Bernstein Type Inequalities Concerning Growth of Polynomials

N.K. Govil and Eze R. Nwaeze

Abstract Let $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$ be a polynomial of degree n, where the coefficients a_j, for $0 \le j \le n$, may be complex, and $p(z) \ne 0$ for $|z| < 1$. Then

$$M(p,R) \le \left(\frac{R^n + 1}{2}\right)||p||, \quad \text{for } R \ge 1, \tag{1}$$

and

$$M(p,r) \ge \left(\frac{r+1}{2}\right)^n ||p||, \quad \text{for } 0 < r \le 1, \tag{2}$$

where $M(p,R) := \max_{|z|=R\ge 1} |p(z)|$, $M(p,r) := \max_{|z|=r\le 1} |p(z)|$, and $||p|| := \max_{|z|=1} |p(z)|$. Inequality (1) is due to Ankeny and Rivlin (Pac. J. Math. **5**, 849–852, 1955), whereas Inequality (2) is due to Rivlin (Am. Math. Mon. **67**, 251–253, 1960). These inequalities, which due to their applications are of great importance, have been the starting point of a considerable literature in Approximation Theory, and in this paper we study some of the developments that have taken place around these inequalities. The paper is expository in nature and would provide results dealing with extensions, generalizations and refinements of these inequalities starting from the beginning of this subject to some of the recent ones.

Keywords Maximum modulus • Complex polynomials • Restricted zeros • Inequalities • Growth

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Introduction

Several years after chemist Mendeleev invented the periodic table of elements he made a study of the specific gravity of a solution as a function of the percentage of the dissolved substance, and for this he needed an answer to the following question.

Question If $p(x)$ is a quadratic polynomial with real coefficients and $|p(x)| \leq 1$ on $-1 \leq x \leq 1$, then how large can $|p'(x)|$ be on $-1 \leq x \leq 1$?

To see how an answer to the above question of Mendeleev helped him in the solution of the problem in Chemistry he was interested in, we refer to the paper of Boas [6].

Note that, even though Mendeleev was a chemist, he was able to show that $|p'(x)| \leq 4$ for $-1 \leq x \leq 1$. This estimate is best possible in the sense that there is a quadratic polynomial $p(x) = 1 - 2x^2$ for which $|p(x)| \leq 1$ on $[-1, 1]$ but $|p'(\pm 1)| = 4$. In the general case when $p(x)$ is a polynomial of degree n with real coefficients the problem was solved by Markov [26], who proved the following result which is known as Markov's Theorem.

Theorem 1.1. *Let* $p(x) = \sum_{j=0}^{n} a_j x^j$ *be an algebraic polynomial of degree n such that* $|p(x)| \leq 1$ *for* $x \in [-1, 1]$. *Then*

$$|p'(x)| \leq n^2, \qquad x \in [-1, 1] \tag{3}$$

The inequality is sharp. Equality holds only if $p(x) = \alpha T_n(x)$, *where* α *is a complex number such that* $|\alpha| = 1$, *and*

$$T_n(x) = \cos(n \cos^{-1} x) = 2^{n-1} \prod_{j=1}^{n} \left[x - \cos((j - \frac{1}{2})\pi/n)\right]$$

is the nth degree Tchebycheff polynomial of the first kind. It can be easily verified that $|T_n(x)| \leq 1$ *for* $x \in [-1, 1]$ *and* $|T_n'(1)| = n^2$.

It would be natural to go on and ask for an upper bound for $|p^{(k)}(x)|$ where $1 \leq k \leq n$. Iterating Markov's Theorem yields $|p^{(k)}(x)| \leq n^{2k} L$ if $|p(x)| \leq L$. However, this inequality is not sharp; the best possible inequality was found by Markov's brother, Markov [27], who proved the following.

Theorem 1.2. *Let* $p(x) = \sum_{j=0}^{n} a_j x^j$ *be an algebraic polynomial of degree n with real coefficients such that* $|p(x)| \leq 1$ *for* $x \in [-1, 1]$. *Then*

$$|p^{(k)}(x)| \leq \frac{(n^2 - 1^2)(n^2 - 2^2) \cdots (n^2 - (k-1)^2)}{1 \cdot 3 \cdots (2k-1)}, \qquad x \in [-1, 1]. \tag{4}$$

The inequality is sharp, and the equality holds again only for $p(x) = T_n(x)$, *where* $T_n(x) = \cos(n \cos^{-1} x)$ *is the Chebyschev polynomial of degree n.*

Several years later, around 1926, Serge Bernstein needed the analogue of the above result Theorem 1.1 of A. A. Markov for polynomials in the complex domain and proved the following, which in the literature is known as Bernstein's Inequality.

Theorem 1.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a complex polynomial of degree at most n. Then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{5}$$

The inequality is best possible and equality holds only for polynomials of the form $p(z) = \alpha z^n$, $\alpha \neq 0$ *being a complex number.*

The above theorem is, in fact, a special case of a more general result due to Riesz [35] for trigonometric polynomials.

For the sake of brevity, throughout in this paper, we shall be using the following notations.

Definition 1. Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree at most n. We will denote

$$M(p, r) := \max_{|z|=r} |p(z)|, \ r > 0,$$

$$||p|| := \max_{|z|=1} |p(z)|,$$

and

$$D(0, K) := \{z : |z| < K\}, \ K > 0.$$

In 1945, S. Bernstein initiated and observed the following result, which in fact is a simple consequence of the maximum modulus principle (see [29] or [35, volume 1, p. 137]). This inequality is also known as the Bernstein's Inequality.

Theorem 1.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n. Then for* $R \geq 1$,

$$M(p, R) \leq R^n ||p||. \tag{6}$$

Equality holds for $p(z) = \alpha z^n$, α *being a complex number.*

If one applies the above inequality to the polynomial $P(z) = z^n p(1/z)$ and use maximum modulus principle, one easily gets

Theorem 1.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n. Then for* $0 < r \leq 1$,

$$M(p, r) \geq r^n ||p||. \tag{7}$$

Equality holds for $p(z) = \alpha z^n$, α *being a complex number.*

The above result is due to Varga [38] who attributes it to E. H. Zarantonello.

By use of the transformation $P(z) = z^n p(1/z)$ and the maximum modulus principle it is not difficult to see that Theorems 1.4 and 1.5 can be obtained from each other. The fact that Theorem 1.3 can be obtained from Theorem 1.4 was proved by Bernstein himself. However, it was not known if Theorem 1.4 can also be obtained from Theorem 1.3, and this has been shown by Govil et al. [22]. Thus all the above three Theorems 1.3, 1.4 and 1.5 are equivalent in the sense that anyone can be obtained from any of the others.

