The Emergence of Fuzzy Sets: A Historical Perspective

Didier Dubois and Henri Prade

Abstract This paper tries to suggest some reasons why fuzzy set theory came to life 50 years ago by pointing out the existence of streams of thought in the first half of the XXth century in logic, linguistics and philosophy, that paved the way to the idea of moving away from the Boolean framework, through the proposal of many-valued logics and the study of the vagueness phenomenon in natural languages. The founding paper in fuzzy set theory can be viewed as the crystallization of such ideas inside the engineering arena. Then we stress the point that this publication in 1965 was followed by several other seminal papers in the subsequent 15 years, regarding classification, ordering and similarity, systems science, decision-making, uncertainty management and approximate reasoning. The continued effort by Zadeh to apply fuzzy sets to the basic notions of a number of disciplines in computer and information sciences proved crucial in the diffusion of this concept from mathematical sciences to industrial applications.

Keywords Fuzzy sets · Many-valued logics · Vagueness · Possibility theory · Approximate reasoning

1 Introduction

The notion of a fuzzy set stems from the observation made by Zadeh [60] fifty years ago in his seminal paper that

> more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership.

By "precisely defined", Zadeh means all-or-nothing, thus emphasizing the continuous nature of many categories used in natural language. This observation emphasizes the gap existing between mental representations of reality and usual

D. Dubois (✉) · H. Prade
IRIT-CNRS, Université Paul Sabatier, 31062 Toulouse Cedex 09, France
e-mail: Didier.Dubois@irit.fr

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mathematical representations thereof, which are traditionally based on binary logic, precise numbers, differential equations and the like. Classes of objects referred to in Zadeh's quotation exist only through such mental representations, e.g., through natural language terms such as *high* temperature, *young* man, *big* size, etc., and also with nouns such as *bird, chair*, etc. Classical logic is too rigid to account for such categories where it appears that membership is a gradual notion rather than an all-or-nothing matter.

The ambition of representing human knowledge in a human-friendly, yet rigorous way might have appeared like a futile exercise not worth spending time on, and even ridiculous from a scientific standpoint, only one hundred years ago. However in the meantime the emergence of computers has significantly affected the landscape of science, and we have now entered the era of information management. The development of sound theories and efficient technology for knowledge representation and automated reasoning has become a major challenge, now that many people possess computers and communicate with them in order to find information that helps them when making decisions. An important issue is to store and exploit human knowledge in various domains where objective and precise data are seldom available. Fuzzy set theory participates to this trend, and, as such, has close connection with Artificial Intelligence. This chapter is meant to account for the history of how the notion of fuzzy set could come to light, and what are the main landmark papers by its founder that stand as noticeable steps towards the construction of the fuzzy set approach to classification, decision, human knowledge representation and uncertainty. Besides, the reader is invited to consult a recent personal account, written by Zadeh [85], of the circumstances in which the founding paper on fuzzy sets was written.

2 A Prehistory of Fuzzy Sets

This section gives some hints to works what can be considered as forerunners of fuzzy sets. Some aspects of the early developments are described in more details by Gottwald [28] and Ostasiewicz [45, 46]. This section freely borrows from [17], previously written with the later author.

2.1 Graded Membership to Sets Before Zadeh

In spite of the considerable interest for multiple-valued logics raised in the early 1900s by Jan Łukasiewicz and his school who developed logics with intermediary truth value(s), it was the American philosopher Black [7] who first proposed so-called "consistency profiles" (the ancestors of fuzzy membership functions) in order to "characterize vague symbols."

As early as in 1946, the philosopher Abraham Kaplan argued in favor of the usefulness of the classical calculus of sets for practical applications. The essential novelty he introduces with respect to the Boolean calculus consists in entities which have a degree of vagueness characteristic of actual (empirical) classes (see Kaplan [33]). The generalization of the traditional characteristic function has been first considered by Weyl [55] in the same year; he explicitly replaces it by a continuous characteristic function. They both suggested calculi for generalized characteristic functions of vague predicates, and the basic fuzzy set connectives already appeared in these works.

