Resolvent Operators for Some Classes of Integro-Differential Equations

I.N. Parasidis and E. Providas

Abstract Explicit representations are constructed for the resolvents of the operators of the form $B = \widehat{A} + Q_1$ and $\mathbf{B} = \widehat{A}^2 + Q_2$, where $\widehat{A}$ and $\widehat{A}^2$ are linear closed operators with known resolvents and Q_1 and Q_2 are perturbation operators embedding inner products of $\widehat{A}$ and $\widehat{A}^2$ as they appear in integro-differential equations and other applications.

Keywords Resolvent Operator • Integro-Differential Equations • Boundary value Problems • Exact Solution

Introduction

The present article is concerned with the study of generalized boundary value problems containing differential or integro-differential operators by means of the resolvent operator. Specifically, we derive explicit representations for the resolvent operators for three classes of problems.

The first problem involves a linear operator defined in the complex Hilbert space H of the form

$$B = \widehat{A} + Q_1 = \widehat{A} - \sum_{i=1}^{m} g_i \langle \widehat{A}\,\cdot, \phi_i \rangle_H, \quad D(B) = D(\widehat{A}), \tag{1}$$

where $\widehat{A}$ is a linear closed, not necessarily bounded, operator, $g_1, g_2, \ldots, g_m$ are linearly independent elements of H, $\phi_1, \phi_2, \ldots, \phi_m \in H$ and $\langle \cdot, \cdot \rangle_H$ denotes the inner product in H. Note that the operator B can be viewed as a perturbation of the operator $\widehat{A}$ by the operator Q_1. Moreover, the operators $\widehat{A}$, B are extensions of

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a minimal operator $A_0 \subset \widehat{A}$ with $D(A_0) = D(\widehat{A}) \cap \ker Q_1$. We prove that when the resolvent set $\rho(\widehat{A})$ and the resolvent operator $R_\lambda(\widehat{A}) = (\widehat{A} - \lambda I)^{-1}$ of $\widehat{A}$ are known then we can find the resolvent set $\rho(B) \cap \rho(\widehat{A})$ and the resolvent operator $R_\lambda(B) = (B - \lambda I)^{-1}$ of B in closed form.

The second problem encompasses an operator of the kind

$$\mathbf{B} = \widehat{A}^2 + Q_2 = \widehat{A}^2 - \sum_{i=1}^{m} s_i \langle \widehat{A}\cdot, \phi_i \rangle_H - \sum_{i=1}^{m} g_i \langle \widehat{A}^2\cdot, \phi_i \rangle_H, \quad D(\mathbf{B}) = D(\widehat{A}^2), \tag{2}$$

where in addition $s_i \in H,\ i = 1, \ldots, m$. We show that the resolvent set $\rho(\mathbf{B}) \cap \rho(\widehat{A}^2)$ and the resolvent operator $R_\lambda(\mathbf{B}) = (\mathbf{B} - \lambda I)^{-1}$ of the operator $\mathbf{B}$ can be evaluated when their counterparts $\rho(\widehat{A}^2)$ and $R_\lambda(\widehat{A}^2) = (\widehat{A}^2 - \lambda I)^{-1}$ of the simpler operator $\widehat{A}^2$ are known.

A special form of the second problem with interest is obtained when we take $g_i \in D(\widehat{A})$ and $s_i = Bg_i,\ i = 1, \ldots, m$. In this case $\mathbf{B} = B^2$, i.e. $\mathbf{B}$ becomes a quadratic operator. The explicit formula for the resolvent operator $R_{\lambda^2}(\mathbf{B})$ for $\lambda^2 \in \rho(\mathbf{B})$ is constructed from the resolvent operators $R_\lambda(\widehat{A})$ and $R_{-\lambda}(\widehat{A})$ for $\pm\lambda \in \rho(\widehat{A})$.

Resolvent operators are associated with the spectral theory and their origin goes as back as to the early days of functional analysis, see, e.g., [13] and [10]. When the resolvent $R_\lambda(B)$ of an operator B exists and is provided in an analytic form, it is valuable for the study of the operator B itself and the solution of the problems $(B - \lambda I)x = f,\ f \in H$ and $Bx = f\ (\lambda = 0)$. Perturbation theory for linear operators was first introduced by Rayleigh and Schrödinger [17] and founded later by Kato [10]. Since then it occupies an important place in theoretical physics, mechanics and applied mathematics. Extension theory was initiated by von Neumann [19] and developed further by [12, 23] and [1], commonly known as Birman–Kreĭn–Vishik theory, as well as [3, 5, 11, 14, 21, 22] and many others. Integro-differential equations appear in the mathematical modeling in biology, engineering, telecommunications and economics. Of interest here are the following works. In [4, 6, 7, 15, 16] and [8], resolvent methods have been employed to study a class of integro-differential equations, occurring in heat conduction and viscoelasticity, where $\widehat{A}$ in (1) is a specific operator and Q_1 is a Volterra operator. By the same means certain integro-differential equations, arising in quantum-mechanical scattering theory, where $\widehat{A}$ is a special operator and Q_1 is a one-dimensional perturbation ($m = 1$), have also been investigated, see, e.g., [2]. The resolvent and the spectrum of perturbed operators of the type (1) where $\widehat{A}$ is a symmetric operator and $\langle \widehat{A}\cdot, \phi_i \rangle_H = a_i \langle \cdot, g_i \rangle_H,\ a_i \in \mathbf{R}$ have been studied by [9] and the references therein. Finally, a Fredholm type boundary integral equation in elasticity with $\langle \widehat{A}\cdot, \phi_i \rangle_H = p_i(\cdot)$ has been considered in [18].

In the rest we make use of the following notation. Namely, $F = (\phi_1, \ldots, \phi_m)$, $G = (g_1, \ldots, g_m)$ and $AF = (A\phi_1, \ldots, A\phi_m)$ are vectors of H^m. We write F^t and $\langle Ax, F^t \rangle_{H^m}$ for the column vectors $\mathrm{col}(\phi_1, \ldots, \phi_m)$ and $\mathrm{col}(\langle Ax, \phi_1 \rangle_H, \ldots, \langle Ax, \phi_m \rangle_H)$, respectively. We denote by $\overline{M}$ $(resp. M^t)$ the conjugate (resp. transpose) matrix of M and by $\langle G^t, F \rangle_{H^m}$ the $m \times m$ matrix whose

i, j-th entry is the inner product $\langle g_i, \phi_j \rangle_H$. Notice that $\langle G^t, F \rangle_{H^m}$ defines the matrix inner product and has the properties:

$$\langle G^t, FC \rangle_{H^m} = \langle G^t, F \rangle_{H^m} \overline{C}, \quad \langle G^t, F \rangle_{H^m} = \overline{\langle F^t, G \rangle}^t_{H^m}, \tag{3}$$

where C is an $m \times k$ constant complex matrix. We denote by I_m the $m \times m$ identity matrix and by 0_m the $m \times m$ zero matrix. It is understood that $D(A)$ and $R(A)$ stand for the domain and the range of A, respectively.

The paper is organized as follows. In sections "Resolvent of Extensions of a Minimal Operator," "Resolvent of Extensions of the Square of a Minimal Operator," and "Resolvent of Quadratic Operators" we develop the theory for acquiring analytic formulas for the resolvent operators corresponding to each of the three classes of problems presented. In section "Resolvent of Extensions of a Minimal Operator" we apply the theory to three Fredholm type, generalized integro-differential boundary value problems to demonstrate the power of the theory developed.

Resolvent of Extensions of a Minimal Operator

In this section we consider the operator B in Eq. (1) and determine the necessary and sufficient conditions for the existence of the resolvent $R_\lambda(B)$ and we find it in an explicit form provided $R_\lambda(\widehat{A})$ is known. In [22] the perturbed operator B has been studied as the extension of the minimal operator A_0, i.e.

$$A_0 x = \widehat{A} x \ \text{ for } \ x \in D(A_0) = \{x \in D(\widehat{A}) : \ \langle \widehat{A} x, F^t \rangle_{H^m} = \mathbf{0}^t\}$$

We begin by giving the definition of the resolvent. Let $A : H \to H$ be a linear operator. We say that $\lambda \in \mathbf{C}$ belongs to the resolvent set $\rho(A)$ of A if there exists the operator $R_\lambda = R_\lambda(A) = (A - \lambda I)^{-1}$ which is bounded and $D(R_\lambda)$ is dense in H ($\overline{D(R_\lambda)} = H$). The operator $R_\lambda(A)$ is called the resolvent of A. We recall from [13, Lemma 7.2-3] the following result.

Lemma 1. *Let X be a complex Banach space, $A : X \to X$ a linear operator, and $\lambda \in \rho(A)$. Assume that (a) A is closed or (b) A is bounded. Then the resolvent operator $R_\lambda(A)$ of A is defined on the whole space X and is bounded.*

The proposition below is well known and is utilized here several times.

Proposition 1. *If in a Hilbert space H, $\widehat{A} : H \to H$ is a closed linear operator and $\lambda \in \rho(\widehat{A})$, then*

$$\widehat{A} R_\lambda(\widehat{A}) = I + \lambda R_\lambda(\widehat{A}) \quad \text{and} \quad \widehat{A} R_\lambda(\widehat{A}) = R_\lambda(\widehat{A}) \widehat{A}. \tag{4}$$

We now prove the key theorem for obtaining the resolvent of the operator B.

Theorem 1. *Let H be a complex Hilbert space, $\widehat{A} : H \to H$ a linear closed operator, $\lambda \in \rho(\widehat{A})$ and $B : H \to H$ the operator defined by*

$$Bx = \widehat{A}x - G\langle \widehat{A}x, F^t\rangle_{H^m}, \quad D(B) = D(\widehat{A}), \tag{5}$$

where the vectors $G = (g_1, \ldots, g_m)$, with $g_1, \ldots, g_m$ being linearly independent elements, $F^t = \mathrm{col}(\phi_1, \ldots, \phi_m) \in H^m$ and $x \in D(B)$. Let the operator

$$B_\lambda x = (B - \lambda I)x = f, \quad D(B_\lambda) = D(\widehat{A}), \tag{6}$$

where I is the identity operator on $D(B)$ and $f \in H$. Then:

(i) $\lambda \in \rho(B)$ *if and only if*

$$\det L_\lambda = \det\left[I_m - \langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle_{H^m}\right] \neq 0. \tag{7}$$

(ii) $\rho(B) \cap \rho(\widehat{A}) = \{\lambda \in \rho(\widehat{A}) : \det L_\lambda \neq 0\}$.