For the sharpening of Theorems 1.3, 1.4 and 1.5 we refer the reader to the paper of Frappier et al. [12] (also, see Sharma and Singh [37]).

For polynomial of degree n not vanishing in the interior of the unit circle, Ankeny and Rivlin [1] proved the following result.

Theorem 1.6. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, 1)$. *Then for* $R \geq 1$,

$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right) ||p||. \tag{8}$$

Here equality holds for $p(z) = \dfrac{\alpha + \beta z^n}{2}$, *where* $|\alpha| = |\beta| = 1$.

For some generalizations of inequalities (6) and (8), see Govil et al. [23].

The analogue of Inequality (7) for polynomials not vanishing in the interior of a unit circle was proved later in 1960 by Rivlin [36], who in fact proved

Theorem 1.7. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, 1)$. *Then for* $0 < r \leq 1$,

$$M(p, r) \geq \left(\frac{r + 1}{2}\right)^n ||p||, \tag{9}$$

and equality holding for $p(z) = \left(\dfrac{\alpha + \beta z}{2}\right)^n$, *where* $|\alpha| = |\beta| = 1$.

The above results, Theorems 1.4–1.5 which are known as Bernstein inequalities concerning growth of polynomials, and Theorems 1.6–1.7 have been the starting point of a considerable literature in Approximation Theory. Several books and research monographs have been written on this subject of inequalities (see, for example, Govil and Mohapatra [20], Milovanović et al. [28], Pinkus [30], and Rahman and Schmeisser [34]) and in this chapter we study some of the developments that have, over a period, taken place around these inequalities.

The present chapter is expository in nature and consists of four sections. Having introduced Bernstein type inequalities concerning the growth of polynomials, Theorems 1.4–1.7 in section "Introduction," we in section "Results Concerning Generalizations, Extensions and Refinements of Theorem 1.6" study inequalities related to the inequality in Theorem 1.6 while section "Results Concerning Generalizations, Extensions and Refinements of Theorem 1.7" will consist of inequalities related to Theorem 1.7. Lastly, the section "Polynomials Having all the Zeros on $S(0, K)$" deals with inequalities concerning the growth of polynomials having all their zeros on a circle.

Results Concerning Generalizations, Extensions and Refinements of Theorem 1.6

We begin with Theorem 1.6 stated in section "Introduction," and which is due to Ankeny and Rivlin [1].

Theorem 2.1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, 1)$. *Then for* $R \geq 1$,

$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right)||p||. \tag{10}$$

Equality holding for $p(z) = \dfrac{\alpha + \beta z^n}{2}$, *where* $|\alpha| = |\beta| = 1$.

We present here the brief outlines of the proof of Theorem 2.1 as given by Ankeny and Rivlin in [1], which makes use of Erdös-Lax Theorem. As is well known, the Erdös-Lax Theorem which is stated below as Lemma 2.2, was conjectured by Erdös and proved by Lax [25]. It may be remarked that a simpler proof of Lemma 2.2 was provided by Aziz and Mohammad [3].

Lemma 2.2 (Lax [25]). *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, 1)$. *Then*

$$M(p', 1) \leq \frac{n}{2}||p||. \tag{11}$$

Proof (Proof of Theorem 2.1). Let us assume that $p(z)$ does not have the form $\dfrac{\alpha + \beta z^n}{2}$. In view of Lemma 2.2

$$|p'(e^{i\theta})| \leq \frac{n}{2}||p||, \qquad 0 \leq \theta < 2\pi, \tag{12}$$

from which we may deduce that

$$|p'(re^{i\theta})| < \frac{n}{2}r^{n-1}||p||, \qquad 0 \leq \theta < 2\pi, \quad r > 1, \tag{13}$$

by applying Theorem 1.4 to the polynomial $p'(z)/(n/2)$ and observing that we have the strict inequality in (13) because $p(z)$ does not have the form $\dfrac{\alpha + \beta z^n}{2}$. But for each $\theta, 0 \leq \theta < 2\pi$, we have

$$p(Re^{i\theta}) - p(e^{i\theta}) = \int_1^R e^{i\theta} p'(re^{i\theta})dr.$$

Hence

$$\left|p(Re^{i\theta}) - p(e^{i\theta})\right| \leq \int_1^R |p'(re^{i\theta})|dr < \frac{n}{2}||p|| \int_1^R r^{n-1}dr = \frac{||p||}{2}(R^n - 1),$$

and

$$\left|p(Re^{i\theta})\right| < \frac{||p||}{2}(R^n - 1) + \left|p(e^{i\theta})\right| \leq \frac{||p||}{2}(1 + R^n).$$

Finally, if $p(z) = \dfrac{\alpha + \beta z^n}{2}$, $|\alpha| = |\beta| = 1$, then clearly

$$M(p, R) = \frac{1 + R^n}{2}, \; R > 1,$$

and the proof of Theorem 2.1 is thus complete.

It may be remarked that later a simpler proof of Theorem 2.1 which does not make use of Erdös-Lax Theorem was given by Dewan [7].

Remark 1. The converse of Theorem 2.1 is false as the simple example $p(z) = (z + \frac{1}{2})(z + 3)$ shows. However, the following result in the converse direction, which is also due to Ankeny and Rivlin [1], is valid.

Theorem 2.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that* $p(1) = 1$ *and*

$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right)||p||$$

for $0 < R - 1 < \delta$, *where* δ *is any positive number. Then* $p(z)$ *does not have all its zeros within the unit circle.*

In 1989, Govil [17] observed that since the equality in (10) holds only for polynomials $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$, which satisfy

$$\left|\text{coefficient of } z^n\right| = \frac{1}{2}||p||, \tag{14}$$

it should be possible to improve upon the bound in (10) for polynomials not satisfying (14), and therefore in this connection he proved the following refinement of (10).

Theorem 2.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)||p|| - \frac{n(||p||^2 - 4|a_n|^2)}{2||p||}\left\{\frac{(R-1)||p||}{||p||+2|a_n|} - \ln\left[1 + \frac{(R-1)||p||}{||p||+2|a_n|}\right]\right\} \tag{15}$$

Equality holding for $p(z) = \alpha + \beta z^n$, *where* $|\alpha| = |\beta|$.