Such calculus has been presented by Kaplan and Schott [34] in more detail, and has been called the calculus of empirical classes (CEC). Instead of notion of "property", Kaplan and Schott prefer to use the term "profile" defined as a type of quality. This means that a profile could refer to a simple property like *red, green*, etc. or to a complex property like *red and 20 cm long, green and 2 years old*, etc. They have replaced the classical characteristic function by an indicator which takes on values in the unit interval. These values are called the weight from a given profile to a specified class. In the work of Kaplan and Schott, the notion of "empirical class" corresponds to the actual notion of "fuzzy set", and a value in the range of the generalized characteristic function (indicator, in their terminology) is already called by Kaplan and Schott a "degree of membership" (Zadehian grade of membership). Indicators of profiles are now called membership functions of fuzzy sets. Strangely enough it is the mathematician of probabilistic metric spaces, Karl Menger, who, in 1951, was the first to use the term "ensemble flou" (the French counterpart of "fuzzy set") in the title of a paper [40] of his.

2.2 Many-Valued Logics

The Polish logician Jan Łukasiewicz (1878–1956) is considered as the main founder of multi-valued logic. This is an important point as multi-valued logic is to fuzzy set theory what classical logic is to set theory. The new system he proposed has been published for the first time in Polish in 1920. However, the meaning of truth-values other than "true" and "false" remained rather unclear until Zadeh introduced fuzzy sets. For instance, Łukasiewicz [38] interpreted the third truth-value of his 3-valued logic as "possible", which refers to a modality rather than a truth-value. Kleene [35] suggests that the third truth-value means "unknown" or "undefined". See Ciucci and Dubois [9] for a overview of such epistemic interpretations of three-valued logics. On the contrary, Zadeh [60] considered intermediate truth-degrees of fuzzy propositions as ontic, that is, being part of the definition of a gradual predicate. Zadeh observes that the case where the unit interval is used as a membership scale "corresponds to a multivalued logic with a continuum of truth values in the interval [0, 1]", acknowledging the link between fuzzy sets and many-valued logics. Clearly, for Zadeh, such degrees of truth do not refer to any kind of uncertainty, contrary to what is often found in more recent texts about fuzzy sets by various authors. Later on, Zadeh [70] would not consider fuzzy logic to be another name for many-valued logic. He soon considered that fuzzy truth-values should be considered as fuzzy sets of the unit interval, and that fuzzy logic should be viewed as a theory of approximate reasoning whereby fuzzy truth-values act as modifiers of the fuzzy statement they apply to.

2.3 The Issue of Vagueness

More than one hundred years ago, the American philosopher Peirce [47] was one of the first scholars in the modern age to point out, and to regret, that

> Logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought.

Some time later, Russell pointed out that the law of excluded middle cannot be applied to vague predicates [49]. Even Wittgenstein [57] pointed out that concepts in natural language do not possess a clear collection of properties defining them, but have extendable boundaries, and that there are central and less central members in a category.

The claim that fuzzy sets are a basic tool for addressing vagueness of linguistic terms has been around for a long time. For instance, Novák [44] insists that fuzzy logic is tailored for vagueness and he opposes vagueness to uncertainty.

Nevertheless, in the last thirty years, the literature dealing with vagueness has grown significantly, and much of it is far from agreeing on the central role played by fuzzy sets in this phenomenon. Following Keefe and Smith [53], vague concepts in natural language display at least one among three features:

