde{f}(x) dx (y \in (1, \infty)).$$

If there exists a positive constant $K \leq \prod_{i=1}^2 k_\lambda^{(i)}(\mu)$, such that (53) is valid when replacing $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ to K, then in particular, we have

$$\widetilde{L}_k = \frac{1}{k} \int_0^\infty \int_0^\infty k_\lambda^{(1)}(x, y) \widetilde{F}_k(y) \tilde{g}(x) dy dx$$
$$< \frac{1}{k} K ||\tilde{f}||_{p,\phi} ||\tilde{g}||_{q,\psi} = \frac{1}{k} K \int_1^\infty x^{-\frac{1}{k} - 1} dx = K.$$

By (50), we find

$$\prod_{i=1}^2 k_\lambda^{(i)}(\mu) + o(1) = \widetilde{L}_k < K,$$

and then $\prod_{i=1}^2 k_\lambda^{(i)}(\mu) \leq K (k \to \infty)$. Hence $K = \prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ is the best possible constant factor of (53).

The constant factor in (54) is the best possible. Otherwise, if the constant factor in (54) is not the best possible, then we would reach a contradiction by (57) that the constant factor in (53) is not the best possible. □

Theorem 5. *Suppose that (a)* $\lambda \in A_i (\neq \Phi) \subset \mathbf{R}$, $k_\lambda^{(i)}(x, y)$ *are homogeneous functions of degree* $-\lambda$ *in* $\mathbf{R}_+^2$,

$$k_\lambda^{(i)}(\mu) = \int_0^\infty k_\lambda^{(i)}(u, 1) u^{\mu - 1} du,$$

there exists a constant $\delta_0 > 0$, *such that* $k_\lambda^{(i)}(\mu \pm \delta_0) \in \mathbf{R}_+ (i = 1, 2)$; *(b)* $k_\lambda^{(1)}(x, y)$ *is a symmetric function. Then for* $0 < p < 1, f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *and*

$$F_\lambda(y) = \begin{cases} y^{\lambda - 1} \int_0^\infty k_\lambda^{(2)}(x, y) f(x) dx, y \in \{y \in \mathbf{R}_+; f(y) > 0\}, \\ 0, y \in \{y \in \mathbf{R}_+; f(y) = 0\}, \end{cases}$$

we have the equivalent reverses of (53) and (54) with the best possible constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$.

Proof. Since $k_\lambda^{(1)}(y,x) = k_\lambda^{(1)}(x,y)$, by the reverse Hölder's inequality, we obtain the reverses of (55) and (56). Then we deduce to the reverse of (54). By the reverse Hölder's inequality (cf. [25]), we have

$$I = \int_0^\infty \left(x^{\sigma - \frac{1}{p}} \int_0^\infty k_\lambda^{(1)}(x,y) F_\lambda(y) dy \right) (x^{-\sigma + \frac{1}{p}} g(x)) dx \geq J ||g||_{q,\psi}. \quad (58)$$

Then by the reverse of (54), we have the reverse of (53).

On the other hand, suppose that the reverse of (53) is valid. Setting $g(x)$ as Theorem 4, we find $||g||_{q,\psi}^q = J^p$. If $J = \infty$, then the reverse of (54) is trivially valid; if $J = 0$, then by reverse of (55), we have $||F_\lambda||_{p,\varphi} = 0$, which contradicts the reverse of (56). Assuming that $0 < J < \infty$, then by the reverse of (53), we have

$$||g||_{q,\psi}^q = J^p = I > \prod_{i=1}^2 k_\lambda^{(i)}(\mu) ||f||_{p,\phi} ||g||_{q,\psi},$$

$$||g||_{q,\psi}^{q-1} = J > \prod_{i=1}^2 k_\lambda^{(i)}(\mu) ||f||_{p,\phi},$$

and the reverse of (54) follows, which is equivalent to reverse of (53).

For any $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\} (k \in \mathbf{N})$, we set $\tilde{f}(x), \tilde{g}(y), \widetilde{F}_\lambda(y)$ as Theorem 4. If there exists a positive constant $K \geq \prod_{i=1}^2 k_\lambda^{(i)}(\mu)$, such that the reverse of (53) is valid when replacing $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ to K, then in particular, we have

$$\begin{aligned} \widetilde{L}_k &= \frac{1}{k} \int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y) \widetilde{F}_k(y) \tilde{g}(x) dy dx \\ &> \frac{1}{k} K ||\tilde{f}||_{p,\phi} ||\tilde{g}||_{q,\psi} = K. \end{aligned}$$

By (51) and (52), we find

$$\prod_{i=1}^2 k_\lambda^{(i)}(\mu) + \tilde{o}(1) = I_1 \geq \widetilde{L}_k > K,$$

and then $\prod_{i=1}^2 k_\lambda^{(i)}(\mu) \geq K (k \to \infty)$. Hence $K = \prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ is the best possible constant factor of the reverse of (53).

By the equivalency, if the constant factor in the reverse of (54) is not the best possible, then by (58), we would reach a contradiction that the constant factor in the reverse of (53) is not the best possible. □

Replacing x to $\frac{1}{x}$ in (53) and (54), setting $\hat{g}(x) := x^{\lambda-2} g(\frac{1}{x})$, $\widehat{\psi}(y) := y^{q(1-\mu)-1}$, by simplification, we have

Theorem 6. *If* $f(x), \hat{g}(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+),\ \hat{g} \in L_{q,\widehat{\psi}}(\mathbf{R}_+),\ ||f||_{p,\phi}, ||\hat{g}||_{q,\widehat{\psi}} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(xy,1)F_\lambda(y)\hat{g}(x)dydx < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi}||\hat{g}||_{q,\widehat{\psi}}, \quad (59)$$

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_0^\infty k_\lambda^{(1)}(xy,1)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi}, \quad (60)$$

where the constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (59) and (60) with the best possible constant factor* $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$.

Replacing y to $\frac{1}{y}$ in (59) and (60), setting

$$\widehat{F}_\lambda(y) := y^{\lambda-2}F_\lambda(\frac{1}{y})$$
$$= \begin{cases} y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(xy,1)f(x)dx, y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\}, \\ 0, y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

by simplification, we have

Theorem 7. *If* $f(x), \hat{g}(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+),\ \hat{g} \in L_{q,\widehat{\psi}}(\mathbf{R}_+),\ ||f||_{p,\phi}, ||\hat{g}||_{q,\widehat{\psi}} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y)\widehat{F}_\lambda(y)\hat{g}(x)dydx < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi}||\hat{g}||_{q,\widehat{\psi}} \quad (61)$$

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_0^\infty k_\lambda^{(1)}(x,y)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < \prod_{i=1}^2 k_\lambda^{(i)}(\mu)||f||_{p,\phi}, \quad (62)$$

where the constant factor $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (61) and (62) with the best possible constant factor* $\prod_{i=1}^2 k_\lambda^{(i)}(\mu)$.

Replacing y to $\frac{1}{y}$ in (53) and (54), by simplification, we have

Theorem 8. *If* $f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+), g \in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)g(x)dydx < \prod_{i=1}^{2} k_\lambda^{(i)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \quad (63)$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_0^\infty k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < \prod_{i=1}^{2} k_\lambda^{(i)}(\mu)||f||_{p,\phi}, \quad (64)$$

where the constant factor $\prod_{i=1}^{2} k_\lambda^{(i)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (63) and (64) with the best possible constant factor* $\prod_{i=1}^{2} k_\lambda^{(i)}(\mu)$.

3.3 Some Corollaries on Theorems 4 and 5

Assuming that $k_\lambda^{(1)}(x,y) = 0 (0 < y \le x)$, we find $k_\lambda^{(1)}(u,1) = 0 (u \ge 1)$, and

$$k_\lambda^{(1)}(\mu) = k_{\lambda,1}^{(1)}(\mu) := \int_0^1 k_\lambda^{(1)}(u,1)u^{\mu-1}du. \quad (65)$$

By Theorems 4 and 5, we have

Corollary 5. *If* $k_{\lambda,1}^{(1)}(\mu), k_\lambda^{(2)}(\mu) \in \mathbf{R}_+, f(x), g(y) \ge 0$, $f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_x^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)g(x)dydx < k_{\lambda,1}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}||g||_{q,\psi} \quad (66)$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_x^\infty k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,1}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}, \quad (67)$$

where the constant factor $k_{\lambda,1}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (66) and (67) with the best possible constant factor* $k_{\lambda,1}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$.

Assuming that $k_\lambda^{(1)}(x,y) = 0 (0 < x \le y)$, we find $k_\lambda^{(1)}(u,1) = 0 (0 < u \le 1)$, and

$$k_\lambda^{(1)}(\mu) = k_{\lambda,2}^{(1)}(\mu) := \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-1}du. \quad (68)$$

By Theorems 4 and 5, we have

Corollary 6. *If* $k_{\lambda,2}^{(1)}(\mu), k_\lambda^{(2)}(\mu) \in \mathbf{R}_+, f(x), g(y) \ge 0$, $f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^x k_\lambda^{(1)}(x,y)F_\lambda(y)g(x)dydx < k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}||g||_{q,\psi} \tag{69}$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_0^x k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}, \tag{70}$$

where the constant factor $k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (69) and (70) with the best possible constant factor* $k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$.

Remark 5. For $x > 0$, setting $A_{x,0} = (0,\infty), A_{x,1} = (x,\infty), A_{x,2} = (0,x)$, and $k_{\lambda,0}^{(1)}(\mu) := k_\lambda^{(1)}(\mu)$, by Theorems 4 and 5, Corollaries 5 and 6, for $p > 1$, we have the following equivalent inequalities:

$$\int_0^\infty \int_{A_{x,i}} k_\lambda^{(1)}(x,y)F_\lambda(y)g(x)dydx < k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}||g||_{q,\psi} \tag{71}$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{A_{x,i}} k_\lambda^{(1)}(x,y)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}, \tag{72}$$

where the constant factors $k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ $(i = 0,1,2)$ are the best possible; for $0 < p < 1$, we have the equivalent reverses of (71) and (72) with the best possible constant factors $k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ $(i = 0,1,2)$.

If $k_\lambda^{(2)}(x,y) = 0(y \in \mathbf{R}_+ \backslash A_{x,1})$, then we find $k_\lambda^{(2)}(u,1) = 0(u \geq 1)$, and

$$k_\lambda^{(2)}(\mu) = k_{\lambda,1}^{(2)}(\mu) := \int_0^1 k_\lambda^{(2)}(u,1)u^{\mu-1}du;$$

if $k_\lambda^{(2)}(x,y) = 0(y \in \mathbf{R}_+ \backslash A_{x,2})$, then we find $k_\lambda^{(2)}(u,1) = 0(0 < u \leq 1)$, and

$$k_\lambda^{(2)}(\mu) = k_{\lambda,2}^{(2)}(\mu) := \int_1^\infty k_\lambda^{(2)}(u,1)u^{\mu-1}du.$$

Assuming that $k_{\lambda,0}^{(2)}(\mu) := k_\lambda^{(2)}(\mu)$, setting

$$F_{\lambda,j}(y) := \begin{cases} y^{\lambda-1}\int_{A_{x,j}} k_\lambda^{(2)}(x,y)f(x)dx, y \in \{y \in \mathbf{R}_+; f(y) > 0\}, \\ 0, y \in \{y \in \mathbf{R}_+; f(y) = 0\}, \end{cases}$$

then it follows that $F_{\lambda,0}(y) = F_\lambda(y)$. In the same way, we have

Corollary 7. *If for* $i, j = 0, 1, 2$, $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+, f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_{A_{x,i}} k_\lambda^{(1)}(x,y) F_{\lambda,j}(y) g(x) dy dx < k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{73}$$

$$\left[\int_0^\infty x^{p\sigma-1} \left(\int_{A_{x,i}} k_\lambda^{(1)}(x,y) F_{\lambda,j}(y) dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) ||f||_{p,\phi}, \tag{74}$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (73) and (74) with the best possible constant factor* $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu)$.

By (34), for $p > 1$, we still can find that

$$\left[\int_0^\infty x^{p\sigma-1} \left(\int_{A_{x,i}} k_\lambda^{(1)}(x,y) F_{\lambda,j}(y) dy\right)^p dx\right]^{\frac{1}{p}} \leq k_{\lambda,i}^{(1)}(\mu) ||F_{\lambda,j}||_{p,\phi}, \tag{75}$$

$$||F_{\lambda,j}||_{p,\phi} \leq k_{\lambda,j}^{(2)}(\mu) ||f||_{p,\phi} (i, j = 0, 1, 2), \tag{76}$$

where the constant factors $k_{\lambda,i}^{(1)}(\mu)$ $(i = 0, 1, 2)$ and $k_{\lambda,j}^{(2)}(\mu)$ $(j = 0, 1, 2)$ are the best possible.