In 1998, Dewan and Bhat [9] sharpened the above Theorem 2.4 as follows:

Theorem 2.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)||p|| - \left(\frac{R^n-1}{2}\right)m - \frac{n}{2}\left[\frac{(||p||-m)^2 - 4|a_n|^2}{(||p||-m)}\right]\left\{\frac{(R-1)(||p||-m)}{(||p||-m)+2|a_n|} - \ln\left[1 + \frac{(R-1)(||p||-m)}{(||p||-m)+2|a_n|}\right]\right\}, \tag{16}$$

where $m = \min_{|z|=1} |p(z)|$. *Here again, equality holds for* $p(z) = \alpha + \beta z^n$, *where* $|\alpha| = |\beta|$.

In 2001, Govil and Nyuydinkong [21] generalized Theorem 2.5, where they considered polynomials not vanishing in $D(0,K)$, $K \geq 1$. More specifically, they proved

Theorem 2.6. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,K),\ K \geq 1$. *Then for* $R \geq 1$,

$$M(p,R) \leq \Big(\frac{R^n + K}{1+K}\Big)||p|| - \Big(\frac{R^n - 1}{1+K}\Big)m - \frac{n}{1+K}\left[\frac{(||p||-m)^2 - (1+K)^2|a_n|^2}{(||p||-m)}\right] \times \left\{\frac{(R-1)(||p||-m)}{(||p||-m)+(1+K)|a_n|} - \ln\left[1 + \frac{(R-1)(||p||-m)}{(||p||-m)+(1+K)|a_n|}\right]\right\}, \tag{17}$$

where $m = \min_{|z|=K} |p(z)|$.

Following immediately, Gardner et al. [14] generalized Theorem 2.6 by considering polynomials of the form $a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \leq t \leq n$, and for this, they proved the following:

Theorem 2.7. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \leq t \leq n$, *be a polynomial of degree* n *and* $p(z) \neq 0$ *in* $D(0,K),\ K \geq 1$. *Then for* $R \geq 1$,

$$M(p,R) \leq \Big(\frac{R^n + K^t}{1+K^t}\Big)||p|| - \Big(\frac{R^n - 1}{1+K^t}\Big)m - \frac{n}{1+K^t}\left[\frac{(||p||-m)^2 - (1+K^t)^2|a_n|^2}{(||p||-m)}\right] \times \left\{\frac{(R-1)(||p||-m)}{(||p||-m)+(1+K^t)|a_n|} - \ln\left[1 + \frac{(R-1)(||p||-m)}{(||p||-m)+(1+K^t)|a_n|}\right]\right\}, \tag{18}$$

where $m = \min_{|z|=K} |p(z)|$.

Clearly, for $t = 1$, Theorem 2.7 gives Theorem 2.6, which for $K = 1$ reduces to Theorem 2.5.

In 2005, Gardner et al. [15] proved the following generalization and sharpening of Theorem 2.4.

Theorem 2.8. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \leq t \leq n$, *be a polynomial of degree* n *and* $p(z) \neq 0$ *in* $D(0,K),\ K \geq 1$. *Then for* $R \geq 1$,

$$M(p,R) \le \Big(\frac{R^n+s_0}{1+s_0}\Big)||p|| - \Big(\frac{R^n-1}{1+s_0}\Big)m - \frac{n}{1+s_0}\left[\frac{(||p||-m)^2-(1+s_0)^2|a_n|^2}{(||p||-m)}\right]$$
$$\times\left\{\frac{(R-1)(||p||-m)}{(||p||-m)+(1+s_0)|a_n|} - \ln\left[1+\frac{(R-1)(||p||-m)}{(||p||-m)+(1+s_0)|a_n|}\right]\right\}, \tag{19}$$

where $m = \min_{|z|=K} |p(z)|$, *and*

$$s_0 = K^{t+1}\frac{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t-1}+1}{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t+1}+1}.$$

Dividing both sides of (19) by R^n, and letting $R \to \infty$, one gets

Corollary 2.9. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \le t \le n$, *be a polynomial of degree* n *and* $p(z) \ne 0$ *in* $D(0,K),\ K \ge 1$. *Then*

$$|a_n| \le \frac{1}{1+s_0}\big(||p||-m\big), \tag{20}$$

where $m = \min_{|z|=K} |p(z)|$.

In case one does not have knowledge of $m = \min_{|z|=K} |p(z)|$, one could use the following result due to Gardner et al. [15] which does not depend on m.

Theorem 2.10. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \le t \le n$, *be a polynomial of degree* n *and* $p(z) \ne 0$ *in* $D(0,K),\ K \ge 1$. *Then for* $R \ge 1$,

$$M(p,R) \le \Big(\frac{R^n+s_1}{1+s_1}\Big)||p|| - \frac{n}{1+s_1}\left[\frac{||p||^2-(1+s_1)^2|a_n|^2}{||p||}\right]$$
$$\times\left\{\frac{(R-1)||p||}{||p||+(1+s_1)|a_n|} - \ln\left[1+\frac{(R-1)||p||}{||p||+(1+s_1)|a_n|}\right]\right\}, \tag{21}$$

where

$$s_1 = K^{t+1} \frac{\frac{t}{n} \cdot \frac{|a_t|}{|a_0|} K^{t-1} + 1}{\frac{t}{n} \cdot \frac{|a_t|}{|a_0|} K^{t+1} + 1}.$$

If, in the above theorem, one divides both sides of (21) by R^n and let $R \to \infty$, one obtains the following

Corollary 2.11. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \le t \le n$, *be a polynomial of degree* n *and* $p(z) \neq 0$ *in* $D(0, K),\ K \ge 1$. *Then*

$$|a_n| \le \frac{1}{1 + s_1} ||p||. \tag{22}$$

Both Corollaries 2.9 and 2.11 generalize and sharpen the well-known inequality, obtainable by an application of Visser's Inequality [39], that if $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n and $p(z) \neq 0$ in $D(0, 1)$ then $|a_n| \le \frac{1}{2}||p||$.

We present some of the examples Gardner et al. [15] gave to illustrate the quality of Theorems 2.7, 2.8 and 2.10.

Example 1. Let $p(z) = 1000 + z^2 + z^3 + z^4$. Clearly, here $t = 2$ and $n = 4$, and one can take $K = 5.4$, since numerically $p \neq 0$ for $|z| < 5.4483$. For this polynomial, the bound for $M(p, 2)$ by Theorem 2.7 comes out to be 1447.503, and by Theorem 2.8, it comes out to be 1101.84, which is a significant improvement over the bound obtained from Theorem 2.7. Numerically, for this polynomial $M(p, 2) \approx$ 1028, which is quite close to the bound 1101.84, that is obtainable by Theorem 2.10. The bound for $M(p, 2)$ obtained by Theorem 2.10 is 1105.05, which is also quite close to the actual bound $\approx$ 1028. However, in this case Theorem 2.8 gives the best bound.