(iii) For $\lambda \in \rho(B) \cap \rho(\widehat{A})$ the resolvent operator $R_\lambda(B)$ is defined on the whole space H, is bounded and has the representation

$$R_\lambda(B)f = R_\lambda(\widehat{A})f + R_\lambda(\widehat{A})G\overline{L_\lambda^{-1}}\langle \widehat{A}R_\lambda(\widehat{A})f, F^t\rangle_{H^m}. \tag{8}$$

Proof. (i)–(iii) Suppose that $\det L_\lambda \neq 0$. Since $D(\widehat{A}) = D(\widehat{A} - \lambda I) = D(B - \lambda I)$ then from (6) for $x \in D(\widehat{A})$ we have

$$(\widehat{A} - \lambda I)x - G\langle \widehat{A}x, F^t\rangle_{H^m} = f, \quad \forall f \in H. \tag{9}$$

Since $\widehat{A}$ is closed, the resolvent operator $R_\lambda(\widehat{A})$ is defined on the whole space H and is bounded as it is implied by Lemma 1. By applying first the operator $R_\lambda(\widehat{A})$ and then the operator $\widehat{A}$ on (9) we get

$$x - R_\lambda(\widehat{A})G\langle \widehat{A}x, F^t\rangle_{H^m} = R_\lambda(\widehat{A})f, \tag{10}$$

$$\widehat{A}x - \widehat{A}R_\lambda(\widehat{A})G\overline{\langle F^t, \widehat{A}x\rangle}_{H^m} = \widehat{A}R_\lambda(\widehat{A})f. \tag{11}$$

By taking the inner products of both sides of (11) with the components of the vector F^t and by observing that the inner product is conjugate linear in the second factor, see Eq. (3), we obtain

$$\langle F^t, \widehat{A}x\rangle_{H^m} - \langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle_{H^m}\langle F^t, \widehat{A}x\rangle_{H^m} = \langle F^t, \widehat{A}R_\lambda(\widehat{A})f\rangle_{H^m},$$

$$\left[I_m - \langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle_{H^m}\right]\langle F^t, \widehat{A}x\rangle_{H^m} = \langle F^t, \widehat{A}R_\lambda(\widehat{A})f\rangle_{H^m},$$

$$L_\lambda\langle F^t, \widehat{A}x\rangle_{H^m} = \langle F^t, \widehat{A}R_\lambda(\widehat{A})f\rangle_{H^m}. \tag{12}$$

Hence, since $\det L_\lambda \neq 0$,

$$\langle F^t, \widehat{A}x\rangle_{H^m} = L_\lambda^{-1}\langle F^t, \widehat{A}R_\lambda(\widehat{A})f\rangle_{H^m},$$
$$\langle \widehat{A}x, F^t\rangle_{H^m} = \overline{L_\lambda^{-1}}\langle \widehat{A}R_\lambda(\widehat{A})f, F^t\rangle_{H^m}. \tag{13}$$

By substituting (13) and $x = (B - \lambda I)^{-1}f$ into (10), we have

$$x = R_\lambda(\widehat{A})f + R_\lambda(\widehat{A})G\overline{L_\lambda^{-1}}\langle \widehat{A}R_\lambda(\widehat{A})f, F^t\rangle_{H^m}, \tag{14}$$

from where Eq. (8) follows. From Proposition 1 we have

$$\widehat{A}R_\lambda(\widehat{A})f = (I + \lambda R_\lambda(\widehat{A}))f$$

and since $R_\lambda(\widehat{A})$ is defined on the whole space H and is bounded, it follows that the resolvent operator $R_\lambda(B)$ is also defined on the whole space H and is bounded. Consequently, $\lambda \in \rho(B)$.

Conversely, let $\lambda \in \rho(B) \cap \rho(\widehat{A})$. We will show that $\det L_\lambda \neq 0$. Assume that $\det L_\lambda = 0$, then $\det \overline{L_\lambda} = 0$. Hence, exists a nonzero vector $\mathbf{a}^t = \mathrm{col}(a_1, \ldots, a_m) \in \mathbf{C}^m$ such that $\overline{L_\lambda}\mathbf{a}^t = \mathbf{0}^t$. We consider the element $x_0 = R_\lambda(\widehat{A})G\mathbf{a}^t \in D(\widehat{A})$. Since the components of the vector G are linearly independent elements we have $G\mathbf{a}^t \neq 0$ and therefore $x_0 \neq 0$. By substituting in (6) we get

$$\begin{aligned}
B_\lambda x_0 &= (\widehat{A} - \lambda I)x_0 - G\overline{\langle F^t, \widehat{A}x_0\rangle}_{H^m} \\
&= (\widehat{A} - \lambda I)R_\lambda(\widehat{A})G\mathbf{a}^t - G\overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A})G\mathbf{a}^t\rangle}_{H^m} \\
&= G\mathbf{a}^t - G\overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle}_{H^m}\mathbf{a}^t \\
&= G\left[I_m - \overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle}_{H^m}\right]\mathbf{a}^t \\
&= G\overline{L_\lambda}\mathbf{a}^t = G\mathbf{0}^t = 0.
\end{aligned} \tag{15}$$

This means that $x_0 \in \ker B_\lambda$ and so $\lambda \notin \rho(B)$, which contradicts the assumption that $\lambda \in \rho(B) \cap \rho(\widehat{A})$. Thus, $\det L_\lambda \neq 0$ and $\rho(B) \cap \rho(\widehat{A}) = \{\lambda \in \rho(\widehat{A}) : \det L_\lambda \neq 0\}$. □

Remark 1. The linear independence of the components of the vector G is required to prove the necessary condition of (i). Hence, if the components of the vector G are not linearly independent, then holds only: $\lambda \in \rho(B)$ if $\det L_\lambda \neq 0$.

Resolvent of Extensions of the Square of a Minimal Operator

In this section we extend the results of the previous section for the case of the operator $\mathbf{B}$ defined in Eq. (2). This operator has been studied in [22] as an extension of the minimal operator A_0^2, namely

$$A_0^2 x = \widehat{A}^2 x \;\text{ for }\; x \in D(A_0^2) = \{x \in D(\widehat{A}^2) : \langle \widehat{A}x, F^t \rangle_{H^m} = \langle \widehat{A}^2 x, F^t \rangle_{H^m} = \mathbf{0}^t\}.$$

We find here the resolvent set $\rho(\mathbf{B}) \cap \rho(\widehat{A}^2)$ and a resolvent $R_\lambda(\mathbf{B})$ when the $\rho(\widehat{A}^2)$ and $R_\lambda(\widehat{A}^2)$ are known.

First, we recall the following proposition from [20, pp. 39]

Proposition 2. *If A is a closed operator on a complex Banach space and its resolvent set $\rho(A)$ is not empty, then A^n is closed too for each $n \in \mathbf{N}$.*

We now prove the theorem for the resolvent of the operator $\mathbf{B}$.

Theorem 2. *Let H be a complex Hilbert space, $\widehat{A} : H \to H$ a linear closed operator, $\rho(\widehat{A})$ is not empty, $\lambda \in \rho(\widehat{A}^2)$ and $\mathbf{B} : H \to H$ the operator defined by*

$$\mathbf{B}x = \widehat{A}^2 x - S\langle \widehat{A}x, F^t \rangle_{H^m} - G\langle \widehat{A}^2 x, F^t \rangle_{H^m}, \quad D(\mathbf{B}) = D(\widehat{A}^2), \tag{16}$$

where $F^t = \mathrm{col}(\phi_1, \ldots, \phi_m)$, $S = (s_1, \ldots, s_m)$ and $G = (g_1, \ldots, g_m) \in H^m$, with the components of the vector (S, G) being linearly independent elements of H, and $x \in D(\mathbf{B})$. Let the operator

$$\mathbf{B}_\lambda x = (\mathbf{B} - \lambda I)x = f, \quad D(\mathbf{B}_\lambda) = D(\widehat{A}^2), \tag{17}$$

where I is the identity operator on $D(\mathbf{B})$ and $f \in H$. Then:

(i) $\lambda \in \rho(\mathbf{B})$ *if and only if*

$$\det W_\lambda = \det \begin{pmatrix} I_m - \langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)S \rangle_{H^m} & -\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)G \rangle_{H^m} \\ -\langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)S \rangle_{H^m} & I_m - \langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)G \rangle_{H^m} \end{pmatrix} \neq 0. \tag{18}$$

(ii) $\rho(\mathbf{B}) \cap \rho(\widehat{A}^2) = \{\lambda \in \rho(\widehat{A}^2) : \det W_\lambda \neq 0\}$.

(iii) For $\lambda \in \rho(\mathbf{B}) \cap \rho(\widehat{A}^2)$ the resolvent operator $R_\lambda(\mathbf{B})$ is defined on the whole space H, is bounded and has the representation

$$\begin{aligned} R_\lambda(\mathbf{B})f = R_\lambda(\widehat{A}^2)f &+ R_\lambda(\widehat{A}^2)(S\,\overline{W_{\lambda 11}^{-1}} + G\,\overline{W_{\lambda 21}^{-1}})\langle \widehat{A}R_\lambda(\widehat{A}^2)f, F^t \rangle_{H^m} \\ &+ R_\lambda(\widehat{A}^2)(S\,\overline{W_{\lambda 12}^{-1}} + G\,\overline{W_{\lambda 22}^{-1}})\langle \widehat{A}^2 R_\lambda(\widehat{A}^2)f, F^t \rangle_{H^m}, \end{aligned} \tag{19}$$

where $W_{\lambda ij}^{-1}$, $i, j = 1, 2$ are the $m \times m$ submatrices of the partition of the $2m \times 2m$ inverse matrix W_λ^{-1}.

