Metric Characterizations of Some Classes of Banach Spaces

Mikhail Ostrovskii

Dedicated to the Memory of Cora Sadosky.

Abstract The main purpose of the paper is to present some recent results on metric characterizations of superreflexivity and the Radon–Nikodým property.

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Introduction

By a *metric characterization* of a class of Banach spaces in the most general sense we mean a characterization which refers only to the metric structure of a Banach space and does not involve the linear structure. Some origins of the idea of a metric characterization can be seen in the classical theorem of Mazur and Ulam [75]: Two Banach spaces (over reals) are isometric as metric spaces if and only if they are linearly isometric as Banach spaces.

However study of metric characterizations became an active research direction only in mid-1980s, in the work of Bourgain [13] and Bourgain–Milman–Wolfson [16]. This study was motivated by the following result of Ribe [106].

Definition 1.1. Let X and Y be two Banach spaces. The space X is said to be *finitely representable* in Y if for any $\varepsilon > 0$ and any finite-dimensional subspace $F \subset X$ there exists a finite-dimensional subspace $G \subset Y$ such that $d(F, G) < 1 + \varepsilon$, where $d(F, G)$ is the Banach–Mazur distance.

The space X is said to be *crudely finitely representable* in Y if there exists $1 \le C < \infty$ such that for any finite-dimensional subspace $F \subset X$ there exists a finite-dimensional subspace $G \subset Y$ such that $d(F, G) \le C$.

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Theorem 1.2 (Ribe [106]). *Let Z and Y be Banach spaces. If Z and Y are uniformly homeomorphic, then Z and Y are crudely finitely representable in each other.*

Three proofs of this theorem are known at the moment:

- The original proof of Ribe [106]. Some versions of it were presented in Enflo's survey [39] and the book by Benyamini and Lindenstrauss [11, pp. 222–224].
- The proof of Heinrich–Mankiewicz [52] based on ultraproduct techniques, also presented in [11].
- The proof of Bourgain [14] containing related quantitative estimates. This paper is a very difficult reading. The proof has been clarified and simplified by Giladi–Naor–Schechtman [44] (one of the steps was simplified earlier by Begun [8]). The presentation of [44] is easy to understand, but some of the ε-δ ends in it do not meet. I tried to fix this when I presented this result in my book [90, Sect. 9.2] (let me know if you find any problems with ε-δ choices there).

These three proofs develop a wide spectrum of methods of the nonlinear Banach space theory and are well worth studying.

The Ribe theorem implies stability under uniform homeomorphisms of each class $\mathcal{P}$ of Banach spaces satisfying the following condition LHI (local, hereditary, isomorphic): if $X \in \mathcal{P}$ and Y crudely finitely representable in X, then $Y \in \mathcal{P}$.

The following well-known classes have the described property:

- superreflexive spaces (see the definition and related results in section "Metric Characterizations of Superreflexivity" of this paper),
- spaces having cotype q, $q \in [2, \infty)$ (the definitions of type and cotype can be found, for example, in [90, Sect. 2.4]),
- spaces having cotype r for each $r > q$ where $q \in [2, \infty)$,
- spaces having type p, $p \in (1, 2]$,
- spaces having type r for each $r < p$ where $p \in (1, 2]$,
- Banach spaces isomorphic to q-convex spaces $q \in [2, \infty)$ (see Definition 2.7 below, more details can be found, for example, in [90, Sect. 8.4]),
- Banach spaces isomorphic to p-smooth spaces $p \in (1, 2]$ (see Definition 1.6 and [90, Sect. 8.4]),
- UMD (unconditional for martingale differences) spaces (recommended source for information on the UMD property is the forthcoming book [103]),
- Intersections of some collections of classes described above,
- One of such intersections is the class of spaces isomorphic to Hilbert spaces (by the Kwapień theorem [62], each Banach space having both type 2 and cotype 2 is isomorphic to a Hilbert space),
- Banach spaces isomorphic to subspaces of the space $L_p(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ), $p \neq 2, \infty$ (for $p = \infty$ we get the class of all Banach spaces, for $p = 2$ we get the class of spaces isomorphic to Hilbert spaces).

Remark 1.3. This list seems to constitute the list of all classes of Banach spaces satisfying the condition LHI which were systematically studied.

By the Ribe Theorem (Theorem 1.2), one can expect that each class satisfying the condition LHI has a metric characterization. At this point metric characterizations are known for all classes listed above except p-smooth and UMD (and some of the intersections involving these classes). Here are the references:

- Superreflexivity—see section "Metric Characterizations of Superreflexivity" of this paper for a detailed account.
- Properties related to type—[76] (see [16, 40, 102] for previous important results in this direction, and [42] for some improvements).
- Properties related to cotype—[77], see [43] for some improvements.
- q-convexity—[78].
- Spaces isomorphic to subspaces of L_p ($p \neq 2, \infty$). Rabinovich noticed that one can generalize results of [71] and characterize the optimal distortion of embeddings of a finite metric space into L_p-space (see [74, Exercise 4 on p. 383 and comment of p. 380] and a detailed presentation in [90, Sect. 4.3]). Johnson, Mendel, and Schechtman (unpublished) found another characterization of the optimal distortion using a modification of the argument of Lindenstrauss and Pełczyński [70, Theorem 7.3]. These characterizations are very close to each other. They are not satisfactory in some respects.

Ribe Program

It should be mentioned that some of the metric characterizations (for example of the class of spaces having some type > 1) can be derived from the known 'linear' theory. Substantially nonlinear characterizations started with the paper of Bourgain [13] in which he characterized superreflexive Banach spaces in terms of binary trees.

This paper of Bourgain and the whole direction of metric characterizations was inspired by the unpublished paper of Joram Lindenstrauss with the tentative title "Topics in the geometry of metric spaces." This paper has never been published (and apparently has never been written, so it looks like it was just a *conversation*, and not a *paper*), but it had a significant impact on this direction of research. The unpublished paper of Lindenstrauss and the mentioned paper of Bourgain [13] initiated what is now known as the *Ribe program*.

