Type-1 to Type-n Fuzzy Logic and Systems

M.H. Fazel Zarandi, R. Gamasaee and O. Castillo

Abstract In this chapter, the motivation for using fuzzy systems, the mathematical concepts of type-1 to type-n fuzzy sets, logic, and systems as well as their applications in solving real world problems are presented.

Keywords Type-n fuzzy sets · Type-2 fuzzy sets · T-norm · S-norm · Takagi-Sugeno-Kang (TSK) fuzzy system

1 Introduction

Fuzzy logic and sets has been proposed by Zadeh [1]. According to Zadeh, fuzzy logic is a precise logic of imprecision and approximate reasoning. It is used to reason and make logical decisions in the presence of uncertainty, imprecision, and imperfect information. In addition, it is capable of modeling problems in case of no measurements [2]. Since real world problems are mostly involved with uncertainty and imperfect information, fuzzy logic and fuzzy sets are required to model and solve those problems.

Therefore, Zadeh [1] introduced the first type of fuzzy logic to be used in the abovementioned problems. In type-1 fuzzy logic, the traditional view to a set which believes that membership of an element to a set is either zero or one is relaxed. In other words, membership of an element to a set is a matter of degree. This mem-

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C. Kahraman et al. (eds.), *Fuzzy Logic in Its 50th Year*,
Studies in Fuzziness and Soft Computing 341,
DOI 10.1007/978-3-319-31093-0_6

bership degree is a certain single value for an element. However, there are some situations in which uncertainty of the data is too high. When the degree of information vagueness is too high, membership functions (MFs) are not certain, and they encounter uncertainty and volatility. Therefore, type-1 fuzzy membership functions are not applicable in this case since they are a certain value.

In order to model and solve problems with higher degree of vagueness and uncertain MFs, Zadeh [3] introduced type-2 fuzzy sets as an extension to type-1 fuzzy sets. In type-2 fuzzy sets, each MF is represented by another MF known as secondary MF. Type-2 fuzzy sets have been proposed by Zadeh [3] to tackle the drawbacks of type-1 fuzzy sets when uncertainty is too high. Three ways in which such uncertainty can occur are: (1) the words that are used in antecedents and consequents of rules can mean different things to different people, (2) consequents obtained by polling a group of experts will often be different for the same rule because the experts will not necessarily be in agreement, and (3) noisy training data (Liang and Mendel [4]). However, due to the various degrees of uncertainty, different kinds of type-2 fuzzy sets are required.

There are two kinds of type-2 fuzzy sets: (I) interval type-2 fuzzy sets (IT2 FSs); (II) general type-2 fuzzy sets (GT2 FSs). In IT2 FSs, MFs are interval instead of a single value. They are special type of GT2 FSs whose secondary membership values for the entire members of the primary domain are one. GT2 FSs are extensions of IT2 FSs, and they are bivariate interval valued [0, 1] supporting more degrees of freedom [5]. However, IT2 FSs are less computationally expensive than GT2 FSs.

In this chapter, fundamental concepts of type-1 fuzzy sets and mathematical operations on type-1 fuzzy sets are briefly reviewed in Sects. 2 and 3 respectively. Then fuzzy logic is described in Sect. 4. Thereafter, since approximate reasoning is required to apply fuzzy set theory in real world problems, it is presented in Sect. 5. In order to model real world problems, the use of fuzzy systems is inevitable, so they are illustrated in Sect. 6. Type-2 fuzzy sets and type-2 fuzzy systems are described in Sects. 7 and 8 respectively. Then, the applications of type-2 fuzzy systems are reviewed in Sect. 9. Finally, type-n fuzzy systems are presented in Sect. 10.

2 Type-1 Fuzzy Sets

Type-1 fuzzy logic assigns a membership degree in the interval of [0, 1] to the objects indicating the degree to which each object belongs to the fuzzy set. Consider two fuzzy sets X and A, where, X is a universe of discourse whose element is shown by x. A membership function of element x is denoted by $\mu_A(x)$ as it is stated in (1), the following equations are mostly adapted from Celikyilmaz and Türksen [6].

$$\mu_A(x): X \rightarrow [0, 1] \tag{1}$$

Fuzzy sets are represented in discrete and continuous forms. If fuzzy set A is defined on a discrete universe of discourse X, it is represented by (2).

$$A = \sum_{x \in X} {\mu_A(x)}/_{x} \tag{2}$$

where, $\sum_{x \in X}$ is an aggregation operator indicating all membership degrees pertain to a fuzzy set A and universe of discourse X. If fuzzy set A is defined on a continuous universe of discourse X, it is defined by (3).

$$A = \int_{x \in X} {\mu_A(x)}/_{x} \tag{3}$$

where, $\int_{x \in X}$ is not an integration operation but an aggregation operation. Other important operations in fuzzy sets are α-cut (α_A) and strong α-cut (α_A^+) as shown in the following equations.

$$\alpha_A = \{x | \mu_A(x) \geq \alpha\} \alpha \in [0, 1] \tag{4}$$

$$\alpha_A^+ = \{x | \mu_A(x) > \alpha\} \alpha \in [0, 1] \tag{5}$$

α-cut decomposition of fuzzy set A on U is shown by (6).

$$\mu_A(U) = sup \min\{\alpha, \mu_{\alpha_A}(u)\}, \forall u \in U \quad \alpha \in [0, 1]$$

where, $\mu_{\alpha_A}(u) \in \{0, 1\}$ is a MF of a crisp set.

Another useful definition is the “support” of a fuzzy set. Let X be a universe of discourse and A be a fuzzy set. “Support” of A shows all the elements of X with non-zero membership grades in A, as it is indicated in (6).

$$Supp(A) = \{x \in X | \mu_A(x) > 0\} \tag{6}$$

3 Mathematical Operations on Type-1 Fuzzy Sets

In this section, mathematical operations on fuzzy sets are reviewed.

Let A and B be two fuzzy sets; then, the “equality of fuzzy sets” is defined by the following equation.

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \quad (7)$$

Equation (8) shows "inclusion of fuzzy sets" A and B.

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \forall x \in S \quad (8)$$

Equation (9) indicates the "union of fuzzy sets" A and B.

$$A \cup B : \mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)] \quad (9)$$

Equation (10) shows the "intersection of fuzzy sets" A and B.

$$A \cap B : \mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)] \quad (10)$$

Equation (11) is the "Complement" of the fuzzy set A (Zadeh [1]).

$$\mu_{AC}(x) = 1 - \mu_A(x) \quad (11)$$

3.1 Fuzzy Intersection (T-norm)

Different types of logical operators other than "min" have been defined in literature known as "t-norm" operators. The following axioms have been defined by Klir and Yuan [7] for fuzzy t-norm in which $a, b, d \in [0, 1]$.

(I) $i(a, 1) = a$ (boundary condition)
(II) $b \leq d$ implies $i(a, b) \leq i(a, d)$ (monotonicity).
(III) $i(a, b) = i(b, a)$ (commutativity)
(IV) $i(a, i(b, d)) = i(i(a, b), d)$ (associativity)
Sometimes considering other restrictions on defining fuzzy t-norms leads to better results. Therefore, the following additional axioms should be satisfied for defining a fuzzy t-norm (Klir and Yuan [7]).
(V) i is a continuous function (continuity).
(VI) $i(a, b) < a$ (subidempotency)
(VII) $a_1 < a_2$ and $b_1 < b_2$ implies $i(a_1, b_1) < i(a_2, b_2)$ (strict monotonicity)

Table 1 shows different classes of t-norms applied in literature, and Fig. 1 depicts some of them.

The boundary condition in axiom (I) means if an argument of a t-norm is 1, membership degree of the t-norm is equal to the other argument. Commutativity means the fuzzy t-norm is symmetric. Monotonicity of t-norm indicates that any decrease in membership degrees in set A or B will not increase the membership value in t-norm. The axiom (IV) shows that order of numbers in t-norm is not important.

