Chapter 12
Exotic Smoothness, Physics and Related Topics

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Abstract In 1854 Riemann, the father of differential geometry, suggested that the geometry of space may be more than just a mathematical tool defining a stage for physical phenomena, and may in fact have profound physical meaning in its own right. Since then various assumptions about the spacetime structure have been put forward. But to what extent the choice of mathematical model for spacetime has important physical significance? With the advent of general relativity physicists began to think of the spacetime in terms a differential manifolds. In this short essay we will discuss to what extent the structure of spacetime can be determined (modelled) and the possible role of differential calculus in the due process. The counterintuitive discovery of exotic four dimensional Euclidean spaces following from the work of Freedman and Donaldson surprised mathematicians. Later, it has been shown that exotic smooth structures are especially abundant in dimension four—the dimension of the physical spacetime. These facts spurred research into possible the physical role of exotic smoothness, an interesting but not an easy task, as we will show.

12.1 Introduction

The outcomes of physical measurements are expressed in rational numbers: all meaningful measurements are performed with certain accuracy and it is even hard to imagine how can they produce an irrational number. Nevertheless, we believe that all possible values of physical variables constitute the set of real numbers $\mathbb{R}$. Most of physical theories, including quantum gravity, use the concept of spacetime as scene of physical processes, at least approximately. We also suppose that the spacetime, whatever it actually is, can be faithfully modeled as a manifold. At this stage the algebra of real continuous functions $C(M)$ on the spacetime manifold M comes to play [16]. This algebra play central rôle in physics, although this fact is not always

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stressed or even perceived. It is tightly intertwined with one of the most important and fundamental open problems in theoretical physics: to explain the origin and structure of spacetime and to analyse how faithful our theoretical models of the spacetime can be. We will suppose that it is possible to determine the algebra of, say continuous, $K-$valued functions $C(M, K)$ defined on the spacetime (assumed to be a topological space) with sufficient for our aim accuracy for various K. Actually, if we confine our aspirations to the analysis of only local properties of M the algebra in question can be substantially smaller. This does not mean that we have to be able to find all elements of $C(M, K)$ "experimentally": some abstraction or inductive construction would be sufficient. With obvious abuse of language, we will call elements of $C(M, K)$ observables. Our aim is to find out to what extent the structure of the mathematical model of spacetime is determined by $C(M, K)$ for M being a topological space. We will also analyse what happens if we admit of M being a differential manifold or to have no topology. Finally, we will show how $C(M, K)$ can be used to construct a field theory of fundamental interactions in the A. Connes' noncommutative geometry formalism [10].

It is a great honor and pleasure to be able to contribute to this volume celebrating Carl Brans eightieth birthday and his scientific achievements that influenced the development of the theory of gravitation in such a significant way.

12.2 The Topology of Spacetime

A lot of information on a given topological space M is encoded in the associated algebras $C(M, \mathbb{R})$ of continuous real functions defined on M. For any set M the family $\mathcal{C}$ of real functions $M \to \mathbb{R}$ determines a minimal topology $\tau_{\mathcal{C}}$ on M such that all function in $\mathcal{C}$ are continuous [16, 28]. It is less known that the reals can be replaced by some other algebraic structures [28, 29]. Therefore, we will also consider $C(M, K)$, the algebras of K-valued functions, K being a topological ring, field, algebra and so on. Suppose that our experimental technique is powerful enough to reconstruct $C(M, K)$ acting on our model of the spacetime M. What information about M provides us with knowledge of $C(M, K)$ information concerning M can be extracted from these data? If M is a set and $\mathcal{C}$ a family of real functions $M \to \mathbb{R}$ then $\mathcal{C}$ determines a (minimal) topology $\tau_{\mathcal{C}}$ on M such that all function in $\mathcal{C}$ are continuous [16]. In general, there would exist real continuous functions on M that do not belong to $\mathcal{C}$ and other families of real functions on M can define the same topology on M. The topological space represented by $(M, \tau_{\mathcal{C}})$ would be a Hausdorff space if and only if for every pair of different points $p_1, p_2 \in M$ there is a function $f \in \mathcal{C}$ such that $f(p_1) \neq f(p_2)$. If the "space" or "time" are continuous by their nature we can hardly imagine any experiment that would be able to discover or distinguish two points "unseparable points".[1] From the physical point of view, to

[1] Actually, one can impose various inequivalent forms of such separation "axioms".

be able to distinguish x from y in our model of spacetime we have to find such an observable $f \in C(M)$ that for $x, y \in M \;\; f(x) \neq f(y)$. Therefore, sooner or later it seems reasonable to accept that

$$f(x) = f(y) \;\; \forall f \in C(M) \quad \Rightarrow \quad x = y. \tag{12.1}$$