Example 7. (a) We set

$$k_\lambda^{(1)}(u, 1) = \frac{1}{(\max\{u, 1\})^\lambda} (\lambda \in A_1 = (0, \infty)),$$

$$k_\lambda^{(2)}(u, 1) = \frac{1}{|u-1|^\lambda} (\lambda \in A_2 = (0, 1)).$$

Then $k_\lambda^{(2)}(u, 1)$ satisfies for using Condition (ii) while $\lambda \in (0, 1) \cap A_1 \cap A_2 = (0, 1)$, and for $i, j = 0, 1, 2$, we obtain

$$k_{\lambda,i}^{(1)}(\mu) = (2-i)(1+i)\frac{1}{2\mu} + (1-i)^2 \frac{1}{\sigma},$$

$$k_{\lambda,j}^{(2)}(\mu) = \frac{1}{2}(2-j)(1+j)B(1-\lambda, \mu) + (1-j)^2 B(1-\lambda, \sigma).$$

(b) We set

$$k_\lambda^{(1)}(u, 1) = \frac{1}{|u-1|^\lambda} (\lambda \in A_1 = (0, 1)),$$

$$k_\lambda^{(2)}(u, 1) = \frac{|\ln u|^\beta}{(\max\{u, 1\})^\lambda} (\beta \geq 0, \lambda \in A_2 = (0, \infty)).$$

Then $k_{\lambda}^{(2)}(u,1)$ satisfies for using Condition (i) while $\lambda \in A_1 \cap A_2 = (0,1)$, and for $i,j = 0,1,2$, we obtain

$$k_{\lambda,i}^{(1)}(\mu) = (2-i)(1+i)B(1-\lambda,\mu) + (1-i)^2 B(1-\lambda,\sigma),$$
$$k_{\lambda,j}^{(2)}(\mu) = \frac{1}{2}(2-j)(1+j)\frac{\Gamma(\beta+1)}{\mu^{\beta+1}} + (1-j)^2\frac{\Gamma(\beta+1)}{\sigma^{\beta+1}}.$$

(c) We set

$$k_{\lambda}^{(1)}(u,1) = \frac{1}{(\max\{u,1\})^{\lambda}}(\lambda \in A_1 = (0,\infty)),$$
$$k_{\lambda}^{(2)}(u,1) = \frac{|\ln u|^{\beta}}{(\max\{u,1\})^{\lambda}}(\beta \geq 0, \lambda \in A_2 = (0,\infty)).$$

Then $k_{\lambda}^{(2)}(u,1)$ satisfies for using Condition (i) while $\lambda \in A_1 \cap A_2 = (0,\infty)$, and for $i,j = 0,1,2$, we obtain

$$k_{\lambda,i}^{(1)}(\mu) = (2-i)(1+i)\frac{1}{2\mu} + (1-i)^2\frac{1}{\sigma},$$
$$k_{\lambda,j}^{(2)}(\mu) = \frac{1}{2}(2-j)(1+j)\frac{\Gamma(\beta+1)}{\mu^{\beta+1}} + (1-j)^2\frac{\Gamma(\beta+1)}{\sigma^{\beta+1}}.$$

(d) We set

$$k_{0}^{(1)}(u,1) = \left(\frac{\min\{u,1\}}{|u-1|}\right)^{\rho}(0 < \rho < 1),$$
$$k_{0}^{(2)}(u,1) = \left(\frac{\min\{u,1\}}{\max\{u,1\}}\right)^{\eta}|\ln u|^{\beta}(\beta \geq 0, \eta > 0).$$

Then for $|\mu| < \min\{\eta,\rho\}, 0 < \delta_0 < \mu+\eta$, $k_0^{(2)}(u,1)$ satisfies for using Condition (i) while $\lambda = 0$ and for $i,j = 0,1,2$, we obtain

$$k_{i}^{(1)}(\mu) = \frac{1}{2}(2-i)(1+i)B(1-\rho,\rho+\mu)$$
$$+(1-i)^2(1-\rho,\rho-\mu),$$
$$k_{j}^{(2)}(\mu) = (2-j)(1+j)\frac{\Gamma(\beta+1)}{(\eta+\mu)^{\beta+1}} + (1-j)^2\frac{\Gamma(\beta+1)}{(\eta-\mu)^{\beta+1}}.$$

3.4 Some Corollaries on Theorem 6

Assuming that $k_\lambda^{(1)}(xy,1) = 0 (0 < \frac{1}{x} \le y)$, then we find $k_\lambda^{(1)}(u,1) = 0 (u \ge 1)$, and $k_\lambda^{(1)}(\mu) = k_{\lambda,1}^{(1)}(\mu)$. By Theorem 6, we have

Corollary 8. *If $k_{\lambda,1}^{(1)}(\mu), k_\lambda^{(2)}(\mu) \in \mathbf{R}_+, f(x), \hat{g}(y) \ge 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $\hat{g} \in L_{q,\widehat{\psi}}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||\hat{g}||_{q,\widehat{\psi}} > 0$, then for $p > 1$, on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^{\frac{1}{x}} k_\lambda^{(1)}(xy,1) F_\lambda(y) \hat{g}(x) dy dx < k_{\lambda,1}^{(1)}(\mu) k_\lambda^{(2)}(\mu) ||f||_{p,\phi} ||\hat{g}||_{q,\widehat{\psi}}, \tag{77}$$

$$\left[\int_0^\infty x^{p\mu-1} \left(\int_0^{\frac{1}{x}} k_\lambda^{(1)}(xy,1) F_\lambda(y) dy \right)^p dx \right]^{\frac{1}{p}} < k_{\lambda,1}^{(1)}(\mu) k_\lambda^{(2)}(\mu) ||f||_{p,\phi}, \tag{78}$$

where the constant factor $k_{\lambda,1}^{(1)}(\mu) k_\lambda^{(2)}(\mu)$ is the best possible; for $0 < p < 1$, on the assumptions of Theorem 5, we have the equivalent reverses of (77) and (78) with the best possible constant factor $k_{\lambda,1}^{(1)}(\mu) k_\lambda^{(2)}(\mu)$.

Assuming that $k_\lambda^{(1)}(xy,1) = 0 (0 < y \le \frac{1}{x})$, then we find $k_\lambda^{(1)}(u,1) = 0 (0 < u \le 1)$, and $k_\lambda^{(1)}(\mu) = k_{\lambda,2}^{(1)}(\mu)$. By Theorem 6, we have

Corollary 9. *If $k_{\lambda,2}^{(1)}(\mu)\, k_\lambda^{(2)}(\mu) \in \mathbf{R}_+, f(x), \hat{g}(y) \ge 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $\hat{g} \in L_{q,\widehat{\psi}}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||\hat{g}||_{q,\widehat{\psi}} > 0$, then for $p > 1$, on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(xy,1) F_\lambda(y) \hat{g}(x) dy dx < k_{\lambda,2}^{(1)}(\mu) k_\lambda^{(2)}(\mu) ||f||_{p,\phi} ||\hat{g}||_{q,\widehat{\psi}}, \tag{79}$$

$$\left[\int_0^\infty x^{p\mu-1} \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(xy,1) F_\lambda(y) dy \right)^p dx \right]^{\frac{1}{p}} < k_{\lambda,2}^{(1)}(\mu) k_\lambda^{(2)}(\mu) ||f||_{p,\phi}, \tag{80}$$

where the constant factor $k_{\lambda,2}^{(1)}(\mu) k_\lambda^{(2)}(\mu)$ is the best possible; for $0 < p < 1$, on the assumptions of Theorem 5, we have the equivalent reverses of (79) and (80) with the best possible constant factor $k_{\lambda,2}^{(1)}(\mu) k_\lambda^{(2)}(\mu)$.

Remark 6. For $x > 0$, setting $B_{x,0} = (0,\infty), B_{x,1} = (0, \frac{1}{x}), B_{x,2} = (\frac{1}{x}, \infty)$, and $k_{\lambda,0}^{(1)}(\mu) = k_\lambda^{(1)}(\mu)$, by Theorem 6, Corollaries 8 and 9, for $p > 1$, we have the following equivalent inequalities:

$$\int_0^\infty \int_{B_{x,i}} k_\lambda^{(1)}(xy,1) F_\lambda(y) \hat{g}(x) dy dx < k_{\lambda,i}^{(1)}(\mu) k_\lambda^{(2)}(\mu) ||f||_{p,\phi} ||\hat{g}||_{q,\widehat{\psi}}, \tag{81}$$

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)F_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}, \quad (82)$$

where the constant factors $k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ $(i = 0,1,2)$ are the best possible; for $0 < p < 1$, we have the equivalent reverses of (81) and (82) with the best possible constant factors $k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ $(i = 0,1,2)$.

If $k_\lambda^{(2)}(x,y) = 0(y \in \mathbf{R}_+\backslash A_{x,1})$, then we find $k_\lambda^{(2)}(u,1) = 0(u \geq 1)$, and $k_\lambda^{(2)}(\mu) = k_{\lambda,1}^{(2)}(\mu)$; if $k_\lambda^{(2)}(x,y) = 0(y \in \mathbf{R}_+\backslash A_{x,2})$, then $k_\lambda^{(2)}(u,1) = 0(0 < u \leq 1), k_\lambda^{(2)}(\mu) = k_{\lambda,2}^{(2)}(\mu)$. In the same way, in view of (81) and (82), we have

Corollary 10. *If for* $i,j = 0,1,2$, $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+, f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_{B_{x,i}} k_\lambda^{(1)}(xy,1)F_{\lambda,j}(y)\hat{g}(x)dydx < k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}||\hat{g}||_{q,\widehat{\psi}}, \quad (83)$$

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)F_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}, \quad (84)$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (83) and (84) with the best possible constant factor* $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)$.

By (26) and (34), for $p > 1$, we still can find that

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)F_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{p}} \leq k_{\lambda,i}^{(1)}(\mu)||F_{\lambda,j}||_{p,\phi}, \quad (85)$$

$$||F_{\lambda,j}||_{p,\phi} \leq k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}(i,j = 0,1,2), \quad (86)$$

where the constant factors $k_{\lambda,i}^{(1)}(\mu)(i = 0,1,2)$ and $k_{\lambda,j}^{(2)}(\mu)(j = 0,1,2)$ are the best possible.

3.5 Some Corollaries on Theorem 7

Assuming that $k_\lambda^{(1)}(x,y) = 0(0 < y \leq x)$, then we find $k_\lambda^{(1)}(u,1) = 0(u \geq 1)$, and $k_\lambda^{(1)}(\mu) = k_{\lambda,1}^{(1)}(\mu)$. By Theorem 7, we have

Corollary 11. *If* $k^{(1)}_{\lambda,1}(\mu) \in \mathbf{R}_+, f(x), \hat{g}(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+),\ \hat{g} \in L_{q,\widehat{\psi}}(\mathbf{R}_+),\ ||f||_{p,\phi},\ ||\hat{g}||_{q,\widehat{\psi}} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_x^\infty k^{(1)}_\lambda(x,y)\widehat{F}_\lambda(y)\hat{g}(x)dydx < k^{(1)}_{\lambda,1}(\mu)k^{(2)}_\lambda(\mu)||f||_{p,\phi}||\hat{g}||_{q,\widehat{\psi}}, \tag{87}$$

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_x^\infty k^{(1)}_\lambda(x,y)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k^{(1)}_{\lambda,1}(\mu)k^{(2)}_\lambda(\mu)||f||_{p,\phi}, \tag{88}$$

where the constant factor $k^{(1)}_{\lambda,1}(\mu)k^{(2)}_\lambda(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (87) and (88) with the best possible constant factor* $k^{(1)}_{\lambda,1}(\mu)k^{(2)}_\lambda(\mu)$.

Assuming that $k^{(1)}_\lambda(x,y)=0 (0 < x \leq y)$, then we find $k^{(1)}_\lambda(u,1)=0 (0 < u \leq 1)$, and $k^{(1)}_\lambda(\mu) = k^{(1)}_{\lambda,2}(\mu)$. By Theorem 7, we have

Corollary 12. *If* $k^{(1)}_{\lambda,2}(\mu), k^{(2)}_\lambda(\mu) \in \mathbf{R}_+,\ f(x), \hat{g}(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+),\ \hat{g} \in L_{q,\widehat{\psi}}(\mathbf{R}_+),\ ||f||_{p,\phi}, ||\hat{g}||_{q,\widehat{\psi}} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^x k^{(1)}_\lambda(x,y)\widehat{F}_\lambda(y)\hat{g}(x)dydx < k^{(1)}_{\lambda,2}(\mu)k^{(2)}_\lambda(\mu)||f||_{p,\phi}||\hat{g}||_{q,\widehat{\psi}}, \tag{89}$$

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_0^x k^{(1)}_\lambda(x,y)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k^{(1)}_{\lambda,2}(\mu)k^{(2)}_\lambda(\mu)||f||_{p,\phi}, \tag{90}$$

where the constant factor $k^{(1)}_{\lambda,2}(\mu)k^{(2)}_\lambda(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (87) and (88) with the best possible constant factor* $k^{(1)}_{\lambda,2}(\mu)k^{(2)}_\lambda(\mu)$.

Remark 7. For $x > 0$, setting $A_{x,0} = (0,\infty), A_{x,1} = (x,\infty), A_{x,2} = (0,x)$, and $k^{(1)}_{\lambda,0}(\mu) = k^{(1)}_\lambda(\mu)$, by Theorem 7, Corollaries 11 and 12, for $p > 1$, we have the following equivalent inequalities:

$$\int_0^\infty \int_{A_{x,i}} k^{(1)}_\lambda(x,y)\widehat{F}_\lambda(y)\hat{g}(x)dydx < k^{(1)}_{\lambda,i}(\mu)k^{(2)}_\lambda(\mu)||f||_{p,\phi}||\hat{g}||_{q,\widehat{\psi}}, \tag{91}$$

$$\left[\int_0^\infty x^{p\mu-1}\left(\int_{A_{x,i}} k^{(1)}_\lambda(x,y)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k^{(1)}_{\lambda,i}(\mu)k^{(2)}_\lambda(\mu)||f||_{p,\phi}, \tag{92}$$

where the constant factors $k^{(1)}_{\lambda,i}(\mu)k^{(2)}_\lambda(\mu)$ $(i = 0,1,2)$ are the best possible; for $0 < p < 1$, we have the equivalent reverses of (91) and (92) with the best possible constant factors $k^{(1)}_{\lambda,i}(\mu)k^{(2)}_\lambda(\mu)(i = 0,1,2)$.

If $k_\lambda^{(2)}(xy,1) = 0 (y \in \mathbf{R}_+ \backslash B_{x,1})$, then we find $k_\lambda^{(2)}(u,1) = 0 (u \geq 1), k_\lambda^{(2)}(\mu) = k_{\lambda,1}^{(2)}(\mu)$; if $k_\lambda^{(2)}(x,y) = 0 (y \in \mathbf{R}_+ \backslash B_{x,2})$, then we find $k_\lambda^{(2)}(u,1) = 0 (0 < u \leq 1)$, $k_\lambda^{(2)}(\mu) = k_{\lambda,2}^{(2)}(\mu)$. In the same way, in view of (91) and (92), we have

Corollary 13. *If for $i,j = 0,1,2$, $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+, f(x), \hat{g}(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $\hat{g} \in L_{q,\widehat{\psi}}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\widehat{\psi}} > 0$, then for $p > 1$, on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_{A_{x,i}} k_\lambda^{(1)}(x,y) \widehat{F}_{\lambda,j}(y) \hat{g}(x) dy dx < k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) ||f||_{p,\phi} ||\hat{g}||_{q,\widehat{\psi}}, \tag{93}$$

$$\left[\int_0^\infty x^{p\mu-1} \left(\int_{A_{x,i}} k_\lambda^{(1)}(x,y) \widehat{F}_{\lambda,j}(y) dy \right)^p dx \right]^{\frac{1}{p}} < k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) ||f||_{p,\phi}, \tag{94}$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu)$ is the best possible; for $0 < p < 1$, on the assumptions of Theorem 5, we have the equivalent reverses of (93) and (94) with the best possible constant factor $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu)$.