Proof. (i)–(iii) Let us assume that (18) is true, specifically $\det W_\lambda \neq 0$. Since $D(\widehat{A}^2) = D(\widehat{A}^2 - \lambda I) = D(\mathbf{B} - \lambda I)$ then from (17) and for $x \in D(\widehat{A}^2)$ we have

$$(\widehat{A}^2 - \lambda I)x - S\langle \widehat{A}x, F^t\rangle_{H^m} - G\langle \widehat{A}^2 x, F^t\rangle_{H^m} = f. \tag{20}$$

Proposition 2 states that the operator $\widehat{A}^2$ is closed and because $\lambda \in \rho(\widehat{A}^2)$, it follows from Lemma 1 that the resolvent operator $R_\lambda(\widehat{A}^2)$ is defined and is bounded on the whole space H. By applying the resolvent operator $R_\lambda(\widehat{A}^2)$ on (20), we get

$$x - R_\lambda(\widehat{A}^2)\left[S\langle \widehat{A}x, F^t\rangle_{H^m} + G\langle \widehat{A}^2 x, F^t\rangle_{H^m}\right] = R_\lambda(\widehat{A}^2)f. \tag{21}$$

By employing the operators $\widehat{A}$ and $\widehat{A}^2$, we have

$$\widehat{A}x - \widehat{A}R_\lambda(\widehat{A}^2)\left[S\langle \widehat{A}x, F^t\rangle_{H^m} + G\langle \widehat{A}^2 x, F^t\rangle_{H^m}\right] = \widehat{A}R_\lambda(\widehat{A}^2)f, \tag{22}$$

$$\widehat{A}^2 x - \widehat{A}^2 R_\lambda(\widehat{A}^2)\left[S\langle \widehat{A}x, F^t\rangle_{H^m} + G\langle \widehat{A}^2 x, F^t\rangle_{H^m}\right] = \widehat{A}^2 R_\lambda(\widehat{A}^2)f. \tag{23}$$

Taking the inner products of (22) and (23) with the components of the vector F^t, we obtain the system

$$\langle F^t, \widehat{A}x\rangle_{H^m} - \langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)\left[S\overline{\langle F^t, \widehat{A}x\rangle}_{H^m} + G\overline{\langle F^t, \widehat{A}^2 x\rangle}_{H^m}\right]\rangle_{H^m} = \langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)f\rangle_{H^m}, \tag{24}$$

$$\langle F^t, \widehat{A}^2 x\rangle_{H^m} - \langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)\left[S\overline{\langle F^t, \widehat{A}x\rangle}_{H^m} + G\overline{\langle F^t, \widehat{A}^2 x\rangle}_{H^m}\right]\rangle_{H^m} = \langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)f\rangle_{H^m}. \tag{25}$$

Exploiting the conjugate linear property of the inner product as in (3), we get

$$\left[I_m - \langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)S\rangle_{H^m}\right]\langle F^t, \widehat{A}x\rangle_{H^m} - \langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)G\rangle_{H^m}\langle F^t, \widehat{A}^2 x\rangle_{H^m} = \langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)f\rangle_{H^m}, \tag{26}$$

$$-\langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)S\rangle_{H^m}\langle F^t, \widehat{A}x\rangle_{H^m} + \left[I_m - \langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)G\rangle_{H^m}\right]\langle F^t, \widehat{A}^2 x\rangle_{H^m} = \langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)f\rangle_{H^m}. \tag{27}$$

Writing Eqs. (26) and (27) in a matrix form by using the matrix W_λ in (18) and inverting, since $\det W_\lambda \neq 0$, we have

$$\begin{pmatrix} \langle F^t, \widehat{A}x\rangle_{H^m} \\ \langle F^t, \widehat{A}^2 x\rangle_{H^m} \end{pmatrix} = \begin{pmatrix} W_{\lambda 11}^{-1} & W_{\lambda 12}^{-1} \\ W_{\lambda 21}^{-1} & W_{\lambda 22}^{-1} \end{pmatrix} \begin{pmatrix} \langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)f\rangle_{H^m} \\ \langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)f\rangle_{H^m} \end{pmatrix}, \tag{28}$$

or, by taking the conjugates,

$$\begin{pmatrix} \langle \widehat{A}x, F^t\rangle_{H^m} \\ \langle \widehat{A}^2 x, F^t\rangle_{H^m} \end{pmatrix} = \overline{\begin{pmatrix} W_{\lambda 11}^{-1} & W_{\lambda 12}^{-1} \\ W_{\lambda 21}^{-1} & W_{\lambda 22}^{-1} \end{pmatrix}} \begin{pmatrix} \langle \widehat{A}R_\lambda(\widehat{A}^2)f, F^t\rangle_{H^m} \\ \langle \widehat{A}^2 R_\lambda(\widehat{A}^2)f, F^t\rangle_{H^m} \end{pmatrix}. \tag{29}$$

Substituting (29) into (21) and using $x = (\mathbf{B}-\lambda I)^{-1}f$ the resolvent operator $R_\lambda(\mathbf{B})$ is obtained as in Eq. (19). Since $\widehat{A}^2 R_\lambda(\widehat{A}^2)f = (I + \lambda R_\lambda(\widehat{A}^2))f$ by Proposition 1 and $R_\lambda(\widehat{A}^2)$ is defined on the whole space H and is bounded, it follows that $R_\lambda(\mathbf{B})$ is also defined on the whole space H and is bounded. Hence $\lambda \in \rho(\mathbf{B})$.

Conversely, let $\lambda \in \rho(\mathbf{B}) \cap \rho(\widehat{A}^2)$. We will show that $\det W_\lambda \neq 0$. Suppose $\det W_\lambda = 0$, then $\det \overline{W}_\lambda = 0$. Hence, there exists a nonzero vector $\mathbf{a}^t = \mathrm{col}(\mathbf{a}_1, \mathbf{a}_2) = \mathrm{col}(a_1, \ldots, a_m, a_{m+1}, \ldots, a_{2m}) \in \mathbf{C}^{2m}$ such that $\overline{W}_\lambda \mathbf{a}^t = \mathbf{0}^t$. We consider the element $x_0 = R_\lambda(\widehat{A}^2)(S\mathbf{a}_1^t + G\mathbf{a}_2^t) \in D(\widehat{A}^2)$. The components of the vector (S, G) are linearly independent and therefore $S\mathbf{a}_1^t + G\mathbf{a}_2^t \neq 0$ and hence $x_0 \neq 0$. Substituting into (17), we have

$$\begin{aligned}
\mathbf{B}_\lambda x_0 &= (\widehat{A}^2 - \lambda I)x_0 - S\overline{\langle F^t, \widehat{A}x_0\rangle}_{H^m} - G\overline{\langle F^t, \widehat{A}^2 x_0\rangle}_{H^m} \\
&= (\widehat{A}^2 - \lambda I)[R_\lambda(\widehat{A}^2)(S\mathbf{a}_1^t + G\mathbf{a}_2^t)] - S\overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)(S\mathbf{a}_1^t + G\mathbf{a}_2^t)\rangle}_{H^m} \\
&\quad - G\overline{\langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)(S\mathbf{a}_1^t + G\mathbf{a}_2^t)\rangle}_{H^m} \\
&= S\mathbf{a}_1^t + G\mathbf{a}_2^t - S\overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)S\rangle}_{H^m}\mathbf{a}_1^t - S\overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)G\rangle}_{H^m}\mathbf{a}_2^t \\
&\quad - G\overline{\langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)S\rangle}_{H^m}\mathbf{a}_1^t - G\overline{\langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)G\rangle}_{H^m}\mathbf{a}_2^t \\
&= S\left[\left(I_m - \overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)S\rangle}_{H^m}\right)\mathbf{a}_1^t - \overline{\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)G\rangle}_{H^m}\mathbf{a}_2^t\right] \\
&\quad + G\left[\left(I_m - \overline{\langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)G\rangle}_{H^m}\right)\mathbf{a}_2^t - \overline{\langle F^t, \widehat{A}^2 R_\lambda(\widehat{A}^2)S\rangle}_{H^m}\mathbf{a}_1^t\right] \\
&= (S, G)\overline{W}_\lambda \mathbf{a}^t = (S, G)\mathbf{0}^t = 0.
\end{aligned} \tag{30}$$

Consequently, $x_0 \in \ker \mathbf{B}_\lambda$ and so $\lambda \notin \rho(\mathbf{B})$, which is not true since $\lambda \in \rho(\mathbf{B}) \cap \rho(\widehat{A}^2)$. Therefore $\det W_\lambda \neq 0$ and $\rho(\mathbf{B}) \cap \rho(\widehat{A}^2) = \{\lambda \in \rho(\widehat{A}^2) : \det W_\lambda \neq 0\}$. □

Remark 2. The linear independence of the components of the vector (S, G) is required to prove the necessary condition of (i). So, if the components of the vector (S, G) are not linearly independent, then holds only: $\lambda \in \rho(\mathbf{B}))$ if $\det W_\lambda \neq 0$.

Resolvent of Quadratic Operators

In reference [22, Theorem 4.6] it is stated that for the operator $\mathbf{B}$ defined in (16) holds $\mathbf{B} = B^2$, where B is as in (5), if and only if

$$G \in D(\widehat{A}) \ \text{ and } \ S = BG = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}_{H^m}. \tag{31}$$

This is important because it provides the means to construct the resolvent $R_{\lambda^2}(B^2)$ of the quadratic operator B^2 from the resolvents $R_\lambda(\widehat{A})$ and $R_{-\lambda}(\widehat{A})$ as it is shown below.

Before we articulate the main theorem we have to prove first the next lemma.