Table 1 Different types of fuzzy t-norms (Klir and Yuan [7])

Authors	Reference	Year	t-norm
Dombi	[8]	1982	$\left\{\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{1/\lambda}\right\}^{-1}$
Frank	[9]	1979	$\log[1+\frac{(s^a-1)(s^b-1)}{s-1}]$
Hamacher	–	1987	$\frac{ab}{r+(1-r)(a+b-ab)}$
Schweizer and Sklar	[10]	1963	$\{\max(0, a^p+b^p-1)\}^{\frac{1}{p}}$
Schweizer and Sklar	[11]		$1-[(1-a)^p+(1-b)^p-(1-a)^p(1-b)^p]^{\frac{1}{p}}$
Schweizer and Sklar	[12]		$\exp(-(\lvert\ln a\rvert^p+\lvert\ln b\rvert^p)^{\frac{1}{p}})$
Schweizer and Sklar	[13]		$\frac{ab}{[a^p+b^p-a^p b^p]^{\frac{1}{p}}}$
Yager	[14]	1980	$1-\min\{1, [(1-a)^{\omega}+(1-b)^{\omega}]^{\frac{1}{\omega}}\}$
Dubois and Prade	[15]	1980	$\frac{ab}{\max(a,b,a)}$
Weber	[16]	1983	$\max\left(0, \frac{a+b+\lambda ab-1}{1+\lambda}\right)$
Yu	[17]	1985	$\max[0, (1+\lambda)(a+b-1)-\lambda ab]$

Some of the common t-norms applied in literature are as follows:

- Standard t-norms: $i(a,b)=\min(a,b)$
- Algebraic product: $i(a,b)=ab$
- Bounded difference: $i(a,b)=\max(0, a+b-1)$
- Drastic t-norm: $i(a,b)=\begin{cases} a & when\ b=1 \\ b & when\ a=1 \\ 0 & otherwise \end{cases}$

3.2 *Fuzzy Union (T-conorm)*

Different types of logical operators other than "max" have been defined in literature known as "t-conorm" operators. The following axioms have been defined by Klir and Yuan [7] for fuzzy t-conorm in which $a, b, d \in [0,1]$.

(I) $u(a,0)=a$ (boundary condition).
(II) $b \le d$ implies $u(a,b) \le u(a,d)$ (monotonicity).
(III) $u(a,b)=u(b,a)$ (commutativity).
(IV) $u(a,u(b,d))=u(u(a,b),d)$ (associativity).
There are some additional axioms that must be satisfied to form a t-conorm as follows. Table 2 shows different types of t-conorms, and Fig. 2 depicts some of them.
(V) u is a continuous function (continuity).
(VI) $u(a,a) > a$ (superidempotency).
(VII) $a_1 < a_2$ and $b_1 < b_2$ implies $u(a_1,b_1) < u(a_2,b_2)$ (strict monotonicity)

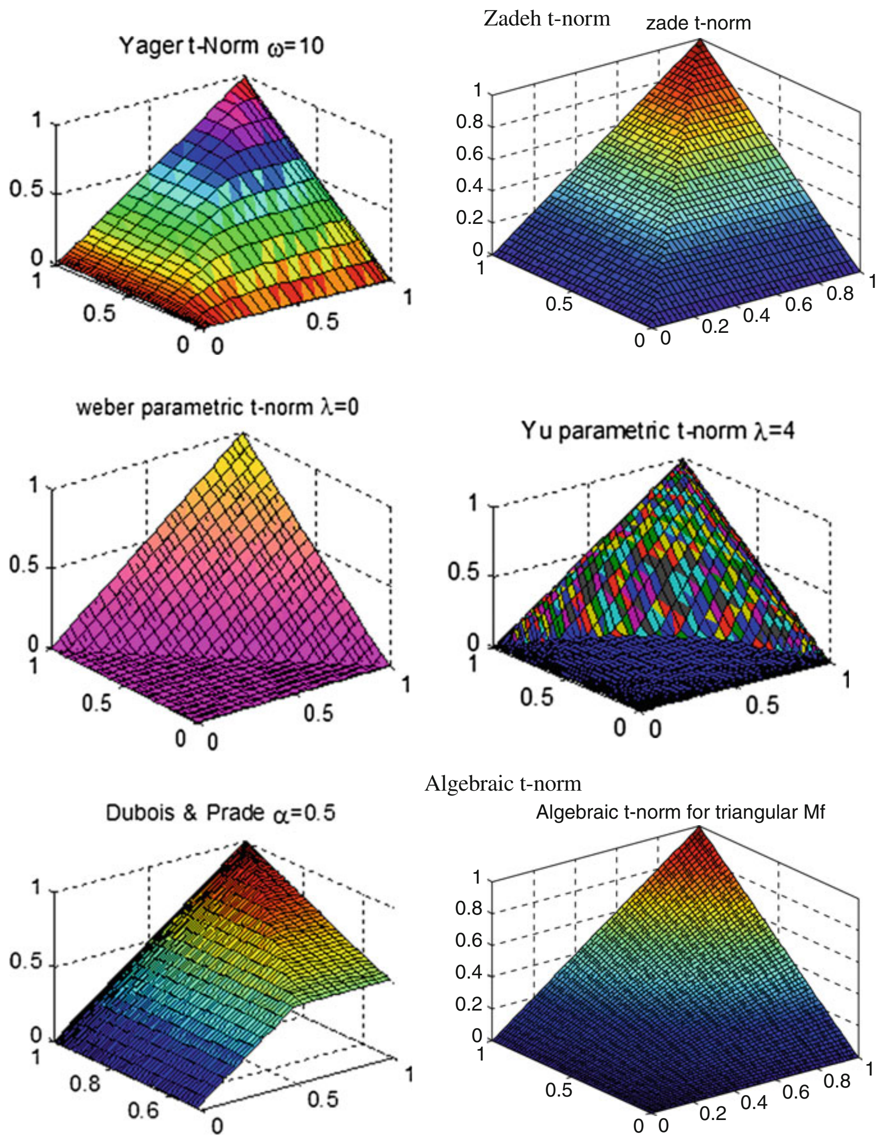

Fig. 1 Different types of t-norms

Some of the common t-conorms applied in literature are as follows:

- Standard t-conorms: $u(a, b) = \max(a, b)$
- Algebraic sum: $u(a, b) = a + b - ab$

Table 2 Different types of fuzzy t-conorms (Klir and Yuan [7])

Authors	Reference	Year	
Dombi	[8]	1982	$\left\{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{\frac{-1}{\lambda}}\right\}^{-1}$
Frank	[9]	1979	$1-\log_s\left[1+\frac{\left(s^{1-a}-1\right)\left(s^{1-b}-1\right)}{s-1}\right]$
Hamacher	–	1978	$\frac{a+b+(r-2)ab}{r+(r-1)ab}$
Schweizer and Sklar	[10]	1963	$1-\{\max(0,(1-a)^p+(1-b)^p-1)\}^{\frac{1}{p}}$
Schweizer and Sklar	[11]		$[a^p+b^p-a^pb^p]^{\frac{1}{p}}$
Schweizer and Sklar	[12]		$1-\exp\left(-(\lvert\ln(1-a)\rvert^p+\lvert\ln(1-b)\rvert^p)^{\frac{1}{p}}\right)$
Schweizer and Sklar	[13]		$1-\frac{(1-a)(1-b)}{[(1-a)^p+(1-b)^p-(1-a)^p(1-b)^p]^{\frac{1}{p}}}$
Yager	[14]	1980	$\min\left[1,\left(a^{\lambda}+b^{\lambda}\right)^{\frac{1}{\lambda}}\right]$
Dubois and Prade	[15]	1980	$1-\frac{(1-a)(1-b)}{\max((1-a),(1-b),a)}$
Weber	[16]	1983	$\min\left(1,a+b-\frac{\lambda}{1-\lambda}ab\right)$
Yu	[17]	1985	$\min(1,a+b+\lambda ab)$

- Bounded sum: $u(a,b)=\min(1,a+b)$
- Drastic t-conorm: $u(a,b)=\begin{cases} a & when\ b=0 \\ b & when\ a=0 \\ 1 & otherwise \end{cases}$

4 Fuzzy Logic

Fuzzy logic is different from classical logic because the truth or falsity of a proposition is stated with a degree of truth represented by a number in the interval [0, 1]. Fuzzy logic was introduced by Zadeh [1] along with fuzzy set theory. According to Turksen [18], fuzzy logic is an application of fuzzy set theory. "Degree of membership" used in fuzzy set theory is applied as "degree of truth" in fuzzy logic. Let A be a fuzzy set. The truth of fuzzy proposition "x is a member of A" is stated in two different ways Turksen [18]: (I) using classical logic, the degree of "truth" of proposition 'x is A' is a single value: $\tau(A(x))=1$; (II) using fuzzy logic, the degree of "truth" is not a single value: $\tau(A(x))=\tau[0,1]$. Fuzziness of propositions, include different combinations of linguistic terms. Fuzzy implications are used generally which includes fuzzy propositions as antecedent and consequent as presented in (12).

