A Review of Hesitant Fuzzy Sets: Quantitative and Qualitative Extensions

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Abstract Since the concept of fuzzy set was introduced, different extensions and generalizations have been proposed to manage the uncertainty in different problems. This chapter is focused in a recent extension so-called hesitant fuzzy set. Many researchers have paid attention on it and have proposed different extensions both in quantitative and qualitative contexts. Several concepts, basic operations and its extensions are revised in this chapter.

Keywords Hesitant fuzzy set · Operations · Extensions

1 Introduction

Fuzzy sets were introduced by Zadeh [69]. Since then theory on fuzzy sets and fuzzy logic has been developed in parallel with a large number of successful applications.

Fuzzy sets permit to represent that elements have partial membership to a set, and they can be modeled to represent graduality between non-membership and complete membership to a set. A fuzzy set is represented mathematically by means of membership functions which generalize characteristic functions. While the latter

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are functions that given an element of the reference set return a value that is either 0 or 1, membership functions return a value in the interval [0,1].

Therefore, the definition of a fuzzy set $\tilde{A}$ requires the definition of the membership function of $\tilde{A}$. This implies that we need to assign a number in the interval [0,1] to all elements of the reference set.

At present there exist several generalizations of fuzzy sets. Some of them have been introduced in order to ease the definition of fuzzy sets by means of relaxing the requirement that the membership function needs a value for each element in the reference set. In particular we can mention type 2 fuzzy sets [27] where the membership of an element is a fuzzy set instead of a single number. In this way, we can model the uncertainty on the number we need to assign. Interval valued fuzzy sets (IFS) [45] and Atanassov's intuitionistic fuzzy sets (A-IFS) [1] are two other examples. In this case, the value assigned to an element is an interval. Differences from the point of view of interpretation and discussion about the terminology between these two extensions of fuzzy sets can be found in [11]. Then, the concept of type 2 fuzzy sets can be further generalized into type n fuzzy set. Informally speaking, a type n fuzzy sets corresponds to a type (n-1) fuzzy set in which the membership values of the type (n-1) fuzzy sets are a fuzzy set.

Goguen introduced [13] L-fuzzy sets which also generalize fuzzy sets. The idea is that while in a fuzzy set the membership assigns values in the range [0,1] which is a total order, we can consider the assignment of membership values in partial orders (posets).

Recently, hesitant fuzzy sets (HFSs) were introduced in [40]. In this type of fuzzy sets, the membership value of an element is a subset of [0,1] and typically a finite set of values in [0,1]. As stated in [43], the motivation for introducing this type of fuzzy sets "is that when defining the membership of an element, the difficulty of establishing the membership degree is not because we have a margin of error (as in A-IFS), or some possibility distribution (as in type 2 fuzzy sets) on the possible values, but because we have a set of possible values".

In this chapter some of the results found in the literature on HFSs are revised. Its structure is as follows. Section 2 reviews the concept of HFS, some basic operations and hesitant fuzzy relations. Sections 3 and 4 revise extensions of HFS in quantitative and qualitative contexts. Section 5 introduces some discussions and trends of the hesitant context, and finally some conclusions are pointed out in Sect. 6.

2 Hesitant Fuzzy Sets

The concept of HFS was recently introduced as an extension of fuzzy sets with the goal of modeling the uncertainty provoked by the hesitation when it is necessary to assign the degree of membership of an element to a fuzzy set. This section revises some basic concepts and operations about HFSs.

2.1 Concepts of Hesitant Fuzzy Sets

As briefly it is stated in the introduction, a HFS is a generalization of a fuzzy set in which the membership function returns a subset of values in [0,1]. This is formalized in the following definition.

Definition 1 [40] Let X be a reference set. Then, a HFS on X is a function h that returns a subset of [0,1] to elements $x \in X$.

$$h : X \to \wp([0,1]) \tag{1}$$

Xia and Xu [61] call $h(x)$ a hesitant fuzzy element (HFE). Note that a hesitant fuzzy element is a set of values in [0,1], and a HFS is a set of HFEs, one for each element in the reference set. That is, if $h(x)$ is the HFE associated to x then $\cup_{x\in X} h(x)$ is a HFS.

A typical hesitant fuzzy set [2] is when $h(x)$ is a finite nonempty subset of [0,1] for all $x \in X$, i.e., HFEs are finite nonempty sets.

The literature presents several papers in which operators on HFS are defined (or can be defined) through operators on HFEs. The extension principle introduced in [44] is one of them.

Definition 2 Let $\{H_1, \ldots, H_n\}$ be n HFSs on a reference set X, let ϕ a function on n HFEs (i.e., ϕ combines n sets into a new set). Then,

$$\phi'(H_1, \ldots, H_n)(x) = \phi(H_1(x), \ldots, H_n(x))$$

defines an operation ϕ' on HFSs.

The extension principle is defined as follows.