Lemma 2. *If $\widehat{A} : H \to H$ is a closed linear operator in a complex Hilbert space H and the resolvent set $\rho(\widehat{A})$ is nonempty, then:*

(i) For $\pm\lambda \in \rho(\widehat{A})$ the resolvent operators $R_\lambda(\widehat{A})$ and $R_{-\lambda}(\widehat{A})$ are bounded and are defined on the whole H and commute.
(ii) The resolvent set $\rho(\widehat{A}^2) = \{\lambda^2 \in \mathbf{C} : \pm\lambda \in \rho(\widehat{A})\}$, i.e. $\lambda^2 \in R_{\lambda^2}(\widehat{A}^2)$ if and only if $\pm\lambda \in \rho(\widehat{A})$. The resolvent operator $R_{\lambda^2}(\widehat{A}^2)$ is bounded and is defined on the whole H. Moreover,

$$R_{\lambda^2}(\widehat{A}^2) = R_\lambda(\widehat{A})R_{-\lambda}(\widehat{A}), \tag{32}$$

$$\widehat{A}R_{\lambda^2}(\widehat{A}^2) = R_{-\lambda}(\widehat{A}) + \lambda R_{\lambda^2}(\widehat{A}^2), \tag{33}$$

$$\widehat{A}R_{\lambda^2}(\widehat{A}^2) = R_\lambda(\widehat{A}) - \lambda R_{\lambda^2}(\widehat{A}^2). \tag{34}$$

(iii) If in addition $\lambda \neq 0$, then

$$R_{\lambda^2}(\widehat{A}^2) = \frac{1}{2\lambda}[R_\lambda(\widehat{A}) - R_{-\lambda}(\widehat{A})], \tag{35}$$

$$\widehat{A}R_{\lambda^2}(\widehat{A}^2) = \frac{1}{2}\left[R_\lambda(\widehat{A}) + R_{-\lambda}(\widehat{A})\right], \tag{36}$$

$$\widehat{A}^2R_{\lambda^2}(\widehat{A}^2) = I + \frac{\lambda}{2}\left[R_\lambda(\widehat{A}) - R_{-\lambda}(\widehat{A})\right]. \tag{37}$$

Proof. (i) Since $\widehat{A}$ is a closed linear operator and $\pm\lambda \in \rho(\widehat{A})$ then the resolvent operators $R_{-\lambda}(\widehat{A})$ and $R_\lambda(\widehat{A})$ are bounded and are defined on the whole H by Lemma 1. The commuting property $R_\lambda(\widehat{A})R_{-\lambda}(\widehat{A}) = R_{-\lambda}(\widehat{A})R_\lambda(\widehat{A})$ follows from the resolvent equation

$$R_{\lambda_1}(\widehat{A}) - R_{\lambda_2}(\widehat{A}) = (\lambda_1 - \lambda_2)R_{\lambda_1}(\widehat{A})R_{\lambda_2}(\widehat{A}) \tag{38}$$

which holds for every closed operator $\widehat{A}$ in a complex Banach space and for every $\lambda_1, \lambda_2 \in \rho(\widehat{A})$, see, e.g., [10].

(ii) Note that for each $\lambda \in \mathbf{C}$ and $D(\widehat{A} \pm \lambda I) = D(\widehat{A})$, $D(\widehat{A}^2 \pm \lambda^2 I) = D(\widehat{A}^2)$ we have

$$\widehat{A}^2 - \lambda^2 I = (\widehat{A} + \lambda I)(\widehat{A} - \lambda I),$$
$$(\widehat{A}^2 - \lambda^2 I)x = (\widehat{A} + \lambda I)(\widehat{A} - \lambda I)x = f, \quad x \in D(\widehat{A}^2), \quad f \in H. \tag{39}$$

Let $\pm\lambda \in \rho(\widehat{A})$. Then because of case (i) we get

$$R_{-\lambda}(\widehat{A})(\widehat{A} + \lambda I)(\widehat{A} - \lambda I)x = R_{-\lambda}(\widehat{A})f,$$
$$R_{\lambda}(\widehat{A})(\widehat{A} - \lambda I)x = R_{\lambda}(\widehat{A})R_{-\lambda}(\widehat{A})f,$$
$$x = R_{\lambda}(\widehat{A})R_{-\lambda}(\widehat{A})f. \tag{40}$$

It follows from (39) that

$$x = (\widehat{A}^2 - \lambda^2 I)^{-1}f = R_{\lambda}(\widehat{A})R_{-\lambda}(\widehat{A})f \tag{41}$$

and hence $R_{\lambda^2}(\widehat{A}^2)$ is bounded and is defined on the whole H. Thus, $\lambda^2 \in \rho(\widehat{A}^2)$.

Conversely, by Proposition 2 the operator $\widehat{A}^2$ is closed. Let $\lambda^2 \in \rho(\widehat{A}^2)$. Then by Lemma 1, the resolvent operator $R_{\lambda^2}(\widehat{A}^2)$ is bounded and defined on the whole H. From (39) and since $\ker(\widehat{A}^2 - \lambda^2 I) = \{0\}$ and the operators $(\widehat{A} - \lambda I)$, $(\widehat{A} + \lambda I)$ commute, it follows that $\ker(\widehat{A} \pm \lambda I) = \{0\}$ and $(\widehat{A}^2 - \lambda^2 I)^{-1} = (\widehat{A} - \lambda I)^{-1}(\widehat{A} + \lambda I)^{-1}$. Because $R(\widehat{A}^2 - \lambda^2 I) = H$ it is implied that $R(\widehat{A} \pm \lambda I) = H$. Since $\widehat{A}$ is closed then $\widehat{A} \pm \lambda I$ and $(\widehat{A} \pm \lambda I)^{-1}$ are closed too. By the Closed Graph theorem the operators $(\widehat{A} \pm \lambda I)^{-1}$ are bounded. Hence $\pm\lambda \in \rho(\widehat{A})$.

Furthermore, the identity (32) follows from Eq. (41) while the identities (33) and (34) are easily proved, viz.

$$\widehat{A}R_{\lambda^2}(\widehat{A}^2) = \widehat{A}R_{\lambda}(\widehat{A})R_{-\lambda}(\widehat{A}) = [I + \lambda R_{\lambda}(\widehat{A})]R_{-\lambda}(\widehat{A}) = R_{-\lambda}(\widehat{A}) + \lambda R_{\lambda^2}(\widehat{A}^2),$$
$$\widehat{A}R_{\lambda^2}(\widehat{A}^2) = \widehat{A}R_{-\lambda}(\widehat{A})R_{\lambda}(\widehat{A}) = [I - \lambda R_{-\lambda}(\widehat{A})]R_{\lambda}(\widehat{A}) = R_{\lambda}(\widehat{A}) - \lambda R_{\lambda^2}(\widehat{A}^2).$$

(iii) From (41) and (38) we get

$$R_{\lambda^2}(\widehat{A}^2) = R_{\lambda}(\widehat{A})R_{-\lambda}(\widehat{A}) = \frac{1}{2\lambda}[R_{\lambda}(\widehat{A}) - R_{-\lambda}(\widehat{A})].$$

By acting $\widehat{A}$ on the last equation we obtain

$$\widehat{A}R_{\lambda^2}(\widehat{A}^2) = \frac{1}{2\lambda}[\widehat{A}R_{\lambda}(\widehat{A}) - \widehat{A}R_{-\lambda}(\widehat{A})]$$
$$= \frac{1}{2\lambda}\{I + \lambda R_{\lambda}(\widehat{A}) - [I + (-\lambda)R_{-\lambda}(\widehat{A})]\} = \frac{1}{2}\left[R_{\lambda}(\widehat{A}) + R_{-\lambda}(\widehat{A})\right].$$

Finally, operating $\widehat{A}$ on this equation we get (37). This completes the proof. □

Now we present the main theorem for the resolvent of the quadratic operator $\mathbf{B}$.

Theorem 3. *Let H be a complex Hilbert space, $\widehat{A} : H \to H$ a linear closed operator, $\pm\lambda \in \rho(\widehat{A})$ and $\mathbf{B}$ the operator defined by (16). Suppose the vectors G, S satisfy (31) and the components of the vector (S, G) are linearly independent elements. Let the operator*

$$\mathbf{B}_{\lambda^2}x = (\mathbf{B} - \lambda^2 I)x = \widehat{A}^2 x - S\langle \widehat{A}x, F^t\rangle_{H^m} - G\langle \widehat{A}^2 x, F^t\rangle_{H^m} - \lambda^2 x = f,$$
$$D(\mathbf{B}_{\lambda^2}) = D(\widehat{A}^2), \tag{42}$$

where $f \in H$. Then:

(i) $\lambda^2 \in \rho(\mathbf{B})$ *if and only if*

$$\det L_\lambda = \det[I_m - \langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle_{H^m}] \neq 0, \tag{43}$$

$$\det L_{-\lambda} = \det[I_m - \langle F^t, \widehat{A}R_{-\lambda}(\widehat{A})G\rangle_{H^m}] \neq 0. \tag{44}$$

(ii) $\rho(\mathbf{B}) \cap \rho(\widehat{A}^2) = \{\lambda^2 \in \rho(\widehat{A}^2) : \det L_\lambda \neq 0,\ \det L_{-\lambda} \neq 0\}$.

(iii) If $\lambda \neq 0$, $\det L_\lambda \neq 0$ *and* $\det L_{-\lambda} \neq 0$*, then there exists the resolvent operator* $R_{\lambda^2}(\mathbf{B})$ *which is defined on the whole space H, is bounded and is given by*

$$R_{\lambda^2}(\mathbf{B})f = \frac{1}{2\lambda}\Big[R_\lambda(\widehat{A})f - R_{-\lambda}(\widehat{A})f + R_\lambda(\widehat{A})G\overline{L_\lambda^{-1}}\langle \widehat{A}R_\lambda(\widehat{A})f,\ F^t\rangle_{H^m}$$
$$- R_{-\lambda}(\widehat{A})G\overline{L_{-\lambda}^{-1}}\langle \widehat{A}R_{-\lambda}(\widehat{A})f,\ F^t\rangle_{H^m}\Big]. \tag{45}$$

Proof. Note that here, for brevity and for ease of presentation, we denote the inner products without their index $_{H^m}$, e.g. $\langle F^t, G\rangle_{H^m} = \langle F^t, G\rangle$. Also, we use some others shorthands which are explained as they appear.