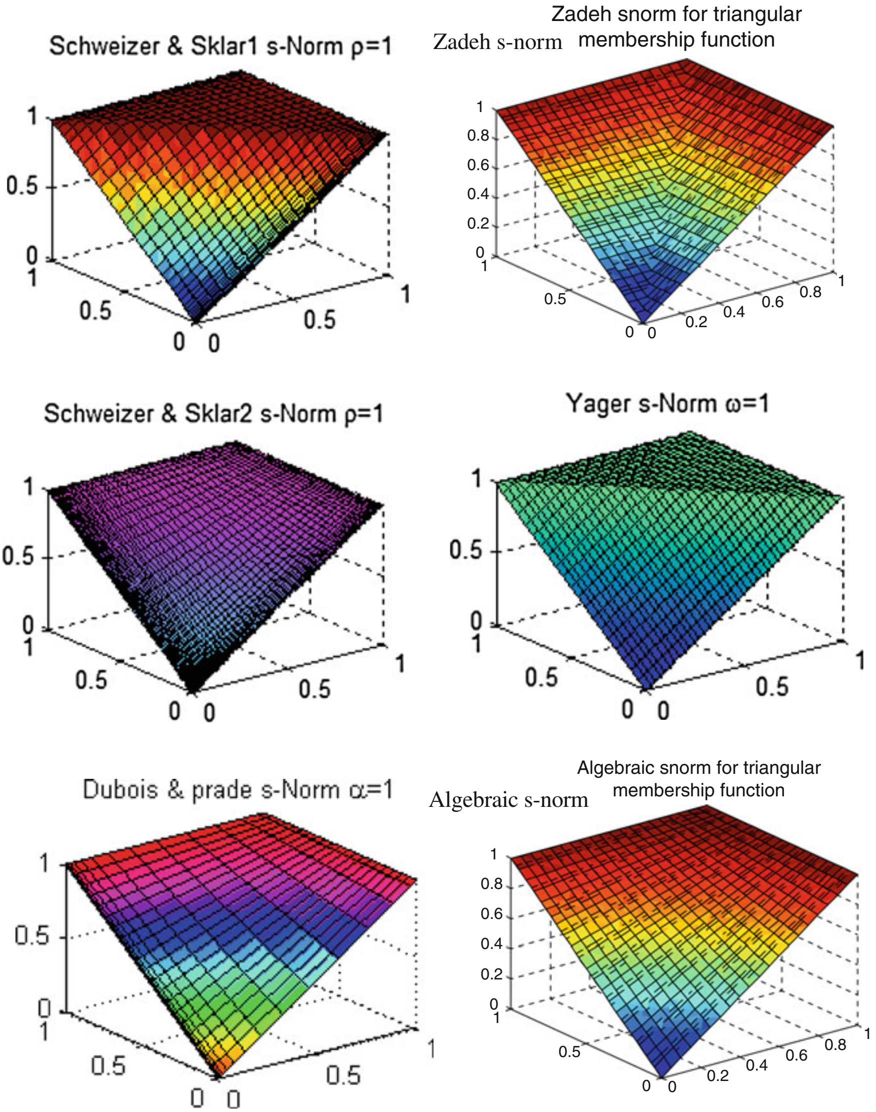

Fig. 2 Different types of t-conorms (s-Norms)

$$P\text{: IF } X \text{ is } A \text{ is true, THEN } Y \text{ is } B \text{ is true} \tag{12}$$

where X and Y are variables that take values x and y from sets X and Y, respectively, A and B are linguistic variables [6]. Implication between $A(x)$ and $B(x)$ are represented as follows:

$$A(x) \Rightarrow B(x) \tag{13}$$

Using an appropriate function, the degree to which the fuzzy proposition is true will be calculated. Lukasiewicz implication is generally used in literature and it is presented in (14).

$$I(P_{xy}) = I[A(x), B(y)] = \min[1, 1 - A(x) + B(x)] \tag{14}$$

Equation (14) indicates that calculating the degree of truth values $A(x)$ and $B(x)$ leads to the degree of truth value of (12). I is used to connect fuzzy sets to fuzzy propositions. The fuzzy truth value is indicated by a value of $\tau \in [0, 1]$ which is shown in (15).

$$P\text{:IF } X \text{ is } A \text{ is true to the degree } \tau_1, \text{ THEN } Y \text{ is } B \text{ is true to the degree } \tau_2 \tag{15}$$

Two states "X is A" is τ_1 and "Y is B" is τ_2 are indicated by $(A(x), \tau_1)$ and $(B(y), \tau_2)$ and the implication of them is presented by (16).

$$(A(x), \tau_1) \Rightarrow (B(y), \tau_2) \text{ is true} \tag{16}$$

Using Lukasiewicz implication the following equation is obtaines.

$$I(P_{xy}) = I[(A(x), \tau_1), (B(y), \tau_2)] = [\min[1, 1 - A(x) + B(x)], \min[1, 1 - \tau_1 + \tau_2]] \tag{17}$$

5 Approximate Reasoning

In order to apply fuzzy set theory in real world problems, approximate reasoning is required. One of the most common techniques of reasoning are inference rules in classical logic which use Modus Ponens for inference process. Generally speaking, Modus Ponens have been extended to be used in fuzzy logic. Zadeh [19] introduced Generalized Modus Ponens (GMP) for fuzzy aet and logic theory [6]. Using Compositional Rule of Inference (CRI) for infering fuzzy consequents with GMP, he developed fuzzy reseaning from classical reasoning. GMP is presented by the following equations [6].

$$\begin{array}{l} \text{Premise 1: A} \rightarrow \text{B} \\ \underline{\text{Premise 2: A}'} \\ \text{Deduction: B}^* \end{array} \tag{18}$$

where A and A′ are two linguistic variableson the universe of discourse of variable x with membership functions $\mu_A(x){:}\, x \in X \rightarrow [0, 1]$. Moreover, B and B* are linguistic terms on the universe of discourse of consequent variable y with membership functions $\mu_B(y)\colon y \in Y \to [0,1]$. Using membership degrees, (18) is calculated by the following equation proposed by Zadeh [1].

$$\mu_{B^*}(y) : sup_{x\in X} AND(\mu_A(x), \mu_{A\to B}(x,y)) \quad (19)$$

Using Zadeh rule base, "AND" operator is used for representing "MIN" and "MAX" operator is utilized as "sup" [6]. Table 3 shows different types of implications.

6 Fuzzy Systems

There are two main types of fuzzy systems: (I) Mamdani fuzzy systems; (II) Takagi-Sugeno-Kang (TSK) fuzzy systems.

6.1 Mamdani Fuzzy System

$$\text{IF } X \text{ isr } B_i \text{ THEN } Y \text{ isr } D_i \;\; i = 1, \ldots, m \quad (20)$$

$$R_i = B_i \cap D_i \text{ and } R = \bigcup_{i=1}^{m} R_i \quad (21)$$

$$\mu_{R_i}(x,y) = \mu_{B_i}(x) \bigwedge \mu_{D_i}(y) \quad (22)$$

$$\mu_R(x,y) = \bigvee_{i=1}^{m} \mu_{R_i}(x,y) \quad (23)$$

$$\mu_F(y) = \bigvee_x \left(\mu_A(x) \bigwedge \mu_R(x,y)\right) = \bigvee_{i=1}^{m} \left[\bigvee_x \left(\mu_A(x) \bigwedge \mu_{B_i}(x)\right)\right] \bigwedge \mu_{D_i}(y) \quad (24)$$

where X and Y are two fuzzy variables; B_i and D_i are two linguistic labels; i is the number of rules.The intersection and union operators are represented by $\cap$ and $\cup$ respectively in (21). Membership functions for antecedent variable (X) and consequent variable (Y) are represented by $\mu_{B_i}(x)$ and $\mu_{D_i}(y)$ respectively in (22); $\wedge$ and $\vee$ are t-norm and t-conorm operators explained in Sects. 3.1 and 3.2 respectively. $\mu_{R_i}(x,y)$ shows the aggregated MFs of antecedents and consequents in each rule. $\mu_R(x,y)$ indicates the aggregated MFs of all rules, and $\mu_F(y)$ represents the final aggregated MF. Figure 3 depicts Mamdani fuzzy inference system.