Definition 3 Let $\{H_1, \ldots, H_n\}$ be n HFSs on a reference set X, and let Θ be a function $\Theta : [0,1]^n \to [0,1]$, we export Θ to HFSs defining the HFS Θ_E as follows

$$\Theta_E(x) = \cup_{\gamma \in H_1(x)\times\cdots\times H_n(x)} \{\Theta(\gamma)\}.$$

Note that for a given function Θ we can define the function,

$$\phi(S_1, \ldots, S_n) = \cup_{\gamma \in S_1\times\cdots\times S_n} \{\Theta(\gamma)\},$$

which permits to express the extension principle in terms of Definition 2.

Both Definition 2 and the extension principle cause that the properties of the operator ϕ and Θ are inherited by ϕ' and Θ_E. As reported by Rodríguez et al. [34], it is trivial to prove the commutativity and associativity of Θ_E from the ones of Θ.

The definition of operations for HFS from operations for HFEs (as in Definition 2) is not the only way to do so. There are operations on HFS that cannot be represented in this way. The following operator is an example that illustrates this fact.

Definition 4 [34] Let $\{H_1, \ldots, H_n\}$ be n HFSs on a reference set X, then

$$\phi(H_1, \ldots, H_n)(x) = \frac{(\max_i \max(H_i) + \min_i \min(H_i))}{2} \wedge \cup_i H_i(x)$$

where for any α in [0,1], $\alpha \wedge h$ corresponds to the set $\{s | s \in h, s \leq \alpha\}$.

This operation ϕ cannot be represented in terms of another function ϕ' on HFEs.

2.2 *Basic Operations of Hesitant Fuzzy Sets*

In [40] were introduced some basic operations to manage HFEs. These definitions follow the approach of the Definition 2, that is, a function for HFSs defined in terms of a function for HFEs.

Definition 5 [40] Given a HFE, h, its lower and upper bounds are:

$$h^- = inf\{\gamma | \gamma \in h\} \tag{2}$$

$$h^+ = sup\{\gamma | \gamma \in h\} \tag{3}$$

Definition 6 [40] Let h be a HFE, the complement of h is defined as follows:

$$h^c = \bigcup_{\gamma \in h} \{1 - \gamma\} \tag{4}$$

Definition 7 [40] Let h_1 and h_2 be two HFEs, the union of two HFEs $h_1 \cup h_2$, is defined as:

$$h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\max\{\gamma_1, \gamma_2\}\} \tag{5}$$

Definition 8 [40] Let h_1 and h_2 be two HFEs, the intersection of two HFEs $h_1 \cap h_2$ is defined as:

$$h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\min\{\gamma_1, \gamma_2\}\} \tag{6}$$

The relation between HFS and A-IFS was discussed in [40]. Let us start recalling the definition of interval valued fuzzy sets (IFS) and A-IFS. As both are mathematically equivalent, we only give one definition.

Definition 9 Let X be a reference set. Then, an IFS on X is represented by means of two functions $\mu : X \rightarrow [0,1]$ and $v : X \rightarrow [0,1]$ such that $0 \leq \mu(x) + v(x) \leq 1$ for all $x \in X$.

It is easy to prove the following.

Proposition 1 [43] *All IFSs are HFS.*

Definition 10 [43] Given a HFE h, we define the envelope of h as the IFS represented by μ and v defined by $\mu(x) = h^-(x)$ and $v(x) = 1 - h^+(x)$, respectively.

It can be proven that this is the smallest IFS that includes the HFE h.

Proposition 2 *All HFS are L-fuzzy sets.*

This follows from the fact that subsets of [0,1] define a partial order. Note that IFS and type n fuzzy sets can also be seen as L-fuzzy sets.

Sometimes, it is necessary to compare two HFEs to establish an order between them. Different proposals have been introduced in the literature to compare HFEs. Xia and Xu defined a score function to compare HFEs [61], however Farhadinia pointed out that this score function could not distinguish between two HFEs in some cases. Thus, a new score function was presented by Farhadinia [12]. Despite the new score function can compare HFEs when the score function proposed by Xia and Xu cannot, Rodríguez et al. shown a counterexample [34] in which the new function cannot either discriminate some HFEs. Recently, Xia and Xu have presented a variance function [23] to improve the comparison law proposed in [61].

Definition 11 [61] Let h be a HFE, the score function $s(h)$ is defined as follows:

$$s(h) = \frac{1}{l(h)} \sum_{\gamma \in h} \gamma, \tag{7}$$

being $l(h)$ the number of elements in h.

Definition 12 [23] Let h be a HFE, the variance function $v(h)$, is defined as follows:

$$v(h) = \frac{1}{l(h)} \sqrt{\sum_{\gamma_i, \gamma_j \in h} (\gamma_i - \gamma_j)^2}. \tag{8}$$

From the Definitions 11 and 12 of $s(\cdot)$ and $v(\cdot)$ respectively, the following comparison law was defined.