Bourgain [13, p. 222] formulated it as the program of search for equivalent definitions of different LHI invariants in terms of metric structure with the next step consisting in studying these metrical concepts in general metric spaces in an attempt to develop an analogue of the linear theory.

Bourgain himself made several important contributions to the Ribe program, now it is a very deep and extensive research direction. In words of Ball [3]: "Within a decade or two the Ribe programme acquired an importance that would have been hard to predict at the outset." In this paper I am going to cover only a very small part of known results on this program. I refer interested people to the surveys of Ball [3] (short survey) and Naor [81] (extensive survey).

Many of the known metric characterizations use the following standard definitions:

Definition 1.4. Let $0 \le C < \infty$. A map $f : (A, d_A) \to (Y, d_Y)$ between two metric spaces is called *C-Lipschitz* if

$$\forall u, v \in A \quad d_Y(f(u), f(v)) \le C d_A(u, v).$$

A map f is called *Lipschitz* if it is C-Lipschitz for some $0 \le C < \infty$.

Let $1 \le C < \infty$. A map $f : A \to Y$ is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in A \quad r d_A(u, v) \le d_Y(f(u), f(v)) \le rC d_A(u, v). \tag{1}$$

A *bilipschitz embedding* is an embedding which is C-bilipschitz for some $1 \le C < \infty$. The smallest constant C for which there exist $r > 0$ such that (1) is satisfied is called the *distortion* of f.

There are at least two directions in which we can seek metric characterizations:

(1) We can try to characterize metric spaces which admit bilipschitz embeddings into some Banach spaces belonging to $\mathcal{P}$.
(2) We can try to find metric structures which are present in each Banach space $X \notin \mathcal{P}$.

Characterizations of type **(1)** would be much more interesting for applications. However, as far as I know such characterizations were found only in the following cases: (1) $\mathcal{P}$ = {the class of Banach spaces isomorphic to a Hilbert space} (it is the Linial–London–Rabinovich [71, Corollary 3.5] formula for distortion of embeddings of a finite metric space into ℓ_2). (2) $\mathcal{P}$ = {the class of Banach spaces isomorphic to a subspace of some L_p-space}, p is a fixed number $p \neq 2, \infty$, see the last paragraph preceding section "Ribe Program."

Local Properties for Which no Metric Characterization is Known

Problem 1.5. *Find a metric characterization of UMD.*

Here *UMD* stays for *unconditional for martingale differences*. The most comprehensive source of information on UMD is the forthcoming book of Pisier [103].

I have not found in the literature any traces of attempts to work on Problem 1.5.

Definition 1.6. A Banach space is called *p-smooth* if its modulus of smoothness satisfies $\rho(t) \le Ct^p$ for $p \in (1, 2]$.

See [90, Sect. 8.4] for information on p-smooth spaces.

Problem 1.7. *Find a metric characterization of the class of Banach spaces isomorphic to p-smooth spaces* $p \in (1, 2]$.

This problem was posed and discussed in the paper by Mendel and Naor [78], where a similar problem is solved for q-convex spaces. Mendel and Naor wrote [78, p. 335]: "Trees are natural candidates for finite metric obstructions to q-convexity, but it is unclear what would be the possible finite metric witnesses to the "non-p-smoothness" of a metric space."

Metric Characterizations of Superreflexivity

Definition 2.1 (James [55, 56]). A Banach space X is called *superreflexive* if each Banach space which is finitely representable in X is reflexive.

It might look like a rather peculiar definition, but, as I understand, introducing it ($\approx$ 1967) James already had a feeling that it is a very natural and important definition. This feeling was shown to be completely justified when Enflo [38] completed the series of results of James by proving that each superreflexive space has an equivalent uniformly convex norm.

Definition 2.2. A Banach space is called *uniformly convex* if for every $\varepsilon > 0$ there is some $\delta > 0$ so that for any two vectors with $\|x\| \le 1$ and $\|y\| \le 1$, the inequality

$$1 - \left\| \frac{x+y}{2} \right\| < \delta$$

implies

$$\|x - y\| < \varepsilon.$$

Definition 2.3. Two norms $||\cdot||_1$ and $||\cdot||_2$ on a linear space X are called *equivalent* if there are constants $0 < c \le C < \infty$ such that

$$c||x||_1 \le ||x||_2 \le C||x||_1$$

for each $x \in X$.

After the pioneering results of James and Enflo numerous equivalent reformulations of superreflexivity were found and superreflexivity was used in many different contexts.

The metric characterizations of superreflexivity which we are going to present belong to the class of the so-called *test-space characterizations*.

Definition 2.4. Let $\mathcal{P}$ be a class of Banach spaces and let $T = \{T_\alpha\}_{\alpha \in A}$ be a set of metric spaces. We say that T is a set of *test-spaces* for $\mathcal{P}$ if the following two conditions are equivalent: **(1)** $X \notin \mathcal{P}$; **(2)** The spaces $\{T_\alpha\}_{\alpha \in A}$ admit bilipschitz embeddings into X with uniformly bounded distortions.

Characterization of Superreflexivity in Terms of Binary Trees

Definition 2.5. A *binary tree of depth* n, denoted T_n, is a finite graph in which each vertex is represented by a finite (possibly empty) sequence of 0 and 1, of length at most n. Two vertices in T_n are adjacent if the sequence corresponding to one of them is obtained from the sequence corresponding to the other by adding one term on the right. (For example, vertices corresponding to $(1, 1, 1, 0)$ and $(1, 1, 1, 0, 1)$ are adjacent.) A vertex corresponding to a sequence of length n in T_n is called a *leaf*.

An *infinite binary tree*, denoted T_∞, is an infinite graph in which each vertex is represented by a finite (possibly empty) sequence of 0 and 1. Two vertices in T_∞ are adjacent if the sequence corresponding to one of them is obtained from the sequence corresponding to the other by adding one term on the right.