Table 3 Different types of implications (Klir and Yuan [7])

Name	References	Function $I(a, b)$	Year
Zadeh	[20]	$\max[1-a, \min(a,b)]$	1973
Gaines Rescher	[21]	$\begin{cases} 1 & a \leq b \\ 0 & a > b \end{cases}$	1969
Godel	–	$\begin{cases} 1 & a \leq b \\ b & a > b \end{cases}$	1976
Goguen	[22]	$\begin{cases} 1 & a \leq b \\ b/a & a > b \end{cases}$	1969
Kleene-Dienes	–	$\max(1-a, b)$	1938–1949
Lukasiewicz	[23]	$\min(1, 1-a+b)$	1920
Smets and Magrez	[24]	$\min[1, \frac{1-a+(1+\lambda)b}{1+\lambda a}]$	1987
Smets and Magrez	[24]	$\min[1, (1-a^{\omega}+b^{\omega})^{\frac{1}{\omega}}]$	1987
Reichenbach	[25, 26]	$1-a+ab$	1935–1949
Willmott	[27]	$\min[\max(1-a,b), \max(a, 1-a), \max(b, 1-b)]$	1980
Wu	[28]	$\begin{cases} 1 & a \leq b \\ \min(1-a, b) & a > b \end{cases}$	1986
Yager	[15]	$\begin{cases} 1 & a = b = 0 \\ b^a & others \end{cases}$	1980
Klir and Yuan	[29]	$1-a+a^2b$	1994
Klir and Yuan	[29]	$\begin{cases} b & a = 1 \\ 1-a & a \neq 1, b \neq 1 \\ 1 & a \neq 1, b = 1 \end{cases}$	1994

After finding the final aggregated MF as an output of the fuzzy rule based system, deffuzification is used for obtaining a crisp and single-valued result of the system. There are several deffuzification methods in literature, and two of them are listed herein. They are the Center-of-Area (CoA) or Center-of-Gravity method, and the Middle-of-Maxima (MoM) defuzzification methods. Figures 4 and 5 show CoA and MoM defuzzification techniques. Equations (25) and (26) indicate CoA method in discrete and continuous cases [30].

$$y^* = \frac{\sum_{i=1}^{l} y_i \mu_Y(y_i)}{\sum_{i=1}^{l} \mu_Y(y_i)} = \frac{\sum_{i=1}^{l} y_i \max_k \mu_{CLY}(k)(y_i)}{\sum_{i=1}^{l} \max_k \mu_{CLY}(k)(y_i)} \tag{25}$$

$$u^* = \frac{\int_y u\mu_Y(y)dy}{\int_y \mu_Y(y)dy} = \frac{\int_y y \max_k \mu_{CLY}(k)(y)dy}{\int_y \max_k \mu_{CLY}(k)(y)dy} \tag{26}$$

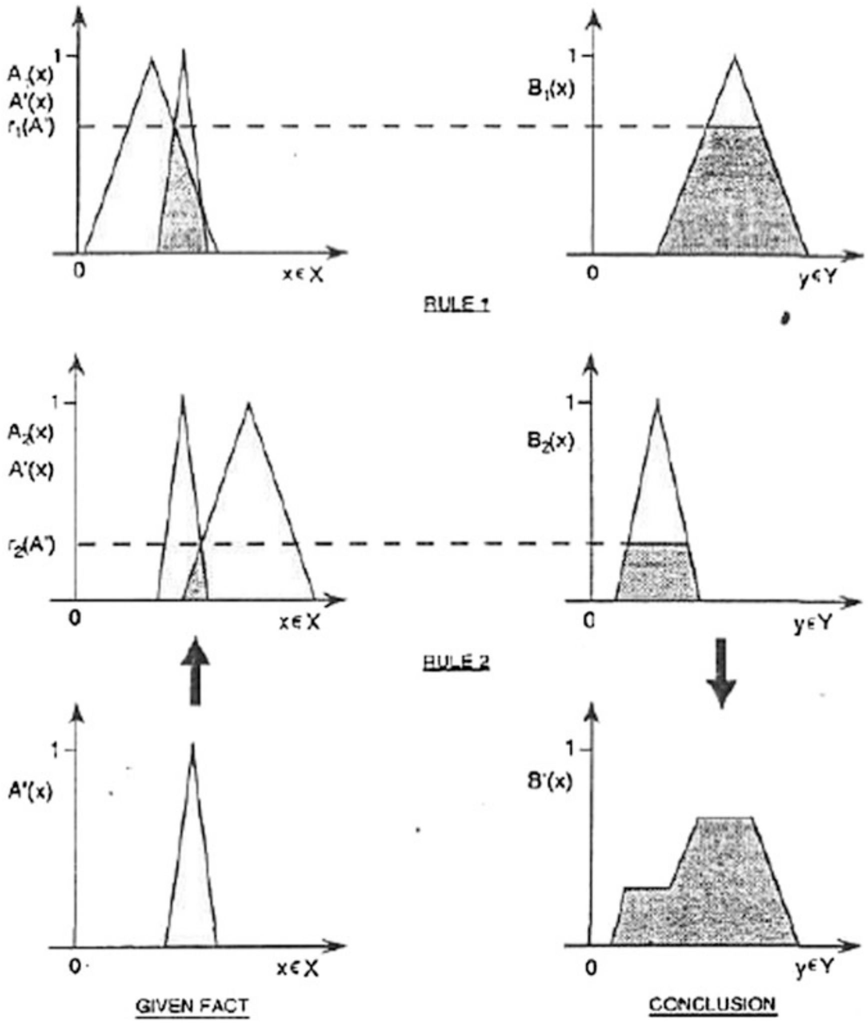

Fig. 3 Mamdani fuzzy inference system (Klir and Yuan [7])

where $\mu_{CLY}(k)$ is defined the same as $\mu_{R_i}(x, y)$ in (22), and "$\int$" is the classical integral.

Equation (27) shows MoM defuzzification technique. It determines the first and the last from of all values where Y has maximal membership degree and then takes the average of these two values [30].

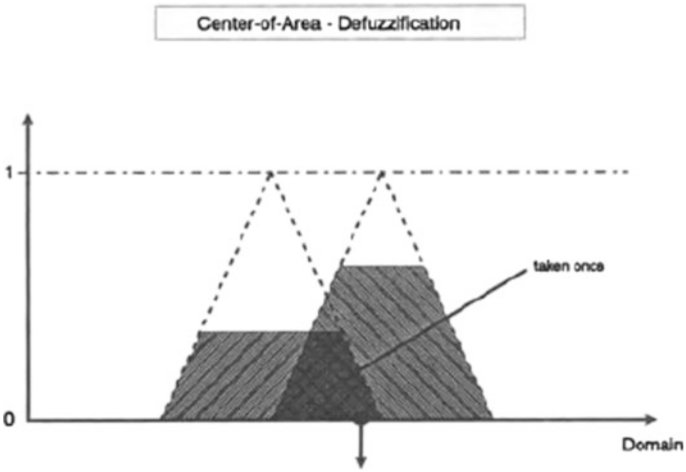

Fig. 4 CoA deffuzification method (Driankov and Saffiotti [30])

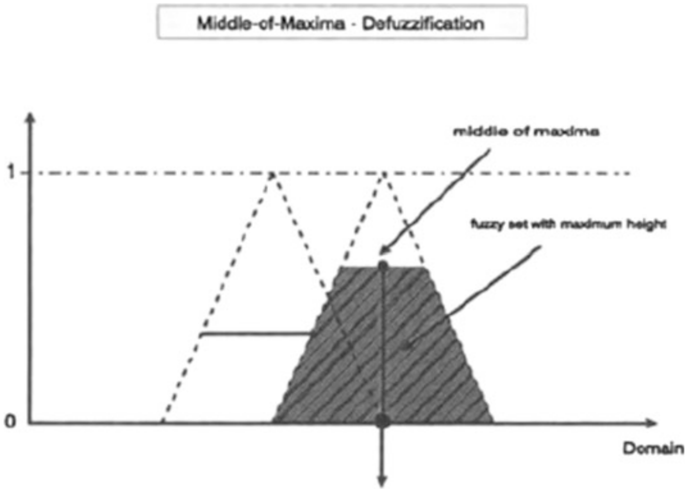

Fig. 5 MoM deffuzification method (Driankov and Saffiotti [30])

$$y^* = \frac{inf_{y \in Y}\{y \in Y | \mu_y(y) = hgt(Y)\} + sup_{y \in Y}\{y \in Y | \mu_y(y) = hgt(Y)\}}{2} \quad (27)$$

where $hgt(Y) = sup_{y \in X} \mu_B(y)$.