Definition 13 Let h_1 and h_2 be two HFEs,

If $s(h_1) < s(h_2)$, then $h_1 < h_2$,
If $s(h_1) = s(h_2)$, then

If $v(h_1) < v(h_2)$, then $h_1 > h_2$,
If $v(h_1) = v(h_2)$, then $h_1 = h_2$.

More operations and their properties have been presented in [29].

2.3 Hesitant Fuzzy Relations

Fuzzy relations have been used in several contexts. They generalize crisp relations permitting fuzzy membership. For example, binary relations on a reference set X are generalized to fuzzy relations by means of a membership degree of each pair $(x_1, x_2) \in X \times X$. When X is finite, say $X = \{x_1, \ldots, x_n\}$, a fuzzy relation R is represented by a matrix $R = \{r_{ij}\}_{ij}$ where r_{ij} is the membership degree of (x_i, x_j) into the relationship R. A fuzzy relation can be understood as a weighted graph with nodes X and weights R.

Literature discusses additive preference relations (APR), which are fuzzy relations where $\mu(x_i, x_j) + \mu(x_j, x_i) = 1$ for all $x_i, x_j \in X$.

In addition, the literature discusses multiplicative preference relations (MPR). They diverge from fuzzy relations because they are functions from $X \times X$ into [1/9,9]. They require $\mu(x_i, x_j) \cdot \mu(x_j, x_i) = 1$. The range [1/9,9] is based on Saaty's scale for the Analytic Hierarchy Process [35, 36]. In multicriteria decision making problems, multiplicative preference relations are used to represent users' preferences on the criteria from which weights on the criteria are extracted using prioritization methods.

Additive and multiplicative relations are isomorphic. Note that given a MPR $R = \{r_{ij}\}_{ij}$, with values in the range [1/9,9], then when R' is defined in terms of $r'_{ij} = 0.5(1 + \log_9 r_{ij})$ as $R' = \{r'_{ij}\}_{ij}$, we have that R' is an APR, and given an APR we can define the corresponding MPR using the inverse of the function $f(x) = 0.5(1 + \log_9 x)$ (see [8] for details).

Fuzzy preference relations and multiplicative preference relations have been extended in order to include hesitancy on the value assigned in the matrix.

A hesitant fuzzy relation is defined by means of a function μ which assigns a finite subset of [0,1] to each pair (x_i, x_j). Constraints are added on the possible values of $\mu(x_i, x_j)$. Formally, if $\mu(x_i, x_j)$ is a finite set, we can denote the hesitant fuzzy relation by a *matrix*-like structure

$$C(i,j) = \{a_{ij}^1, \ldots, a_{ij}^{k_{ij}}\},$$

for all $i,j = 1, \ldots, |X|$ and where k_{ij} is the number of elements in the pair (x_i, x_j). A hesitant fuzzy relation is defined requiring $k_{ij} = k_{ji}$, $C(i,i) = \{1/2\}$ and that if elements of $C(i,j)$ and $C(j,i)$ are ordered the first set in increasing order, and the second one in decreasing order the pairs in the kth positions should sum one. That is, assume that elements are ordered

$$C(i,j) = \left(a_{ij}^1 \leq \cdots \leq a_{ij}^{k_{ij}}\right)$$
$$C(j,i) = \left(a_{ji}^1 \geq \cdots \geq a_{ji}^{k_{ji}}\right)$$

then, $a_{ij}^k + a_{ji}^k = 1$ (see [76] for details).

A hesitant multiplicative preference relation (HMPR) was defined in [62, 71] requiring that values a_{ij}^k are in the [1/9,9] interval, that $C(i,j)$ and $C(j,i)$ have the same number of elements (i.e., $k_{ij} = k_{ji}$), and that the elements in $C(i,j)$ and $C(j,i)$ can be matched, so that the multiplication of any matched pair is one.

In [41, 42] the conditions on the number of elements and pairing elements are not considered. This definition is isomorphic to the definition of numerical preference relations introduced in [78].

A consistent hesitant multiplicative matrix is then defined as follows. Here, a $n \times n$ structure is the function $C(i,j)$ defined above where $C(i,j)$ is a finite set of values. No constraints are given on the possible values in $C(i,j)$.

Definition 14 Let M be a $n \times n$ structure (or hesitant matrix). We say that M is a consistent hesitant multiplicative preference relation (cHMPR) if it satisfies

C1. $C(i,i) = \{1\}$ for all i,
C2. For all i, j if $a_{ij} \in C(i,j)$ then there is $1/a_{ij} \in C(j,i)$,
C3. For all i, j if $a_{ij} \in C(i,j)$ then there exists k such that $a_{ij} = a_{ik}a_{kj}$ and $a_{ik} \in C(i,k)$ and $a_{kj} \in C(k,j)$.

Prioritization methods have been obtained to derive weights from hesitant matrices. See e.g. [41, 42] inspired in the geometric mean approach introduced in [9].