From the mathematical point of view,we have to identify all points that are not distinguished by $C(M)$ in the above sense. It is then easy to show that such spaces are Hausdorff spaces. This means that we should look for the topological representation of the spacetime in the class of Hausdorff spaces. Note, that it is standard to postulate even more—one assumes paracompactness of space time manifold—this gives us the powerful tools of differential geometry. To proceed, let us define [16, 19, 28, 29]:

Definition 12.1 Let E be a topological space. A topological Hausdorff space X is called E-compact (E-regular) if it is homeomorphic to a closed (arbitrary) subspace of E^Y, for some Y.

Here E^Y is the space mappings $Y \to E$ (Tychonoff power). One can prove that for a topological space X (not necessarily a Hausdorff one!) we can construct an E-regular space $\tau_E X$ and its E-compact extension $\upsilon_E X$ so that we have [16, 28]

$$C(X, E) \cong C(\tau_E X, E) \cong C(\upsilon_E X, E) \cong C(\upsilon_E \tau_E X, E), \tag{12.2}$$

where $\cong$ denotes isomorphism. The spaces $\tau_E X$ and $\upsilon_E \tau_E X$ have the assumed property (1)! It should be obvious that, in general, our theoretical model of the spacetime would not be unique. This important result also says that we can always model our spacetime as a subset of some Tychonoff power of $\mathbb{R}$ provided $C(M)$ is known. But it also says that we can model it on a subset of a Tychonoff power of a different topological space e.g. the rational numbers $\mathbb{Q}$. Actually, it is our choice. The topological number fields $\mathbb{R}$ and $\mathbb{Q}$ have the very important property of determining uniquely (that is up to a homeomorphism) $\mathbb{R}$- and $\mathbb{Q}$-compact sets provided the appropriate algebras of continuous functions are known:

$$C(X, E) \cong C(Y, E) \iff X \textit{ is homeomorphic to } Y, \quad E = \mathbb{R} \textit{ or } \mathbb{Q}. \tag{12.3}$$

Other topological rings might also have this property but it is far from being a rule. But this does not mean that the spacetime modelled on $C(M, E)$ is homeomorphic to the one modelled on $C(M, E')$. Hewitt have shown that $\mathbb{R}$-compact spaces are determined up to a homeomorphism by $C(X, E)$, where $E = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ are the topological fields of real, complex numbers and quaternions, respectively [19]. This means that if we are interested in modelling spacetime on an $\mathbb{R}$-compact space then we can use $C(M, \mathbb{R})$, $C(M, \mathbb{C})$ or $C(M, \mathbb{H})$ to determine it. Unfortunately, such conclusion is false for rational numbers. It a serious obstacle as we probable never get more than rational numbers out of any feasible experiment.

To sum up, $\mathbb{R}$-compact are determined up-to homomorphism from $C(M)$. Consider any space M. There exists the smallest $\mathbb{R}$-compact space υM in which M is dense and υM would be the actual spacetime that we would probably reconstruct. υM need not to be compact—every subset of euclidean space is $\mathbb{R}$-compact, e.g. $\mathbb{R}^4$. $\mathbb{R}$-compact for a sufficiently wide class of functions for our aims: it is not easy to construct a space that is not $\mathbb{R}$-compact. Discrete spaces are $\mathbb{R}$-compact, practically all metrizable spaces are $\mathbb{R}$-compact. The reader is referred to [16, 28, 29] for details.

We will also need to deal with the technical problem of deciding whether we are dealing with the algebra $C(X, E)$ or only with the algebra of all bounded E-valued functions on X, $C^*(X, E)$ (if this concept of boundness make any sense). For a compact space X we have $C(X, E) = C^*(X, E)$, but in general, they are distinct. Spaces on which all continuous real functions are bounded are called pseudocompact. An $\mathbb{R}$-compact pseudocompact space is compact. We might get hints that some observables may in fact be unbounded but we are unlikely to be able to observe infinities. Moreover, with a high probability physical resources are not unlimited but an unbounded observable would be necessary for spacetime to be noncompact. Therefore, if we suppose that we can only recover $C^*(M, \mathbb{R}) \equiv C^*(M)$, then we can as well suppose that M is compact (for an $\mathbb{R}$-compact M). We often compactify configuration spaces by adding extra points or imposing appropriate boundary conditions (e.g. by demanding that all relevant fields vanish at infinity is practically equivalent to the one point compactification of the spacetime[2] A topological space X has more then one such an extension (compactification). Although mathematically one compactification can be differentiated from another with help of regular subrings this is unlikely to be done on the physical ground. Therefore, we will be forced to make various assumptions to choose one among the possible compactifications.