- **The existence of borderline cases**: That is, there are some objects such that neither a concept nor its negation can be applied to them. For a borderline object, it is difficult to make a firm decision as to the truth or the falsity of a proposition containing a vague predicate applied to this object, even if a precise description of the latter is available. The existence of borderline cases is sometimes seen as a violation of the law of excluded middle.
- **Unsharp boundaries**: The extent to which a vague concept applies to an object is supposed to be a matter of degree, not an all-or-nothing decision. It is relevant for predicates referring to continuous scales, like *tall, old*, etc. This idea can be viewed as a specialization of the former, if we regard as borderline cases objects for which a proposition is neither totally true nor totally false. In the following we shall speak of "gradualness" to describe such a feature. Using degrees of appropriateness of concepts to objects as truth degrees of statements involving these concepts goes against the Boolean tradition of classical logic.
- **Susceptibility to Sorites paradoxes**. This is the idea that the presence of vague propositions make long inference chains inappropriate, yielding debatable results. The well-known examples deal with heaps of sand (whereby, since adding a grain of sand to a small heap keeps its small, all heaps of sand should

be considered small), young persons getting older by one day, bald persons that are added one hair, etc.

Since their inception, fuzzy sets have been controversial for philosophers, many of whom are reluctant to consider the possibility of non-Boolean predicates, as it questions the usual view of truth as an absolute entity. A disagreement opposes those who, like Williamson, claim a vague predicate has a standard, though ill-known, extension [56], to those who, like Kit Fine, deny the existence of a decision threshold and just speak of a truth value gap [24]. However, the two latter views reject the concept of gradual truth, and concur on the point that fuzzy sets do not propose a good model for vague predicates. One of the reasons for the misunderstanding between fuzzy sets and the philosophy of vagueness may lie in the fact that Zadeh was trained in engineering mathematics, not in the area of philosophy. In particular, vagueness is often understood as a defect of natural language (since it is not appropriate for devising formal proofs, it questions usual rational forms of reasoning). Actually, vagueness of linguistic terms was considered as a logical nightmare for early 20th century philosophers. In contrast, for Zadeh, going from Boolean logic to fuzzy logic is viewed as a positive move: it captures tolerance to errors (softening blunt threshold effects in algorithms) and may account for the flexible use of words by people [73]. It also allows for information summarization: detailed descriptions are sometimes hard to make sense of, while summaries, even if imprecise, are easier to grasp [69].

However, the epistemological situation of fuzzy set theory itself may appear kind of unclear. Fuzzy sets and their extensions have been understood in various ways in the literature: there are several notions that are appealed to in connection with fuzzy sets, like similarity, uncertainty and preference [19]. The concept of similarity to prototypes has been central in the development of fuzzy sets as testified by numerous works on fuzzy clustering. It is also natural to represent incomplete knowledge by fuzzy sets (of possible models of a fuzzy knowledge base, or fuzzy error intervals, for instance), in connection to possibility theory [18, 74]. Utility functions in decision theory also appear as describing fuzzy sets of good options. These topics are not really related to the issue of vagueness.

Indeed, in his works, Zadeh insists that, even when applied to natural language, fuzziness is not vagueness. The term fuzzy is restricted to sets where the transition between membership and non-membership is gradual rather than abrupt, not when it is crisp but unknown. Zadeh [73] argues as follows:

> Although the terms fuzzy and vague are frequently used interchangeably in the literature, there is, in fact, a significant difference between them. Specifically, a proposition, p, is fuzzy if it contains words which are labels of fuzzy sets; and p is vague if it is both fuzzy and insufficiently specific for a particular purpose. For example, "Bob will be back in a few minutes" is fuzzy, while "Bob will be back sometime" is vague if it is insufficiently informative as a basis for a decision. Thus, the vagueness of a proposition is a decision-dependent characteristic whereas its fuzziness is not.

Of course, the distinction made by Zadeh may not be so strict as he claims. While "in a few minutes" is more specific than "sometime" and sounds less vague, one may argue that there is some residual vagueness in the former, and that the latter does not sound very crisp after all. Actually, one may argue that the notion of non-Boolean linguistic categories proposed by Zadeh from 1965 on is capturing the idea of gradualness, not vagueness in its philosophical understanding. Zadeh repetitively claims that gradualness is pervasive in the representation of information, especially human-originated.