Both for finite and infinite binary trees we use the following terminology. The vertex corresponding to the empty sequence is called a *root*. If a sequence τ is an initial segment of the sequence σ we say that σ is a *descendant* of τ and that τ is an *ancestor* of σ. If a descendant σ of τ is adjacent to τ, we say that σ is a *child* of τ and that τ is a *parent* of σ. Two children of the same parent are called *siblings*. Child of a child is called a *grandchild*. (It is clear that each vertex in T_∞ has exactly two children, the same is true for all vertices of T_n except leaves.)

Theorem 2.6 (Bourgain [13]). *A Banach space X is nonsuperreflexive if and only if it admits bilipschitz embeddings with uniformly bounded distortions of finite binary trees $\{T_n\}_{n=1}^\infty$ of all depths, see Fig. 1.*

In Bourgain's proof the difficult direction is the "if" direction, the "only if" is an easy consequence of the theory of superreflexive spaces. Recently Kloeckner [61] found a very simple proof of the "if" direction. I plan to describe the proofs of Bourgain and Kloeckner after recalling the results on superreflexivity which we need.

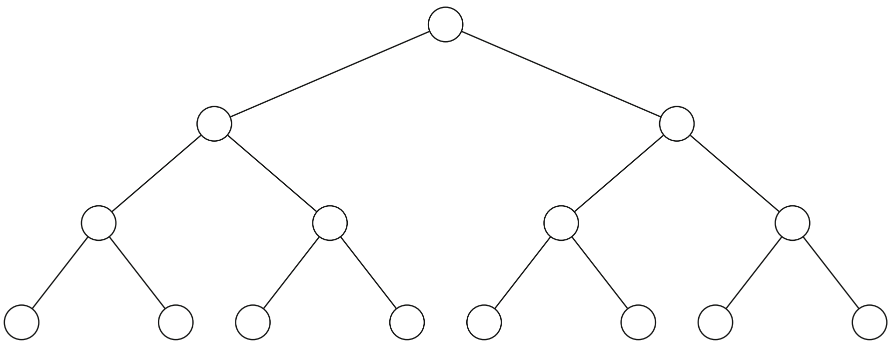


Fig. 1 The binary tree of depth 3, that is, T_3

Definition 2.7. The *modulus of* (uniform) *convexity* $\delta_X(\varepsilon)$ of a Banach space X with norm $\|\cdot\|$ is defined as

$$\inf\left\{1-\left\|\frac{x+y}{2}\right\| \,\middle|\, \|x\|=\|y\|=1 \text{ and } \|x-y\|\geqslant\varepsilon\right\}$$

for $\varepsilon\in(0,2]$. The space X or its norm is said to be *q-convex*, $q\in[2,\infty)$ if $\delta_X(\varepsilon)\geqslant c\varepsilon^q$ for some $c>0$.

Remark 2.8. It is easy to see that the definition of the uniform convexity given in Definition 2.2 is equivalent to: X is *uniformly convex* if and only if $\delta_X(\varepsilon)>0$ for each $\varepsilon\in(0,2]$.

Theorem 2.9 (Pisier [101]). *The following properties of a Banach space Y are equivalent:*

1. *Y is superreflexive.*
2. *Y has an equivalent q-convex norm for some $q\in[2,\infty)$.*

Using Theorem 2.9 we can prove the "if" part of Bourgain's characterization. Denote by $c_X(T_n)$ the infimum of distortions of embeddings of the binary tree T_n into a Banach space X.

By Theorem 2.9, for the "if" part of Bourgain's theorem it suffices to prove that if X is q-convex, then for some $c_1>0$ we have

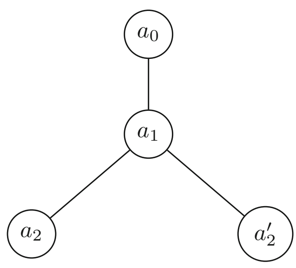
$$c_X(T_n)\geq c_1(\log_2 n)^{\frac{1}{q}}.$$

Proof (Kloeckner [61]). Let F be the four-vertex tree with one root a_0 which has one child a_1 and two grandchildren a_2, a_2'. Sometimes such tree is called a *fork*, see Fig. 2. The following lemma is similar to the corresponding results in [73].

Lemma 2.10. *There is a constant $K=K(X)$ such that if $\varphi: F\to X$ is D-Lipschitz and distance non-decreasing, then either*

$$\|\varphi(a_0)-\varphi(a_2)\|\leqslant 2\left(D-\frac{K}{D^{q-1}}\right)$$

Fig. 2 A fork



or

$$\|\varphi(a_0) - \varphi(a_2')\| \leqslant 2\left(D - \frac{K}{D^{q-1}}\right)$$

First we finish the proof of $c_X(T_n) \geq c_1(\log_2 n)^{\frac{1}{q}}$ using Lemma 2.10. So let $\varphi : T_n \to X$ be a map of distortion D. Since X is a Banach space, we may assume that φ is D-Lipschitz, distance non-decreasing map, that is

$$d_{T_n}(u, v) \leq ||\varphi(u) - \varphi(v)|| \leq D d_{T_n}(u, v).$$

The main idea of the proof is to construct a less-distorted embedding of a smaller tree.

Given any vertex of T_n which is not a leaf, let us name arbitrarily one of its two children its *daughter*, and the other its *son*. We select two grandchildren of the root in the following way: we pick the grandchild mapped by φ closest to the root among its daughter's children and the grandchild mapped by φ closest to the root among its son's children (ties are resolved arbitrarily). Then we select inductively, in the same way, two grandchildren for all previously selected vertices up to generation $n-2$.

The set of selected vertices, endowed with half the distance induced by the tree metric is isometric to $T_{\lfloor \frac{n}{2} \rfloor}$, and Lemma 2.10 implies that the restriction of φ to this set has distortion at most $f(D) = D - \frac{K}{D^{q-1}}$.

We can iterate such restrictions $\lfloor \log_2 n \rfloor$ times to get an embedding of T_1 whose distortion is at most

$$D - \lfloor \log_2 n \rfloor \frac{K}{D^{q-1}}$$

since each iteration improves the distortion by at least K/D^{q-1}. Since the distortion of any embedding is at least 1, we get the desired inequality. □

Remark 2.11. Kloeckner borrowed the approach based on "controlled improvement for embeddings of smaller parts" from the Johnson–Schechtman paper [58] in which it is used for diamond graphs (Kloeckner calls this approach a *self-improvement argument*). Arguments of this type are well known and widely used in the linear theory, where they go back at least to James [53]; but these two examples (Johnson–Schechtman [58] and Kloeckner [61]) seem to be the only two known results of this type in the nonlinear theory. It would be interesting to find further nonlinear arguments of this type.