We have argued for looking for the spacetime model in the class of $\mathbb{R}$-compact M spaces. But what if we consider a more general field? It is often conjectured that at the sub-Plankian scales spacetime is non-Archimedean. Such spaces are much less tame do deal with. The Archimedean axiom says that, for any $x \in K$, there is a natural number n such that $|nx| > 1$. An Archimedean field is one in which this axiom holds. Examples are the real numbers and the complex numbers. Technically, there are no other examples: the only complete Archimedean fields are $\mathbb{R}$ and $\mathbb{C}$.[3] A non-Archimedean field is a complete normed field. For a given a non-Archimedean K, the cartesian product K^n is disconnected and it is not easy to follow the insight gained from $\mathbb{R}$ or $\mathbb{C}$. Note that filling in gaps between points (completion) of $\mathbb{Q}$ results in $\mathbb{R}$ but the initial set $\mathbb{Q}$ becomes practically negligible: we can more or less ignore the rational numbers when we do analysis. Therefore, the gap-filling process is performed in a more abstract way by introducing a Grothendieck topology on K^n.

The problems of dimension, density and "tightness" of the spacetime can also be addressed in terms of rings of real continuous functions with various topologies

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[2]That is we impose that the field in question vanish at the extra point glued to the space.

[3]The Archimedean axiom is also satisfied by C, with powers of the usual norm, and restrictions of these norms to subfields.

although experimental verification of these properties except dimension seems to be unlikely. The reader is referred to [1, 2] for details. The cardinality of the spacetime seem to be to abstract to have any practical significance, there are suggestions that such conclusions may be wrong [31]. Such problems might lead us outside the standard axioms of set theory.

One may also wonder if the knowledge of some symmetries might be of any help. Unfortunately, a topological space X is not determined by its symmetries considered as continuous maps $X \to X$ [15, 22, 27, 30]. Of course it sometimes can provide us with useful information. For example, if we know that some group G acts transitively on X then the cardinality of X cannot be greater than that of G [6]. For example, if we are pretty sure that the Poincare or Galilean symmetry groups act transitively on the spacetime we have got an upper bound on the cardinality of the spacetime. The situation is better if some extra structures are imposed, e.g. demanding existence of lorentzian structure is quite restrictive.

12.3 What if There Is No Topology on the Spacetime?

Up to now we have considered the arbitrariness of our mathematical model X of the spacetime as determined by $C(X, \mathbb{R})$. This means that we assume that M is a topological space. We can also ask to what extent algebras that we identify as an algebra of physical observables on the spacetime actually define a topological space. A commutative algebra must fulfill various sets of conditions to represent a $C(X, \mathbb{R})$ of some topological space X. If we suppose that our model of the spacetime is not a topological space we can deal with $\mathbb{R}^X$, the algebra of all real functions on X. But to have any "selective" power we have to demand the existence of some additional structure on X, for example to distinguish a collection of subsets of X or fix an algebraic structure on the class of functions we consider [22]. If (X, τ) is a pair consisting of a set X and a family τ of its subsets then we can define sort of "continuity" and "homeomorphisms" by replacing topology by the family τ. In this such cases the following theorem holds [16, 28].

Theorem 12.1 *Let X and Y be two sets and τ and σ families of their subsets containing the empty set, closed with respect to finite intersections and summing up to X and Y, respectively. Then X and Y are "homeomorphic" if and only if there is an isomorphism of the semigroups D^X and D^Y such that "$C(X, D)$" is mapped onto "$C(Y, D)$".*

Such generalized space are more difficult to deal with than ordinary topological spaces therefore we think that spacetime should be modelled in the class of topological spaces.