(i) and (ii) Since $\widehat{A}$ is a closed, so is $\widehat{A}^2$ by Proposition 2. By hypothesis the vectors S and G satisfy (31) and hence by [22, Theorem 4.6] the operator $\mathbf{B} = B^2$, where B as in (5). Theorem 2 affirms that $\lambda^2 \in \rho(\mathbf{B})$ if and only if $\det W_{\lambda^2} \neq 0$. By introducing the notations $T_1 = \widehat{A}R_{\lambda^2}(\widehat{A}^2)$ and $T_2 = \widehat{A}^2 R_{\lambda^2}(\widehat{A}^2)$ for convenience, the $\det W_{\lambda^2}$ in Eq. (18) is written as follows:

$$\det W_{\lambda^2} = |W_{\lambda^2}| = \begin{vmatrix} I_m - \langle F^t, T_1 S\rangle & -\langle F^t, T_1 G\rangle \\ -\langle F^t, T_2 S\rangle & I_m - \langle F^t, T_2 G\rangle \end{vmatrix}. \tag{46}$$

Substituting $S = \widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}$ and utilizing property (3), we have

$$|W_{\lambda^2}| = \begin{vmatrix} I_m - \langle F^t, T_1\Big[\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}\Big]\rangle & -\langle F^t, T_1 G\rangle \\ -\langle F^t, T_2\Big[\widehat{A}G - G\overline{\langle F^t, \widehat{A}G\rangle}\Big]\rangle & I_m - \langle F^t, T_2 G\rangle \end{vmatrix}$$

$$= \begin{vmatrix} I_m - \langle F^t, T_1\widehat{A}G\rangle + \langle F^t, T_1G\rangle\langle F^t, \widehat{A}G\rangle & -\langle F^t, T_1G\rangle \\ -\langle F^t, T_2\widehat{A}G\rangle + \langle F^t, T_2G\rangle\langle F^t, \widehat{A}G\rangle & I_m - \langle F^t, T_2G\rangle \end{vmatrix}. \tag{47}$$

Multiplying from the right the elements of the second column by $\langle F^t, \widehat{A}G\rangle$ and adding to the matching elements of the first column, and replacing $T_1 = \widehat{A}R_{\lambda^2}(\widehat{A^2})$ and $T_2 = \widehat{A}^2R_{\lambda^2}(\widehat{A^2})$, we obtain

$$|W_{\lambda^2}| = \begin{vmatrix} I_m - \langle F^t, \widehat{A}R_{\lambda^2}(\widehat{A^2})\widehat{A}G\rangle & -\langle F^t, \widehat{A}R_{\lambda^2}(\widehat{A^2})G\rangle \\ \langle F^t, \widehat{A}G\rangle - \langle F^t, \widehat{A}^2R_{\lambda^2}(\widehat{A^2})\widehat{A}G\rangle & I_m - \langle F^t, \widehat{A}^2R_{\lambda^2}(\widehat{A^2})G\rangle \end{vmatrix}. \tag{48}$$

Now, by using Proposition 1, Eqs. (33), (34) and the commuting property $R_\lambda(\widehat{A})\widehat{A}G = \widehat{A}R_\lambda(\widehat{A})G$, and denoting $R_{\pm\lambda} = R_{\pm\lambda}(\widehat{A})$ and $P = R_\lambda R_{-\lambda}$ for ease of presentation, we get

$$\begin{aligned} |W_{\lambda^2}| &= \begin{vmatrix} I_m - \langle F^t, (I + \lambda^2 P)G\rangle & -\langle F^t, (R_{-\lambda} + \lambda P)G\rangle \\ \langle F^t, \widehat{A}G\rangle - \langle F^t, \widehat{A}(I + \lambda^2 P)G\rangle & I_m - \langle F^t, (I + \lambda^2 P)G\rangle \end{vmatrix} \\ &= \begin{vmatrix} I_m - \langle F^t, G\rangle - \overline{\lambda^2}\langle F^t, PG\rangle & -\langle F^t, R_{-\lambda}G\rangle - \overline{\lambda}\langle F^t, PG\rangle \\ -\overline{\lambda^2}\langle F^t, R_{-\lambda}G\rangle - \overline{\lambda^3}\langle F^t, PG\rangle & I_m - \langle F^t, G\rangle - \overline{\lambda^2}\langle F^t, PG\rangle \end{vmatrix}. \end{aligned} \tag{49}$$

Multiplying the elements of the first row by $-\overline{\lambda}$ and adding to the corresponding elements of the second row, we obtain

$$|W_{\lambda^2}| = \begin{vmatrix} I_m - \langle F^t, G\rangle - \overline{\lambda^2}\langle F^t, PG\rangle & -\langle F^t, R_{-\lambda}G\rangle - \overline{\lambda}\langle F^t, PG\rangle \\ -\overline{\lambda}I + \overline{\lambda}\langle F^t, G\rangle - \overline{\lambda^2}\langle F^t, R_{-\lambda}G\rangle & I_m - \langle F^t, G\rangle + \overline{\lambda}\langle F^t, R_{-\lambda}G\rangle \end{vmatrix}. \tag{50}$$

Furthermore, multiplying the elements of the second column by $\overline{\lambda}$ and adding to the matching elements of the first column, and replacing $P = R_\lambda R_{-\lambda}$, we get

$$|W_{\lambda^2}| = \begin{vmatrix} I_m - \langle F^t, (I + \lambda R_{-\lambda} + 2\lambda^2 R_\lambda R_{-\lambda})G\rangle & -\langle F^t, (I + \lambda R_\lambda)R_{-\lambda}G\rangle \\ 0_m & I_m - \langle F^t, (I - \lambda R_{-\lambda})G\rangle \end{vmatrix}. \tag{51}$$

Using Proposition 1 and (32)–(34) we get $I - \lambda R_{-\lambda} = \widehat{A}R_{-\lambda}$ and

$$\begin{aligned} I + \lambda R_{-\lambda} + 2\lambda^2 R_\lambda R_{-\lambda} &= I + \lambda(\widehat{A}R_\lambda R_{-\lambda} - \lambda R_\lambda R_{-\lambda}) + 2\lambda^2 R_\lambda R_{-\lambda} \\ &= I + \lambda\widehat{A}R_\lambda R_{-\lambda} + \lambda^2 R_\lambda R_{-\lambda} \\ &= I + \lambda(R_\lambda - \lambda R_{-\lambda}R_\lambda) + \lambda^2 R_\lambda R_{-\lambda} \\ &= I + \lambda R_\lambda = \widehat{A}R_\lambda. \end{aligned} \tag{52}$$

Substituting these results in (51), we acquire

$$\det W_{\lambda^2} = \det\left[I_m - \langle F^t, \widehat{A}R_\lambda G\rangle\right] \cdot \det\left[I_m - \langle F^t, \widehat{A}R_{-\lambda} G\rangle\right], \tag{53}$$

Hence, $\lambda^2 \in \rho(\mathbf{B}) \cap \rho(\widehat{A}^2)$ if and only if $\det L_{\pm\lambda} \neq 0$.

(iii) It has been proven that $\widehat{A}^2$ is a closed operator and $\lambda^2 \in \rho(\mathbf{B}) \cap \rho(\widehat{A}^2)$. Then $\pm\lambda \in \rho(\widehat{A})$ by Lemma 2 and also $\pm\lambda \in \rho(B)$ by Theorem 1. From Lemma 2 for $\lambda \neq 0$ we find

$$R_{\lambda^2}(\mathbf{B}) = \frac{1}{2\lambda}[R_\lambda(B) - R_{-\lambda}(B)], \tag{54}$$

where the resolvent operator $R_\lambda(B)$ is set out in (8) and $R_{-\lambda}(B)$ is the same as in (8) except that λ is replaced by $-\lambda$. Substituting these formulas into (54) we obtain (45). The resolvent operators $R_{\pm\lambda}(B)$ are bounded and are defined on the whole H and so is $R_{\lambda^2}(\mathbf{B})$ by (54). This completes the proof.

Applications

In this section we apply the theory presented in the previous sections to boundary value problems involving integro-differential equations of the Fredholm type. In particular, we find the resolvent sets and provide closed form representations for the resolvent operators. Some auxiliary results needed are quoted in Appendix for ease of reference. By $H^1(0,1)$ (resp. $H^2(0,1)$) is denoted the Sobolev space of all complex functions of $L_2(0,1)$ which have generalized derivatives up to the first (resp. second) order that are Lebesgue integrable.

First Order Integro-differential Equation

Consider the following integro-differential boundary value problem

$$\begin{aligned} &iu'(t) - ie^{i\pi t}\int_0^1 xu'(x)dx - \lambda u(t) = f(t), \\ &u(0) + u(1) = 0, \quad u(t) \in H^1(0,1). \end{aligned} \tag{55}$$

In Problem 1 in the Appendix it is quoted that the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ defined by

$$\widehat{A}u = iu' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) + u(1) = 0\}, \tag{56}$$

is a linear closed operator and that $\rho(\widehat{A}) = \{\lambda \in \mathbf{C} : \lambda \neq (2k+1)\pi,\ k \in \mathbf{Z}\}$, while for $\lambda \in \rho(\widehat{A})$ the resolvent operator $R_\lambda(\widehat{A})$ is defined on the whole space $L_2(0,1)$, is bounded and is given explicitly by the formula (98). Let $B : L_2(0,1) \to L_2(0,1)$ be the operator

$$\begin{aligned} Bu(t) &= iu'(t) - ie^{i\pi t}\int_0^1 xu'(x)dx \\ &= \widehat{A}u(t) - G\langle \widehat{A}u(t), F^t\rangle, \quad D(B) = D(\widehat{A}), \end{aligned} \tag{57}$$

where $G = e^{i\pi t}$ and $F = t$. We express the boundary value problem (55) in the operator form

$$B_\lambda u(t) = (B - \lambda I)u(t) = f(t), \quad D(B_\lambda) = D(\widehat{A}). \tag{58}$$

In applying Theorem 1 we have first to compute the determinant

$$\det L_\lambda = \det\left[I_m - \langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle_{H^m}\right] \neq 0. \tag{59}$$

Acting $R_\lambda(\widehat{A})$ on G, operating by $\widehat{A}$ and taking the inner products, we find

$$\begin{aligned} &R_\lambda(\widehat{A})G = ie^{-i\lambda t}\left[(e^{i\lambda}+1)^{-1}\int_0^1 e^{i\pi x}e^{i\lambda x}dx - \int_0^t e^{i\pi x}e^{i\lambda x}dx\right] = -\frac{e^{i\pi t}}{\pi+\lambda}, \\ &\widehat{A}R_\lambda(\widehat{A})G = \frac{\pi}{\pi+\lambda}e^{i\pi t}, \\ &\langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle_{H^m} = \frac{\pi}{\pi+\overline{\lambda}}\int_0^1 xe^{-i\pi x}dx = -\frac{2+i\pi}{(\pi+\overline{\lambda})\pi}. \end{aligned} \tag{60}$$

Substituting Eq. (60) into (59) we have

$$\det L_\lambda = \det\left[1 + \frac{2+i\pi}{(\pi+\overline{\lambda})\pi}\right] = \frac{\pi^2+\overline{\lambda}\pi+i\pi+2}{(\pi+\overline{\lambda})\pi} \neq 0, \tag{61}$$

which implies that $\det L_\lambda \neq 0$ if and only if $\lambda \neq -(\pi^2+2)/\pi + i$. Then from Theorem 1 it follows that $\lambda \in \rho(B)$ if $\lambda \neq (2k+1)\pi,\ k \in \mathbf{Z}$ and $\lambda \neq -(\pi^2+2)/\pi + i$. Moreover, the resolvent operator $R_\lambda(B)$ is bounded and defined in all $L_2(0,1)$ and is given by

$$R_\lambda(B)f = R_\lambda(\widehat{A})f + R_\lambda(\widehat{A})G\overline{L_\lambda^{-1}}\langle \widehat{A}R_\lambda(\widehat{A})f, F^t\rangle_{H^m}. \tag{62}$$