In [42] an algorithm was introduced to build a cHMPR for any given $n \times n$ structure (or hesitant matrix). The algorithm can also be applied to any standard real-valued matrix which is not consistent (i.e., which is not a multiplicative preference relation) obtaining a hesitant matrix. Algorithm 1 corresponds to this process.

Algorithm 1: Reconcile

Data: $C : n \times n$ structure
Result: $n \times n$ structure
$C_0 = C(i,j) \cup_{a_{ij} \in C(j,i)} 1/a_{ji}$;
for *all* i, j *such that* $i > j$ **do**
  $mult(i,j) =$ there exists k such that $a_{ij} = a_{ik}a_{kj}$? ;
  if $false(mult(i,j))$ **then**
    for some k (the selection of k is arbitrary) ;
      $C_0(k,j) = C_0(k,j) \cup a_{ij}/a_{ik}$;
      $C_0(j,k) = C_0(j,k) \cup a_{ik}/a_{ij}$;
  return $(C' = C_0)$

3 Extensions of Hesitant Fuzzy Sets in Quantitative Settings

We have stated in Sect. 2.1 that a typical hesitant fuzzy set [2] is when $h(x)$ is a finite nonempty subset of [0,1] for all $x \in X$. Several extensions and generalizations of HFS which diverge from this typical type of HFS have been proposed to deal with the hesitation in quantitative settings. These extensions are introduced in this section.

3.1 Dual Hesitant Fuzzy Sets

The concept of Dual Hesitant Fuzzy Set (DHFS) [79] is an extension of HFS based on A-IFS that deals with the hesitation both for the membership and non-membership degrees. Therefore, a DHFS is defined in terms of two functions that return two sets of membership and non-membership values respectively for each element in the domain:

Definition 15 [79] Let X be a set, a DHFS D on X is defined as:

$$D = \{\langle x, h(x), g(x)\rangle | x \in X\} \tag{9}$$

being $h(x)$ and $g(x)$ two sets of values in the interval [0,1], that denote the possible membership and non-membership degrees of the element $x \in X$ to the set D respectively, with the following conditions,

$$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1$$

where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \max_{\gamma \in h(x)}\{\gamma\}$, and $\eta^+ = \max_{\eta \in g(x)}\{\eta\} \forall x \in X$.

The pair $d(x) = (h(x), g(x))$ is called Dual Hesitant Fuzzy Element (DHFE) and by simplicity it is noted $d = (h, g)$.

Example 1 Let $X = \{x_1, x_2\}$ be a reference set, a DHFS D, is defined as follows:

$$D = \{\langle x_1, \{0.4, 0.5\}, \{0.3\}\rangle, \langle x_2, \{0.2, 0.4\}, \{0.3, 0.5\}\rangle\}$$

Some basic operations, such as the complement of a DHFE, the union and intersection of two DHFEs were introduced in [79]. A score function and accuracy function were also defined with the goal of proposing a comparison law to compare DHFEs. Recently, different aggregation operators to aggregate DHFEs have been defined. In [50] it has been introduced Dual Hesitant Fuzzy Weighted Average (DHFWA), Dual Hesitant Fuzzy Weighted Geometric (DHFWG), Dual Hesitant

Fuzzy Ordered Weighted Average (DHFOWA), Dual Hesitant Fuzzy Ordered Weighted Geometric (DHFOWG), Dual Hesitant Fuzzy Hybrid Average (DHFHA) and Dual Hesitant Fuzzy Hybrid Geometric (DHFHG). These operators have been used to propose some generalized dual hesitant fuzzy aggregation operators [46, 67, 68]. The Hamacher operations have been extended to propose some aggregation operators for DHFEs [17, 70]. The Choquet integral has been also used to develop several aggregation operators for DHFEs [16].

Several approaches to compute the correlation coefficient of DHFEs have been defined [7, 53, 65] and some properties have been studied.

A similarity measure that considers the membership and non-membership degrees of DHFEs has been introduced in [39].

3.2 Interval-Valued Hesitant Fuzzy Sets

Sometimes, in real-world decision making problems, it is difficult for experts to express their assessments by using crisp values, because of the lack of information about the problem. In these situations, an interval value belonging to [0,1] could be used. Keeping in mind the concept of HFS, Chen et al. introduced the definition of Interval-Valued Hesitant Fuzzy Set (IVHFS) [5] where the membership degrees are given by several possible interval values.

An IVHFS is defined as follows.

Definition 16 [5] Let X be a reference set, and I([0,1]) be a set of all closed subintervals of [0,1]. An IVHFS on X is,

$$\tilde{A} = \{\langle x_i, \tilde{h}_A(x_i)\rangle | x_i \in X, \quad i = 1, \ldots, n\} \tag{10}$$

where $\tilde{h}_A(x_i) : X \to \wp(I([0,1]))$ denotes all possible interval-valued membership degrees of the element $x_i \in X$ to the set $\tilde{A}$.