6.2 Takagi-Sugeno-Kang (TSK) Fuzzy System

In TSK fuzzy systems, consequent of each rule includes a function instead of the linguistic variable(s). Equation (28) shows the TS fuzzy model including R rules.

$$\begin{aligned} &\text{Rule } i\text{:if } z_1 \text{ is } A_1^{i,k_1}, z_2 \text{ is } A_2^{i,k_2}, \ldots, \text{ and } z_m \text{ is } A_m^{i,k_m} \\ &\quad \text{then } y^i = a_1^i x_1 + a_2^i x_2 + \cdots + a_q^i x_q \\ &\quad i = 1, 2, \ldots, R.\ k_j = 1, 2, \ldots, r_j. \end{aligned} \tag{28}$$

where R is the number of rules in the TS fuzzy model. $z_j\ (j = 1, 2, \ldots, m)$ is the jth variable. $x_l\ (l = 1, 2, \ldots, q)$ is the lth model input. y^i is the output of the ith rule. For the ith rule, A_j^{i,k_j} is the k_j^{th} fuzzy subset of z_j. a_l^i is the coefficient of the consequent. r_j is the fuzzy partition number of z_j.

The output of the TSK model is calculated by taking the weighted-average of output of each rule as follows:

$$y = \sum_{i=1}^{R} \mu^i y^i \Bigg/ \sum_{i=1}^{R} \mu^i \tag{29}$$

where y^i is determined by the equation in the consequent of the ith rule. Moreover, (29) can be restated by (30).

$$y = \left(\sum_{i=1}^{R} \mu^i a_1^i x_1 + \cdots + \sum_{i=1}^{R} \mu^i a_q^i x_q \right) \Bigg/ \sum_{i=1}^{R} \mu^i \tag{30}$$

7 Type-2 Fuzzy Sets

In order to model and solve problems with higher degree of vagueness and uncertain MFs, Zadeh [3] introduced type-2 fuzzy sets as an extension to type-1 fuzzy sets. In type-2 fuzzy sets, each MF is represented by another MF known as secondary membership function. Type-2 fuzzy sets have been proposed by Zadeh [3] to tackle the drawbacks of type-1 fuzzy sets when uncertainty is too high. Three ways in which such uncertainty can occur are: (1) the words that are used in antecedents and consequents of rules can mean different things to different people, (2) consequents obtained by polling a group of experts will often be different for the same rule because the experts will not necessarily be in agreement, and (3) noisy training data (Liang and Mendel [4]).

Type-1 fuzzy sets are unable to handle higher degrees of uncertainty. Moreover, some datasets are very changeable over time. This variability causes MFs to change and include more than one single-valued membership degree for each point in a universe of discourse. Because type-1 MFs assign only one single-valued membership degree to each point they are incapable of modeling systems using those kinds of uncertain data sets. Hence, type-2 fuzzy sets which assign another membership degree to the primary membership values are used in such systems. Figure 6 shows type-1 and type-2 fuzzy MFs (interval type-2 fuzzy MF and general type-2 fuzzy MF). Comparing both MFs in Fig. 6 indicates that type-2 fuzzy MFs are capable of including more information and handling variable data sets using two MFs.

A type-2 fuzzy set $\tilde{A}$ is represented by a type-2 membership function $\mu_{\tilde{A}}$ as follows.

$$\mu_{\tilde{A}} : X \rightarrow [0, 1] \tag{31}$$

Mendel [32] defines the type-2 fuzzy set as it is shown in (32).

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | \forall x \in X,\ \forall u \in J_x \subseteq [0, 1]\} \tag{32}$$

where $0 \le \mu_{\tilde{A}}(x, u) \le 1$, J_{x} is a primary MF in the interval [0, 1], and u is the primary membership value. Another form of representing type-2 fuzzy set $\tilde{A}$ is as follows.

$$\tilde{A} = \int\limits_{x \in X} \int\limits_{u \in Jx} \mu_{\tilde{A}}(x, u)/(x, u) = \int\limits_{x \in X} \int\limits_{u \in Jx} f_x(u)/(x, u) \tag{33}$$

where $\iint$ shows the union of admissible x and u, $f_x(u)$ is the secondary MF [6].

Since type-2 fuzzy sets are presented as the union of secondary MFs ($f_x(u)$), type-2 fuzzy MFs are also represented with the secondary MFs for continuous and discrete universe of discourse as shown in (34) and (35) respectively.

$$\mu_{\tilde{A}}(x, u) = \int\limits_{u \in Jx} f_x(u)/u \tag{34}$$

$$\mu_{\tilde{A}}(x, u) = \sum_{u \in Jx} f_x(u)/u \tag{35}$$

However, due to the various degrees of uncertainty, different kinds of type-2 fuzzy sets are required. There are two kinds of type-2 fuzzy sets: (I) interval type-2 fuzzy sets (IT2 FSs); (II) general type-2 fuzzy sets (GT2 FSs). In IT2 FSs, MFs are interval instead of a single value. They are special type of GT2 FSs whose secondary membership values for the entire members of the primary domain are one. GT2 FSs are extensions of IT2 FSs, and they are bivariate interval valued [0,1] supporting more degrees of freedom [5]. However, IT2 FSs are less computationally expensive than GT2 FSs.

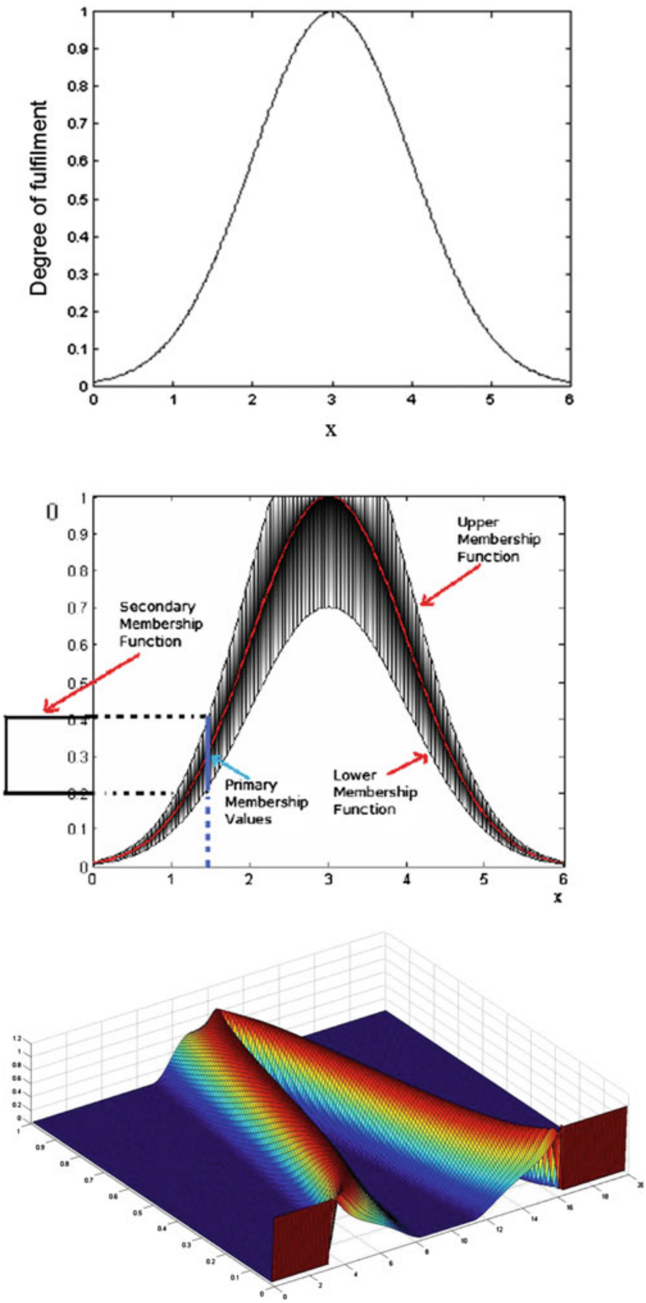

Fig. 6 Type1 fuzzy MF, interval type-2 fuzzy MF [6], and general type-2 fuzzy MF [31]