Sketch of the Proof of Lemma 2.10. Assume $\varphi(a_0) = 0$ and let $x_1 = \varphi(a_1)$, $x_2 = \varphi(a_2)$ and $x_2' = \varphi(a_2')$. Recall that we assumed

$$d_{T_n}(u, v) \leq ||\varphi(u) - \varphi(v)|| \leq D d_{T_n}(u, v). \qquad (*)$$

Consider the (easy) case where $\|x_2\| = 2D$ and $\|x_2'\| = 2D$ (that is, the distortion D is attained on these vectors). We claim that this implies that $x_2 = x_2'$. In fact, it is easy to check that this implies $\|x_1\| = D$, $\|x_2 - x_1\| = D$, and $\|x_2' - x_1\| = D$. Also $\left\|\frac{x_1+(x_2-x_1)}{2}\right\| = D$ and $\left\|\frac{x_1+(x_2'-x_1)}{2}\right\| = D$. By the uniform convexity we get $||x_1 - (x_2 - x_1)|| = 0$ and $||x_1 - (x_2' - x_1)|| = 0$. Hence $x_2 = x_2'$, and we get that the conditions $\|x_2\| = 2D$ and $\|x_2'\| = 2D$ cannot be satisfied simultaneously.

The proof of Lemma 2.10 goes as follows. We start by letting $\|x_2\| \geq 2(D - \eta)$ and $\|x_2'\| \geq 2(D - \eta)$ for some $\eta > 0$. Using a perturbed version of the argument just presented, the definition of the modulus of convexity, and our assumption $\delta_X(\varepsilon) \geq c\varepsilon^p$, we get an estimate of $||x_2 - x_2'||$ from above in terms of η. Comparing this estimate with the assumption $||x_2 - x_2'|| \geq 2$ (which follows from $d_{T_n}(u, v) \leq ||\varphi(u) - \varphi(v)||$), we get the desired estimate for η from below, see [61] for details. □

Remark 2.12. The approach of Kloeckner can be used for any uniformly convex space, it is not necessary to combine it with the Pisier Theorem (Theorem 2.9), see [103].

To prove the "only if" part of Bourgain's theorem we need the following characterization of superreflexivity, one of the most suitable sources for this characterization of superreflexivity is [103].

Theorem 2.13 (James [53, 55, 108]). *Let X be a Banach space. The following are equivalent:*

1. *X is not superreflexive*
2. *There exists* $\alpha \in (0, 1]$ *such that for each* $m \in \mathbb{N}$ *the unit ball of the space X contains a finite sequence* $x_1, x_2, \ldots, x_m$ *of vectors satisfying, for any* $j \in \{1, \ldots, m-1\}$ *and any real coefficients* $a_1, \ldots, a_m$*, the condition*

$$\left\|\sum_{i=1}^{m} a_i x_i\right\| \geq \alpha \left(\left|\sum_{i=1}^{j} a_i\right| + \left|\sum_{i=j+1}^{m} a_i\right|\right). \tag{2}$$

3. *For each* $\alpha \in (0, 1)$ *and each* $m \in \mathbb{N}$ *the unit ball of the space X contains a finite sequence* $x_1, x_2, \ldots, x_m$ *of vectors satisfying, for any* $j \in \{1, \ldots, m-1\}$ *and any real coefficients* $a_1, \ldots, a_m$*, the condition* (2)*.*

Remark 2.14. It is worth mentioning that the proof of (1)⇒(3) in the case where $\alpha \in [\frac{1}{2}, 1)$ is much more difficult than in the case $\alpha \in (0, \frac{1}{2})$. A relatively easy proof in the case $\alpha \in [\frac{1}{2}, 1)$ was found by Brunel and Sucheston [20], see also its presentation in [103].

Remark 2.15. To prove the Bourgain's theorem it suffices to use (1)⇒(3) in the 'easy' case $\alpha \in (0, \frac{1}{2})$. The case $\alpha \in [\frac{1}{2}, 1)$ is needed only for "almost-isometric" embeddings of trees into nonsuperreflexive spaces.

Remark 2.16. The equivalence of (2)⇔(3) in Theorem 2.13 can be proved using a "self-improvement argument," but the proof of James is different. A proof of (2)⇔(3) using a "self-improvement argument" was obtained by Wenzel [114], it is based on the Ramsey theorem, so it requires a very lengthy sequence to get a better α. In [84] it was proved that to some extent the usage of 'very lengthy' sequences is necessary.

Proof of the "Only If" Part. There is a natural partial order on T_n: we say that $s < t$ ($s, t \in T_n$) if the sequence corresponding to s is the initial segment of the sequence corresponding to t.

An important observation of Bourgain is that there is a bijective mapping

$$\varphi : T_n \to [1, \ldots, 2^{n+1} - 1]$$

such that φ maps two disjoint intervals of the ordering of T_n, starting at the same vertex and going "down" into disjoint intervals of $[1, \ldots, 2^{n+1} - 1]$. The existence of φ can be seen from a suitably drawn picture of T_n (see Fig. 3), or using the expansion of numbers in base 2. To use the expansion of numbers, we observe that the map $\{\theta_i\}_{i=1}^n \to \{2\theta_i - 1\}_{i=1}^n$ maps a $0,1-$sequence onto the corresponding $\pm 1-$sequence. Now we introduce a map $\psi : T_n \to [-1, 1]$ by letting $\psi(\emptyset) = 0$ and

$$\psi(\theta_1, \ldots, \theta_n) = \sum_{i=1}^{n} 2^{-i} (2\theta_i - 1).$$

To construct φ we relabel the range of ψ in the increasing order using numbers $[1, \ldots, 2^{n+1} - 1]$.