$\tilde{h}_A(x_i)$ is called an Interval-Valued Hesitant Fuzzy Element (IVHFE), where each $\tilde{\gamma} \in \tilde{h}_A(x_i)$ is an interval and $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$, being $\tilde{\gamma}^L$ and $\tilde{\gamma}^U$ the lower and upper limits of $\tilde{\gamma}$, respectively.

Example 2 Let $X = \{x_1, x_2\}$ be a reference set, a IVHFS $\tilde{A}$, could be as follows,

$$\tilde{A} = \{\langle x_1, \{[0.2, 0.3], [0.4, 0.5]\}\rangle, \langle x_2, \{[0.1, 0.4], [0.5, 0.6], [0.8, 0.9]\}\rangle\}$$

When the upper and lower limits of all the interval values are equal, the IVHFS is a HFS.

Some basic operations, such as the union, intersection and complement were introduced in [5]. A score function to compare two IVHFEs was also defined [5].

Several aggregation operators for IVHFEs such as, the Interval-Valued Hesitant Fuzzy Weighted Averaging (IVHFWA), Interval-Valued Hesitant Fuzzy Weighted Geometric (IVHFWG), Interval-Valued Hesitant Fuzzy Ordered Weighted Averaging (IVHFOWA), Interval-Valued Hesitant Fuzzy Ordered Weighted Geometric (IVHFOWG) and their generalizations were defined in [5, 59]. Different Einstein aggregation operators for IVHFEs have been presented in [58, 80]. In [20] was studied the Hamacher t-norms to extend and generalize the Hamacher operations for IVHFEs. Two induced generalized hybrid operators based on Shapley for IVHFEs have been defined in [28]. A set of continuous aggregation operators for IVHFEs are introduced in [30]. Some operations for IVHFEs based on Archimedean t-norms and t-conorms are presented in [4] as well as their properties.

Different correlations coefficient for IVHFEs have been introduced in [60].

In order to calculate the distance between two IVHFEs the Hamming, Euclidean and Hausdorff distances are extended to propose a variety of distance measures for IVHFEs [5, 56].

3.3 Generalized Hesitant Fuzzy Sets

Another extension of HFS is the Generalized Hesitant Fuzzy Set (GHFS) [31] which consists of representing the membership as the union of some A-IFS [1].

Definition 17 [31] Given a set of n membership functions:

$$M = \{\alpha_i = (\mu_i, \upsilon_i) | 0 \leq \mu_i, \upsilon_i \leq 1, 0 \leq \mu_i + \upsilon_i \leq 1, \quad i = \{1, \ldots, n\}\}, \tag{11}$$

the GHFS associated to M, $\mathfrak{h}_M$, is defined as follows:

$$\mathfrak{h}_M(x) = \cup_{(\mu_i(x), \upsilon_i(x)) \in M} (\mu_i(x), \upsilon_i(x)). \tag{12}$$

Remark 1 Notice that a GHFS extends slightly the concept of DHFS [79] as we can see in the following example.

Example 3 Let $X = \{x_1\}$ be a reference set, then

$$\mathfrak{h}_M(x_1) = \{(0.5, 0.3), (0.6, 0.3), (0.4, 0.5)\}$$

is a GHFS.

In this example, $\gamma^+ = 0.6$ and $\eta^+ = 0.5$, therefore $0.6 + 0.5 > 1$, it does not achieve the restriction to be a DHFS.

The complement, union and intersection of GHFSs, as well as, the envelope of a GHFS were presented in [31]. Some properties and relationships with HFSs were also discussed [31]. A comparison law was introduced to compare two GHFSs according to the score and consistency functions defined for this type of information. It was also proposed an extension principle which extends the operations for A-IFSs to GHFSs.

3.4 Hesitant Triangular Fuzzy Sets

Some authors [57, 66, 74] point out that in many decision making problems due to the increasing complexity of the socioeconomic environment and the uncertain information, it is difficult for experts to express the membership degrees of an element to a given set only by means of crisp values. Therefore, the concept of Hesitant Triangular Fuzzy Set (HTFS) was introduced as an extension of HFS where the membership degrees of an element to a fuzzy set are expressed by several triangular fuzzy numbers [18]. This concept has been proposed by different authors [57, 66, 74] with different names. Here it will be used HTFS.

Definition 18 [57, 66, 74] Let X be a fixed set, a HTFS $\tilde{E}$ on X is defined in terms of a function $\tilde{f}_{\tilde{E}}(x)$ that returns several triangular fuzzy values,

$$\tilde{E} = \{\langle x, \tilde{f}_{\tilde{E}}(x)\rangle | x \in X\} \quad (13)$$

where $\tilde{f}_{\tilde{E}}(x)$ is a set of several triangular fuzzy numbers which express the possible membership degrees of an element $x \in X$ to a set $\tilde{E}$. $\tilde{f}_{\tilde{E}}(x)$ is called Hesitant Triangular Fuzzy Element (HTFE) and noted $(\tilde{f})_{\tilde{E}}(x_i) = \{(\tilde{\xi}^L, \tilde{\xi}^M, \tilde{\xi}^U) | \tilde{\xi} \in \tilde{f}_{\tilde{E}}(x_i)\}$.