By (34) and (26), for $p > 1, \varphi(x) = x^{p(1-\sigma)-1}$, we still can find that

$$\left[\int_0^\infty x^{p\mu-1} \left(\int_{A_{x,i}} k_\lambda^{(1)}(x,y) \widehat{F}_{\lambda,j}(y) dy \right)^p dx \right]^{\frac{1}{p}} \leq k_{\lambda,i}^{(1)}(\mu) ||\widehat{F}_{\lambda,j}||_{p,\varphi}, \tag{95}$$

$$||\widehat{F}_{\lambda,j}||_{p,\varphi} \leq k_{\lambda,j}^{(2)}(\mu) ||f||_{p,\phi} (j = 0,1,2), \tag{96}$$

where the constant factors $k_{\lambda,i}^{(1)}(\mu)$ and $k_{\lambda,j}^{(2)}(\mu)$ are the best possible.

3.6 Some Corollaries on Theorem 8

Assuming that $k_\lambda^{(1)}(xy,1) = 0 (0 < \frac{1}{x} \leq y)$, then we find $k_\lambda^{(1)}(u,1) = 0 (u \geq 1)$, and $k_\lambda^{(1)}(\mu) = k_{\lambda,1}^{(1)}(\mu)$. By Theorem 8, we have

Corollary 14. *If $k_{\lambda,1}^{(1)}(\mu), k_\lambda^{(2)}(\mu) \in \mathbf{R}_+ f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+), g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then for $p > 1$, on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^{\frac{1}{x}} k_\lambda^{(1)}(xy,1) \widehat{F}_\lambda(y) g(x) dy dx < k_{\lambda,1}^{(1)}(\mu) k_\lambda^{(2)}(\mu) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{97}$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_0^{\frac{1}{x}} k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,1}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}, \quad (98)$$

where the constant factor $k_{\lambda,1}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (97) and (98) with the best possible constant factor* $k_{\lambda,1}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$.

Assuming that $k_\lambda^{(1)}(xy,1)=0(0 < y \le \frac{1}{x})$, then we find $k_\lambda^{(1)}(u,1)=0(0 < u \le 1)$, and $k_\lambda^{(1)}(\mu) = k_{\lambda,2}^{(1)}(\mu)$. By Theorem 8, we have

Corollary 15. *If* $k_{\lambda,2}^{(1)}(\mu), k_\lambda^{(2)}(\mu) \in \mathbf{R}_+, f(x), g(y) \ge 0, f \in L_{p,\phi}(\mathbf{R}_+), g \in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)G(x)dydx < k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}||G||_{q,\psi}, \quad (99)$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}, \quad (100)$$

where the constant factor $k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (99) and (100) with the best possible constant factor* $k_{\lambda,2}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$.

Remark 8. For $x > 0$, setting $B_{x,0} = (0,\infty), B_{x,1} = (0,\frac{1}{x}), B_{x,2} = (\frac{1}{x},\infty)$, and $k_{\lambda,0}^{(1)}(\mu) = k_\lambda^{(1)}(\mu)$, by Theorem 8, Corollaries 14 and 15, for $p > 1$, we have the following equivalent inequalities

$$\int_0^\infty \int_{B_{x,i}} k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)g(x)dydx < k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \quad (101)$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)||f||_{p,\phi}, \quad (102)$$

where the constant factors $k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ $(i = 0,1,2)$ are the best possible; for $0 < p < 1$, we have the equivalent reverses of (101) and (102) with the best possible constant factor $k_{\lambda,i}^{(1)}(\mu)k_\lambda^{(2)}(\mu)$ $(i = 0,1,2)$.

If $k_\lambda^{(2)}(xy,1) = 0(y \in \mathbf{R}_+\backslash B_{x,1})$, then we find $k_\lambda^{(2)}(u,1) = 0(u \ge 1), k_\lambda^{(2)}(\mu) = k_{\lambda,1}^{(2)}(\mu)$; if $k_\lambda^{(2)}(x,y) = 0(y \in \mathbf{R}_+\backslash B_{x,2})$, then we find $k_\lambda^{(2)}(u,1) = 0(0 < u \le 1), k_\lambda^{(2)}(\mu) = k_{\lambda,2}^{(2)}(\mu)$. In the same way, in view of (91) and (92), we have

Corollary 16. *If for* $i, j = 0, 1, 2$, $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+, f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *then for* $p > 1$, *on the assumptions of Theorem 4, we have the following equivalent inequalities:*

$$\int_0^\infty \int_{B_{x,i}} k_\lambda^{(1)}(xy, 1)\widehat{F}_{\lambda,j}(y)g(x)dydx < k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \quad (103)$$

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy, 1)\widehat{F}_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{p}} < k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}, \quad (104)$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)$ *is the best possible; for* $0 < p < 1$, *on the assumptions of Theorem 5, we have the equivalent reverses of (103) and (104) with the best possible constant factor* $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)$.

By (26), for $p > 1, \varphi(x) = x^{p(1-\sigma)-1}$, we still can find that

$$\left[\int_0^\infty x^{p\sigma-1}\left(\int_{B_{x,i}} k_\lambda^{(1)}(xy, 1)\widehat{F}_{\lambda,j}(y)dy\right)^p dx\right]^{\frac{1}{p}} \leq k_{\lambda,i}^{(1)}(\mu)||\widehat{F}_{\lambda,j}||_{p,\varphi}, \quad (105)$$

$$||\widehat{F}_{\lambda,j}||_{p,\varphi} \leq k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}(i, j = 0, 1, 2), \quad (106)$$

where the constant factors $k_{\lambda,i}^{(1)}(\mu)(i = 0, 1, 2)$ and $k_{\lambda,j}^{(2)}(\mu)(j = 0, 1, 2)$ are the best possible.

4 Related Operators and Composition Formulas

In this section, we agree on that $p > 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda \in A_i(\neq \Phi), k_\lambda^{(i)}(x, y)$ are homogeneous functions of degree $-\lambda$ in $\mathbf{R}_+^2$,

$$k_\lambda^{(i)}(\mu) = \int_0^\infty k_\lambda^{(i)}(u, 1)u^{\mu-1}du \in \mathbf{R}_+(i = 1, 2),$$

$k_\lambda^{(1)}(x, y)$ is a symmetric function, there exists a constant $\delta_0 > 0$, such that $k_\lambda^{(i)}(\mu \pm \delta_0) \in \mathbf{R}_+(i = 1, 2)$ and $k_\lambda^{(2)}(u, 1)$ satisfies Condition (i) for $\lambda \in A_1 \cap A_2$ or Condition (ii) for $\lambda \in (0, 1) \cap A_1 \cap A_2$, where,

Condition (i). For $\lambda \in A_1 \cap A_2(\neq \Phi)$, there exist constants $\delta_1 \in (0, \delta_0)$ and $L_1 > 0$, such that

$$k_\lambda^{(2)}(u, 1)u^{\mu-\delta_1} \leq L_1(u \in (0, 1)).$$

Condition (ii). For $\lambda \in (0, 1) \cap A_1 \cap A_2 (\neq \Phi)$, there exists a constant $L_2 > 0$, such that

$$k_{\lambda}^{(2)}(u, 1)(1-u)^{\lambda} \leq L_2 (u \in (0, 1)).$$

4.1 A Composition Formula of the Operators Related to Corollary 7

For any $i = 0, 1, 2, F \in L_{p,\phi}(\mathbf{R}_+)$, we set

$$h(x) := x^{\lambda-1} \int_{A_{x,i}} k_{\lambda}^{(1)}(x, y) F(y) dy (x \in \mathbf{R}_+).$$

Then by (75), we have

$$||h||_{p,\phi} \leq k_{\lambda,i}^{(1)}(\mu) ||F||_{p,\phi}. \tag{107}$$

Definition 6. For any $i = 0, 1, 2$, we define an operator

$$T_1^{(i)} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\phi}(\mathbf{R}_+)$$

as follows:

For any $F \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $T_1^{(i)} F = h \in L_{p,\phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+, T_1^{(i)} F(x) = h(x)$.

By (107), we have

$$||T_1^{(i)} F||_{p,\phi} \leq k_{\lambda,i}^{(1)}(\mu) ||F||_{p,\phi}.$$

Hence $T_1^{(i)}$ is a bounded linear operator with

$$||T_1^{(i)}|| := \sup_{F(\neq\theta) \in L_{p,\phi}(\mathbf{R}_+)} \frac{||T_1^{(i)} F||_{p,\phi}}{||F||_{p,\phi}} \leq k_{\lambda,i}^{(1)}(\mu).$$

Since the constant factor in (107) is the best possible, we have

$$||T_1^{(i)}|| = k_{\lambda,i}^{(1)}(\mu)(i = 0, 1, 2).$$

Definition 7. For any $j = 0, 1, 2$, we define an operator

$$T_2^{(j)} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\phi}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $T_2^{(j)}f = F_{\lambda,j} \in L_{p,\phi}(\mathbf{R}_+)$, such that for any $y \in \mathbf{R}_+$, $T_2^{(j)}f(y) = F_{\lambda,j}(y)$.

By (76), we have

$$||T_2^{(j)}f||_{p,\phi} = ||F_{\lambda,j}||_{p,\phi} \leq k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}.$$

Hence $T_2^{(j)}$ is a bounded linear operator with

$$||T_2^{(j)}|| := \sup_{f(\neq\theta)\in L_{p,\phi}(\mathbf{R}_+)} \frac{||T_2^{(j)}f||_{p,\phi}}{||f||_{p,\phi}} \leq k_{\lambda,j}^{(2)}(\mu).$$

Since the constant factor in (76) is the best possible, we have

$$||T_2^{(j)}|| = k_{\lambda,j}^{(2)}(\mu)(j = 0, 1, 2).$$

Definition 8. For any $i, j = 0, 1, 2$, we define a compositional Yang-Hilbert-type operator

$$T_{i,j} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\phi}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $T_{i,j}f = T_1^{(i)}F_{\lambda,j} \in L_{p,\phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+$,

$$T_{i,j}f(x) = T_1^{(i)}F_{\lambda,j}(x) = x^{\lambda-1}\int_{A_{x,i}} k_\lambda^{(1)}(x,y)F_{\lambda,j}(y)dy. \tag{108}$$

It is evident that

$$T_{i,j}f = T_1^{(i)}F_{\lambda,j} = T_1^{(i)}(T_2^{(j)}f) = (T_1^{(i)}T_2^{(j)})f,$$

and then $T_{i,j} = T_1^{(i)}T_2^{(j)}$. Hence, $T_{i,j}$ is a composition of $T_1^{(i)}$ and $T_2^{(j)}$, and (cf. [30])

$$||T_{i,j}|| = ||T_1^{(i)}T_2^{(j)}|| \leq ||T_1^{(i)}|| \cdot ||T_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu).$$

By (74), we have

$$||T_{i,j}f||_{p,\varphi} = ||T_1^{(i)}F_{\lambda,j}||_{p,\phi} \leq k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}.$$

Since the constant factor in (74) is the best possible, it follows that

Theorem 9. *For any $i, j = 0, 1, 2$, if $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, then we have*

$$||T_{i,j}|| = ||T_1^{(i)}T_2^{(j)}|| = ||T_1^{(i)}|| \cdot ||T_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu). \tag{109}$$

4.2 A Composition Formula of the Operators Related to Corollary 10

For any $i = 0, 1, 2, F \in L_{p,\phi}(\mathbf{R}_+)$, we set

$$\tilde{h}(x) := x^{\lambda-1} \int_{B_{x,i}} k_\lambda^{(1)}(xy, 1)F(y)dy(x \in \mathbf{R}_+).$$

Then by (85), we have

$$||\tilde{h}||_{p,\phi} \leq k_{\lambda,i}^{(1)}(\mu)||F||_{p,\phi}. \tag{110}$$

Definition 9. For any $i = 0, 1, 2$, we define an operator

$$\widetilde{T}_1^{(i)} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\phi}(\mathbf{R}_+)$$

as follows:

For any $F \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $\widetilde{T}_1^{(i)}F = \tilde{h} \in L_{p,\phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+, \widetilde{T}_1^{(i)}F(x) = \tilde{h}(x)$.

By (110), we have

$$||\widetilde{T}_1^{(i)}F||_{p,\phi} \leq k_{\lambda,i}^{(1)}(\mu)||F||_{p,\phi}.$$

Hence, $\widetilde{T}_1^{(i)}$ is a bounded linear operator with

$$||\widetilde{T}_1^{(i)}|| := \sup_{F(\neq\theta)\in L_{p,\phi}(\mathbf{R}_+)} \frac{||\widetilde{T}_1^{(i)}F||_{p,\phi}}{||F||_{p,\phi}} \leq k_{\lambda,i}^{(1)}(\mu).$$

Since the constant factor in (110) is the best possible, we have

$$||\widetilde{T}_1^{(i)}|| = k_{\lambda,i}^{(1)}(\mu)(i = 0, 1, 2).$$

For any $j = 0, 1, 2$, we define $T_2^{(j)}$ as Definition 7 and then obtain

$$||T_2^{(j)}|| = k_{\lambda,j}^{(2)}(\mu).$$

Definition 10. For any $i, j = 0, 1, 2$, we define a Yang-Hilbert-type operator

$$\widetilde{T}_{i,j} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\phi}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $\widetilde{T}_{i,j}f = \widetilde{T}_1^{(i)}F_{\lambda,j} \in L_{p,\phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+$,

$$\widetilde{T}_{i,j}f(x) = \widetilde{T}_1^{(i)}F_{\lambda,j}(x) = x^{\lambda-1}\int_{B_{x,i}} k_\lambda^{(1)}(xy,1)F_{\lambda,j}(y)dy. \tag{111}$$

It is evident that

$$\widetilde{T}_{i,j}f = \widetilde{T}_1^{(i)}F_{\lambda,j} = \widetilde{T}_1^{(i)}(T_2^{(j)}f) = (\widetilde{T}_1^{(i)}T_2^{(j)})f,$$

and then $\widetilde{T}_{i,j} = \widetilde{T}_1^{(i)}T_2^{(j)}$. Hence, $\widetilde{T}_{i,j}$ is a composition of $\widetilde{T}_1^{(i)}$ and $T_2^{(j)}$, and

$$||\widetilde{T}_{i,j}|| = ||\widetilde{T}_1^{(i)}T_2^{(j)}|| \le ||\widetilde{T}_1^{(i)}|| \cdot ||T_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu).$$

By (84), we have

$$||\widetilde{T}_{i,j}f||_{p,\varphi} = ||\widetilde{T}_1^{(i)}F_{\lambda,j}||_{p,\varphi} \le k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}.$$

Since the constant factor in (84) is the best possible, it follows that

Theorem 10. *For any $i,j = 0,1,2$, if $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, then we have*

$$||\widetilde{T}_{i,j}|| = ||\widetilde{T}_1^{(i)}T_2^{(j)}|| = ||\widetilde{T}_1^{(i)}|| \cdot ||T_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu). \tag{112}$$

4.3 A Composition Formula of the Operators Related to Corollary 13

For any $i = 0,1,2$, we define $T_1^{(i)}$ as Definition 6 and then obtain

$$||T_1^{(i)}|| = k_{\lambda,i}^{(1)}(\mu).$$

Definition 11. For any $j = 0,1,2$, we define an operator

$$\widehat{T}_2^{(j)} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\varphi}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $\widehat{T}_2^{(j)}f = \widehat{F}_{\lambda,j} \in L_{p,\varphi}(\mathbf{R}_+)$, such that for any $y \in \mathbf{R}_+$, $\widehat{T}_2^{(j)}f(y) = \widehat{F}_{\lambda,j}(y)$.