Let $\{x_1, x_2, \ldots, x_{2^{n+1}-1}\}$ be a sequence in a nonsuperreflexive Banach space X whose existence is guaranteed by Theorem 2.13 ((1)⇒(3)). We introduce an embedding $F_n : T_n \to X$ by

$$F_n(t) = \sum_{s \le t} x_{\varphi(s)},$$

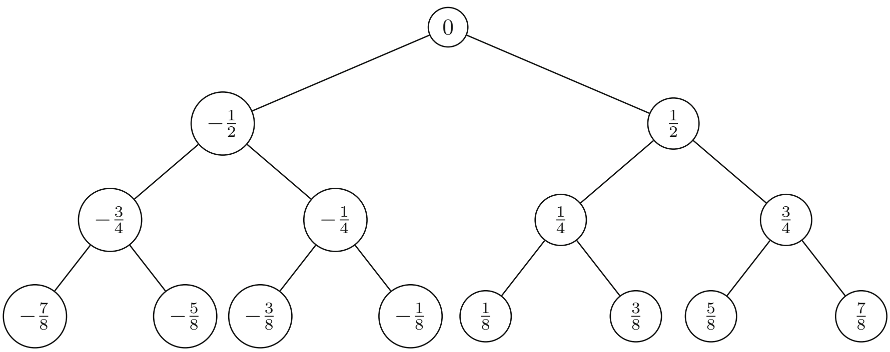


Fig. 3 The map of T_3 into $[-1, 1]$

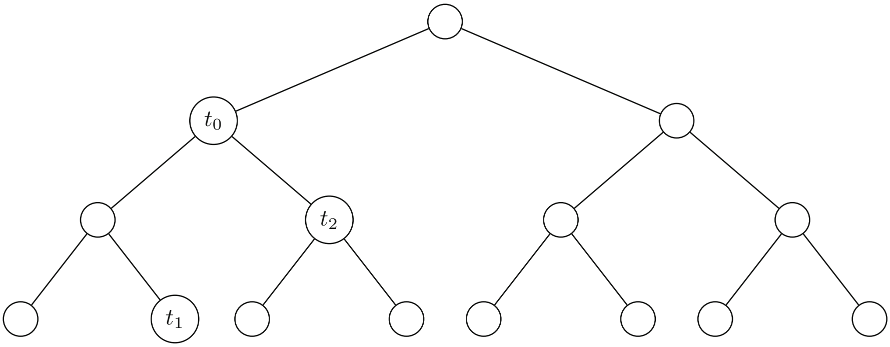


Fig. 4 t_0 is the closest common ancestor of t_1 and t_2

where $s \leq t$ for vertices of a binary tree means that s is the initial segment of the sequence t. Then $F_n(t_1) - F_n(t_2) = \sum_{t_0 < s \leq t_1} x_{\varphi(s)} - \sum_{t_0 < s \leq t_2} x_{\varphi(s)}$, where t_0 is the vertex of T_n corresponding to the largest initial common segment of t_1 and t_2, see Fig. 4. The condition in (2) and the choice of φ imply that

$$||F_n(t_1) - F_n(t_2)|| \geq \alpha(d_T(t_1, t_0) + d_T(t_2, t_0)) = \alpha d_{T_n}(t_1, t_2).$$

The estimate $||F_n(t_1) - F_n(t_2)||$ from above is straightforward. This completes the proof of bilipschitz embeddability of $\{T_n\}$ into any nonsuperreflexive Banach space with uniformly bounded distortions. □

Characterization of Superreflexivity in Terms of Diamond Graphs

Johnson and Schechtman [58] proved that there are some other sequences of graphs (with their graph metrics) which also can serve as test-spaces for superreflexivity. For example, binary trees in Bourgain's theorem can be replaced by the diamond graphs or by Laakso graphs.

Definition 2.17. Diamond graphs $\{D_n\}_{n=0}^{\infty}$ are defined as follows: The *diamond graph* of level 0 is denoted D_0. It has two vertices joined by an edge of length 1. The *diamond graph* D_n is obtained from D_{n-1} as follows. Given an edge $uv \in E(D_{n-1})$, it is replaced by a quadrilateral u, a, v, b, with edges ua, av, vb, bu. (See Fig. 5.)

Two different normalizations of the graphs $\{D_n\}_{n=1}^{\infty}$ are considered

- *Unweighted diamonds:* Each edge has length 1.
- *Weighted diamonds:* Each edge of D_n has length 2^{-n}

In both cases we endow vertex sets of $\{D_n\}_{n=0}^{\infty}$ with their shortest path metrics.

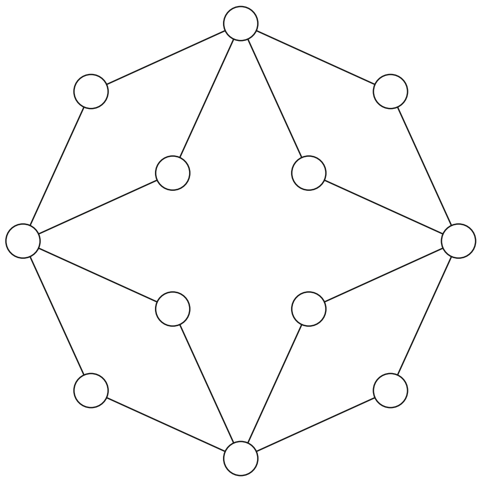


Fig. 5 Diamond D_2

In the case of weighted diamonds the identity map $D_{n-1} \mapsto D_n$ is an isometry. In this case the union of D_n, endowed with its the metric induced from $\{D_n\}_{n=0}^{\infty}$ is called the *infinite diamond* and is denoted D_ω.

To the best of my knowledge the first paper in which diamond graphs $\{D_n\}_{n=0}^{\infty}$ were used in Metric Geometry is [50] (a conference version was published in 1999).

Definition 2.18. Laakso graphs $\{L_n\}_{n=0}^{\infty}$ are defined as follows: The *Laakso graph* of level 0 is denoted L_0. It has two vertices joined by an edge of length 1. The *Laakso graph* L_n is obtained from L_{n-1} as follows. Given an edge $uv \in E(L_{n-1})$, it is replaced by the graph L_1 shown in Fig. 6, the vertices u and v are identified with the vertices of degree 1 of L_1.