The connection from gradualness to vagueness does exist in the sense that, insofar as vagueness refers to uncertainty about meaning of natural language categories, gradual predicates tend to be more often vague than Boolean ones: indeed, it is more difficult to precisely measure the membership function of a fuzzy set representing a gradual category than to define the characteristic function of a set representing the extension of a Boolean predicate [13]. In fact, the power of expressiveness of real numbers is far beyond the limited level of precision perceived by the human mind. Humans basically handle meaningful summaries. Analytical representations of physical phenomena can be faithful as models of reality, but remain esoteric to lay people; the same may hold for real-valued membership grades. Indeed, mental representations are tainted with vagueness, which encompasses at the same time the lack of specificity of linguistic terms, and the lack of well-defined boundaries of the class of objects they refer to, as much as the lack of precision of membership grades. So moving from binary membership to continuous is a bold step, and real-valued membership grades often used in fuzzy sets are just another kind of idealization of human perception, that leaves vagueness aside.

3 The Development of Fuzzy Sets and Systems

Having discussed the various streams of ideas that led to the invention of fuzzy sets, we now outline the basic building blocks of fuzzy set theory, as they emerged from 1965 all the way to the early 1980s, under the impulse of the founding father, via several landmark papers, with no pretense to exhaustiveness. Before discussing the landmark papers that founded the field, it is of interest to briefly summarize how L. A. Zadeh apparently came to the idea of developing fuzzy sets and more generally fuzzy logic. See also [50] for historical details. First, it is worth mentioning that, already in 1950, after commenting the first steps towards building thinking machines (a recently hot topic at the time), he indicated in his conclusion [58]:

> Through their association with mathematicians, the electronic engineers working on thinking machines have become familiar with such hitherto remote subjects as Boolean algebra, multivalued logic, and so forth.

which shows an early concern for logic and many-valued calculi. Twelve years later, when providing "a brief survey of the evolution of system theory [59] he wrote (p. 857)

> There are some who feel that this gap reflects the fundamental inadequacy of the conventional mathematics - the mathematics of precisely-defined points, functions, sets,

> probability measures, etc. - for coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitude more complex than man-made systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions.

This quotation shows that Zadeh was first motivated by an attempt at dealing with complex systems rather than with man-made systems, in relation with the current trends of interest in neuro-cybernetics in that time (in that respect, he pursued the idea of applying fuzzy sets to biological systems at least until 1969 [64]).

3.1 Fuzzy Sets: The Founding Paper and Its Motivations

The introduction of the notion of a fuzzy set by Zadeh [60] was motivated by the fact that, quoting the founding paper:

> imprecisely defined “classes” play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.

This seems to have been a recurring concern in all of Zadeh’s fuzzy set papers since the beginning, as well as the need to develop a sound mathematical framework for handling this kind of “classes”. This purpose required an effort to go beyond classical binary-valued logic, the usual setting for classes. Although many-valued logics had been around for a while, what is really remarkable is that due to this concern, Zadeh started to think in terms of *sets* rather than only in terms of degrees of truth, in accordance with intuitions formalized by Kaplan but not pursued further. Since a set is a very basic notion, it was opening the road to the introduction of the fuzzification of any set-based notions such as relations, events, or intervals, while sticking to the many-valued logic point of view only does not lead you to consider such generalized notions. In other words, while Boolean algebras are underlying both propositional logic and naive set theory, the set point of view may be found richer in terms of mathematical modeling, and the same thing takes place when moving from many-valued logics to fuzzy sets.

A fuzzy set can be understood as a class equipped with an ordering of elements expressing that some objects are more inside the class than others. However, in order to extend the Boolean connectives, we need more than a mere relation in order to extend intersection, union and complement of sets, let alone implication. The set of possible membership grades has to be a complete lattice [27] so as to capture union and intersection, and either the concept of residuation or an order-reversing function in order to express some kind of negation and implication. The study of set operations on fuzzy sets has in return strongly contributed to a renewal of many-valued logics under the impulse of Hájek [29] (see [14] for an introductory overview).