Applying $\widehat{A}$ on Eq. (98) and then forming the inner products we get

$$\widehat{A}R_\lambda(\widehat{A})f = e^{-i\lambda t} i\lambda \left[(e^{i\lambda}+1)^{-1} \int_0^1 f(x)e^{i\lambda x}dx - \int_0^t f(x)e^{i\lambda x}dx \right] + f(t),$$

$$\begin{aligned}\langle \widehat{A}R_\lambda(\widehat{A})f, F^t\rangle_{H^m} &= \int_0^1 \left\{ e^{-i\lambda t} i\lambda \left[(e^{i\lambda}+1)^{-1} \int_0^1 f(x)e^{i\lambda x}dx \right.\right. \\ &\quad \left.\left. - \int_0^t f(x)e^{i\lambda x}dx \right] + f(t) \right\} tdt \\ &= i\lambda(e^{i\lambda}+1)^{-1} \int_0^1 e^{-i\lambda t}tdt \int_0^1 f(x)e^{i\lambda x}dx \\ &\quad -i\lambda \int_0^1 e^{-i\lambda t} \left[\int_0^t f(x)e^{i\lambda x}dx \right] tdt + \int_0^1 tf(t)dt. \qquad (63)\end{aligned}$$

Exploiting the Fubini theorem we find

$$\begin{aligned}\langle \widehat{A}R_\lambda(\widehat{A})f, F^t\rangle_{H^m} &= i\lambda(e^{i\lambda}+1)^{-1} \int_0^1 f(x)e^{i\lambda x}dx \frac{1}{\lambda^2}\left[e^{-i\lambda}(i\lambda+1)-1\right] \\ &\quad - i\lambda \int_0^1 f(x)e^{i\lambda x}dx \int_x^1 e^{-i\lambda t}tdt + \int_0^1 f(t)tdt \\ &= \frac{i\left[e^{-i\lambda}(i\lambda+1)-1\right]}{\lambda(e^{i\lambda}+1)} \int_0^1 f(x)e^{i\lambda x}dx \\ &\quad - \frac{i(1+i\lambda)}{\lambda} \int_0^1 f(x)e^{i\lambda(x-1)}dx \\ &\quad + \int_0^1 \frac{i(i\lambda x+1)}{\lambda} f(x)dx + \int_0^1 f(x)xdx \\ &= \frac{i\left[e^{-i\lambda}(i\lambda+1)-1\right]}{\lambda(e^{i\lambda}+1)} \int_0^1 f(x)e^{i\lambda x}dx \\ &\quad - \frac{i(1+i\lambda)}{\lambda} \int_0^1 f(x)e^{i\lambda(x-1)}dx + \frac{i}{\lambda}\int_0^1 f(x)dx \\ &= \frac{i\left[e^{-i\lambda}(i\lambda+1)-1\right]}{\lambda(e^{i\lambda}+1)} \int_0^1 e^{i\lambda x}f(x)dx \\ &\quad - i\int_0^1 \frac{(1+i\lambda)e^{i\lambda(x-1)}-1}{\lambda} f(x)dx. \qquad (64)\end{aligned}$$

Finally, by substituting (98), (60), (64) and the inverse $\overline{L_\lambda^{-1}}$ from (61) into (62), we get

$$\begin{aligned}R_\lambda(B)f(t) &= R_\lambda(\widehat{A})f(t) \\ &\quad - \frac{i\pi e^{i\pi t}}{\pi^2+\lambda\pi-i\pi+2} \left\{ \frac{e^{-i\lambda}(i\lambda+1)-1}{\lambda(e^{i\lambda}+1)} \int_0^1 f(x)e^{i\lambda x}dx \right.\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^1 \frac{(1+i\lambda)e^{i\lambda(x-1)} - 1}{\lambda} f(x)dx \Big\} \\
&= R_\lambda(\widehat{A})f(t) \\
&\quad - \frac{i\pi e^{i\pi t}(e^{i\lambda}+1)^{-1}}{\pi^2 + (\lambda - i)\pi + 2} \int_0^1 \frac{e^{i\lambda} + 1 - (2+i\lambda)e^{i\lambda x}}{\lambda} f(x)dx. \qquad (65)
\end{aligned}
$$

Second Order Integro-differential Equation

Consider the following boundary value problem involving a second order Fredholm integro-differential equation

$$
\begin{aligned}
&u''(t) - t\int_0^1 u'(x)\cos 2\pi x dx - \int_0^1 u''(x)\cos 2\pi x dx - \lambda u(t) = f(t), \\
&u(0) = u(1), \quad u'(0) = u'(1), \quad u(t) \in H^2(0,1). \qquad (66)
\end{aligned}
$$

We define the operators $\widehat{A},\ \widehat{A}^2 : L_2(0,1) \to L_2(0,1)$ as follows:

$$
\widehat{A}u = u'(t), \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) = u(1)\}, \qquad (67)
$$

$$
\widehat{A}^2 u = u''(t), \quad D(\widehat{A}^2) = \{u(t) \in H^2(0,1) : u(0) = u(1),\, u'(0) = u'(1)\}. \qquad (68)
$$

In Problem 2 in the Appendix it is given that the operator $\widehat{A}^2$ is closed and that the resolvent set is $\rho(\widehat{A}^2) = \{\lambda \in \mathbf{C} : \lambda \neq -4k^2\pi^2,\ k \in \mathbf{Z}\}$, whereas for $\lambda \in \rho(\widehat{A}^2)$ the resolvent operator $R_\lambda(\widehat{A}^2)$ is defined on the whole space $L_2(0,1)$, is bounded and is expressed analytically in (105). Additionally, we define the operator $\mathbf{B} : L_2(0,1) \to L_2(0,1)$ as

$$
\begin{aligned}
\mathbf{B}u(t) &= u''(t) - t\int_0^1 u'(x)\cos 2\pi x dx - \int_0^1 u''(x)\cos 2\pi x dx \\
&= \widehat{A}^2 x - S\langle \widehat{A}x, F^t\rangle_{H^m} - G\langle \widehat{A}^2 x, F^t\rangle_{H^m}, \quad D(\mathbf{B}) = D(\widehat{A}^2), \qquad (69)
\end{aligned}
$$

where $S = t$, $G = 1$ and $F = \cos 2\pi t$. We reformulate the integro-differential equation (66) as

$$
\mathbf{B}_\lambda u(t) = (\mathbf{B} - \lambda I)u(t) = f(t), \quad D(\mathbf{B}_\lambda) = D(\widehat{A}^2). \qquad (70)
$$

According to Theorem 2, any λ from $\rho(\widehat{A}^2)$ belongs to $\rho(\mathbf{B})$ if and only if Eq. (18) is satisfied, i.e. $\det W_\lambda \neq 0$. By acting $R_\lambda(\widehat{A}^2)$ from Eq. (105) on S, applying $\widehat{A}$ and $\widehat{A}^2$ and taking the inner products, we have

$$R_\lambda(\widehat{A}^2)S = \frac{1}{2\lambda}\left[\frac{e^{t\sqrt{\lambda}}}{e^{\sqrt{\lambda}}-1} + \frac{e^{(1-t)\sqrt{\lambda}}}{1-e^{\sqrt{\lambda}}} - 2t\right], \tag{71}$$

$$\widehat{A}R_\lambda(\widehat{A}^2)S = \frac{1}{2\sqrt{\lambda}}\left[\frac{e^{t\sqrt{\lambda}}}{e^{\sqrt{\lambda}}-1} + \frac{e^{(1-t)\sqrt{\lambda}}}{e^{\sqrt{\lambda}}-1} - \frac{2}{\sqrt{\lambda}}\right], \tag{72}$$

$$\widehat{A}^2R_\lambda(\widehat{A}^2)S = \frac{e^{t\sqrt{\lambda}}}{2(e^{\sqrt{\lambda}}-1)} + \frac{e^{(1-t)\sqrt{\lambda}}}{2(1-e^{\sqrt{\lambda}})}, \tag{73}$$

$$\begin{aligned}\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)S\rangle_H &= \frac{1}{2\sqrt{\overline{\lambda}}}\int_0^1\left[\frac{e^{t\sqrt{\overline{\lambda}}}}{e^{\sqrt{\overline{\lambda}}}-1} + \frac{e^{(1-t)\sqrt{\overline{\lambda}}}}{e^{\sqrt{\overline{\lambda}}}-1} - \frac{2}{\sqrt{\overline{\lambda}}}\right]\cos 2\pi t dt \\ &= \frac{1}{\overline{\lambda}+4\pi^2},\end{aligned} \tag{74}$$

$$\langle F^t, \widehat{A}^2R_\lambda(\widehat{A}^2)S\rangle_H = 0. \tag{75}$$

Imitating the same procedure for G, we get

$$R_\lambda(\widehat{A}^2)G = -\frac{1}{\lambda}, \tag{76}$$

$$\widehat{A}R_\lambda(\widehat{A}^2)G = \widehat{A}^2R_\lambda(\widehat{A}^2)G = 0, \tag{77}$$

$$\langle F^t, \widehat{A}R_\lambda(\widehat{A}^2)G\rangle_H = \langle F^t, \widehat{A}^2R_\lambda(\widehat{A}^2)G\rangle_H = 0. \tag{78}$$

Substituting Eqs. (74), (75) and (78) into (18), we obtain

$$\det W_\lambda = \det\begin{pmatrix} 1-\frac{1}{(\overline{\lambda}+4\pi^2)} & 0 \\ 0 & 1\end{pmatrix} = \frac{\overline{\lambda}+4\pi^2-1}{\overline{\lambda}+4\pi^2} \neq 0. \tag{79}$$

Hence $\lambda \in \rho(\mathbf{B})$ if $\lambda \neq -4k^2\pi^2,\ k \in \mathbf{Z}$ and $\lambda \neq 1-4\pi^2$. Moreover, Theorem 2 provides the resolvent operator $R_\lambda(\mathbf{B})$ as in formula (19).