By (96), we have

$$||\widehat{T}_2^{(j)}f||_{p,\varphi} = ||\widehat{F}_{\lambda,j}||_{p,\varphi} \le k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}.$$

Hence $\widehat{T}_2^{(j)}$ is a bounded linear operator with

$$||\widehat{T}_2^{(j)}|| := \sup_{f(\neq\theta)\in L_{p,\phi}(\mathbf{R}_+)} \frac{||\widehat{T}_2^{(j)}f||_{p,\varphi}}{||f||_{p,\phi}} \leq k_{\lambda,j}^{(2)}(\mu).$$

Since the constant factor in (96) is the best possible, we have

$$||\widehat{T}_2^{(j)}|| = k_{\lambda,j}^{(2)}(\mu)(j = 0, 1, 2).$$

Definition 12. For any $i, j = 0, 1, 2$, we define a compositional Yang-Hilbert-type operator

$$\widehat{T}_{i,j} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\phi}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $\widehat{T}_{i,j}f = T_1^{(i)}\widehat{F}_{\lambda,j} \in L_{p,\phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+$,

$$\widehat{T}_{i,j}f(x) = T_1^{(i)}\widehat{F}_{\lambda,j}(x) = x^{\lambda-1}\int_{A_{x,i}} k_\lambda^{(1)}(x,y)\widehat{F}_{\lambda,j}(y)dy. \tag{113}$$

It is evident that

$$\widehat{T}_{i,j}f = T_1^{(i)}\widehat{F}_{\lambda,j} = T_1^{(i)}(\widehat{T}_2^{(j)}f) = (T_1^{(i)}\widehat{T}_2^{(j)})f,$$

and then $\widehat{T}_{i,j} = T_1^{(i)}\widehat{T}_2^{(j)}$. Hence, $\widehat{T}_{i,j}$ is a composition of $T_1^{(i)}$ and $\widehat{T}_2^{(j)}$, and (cf. [30])

$$||\widehat{T}_{i,j}|| = ||T_1^{(i)}\widehat{T}_2^{(j)}|| \leq ||T_1^{(i)}|| \cdot ||\widehat{T}_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu).$$

By (94), we have

$$||\widehat{T}_{i,j}f||_{p,\phi} = ||T_1^{(i)}\widehat{F}_{\lambda,j}||_{p,\phi} \leq k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}.$$

Since the constant factor in (94) is the best possible, it follows that

Theorem 11. *For any $i, j = 0, 1, 2$, if $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, then we have*

$$||\widehat{T}_{i,j}|| = ||T_1^{(i)}\widehat{T}_2^{(j)}|| = ||T_1^{(i)}|| \cdot ||\widehat{T}_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu). \tag{114}$$

4.4 A Composition Formula of the Operators Related to Corollary 16

For any $i = 0, 1, 2$, we define $\widetilde{T}_1^{(i)}$ as Definition 9, and then obtain

$$||\widetilde{T}_1^{(i)}|| = k_{\lambda,i}^{(1)}(\mu).$$

Also for any $j = 0, 1, 2$, we define $\widehat{T}_2^{(j)}$ as Definition 11, and then obtain

$$||\widehat{T}_2^{(j)}|| = k_{\lambda,j}^{(2)}(\mu).$$

Definition 13. For any $i, j = 0, 1, 2$, we define a compositional Yang-Hilbert-type operator

$$\overline{T}_{i,j} : L_{p,\phi}(\mathbf{R}_+) \to L_{p,\phi}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unified expression $\overline{T}_{i,j}f = \widetilde{T}_1^{(i)}\widehat{F}_{\lambda,j} \in L_{p,\phi}(\mathbf{R}_+)$, such that for any $x \in \mathbf{R}_+$,

$$\overline{T}_{i,j}f(x) = \widetilde{T}_1^{(i)}\widehat{F}_{\lambda,j}(x) = x^{\lambda-1}\int_{B_{x,i}} k_\lambda^{(1)}(xy, 1)\widehat{F}_{\lambda,j}(y)dy. \tag{115}$$

It is evident that

$$\overline{T}_{i,j}f = \widetilde{T}_1^{(i)}\widehat{F}_{\lambda,j} = \widetilde{T}_1^{(i)}(\widehat{T}_2^{(j)}f) = (\widetilde{T}_1^{(i)}\widehat{T}_2^{(j)})f,$$

and then $\overline{T}_{i,j} = \widetilde{T}_1^{(i)}\widehat{T}_2^{(j)}$. Hence, $\overline{T}_{i,j}$ is a composition of $\widetilde{T}_1^{(i)}$ and $\widehat{T}_2^{(j)}$, and

$$||\overline{T}_{i,j}|| = ||\widetilde{T}_1^{(i)}\widehat{T}_2^{(j)}|| \leq ||\widetilde{T}_1^{(i)}|| \cdot ||\widehat{T}_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu).$$

By (104), we have

$$||\overline{T}_{i,j}f||_{p,\phi} = ||\widetilde{T}_1^{(i)}\widehat{F}_{\lambda,j}||_{p,\phi} \leq k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)||f||_{p,\phi}.$$

Since the constant factor in (104) is the best possible, it follows that

Theorem 12. *For any $i, j = 0, 1, 2$, if $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, then we have*

$$||\overline{T}_{i,j}|| = ||\widetilde{T}_1^{(i)}\widehat{T}_2^{(j)}|| = ||\widetilde{T}_1^{(i)}|| \cdot ||\widehat{T}_2^{(j)}|| = k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu). \tag{116}$$

5 A Second Kind of Compositional Yang-Hilbert-Type Integral Inequality

In this section, by introducing a few lemmas and four conditions, we obtain a second kind of compositional Yang-Hilbert-type integral inequality with a best possible constant factor and the reverse. Some corollaries are also deduced. We agree on that $p > 0 (p \neq 1), \frac{1}{p} + \frac{1}{q} = 1,\ \mu, \sigma \in \mathbf{R},\ \mu + \sigma = \lambda$ in the following.

5.1 Some Lemmas

Similarly, by Lemma 3, we have

Lemma 5. *Suppose that* $\lambda \in A_3 (\neq \Phi) \subset \mathbf{R},\ k_\lambda^{(3)}(x, y)$ *is a homogeneous function of degree* $-\lambda$ *in* $\mathbf{R}_+^2$,

$$k_\lambda^{(3)}(\mu) := \int_0^\infty k_\lambda^{(3)}(u, 1) u^{\mu - 1} du, \tag{117}$$

there exists a constant $\delta_0 > 0$, *such that* $k_\lambda^{(3)}(\mu \pm \delta_0) \in \mathbf{R}_+$. *Then for any* $\delta \in [0, \delta_0)$, *we have* $k_\lambda^{(3)}(\mu \pm \delta) \in \mathbf{R}_+$, *and*

$$\lim_{\delta \to 0^+} k_\lambda^{(3)}(\mu \pm \delta) = k_\lambda^{(3)}(\mu). \tag{118}$$

On the assumptions of Lemmas 3 and 5, we set the following four conditions:

Condition (a). For any $\lambda \in \bigcap_{i=1}^3 A_i (\neq \Phi)$, there exist constants $\delta_1 \in (0, \delta_0)$ and $L_1 > 0$, such that

$$k_\lambda^{(2)}(u, 1) u^{\mu - \delta_1} \leq L_1 (u \in (0, 1)),$$

$$k_\lambda^{(3)}(u, 1) u^{\mu + \delta_1} \leq L_1 (u \in (1, \infty)). \tag{119}$$

Condition (b). For any $\lambda \in \bigcap_{i=1}^3 A_i \cap (0, 1) (\neq \Phi)$, there exists a constant $L_2 > 0$, such that

$$k_\lambda^{(2)}(u, 1)(1 - u)^\lambda \leq L_2 (u \in (0, 1)),$$

$$k_\lambda^{(3)}(u, 1)(u - 1)^\lambda \leq L_2 (u \in (1, \infty)). \tag{120}$$

Condition (c). For any $\lambda \in \bigcap_{i=1}^3 A_i \cap (0, 1) (\neq \Phi)$, there exist constants $a \in (0, \lambda)$ and $L_3 > 0$, such that

$$k_\lambda^{(1)}(u, 1) u^a \leq L_3 (u \in (0, \infty)). \tag{121}$$

Condition (d). For any $\lambda \in \bigcap_{i=1}^{3} A_i \cap (0, \frac{2}{3})(\neq \Phi)$, there exists a constant $L_4 > 0$, such that

$$k_{\lambda}^{(1)}(u,1)|1-u|^{\lambda} \leq L_4(u \in (0,\infty)). \tag{122}$$

Example 8. (i) For $\lambda \in A_1 = A_2 = \mathbf{R}_+$, $0 < \delta_0 < \min\{\mu, \sigma\}(\mu, \sigma > 0)$, the functions

$$k_{\lambda}^{(3)}(u,1) = \frac{1}{(u+1)^{\lambda}}, \frac{1}{u^{\lambda}+1}, \frac{\ln u}{u^{\lambda}-1}, \frac{|\ln u|^{\beta}}{(\max\{u,1\})^{\lambda}}(\beta \geq 0)$$

satisfy for using Condition (a); for $\lambda = 0, 0 < \delta_0 < \eta - \mu(|\mu| < \eta), \beta \geq 0$, the function

$$k_0^{(3)}(u,1) = \left(\frac{\min\{u,1\}}{\max\{u,1\}}\right)^{\eta} |\ln u|^{\beta}$$

also satisfies for using Condition (a). In fact, for $\delta_1 \in (0, \delta_0)$, we obtain

$$\lim_{u \to \infty} k_{\lambda}^{(3)}(u,1)u^{\mu+\delta_1} = 0.$$

In view of the continuity, $k_{\lambda}^{(3)}(u,1)u^{\mu+\delta_1}$ is bounded in $(1,\infty)$.

(ii) For $\lambda \in A_1 \cap A_2 = (0,1)$, the function

$$k_{\lambda}^{(3)}(u,1) = \frac{1}{|u-1|^{\lambda}}$$

satisfies for using Condition (b).

Definition 14. On the assumptions of Lemma 3 and Lemma 5, define the following real functions:

$$\widetilde{F}_k(y) = \begin{cases} y^{\lambda-1} \int_1^{\infty} k_{\lambda}^{(2)}(x,y)x^{\mu-\frac{1}{pk}-1}dx, y \in (1,\infty), \\ 0, y \in (0,1], \end{cases}$$

$$\widetilde{G}_k(x) := \begin{cases} x^{\lambda-1} \int_1^{\infty} k_{\lambda}^{(3)}(x,y)y^{\sigma-\frac{1}{qk}-1}dy, x \in (1,\infty), \\ 0, x \in (0,1], \end{cases}$$

where $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\}(k \in \mathbf{N})$.

Setting $u = x/y(y > 1)$, we find

$$\widetilde{F}_k(y) = y^{\mu-\frac{1}{pk}-1}k_{\lambda}^{(2)}(\mu - \frac{1}{pk}) - F_k(y),$$

$$F_k(y) = y^{\mu-\frac{1}{pk}-1} \int_0^{\frac{1}{y}} k_{\lambda}^{(2)}(u,1)u^{\mu-\frac{1}{pk}-1}du(y \in (1,\infty)). \tag{123}$$

Still setting $u = x/y (x > 1)$, we obtain

$$\begin{aligned}\widetilde{G}_k(x) &= x^{\sigma-\frac{1}{qk}-1}\int_0^x k_\lambda^{(3)}(u,1)u^{\mu+\frac{1}{qk}-1}du \\ &= x^{\sigma-\frac{1}{qk}-1}\left[\int_0^\infty k_\lambda^{(3)}(u,1)u^{(\mu+\frac{1}{qk})-1}du - \int_x^\infty k_\lambda^{(3)}(u,1)u^{\mu+\frac{1}{qk}-1}du\right] \\ &= x^{\sigma-\frac{1}{qk}-1}k_\lambda^{(3)}(\mu+\frac{1}{qk}) - G_k(x),\end{aligned}$$

$$G_k(x) := x^{\sigma-\frac{1}{qk}-1}\int_x^\infty k_\lambda^{(3)}(u,1)u^{\mu+\frac{1}{qk}-1}du (x \in (1,\infty)). \tag{124}$$

In the following, we use Condition (a) and Condition (b) to estimate $F_k(y)$ and $G_k(x)$:

(i) If $k_\lambda^{(2)}(u,1)$ satisfies Condition (a), then by (119), we have

$$\begin{aligned}0 \le F_k(y) &\le L_1 y^{\mu-\frac{1}{pk}-1}\int_0^{\frac{1}{y}} u^{-\mu+\delta_1}u^{\mu-\frac{1}{pk}-1}du \\ &= \frac{L_1 y^{\mu-\delta_1-1}}{\delta_1-\frac{1}{pk}}(y \in (1,\infty))\end{aligned}$$

(ii) If $k_\lambda^{(2)}(u,1)$ satisfies Condition (b), then by (120), we have

$$\begin{aligned}0 \le F_k(y) &\le L_2 y^{\mu-\frac{1}{pk}-1}\int_0^{\frac{1}{y}} \frac{u^{\mu-\frac{1}{pk}-1}}{(1-u)^\lambda}du \\ &= L_2 y^{\lambda-1}\int_0^1 \frac{v^{\mu-\frac{1}{pk}-1}}{(y-v)^\lambda}dv \le \frac{L_2 y^{\lambda-1}}{(y-1)^\lambda}\int_0^1 v^{\mu-\frac{1}{pk}-1}dv \\ &= \frac{L_2}{\mu-\frac{1}{pk}}\frac{y^{\lambda-1}}{(y-1)^\lambda}(y \in (1,\infty))\end{aligned}$$