7.1 Interval Type-2 Fuzzy Sets

IT2 FSs have been proposed by Karnik et al. [33]. Since they are less complex and computationally expensive than GT2 FSs, researchers are most willing to use them instead of GT2 FSs. However, GT2 FSs are more capable of representing wider range of information and being used in face of variable data sets. MFs of interval type-2 fuzzy sets are defined by the following equations.

$$\mu_{\tilde{A}}(x){:}\, X \to 1/u, \quad u \in J_x, J_x \subseteq [0,1] \tag{36}$$

$$f_x(u) = \forall x \in X, \quad u \in J_x, J_x \subseteq [0,1] \tag{37}$$

Equation (37) indicates that at each value of *x,* all the secondary membership values are equal to 1 as it is shown in Fig. 6. There is more than one MF for each value of x. The MFs includes the lower and upper MFs. The upper and lower MFs of IT2 FSs are defined by (38) and (39).

$$\mu_{\tilde{A}}^{U}(X) = \max_{u \in J_x}(u),\ \forall x \in X \tag{38}$$

$$\mu_{\tilde{A}}^{U}(X) = \min_{u \in J_x}(u),\ \forall x \in X \tag{39}$$

where $\mu_{\tilde{A}}^{U}(x)$ is the upper MF and $\mu_{\tilde{A}}^{L}(x)$ is the lower MF of the IT2 FS. Therefore, the IT2 FS is defined using upper and lower MFs by the following equation.

$$\mu_{\tilde{A}}(x){:}X \to 1/u, \quad u \in \left[\mu_{\tilde{A}}^{L}(x), \mu_{\tilde{A}}^{U}(x)\right] \tag{40}$$

7.2 General Type-2 Fuzzy Sets

Using a type-2 fuzzy MF ($\mu_{\tilde{A}}(x,y)$), a GT2 FS $\tilde{A}$ is represented by the following equation [32].

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u)/(x,u), J_x \subseteq [0,1] \tag{41}$$

where $x \in X$ is the primary variable, $u \in J_x$ is the primary MF, J_x indicates an interval between the lower and upper MFs, and $\mu_{\tilde{A}}(x,u)$ is the secondary MF. GT2 FSs are expressed in two different ways: the vertical slice and the wavy slice [34].

Let $x = x_0$ be a specific point in the universe of discourse X; then, vertical slice $\mu_{\tilde{A}}(x', u)$ of the fuzzy MF $\mu_{\tilde{A}}(x, u)$ is obtained. In other words, a secondary MF $\mu_{\tilde{A}}(x = x', u)$ is denoted by vertical slices at each point for $x' \in x$ and $\forall u \in J_{x'} \subseteq [0, 1]$ as follows.

$$\mu_{\tilde{A}}(x = x', u) \equiv \int\limits_{u \in J_{x'}} f_{x'}(u)/u, J_x \subseteq [0, 1] \tag{42}$$

where $f_{x'}(u)$ is the secondary MF and $f_{x'}(u) \subseteq [0, 1]$. Discretizing the entire universe of discourse into N samples, then the GT2 FS $\tilde{A}$ is denoted by aggregating the whole vertical slices as it is shown in (43) [35].

$$\tilde{A} = \sum_{i=1}^{N} \left[\int_{u \in J_{xi}} f_{x_i}(u)/u \right] \Big/ x_i \tag{43}$$

$\tilde{A}$ is also denoted by the union of all its embedded T2 FSs as follows.

$$\tilde{A} = \bigcup_{\forall \tilde{A}_e} \tilde{A}_e \tag{44}$$

where $\tilde{A}_e$ is represented as follows.

$$\tilde{A}_e = \int\limits_{x \in X} u/x, \quad u \in J_x \tag{45}$$

The embedded T1 FS $\tilde{A}_e$, corresponding to an embedded T2 FS includes the primary MF of $\tilde{A}_e$. The centroid of $\tilde{A}_e$, i.e., $C_{\tilde{A}}$, is the union of the centroids of its entire embedded IT2 FSs. Its MF $C_{\tilde{A}}(\xi)$ ($\forall \xi \in X \subset R$), is defined as follows [36].

$$C_{\tilde{A}}(\xi) = \bigcup_{\forall \tilde{A}_e} \left\{ \min_{(x,u) \in A_e} f_x(u) \Big/ \xi = \frac{\int_{-\infty}^{+\infty} x u_{A_e}(x) dx}{\int_{-\infty}^{+\infty} u_{A_e}(x) dx} \right\} \tag{46}$$

The centroid of the GT2 FS $\tilde{A}$ is calculated by aggregating centroids of each α-plane ($C_{\tilde{A}}$) [35]. Hence, the center of the GT2 FS $\tilde{A}$ is presented by (47).

$$C_{\tilde{A}}(\xi) = \bigcup_{\alpha \in [0,1]} \alpha / C_{\tilde{A}\alpha}(\xi) (\forall \xi \in X) \tag{47}$$

where ${}^{\alpha}/C_{\tilde{A}}(\xi)$ is the centroid of $\tilde{A}_\alpha$.

$$C_{\tilde{A}_\alpha}(\xi) = \bigcup_{\forall \tilde{A}_e(\alpha)} \left[\xi = \frac{\int_{-\infty}^{+\infty} x u_{A_e(\alpha)}(x) dx}{\int_{-\infty}^{+\infty} u_{A_e(\alpha)}(x) dx} \right] = \left[C_l(\tilde{A}_\alpha), C_r(\tilde{A}_\alpha) \right] \quad (48)$$

where $C_l(\tilde{A}_\alpha)$ and $C_r(\tilde{A}_\alpha)$ are calculated by KM [37] and EKM [38] algorithms. $f_x(u)$ is considered as the combination of two increasing and decreasing functions called $g_x(u)$ and $h_x(u)$.

$$f_{x(u)} = \begin{cases} g_x(u) & u \in [S_L(x|0), S_L(x|1)] \\ h_x(u) & u \in [S_R(x|1), S_R(x|0)] \\ 1, & u \in [S_L(x|1), S_R(x|1)] \\ 0, & \text{Otherwise} \end{cases} \quad (49)$$

The upper and lower MF of GT2 FS with the secondary membership value of α are defined as $S_R(x|\alpha)$ and $S_L(x|\alpha)$ respectively. The slopes of the secondary MFs at α for $g_x(u)$ and $h_x(u)$ are equal to $g'_x(u)$ and $h'_x(u)$. Then in order to calculate upper and lower MFs at the plane "$\alpha + TS$" the following equations are required.

$$\begin{aligned} S_R(x|\alpha + Ts) &= S_R(x|\alpha) + \frac{T_S}{h'_x(u)} \\ S_L(x|\alpha + Ts) &= S_L(x|\alpha) + \frac{T_S}{g'_x(u)} \end{aligned} \quad (50)$$

8 Type-2 Fuzzy Systems

An interval type-2 fuzzy rule based system is presented in the following equation (Celikyilmaz and Turksen [6]).

$$\tilde{R}_i{:}IF \underset{j=1}{\overset{nv}{AND}} \left(x_j \in X_j \ isr \ \tilde{A}_{ij} \right) THEN \ y \in Y \ isr \ \tilde{B}_i \quad (51)$$

where,

- c is the number of rules in system model,
- x_j is jth input variable, $j = 1, \ldots, nv$, "nv" is total number of input variables,
- X_j is the domain of x_j,
- $\tilde{A}_{ij}$ is the linguistic label associated with jth input variable in the ith rule represented by a type-2 membership function, $\tilde{\mu}_i(x_j) \in X_j \rightarrow f_{x_j}(u)/u$, $u \in J_{x_i}, J_{x_i} \subseteq [0, 1]$.