Besides, from the beginning, it was made clear that fuzzy sets were not meant as probabilities in disguise, since one can read [60] that

> the notion of a fuzzy set is completely non-statistical in nature

and that it provides

> a natural way of dealing with problems where the source of imprecision is the absence of sharply defined criteria of membership rather than the presence of random variables.

Presented as such, fuzzy sets are prima facie not related to the notion of uncertainty. The point that typicality notions underlie the use of gradual membership functions of linguistic terms is more connected to similarity than to uncertainty. As a consequence,

- originally, fuzzy sets were designed to formalize the idea of soft classification, which is more in agreement with the way people use categories in natural language.
- fuzziness is just implementing the concept of gradation in all forms of reasoning and problem-solving, as for Zadeh, everything is a matter of degree.
- a degree of membership is an abstract notion to be interpreted in practice.

Important definitions appear in the founding paper such as cuts (a fuzzy set can be viewed as a family of nested crisp sets called its cuts), the basic fuzzy set-theoretic connectives (e.g. minimum and product as candidates for intersection, inclusion via inequality of membership functions) and the extension principle whereby the domain of a function is extended to fuzzy set-valued arguments. As pointed out earlier, according to the area of application, several interpretations can be found such as degree of similarity (to a prototype in a class), degree of plausibility, or degree of preference [19]. However, in the founding paper, membership functions are considered in connection with the representation of human categories only. The three kinds of interpretation would become patent in subsequent papers.

3.2 Fuzzy Sets and Classification

A popular part of the fuzzy set literature deals with fuzzy clustering where gradual transitions between classes and their use in interpolation are the basic contribution of fuzzy sets. The idea that fuzzy sets would be instrumental to avoid too rough classifications was provided very early by Zadeh, along with Bellman and Kalaba [3]. They outline how to construct membership functions of classes from examples thereof. Intuitively speaking, a cluster gathers elements that are rather close to each other (or close to some core element(s)), while they are well-separated from the elements in the other cluster(s). Thus, the notions of graded proximity, similarity (dissimilarity) are at work in fuzzy clustering. With gradual clusters, the key issue is to define fuzzy partitions. The most widely used definition of a fuzzy partition originally due to Ruspini [48], where the sum of membership grades of one element to the various classes is 1. This was enough to trigger the fuzzy clustering literature, that culminated with the numerous works by Bezdek and colleagues, with applications to image processing for instance [5].

3.3 Fuzzy Events

The idea of replacing sets by fuzzy sets was quickly applied by Zadeh to the notion of event in probability theory [63]. The probability of a fuzzy event is just the expectation of its membership function. Beyond the mathematical exercise, this definition has the merit of showing the complementarity between fuzzy set theory and probability theory: while the latter models uncertainty of events, the former modifies the notion of an event admitting it can occur to some degree when observing a precise outcome. This membership degree is ontic as it is part of the definition of the event, as opposed to probability that has an epistemic flavor, a point made very early by De Finetti [12], commenting Łukasiewicz logic. However, it took 35 years before a generalization of De Finetti's theory of subjective probability to fuzzy events was devised by Mundici [42]. Since then, there is an active mathematical area studying probability theory on algebras of fuzzy events.

3.4 Decision-Making with Fuzzy Sets

Fuzzy sets can be useful in decision sciences. This is not surprising since decision analysis is a field where human-originated information is pervasive. While the suggestion of modelling fuzzy optimization as the (product-based) aggregation of an objective function with a fuzzy constraint first appeared in the last section of [61], the full-fledged seminal paper in this area was written by Bellman and Zadeh [4] in 1970, highlighting the role of fuzzy set connectives in criteria aggregation. That pioneering paper makes three main points:

1. Membership functions can be viewed as a variant of utility functions or rescaled objective functions, and optimized as such.
2. Combining membership functions, especially using the minimum, can be one approach to criteria aggregation.
3. Multiple-stage decision-making problems based on the minimum aggregation connective can then be stated and solved by means of dynamic programming.