(iii) If $k_\lambda^{(3)}(u,1)$ satisfies Condition (a), then by (119), we have

$$\begin{aligned}0 \le G_k(x) &\le L_1 x^{\sigma-\frac{1}{qk}-1}\int_x^\infty u^{-\mu-\delta_1}u^{\mu+\frac{1}{qk}-1}du \\ &= \frac{L_1 x^{\sigma-\delta_1-1}}{\delta_1-\frac{1}{qk}}(x \in (1,\infty))\end{aligned}$$

(iv) If $k_\lambda^{(3)}(u,1)$ satisfies Condition (b), then by (120), we have

$$0 \leq G_k(x) \leq L_2 x^{\sigma-\frac{1}{qk}-1}\int_x^\infty \frac{u^{\mu+\frac{1}{qk}-1}}{(u-1)^\lambda}du$$
$$= L_2 x^{\lambda-1}\int_0^1 \frac{v^{\sigma-\frac{1}{qk}-1}}{(x-v)^\lambda}dv \leq \frac{L_2 x^{\lambda-1}}{(x-1)^\lambda}\int_0^1 v^{\sigma-\frac{1}{qk}-1}dv$$
$$= \frac{L_2 x^{\lambda-1}}{(\sigma-\frac{1}{qk})(x-1)^\lambda}(x\in(1,\infty)).$$

Remark 9. In view of the cases (i)–(iv), there exists a large constant $L>0$, such that

(i) $F_k(y) \leq L y^{\mu-\delta_1-1}(y\in(1,\infty);\lambda\in\bigcap_{i=1}^3 A_i)$
(ii) $F_k(y) \leq L\frac{y^{\lambda-1}}{(y-1)^\lambda}(y\in(1,\infty);\lambda\in\bigcap_{i=1}^3 A_i\bigcap(0,1))$
(iii) $G_k(x) \leq L x^{\sigma-\delta_1-1}(x\in(1,\infty);\lambda\in\bigcap_{i=1}^3 A_i)$
(iv) $G_k(x) \leq L\frac{x^{\lambda-1}}{(x-1)^\lambda}(x\in(1,\infty);\lambda\in\bigcap_{i=1}^3 A_i\bigcap(0,1))$

Lemma 6. *On the assumptions of Lemma 3 and Lemma 5, if $k_\lambda^{(1)}(x,y)$ is a symmetric function and $k_\lambda^{(2)}(u,1)(k_\lambda^{(3)}(u,1))$ satisfies Condition (a) for $\lambda\in\bigcap_{i=1}^3 A_i(\neq\Phi)$ or Condition (b) for $\lambda\in\cap_{i=1}^3 A_i\bigcap(0,1)(\neq\Phi)$, then we have*

$$\widetilde{L}_k := \frac{1}{k}\int_0^\infty\int_0^\infty k_\lambda^{(1)}(x,y)\widetilde{F}_k(y)\widetilde{G}_k(x)dydx$$
$$\geq \prod_{i=1}^3 k_\lambda^{(i)}(\mu) + o(1)(k\to\infty). \tag{125}$$

Proof. In view of (123) and (124), we have

$$\widetilde{L}_k = \frac{1}{k}\int_1^\infty\int_1^\infty k_\lambda^{(1)}(x,y)\left(y^{\mu-\frac{1}{pk}-1}k_\lambda^{(2)}(\mu-\frac{1}{pk})-F_k(y)\right)$$
$$\times\left(x^{\sigma-\frac{1}{qk}-1}k_\lambda^{(3)}(\mu+\frac{1}{qk})-G_k(x)\right)dydx$$
$$= I_1 - I_2 - I_3 + I_4, \tag{126}$$

where we define

$$I_1 := \frac{1}{k}k_\lambda^{(2)}(\mu-\frac{1}{pk})k_\lambda^{(3)}(\mu+\frac{1}{qk})$$
$$\times\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right)x^{\sigma-\frac{1}{qk}-1}dx,$$
$$I_2 := \frac{1}{k}k_\lambda^{(3)}(\mu+\frac{1}{qk})\int_1^\infty\left(\int_1^\infty k_\lambda^{(1)}(x,y)F_k(y)dy\right)x^{\sigma-\frac{1}{qk}-1}dx,$$

$$I_3 := \frac{1}{k}k_\lambda^{(2)}(\mu - \frac{1}{pk})\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right) G_k(x)dx,$$
$$I_4 := \frac{1}{k}\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)F_k(y)dy\right) G_k(x)dx.$$

It is evident that

$$I_1 - I_2 - I_3 \leq \widetilde{L}_k \leq I_1 + I_4. \tag{127}$$

By the proof of Lemmas 4 and 5, we find

$$\begin{aligned} I_1 &= k_\lambda^{(2)}(\mu - \frac{1}{pk})k_\lambda^{(3)}(\mu + \frac{1}{qk}) \\ &\quad \times \left(\int_0^1 k_\lambda^{(1)}(u,1)u^{\mu+\frac{1}{qk}-1}du + \int_1^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right) \\ &\quad \to \prod_{i=1}^3 k_\lambda^{(i)}(\mu)(k \to \infty), \end{aligned} \tag{128}$$

and $I_2 \to 0(k \to \infty)$.

We estimate I_3. (i) If $k_\lambda^{(3)}(u,1)$ satisfies Condition (a), then by Remark 9(iii), we have

$$\begin{aligned} 0 \leq J_3 &:= \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x,y)y^{\mu-\frac{1}{pk}-1}dy\right) G_k(x)dx \\ &\leq L\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\frac{1}{pk}-1}dy\right) x^{\sigma-\delta_1-1}dx \\ &= L\int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right) x^{-\delta_1-\frac{1}{pk}-1}dx \\ &\leq L\int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right) x^{-\delta_1-\frac{1}{pk}-1}dx \\ &= \frac{L}{\delta_1 + \frac{1}{pk}}k_\lambda^{(1)}(\mu - \frac{1}{pk}) < \infty; \end{aligned}$$

(ii) If $k_\lambda^{(3)}(u,1)$ satisfies Condition (b), then by Remark 9(iv), we have

$$\begin{aligned} 0 \leq J_3 &\leq L\int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x)y^{\mu-\frac{1}{pk}-1}dy\right) \frac{x^{\lambda-1}}{(x-1)^\lambda}dx \\ &= L\int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1)u^{\mu-\frac{1}{pk}-1}du\right) \frac{x^{\mu-\frac{1}{pk}-1}}{(x-1)^\lambda}dx \end{aligned}$$

$$\leq L \int_0^1 \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu - \frac{1}{pk} - 1} du \right) \frac{v^{\sigma + \frac{1}{pk} - 1}}{(1-v)^\lambda} dv$$
$$= L k_\lambda^{(1)}(\mu - \frac{1}{pk}) B(1-\lambda, \sigma + \frac{1}{pk}) < \infty.$$

Therefore, in view of (i) and (ii), we have $I_3 \to 0 (k \to \infty)$.
By (127) and the above results, (125) follows.
The lemma is proved. □

Lemma 7. *On the assumptions of Lemma 3 and Lemma 5, if* $k_\lambda^{(1)}(x, y)$ *is a symmetric function,* $k_\lambda^{(2)}(u, 1)(k_\lambda^{(3)}(u, 1))$ *satisfies Condition (a) for* $\lambda \in \bigcap_{i=1}^3 A_i (\neq \Phi)$ *or Condition (b) for* $\lambda \in \bigcap_{i=1}^3 A_i \bigcap (0,1) (\neq \Phi)$ *and if both* $k_\lambda^{(2)}(u, 1)$ *and* $k_\lambda^{(3)}(u, 1)$ *satisfy Condition (b), then* $k_\lambda^{(1)}(u, 1)$ *satisfies Condition (c) for* $\lambda \in \bigcap_{i=1}^3 A_i \bigcap (0,1) (\neq \Phi)$ *or Condition (d) for* $\lambda \in \bigcap_{i=1}^3 A_i \bigcap (0, \frac{2}{3}) (\neq \Phi)$*, then we have the reverse of (125), namely,*

$$\widetilde{L}_k = \frac{1}{k} \int_0^\infty \int_0^\infty k_\lambda^{(1)}(x, y) \widetilde{F}_k(y) \widetilde{G}_k(x) dy dx$$
$$= \prod_{i=1}^3 k_\lambda^{(i)}(\mu) + o(1)(k \to \infty). \tag{129}$$

Proof. By Remark 9, Condition (c) and Condition (d), we divide five cases to show that $I_4 \to 0 (k \to \infty)$.

Case (i). For $F_k(y) \leq L y^{\mu - \delta_1 - 1}$, $G_k(x) \leq L x^{\sigma - \delta_1 - 1} (y, x \in (1, \infty))$, we have

$$J_4 := \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(x, y) F_k(y) dy \right) G_k(x) dx$$
$$\leq L^2 \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y, x) y^{\mu - \delta_1 - 1} dy \right) x^{\sigma - \delta_1 - 1} dx$$
$$= L^2 \int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u, 1) u^{\mu - \delta_1 - 1} du \right) x^{-2\delta_1 - 1} dx$$
$$\leq L^2 \int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u, 1) u^{\mu - \delta_1 - 1} du \right) x^{-2\delta_1 - 1} dx$$
$$= \frac{L^2}{2\delta_1} k_\lambda^{(1)}(\mu - \delta_1) < \infty.$$

Case (ii). For $F_k(y) \le Ly^{\mu-\delta_1-1},\ G_k(x) \le L\frac{x^{\lambda-1}}{(x-1)^\lambda}(y, x \in (1,\infty))$, we have

$$\begin{aligned}
J_4 &\le L^2 \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x) y^{\mu-\delta_1-1} dy \right) \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&= L^2 \int_1^\infty \left(\int_{\frac{1}{x}}^\infty k_\lambda^{(1)}(u,1) u^{\mu-\delta_1-1} du \right) \frac{x^{\mu-\delta_1-1}}{(x-1)^\lambda} dx \\
&\le L^2 \int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu-\delta_1-1} du \right) \frac{x^{\mu-\delta_1-1}}{(x-1)^\lambda} dx \\
&= L^2 k_\lambda^{(1)}(\mu-\delta_1) B(1-\lambda, \sigma+\delta_1) < \infty (0 < \lambda, \sigma < 1).
\end{aligned}$$

Case (iii). For $F_k(y) \le L\frac{y^{\lambda-1}}{(y-1)^\lambda},\ G_k(x) \le Lx^{\sigma-\delta_1-1}(y, x \in (1,\infty))$, we have

$$\begin{aligned}
J_4 &\le L^2 \int_1^\infty \left(\int_1^\infty k_\lambda^{(1)}(y,x) x^{\sigma-\delta_1-1} dx \right) \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \\
&= L^2 \int_1^\infty \left(\int_0^y k_\lambda^{(1)}(u,1) u^{\mu+\delta_1-1} du \right) \frac{y^{\sigma-\delta_1-1}}{(y-1)^\lambda} dy \\
&\le L^2 \int_1^\infty \left(\int_0^\infty k_\lambda^{(1)}(u,1) u^{\mu+\delta_1-1} du \right) \frac{y^{\sigma-\delta_1-1}}{(y-1)^\lambda} dy \\
&= L^2 k_\lambda^{(1)}(\mu+\delta_1) B(1-\lambda, \mu+\delta_1) < \infty.
\end{aligned}$$

Case (iv). For $F_k(y) \le L\frac{y^{\lambda-1}}{(y-1)^\lambda}, G_k(x) \le L\frac{x^{\lambda-1}}{(x-1)^\lambda}(y, x \in (1,\infty)), k_\lambda^{(1)}(u,1)$ satisfies Condition (c), we have

$$\begin{aligned}
J_4 &\le L^2 \int_1^\infty \left[\int_1^\infty x^{-\lambda} k_\lambda^{(1)}(\frac{y}{x},1) \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right] \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&\le L^2 L_3 \int_1^\infty \left[\int_1^\infty x^{-\lambda} (\frac{y}{x})^{-a} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right] \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&= L^2 L_3 \int_1^\infty \left[\int_1^\infty \frac{y^{\lambda-a-1}}{(y-1)^\lambda} dy \right] \frac{x^{a-1}}{(x-1)^\lambda} dx \\
&= L^2 L_3 B(1-\lambda, a) B(1-\lambda, \lambda-a) < \infty.
\end{aligned}$$

Case (v). For $F_k(y) \le L\frac{y^{\lambda-1}}{(y-1)^\lambda}, G_k(x) \le L\frac{x^{\lambda-1}}{(x-1)^\lambda}(y, x \in (1,\infty)), k_\lambda^{(1)}(u,1)$ satisfies Condition (d), we have

$$
\begin{aligned}
J_4 &\leq L^2 L_4 \int_1^\infty \left[\int_1^\infty \frac{1}{|x-y|^\lambda} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right] \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&= L^2 L_4 \int_1^\infty \left[\int_1^x \frac{1}{(x-y)^\lambda} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right] \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&\quad + L^2 L_4 \int_1^\infty \left[\int_x^\infty \frac{1}{(y-x)^\lambda} \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right] \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&= L^2 L_4 \int_1^\infty \left[\int_y^\infty \frac{1}{(x-y)^\lambda} \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \right] \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \\
&\quad + L^2 L_4 \int_1^\infty \left[\int_x^\infty \frac{1}{(y-x)^\lambda} \cdot \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \right] \frac{x^{\lambda-1}}{(x-1)^\lambda} dx \\
&= 2L^2 L_4 \int_1^\infty \left[\int_1^\infty \frac{1}{(u-1)^\lambda} \frac{u^{\lambda-1}}{(yu-1)^\lambda} du \right] \frac{y^{\lambda-1}}{(y-1)^\lambda} dy \\
&= 2L^2 L_4 \int_1^\infty \left[\int_1^\infty \frac{1}{(u-1)^\lambda} \frac{u^{\lambda-1} du}{(yu-1)^{\lambda/2} (yu-1)^{\lambda/2}} \right] \frac{y^{\lambda-1} dy}{(y-1)^\lambda} \\
&\leq 2L^2 L_4 \int_1^\infty \left[\int_1^\infty \frac{1}{(u-1)^\lambda} \frac{u^{\lambda-1} du}{(u-1)^{\lambda/2} (y-1)^{\lambda/2}} \right] \frac{y^{\lambda-1} dy}{(y-1)^\lambda} \\
&= 2L^2 L_4 \left[\int_1^\infty \frac{y^{\lambda-1}}{(y-1)^{(3\lambda)/2}} dy \right]^2 \\
&= 2L^2 L_4 (B(1 - \frac{3\lambda}{2}, \frac{\lambda}{2}))^2 < \infty.
\end{aligned}
$$

Hence, in the above any case, $I_4 = \frac{1}{k} J_4 \to 0 (k \to \infty)$.