Example 2. Let $p(z) = 1000 + z^2 - z^3 - z^4$. Here also, $t = 2$ and $n = 4$. Again, numerically $p(z) \neq 0$ for $|z| < 5.43003$, and thus take $K = 5.4$. If $R = 3$, then for this polynomial the bound for $M(p, 3)$ obtained by Theorem 2.7 comes out to be 3479.408, while by Theorem 2.10 it comes out to be 1545.3, and by Theorem 2.8 it comes out to be 1534.5, a considerable improvement. Thus again the bounds obtained from Theorem 2.8 and Theorem 2.10 are considerably smaller than the bound obtained from Theorem 2.7, and the bound 1534.5 obtained by Theorem 2.8 is much closer to the actual bound $M(p, 3) \approx$ 1100.6 than the bound 3479.408, obtained from Theorem 2.7.

In 1981, Aziz and Mohammad [4] sharpened Theorem 2.1 by proving the following.

Theorem 2.12. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n and* $p(z) \neq 0$ *in* $D(0, K), K \geq 1$. *Then for* $R \geq 1$,

$$M(p, R) \leq \frac{(R^n + 1)(R + K)^n}{(R + K)^n + (1 + RK)^n} ||p|| \tag{23}$$

Theorem 2.12 is a generalization of Theorem 2.1 in a compact form but unfortunately with the exception of $n = 1$, the Inequality (23) does not appear to be sharp for $K > 1$. However, a precise estimate is given by the following theorem, which is also due to Aziz and Mohammad [4].

Theorem 2.13. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n and* $p(z) \neq 0$ *in* $D(0, K), K \geq 1$. *Then for* $R \geq 1$,

$$M(p, R) \leq \frac{(R + K)^n}{(1 + K)^n} ||p|| \text{ for } 1 \leq R \leq K^2 \tag{24}$$

and

$$M(p, R) \leq \frac{R^n + K^n}{1 + K^n} ||p|| \text{ for } R \geq K^2 \tag{25}$$

The result is best possible with equality in (25) for $p(z) = (z^n + K^n)/(1 + K^n)$ *and in (24) for* $p(z) = (z + K)^n/(1 + K)^n$.

The following theorem which is due to Govil et al. [22] sharpens Inequality (24) in the above Theorem 2.13.

Theorem 2.14. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, K),\ K \geq 1$. *Then for* $1 \leq R \leq K^2$,

$$M(p, R) \leq \left(\frac{K^2 + 2R|\lambda|K + R^2}{K^2 + 2|\lambda|K + 1} \right)^{n/2} M(p, 1), \tag{26}$$

where $\lambda = \lambda(K) := \dfrac{Ka_1}{na_0}$.

The fact that the Inequality (26) sharpens Inequality (24) follows because if $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ for $|z| < K$, where $K \geq 1$, then $|\lambda| = |Ka_1/na_0| \leq 1$, and therefore

$$\left(\frac{K^2+2R|\lambda|K+R^2}{K^2+2|\lambda|K+1}\right) \le \left(\frac{K^2+2RK+R^2}{K^2+2K+1}\right) = \left(\frac{R+K}{K+1}\right)^2,$$

from which the conclusion follows.

In the case $R \ge K^2$, Govil et al. [22] proved

Theorem 2.15. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \ne 0$ *in* $D(0,K),\ K > 1$. *Then for* $R \ge K^2$,

$$M(p,R) \le \frac{R^n}{K^n}\left(\frac{K^n}{K^n+1}\right)^{(R-K^2)/(R+K^2)} M(p,1). \tag{27}$$

About 6 years after Aziz and Mohammad [4] proved Theorem 2.13, Aziz [2] obtained more results in this direction, some of which are presented below.

Theorem 2.16. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n and* $p(z) \ne 0$ *in* $D(0,K), K \ge 1$. *Then for* $R \ge 1$,

$$M(p,R) \le \frac{R^n+1}{2}||p|| - \frac{R^n-1}{2}\min_{|z|=1}|p(z)|. \tag{28}$$

The result is best possible and equality in (28) holds for the polynomial $p(z) = \alpha z^n + \beta K^n,\ |\alpha| = |\beta| = 1,\ K \ge 1.$

As an application of Theorem 2.16, Aziz [2] established

Theorem 2.17. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n and* $p(z) \ne 0$ *in* $D(0,K), K \le 1$. *Then for* $0 \le r \le K$,

$$(1+r^n)M(p,r) - (1-r^n)\min_{|z|=1}|p(z)| \ge 2r^n||p||. \tag{29}$$

The result is best possible and equality in (29) holds for the polynomial $p(z) = \alpha z^n + \beta K^n$, *where* $|\alpha| = |\beta| = 1$, *and* $K \le 1$.

In 2002, Aziz and Zargar [5] proved the following refinement of Theorem 2.12 which includes Theorem 2.16 as a special case.

Theorem 2.18. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n and* $p(z) \ne 0$ *in* $D(0,K), K \ge 1$. *Then for* $R \ge 1$,

$$M(p,R) \le F(R)\left\{(R^n+1)||p|| - \left[R^n - \left(\frac{1+RK}{R+K}\right)^n\right]m\right\}, \tag{30}$$

where $m := \min_{|z|=K} |p(z)|$ *and* $F(R) := \dfrac{(R+K)^n}{(R+K)^n+(1+RK)^n}$.

For $K=1$, the above theorem reduces to Theorem 2.16. In the same paper, they [5] proved the following which is a refinement of (24).

Theorem 2.19. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *and* $p(z) \ne 0$ *in* $D(0,K), K \ge 1$. *Then for* $1 \le R \le K^2$,

$$M(p,R) \le \left(\frac{R+K}{1+K}\right)^n ||p|| - \left\{\left(\frac{R+K}{1+K}\right)^n - 1\right\} m, \tag{31}$$

where $m := \min_{|z|=K} |p(z)|$.

Theorem 2.18, in some special cases, can provide much better information than Theorem 2.12 regarding $M(p,R),\ R>1$, and thus they [5] illustrate with the help of the following examples.