This view was taken over by Tanaka et al. [54] and Zimmermann [86] who developed popular multicriteria linear optimisation techniques in the seventies. The idea is that constraints are soft. Their satisfaction is thus a matter of degree. They can thus be viewed as criteria. The use of the minimum operation instead of the sum for aggregating partial degrees of satisfaction preserves the semantics of constraints since it enforces all of them to be satisfied to some degree. Then any multi-objective linear programming problem becomes a max-min fuzzy linear programming problem.

3.5 Fuzzy Relations

Relations being subsets of a Cartesian product of sets, it was natural to make them fuzzy. There are two landmark papers by Zadeh on this topic in the early 1970s. One published in 1971 [65] makes the notions of equivalence and ordering fuzzy. A similarity relation extends the notion of equivalence, preserving the properties of reflexivity and symmetry and turning transitivity into maxmin transitivity. Similarity relations come close to the notion of ultrametrics and correspond to nested equivalence relations. As to fuzzy counterparts of order relations, Zadeh introduces first definitions of what will later be mathematical models of a form of fuzzy preference relations [25]. In his attempt the difficult part is the extension of antisymmetry that will be shown to be problematic. The notion of fuzzy preorder, involving a similarity relation turns out to be more natural than the one of a fuzzy order [6].

The other paper on fuzzy relations dates back to 1975 [71] and consists in a fuzzy generalization of the relational algebra. This seminal paper paved the way to the application of fuzzy sets to databases (see [8] for a survey), and to flexible constraint satisfaction in artificial intelligence [16, 41].

3.6 Fuzzy Systems

The application of fuzzy sets to systems was not obvious at all, as traditionally systems were described by numerical equations. Zadeh seems to have tried several solutions to come up with a notion of fuzzy system. Fuzzy systems [61] were initially viewed as systems whose state equations involve fuzzy variables or parameters, giving birth to fuzzy classes of systems. Another idea, hinted in 1965 [61], was that a system is fuzzy if either its input, its output or its states would range over a family of fuzzy sets. Later in 1971 [67], he suggested that a fuzzy system could be a generalisation of a non-deterministic system, that moves from a state to a fuzzy set of states. So the transition function is a fuzzy mapping, and the transition equation can be captured by means of fuzzy relations, and the sup-min combination of fuzzy sets and fuzzy relations. These early attempts were outlined before the emergence of the idea of fuzzy control [68]. In 1973, though, there was what looks like a significant move, since for the first time it was suggested that fuzziness lies in the description of approximate rules to make the system work. That view, developed first in [69], was the result of a convergence between the idea of system with the ones of fuzzy algorithms introduced earlier [62], and his increased focus of attention on the representation of natural language statements via linguistic variables. In this very seminal paper, systems of fuzzy if-then rules were first described, which paved the way to fuzzy controllers, built from human information, with the tremendous success met by such line of research in the early 1980s. To-day, fuzzy rule-based systems are extracted from data and serve as models of systems more than as controllers. However the linguistic connection is often lost, and such fuzzy systems are rather standard precise systems using membership function for interpolation, than approaches to the handling of poor knowledge in system descriptions.

3.7 Linguistic Variables and Natural Language Issues

When introducing fuzzy sets, Zadeh seems to have been chiefly motivated by the representation of human information in natural language. This focus explains why many of Zadeh's papers concern fuzzy languages and linguistic variables. He tried to combine results on formal languages and the idea that the term sets that contain the atoms from which this language was built contain fuzzy sets representing the meaning of elementary concepts [66]. This is the topic where the most numerous and extended papers by Zadeh can be found, especially the large treatise devoted to linguistic variables in 1975 [72] and the papers on the PRUF language [73, 78]. Basically, these papers led to the "computing with words" paradigm, which takes the opposite view of say, logic-based artificial intelligence, by putting the main emphasis, including the calculation method, on the semantics rather than the syntax. Starting with natural language sentences, fuzzy words are precisely modelled by fuzzy sets, and inference comes down to some form of non-linear optimisation. In a later step, numerical results are translated into verbal terms, using the so-called linguistic approximation. While this way of reasoning seems to have been at the core of Zadeh's approach, it is clear that most applications of fuzzy sets only use terms sets and linguistic variables in rather elementary ways. For instance, quite a number of authors define linguistic terms in the form of trapezoidal fuzzy sets on an abstract universe which is not measurable and where addition makes no sense, nor linear membership functions. In [72], Zadeh makes it clear that linguistic variables refer to objective measurable scales: only the linguistic term has a subjective meaning, while the universe of discourse contains a measurable quantity like height, age, etc.