The inverse matrix W_λ^{-1} is easily computed as

$$W_\lambda^{-1} = \begin{pmatrix} W_{\lambda 11}^{-1} & W_{\lambda 12}^{-1} \\ W_{\lambda 21}^{-1} & W_{\lambda 22}^{-1}\end{pmatrix} = \begin{pmatrix} \frac{\bar{\lambda}+4\pi^2}{\bar{\lambda}+4\pi^2-1} & 0 \\ 0 & 1\end{pmatrix}. \tag{80}$$

Using Eqs. (71), (76) and (80) as well as the fact that the operator $R_\lambda(\widehat{A}^2)$ is linear, we find

$$R_\lambda(\widehat{A}^2)(S\overline{W_{\lambda 11}^{-1}} + G\overline{W_{\lambda 21}^{-1}}) = \frac{\lambda+4\pi^2}{2\lambda(\lambda+4\pi^2-1)}\left[\frac{e^{t\sqrt{\lambda}}}{e^{\sqrt{\lambda}}-1} + \frac{e^{(1-t)\sqrt{\lambda}}}{1-e^{\sqrt{\lambda}}} - 2t\right], \tag{81}$$

$$R_\lambda(\widehat{A}^2)(S\overline{W_{\lambda 12}^{-1}}+G\overline{W_{\lambda 22}^{-1}})=-\frac{1}{\lambda}. \tag{82}$$

Acting $\widehat{A}$ on (105), taking the inner products and utilizing the Fubini theorem, we obtain

$$\widehat{A}R_\lambda(\widehat{A}^2)f(t)=\frac{1}{2(1-e^{\sqrt{\lambda}})}\int_0^1\left[e^{\sqrt{\lambda}(t-x+1)}-e^{-\sqrt{\lambda}(t-x)}\right]f(x)dx$$
$$+\frac{1}{2}\int_0^t\left[e^{\sqrt{\lambda}(t-x)}+e^{-\sqrt{\lambda}(t-x)}\right]f(x)dx,$$

$$\begin{aligned}
&\langle\widehat{A}R_\lambda(\widehat{A}^2)f(t),F^t\rangle_H\\
&=\frac{1}{2(1-e^{\sqrt{\lambda}})}\int_0^1\cos 2\pi t dt\int_0^1\left[e^{\sqrt{\lambda}(t-x+1)}-e^{-\sqrt{\lambda}(t-x)}\right]f(x)dx\\
&+\frac{1}{2}\int_0^1\cos 2\pi t dt\int_0^t\left[e^{\sqrt{\lambda}(t-x)}+e^{-\sqrt{\lambda}(t-x)}\right]f(x)dx\\
&=\frac{1}{2(1-e^{\sqrt{\lambda}})}\int_0^1 f(x)dx\int_0^1\cos 2\pi t\left[e^{\sqrt{\lambda}(t-x+1)}-e^{-\sqrt{\lambda}(t-x)}\right]dt+\\
&+\frac{1}{2}\int_0^1 f(x)dx\int_x^1\cos 2\pi t\left[e^{\sqrt{\lambda}(t-x)}+e^{-\sqrt{\lambda}(t-x)}\right]dt\\
&=\frac{1}{2(1-e^{\sqrt{\lambda}})}\int_0^1\frac{\sqrt{\lambda}\left(e^{\sqrt{\lambda}(x-1)}-e^{\sqrt{\lambda}x}+e^{\sqrt{\lambda}(2-x)}-e^{\sqrt{\lambda}(1-x)}\right)}{\lambda+4\pi^2}f(x)dx+\\
&+\frac{1}{2}\int_0^1\frac{-\sqrt{\lambda}e^{\sqrt{\lambda}(x-1)}+\sqrt{\lambda}e^{\sqrt{\lambda}(1-x)}-4\pi\sin 2\pi x}{\lambda+4\pi^2}f(x)dx\\
&=\frac{-2\pi}{\lambda+4\pi^2}\int_0^1\sin 2\pi x f(x)dx.
\end{aligned} \tag{83}$$

Working in the same way with $\widehat{A}^2$, we get

$$\begin{aligned}
&\widehat{A}^2R_\lambda(\widehat{A}^2)f(t)=f(t)+\lambda R_\lambda(\widehat{A}^2)f(t),\\
&\langle\widehat{A}^2R_\lambda(\widehat{A}^2)f(t),F^t\rangle_H=\langle f(t)+\lambda R_\lambda(\widehat{A}^2)f(t),F^t\rangle_H\\
&=\langle f(t),F^t\rangle_H+\lambda\langle R_\lambda(\widehat{A}^2)f(t),F^t\rangle_H\\
&=\langle f(t),F^t\rangle_H\\
&\quad+\frac{\sqrt{\lambda}}{2(1-e^{\sqrt{\lambda}})}\int_0^1\cos 2\pi t dt\int_0^1\left[e^{\sqrt{\lambda}(t-x+1)}+e^{\sqrt{\lambda}(x-t)}\right]f(x)dx
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sqrt{\lambda}}{2} \int_0^1 \cos 2\pi t dt \int_0^t \left[e^{\sqrt{\lambda}(t-x)} - e^{\sqrt{\lambda}(x-t)} \right] f(x) dx. \\
&= \langle f(t), F^t \rangle_H \\
&+ \frac{\sqrt{\lambda}}{2(1-e^{\sqrt{\lambda}})} \int_0^1 f(x) dx \int_0^1 \left[e^{\sqrt{\lambda}(t-x+1)} + e^{\sqrt{\lambda}(x-t)} \right] \cos 2\pi t dt \\
&+ \frac{\sqrt{\lambda}}{2} \int_0^1 f(x) dx \int_x^1 \left[e^{\sqrt{\lambda}(t-x)} - e^{\sqrt{\lambda}(x-t)} \right] \cos 2\pi t dt \\
&= \langle f(t), F^t \rangle_H \\
&+ \frac{\lambda}{2(1-e^{\sqrt{\lambda}})(\lambda+4\pi^2)} \int_0^1 \left[e^{\sqrt{\lambda}x} - e^{\sqrt{\lambda}(x-1)} + e^{\sqrt{\lambda}(2-x)} - e^{\sqrt{\lambda}(1-x)} \right] f(x) dx \\
&+ \frac{\lambda}{2(\lambda+4\pi^2)} \int_0^1 \left[e^{\sqrt{\lambda}(x-1)} + e^{\sqrt{\lambda}(1-x)} - 2\cos 2\pi x \right] f(x) dx \\
&= \int_0^1 \cos 2\pi x f(x) dx - \frac{\lambda}{\lambda+4\pi^2} \int_0^1 \cos 2\pi x f(x) dx \\
&= \frac{4\pi^2}{\lambda+4\pi^2} \int_0^1 \cos 2\pi x f(x) dx.
\end{aligned} \tag{84}
$$

Lastly, by substituting (105) and (81)–(84) into (19) we obtain the resolvent operator $R_\lambda(\mathbf{B})$ in the following closed form

$$
\begin{aligned}
R_\lambda(\mathbf{B}) f = & R_\lambda(\widehat{A}^2) f(t) \\
& - \frac{\pi(e^{\sqrt{\lambda}t} - e^{\sqrt{\lambda}(1-t)} + 2t(1-e^{\sqrt{\lambda}})}{\lambda(\lambda+4\pi^2-1)(e^{\sqrt{\lambda}}-1)} \int_0^1 \sin 2\pi x f(x) dx \\
& - \frac{4\pi^2}{\lambda(\lambda+4\pi^2)} \int_0^1 \cos 2\pi x f(x) dx.
\end{aligned} \tag{85}
$$

Integro-Differential Equation with a Quadratic Operator

Consider the following boundary value problem which can be expressed in terms of a quadratic operator

$$
\begin{aligned}
u(t) = & u''(t) - \pi(2\cos 2\pi t - \sin 2\pi t) \int_0^1 u'(x) \cos 2\pi x dx \\
& - \sin 2\pi t \int_0^1 u''(x) \cos 2\pi x dx - \lambda^2 u(t) = f(t), \\
u(0) = & u(1), \quad u'(0) = u'(1), \quad u(t) \in H^2(0,1).
\end{aligned} \tag{86}
$$

Let $\widehat{A},\ \widehat{A}^2 : L_2(0,1) \to L_2(0,1)$ be the operators

$$\widehat{A}u = u'(t),\quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) = u(1)\}, \tag{87}$$

$$\widehat{A}^2 u = u''(t),\quad D(\widehat{A}^2) = \{u(t) \in H^2(0,1) : u(0) = u(1),\ u'(0) = u'(1)\}. \tag{88}$$

Note that, from Problem 2 in the Appendix, $\lambda \in \rho(\widehat{A})$ if and only if $\lambda \neq 2k\pi i,\ k \in \mathbf{Z}$. Then $\lambda^2 \in \rho(\widehat{A}^2)$ iff $\pm\lambda \in \rho(\widehat{A})$ by Lemma 2. Consequently, $\lambda^2 \in \rho(\widehat{A}^2)$ if and only if $\lambda \neq \pm 2k\pi i,\ k \in \mathbf{Z}$. In addition, we define the operator $\mathbf{B} : L_2(0,1) \to L_2(0,1)$ as follows:

$$\begin{aligned}
\mathbf{B}u(t) &= u''(t) - \pi(2\cos 2\pi t - \sin 2\pi t)\int_0^1 u'(x)\cos 2\pi x dx \\
&\qquad - \sin 2\pi t \int_0^1 u''(x)\cos 2\pi x dx \\
&= \widehat{A}^2 u(t) - S\langle \widehat{A}u(t), F^t\rangle_{H^m} - G\langle \widehat{A}^2 u(t), F^t\rangle_{H^m},\quad D(\mathbf{B}) = D(\widehat{A}^2),
\end{aligned} \tag{89}$$

where $S = \pi(2\cos 2\pi t - \sin 2\pi t)$, $G = \sin 2\pi t$ and $F = \cos 2\pi t$. We observe that the components of the vector (S, G) are linearly independent and most important S and G satisfy (31). Therefore the operator $\mathbf{B}$ is quadratic and hence we apply Theorem 3. Accordingly, we rewrite the integro-differential equation (86) in the operator form

$$\mathbf{B}_{\lambda^2}u(t) = (\mathbf{B} - \lambda^2 I)u(t) = f(t),\quad D(\mathbf{B}_{\lambda^2}) = D(\widehat{A}^2). \tag{90}$$