- y is the output variable,
- Y is the domain of y,
- B_i is the linguistic label associated with output variable in the ith rule with type-2 membership, $\tilde{\mu}_i(y) \in Y \to f_y(w)/w, w \in J_y, J_y \subseteq [0, 1]$.
- AND is the logical connective used to aggregate membership values of input variables for a given observation in order to find the degree of fire of each rule,
- THEN ($\to$, $\Rightarrow$) is the logical IMPLICATION connective,
- ALSO is the logical connective used to aggregate model outputs of fuzzy rules, '*isr*' is introduced by Zadeh [1] and it represents that the definition or assignment is not crisp, it is fuzzy.

In order to find the result of the interval type-2 fuzzy rule based system (51), two inference methods have been proposed in literature. The first one is to considerupper and lower membership functions of interval type-2 fuzzy sets as lower and upper points of an interval (Liang and Mendel [4]).

Let $\mu^U_{\tilde{A}}(x)$ be the upper MF, and $\mu^L_{\tilde{A}}(x)$ be the lower MF. Equation (52) shows the inference mechanism proposed by Liang and Mendel [4].

$$\mu_{\tilde{A}}(x)\colon x \to {}^{1}\!/_{u}, u \in [\mu^L_{\tilde{A}}(x), \mu^U_{\tilde{A}}(x)] \tag{52}$$

Membership values of antecedents are aggregated for each rule as it is shown in (53).

$$\tilde{\mu}^L_i(x) = T^{nv}_{j=1}\left(\tilde{\mu}^L_i(x_j)\right), \tilde{\mu}^U_i(x) = T^{nv}_{j=1}\left(\tilde{\mu}^U_i(x_j)\right) \tag{53}$$

where, T denotes the T-norm connective. In (10), $\tilde{\mu}^{*L}_i(y)$ and $\tilde{\mu}^{*U}_i(y)$ are upper and lower memberships of output fuzzy set $\tilde{\mu}^*_i(y)$.

$$\tilde{\mu}^*_i(y)\colon Y \to {}^{1}\!/_{w}, w \in [\tilde{\mu}^{*L}_i(y), \tilde{\mu}^{*U}_i(y)] \tag{54}$$

The aggregated antecedents and consequents for lower and upper MFs are presented in (55) and (56).

$$\tilde{\mu}^{*L}_i(y) = T\left(\tilde{\mu}^L_i(x), \tilde{\mu}^L_i(y)\right) \tag{55}$$

$$\tilde{\mu}^{*U}_i(y) = T\left(\tilde{\mu}^U_i(x), \tilde{\mu}^U_i(y)\right) \tag{56}$$

Then, all rules are aggregated using (57) and (58).

$$\tilde{\mu}^{*L}(y) = S^{c^*}_{i=1}(\tilde{\mu}^L_i(y)) \tag{57}$$

$$\tilde{\mu}^{*U}(y) = S^{c^*}_{i=1}(\tilde{\mu}^U_i(y)) \tag{58}$$

where "S" is a s-norm or t-conorm operator, and "c" is the number of rules. At the end of the inference process, a crisp output is required. The following equation has been presented by Liang and Mendel [4] for calculating the crisp output (y^*).

$$(y^*) = \left[y^{*L} + y^{*U}\right]/2 \tag{59}$$

where, y^{*L} and y^{*U} are lower and upper bounds, i.e., $y^* \in \left[y^{*L}, y^{*U}\right]$.

The second inference method has been proposed by Turksen [39] using fuzzy disjunctive normal forms (FDNF) and fuzzy conjuctive normal forms (FCNF). Turksen's inference process with a rule set is represented as follows [39].

$$\mu_B^*(y) = [\bigvee_{x \in X} \mu'_A(x) T[\mu_{FDNF}(A \to B)(x, y)] \bigvee_{x \in X} \mu'_A(x) T[\mu_{FCNF}(A \to B)(x, y)]], \forall \mathrm{y} \in \mathrm{Y} \tag{60}$$

where "T" is used to indicate t-norm between two MFs. $\mu_{FDNF}(A \to B)(x, y)$ and $\mu_{FCNF}(A \to B)(x, y)$ are two boundaries of type-2 fuzzy systems based on fuzzy normal forms. $\mu_{\mathrm{B}}^*(y)$ is an inference result, and $\mu'_A(x)$ is the observed membership value.

$$\begin{cases} A\left(x^*_{D,i,r}(r)\right) = \mu(x^*_{FDNF(A(i,r))}(r)), \\ A\left(x^*_{C,i,r}(r)\right) = \mu(x^*_{FCNF(A(i,r))}(r)) \end{cases} \tag{61}$$

Equation (61) is the membership value of the FDNF of the left-hand side of the ith rule which has "r" input variables evaluated at $x^*_{(r)} = (x^*_1, x^*_2, \ldots, x^*_r)$. $A_{D,i,r}(x^*(r))$ is the membership value of the FDNF in the left-hand side of the ith rule which has "r" input variables evaluated at $x^*_{(r)} = (x^*_1, x^*_2, \ldots, x^*_r)$. $A\left(x^*_{C,i,r}(r)\right)$ is the membership value of the FCNF [39].

$A\left(x^*_{D,i,r}(r)\right)$ is computed recursively as:

$$\begin{cases} A_{D,i,\rho}(x^*(\rho)) = \left[A_{D,i,\rho-1}(x^*(\rho-1)) T A_{i,\rho}\left(x^*_\rho\right)\right] S \left[A_{D,i,\rho-1}(x^*(\rho-1)) T A_{i,\rho}\left(x^*_\rho\right)\right] \\ For\ \rho = 2, 3, 4, \ldots, r \end{cases} \tag{62}$$

Such that $X^*(2) = (X^*_1, X^*_2)$ and $A(i, 2) = A_{i1}\,\mathrm{AND}\,A_{i2}$. "$S$" in the definition of "$A_{D,i,\rho}(x^*(\rho))$" indicates S-norm of two MFs.

$A_{C,i,r}(x^*(r))$ is computed recursively as [39]:

$$\begin{aligned} A_{C,i,\rho}(x^*(\rho)) = {} & [A(x^*_{C,i,\rho-1}(\rho-1)SA_{i\rho}\left(x^*_\rho\right)]T[A(x^*_{C,i,\rho-1}(\rho-1)SA_{i\rho}\left(x^*_\rho\right)] \\ & T[A(x^*_{C,i,\rho-1}(\rho-1)Sn(A_{i\rho}\left(x^*_\rho\right))]T[A(x^*_{C,i,\rho-1}(\rho-1)Sn(A_{i\rho}\left(x^*_\rho\right))] \\ & T[n(A(x^*_{C,i,\rho-1}(\rho-1))S(A_{i\rho}\left(x^*_\rho\right))]T\left[n\left(A\left(x^*_{C,i,\rho-1}(\rho-1)\right)S\left(A_{i\rho}\left(x^*_\rho\right)\right)\right], \\ \rho = {} & 2,3,4,\ldots,r \end{aligned} \tag{63}$$

In (64), $B^*(y)$ is the final output of the model, which is obtained by combining the final output of FDNF and FCNF.

$$B^*(y) = \beta B^*_D(y) + (1-\beta)B^*_C(y), \quad \forall y \in Y, 0 \le \beta \le 1, \tag{64}$$

where $\beta^*_D(y)$ is the final result for FDNF and $\beta^*_C(y)$ is the final result for FCNF obtained after completing the inference process.

9 Applications of Type-2 Fuzzy Systems

In this section, some of the applications of type-2 fuzzy systems are briefly reviewed. These applications include but are not limited to clustering, classification, pattern recognition, scheduling, forecasting, and supply chain management (SCM).

9.1 Type-2 Fuzzy Systems in Clustering, Classification, and Pattern Recognition

In this subsection, a brief review on the first categories of type-2 fuzzy applications including clustering, classification, and pattern recognition techniques is provided. Type-2 fuzzy systems have been used in clustering and classification models for managing real world problems which encounter higher degrees of uncertainty.