Example 4 Let $X = \{x_1, x_2\}$ be a reference set, a HTFS $\tilde{E}$, is defined by

$$\tilde{E} = \{\langle x_1, \{(0.1, 0.3, 0.5), (0.4, 0.6, 0.8)\}\rangle, \langle x_2, \{(0.1, 0.2, 0.3)\}\rangle\}.$$

Note that if $\tilde{\xi}^L = \tilde{\xi}^M = \tilde{\xi}^U$, then the HTFS is a HFS.

Some basic operations such as, the addition and multiplication of HTFEs were defined in [66]. A score function and an accuracy function were defined to propose a comparison law for HTFEs [57, 66].

Different aggregation operators for HTFEs such as, Hesitant Triangular Fuzzy Weighted Averaging (HTFWA), Hesitant Triangular Fuzzy Ordered Weighted Averaging (HTFOWA), Hesitant Triangular Fuzzy Weighted Geometric (HTFWG), Hesitant Triangular Fuzzy Ordered Weighted Geometric (HTFOWG), Hesitant Triangular Fuzzy Hybrid Average (HTFHA), Hesitant Triangular Fuzzy Hybrid Geometric (HTFHG) have been defined [57, 66]. A set of aggregation operators based on Bonferroni Mean have been introduced in [47]. The Einstein operation has been extended to propose a family of aggregation operators for HTFEs [38, 74]. Two different aggregation operators based on Choquet integral have been also proposed for HTFEs [21, 75].

This type of information has been applied to solve evaluation problems [21, 66].

4 Extensions of Hesitant Fuzzy Sets in Qualitative Settings

The previous section revises extensions of HFS defined in quantitative contexts, but the use of numbers to represent uncertain information is not always appropriate, and usually it is difficult to provide numerical values when the knowledge is vague and imprecise. Usually, experts involved in this type of problems use linguistic information to express their assessments regarding the uncertain knowledge that they have about the problem [26]. Therefore, different extensions about HFS have been proposed to model the experts' hesitancy in qualitative contexts. This section revises such extensions.

4.1 Hesitant Fuzzy Linguistic Term Sets

Most the linguistic approaches model the information by means of just one linguistic term, but sometimes experts might hesitate among several values to express their assessments because of the lack of information and knowledge about the problem. In order to cope with these hesitant situations Rodríguez et al. proposed the concept of Hesitant Fuzzy Linguistic Term Set (HFLTS) [32].

A HFLTS is defined as follows

Definition 19 [32] Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, a HFLTS H_s, is defined as an ordered finite subset of consecutive linguistic terms of S:

$$H_S = \{s_i, s_{i+1}, \ldots, s_j\} \quad such\ that \quad s_k \in S, \quad k \in \{i, \ldots, j\} \tag{14}$$

Example 5 Let $S = \{s_0 : nothing, s_1 : very\ low, s_2 : low, s_3 : medium, s_4 : high, s_5 : very\ high, s_6 : perfect\}$ be a linguistic term set and ϑ be a linguistic variable, then $H_S(\vartheta)$ defined by

$$H_S(\vartheta) = \{very\ low, low, medium\}$$

is a HFLTS.

Remark 2 The use of consecutive linguistic terms in HFLTS is because of a cognitive point of view in which in a discrete domain with a short number of terms (usually not more than 9) makes not sense to hesitate among arbitrary and total different linguistic terms, $\{low, high, very\ high\}$, and not hesitate in their middle terms. The use of comparative linguistic expressions [33] is a clear example of human beings' hesitancy. The natural representation of such comparative linguistic expressions in decision making is HFLTS.

Some basic operations for HFLTS, such as the complement, union and intersection and diverse properties were defined in [32]. It was also introduced the envelope of a HFLTS that was used to propose a comparison law for HFLTSs. Two symbolic aggregation operators, *min_upper* and *max_lower* were developed to aggregate HFLTSs [32].

The concept of HFLTS was introduced as something that can be used directly by experts to elicit several linguistic terms, but usually human beings do not provide their assessments in such a way. Therefore, Rodríguez et al. proposed the use of context-free grammars to generate linguistic expressions close to the natural language used by human beings that are easily represented by HFLTS. A context-free grammar G_H, that generates comparative linguistic expressions similar to the expressions used by experts in decision making problems was proposed in [33].