Therefore, by (127) and (128), we have the reverse of (125), and then (129) follows.

The lemma is proved. □

5.2 Main Results

We set functions $\phi(x) := x^{p(1-\mu)-1}, \psi(y) := y^{q(1-\sigma)-1} (x, y \in \mathbf{R}_+)$ in the following.

Theorem 13. *Suppose that (i) $\lambda \in A_i (\neq \Phi) \subset \mathbf{R}$, $k_\lambda^{(i)}(x, y)$ are homogeneous functions of degree $-\lambda$ in $\mathbf{R}_+^2$,*

$$
k_\lambda^{(i)}(\mu) = \int_0^\infty k_\lambda^{(i)}(u, 1) u^{\mu-1} du,
$$

there exists a constant $\delta_0 > 0$*, such that* $k_\lambda^{(i)}(\mu \pm \delta_0) \in \mathbf{R}_+ (i = 1,2,3)$*; (ii)* $k_\lambda^{(i)}(x,y)(i = 1,3)$ *are symmetric functions; (iii)* $k_\lambda^{(2)}(u,1)(k_\lambda^{(3)}(u,1))$ *satisfies Condition (a) for* $\lambda \in \bigcap_{i=1}^3 A_i (\neq \Phi)$ *or Condition (b) for* $\lambda \in \bigcap_{i=1}^3 A_i \bigcap (0,1)$ $(\neq \Phi)$*. Then for* $p > 1, f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+),\ g \in L_{q,\psi}(\mathbf{R}_+),\ ||f||_{p,\phi}, ||g||_{q,\psi} > 0$*, setting*

$$F_\lambda(y) = \begin{cases} y^{\lambda-1} \int_0^\infty k_\lambda^{(2)}(x,y) f(x)dx, y \in \{y \in \mathbf{R}_+; f(y) > 0\}, \\ 0, y \in \{y \in \mathbf{R}_+; f(y) = 0\}, \end{cases}$$

$$G_\lambda(x) := \begin{cases} x^{\lambda-1} \int_0^\infty k_\lambda^{(3)}(x,y) g(y)dy, x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

we have the following second kind of compositional Yang-Hilbert-type inequality:

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y) F_\lambda(y) G_\lambda(x) dydx < \prod_{i=1}^3 k_\lambda^{(i)}(\mu) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{130}$$

where the constant factor $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ *is the best possible.*

Proof. By (56), since $k_\lambda^{(3)}(x,y)$ is symmetric, we find

$$\begin{aligned} ||G_\lambda||_{q,\psi} &\leq \left[\int_0^\infty x^{q(1-\sigma)-1} \left(x^{\lambda-1} \int_0^\infty k_\lambda^{(3)}(x,y) g(y)dy\right)^q dx\right]^{\frac{1}{q}} \\ &= \left[\int_0^\infty x^{q\mu-1} \left(\int_0^\infty k_\lambda^{(3)}(x,y) g(y)dy\right)^q dx\right]^{\frac{1}{q}} \\ &= \left[\int_0^\infty x^{q\mu-1} \left(\int_0^\infty k_\lambda^{(3)}(y,x) g(y)dy\right)^q dx\right]^{\frac{1}{q}} \\ &< k_\lambda^{(3)}(\mu) ||g||_{q,\psi}. \end{aligned} \tag{131}$$

By (53), we still have

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y) F_\lambda(y) G_\lambda(x) dydx \leq \prod_{i=1}^2 k_\lambda^{(i)}(\mu) ||f||_{p,\phi} ||G_\lambda||_{q,\psi}. \tag{132}$$

Then by (132) and (131), we have (130).

For any $k > \max\{\frac{1}{q\delta_1}, \frac{1}{p\delta_1}\} (k \in \mathbf{N})$, we set $\tilde{f}(x) = \tilde{g}(y) = 0 (x, y \in (0,1]); \tilde{f}(x) = x^{\mu - \frac{1}{pk} - 1},\ \tilde{g}(y) = y^{\sigma - \frac{1}{qk} - 1} (x, y \in (1, \infty))$. Then we have $\widetilde{F}_k(y) = \widetilde{G}_k(x) = 0 (x, y \in (0,1])$;

$$\widetilde{F}_k(y) = y^{\lambda-1}\int_1^\infty k_\lambda^{(2)}(x,y)x^{\mu-\frac{1}{pk}-1}dx$$
$$= y^{\lambda-1}\int_0^\infty k_\lambda^{(2)}(x,y)\tilde{f}(x)dx(y \in (1,\infty)),$$
$$\widetilde{G}_k(x) = x^{\lambda-1}\int_1^\infty k_\lambda^{(3)}(x,y)y^{\sigma-\frac{1}{qk}-1}dy$$
$$= x^{\lambda-1}\int_0^\infty k_\lambda^{(3)}(x,y)\tilde{g}(y)dy(x \in (1,\infty)).$$

If there exists a positive constant $K \leq \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$, such that (130) is valid when replacing $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ to K, then in particular, we have

$$\widetilde{L}_k = \frac{1}{k}\int_0^\infty\int_0^\infty k_\lambda^{(1)}(x,y)\widetilde{F}_k(y)\widetilde{G}_k(x)dydx$$
$$< \frac{1}{k}K||\tilde{f}||_{p,\phi}||\tilde{g}||_{q,\psi} = \frac{1}{k}K\int_1^\infty x^{-\frac{1}{k}-1}dx = K.$$

By (125), we find

$$\prod_{i=1}^3 k_\lambda^{(i)}(\mu) + o(1) \leq \widetilde{L}_k < K,$$

and then $\prod_{i=1}^3 k_\lambda^{(i)}(\mu) \leq K(k \to \infty)$. Hence, $K = \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ is the best possible constant factor of (130).

The theorem is proved. □

Theorem 14. *Suppose that (i)* $\lambda \in A_i(\neq \Phi) \subset \mathbf{R}$, $k_\lambda^{(i)}(x,y)$ *are homogeneous functions of degree* $-\lambda$ *in* $\mathbf{R}_+^2$, *there exists a constant* $\delta_0 > 0$, *such that* $k_\lambda^{(i)}(\mu \pm \delta_0) \in \mathbf{R}_+(i = 1,2,3)$; *(ii)* $k_\lambda^{(i)}(x,y)(i = 1,3)$ *are symmetric functions; (iii)* $k_\lambda^{(2)}(u,1)(k_\lambda^{(3)}(u,1))$ *satisfies Condition (a) for* $\lambda \in \bigcap_{i=1}^3 A_i(\neq \Phi)$ *or Condition (b) for* $\lambda \in \bigcap_{i=1}^3 A_i \bigcap (0,1)(\neq \Phi)$ *; (iv) if both* $k_\lambda^{(2)}(u,1)$ *and* $k_\lambda^{(3)}(u,1)$ *satisfy Condition (b), then* $k_\lambda^{(1)}(u,1)$ *satisfies Condition (c) for* $\lambda \in \bigcap_{i=1}^3 A_i \cap (0,1)(\neq \Phi)$ *or Condition (d) for* $\lambda \in \bigcap_{i=1}^3 A_i \bigcap (0,\frac{2}{3})(\neq \Phi)$. *Then for* $0 < p < 1, f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}, ||g||_{q,\psi} > 0$, *we have the reverse of (130) with the same best possible constant factor* $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$.

Proof. Since $k_\lambda^{(i)}(y,x) = k_\lambda^{(i)}(x,y)(i = 1,3)$, by the reverse Hölder's inequality, we obtain the reverses of (131) and (132). Then we deduce to the reverse of (130).

For any $k > \max\{\frac{1}{|q|\delta_1}, \frac{1}{p\delta_1}\}(k \in \mathbf{N})$, we set $\tilde{f}(x), \tilde{g}(y), \widetilde{F}_\lambda(y), \widetilde{G}_\lambda(x)$ as Theorem 13. If there exists a positive constant $K \geq \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$, such that the reverse of (130) is valid when replacing $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ to K, then in particular, we have

$$\widetilde{L}_k = \frac{1}{k}\int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y)\widetilde{F}_k(y)\widetilde{G}_k(x)dydx$$
$$> \frac{1}{k}K||\tilde{f}||_{p,\phi}||\tilde{g}||_{q,\psi} = \frac{1}{k}K\int_1^\infty x^{-\frac{1}{k}-1}dx = K.$$

By (129), we find

$$\prod_{i=1}^3 k_\lambda^{(i)}(\mu) + o(1) = \widetilde{L}_k > K,$$

and then $\prod_{i=1}^3 k_\lambda^{(i)}(\mu) \geq K(k \to \infty)$. Hence $K = \prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ is the best possible constant factor of the reverse of (130).

The theorem is proved. □

5.3 Some Corollaries

By (73) and (130), for $p > 1,\ i,j = 0,1,2,\ k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, we still have

$$\int_0^\infty \int_{A_{x,i}} k_\lambda^{(1)}(x,y)F_{\lambda,j}(y)G_\lambda(x)dydx$$
$$< k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_\lambda^{(3)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \tag{133}$$

where $A_{x,0} = (0,\infty), A_{x,1} = (x,\infty), A_{x,2} = (0,x)\ (x > 0)$, and

$$F_{\lambda,j}(y) = \begin{cases} y^{\lambda-1}\int_{A_{x,j}} k_\lambda^{(2)}(x,y)f(x)dx, y \in \{y \in \mathbf{R}_+; f(y) > 0\}, \\ 0, y \in \{y \in \mathbf{R}_+; f(y) = 0\}. \end{cases}$$

If $k_\lambda^{(3)}(x,y) = 0(y \in \mathbf{R}_+\backslash A_{x,1})$, then we find $k_\lambda^{(3)}(u,1) = 0(u \geq 1)$, and

$$k_\lambda^{(3)}(\mu) = k_{\lambda,1}^{(3)}(\mu) := \int_0^1 k_\lambda^{(3)}(u,1)u^{\mu-1}du;$$

if $k_\lambda^{(3)}(x,y) = 0(y \in \mathbf{R}_+\backslash A_{x,2})$, then we find $k_\lambda^{(3)}(u,1) = 0(0 < u \leq 1)$, and

$$k_\lambda^{(3)}(\mu) = k_{\lambda,2}^{(3)}(\mu) := \int_1^\infty k_\lambda^{(3)}(u,1)u^{\mu-1}du.$$

Assuming that $k_{\lambda,0}^{(3)}(\mu) := k_{\lambda}^{(3)}(\mu)$, for $s = 0, 1, 2$, setting

$$G_{\lambda,s}(x) := \begin{cases} x^{\lambda-1} \int_{A_{x,s}} k_{\lambda}^{(3)}(x,y)g(y)dy, x \in \{x \in \mathbf{R}_+; g(x) > 0\}, \\ 0, x \in \{x \in \mathbf{R}_+; g(x) = 0\}, \end{cases}$$

then it follows that $G_{\lambda,0}(x) = G_{\lambda}(x)$. By (133), we have

Corollary 17. *Suppose that for* $i, j, s = 0, 1, 2,\ k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu), k_{\lambda,s}^{(3)}(\mu) \in \mathbf{R}_+$. *(i) For* $p > 1$, *with the assumptions of Theorem 13, we have*

$$\int_0^\infty \int_{A_{x,i}} k_{\lambda}^{(1)}(x,y)F_{\lambda,j}(y)G_{\lambda,s}(x)dydx < k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_{\lambda,s}^{(3)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \tag{134}$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_{\lambda,s}^{(3)}(\mu)$ *is the best possible; (ii) for* $0 < p < 1$, *with the assumptions of Theorem 14, we have the reverse of (134) with the best possible constant factor* $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_{\lambda,s}^{(3)}(\mu)$.

Replacing x to $\frac{1}{x}$ in (130) and the reverse, setting

$$\widehat{G}_{\lambda}(x) := x^{\lambda-2}G_{\lambda}(\frac{1}{x}) = \begin{cases} x^{\lambda-1} \int_0^\infty k_{\lambda}^{(3)}(1,xy)g(y)dy, x \in \{x \in \mathbf{R}_+; g(\frac{1}{x}) > 0\}, \\ 0, x \in \{x \in \mathbf{R}_+; g(\frac{1}{x}) = 0\}, \end{cases}$$

it follows that

Corollary 18. *(i) For* $p > 1$, *with the assumptions of Theorem 13, we have*

$$\int_0^\infty \int_0^\infty k_{\lambda}^{(1)}(xy,1)F_{\lambda}(y)\widehat{G}_{\lambda}(x)dydx < \prod_{i=1}^{3} k_{\lambda}^{(i)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \tag{135}$$

where the constant factor $\prod_{i=1}^{3} k_{\lambda}^{(i)}(\mu)$ *is still the best possible; (ii) for* $0 < p < 1$, *with the assumptions of Theorem 14, we have the reverse of (135) with the best possible constant factor* $\prod_{i=1}^{3} k_{\lambda}^{(i)}(\mu)$.

By (83) and (135), for $p > 1,\ i, j = 0, 1, 2,\ k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, we still have

$$\int_0^\infty \int_{B_{x,i}} k_{\lambda}^{(1)}(xy,1)F_{\lambda,j}(y)\widehat{G}_{\lambda}(x)dydx < k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_{\lambda}^{(3)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \tag{136}$$

where $B_{x,0} = (0, \infty), B_{x,1} = (0, \frac{1}{x}), B_{x,2} = (\frac{1}{x}, \infty)(x > 0)$.