3.8 Fuzzy Intervals

In his longest 3-part paper in 1975 [72], Zadeh points out that a trapezoidal fuzzy set of the real line can model imprecise quantities. These trapezoidal fuzzy sets, called fuzzy numbers, generalize intervals on the real line and appear as the mathematical rendering of fuzzy terms that are the values of linguistic variables on numerical scales. The systematic use of the extension principle to such fuzzy numbers and the idea of applying it to basic operations of arithmetics is a key-idea that will also turn out to be seminal. Since then, numerous papers have developed methods for the calculation with fuzzy numbers. It has been shown that the extension principle enable a generalization of interval calculations, hence opening a whole area of fuzzy sensitivity analysis that can cope with incomplete information in a gradual way. The calculus of fuzzy intervals is instrumental in various areas including:

- systems of linear equations with fuzzy coefficients (a critical survey is [36]) and differential equations with fuzzy initial values, and fuzzy set functions [37];
- fuzzy random variables for the handling of linguistic or imprecise statistical data [10, 26];
- fuzzy regression methods [43];
- operations research and optimisation under uncertainty [15, 31, 32].

3.9 Fuzzy Sets and Uncertainty: Possibility Theory

Fuzzy sets can represent uncertainty not because they are fuzzy but because crisp sets are already often used to represent ill-known values or situations, albeit in a crisp way like in interval analysis or in propositional logic. Viewed as representing uncertainty, a set just distinguishes between values that are considered possible and values that are impossible, and fuzzy sets just introduce grades to soften boundaries of an uncertainty set. So in a fuzzy set it is the set that captures uncertainty [20, 21]. This point of view echoes an important distinction made by Zadeh himself [73, 83] between

- conjunctive (fuzzy) sets, where the set is viewed as the conjunction of its elements. This is the case for clusters discussed in the previous section. But also with set-valued attributes like the languages spoken more or less fluently by an individual.
- and disjunctive fuzzy sets which corresponds to mutually exclusive possible values of an ill-known single-valued attribute, like the ill-known birth nationality of an individual.

In the latter case, membership functions of fuzzy sets are called possibility distributions [74] and act as elastic constraints on a precise value. Possibility distributions have an epistemic flavor, since they represent the information we have at our disposal about what values remain more or less possible for the variable under consideration, and what values are (already) known as impossible. Associated with a possibility distribution, is a possibility measure [74], which is a max-decomposable set function. Thus, one can evaluate the possibility of a crisp, or fuzzy, statement of interest, given the available information supposed to be represented by a possibility distribution. It is also important to notice that the introduction of possibility theory by Zadeh was part of the modeling of fuzzy information expressed in natural language [77]. This view contrasts with the motivations of the English economist Shackle [51, 52], interested in a non-probabilistic view of expectation, who had already designed a formally similar theory in the 1940s, but rather based on the idea of degree of impossibility understood as a degree of surprise (using profiles of the form $1\text{-}\mu$, where μ is a membership function). Shackle can also appear as a forerunner of fuzzy sets, of possibility theory, actually.

Curiously, apart from a brief mention in [76], Zadeh does not explicitly use the notion of necessity (the natural dual of the modal notion of possibility) in his work on possibility theory and approximate reasoning. Still, it is important to distinguish between statements that are *necessarily* true (to some extent), i.e. whose negation is almost impossible, from the statements that are only *possibly* true (to some extent) depending on the way the fuzzy knowledge would be made precise. The simultaneous use of the two notions is often required in applications of possibility theory [23] and in possibilistic logic [22].