12.4 Differential Structure?

Existence of a differential structure on the spacetime manifold is a nice property. It certainly is not indispensable. Not every topological space or even topological manifold can support differential structures and demanding the existence of such structures severely restricts our choice of models. A differential structure of class k on a manifold M can be defined by specifying the (sub-)algebra of differentiable functions $C^k(M, \mathbb{R})$ of the algebra $C(M, \mathbb{R})$. The algebra $C^\infty(M)$ of smooth real functions on M determines M up to a diffeomophism (the points of M are in one-to-one correspondence with maximal ideals in $C^\infty(M)$). The algebra of continuous function on M is much larger than $C^k(M, \mathbb{R})$. If the laws of physics are "smooth" then the spacetime should be modelled as a smooth manifold. Therefore, in this case then $C^\infty(M, \mathbb{R})$ is be sufficient to construct M. As any real manifold M can be embedded in $\mathbb{R}^n$ for some n M is $\mathbb{R}$-compact. The most popular models of spacetime are pseudo-Riemannian manifolds. Such spaces are metrizable. This means that these manifolds are as topological spaces determined by $C(X, E)$, where $E = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ but additional knowledge of the subalgebra of differentiable functions is needed to determine the differential structure. Differential structures might not be unique If this is the case the "additional" differential structures are usually referred to as *fake* or *exotic* ones. Surprisingly, they are specially abundant in the four dimensions (one only needs to remove one point from a given manifold to get a manifold with exotic structures [12–14, 17, 18]. More astonishing is the fact that the four dimensional euclidean space $\mathbb{R}^4$ supports uncountably many exotic structures. We have to interpret these mathematical results in physical terms. H. Brans was first who realized this fact. He has conjectured that exoticness can be a source for some gravitational field, just as standard matter can [7–9]. One can put forward many arguments that exotic smoothness might have physical sense [7–9, 24, 25], the actual lack of any tractable description hinders physical predictions. Nevertheless, recent results of Asselmeyer-Maluga, Brans and Król shed a new interesting light on this important issue [3–5]. There are suggestions that existence of exotic smoothness has "something" to do with quantization of various models. Inflation can also be spurred by exoticness. In this context one can also ask if there is an gravitational analogue of the Bohm-Aharonov effect. That is if some points are removed or excluded from the spacetime exotic differential structures do emerge. The "standard" metric tensor and related structures defined by the distribution of matter might not be smooth with respect to some exotic differential structures. Such effect could a priori be detected say in cosmolological/astrophysical observations. This would mean that there is "additional" curvature required by consistency of differential structures.

The existence of exotic differential structures is certainly a challenge to physicists. These problems are involved and it is very difficult to distinguish between cause and effect.

It should be noted here non-Archimedean (p-adic, nonstandard) analysis is also considered to be the correct mathematical formalism to cope with dynamics at sub-Planckian levels [23]. Presently, no concrete data can be used for or against such ideas.

12.5 What if the Spacetime Is Pointless?

We have seen that topology and differential structures can be reconstructed from algebras of functions. These algebras are necessary commutative. Connes [11] made a step forward and showed how to differential geometry without the topological background. The basic structures are a C^*-algebra $\mathcal{A}$ represented in some Hilbert space H and an operator $\mathcal{D}$ acting in H. The differential da of an $a \in \mathcal{A}$ is defined by the commutator $[\mathcal{D}, a]$ and the integral is replaced by the Diximier trace, Tr_ω, with an appropriate inverse n-th power of $|\mathcal{D}|$ performing the role of the volume element $d^n x$:

$$\int a = \frac{Tr_\omega a|\mathcal{D}|^{-n}}{V}, \tag{12.4}$$

where V is some constant majorizing the eigenvaules λ_j of $\mathcal{D}$ ($\lambda_j < \frac{V}{j}$, $j \to \infty$). The Diximier trace of an operator O is, roughly speaking, the logarithmic divergence of the trace:

$$Tr_\omega O = \lim_{n\to\infty} \frac{\lambda_1 + \cdots + \lambda_n}{log\ n}, \tag{12.5}$$

where λ_i is the i-th eigenvalue of O. See Ref. [10, 11] for details. The differential geometry is build up as follows. The notions of covariant derivative (∇), connection (A) and curvature (F) forms are defined so that standard properties are conserved:

$$\nabla = d + A, \quad F = \nabla^2 = dA + A^2, \tag{12.6}$$

where $A \in \Omega^1_{\mathcal{D}}$ is the algebra of one forms defined with respect to d. Fiber bundles became projective modules on $\mathcal{A}$ in this language. For example, an n-dimensional Yang-Mills action can be given by the formula:

$$\mathcal{L}(A, \psi, \mathcal{D}) = Tr_\omega \left(F^2 \mid \mathcal{D} \mid^{-n}\right) + \langle \psi \mid \mathcal{D} + A \mid \psi \rangle, \tag{12.7}$$