Example 3. Let $p(z) = (z^2+9)(z-19)$. Then $p(z)$ is a polynomial of degree 3 which does not vanish in $D(0,t)$, where $t \in (0,3]$. Clearly

$$|p(z)| \ge \left(9-|z|^2\right)\left(19-|z|\right)$$

which in particular gives $M(p,2) \ge 85$ and $||p|| = 200$. Using Theorem 2.12 with $K=t=3,\ R=2$, it follows that

$$M(p,2) \le 480.8, \tag{32}$$

whereas using Theorem 2.18 with $K=2$, and $R=2$, we get

$$M(p,2) \le 435.5, \tag{33}$$

which is much better than (32).

Example 4. Let $p(z) = z^3 + 3^3$. Then $p(z)$ does not vanish in $D(0,t)$, where $t \in (0,3]$. Evidently,

$$M(p,2) \ge 19 \quad \text{and} \quad M(p,1) = 28. \tag{34}$$

Using Theorem 2.12 with $K=t=3,\ R=2$, it follows that

$$M(p,2) \le 67.4. \tag{35}$$

Using Theorem 2.18 with $K = t = 2,\ R = 2$, gives

$$M(p, 2) \leq 46.5, \tag{36}$$

which is much better than (35).

For more results in this direction we refer the reader to [18, 19, 33, 37]. We wrap up this part by presenting the following results where the maximum modulus is taken on an ellipse rather than on a circle.

Theorem 2.20 (Duffin and Schaeffer [10]). *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that* $\max_{-1\leq x\leq 1} |p(x)| \leq 1$ *and* $p(z)$ *is real for real z,. Then for* $R > 1$*,*

$$\max_{z\in\mathscr{E}_{\mathscr{R}}} |p(z)| \leq \frac{R^n + R^{-n}}{2},$$

where $\mathscr{E}_{\mathscr{R}} := \left\{ z = x + iy : \frac{x^2}{\left(\frac{R+R^{-1}}{2}\right)^2} + \frac{y^2}{\left(\frac{R-R^{-1}}{2}\right)^2} = 1 \right\}$

In the following result the hypothesis on the polynomial $p(z)$ that $p(z)$ is real for real z has been dropped.

Theorem 2.21 (Frappier and Rahman [11]). *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n such that* $\max_{-1\leq x\leq 1} |p(x)| \leq 1$*, then for* $R > 1$*, we have*

$$\max_{z\in\mathscr{E}_{\mathscr{R}}} |p(z)| \leq \frac{R^n}{2} + \frac{5+\sqrt{17}}{4} R^{n-2},$$

where $\mathscr{E}_{\mathscr{R}}$ *is the same as above.*

Results Concerning Generalizations, Extensions and Refinements of Theorem 1.7

So far we have been dealing with improvements and generalizations of Inequality (10), and now we turn our attention to Inequality (9), given in Theorem 1.7. In this regard, Govil [16] generalized this Theorem 1.7 by proving

Theorem 3.1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, 1)$*. Then for* $0 < r \leq \rho \leq 1$*,*

$$M(p,r) \geq \left(\frac{1+r}{1+\rho}\right)^n M(p,\rho). \tag{37}$$

The result is best possible and equality holds for the polynomial $p(z) = \left(\frac{1+z}{1+\rho}\right)^n$.

If polynomial $p(z)$ has all its zeros on $|z| = 1$, the polynomial $q(z) = z^n p(\frac{1}{z})$ also has its zeros on $|z| = 1$. Further, if $1 \leq \rho \leq r$, then $\frac{1}{r} \leq \frac{1}{\rho} \leq 1$, and when (37) is applied to $q(z)$, it yields

$$M\left(q, \frac{1}{r}\right) \geq \left(\frac{1+\frac{1}{r}}{1+\frac{1}{\rho}}\right)^n M\left(q, \frac{1}{\rho}\right),$$

which is equivalent to (37).

The above explanation thus leads to the following corollary.

Corollary 2.2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n having all its zeros on the unit circle. Then for* $0 < r \leq \rho \leq 1$, *and for* $1 \leq \rho \leq r$,

$$M(p,r) \geq \left(\frac{1+r}{1+\rho}\right)^n M(p,\rho). \tag{38}$$

The result is best possible and equality holds for the polynomial $p(z) = (1+z)^n$.

If in Theorem 3.1 one also assumes that $p'(0) = 0$, the bound in (37) can be considerably improved. Govil [16] in the same paper obtained the following in this direction.

Theorem 3.2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Let* $p'(0) = 0$. *Then for* $0 < r \leq \rho \leq 1$,

$$M(p,r) \geq \left(\frac{1+r}{1+\rho}\right)^n \left\{\frac{1}{1 - \frac{(1-\rho)(\rho-r)n}{4}\left(\frac{1+r}{1+\rho}\right)^{n-1}}\right\} M(p,\rho). \tag{39}$$

Theorem 3.1 is best possible, however, if $0 < r < \rho < 1$, then for any polynomial $p(z)$ having no zeros in $D(0,1)$, and $p'(0) = 0$, the bound obtained by Theorem 3.2 can be considerably sharper than the bound obtained by Theorem 3.1. Govil [16] illustrated this by means of the following examples.

Example 5. Let $p(z) = 1 + z^3$, $\rho = 0.5$, $r = 0.1$. Theorem 3.1 gives $M(p,r) \geq (0.3943704)M(p,\rho)$, while by Theorem 3.2, $M(p,r) \geq (0.4289743)M(p,\rho)$.

Example 6. Let $p(z) = 1+z^7$, $\rho = 0.168$, $r = 0.022$. Theorem 3.1 gives $M(p,r) \geq (0.3926959)M(p,\rho)$, while by Theorem 3.2, $M(p,r) \geq (0.4341115)M(p,\rho)$.

In 1992, Qazi [32] extended Theorem 3.1 to polynomials with gaps. Specifically he proved

Theorem 3.4. *Let* $1 \leq m \leq n$ *and* $p(z) = a_0 + \sum_{j=m}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $0 < r < \rho \leq 1$,

$$M(p,r) \geq \left(\frac{1+r^m}{1+\rho^m}\right)^{n/m} M(p,\rho); \tag{40}$$

more precisely

$$M(p,r) \geq \exp\left(-n\int_r^\rho \frac{t^m + (m/n)|a_m/a_0|t^{m-1}}{t^{m+1} + (m/n)|a_m/a_0|(t^m+t)+1}dt\right)M(p,\rho). \tag{41}$$

If $m = 1$, Theorem 3.4 gives

Corollary 2.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $0 < r < \rho \leq 1$,

$$M(p,r) \geq \left(\frac{1+2|\lambda|r+r^2}{1+2|\lambda|\rho+\rho^2}\right)^{n/2} M(p,\rho), \tag{42}$$

where $\lambda := \dfrac{a_1}{na_0}$.