If $k_\lambda^{(3)}(xy,1) = 0 (y \in \mathbf{R}_+ \backslash B_{x,1})$, then we find $k_\lambda^{(3)}(u,1) = 0 (u \geq 1)$, and $k_\lambda^{(3)}(\mu) = k_{\lambda,1}^{(3)}(\mu)$; if $k_\lambda^{(3)}(xy,1) = 0 (y \in \mathbf{R}_+ \backslash B_{x,2})$, then we find $k_\lambda^{(3)}(u,1) = 0 (0 < u \leq 1)$, and $k_\lambda^{(3)}(\mu) = k_{\lambda,2}^{(3)}(\mu)$. Assuming that $k_{\lambda,0}^{(3)}(\mu) := k_\lambda^{(3)}(\mu)$, for $s = 0,1,2$, setting

$$\widehat{G}_{\lambda,s}(x) := \begin{cases} x^{\lambda-1} \int_{B_{x,s}} k_\lambda^{(3)}(xy,1) g(y) dy, x \in \{x \in \mathbf{R}_+; g(\frac{1}{x}) > 0\}, \\ 0, x \in \{x \in \mathbf{R}_+; g(\frac{1}{x}) = 0\}, \end{cases}$$

then it follows that $\widehat{G}_{\lambda,0}(x) = \widehat{G}_\lambda(x)$. By (136), we have

Corollary 19. *Suppose that for* $i,j,s = 0,1,2$, $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu), k_{\lambda,s}^{(3)}(\mu) \in \mathbf{R}_+$. *(i) For* $p > 1$, *with the assumptions of Theorem 13, we have*

$$\int_0^\infty \int_{B_{x,i}} k_\lambda^{(1)}(xy,1) F_{\lambda,j}(y) \widehat{G}_{\lambda,s}(x) dy dx < k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) k_{\lambda,s}^{(3)}(\mu) ||f||_{p,\phi} ||g||_{q,\psi}, \quad (137)$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) k_{\lambda,s}^{(3)}(\mu)$ *is the best possible; (ii) for* $0 < p < 1$, *with the assumptions of Theorem 14, we have the reverse of (137) with the best possible constant factor* $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) k_{\lambda,s}^{(3)}(\mu)$.

Replacing y to $\frac{1}{y}$ in (135), and the reverse, setting

$$\widehat{F}_\lambda(y) := y^{\lambda-2} F_\lambda(\frac{1}{y}) = \begin{cases} y^{\lambda-1} \int_0^\infty k_\lambda^{(2)}(xy,1) f(x) dx, y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) > 0\}, \\ 0, y \in \{y \in \mathbf{R}_+; f(\frac{1}{y}) = 0\}, \end{cases}$$

it follows that

Corollary 20. *(i) For* $p > 1$, *with the assumptions of Theorem 13, we have*

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(x,y) \widehat{F}_\lambda(y) \widehat{G}_\lambda(x) dy dx < \prod_{i=1}^3 k_\lambda^{(i)}(\mu) ||f||_{p,\phi} ||g||_{q,\psi}, \quad (138)$$

where the constant factor $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$ *is still the best possible; (ii) for* $0 < p < 1$, *with the assumptions of Theorem 14, we have the reverse of (138) with the best possible constant factor* $\prod_{i=1}^3 k_\lambda^{(i)}(\mu)$.

By (93) and (138), for $p > 1,\ i,j = 0, 1, 2,\ k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, we still have

$$\int_0^\infty \int_{A_{x,i}} k_\lambda^{(1)}(x,y)\widehat{F}_{\lambda,j}(y)\widehat{G}_\lambda(x)dydx$$
$$< k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_\lambda^{(3)}(\mu)||f||_{p,\phi}||g||_{q,\psi}. \tag{139}$$

If $k_\lambda^{(3)}(xy,1) = 0(y \in \mathbf{R}_+\backslash B_{x,1})$, then we find $k_\lambda^{(3)}(u,1) = 0(u \geq 1)$, and $k_\lambda^{(3)}(\mu) = k_{\lambda,1}^{(3)}(\mu)$; if $k_\lambda^{(3)}(xy,1) = 0(y \in \mathbf{R}_+\backslash B_{x,2})$, then we find $k_\lambda^{(3)}(u,1) = 0(0 < u \leq 1)$, and $k_\lambda^{(3)}(\mu) = k_{\lambda,2}^{(3)}(\mu)$. By (139), it follows that

Corollary 21. *Suppose that for* $i,j,s = 0, 1, 2,\ k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu), k_{\lambda,s}^{(3)}(\mu) \in \mathbf{R}_+$. *(i) For* $p > 1$, *with the assumptions of Theorem 13, we have*

$$\int_0^\infty \int_{A_{x,i}} k_\lambda^{(1)}(x,y)\widehat{F}_{\lambda,j}(y)\widehat{G}_{\lambda,s}(x)dydx$$
$$< k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_{\lambda,s}^{(3)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \tag{140}$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_{\lambda,s}^{(3)}(\mu)$ *is still the best possible; (ii) for* $0 < p < 1$, *with the assumptions of Theorem 14, we have the reverse of (140) with the best possible constant factor* $k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_{\lambda,s}^{(3)}(\mu)$.

Replacing y to $\frac{1}{y}$ in (130), and the reverse, we have

Corollary 22. *(i) For* $p > 1$, *with the assumptions of Theorem 13, we have*

$$\int_0^\infty \int_0^\infty k_\lambda^{(1)}(xy,1)\widehat{F}_\lambda(y)G_\lambda(x)dydx < \prod_{i=1}^{3} k_\lambda^{(i)}(\mu)||f||_{p,\phi}||g||_{q,\psi}, \tag{141}$$

where the constant factor $\prod_{i=1}^{3} k_\lambda^{(i)}(\mu)$ *is still the best possible; (ii) for* $0 < p < 1$, *with the assumptions of Theorem 14, we have the reverse of (141) with the best possible constant factor* $\prod_{i=1}^{3} k_\lambda^{(i)}(\mu)$.

By (93) and (141), for $p > 1,\ i,j = 0, 1, 2,\ k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu) \in \mathbf{R}_+$, we still have

$$\int_0^\infty \int_{B_{x,i}} k_\lambda^{(1)}(xy,1)\widehat{F}_{\lambda,j}(y)G_\lambda(x)dydx$$
$$< k_{\lambda,i}^{(1)}(\mu)k_{\lambda,j}^{(2)}(\mu)k_\lambda^{(3)}(\mu)||f||_{p,\phi}||g||_{q,\psi}. \tag{142}$$

If $k_\lambda^{(3)}(x, y) = 0(y \in \mathbf{R}_+ \backslash A_{x,1})$, then we find $k_\lambda^{(3)}(u, 1) = 0(u \geq 1)$, and $k_\lambda^{(3)}(\mu) = k_{\lambda,1}^{(3)}(\mu)$; if $k_\lambda^{(3)}(x, y) = 0(y \in \mathbf{R}_+ \backslash A_{x,2})$, then we find $k_\lambda^{(3)}(u, 1) = 0(0 < u \leq 1)$, and $k_\lambda^{(3)}(\mu) = k_{\lambda,2}^{(3)}(\mu)$. By (142), it follows that

Corollary 23. *Suppose that for* $i, j, s = 0, 1, 2,$ $k_{\lambda,i}^{(1)}(\mu), k_{\lambda,j}^{(2)}(\mu), k_{\lambda,s}^{(3)}(\mu) \in \mathbf{R}_+$. *(i) For* $p > 1$, *with the assumptions of Theorem 13, we have*

$$\int_0^\infty \int_{B_{x,i}} k_\lambda^{(1)}(xy, 1) \widehat{F}_{\lambda,j}(y) G_{\lambda,s}(x) dy dx$$
$$< k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) k_{\lambda,s}^{(3)}(\mu) ||f||_{p,\phi} ||g||_{q,\psi}, \tag{143}$$

where the constant factor $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) k_{\lambda,s}^{(3)}(\mu)$ *is still the best possible; (ii) for* $0 < p < 1$, *with the assumptions of Theorem 14, we have the reverse of (143) with the best possible constant factor* $k_{\lambda,i}^{(1)}(\mu) k_{\lambda,j}^{(2)}(\mu) k_{\lambda,s}^{(3)}(\mu)$.

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Opial Inequalities Involving Higher-Order Partial Derivatives

Chang-Jian Zhao and Wing-Sum Cheung

In Honor of Constantin Carathéodory

Abstract In the present paper, we establish some new Opial's type inequalities involving higher-order partial derivatives. Our results provide new estimates on inequalities of these type.

1 Introduction

In 1960, Opial [21] established the following integral inequality:

Theorem A. *Suppose* $f \in C^1[0,h]$ *satisfies* $f(0) = f(h) = 0$ *and* $f(x) > 0$ *for all* $x \in (0,h)$. *Then the integral inequality*

$$\int_0^h \left| f(x)f'(x) \right| dx \le \frac{h}{4} \int_0^h (f'(x))^2 dx \tag{1}$$

holds, where the constant $\frac{h}{4}$ *is best possible.*

The first natural extension of Opial's inequality (1) to the case involving higher-order derivatives $x^{(n)}(s)(n \ge 1)$ is embodied in the following:

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Theorem B ([3]). *Let $x(t) \in C^{(n)}[0,a]$ be such that $x^{(i)}(t) = 0, 0 \le i \le n-1$ $(n \ge 1)$. Then the following inequality holds*

$$\int_0^a \left|x(t)x^{(n)}(t)\right| dt \le \frac{1}{2}a^n \int_0^a \left|x^{(n)}(t)\right|^2 dt. \tag{2}$$

Some sharp versions of (2) are the following forms established by Das [14]:

Theorem C ([14]). *Let l and m be positive numbers satisfying $l+m=1$ and let $x(t) \in C^{(n-1)}[0,a]$ be such that $x^{(i)}(t) = 0, 0 \le i \le n-1 (n \ge 1)$. Further, let $x^{(n-1)}(t)$ be absolutely continuous, and $\int_0^a \left|(x^{(n)}(t)\right| dt < \infty$. Then the following inequality holds*

$$\int_0^a |x(t)|^l \left|x^{(n)}(t)\right|^m dt \le \frac{m^m}{(n!)^l} a^{nl} \int_0^a \left|x^{(n)}(t)\right| dt. \tag{3}$$

Theorem D ([14]). *Let l and m be positive numbers satisfying $l+m > 1$. Let $x(t) \in C^{(n-1)}[0,a]$ be such that $x^{(i)}(t) = 0, 0 \le i \le n-1 (n \ge 1)$. Further, let $x^{(n-1)}(t)$ be absolutely continuous, and $\int_0^a \left|(x^{(n)}(t)\right|^{l+m} dt < \infty$. Then for $\xi = 1/(l+m)$,*

$$\int_0^a |x(t)|^l \left|x^{(n)}(t)\right|^m dt \le \xi m^{m\xi} \left(\frac{n(1-\xi)}{n-\xi}\right)^{l(1-\xi)} (n!)^{-l} a^{nl} \int_0^a \left|x^{(n)}(t)\right|^{l+m} dt. \tag{4}$$

Opial's inequality and its generalizations, extensions, and discretizations play a fundamental role in the study of the existence and uniqueness problems of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2, 3, 7, 18, 20]. Over the years, this type of inequalities has received considerable attention, and a large number of papers dealing with new proofs, extensions, generalizations, variants, and discrete analogues of Opial's inequality have appeared in the literature [8–13, 15, 16, 19, 22]. For an extensive survey on these inequalities, see [3, 20]. For Opial-type integral inequalities involving high-order partial derivatives, see [1, 4–6, 14, 17].

The main purpose of the present paper is to establish some new Opial-type inequalities involving higher-order partial derivatives. Our main results are the following Theorems:

Theorem 1.1. *Let l and p be positive numbers satisfying $l+p=1$ and let $x(t_1,\ldots,t_n) \in C^{(n_1-1,\ldots,n_n-1)}([0,a_1]\times\cdots\times[0,a_n])$ be such that $\frac{\partial^{\kappa_i}}{\partial t_i^{\kappa_i}} x(t_1,\ldots,t_i,\ldots,t_n)|_{t_i=0} = 0,\ 0 \le \kappa_i \le n_i - 1,\ i = 1,\ldots,n,$ $\frac{\partial^{n_i}}{\partial t_i^{n_i}}\left(\frac{\partial^{n_i-1}}{\partial t_i^{n_i-1}} x(t_1,\ldots,t_i,\ldots,t_n)\right)$ and*

$\frac{\partial^{n_i-1}}{\partial t_i^{n_i-1}}\left(\frac{\partial^{n_i}}{\partial t_i^{n_i}} x(t_1,\ldots,t_i,\ldots,t_n)\right)$ are absolutely continuous on $[0,a_1]\times\cdots\times[0,a_n]$,

and $\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}} x(t_1,\ldots,t_n)\right| dt_1\cdots dt_n$ exists. Then

$$\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|^l\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|^p dt_1\cdots dt_n$$
$$\leq \frac{p^p}{\prod_{i=1}^n[(n_i-\kappa_i)]^l}\prod_{i=1}^n a_i^{l(n_i-\kappa_i)}\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right| dt_1\cdots dt_n. \tag{5}$$

Remark 1.1. Taking $\kappa_i = 0, i = 1,\ldots,n$, (5) reduces to

$$\int_0^{a_1}\cdots\int_0^{a_n}|x(t_1,\ldots,t_n)|^l\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|^p dt_1\cdots dt_n$$
$$\leq \frac{p^p}{\prod_{i=1}^n(n_i)^l}\prod_{i=1}^n a_i^{ln_i}\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right| dt_1\cdots dt_n. \tag{6}$$

Taking $n = 2$, (6) reduces to

$$\int_0^a\int_0^b|x(s,t)|^l\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right|^p dsdt$$
$$\leq \frac{p^p}{(n!m!)^l}a^{nl}b^{ml}\int_0^a\int_0^b\left|\frac{\partial^{n+m}}{\partial s^n\partial t^m}x(s,t)\right| dsdt. \tag{7}$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, and then (7) becomes the following inequality given by Das [14]:

$$\int_0^a|x(t)|^l\left|x^{(n)}(t)\right|^m dt \leq \frac{m^m}{(n!)^l}a^{nl}\int_0^a\left|x^{(n)}(t)\right| dt.$$