Definition 20 [33] Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$$V_N = \{\langle \textit{primary term} \rangle, \langle \textit{composite term} \rangle, \langle \textit{unary relation} \rangle, \langle \textit{binary relation} \rangle, \langle \textit{conjunction} \rangle\}$$

$$V_T = \{\textit{lower than}, \textit{greater than}, \textit{at least}, \textit{at most}, \textit{between}, \textit{and}, s_0, s_1, \ldots, s_g\}$$

$$I \in V_N$$

$$P = \{I ::= \langle \textit{primary term} \rangle | \langle \textit{composite term} \rangle$$

$$\langle \textit{composite term} \rangle ::= \langle \textit{unary relation} \rangle \langle \textit{primary term} \rangle | \langle \textit{binary relation} \rangle \langle \textit{primary term} \rangle \langle \textit{conjunction} \rangle \langle \textit{primary term} \rangle$$

$$\langle \textit{primary term} \rangle ::= s_0 | s_1 | \ldots | s_g$$

$$\langle \textit{unary relation} \rangle ::= \textit{lower than} | \textit{greater than} | \textit{at least} | \textit{at most}$$

$$\langle \textit{binary relation} \rangle ::= \textit{between}$$

$$\langle \textit{conjunction} \rangle ::= \textit{and}\}$$

A transformation function E_{G_H} to obtain HFLTS from the comparative linguistic expressions was defined [32].

Even though the concept of HFLTS is quite novel, it has received a lot of attention by other researchers and different proposals based on this concept have been already presented in the literature. Wei et al. [55] have proposed a comparison method for HFLTS and two aggregation operators, the Hesitant Fuzzy Linguistic Weighted Averaging (HFLWA) and the Hesitant Fuzzy Linguistic Ordered Weighted Averaging (HFLOWA). A different comparison method and more aggregation operators were introduced in [19]. Some authors have studied consistency measures for Hesitant Fuzzy Linguistic Preference Relation (HFLPR) [73, 77]. A variety of distance and similarity measures for HFLTS has been also defined [14, 15, 22] and applied to multicriteria decision making problems.

Liu and Rodríguez pointed out [25] that the semantics of the comparative linguistic expressions based on a context-free grammar and HFLTSs should be represented by fuzzy membership functions instead of linguistic intervals [32]. Therefore, a new fuzzy representation for comparative linguistic expressions based on a fuzzy envelope has been introduced [25]. By using a consensus measure for

HFLTS, an optimization-based consensus model that minimizes the number of adjusted single terms in the consensus process has been recently proposed in [10].

Furthermore, different decision making approaches dealing with HFLTS have been proposed, such as TOPSIS [3], outranking [51], TODIM [54] and so on [6, 33]. And some real applications have been already presented, Sahu et al. [37] used HFLTS to classify documents and Yavuz et al. [64] presented a hierarchical multi-criteria decision making approach using HFLTS to manage complex problems, such as alternative-fuel vehicle selection.

4.2 Extended Hesitant Fuzzy Linguistic Term Sets

Recently, the concept of HFLTS has been generalized to deal with non-consecutive linguistic terms. This generalization has been presented in [48, 72] with different names, Extended Hesitant Fuzzy Linguistic Term Set (EHFLTS) and Hesitant Fuzzy Linguistic Set (HFLS) respectively. Although the names are different its definition is the same. Here we will use EHFLTS to refer this extension.

An EHFLTS is built by the union of several HFLTS. It is formally defined as follows.

Definition 21 [48] Let S be a linguistic term set, an ordered subset of linguistic terms of S, that is,

$$EH_S = \{s_i | s_i \in S\}, \tag{15}$$

is an EHFLTS.

Example 6 Let $S = \{s_{-3} : \textit{very poor}, s_{-2} : \textit{poor}, s_{-1} : \textit{slightly poor}, s_0 : \textit{fair}, s_1 : \textit{slightly good}, s_2 : \textit{good}, s_3 : \textit{very good}\}$ be a linguistic term set and ϑ be a linguistic variable, then $EH_S(\vartheta)$ defined by

$$EH_S(\vartheta) = \{\textit{fair}, \textit{good}, \textit{very good}\}$$

is an EHFLTS.

Some basic operations, such as the union and intersection of EHFLTSs, the complement of an EHFLTS and its envelope have been defined in [48]. Two families of aggregation operators to aggregate a set of EHFLTSs where weighting vectors take the form of real numbers and linguistic terms are also proposed [48]. Wang and Xu have studied the additive and weak consistency of a Extended Hesitant Fuzzy Linguistic Preference Relation (EHFLPR) [49].

Remark 3 It is worthy to note that the concept of EHFLTS is not used directly by experts, but experts involved in a decision making problem provide their preferences by using HFLTS and instead of carrying out an aggregation process, the group's preferences are formed by the union of such HFLTSs obtaining as result a EHFLTS.

4.3 Other Linguistic Extensions

Recently, different linguistic extensions of HFS, DHFS and IVHFS have been introduced. Although they are not well known, their concepts and an example to understand them easily are presented.

In [24] was presented the concept of Hesitant Fuzzy Linguistic Set (HFLS) as follows.