Proof. It is easy to see that

$$\exp\left(-n\int_r^\rho \frac{t+(1/n)|a_1/a_0|}{t^2+(2/n)|a_1/a_0|t+1}dt\right) = \left(\frac{1+2|a_1/na_0|r+r^2}{1+2|a_1/na_0|\rho+\rho^2}\right)^{n/2},$$

and on applying Inequality (41), the corollary follows.

For $\rho = 1$, the above corollary reduces to

Corollary 2.6. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $0 < r < 1$,

$$M(p,r) \geq \left(\frac{1+2|\lambda|r+r^2}{2+2|\lambda|}\right)^{n/2} M(p,1), \tag{43}$$

where $\lambda := \dfrac{a_1}{na_0}$.

In 2003, Govil et al. [22] extended Corollary 2.6 to polynomials not vanishing in $D(0, K),\ K \geq 1$. They, in fact, proved

Theorem 3.7. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, K),\ K \geq 1$. *Then for* $0 < r < 1$,

$$M(p, r) \geq \left(\frac{K^2 + 2r|\lambda|K + r^2}{K^2 + 2|\lambda|K + 1} \right)^{n/2} M(p, 1), \tag{44}$$

where $\lambda = \lambda(K) := \dfrac{Ka_1}{na_0}$.

In the case where n is even, (44) becomes an equality for polynomials of the form $c(K^2 + 2Kze^{i\beta}\cos\alpha + z^2e^{2i\beta})^{n/2},\ c \in \mathbf{C},\ c \neq 0,\ \alpha \in \mathbf{R},\ \beta \in \mathbf{R}$.

For any n, Inequality (44) may be replaced by

$$M(p, r) \geq \left(\frac{r + K}{K + 1} \right)^{n} M(p, 1), \quad 0 < r < 1, \tag{45}$$

where the bound is attained if $p(z) = c(ze^{i\beta} + K)^n,\ c \in \mathbf{C},\ c \neq 0,\ \beta \in \mathbf{R}$. It may be noted that even (45) is a generalization of (9).

Assuming that $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ in $D(0, K),\ K \leq 1$, Govil et al. [22] proved the following result.

Theorem 3.8. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0, K),\ K \leq 1$. *Then for* $0 < r \leq K^2$,

$$M(p, r) \geq \left(\frac{K^2 + 2r|\lambda|K + r^2}{K^2 + 2|\lambda|K + 1} \right)^{n/2} M(p, 1), \tag{46}$$

where $\lambda = \lambda(K) := \dfrac{Ka_1}{na_0}$.

In the case where n is even, (46) becomes an equality for polynomials of the form $c(K^2 + 2Kze^{i\beta}\cos\alpha + z^2e^{2i\beta})^{n/2},\ c \in \mathbf{C},\ c \neq 0,\ \alpha \in \mathbf{R},\ \beta \in \mathbf{R}$.

For any n, Inequality (46) may be replaced by

$$M(p, r) \geq \left(\frac{r + K}{K + 1} \right)^{n} M(p, 1), \quad 0 < r \leq K^2, \tag{47}$$

where the bound is attained if $p(z) = c(ze^{i\beta} + K)^n,\ c \in \mathbf{C},\ c \neq 0,\ \beta \in \mathbf{R}$.

Inequality (47) extends and refines a result of Jain [24, Inequality (1.4)], who had obtained it under the assumption that all the zeros of p lie on the circle $S(0,K) := \{z : |z| = K\},\ K > 0$.

Polynomials Having all the Zeros on $S(0, K)$

While trying to obtain inequality analogous to (10) for polynomials not vanishing in $D(0,K), K \le 1$, Dewan and Ahuja [8] were able to prove this only for polynomials having all the zeros on the circle $S(0,K) := \{z : |z| = K\},\ 0 < K \le 1$.

Theorem 4.1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n having all its zeros on* $S(0,K),\ K \le 1$. *Then for* $R \ge 1$ *and for every positive integer s,*

$$\{M(p,R)\}^s \le \left[\frac{K^{n-1}(1+K) + (R^{ns} - 1)}{K^{n-1} + K^n}\right]\{M(p,1)\}^s. \tag{48}$$

For $s = 1$, the Theorem 4.1 yields

Corollary 4.2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n having all its zeros on* $S(0,K),\ K \le 1$. *Then for* $R \ge 1$,

$$M(p,R) \le \left[\frac{K^{n-1}(1+K) + (R^{n} - 1)}{K^{n-1} + K^n}\right]M(p,1). \tag{49}$$

The following corollary immediately follows from Inequality (49), if one takes $K = 1$.

Corollary 4.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n having all its zeros on* $S(0,1)$. *Then for* $R \ge 1$,

$$M(p,R) \le \left(\frac{R^n + 1}{2}\right)M(p,1). \tag{50}$$

In the same paper, Dewan and Ahuja [8] also proved

Theorem 4.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n having all its zeros on* $S(0,K),\ K \le 1$. *Then for* $R \ge 1$ *and for every positive integer s,*

$$\{M(p,R)\}^s \leq \frac{1}{K^n}\left[\frac{n|a_n|\{K^n(1+K^2)+K^2(R^{ns}-1)\}+|a_{n-1}|\{2K^n+R^{ns}-1\}}{2|a_{n-1}|+n|a_n|(1+K^2)}\right]$$
$$\{M(p,1)\}^s. \tag{51}$$

The following example illustrates that in some case the bound obtained in Theorem 4.4 is considerably better than the bound obtained in Theorem 4.1.

Example 7. Let $p(z) = z^4 - \frac{1}{50}z^2 + \frac{1}{100^2}$ and $K = \frac{1}{10}$, $R = 1.5$ and $s = 2$. Using Theorem 4.1, one gets

$$\{M(p,R)\}^s \leq 22390.91477\{M(p,1)\}^s,$$

while Theorem 4.4 gives

$$\{M(p,R)\}^s \leq 2439.505569\{M(p,1)\}^s,$$

showing that the bound obtained by Theorem 4.4 can be considerably sharper than what one gets from Theorem 4.1.