Theorem 3 claims that the resolvent operator $R_{\lambda^2}(\mathbf{B})$ exists if and only if Eqs. (43) and (44) are satisfied, namely $\det L_\lambda \neq 0$ and $\det L_{-\lambda} \neq 0$. In computing these determinants we need the resolvent operators $R_\lambda(\widehat{A})$ and $R_{-\lambda}(\widehat{A})$ which are set out in (103) and (104) in the Appendix. By applying $R_\lambda(\widehat{A})$ on G, employing $\widehat{A}$ and taking the inner products, we obtain

$$\begin{aligned}
R_\lambda(\widehat{A})G &= \frac{1}{1-e^\lambda}\int_0^1 e^{\lambda(t-x+1)}\sin 2\pi x dx + \int_0^t e^{\lambda(t-x)}\sin 2\pi x dx \\
&= \frac{-\lambda \sin 2\pi t - 2\pi\cos 2\pi t}{\lambda^2 + 4\pi^2}, \\
\widehat{A}R_\lambda(\widehat{A})G &= \frac{-2\pi\lambda\cos 2\pi t + 4\pi^2 \sin 2\pi t}{\lambda^2 + 4\pi^2}, \\
\langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle &= \frac{-\pi\overline{\lambda}}{\overline{\lambda}^2 + 4\pi^2}, \\
L_\lambda &= \left[I_m - \langle F^t, \widehat{A}R_\lambda(\widehat{A})G\rangle\right] = \frac{\overline{\lambda}^2 + 4\pi^2 + \pi\overline{\lambda}}{\overline{\lambda}^2 + 4\pi^2}.
\end{aligned} \tag{91}$$

Repeating the same sequence of operations for $R_{-\lambda}(\widehat{A})$, we have

$$\begin{aligned}
R_{-\lambda}(\widehat{A})G &= \frac{1}{1-e^{-\lambda}}\int_0^1 e^{-\lambda(t-x+1)}\sin 2\pi x dx + \int_0^t e^{-\lambda(t-x)}\sin 2\pi x dx,\\
\widehat{A}R_{-\lambda}(\widehat{A})G &= \frac{2\pi\lambda\cos 2\pi t + 4\pi^2\sin 2\pi t}{\lambda^2+4\pi^2},\\
\langle F^t, \widehat{A}R_{-\lambda}(\widehat{A})G\rangle &= \frac{\pi\overline{\lambda}}{\overline{\lambda}^2+4\pi^2},\\
L_{-\lambda} &= \left[I_m - \langle F^t, \widehat{A}R_{-\lambda}(\widehat{A})G\rangle\right] = \frac{\overline{\lambda}^2+4\pi^2-\pi\overline{\lambda}}{\overline{\lambda}^2+4\pi^2}. \qquad (92)
\end{aligned}$$

It is evident that $\det L_{\lambda} \neq 0$ if and only if $\lambda \neq \pi(-1 \pm i\sqrt{15})/2$ and $\det L_{-\lambda} \neq 0$ if and only if $\lambda \neq \pi(1 \pm i\sqrt{15})/2$. From Theorem 3, $\lambda^2 \in \rho(\mathbf{B})$ if $\lambda \neq \pm 2k\pi i,\ k \in \mathbf{Z}$ and $\lambda \neq \pi(\pm 1 \pm i\sqrt{15})/2\}$ and the resolvent operator $R_{\lambda^2}(\mathbf{B})$ exists and has the representation as in (45).

By acting $\widehat{A}$ on Eq. (103), making use of Proposition 1 and applying the Fubini theorem, we get

$$\begin{aligned}
&\langle \widehat{A}R_{\lambda}(\widehat{A})f(t), F^t\rangle = \langle f + \lambda R_{\lambda}(\widehat{A})f(t), F^t\rangle\\
&= \int_0^1 \cos 2\pi x f(x)dx\\
&\quad + \int_0^1 \cos 2\pi t\left[\frac{\lambda}{1-e^{\lambda}}\int_0^1 e^{\lambda(t-x+1)}f(x)dx + \lambda\int_0^t e^{\lambda(t-x)}f(x)dx\right]dt\\
&= \int_0^1 \cos 2\pi x f(x)dx + \frac{\lambda}{1-e^{\lambda}}\int_0^1 \cos 2\pi t e^{\lambda t}dt\int_0^1 e^{\lambda(-x+1)}f(x)dx\\
&\quad +\lambda\int_0^1 e^{-\lambda x}f(x)dx\int_x^1 e^{\lambda t}\cos 2\pi t dt\\
&= \int_0^1 \cos 2\pi x f(x)dx - \frac{\lambda^2}{\lambda^2+4\pi^2}\int_0^1 e^{\lambda(1-x)}f(x)dx\\
&\quad +\frac{\lambda}{\lambda^2+4\pi^2}\int_0^1 [\lambda e^{\lambda(1-x)} - (\lambda\cos 2\pi x + 2\pi\sin 2\pi x)]f(x)dx\\
&= -\frac{2\pi\lambda}{\lambda^2+4\pi^2}\int_0^1 \sin 2\pi x f(x)dx + \frac{4\pi^2}{\lambda^2+4\pi^2}\int_0^1 \cos 2\pi x f(x)dx\\
&= \frac{2\pi}{\lambda^2+4\pi^2}\int_0^1 (2\pi\cos 2\pi x - \lambda\sin 2\pi x)f(x)dx. \qquad (93)
\end{aligned}$$

By operating alike on (104), we acquire

$$\langle \widehat{A}R_{-\lambda}(\widehat{A})f(t), F^t\rangle = \frac{2\pi}{\lambda^2+4\pi^2}\int_0^1 (2\pi\cos 2\pi x + \lambda \sin 2\pi x)f(x)dx. \tag{94}$$

Substituting (103), (104), $\overline{L_{\pm\lambda}^{-1}}$ from (91) and (92), (93) and (94) into (45), we get

$$\begin{aligned} R_{\lambda^2}(\mathbf{B})f &= \frac{1}{2\lambda}\left\{\frac{1}{1-e^{\lambda}}\int_0^1 \left[e^{\lambda(t-x+1)} + e^{-\lambda(t-x)}\right]f(x)dx \right. \\ &+ \int_0^t \left[e^{\lambda(t-x)} - e^{-\lambda(t-x)}\right]f(x)dx \\ &- \frac{2\pi(\lambda\sin 2\pi t + 2\pi\cos 2\pi t)}{(\lambda^2+4\pi^2)(\lambda^2+4\pi^2+\pi\lambda)}\int_0^1 (2\pi\cos 2\pi x - \lambda\sin 2\pi x)f(x)dx \\ &\left. - \frac{2\pi(\lambda\sin 2\pi t - 2\pi\cos 2\pi t)}{(\lambda^2+4\pi^2)(\lambda^2+4\pi^2-\pi\lambda)}\int_0^1 (2\pi\cos 2\pi x + \lambda\sin 2\pi x)f(x)dx\right\}. \end{aligned} \tag{95}$$

The resolvent operator $R_{\lambda^2}(\mathbf{B})$ for every $\lambda^2 \in \rho(\mathbf{B})$ is defined on the whole space $L_2(0,1)$ and is bounded.

Appendix

Problem 1. Let the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ be defined by

$$\widehat{A}u = iu' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : u(0) + u(1) = 0\} \tag{96}$$

Then $\hat{A}$ is closed and:

(i) $\lambda \in \rho(\widehat{A})$ if and only if $\lambda \neq (2k+1)\pi, \quad k \in \mathbf{Z}$, i.e.

$$\rho(\widehat{A}) = \{\lambda \in \mathbf{C} : \lambda \neq (2k+1)\pi, \quad k \in \mathbf{Z}\}. \tag{97}$$

(ii) For $\lambda \in \rho(\widehat{A})$ the resolvent operator $R_\lambda(\widehat{A})$ is bounded and defined on the whole space $L_2(0,1)$ by the formula

$$R_\lambda(\widehat{A})f(t) = i\int_0^1 e^{i\lambda(x-t)}\left[(e^{i\lambda}+1)^{-1} - \eta(t-x)\right]f(x)dx, \tag{98}$$

where

$$\eta(t-x) = \begin{cases} 1, & x \le t \\ 0, & x > t \end{cases} \text{ is the Heaviside's function.}$$

Problem 2. Let the operator $\widehat{A} : L_2(0,1) \to L_2(0,1)$ be defined by

$$\widehat{A}u = u' = f, \quad D(\widehat{A}) = \{u(t) \in H^1(0,1) : \ u(0) = u(1)\}. \tag{99}$$

Then the quadratic operator $\widehat{A}^2 : L_2(0,1) \to L_2(0,1)$ is closed and defined by

$$\widehat{A}^2 u = u'' = f, \quad D(\widehat{A}^2) = \{u(t) \in H^2(0,1) : u(0) = u(1),\, u'(0) = u'(1)\}, \tag{100}$$

the resolvent sets of $\widehat{A}$ and $\widehat{A}^2$ are

$$\rho(\widehat{A}) = \ \{\lambda \in \mathbf{C} : \lambda \neq 2k\pi i, \ k \in \mathbf{Z}\}, \tag{101}$$

$$\rho(\widehat{A}^2) = \{\lambda \in \mathbf{C} : \lambda \neq -4k^2\pi^2, \, k \in \mathbf{Z}\} \tag{102}$$

and the resolvent operators $R_{\pm\lambda}(\widehat{A}),\, R_\lambda(\widehat{A}^2)$ are bounded and defined on the whole space $L_2(0,1)$ by

$$R_\lambda(\widehat{A})f(t) \ = \frac{1}{1-e^\lambda}\int_0^1 e^{\lambda(t-x+1)}f(x)dx + \int_0^t e^{\lambda(t-x)}f(x)dx \tag{103}$$

$$R_{-\lambda}(\widehat{A})f(t) \ = \frac{1}{1-e^{-\lambda}}\int_0^1 e^{-\lambda(t-x+1)}f(x)dx + \int_0^t e^{-\lambda(t-x)}f(x)dx \tag{104}$$

$$R_\lambda(\widehat{A}^2)f(t) = \frac{1}{2\sqrt{\lambda}(1-e^{\sqrt{\lambda}})}\int_0^1 \left[e^{\sqrt{\lambda}(t-x+1)} + e^{-\sqrt{\lambda}(t-x)}\right]f(x)dx$$

$$+\frac{1}{2\sqrt{\lambda}}\int_0^t \left[e^{\sqrt{\lambda}(t-x)} - e^{-\sqrt{\lambda}(t-x)}\right]f(x)dx. \tag{105}$$

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