3.10 Approximate Reasoning

The first illustration of the power of possibility theory proposed by Zadeh was an original theory of approximate reasoning [75, 79, 80], later reworked for emphasizing new points [82, 84], where pieces of knowledge are represented by possibility distributions that fuzzily restrict the possible values of variables, or tuples of variables. These possibility distributions are combined conjunctively, and then projected in order to compute the fuzzy restrictions acting on the variables of interest. This view is the one at work in his calculus of fuzzy relations in 1975. One research direction, quite in the spirit of the objective of "computing with words" [82], would be to further explore the possibility of a syntactic (or symbolic) computation of the inference step (at least for some noticeable fragments of this general approximate reasoning theory), where the obtained results are parameterized by fuzzy set membership functions that would be used only for the final interpretation of the results. An illustration of this idea is at work in possibilistic logic [22], a very elementary formalism handling pairs made of a Boolean formula and a certainty weight, that captures a tractable form of non-monotonic reasoning. Such pairs syntactically encode simple possibility distributions, combined and reasoned from in agreement with Zadeh's theory of approximate reasoning, but more in the tradition of symbolic artificial intelligence than in conformity with the semantic-based methodology of computing with words.

4 Conclusion

The seminal paper on fuzzy sets by Lotfi Zadeh spewed out a large literature, despite its obvious marginality at the time it appeared. There exist many forgotten original papers without off-springs. Why has Zadeh's paper encountered an eventual dramatic success? One reason is certainly that Zadeh, at the time when the fuzzy set paper was released, was already a renowned scientist in systems engineering. So, his paper, published in a good journal, was visible. However another reason for success lies in the tremendous efforts made by Zadeh in the seventies and the eighties to develop his intuitions in various directions covering many topics from theoretical computer sciences to computational linguistics, from system sciences to decision sciences.

It is not clear that the major successes of fuzzy set theory in applications fully meet the expectations of its founder. Especially there was almost no enduring impact of fuzzy sets on natural language processing and computational linguistics, despite the original motivation and continued effort about the computing with words paradigm. The contribution of fuzzy sets and fuzzy logic was to be found elsewhere, in systems engineering, data analysis, multifactorial evaluation, uncertainty modeling, operations research and optimisation, and even mathematics. The notion of fuzzy rule-based system (Takagi-Sugeno form, not Mamdani's, nor even the view developed in [81]) has now been integrated in both the neural net literature and the non-linear control one. These fields, just like in clustering, only use the notion of fuzzy partition and possibly interpolation between subclasses, and bear almost no connection to the issue of fuzzy modeling of natural language. These fields have come of age, and almost no progress can be observed that concern their fuzzy set ingredients. In optimisation, the fuzzy linear programming method is now used in applications, with only minor variants (if we set apart the handling of uncertainty proper, via fuzzy intervals).

However there are some topics where basic research seems to still be active, with high potential. See [11] for a collection of position papers highlighting various perspectives for the future of fuzzy sets. Let us cite two such topics. The notion of fuzzy interval or fuzzy number, introduced in 1975 by Zadeh [72], and considered with the possibility theory lenses, seems to be more promising in terms of further developments because of its connection with non-Bayesian statistics and imprecise probability [39] and its potential for handling uncertainty in risk analysis [1]. Likewise, the study of fuzzy set connectives initiated by Zadeh in 1965, has given rise to a large literature, and significant developments bridging the gap between many-valued logics and multicriteria evaluation, with promising applications (for instance [2]).

Last but not least, we can emphasize the influence of fuzzy sets on some even more mathematically oriented areas, like the strong impact of fuzzy logic on many-valued logics (triggered by Hajek [29]) and topological and categorical studies of lattice-valued sets [30]. There are very few papers in the literature that could influence such various areas of scientific investigation to that extent.

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