For $s = 1$ in Theorem 4.4 yields

Corollary 4.5. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros on $S(0,K)$, $K \leq 1$. Then for $R \geq 1$,*

$$M(p,R) \leq \frac{1}{K^n}\left[\frac{n|a_n|\{K^n(1+K^2)+K^2(R^n-1)\}+|a_{n-1}|\{2K^n+R^n-1\}}{2|a_{n-1}|+n|a_n|(1+K^2)}\right]M(p,1). \tag{52}$$

By restricting ourselves to polynomials of degree $n \geq 2$, Pukhta [30] obtained an improvement and generalization of Theorem 4.1 and Theorem 4.4. More precisely, he proved

Theorem 4.6. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros on $S(0,K)$, $K \leq 1$. Then for $R \geq 1$ and for every positive integer s,*

$$\{M(p,R)\}^s \leq \left[\frac{K^{n-1}(1+K)+(R^{ns}-1)}{K^{n-1}+K^n}\right]\{M(p,1)\}^s \tag{53}$$
$$-s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right]\{M(p,1)\}^{s-1},$$

for $n > 2$*; and*

$$\{M(p,R)\}^s \leq \left[\frac{K^{n-1}(1+K)+(R^{ns}-1)}{K^{n-1}+K^n}\right]\{M(p,1)\}^s \\ -s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right]\{M(p,1)\}^{s-1}, \tag{54}$$

for $n = 2$.

Setting $s = 1$ in Theorem 4.6 reduces to

Corollary 4.7. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n having all its zeros on* $S(0,K)$, $K \leq 1$. *Then for* $R \geq 1$,

$$M(p,R) \leq \left[\frac{K^{n-1}(1+K)+(R^n-1)}{K^{n-1}+K^n}\right]M(p,1) \\ -|a_1|\left[\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\right], \tag{55}$$

for $n > 2$*; and*

$$M(p,R) \leq \left[\frac{K^{n-1}(1+K)+(R^n-1)}{K^{n-1}+K^n}\right]M(p,1) \\ -|a_1|\left[\frac{R^n-1}{n}-\frac{R^{n-1}-1}{n-1}\right], \tag{56}$$

for $n = 2$.

The following result which deals for polynomials having all the zeros on the circle $S(0,K), K \leq 1$ is also due to Pukhta [30].

Theorem 4.8. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n having all its zeros on* $S(0,K)$, $K \leq 1$. *Then for* $R \geq 1$ *and for every positive integer s,*

$$\{M(p,R)\}^s \leq \frac{1}{K^n}\left[\frac{n|a_n|\{K^n(1+K^2)+K^2(R^{ns}-1)\}+|a_{n-1}|\{2K^n+R^{ns}-1\}}{2|a_{n-1}|+n|a_n|(1+K^2)}\right]$$
$$\{M(p,1)\}^s - s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right]\{M(p,1)\}^{s-1}, \quad \textit{for } n>2$$

and

$$\{M(p,R)\}^s \leq \frac{1}{K^n}\left[\frac{n|a_n|\{K^n(1+K^2)+K^2(R^{ns}-1)\}+|a_{n-1}|\{2K^n+R^{ns}-1\}}{2|a_{n-1}|+n|a_n|(1+K^2)}\right]$$
$$\{M(p,1)\}^s - s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right]\{M(p,1)\}^{s-1}, \quad \textit{for } n=2.$$

Choosing $s=1$ in Theorem 4.8 gives

Corollary 4.9. *Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros on $S(0,K)$, $K\leq 1$. Then for $R\geq 1$,*

$$M(p,R) \leq \frac{1}{K^n}\left[\frac{n|a_n|\{K^n(1+K^2)+K^2(R^{n}-1)\}+|a_{n-1}|\{2K^n+R^{n}-1\}}{2|a_{n-1}|+n|a_n|(1+K^2)}\right]M(p,1)$$
$$-|a_1|\left[\frac{R^{n}-1}{n}-\frac{R^{n-2}-1}{n-2}\right], \quad \textit{for } n>2$$

and

$$M(p,R) \leq \frac{1}{K^n}\left[\frac{n|a_n|\{K^n(1+K^2)+K^2(R^{n}-1)\}+|a_{n-1}|\{2K^n+R^{n}-1\}}{2|a_{n-1}|+n|a_n|(1+K^2)}\right]M(p,1)$$
$$-|a_1|\left[\frac{R^{n}-1}{n}-\frac{R^{n-1}-1}{n-1}\right], \quad \textit{for } n=2.$$

We close this section by stating the following two recent results due to Pukhta [31] which are for polynomials with gaps, and therefore generalize the above results.

Theorem 4.10. *Let $p(z)=a_n z^n+\sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1\leq\mu<n$ be a polynomial of degree n having all its zeros on $S(0,K)$, $K\leq 1$. Then for $R\geq 1$ and for every positive integer s,*

$$\{M(p,R)\}^s \le \left[\frac{K^{n-2\mu+1}(1+K^{\mu})+(R^{ns}-1)}{K^{n-2\mu+1}+K^{n-\mu+1}}\right]\{M(p,1)\}^s - s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right]\{M(p,1)\}^{s-1},$$

for $n>2$; *and*

$$\{M(p,R)\}^s \le \left[\frac{K^{n-2\mu+1}(1+K^{\mu})+(R^{ns}-1)}{K^{n-2\mu+1}+K^{n-\mu+1}}\right]\{M(p,1)\}^s - s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right]\{M(p,1)\}^{s-1},$$

for $n=2$.

Note that for $\mu=1$, the above Theorem 4.10 reduces to Theorem 4.6.