Data clustering is one of the main methods of structure identification. It is an unsupervised technique for putting similar data in a cluster [40]. Bezdek presented fuzzy c-means clustering (FCM) model. For structure identification of type-2 fuzzy systems different methods have been introduced in the literature. For more information please refer to [41–47]. In order to design a rule based system, Aliev et al. [48] proposed a type-2 FCM clustering model. Rhee and Choi [42] introduced three methods for interval type-2 fuzzy membership function generation: (I) Histogram based method; (II) Heuristics methods; (III) Interval type-2 fuzzy C-means (IT2 FCM). Fazel Zarandi, et al. [49] proposed a new interval type-2 fuzzy clustering method for functional systems called interval type-2 fuzzy c-regression clustering model (IT2 FCRM).

Type-2 fuzzy systems are also applied in classification problems. Melin and Castillo [50] reviewed the literature of clustering, classification, and pattern recognition. Tables 4 and 5 show the results of their work. Sharma and Bajaj [51, 52] proposed an interval type-2 fuzzy system for vehicle classification. They assigned each vehicle to its relevant class by investigating its characteristics such as the wheel base, ground clearance, and body length [50]. Tan et al. [53] proposed a type-2 fuzzy system for ECG arrhythmic beat classification. Three classes of ECG signals, called the normal sinusrhythm (NSR), ventricular fibrillation (VF) and ventricular tachycardia (VT) have been investigated in that research.

9.2 Type-2 Fuzzy Systems in Scheduling, Forecasting, and SCM

In this subsection, a brief review on the second categories of type-2 fuzzy applications including scheduling, forecasting, and SCM is provided.

Fazel Zarandi and Gamasaee [82] proposed a type-2 fuzzy hybrid expert system, which uses a combination of Mamdani and Sugeno methods, for tardiness forecasting in scheduling of steel continuous casting process. In that study, tardiness variables are represented by interval type-2 fuzzy MFs, and interval type-2 FDNF and FCNF proposed by Turksen [39] are used in the inference engine. The results of the method proposed by Fazel Zarandi and Gamasaee [82] show that using type-2 fuzzy system leads to the more accurate forecasted outputs in comparison to type-1 fuzzy system and other methods in literature.

Karnik and Mendel [83] applied type-2 fuzzy systems for forecasting of time-series. They showed that noisy data causes type-1 fuzzy system to be incapable of modeling the problem efficiently, so type-2 fuzzy systems are required to handle noisy data. Fazel Zarandi et al. [84] presented a type-2 fuzzy hybrid expert system for forecasting the amount of reagents in desulphurization process of a steel manufacturing company. The results of that model indicate that type-2 fuzzy system has less forecasting error in comparison to type-1 fuzzy system. Another research that uses type-2 fuzzy systems in time series prediction has been conducted by Gaxiola et al. [85]. They presented a generalized type-2 fuzzy weight adjustment for back propagation neural networks in time series prediction.

Pramanik et al. [86] used type-2 fuzzy systems for modeling and solving a fixed-charge transportation problem in a two-stage supply chain network. Since unit transportation costs, fixed charges, availabilities, and demands are imprecise, they have been indicated by Gaussian type-2 fuzzy numbers. The problem has been categorized in class of profit maximization problems, and retailers' demands are satisfied by selecting distribution centers. For solving the problem, type-2 fuzziness has been eliminated using generalized credibility measures. Fazel Zarandi and

Table 4 Type-2 fuzzy systems in clustering and classification [50]

Author(s) (pub. year)	References	Domain of the problem	Comparison with type-1	Why type-2 is required for the problem?
Sharma and Bajaj (2009 and 2010)	[51, 52]	Vehicle classification	Yes	Uncertainty and imperfection of data
Pimenta and Camargo (2010)	[54]	Classification	Yes	Imprecision in classification
Chumklin et al. (2010)	[55]	Cancer detection	Yes	Uncertainty in medical classification
Sanz et al. (2010)	[56]	Classification	No	Imprecision in classification
Wu and Mendel (2010)	[57]	Vehicle classification	Yes	Uncertainty in information
Phong and Thien (2009)	[58]	Arrhythmia classification	No	Uncertainty in medical classification
Aliev et al. (2011)	[48]	Clustering	Yes	Uncertainty in clustering
Abiyev et al. (2011)	[59]	Clustering	No	Uncertainty in clustering
Zheng et al. (2010)	[60]	Classification	No	Uncertainty in classification
Abiyev and Kaynak (2010)	[61]	Clustering	No	Uncertainty in clustering
Zhengetal. (2010)	[62]	Clustering	No	Uncertainty in clustering
Albarracin and Melgarejo (2010)	[63]	Signal clustering	Yes	Uncertainty in clustering
Ozkan and Turksen (2010)	[64]	Clustering	No	Uncertainty in clustering
Pedrycz (2010)	[65]	Clustering	Yes	Uncertainty in granulation
Juang et al. (2009)	[66]	Clustering	Yes	Uncertainty in clustering
Türkşen (2009)	[67]	Clustering	Yes	Uncertainty in clustering
Ren et al. (2010)	[68]	Clustering	No	Uncertainty in clustering
Qun et al. (2010)	[69]	Clustering	Yes	Uncertainty in clustering

Gamasaee [87] proposed a type-2 fuzzy system model for reducing bullwhip effect in supply chains. They concentrated on demand forecasting techniques where all demands, lead times, and orders have been represented by type-2 fuzzy sets. Using interval type-2 fuzzy c-regression clustering technique, demand data have been assigned to appropriate clusters, and structure of type-2 fuzzy expert system has

Table 5 Type-2 fuzzy systems in pattern recognition [50]

Author(s) (pub. year)	Ref. no.	Domain of the problem	Comparison with type-1	Why type-2 is required for the problem?
Melin (2010)	[70]	Edge detection	Yes	Uncertainty in edge detection
Lopez et al. (2010)	[71]	Finger print recognition	Yes	Uncertainty in finger print recognition
Li and Zhang (2010)	[72]	Pattern recognition	No	Uncertainty in pattern recognition
Own (2009)	[73]	Medical diagnosis	No	Uncertainty in diagnosis
Mendoza et al. (2009)	[74]	Face recognition	Yes	Uncertainty in face recognition
Kim et al. (2009)	[75]	Pattern recognition	No	Uncertainty in pattern recognition
Hidalgo et al. (2009)	[76]	Multimodal recognition	Yes	Uncertainty in multimodal pattern recognition
Lopez et al. (2008)	[77]	Finger print recognition	Yes	Uncertainty in finger print recognition
Ozkan and Turksen (2004)	[78]	Pattern recognition	Yes	Uncertainty in pattern recognition
Mitchell (2005)	[79]	Pattern recognition	No	Uncertainty in pattern recognition
Madasu et al. (2008)	[80]	Edge detection	Yes	Uncertainty in edge detection
Tizhoosh (2005)	[81]	Image thresholding	No	Uncertainty in image thresholding

been identified. Then, an interval type-2 fuzzy hybrid expert system was developed for demand forecasting. An interval type-2 fuzzy ordering policy was designed to determine orders in supply chain. Results of that model demonstrated that using type-2 fuzzy system leads to more bullwhip reduction and better forecasted results in comparison to type-1 fuzzy systems.

10 Type-n Fuzzy Systems

There are not too much works in the literature in type-3 and higher level fuzzy sets and systems. Transformation from full type-2 fuzzy sets and systems into higher level ones has serious modeling, representing, and computational complexity problems. First, we should find the real cases with for example type-3 fuzzy syntax and semantic. It is very difficult to show these cases in geometric configuration. Also, modeling type-n fuzzy needs lots of nonlinear variables, with high correlations with each other. Finally, the computational complexity of type-n fuzzy is too much. This fact needs some high level heuristic algorithms to manage it. So, there are rooms of potential scientific and applied works that should be done in the future to resolve these serious problems. We started it from three years ago and hope we can have some preliminary outputs in the near future.

11 Conclusions

In this chapter, the motivation for using fuzzy systems and the mathematical concepts of type-1 to type-n fuzzy sets, logic, and systems were discussed. This chapter has reviewed the most useful knowledge for researchers on several areas of fuzzy logic such as type-1 fuzzy sets, mathematical operations on type-1 fuzzy sets, fuzzy logic, approximate reasoning, fuzzy systems including mamdani and sugeno inference systems, interval type-2 fuzzy sets, general type-2 fuzzy sets, and type-2 fuzzy systems. Moreover, applications of those fuzzy approaches in solving real world problems were reviewed. Thus, this chapter is a reference for researchers to become more familiar with theory and applications of fuzzy logic.

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