Definiton 22 Let X be a reference set, a HFLS on X is a function that returns a subset of values in [0,1]. It is expressed by a mathematical symbol as follows:

$$A = (\langle x, s_{\theta(x)}, h_A(x)\rangle | x \in X) \tag{16}$$

where $h_A(x)$ is a set of some values in [0,1] denoting the possible membership degrees of the element $x \in X$ to the linguistic term $s_{\theta(x)}$.

Example 7 Let $X = \{x_1, x_2\}$ be a reference set and $S = \{s_0 : nothing, s_1 : very\ low, s_2 : low, s_3 : medium, s_4 : high, s_5 : very\ high, s_6 : perfect\}$ be a linguistic term set, then A defined by

$$A = \{\langle x_1, s_1, \{0.3, 0.4, 0.5\}\rangle, \langle x_2, s_3, \{0.3, 0.5\}\rangle\}$$

is a HFLS.

The concept of Interval-Valued Hesitant Fuzzy Linguistic Set (IVHFLS) has been proposed as an extension of IVHFS based on linguistic term sets.

Definition 23 [52] Let X be a reference set, a IVHFLS on X is an object:

$$B = (\langle x, s_{\theta(x)}, \Gamma_B(x)\rangle | x \in X) \tag{17}$$

where $\Gamma_B(x)$ is a set of finite numbers of closed intervals belonging to (0,1] and it denotes the possible interval-valued membership degrees that x belongs to $s_{\theta(x)}$.

Example 8 Let $X = \{x_1, x_2\}$ be a reference set and $S = \{s_0 : nothing, s_1 : very\ low, s_2 : low, s_3 : medium, s_4 : high, s_5 : very\ high, s_6 : perfect\}$ be a linguistic term set, a IVHFLS might be

$$B = \{\langle x_1, s_5, \{[0.4, 0.5], [0.6, 0.7], [0.7, 0.8]\}\rangle, \langle x_2, s_6, \{[0.1, 0.3], [0.5, 0.6]\}\rangle\}.$$

Yang and Ju have extended also the concept DHFS by using linguistic terms.

Definition 24 [63] Let X be a reference set, a DHFLS on X is described as:

$$C = (\langle x, s_{\theta(x)}, h(x), g(x)\rangle | x \in X) \tag{18}$$

where $s_{\theta(x)} \in S$, $h(x)$ and $g(x)$ are two sets of some values in [0,1] denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the linguistic term $s_{\theta(x)}$ with the following conditions:

$$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1,$$

where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \max_{\gamma \in h(x)}\{\gamma\}$, and $\eta^+ = \max_{\eta \in g(x)}\{\eta\} \forall x \in X$.

Example 9 Let $X = \{x_1, x_2\}$ be a reference set and $S = \{s_0 : \textit{nothing}, s_1 : \textit{very low}, s_2 : \textit{low}, s_3 : \textit{medium}, s_4 : \textit{high}, s_5 : \textit{very high}, s_6 : \textit{perfect}\}$ be a linguistic term set, a DHFLS might be

$$C = \{\langle x_1, s_3, \{0.4, 0.5\}, \{0.3, 0.4\}\rangle, \langle x_2, s_4, \{0.3, 0.5\}, \{0.2, 0.3\}\}\rangle\}.$$

5 Trends and Discussions

Due to the usefulness of modelling hesitancy uncertainty in real-world problems, the research of HFS and its extensions have been intensive and extensive researched in the recent years despite the young of this concept. Because of this interest many proposals related with its use and extensions have been developed in the literature. However, some critical points and comments about the new concepts and tools based on HFS must be pointed out in order to clarify better the trends and future direction on this topic:

- Any HFS extension should be clearly justified from a theoretical or practical point of view and solve real-problems with uncertainty. So far, the usefulness of some extensions of HFS is debatable because it is not clear either why are necessary? or in what type of real-world problems can be used?
- It is remarkable that the same concepts have been published by different authors in different papers. Despite the quick growth of research on this topic, it is necessary to make a deep revision of the related literature in order to avoid repeating several times the same concepts, operators and so forth.
- As it has been argued in Sect. 2, some operators for HFS defined in a straightforward way from operators on fuzzy sets inherit their properties. The study of the properties have to take into account this fact. In addition, it is possible to define new operators that are not just extensions of operators on fuzzy sets. This line has not yet been much explored.

6 Conclusions

The introduction of HFS by Torra (see Sect. 2) has attracted the attention of many researchers that have found a new way to model the uncertainty related to the hesitation of human beings in real-world problems when they are not sure about their knowledge.

This chapter reviews the ground concepts and ideas of HFS, its basic operations and the most important extensions that have been presented in the literature about it both quantitative and qualitative way. After such a revision, it points out some important aspects that must be taken into account when new operators and extensions about HFS are developed in order to avoid some disfunctions that have been found in some of the current proposals. Eventually some hints about future directions in HFS research are sketched.

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