Theorem 4.11. *Let* $p(z)=a_nz^n+\sum_{j=\mu}^{n}a_{n-j}z^{n-j}$, $1\le\mu<n$ *be a polynomial of degree* n *having all its zeros on* $S(0,K)$, $K\le 1$. *Then for* $R\ge 1$ *and for every positive integer* s,

$$\{M(p,R)\}^s \le \frac{G(s,k,\mu)}{K^q}\{M(p,1)\}^s - s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-2}-1}{ns-2}\right]\{M(p,1)\}^{s-1}, \quad n>2$$

and

$$\{M(p,R)\}^s \le \frac{G(s,k,\mu)}{K^q}\{M(p,1)\}^s - s|a_1|\left[\frac{R^{ns}-1}{ns}-\frac{R^{ns-1}-1}{ns-1}\right]\{M(p,1)\}^{s-1}, \quad n=2,$$

where

$$G(s,k,\mu)=\frac{n|a_n|\{K^q(K^{\mu-1}+K^{2\mu})+K^{2\mu}(R^{ns}-1)\}+|a_{n-\mu}|\{\mu(K^n+K^q+K^{\mu-1}(R^{ns}-1))\}}{\mu|a_{n-\mu}|(K^{\mu-1}+1)+n|a_n|(K^{\mu-1}+K^{2\mu})},$$

$$q=n-\mu+1.$$

References

1. Ankeny, N.C., Rivlin, T.J.: On a theorem of S. Bernstein. Pac. J. Math. **5**, 849–852 (1955)
2. Aziz, A.: Growth of polynomials whose zeros are within or outside a circle. Bull. Aust. Math. Soc. **35**, 247–256 (1987)
3. Aziz, A., Mohammad, Q.G.: Simple proof of a theorem of Erdös and Lax. Proc. Am. Math. Soc. **80**, 119–122 (1980)
4. Aziz, A., Mohammad, G.: Growth of polynomials with zeros outside a circle. Proc. Am. Math. Soc. **81**, 549–553 (1981)
5. Aziz, A., Zargar, B.A.: Growth of maximum modulus of polynomials with prescribed zeros. Glas. Mat. **37**, 73–81 (2002)
6. Boas, R.P.: Inequalities for the derivatives of polynomials. Math. Mag. **42**, 165–174 (1969)
7. Dewan, K.K.: Another proof of a theorem of Ankeny and Rivlin. Glas. Math. **18**, 291–293 (1983)
8. Dewan, K.K., Ahuja, A.: Growth of polynomials with prescribed zeros. J. Math. Inequal. **5**, 355–361 (2011)
9. Dewan, K.K., Bhat, A.A.: On the maximum modulus of polynomials not vanishing inside the unit circle. J. Interdisciplinary Math. **1**, 129–140 (1998)
10. Duffin, R.J., Schaeffer, C.: Some properties of functions of exponential type. Bull. Am. Math. Soc. **44**, 236–240 (1938)
11. Frappier, C., Rahman, Q.I.: On an Inequality of S. Bernstein. Can. J. Math. **34**(4), 932–944 (1982)
12. Frappier, C., Rahman, Q.I., Ruscheweyh, S.: New inequalities for polynomials. Trans. Am. Math. Soc. **288**, 69–99 (1985)
13. Frappier, C., Rahman, Q. I., Ruscheweyh, S. : New Inequalities for Polynomials. Trans. Am. Math. Soc. **288**, 69–99 (1985)
14. Gardner, R.B., Govil, N.K., Weems, A.: Some results concerning rate of growth of polynomials. East J. Approx. **10**, 301–312 (2004)
15. Gardner, R.B., Govil, N.K., Musukula, S.R.: Rate of growth of polynomials not vanishing inside a circle. J. Inequal. Pure Appl. Math. **6**(2), article 53 (2005)
16. Govil, N.K.: On the maximum modulus of polynomials. J. Math. Anal. Appl. **112**, 253–258 (1985)
17. Govil, N.K.: On the maximum modulus of polynomials not vanishing inside the unit circle. Approx. Theory Appl. **5**, 79–82 (1989)
18. Govil, N.K.: On a theorem of Ankeny and Rivlin concerning maximum modulus of polynomials. Complex Variables **40**, 249–259 (2000)
19. Govil, N.K.: On Growth of polynomials. J. Inequal. Appl. **7**(5), 623–631 (2002)
20. Govil, N.K., Mohapatra, R.N. : Markov and Bernstein type inequalities for polynomials. J. Inequal. Appl. **3**, 349–387 (1999)
21. Govil, N.K., Nyuydinkong, G.: On maximum modulus of polynomials not vanishing inside a circle. J. Interdisciplinary Math. **4**, 93–100 (2001)
22. Govil, N.K., Qazi, M.A., Rahman, Q.I.: Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius. Math. Inequal. Appl. **6**(3), 453–467 (2003)
23. Govil, N.K., Liman, A., Shah, W.M.: Some inequalities concerning derivative and maximum modulus of polynomials. Aust. J. Math. Anal. Appl. **8**, 1–8 (2011)
24. Jain, V.K.: Certain interesting implications of T. J. Rivlins's result on maximum modulus of a polynomial. Glas. Math. **33**, 33–36 (1998)
25. Lax, P.D.: Proof of a conjecture of P. Erdös on the derivatives of a polynomial. Bull. Am. Math. Soc. **50**, 509–513 (1944)
26. Markov, A.A.: On a problem of D. I. Mendeleev (Russian), Zapiski Imp. Akad. Nauk **62**, 1–24 (1889)
27. Markov, V.A.: Über Polynome die in einem gegebenen Intervalle möglichst wenig von null abweichen, Math. Ann. **77**, 213–258 (1916)

28. Milovanović, G.V., Mitrinović, D.S., Rassias, T.M.: Topics in Polynomials: Extremal Problems, Inequalities, Zeros. World Scientific Publishing Co. Pte. Ltd., Singapore (1994)
29. Nehari, Z.: Conformal Mapping. McGraw Hill, New York (1952)
30. Pukhta, M.S.: Rate of growth of polynomials with zeros on the unit Disc. Prog. Appl. Math. **6**(2), 50–58 (2013)
31. Pukhta, M.S.: Growth of the maximum modulus of polynomials with prescribed zeros. J. Class. Anal. **5**(2), 107–113 (2014)
32. Qazi, M.A.: On the maximum modulus of polynomials. Proc. Am. Math. Soc. **115**(2), 337–343 (1992)
33. Rahman, Q.I., Schmeisser, G.: Les inégalités de Markoff et de Bernstein. Les Presses de l' Université de Montréal, Québec (1983)
34. Rahman, Q.I., Schmeisser, G.: Analytic Theory of Polynomials. Oxford University Press, New York (2002)
35. Riesz, M.: Über einen Satz des Herrn Serge Bernstein. Acta. Math. **40**, 337–347 (1916)
36. Rivlin, T.J.: On the maximum modulus of polynomials. Am. Math. Mon. **67**, 251–253 (1960)
37. Sharma, A., Singh, V.: Some Bernstein type inequalities for polynomials. Analysis **5**, 321–341 (1985)
38. Varga, R.S.: A comparison of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomials. J. Soc. Ind. Appl. Math. **5**, 39–46 (1957)
39. Visser, C.: A simple proof of certain inequalities concerning polynomials. Nederl. Akad. Wetensch. Proc. **47**, 276–281 (1945)