Two different normalizations of the graphs $\{L_n\}_{n=1}^{\infty}$ are considered

- *Unweighted Laakso graphs:* Each edge has length 1.
- *Weighted Laakso graphs:* Each edge of L_n has length 4^{-n}

In both cases we endow vertex sets of $\{L_n\}_{n=0}^{\infty}$ with their shortest path metrics.

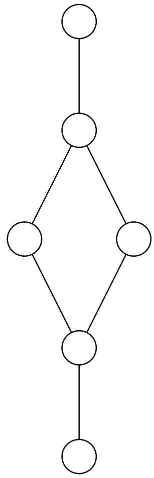
In the case of weighted Laakso graphs the identity map $L_{n-1} \mapsto L_n$ is an isometry. In this case the union of L_n endowed with its the metric induced from $\{L_n\}_{n=0}^{\infty}$ is called the *Laakso space* and is denoted L_ω.

The Laakso graphs were introduced in [65], but they were inspired by the construction of Laakso in [63].

Theorem 2.19 (Johnson–Schechtman [58]). *A Banach space X is nonsuperreflexive if and only if it admits bilipschitz embeddings with uniformly bounded distortions of diamonds $\{D_n\}_{n=1}^{\infty}$ of all sizes.*

Theorem 2.20 (Johnson–Schechtman [58]). *A similar result holds for $\{L_n\}_{n=1}^{\infty}$.*

Fig. 6 Laakso graph L_1



Without Proof. These results, whose original proofs (especially for diamond graphs) are elegant in both directions, are loved by expositors. Proof of Theorem 2.19 is presented in the lecture notes of Lancien [64], in the book of Pisier [103], and in my book [90]. □

Remark 2.21. In the "if" direction of Theorem 2.19, in addition to the original (controlled improvement for embeddings of smaller parts) argument of Johnson–Schechtman [58], there are two other arguments:

(1) The argument based on Markov convexity (see Definition 2.26). It is obtained by combining results of Lee–Naor–Peres [67] (each superreflexive Banach space is Markov p-convex for some $p \in [2, \infty)$) and Mendel–Naor [78] (Markov convexity constants of diamond graphs are not uniformly bounded from below, actually in [78] this statement is proved for Laakso graphs, but similar argument works for diamond graphs).
(2) The argument of [87, Sect. 3.1] showing that bilipschitz embeddability of diamond graphs with uniformly bounded distortions implies the finite tree property of the space, defined as follows:

Definition 2.22 (James [55]). Let $\delta > 0$. A *δ-tree* in a Banach space X is a subset $\{x_\tau\}_{\tau \in T_\infty}$ of X labelled by elements of the infinite binary tree T_∞, such that for each $\tau \in T_\infty$ we have

$$x_\tau = \frac{1}{2}(x_{\sigma_1} + x_{\sigma_2}) \quad \text{and } ||x_\tau - x_{\sigma_1}|| = ||x_\tau - x_{\sigma_2}|| \geq \delta, \tag{3}$$

where σ_1 and σ_2 are the children of τ. A Banach space X is said to have the *infinite tree property* if it contains a bounded δ-tree.

A *δ-tree of depth n* in a Banach space X is a finite subset $\{x_\tau\}_{\tau\in T_n}$ of X labelled by the binary tree T_n of depth n, such that the condition (3) is satisfied for each $\tau \in T_n$, which is not a leaf. A Banach space X has the *finite tree property* if for some $\delta > 0$ and each $n \in \mathbb{N}$ the unit ball of X contains a δ-tree of depth n.

Remark 2.23. In the "only if" direction of Theorem 2.19 there is a different (and more complicated) proof in [91, 96], which consists in a combination of the following two results:

(i) Existence of a bilipschitz embedding of the infinite diamond D_ω into any non-separable dual of a separable Banach space (using Stegall's [111] construction), see [91].
(ii) Finite subsets of a metric space which admits a bilipschitz embedding into any nonseparable dual of a separable Banach space, admit embeddings into any nonsuperreflexive Banach space with uniformly bounded distortions, see [96]. (The proof uses transfinite duals [9, 33, 34] and the results of Brunel–Sucheston [20, 21] and Perrott [100] on equal-signs-additive sequences.)

I would like to turn your attention to the fact that the Johnson–Schechtman Theorem 2.19 shows some obstacles on the way to a solution the (mentioned above) problem for superreflexivity:

Characterize metric spaces which admit bilipschitz embeddings into some superreflexive Banach spaces.

We need the following definitions and results. Let $\{M_n\}_{n=1}^\infty$ and $\{R_n\}_{n=1}^\infty$ be two sequences of metric spaces. We say that $\{M_n\}_{n=1}^\infty$ admits *uniformly bilipschitz embeddings* into $\{R_n\}_{n=1}^\infty$ if for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ and a bilipschitz map $f_n : M_n \to R_{m(n)}$ such that the distortions of $\{f_n\}_{n=1}^\infty$ are uniformly bounded.

Theorem 2.24 ([92]). *Binary trees $\{T_n\}_{n=1}^\infty$ do not admit uniformly bilipschitz embeddings into diamonds $\{D_n\}_{n=1}^\infty$.*

Without Proof. The proof is elementary, but rather lengthy combinatorial argument. □

There is also a non-embeddability in the other direction: The fact that diamonds $\{D_n\}$ do not admit uniformly bilipschitz embeddings into binary trees $\{T_n\}$ is well known, it follows immediately from the fact that D_n $(n \geq 1)$ contains a cycle of length 2^{n+1} isometrically, and the well-known observation of Rabinovich and Raz [105] stating that the distortion of any embedding of an m-cycle into any tree is $\geq \frac{m}{3} - 1$.

Remark 2.25. Mutual non-embeddability of Laakso graphs and binary trees is much simpler: (1) Laakso graphs are non-embeddable into trees because large Laakso graphs contain large cycles isometrically. (2) Binary trees are not embeddable into Laakso graphs because the Laakso graphs are uniformly doubling (see [51, p. 81] for the definition of a doubling metric space), but binary trees are not uniformly doubling.