where $\langle | \rangle$ denotes the inner product in the corresponding Hilbert space. For $\mathcal{A} = C^\infty(M, \mathbb{R})$ and $\mathcal{D}$ being the ordinary Dirac operator we recover the ordinary Riemannian geometry of the (spin-) manifold M. In, general, noncommutative C^*-algebras do not "produce" topological spaces. In some "almost trivial" cases M be multiplied by some discrete space. In this approach gravitation is hidden in the metric tensor that "enters" the Dirac operator. This means that we may not know the structure of the spacetime with satisfactory precision but nevertheless fundamental interactions determine it in a quite unique way. Of course, it is possible that the C^* algebra $\mathcal{A}$ that describes correctly fundamental interactions do not correspond to any topological space and spacetime can only approximately be described as a topological space or that fundamental interactions does not determine it uniquely. It should be stressed here that matter fields (fermions) and their interactions are essential in the process determining the spacetime structure and the notion of spacetime is not a fundamental one. The noncommutative geometry formalism actually says that fermions create the spacetime at least on the theoretical level. The pure gauge sector is insufficient because two E-compact spaces X and Y are homeomorphic if and only if the categories of all modules over $C(X, E)$ and $C(Y, E)$ are equivalent. Some models of set theory can be constructed which admit "sets" that have no points. They might not be empty but still are pointless [21]. Toposes are regarded as the simplest generalization of classical set theory. They can be used as the stage for gravitational interactions [20]. Such objects are much more difficult to identify and differentiating among them might be possible only at the aesthetic or philosophical levels.

12.6 Gravitational Interactions

Up to now, we have deliberately avoided any physical interpretation of its points. Although spacetime plays the role of stage in almost all physical models it is inextricably linked with gravity. Since the birth of general relativity, spacetime is understood as the set of all physical events and its geometry is governed by distribution of matter and its dynamics. The metric field $\mathbf{g}$ unlike other physical fields represents nothing else but a class of properties a more or less substantial spacetime M. Mathematically, the spacetime geometry is an additional structure over the topological space M, e.g. for defining the metric field $\mathbf{g}$ and deriving from it the curvature (connection) and so on. Einstein's general relativity requires that M is a smooth manifold and the metric $\mathbf{g}$ is from the Hilbert-Einstein action:

$$S = \int d^4x\sqrt{-g}[\frac{R}{2\kappa^2} + L_m(\mathbf{g}, \psi)], \tag{12.8}$$

where R is the curvature scalar and $L_m(\mathbf{g}, \psi)$ is the matter lagrangian with matter fields collectively denoted by ψ. This can be generalized to

$$S = \int d^4x\sqrt{-g}[\frac{f(\mathfrak{R})}{2\kappa^2} + L_m(\mathbf{g}, \psi)], \tag{12.9}$$

where $f(\mathfrak{R})$ is an arbitrary function of the curvature tensor $\mathfrak{R}$. Therefore, we have a plethora of acceptable theories of gravitational interaction and among them a wide class of virtually indistinguishable theories with local observations. This class would probably be extended by some generalized noncommutative geometry models. We can hardly believe that we would ever be able recover the correct theory of gravity from observational data, cg the discussion in [26]

12.7 Conclusions

We have described the problem of mathematical modelling of the spacetime structure. A priori we should be able to build a faithful and unique model of the spacetime in the class of $\mathbb{R}$-compact spaces. Nevertheless, some of the features would have to be conjectured. We have to find a phenomenon that cannot be described in terms of the algebra $C(M, \mathbb{R})$ to reject the assumption of $\mathbb{R}$-compactness. If we restrict ourselves only to topological methods, we will not be able to construct the topological model M of the spacetime uniquely—extra assumptions of, say, "minimality" should be made (Occam's razor). Some of popular assumptions about might never be provable. For example, an unbounded observable is necessary to prove noncompactness of spacetime manifold. In the general case, we will be able to construct only the Stone-Čech compactification βM of the space M and such spaces are in some sense maximal and could be enormous.[4] The existence of a differential structure on M allows for the identification of M with the set of maximal ideals of $C^\infty(M, \mathbb{R})$, although we anticipate that the determination of the differential structure may be problematic, especially if there is a lot of them. Note that the construction of the standard model of electroweak interactions imply that fundamental interactions determine the model of spacetime in the class $\mathbb{R}$-compact space in a unique way because they are specified by $C(M, \mathbb{H})$. This is not true for other symmetry groups, e.g. GUTs, are lacking in such a determinative power. Matter fields are fundamental for defining and determining the spacetime properties and the associated geometries. If we are not able to determine $C(M, \mathbb{R})$ or $C(M, \mathbb{Q})$ then our knowledge of the spacetime structure is significantly less accurate. In general, we have a bigger class of spaces "at our disposal" and we are more free in making assumptions about the topology and even about the cardinality of the model of spacetime.

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[4] For example, $\beta\mathbb{N}$ has the cardinality $2^{2^{\aleph_0}}$ and still is totally disconnected; any Hausdorff space with a basis of cardinality $\leq\aleph_1$ is a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$.

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