Theorem 1.2. *Let l and p be positive numbers satisfying $l+p>1$ and let $x(t_1,\ldots,t_n) \in C^{(n_1-1,\ldots,n_n-1)}([0,a_1]\times\cdots\times[0,a_n])$ be such that $\frac{\partial^{\kappa_i}}{\partial t_i^{\kappa_i}}x(t_1,\ldots,t_i,\ldots,t_n)|_{t_i=0} = 0,\ 0\leq\kappa_i\leq n_i-1,\ i=1,\ldots,n,$ $\frac{\partial^{n_i}}{\partial t_i^{n_i}}\left(\frac{\partial^{n_i-1}}{\partial t_i^{n_i-1}}x(t_1,\ldots,t_i,\ldots,t_n)\right)$ and $\frac{\partial^{n_i-1}}{\partial t_i^{n_i-1}}\left(\frac{\partial^{n_i}}{\partial t_i^{n_i}}x(t_1,\ldots,t_i,\ldots,t_n)\right)$ are absolutely continuous on $[0,a_1]\times\cdots\times[0,a_n]$, and $\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|^{l+p} dt_1\cdots dt_n$ exists. Then*

$$\int_0^{a_1}\cdots\int_0^{a_n}q(t_1,\ldots,t_n)\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|^l\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^p dt_1\cdots dt_n$$
$$\leq C\prod_{i=1}^n a_i^{l(n_i-\kappa_i)}\int_0^{a_1}\cdots\int_0^{a_n}q(t_1,\ldots,t_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^{l+p} dsdt, \tag{8}$$

where

$$C = \xi^{1+l\xi} p^{p\xi} \left(\frac{\prod_{i=1}^{n} (n_i - \kappa_i)(1-\xi)^n}{\prod_{i=1}^{n} (n_i - \kappa_i - 1)} \right)^{l(1-\xi)} \frac{1}{[\prod_{i=1}^{n} (n_i - \kappa_i)!]^l}, \quad \xi = \frac{1}{l+p}.$$

Remark 1.2. Taking $n = 2, q(t_1, \ldots, t_n) = 1$ and $\kappa_i = 0, i = 1, \ldots, n$, (8) reduces to

$$\int_0^a \int_0^b |x(s,t)|^l \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|^p dsdt \leq C_{n,m} a^{nl} b^{ml} \int_0^a \int_0^b \left| \frac{\partial^{n+m}}{\partial s^n \partial t^m} x(s,t) \right|^{l+p} dsdt, \tag{9}$$

where

$$C_{n,m,l,p} = \xi^{l\xi+1} p^{\xi p} \left(\frac{mn(1-\xi)^2}{(n-\xi)(m-\xi)} \right)^{l(1-\xi)} \frac{1}{(n!m!)^l}, \quad \xi = \frac{1}{l+p}. \tag{10}$$

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, and then (9) becomes the following inequality:

$$\int_0^a |x(t)|^l \left| x^{(n)}(t) \right|^m dt \leq \xi m^{m\xi} \left(\frac{n(1-\xi)}{n-\xi} \right)^{l(1-\xi)} (n!)^{-l} a^{nl} \int_0^a \left| x^{(n)}(t) \right|^{l+m} dt, \quad \xi = \frac{1}{l+m}. \tag{11}$$

This is an inequality given by Das [14].

Taking $n = 1$ in (11), we have

$$\int_0^a |x(t)|^l |x'(t)|^m dt \leq \frac{m^{m/(l+m)}}{l+m} a^l \int_0^a |x'(t)|^{m+l} dt. \tag{12}$$

For $m, l \geq 1$ Yang [23] established the following inequality:

$$\int_0^a |x(t)|^l |x'(t)|^m dt \leq \frac{m}{l+m} a^l \int_0^a |x'(t)|^{m+l} dt. \tag{13}$$

Obviously, for $m, l \geq 1$, (12) is sharper than (13).

Remark 1.3. Taking $n = m = 1$, $q(s,t) = 1$ and $\kappa = \lambda = 0$, (7) reduces to

$$\int_0^a \int_0^b |x(s,t)|^l \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^p dsdt$$
$$\leq C^*_{1,1,l,p} (ab)^l \int_0^a \int_0^b \left| \frac{\partial^2}{\partial s \partial t} x(s,t) \right|^{p+l} dsdt, \tag{14}$$

where $C^*_{1,1,l,p}$ is as in (10).

Let $x(s,t)$ reduce to $s(t)$ and with suitable modifications, then (14) becomes the following inequality:

$$\int_0^a |x(t)|^l |x'(t)|^m dt \leq \xi m^{m\xi} a^l \int_0^a |x'(t)|^{m+l} dt, \quad \xi = \frac{1}{l+m}.$$

This is just an inequality established by Yang [23].

2 Proof of Main Results

Proof of Theorem 1.1. From the hypotheses of the Theorem 1.1, we have for $0 \leq \kappa_i \leq n_i - 1, 0 \leq i \leq n$,

$$\left| \frac{\partial^{\kappa_1 + \cdots + \kappa_n}}{\partial t^{\kappa_1} \cdots \partial t^{\kappa_n}} x(t_1, \ldots, t_n) \right|$$
$$\leq \frac{\prod_{i=1}^n t_i^{n_i - \kappa_i - 1}}{\prod_{i=1}^n (n_i - \kappa_i - 1)!} \int_0^{t_1} \cdots \int_0^{t_n} \left| \frac{\partial^{n_1 + \cdots + n_n}}{\partial \sigma_1^{n_1} \cdots \partial \sigma_n^{n_n}} x(\sigma_1, \ldots, \sigma_n) \right| d\sigma_1 \cdots d\sigma_n. \tag{15}$$

Multiplying both sides of (15) by $\left| \frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1} \cdots \partial t_n^{n_n}} x(t_1, \ldots, t_n) \right|^p$ and integrating both sides over t_i from 0 to a_i, $i = 1, \ldots, n$, respectively, we obtain

$$\int_0^{a_1} \cdots \int_0^{a_n} \left| \frac{\partial^{\kappa_1 + \cdots + \kappa_n}}{\partial t^{\kappa_1} \cdots \partial t^{\kappa_n}} x(t_1, \ldots, t_n) \right|^l \left| \frac{\partial^{n_1 + \cdots + n_n}}{\partial t_1^{n_1} \cdots \partial t_n^{n_n}} x(t_1, \ldots, t_n) \right|^p dt_1 \cdots dt_n$$
$$\leq \frac{1}{\prod_{i=1}^n [l(n_i - \kappa_i - 1)!]^l} \int_0^{a_1} \cdots \int_0^{a_n} \prod_{i=1}^n t_i^{l(n - \kappa_i - 1)} \left| \frac{\partial^{n_1 + \cdots + n_n}}{\partial t_1^{n_1} \cdots \partial t_n^{n_n}} x(t_1, \ldots, t_n) \right|^p$$
$$\times \left\{ \int_0^{t_1} \cdots \int_0^{t_n} \left| \frac{\partial^{n_1 + \cdots + n_n}}{\partial \sigma_1^{n_1} \cdots \partial \sigma_n^{n_n}} x(\sigma_1, \ldots, \sigma_n) \right| d\sigma_1 \cdots d\sigma_n \right\}^l dt_1 \ldots dt_n. \tag{16}$$

Now, applying Hölder's inequality with indices $\frac{1}{l}$ and $\frac{1}{p}$ to the integral on the right side of (16), we obtain

$$\begin{aligned}
&\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|^l\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|^p dt_1\cdots dt_n\\
&\le\frac{1}{\prod_{i=1}^n[(n_i-\kappa_i-1)!]^l}\\
&\quad\times\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n t_i^{n-\kappa_i-1}dt_1\cdots dt_n\right)^l\left\{\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|\right.\\
&\quad\left.\times\left(\int_0^{t_1}\cdots\int_0^{t_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|d\sigma_1\cdots d\sigma_n\right)^{l/p}dt_1\cdots dt_n\right\}^p. \qquad (17)
\end{aligned}$$

On the other hand, from the hypotheses of Theorem 1.1 and in view of the following facts

$$\begin{aligned}
&\frac{\partial^n}{\partial t_1\cdots\partial t_n}\left[\left(\int_0^{t_1}\cdots\int_0^{t_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|d\sigma_1\cdots d\sigma_n\right)^{l/p+1}\right]\\
&\quad=\frac{1}{p}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|\\
&\qquad\times\left(\int_0^{t_1}\cdots\int_0^{t_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|d\sigma_1\cdots d\sigma_n\right)^{l/p}, \qquad (18)
\end{aligned}$$

and

$$\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n t_i^{n_i-\kappa_i-1}dt_1\cdots dt_n\right)^l=\frac{\prod_{i=1}^n a_i^{l(n_i-\kappa_i)}}{\prod_{i=1}^n[(n_i-\kappa_i)]^l}, \qquad (19)$$

and by (17)–(19), we have

$$\begin{aligned}
&\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|^l\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|^p dt_1\cdots dt_n\\
&\le\frac{p^p}{\prod_{i=1}^n[(n_i-\kappa_i)]^l}\prod_{i=1}^n a_i^{l(n_i-\kappa_i)}\int_0^{a_1}\cdots\int_0^{a_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t_1^{n_1}\cdots\partial t_n^{n_n}}x(t_1,\ldots,t_n)\right|dt_1\cdots dt_n.
\end{aligned}$$

This completes the proof.

Proof of Theorem 1.2. From the hypotheses of Theorem 1.2, we have for $0 \le \kappa_i \le n_i - 1$, $0 \le i \le n$

$$\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right| \le \frac{1}{\prod_{i=1}^{n}(n_i-\kappa_i-1)!}$$
$$\times \int_0^{t_1}\cdots\int_0^{t_n}\prod_{i=1}^{n}(t_i-\sigma_i)^{n_i-\kappa_i-1}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right| d\sigma_1\cdots d\sigma_n. \tag{20}$$

By Hölder's inequality with indices $l+p$ and $\frac{l+p}{l+p-1}$, it follows that

$$\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|$$
$$\le \frac{1}{\prod_{i=1}^{n}(n_i-\kappa_i-1)!}\left(\int_0^{t_1}\cdots\int_0^{t_n}\left[\prod_{i=1}^{n}(t_i-\sigma_i)^{n_i-\kappa_i-1}\right]^{\frac{l+p}{l+p-1}} d\sigma_1\cdots d\sigma_n\right)^{\frac{l+p-1}{l+p}}$$
$$\times\left(\int_0^{t_1}\cdots\int_0^{t_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|^{l+p} d\sigma_1\cdots d\sigma_n\right)^{\frac{1}{l+p}}$$
$$= D\prod_{i=1}^{n}t_i^{n_i-\kappa_i-\xi}\left(\int_0^{t_1}\cdots\int_0^{t_n}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|^{l+p} d\sigma_1\cdots d\sigma_n\right)^{\frac{1}{l+p}},$$

where

$$D = \left(\frac{(1-\xi)^n}{\prod_{i=1}^{n}(n_i-\kappa_i-\xi)}\right)^{1-\xi}\frac{1}{\prod_{i=1}^{n}(n_i-\kappa_i-1)!},\ \xi = (l+p)^{-1}.$$

Hence, in view of the nonincreasing nature of $q(t_1,\ldots,t_n)$, we have

$$\int_0^{a_1}\cdots\int_0^{a_n}q(t_1,\ldots,t_n)\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|^{l}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^{p}dt_1\cdots dt_n$$
$$\le D^l\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^{n}t_i^{l(n_i-\kappa_i-\xi)}(q(t_1,\ldots,t_n))^{\xi p}\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^{p}$$
$$\times\left(\int_0^{t_1}\cdots\int_0^{t_n}q(\sigma_1,\ldots,\sigma_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|^{l+p}d\sigma_1\cdots d\sigma_n\right)^{l\xi}dt_1\cdots dt_n.$$

Now, applying Hölder's inequality with indices $\frac{l+p}{l}$ and $\frac{l+p}{p}$ to the integral on the right side, we obtain

$$\int_0^{a_1}\cdots\int_0^{a_n} q(t_1,\ldots,t_n)\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|^l\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^p dt_1\cdots dt_n$$
$$\leq D^l\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n t^{(n_i-\kappa_i-\xi)(l+p)}dt_1\cdots dt_n\right)^{l/(l+p)}$$
$$\times\left\{\int_0^{a_1}\cdots\int_0^{a_n} q(t_1,\ldots,t_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^{l+p}\right.$$
$$\left.\times\left(\int_0^{t_1}\cdots\int_0^{t_n} q(\sigma_1,\ldots,\sigma_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|^{l+p}d\sigma_1\cdots d\sigma_n\right)^{l/p}dt_1\cdots dt_n\right\}^{p/(l+p)}. \tag{21}$$

On the other hand, from the hypotheses of Theorem 1.2 and in view of the following facts

$$\frac{\partial^n}{\partial t_1\cdots\partial t_n}\left[\left(\int_0^{t_1}\cdots\int_0^{t_n} q(\sigma_1,\ldots,\sigma_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|^{l+p}d\sigma_1\cdots d\sigma_n\right)^{l/p+1}\right]$$
$$=\frac{l+p}{p}q(t_1,\ldots,t_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^{l+p}$$
$$\times\left(\int_0^{t_1}\cdots\int_0^{t_n} q(\sigma_1,\ldots,\sigma_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial\sigma_1^{n_1}\cdots\partial\sigma_n^{n_n}}x(\sigma_1,\ldots,\sigma_n)\right|^{l+p}d\sigma_1\cdots d\sigma_n\right)^{l/p} \tag{22}$$

and

$$\left(\int_0^{a_1}\cdots\int_0^{a_n}\prod_{i=1}^n t_i^{(n_i-\kappa_i-\xi)(l+p)}dt_1\cdots dt_n\right)^{l/(l+p)}=\left(\frac{\xi^n}{\prod_{i=1}^n(n_i-\kappa_i)}\right)^{l\xi}\prod_{i=1}^n a_i^{l(n_i-\kappa_i)}, \tag{23}$$

and by (21)–(23), we have

$$\int_0^{a_1}\cdots\int_0^{a_n} q(t_1,\ldots,t_n)\left|\frac{\partial^{\kappa_1+\cdots+\kappa_n}}{\partial t^{\kappa_1}\cdots\partial t^{\kappa_n}}x(t_1,\ldots,t_n)\right|^l\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^p dt_1\cdots dt_n$$
$$\leq C\prod_{i=1}^n a_i^{l(n_i-\kappa_i)}\int_0^{a_1}\cdots\int_0^{a_n} q(t_1,\ldots,t_n)\left|\frac{\partial^{n_1+\cdots+n_n}}{\partial t^{n_1}\cdots\partial t^{n_n}}x(t_1,\ldots,t_n)\right|^{l+p}dsdt,$$

where

$$C=\xi^{1+l\xi}p^{p\xi}\left(\frac{\prod_{i=1}^n(n_i-\kappa_i)(1-\xi)^n}{\prod_{i=1}^n(n_i-\kappa_i-1)}\right)^{l(1-\xi)}\frac{1}{[\prod_{i=1}^n(n_i-\kappa_i)!]^l},\quad \xi=\frac{1}{l+p}.$$

This completes the proof.

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