Let us show that these results, in combination with some other known results, imply that it is impossible to find a sequence $\{C_n\}_{n=1}^{\infty}$ of finite metric spaces which admits uniformly bilipschitz embeddings into a metric space M if and only if M does not admit a bilipschitz embedding into a superreflexive Banach space. Assume the contrary: Such sequence $\{C_n\}_{n=1}^{\infty}$ exists. Then $\{C_n\}$ admits uniformly bilipschitz embeddings into the infinite binary tree. Therefore, by the result of Gupta [49], the spaces $\{C_n\}_{n=1}^{\infty}$ are uniformly bilipschitz-equivalent to weighted trees $\{W_n\}_{n=1}^{\infty}$. The trees $\{W_n\}_{n=1}^{\infty}$ should admit, by a result Lee–Naor–Peres [67] uniformly bilipschitz embeddings of increasing binary trees (these authors proved that $\{W_n\}_{n=1}^{\infty}$ would admit uniformly bilipschitz embeddings into ℓ_2 otherwise). Therefore, by Theorem 2.24 the spaces $\{C_n\}_{n=1}^{\infty}$ cannot be embeddable into diamonds with uniformly bounded distortion. Therefore they do not admit uniformly bilipschitz embeddings into D_ω (since the union of $\{D_i\}_{i=0}^{\infty}$ is dense in D_ω). On the other hand, Theorem 2.19 implies that D_ω does not admit a bilipschitz embedding into a superreflexive space, a contradiction.

One can try to find a characterization of metric spaces which are embeddable into superreflexive spaces in terms of some inequalities for distances. Some hope for such characterization was given by the already mentioned Markov convexity introduced by Lee–Naor–Peres [67], because it provides a reason for non-embeddability into superreflexive Banach spaces of both binary trees and diamonds (and many other trees and diamond-like spaces).

Definition 2.26 (Lee–Naor–Peres [67]). Let $\{X_t\}_{t\in\mathbb{Z}}$ be a Markov chain on a state space Ω. Given an integer $k \geq 0$, we denote by $\{\widetilde{X}_t(k)\}_{t\in\mathbb{Z}}$ the process which equals X_t for time $t \leq k$, and evolves independently (with respect to the same transition probabilities) for time $t > k$. Fix $p > 0$. A metric space (X, d_X) is called *Markov p-convex with constant* Π if for every Markov chain $\{X_t\}_{t\in\mathbb{Z}}$ on a state space Ω, and every $f : \Omega \to X$,

$$\sum_{k=0}^{\infty}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\left[d_X\left(f(X_t), f\left(\widetilde{X}_t\left(t-2^k\right)\right)\right)^p\right]}{2^{kp}} \leq \Pi^p \cdot \sum_{t\in\mathbb{Z}} \mathbb{E}\big[d_X(f(X_t), f(X_{t-1}))^p\big]. \qquad (4)$$

The least constant Π for which (4) holds for all Markov chains is called the *Markov p-convexity constant* of X, and is denoted $\Pi_p(X)$. We say that (X, d_X) is *Markov p-convex* if $\Pi_p(X) < \infty$.

Remark 2.27. The choice of the rather complicated left-hand side in (4) is inspired by the original Bourgain's proof [13] of the "if" part of Theorem 2.6.

Remark 2.28. It is unknown whether for general metric spaces Markov p-convexity implies Markov q-convexity for $q > p$. (This is known to be true for Banach spaces.)

Lee–Naor–Peres [67] showed that Definition 2.26 is important for the theory of metric embeddings by proving that each superreflexive space X is Markov q-convex for sufficiently large q. More precisely, it suffices to pick q such that X has an equivalent q-convex norm (see Definition 2.7), and by Theorem 2.9 of Pisier, such $q \in [2, \infty)$ exists for each superreflexive space.

On the other hand, Lee–Naor–Peres have shown that for any $0 < p < \infty$ the Markov p-convexity constants of binary trees $\{T_n\}$ are not uniformly bounded. Later Mendel and Naor [78] verified that the Markov p-convexity constants of Laakso graphs are not uniformly bounded. Similar proof works for diamonds $\{D_n\}$. See Theorem 3.11 and Remark 3.13 for a more general result.

Example 2.29 (Lee–Naor–Peres [67]). For every $m \in \mathbb{N}$, we have $\Pi_p(T_{2^m}) \geq 2^{1-\frac{2}{p}} \cdot m^{\frac{1}{p}}$.

Proof. Simplifying the description of the chain somewhat (precise description of Ω and the map f requires some formalities), we consider only times $t = 1, \dots, 2^m$ and let $\{X_t\}_{t=0}^m$ be the downward random walk on T_{2^m} which is at the root at time $t = 0$ and X_{t+1} is obtained from X_t by moving down-left or down-right with probability $\frac{1}{2}$ each, see Fig. 7. We also assume that X_t is at the root with probability 1 if $t < 0$ (here more formal description of the chain is needed) and that for $t > 2^m$ we have $X_{t+1} = X_t$ (this is usually expressed by saying that *leaves are absorbing states*). Then

$$\sum_{t=1}^{2^m} \mathbb{E}\left[d_{T_{2^m}}(X_t, X_{t-1})^p\right] = 2^m.$$

Moreover, in the downward random walk, after splitting at time $r \leq 2^m$ with probability at least $\frac{1}{2}$ two independent walks will accumulate distance which is at least twice the number of steps (until a leaf is encountered). Thus

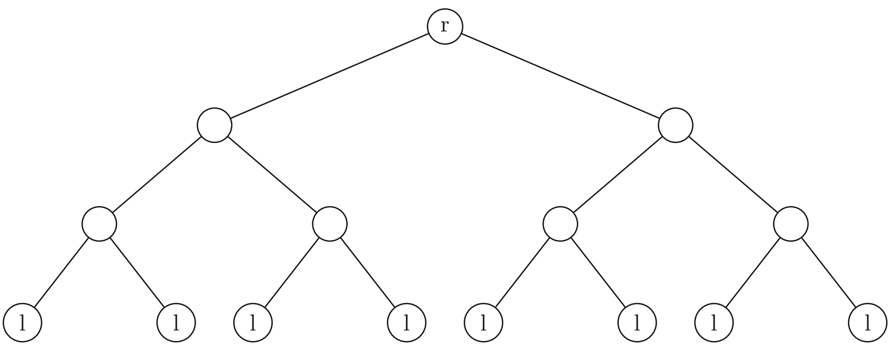


Fig. 7 T_3, with the root (r) and leaves (l) marked

$$\sum_{k=0}^{m}\sum_{t=1}^{2^m}\frac{\mathbb{E}\left[d_{T_{2^m}}\left(X_t,\widetilde{X}_t\left(t-2^k\right)\right)^p\right]}{2^{kp}} \geq \sum_{k=0}^{m-1}\sum_{t=2^k}^{2^m}\frac{1}{2^{kp}}\cdot\frac{1}{2}\cdot 2^{(k+1)p}$$
$$\geq m2^{m-1}2^{p-1}$$
$$= 2^{p-2}\cdot m\cdot 2^m.$$

The claim follows. □

For diamond graphs and Laakso graphs the argument is similar, but more complicated, because in such graphs the trajectories can come close after separation.

After uniting the reasons for non-embeddability for diamonds and trees one can hope to show that Markov convexity characterizes metric spaces which are embeddable into superreflexive spaces. It turns out that this is not the case. It was shown by Li [68, 69] that the Heisenberg group $\mathbb{H}(\mathbb{R})$ (see Definition 3.9) is Markov convex. On the other hand, it is known that the Heisenberg group does not admit a bilipschitz embedding into any superreflexive Banach space [26, 66]. (It is worth mentioning that in the present context we may consider the discrete Heisenberg group $\mathbb{H}(\mathbb{Z})$ consisting of the matrices with integer entries of the form shown in Definition 3.9 endowed with its word distance, see Definition 2.35.)

I suggest the following problem which is open as far as I know:

Problem 2.30. *Does there exist a test-space for superreflexivity which is Markov p-convex for some $0 < p < \infty$? (Or a sequence of test-spaces with uniformly bounded Markov p-convexity constants?)*

Remark 2.31. The Heisenberg group $H(\mathbb{Z})$ (with integer entries) has two properties needed for the test-space in Problem 2.30: it is not embeddable into any superreflexive space and is Markov convex. The only needed property which it does not have is: embeddability into each nonsuperreflexive space. Cheeger and Kleiner [28] proved that $H(\mathbb{Z})$ is not embeddable into some nonsuperreflexive Banach spaces, for example, into $L_1(0,1)$.

One more problem which I would like to mention here is

Problem 2.32 (Naor, July 2013). *Does there exist a sequence of finite metric spaces $\{M_i\}_{i=1}^{\infty}$ which is a sequence of test-spaces for superreflexivity with the following universality property: if $\{A_i\}_{i=1}^{\infty}$ is a sequence of test-spaces for superreflexivity, then there exist uniformly bilipschitz embeddings $E_i : A_i \to M_{n(i)}$, where $\{n(i)\}_{i=1}^{\infty}$ is some sequence of positive integers?*

Characterization of Superreflexivity in Terms of One Test-Space

Baudier [4] strengthened the "only if" direction of Bourgain's characterization and proved

Theorem 2.33 (Baudier [4]). *A Banach space X is nonsuperreflexive if and only if it admits bilipschitz embedding of the infinite binary tree T_∞.*

The following result hinted that possibly the Cayley graph of any nontrivially complicated hyperbolic group is the test-space for superreflexivity:

Theorem 2.34 (Buyalo–Dranishnikov–Schroeder [22]). *Every Gromov hyperbolic group admits a quasi-isometric embedding into the product of finitely many copies of the binary tree.*

Let us introduce notions used in this statement.

Definition 2.35. Let G be a group generated by a finite set S.

- The *Cayley graph* $\mathrm{Cay}(G,S)$ is defined as a graph whose vertex set is G and whose edge set is the set of all pairs of the form (g, sg), where $g \in G$, $s \in S$.
- In this context we consider each edge as a line segment of length 1 and endow $\mathrm{Cay}(G,S)$ with the shortest path distance. The restriction of this distance to G is called the *word distance*.
- Let u and v be two elements in a metric space (M, d_M). A *uv-geodesic* is a distance-preserving map $g : [0, d_M(u,v)] \to M$ such that $g(0) = u$ and $g(d_M(u,v)) = v$ (where $[0, d_M(u,v)]$ is an interval of the real line with the distance inherited from $\mathbb{R}$).
- A metric space M is *geodesic* if for any two points u and v in M, there is a uv-geodesic in M; $\mathrm{Cay}(G,S)$, with edges identified with line segments and with the shortest path distance is a geodesic metric space.
- A geodesic metric space M is called *δ-hyperbolic*, if for each triple $u, v, w \in M$ and any choice of a uv-geodesic, vw-geodesic, and wu-geodesics, each of these geodesics is in the δ-neighborhood of the union of the other two.
- A group is *word hyperbolic* or *Gromov hyperbolic* if $\mathrm{Cay}(G,S)$ is δ-hyperbolic for some $\delta < \infty$.

Remark 2.36.
- It might seem that the definition of hyperbolicity depends on the choice of the generating set S.
- It turns out that the value of δ depends on S, but its existence does not.
- The theory of hyperbolic groups was created by Gromov [46], although some related results were known before. The theory of hyperbolic groups plays an important role in group theory, geometry, and topology.
- Theory of hyperbolic groups is presented in many sources, see [1, 18].

Remark 2.37. It is worth mentioning that the identification of edges of $\mathrm{Cay}(G,S)$ with line segments is useful and important when we study geodesics and introduce the definition of hyperbolicity. In the theory of embeddings it is much more convenient to consider $\mathrm{Cay}(G,S)$ as a countable set (it is countable because we consider groups generated by finite sets), endowed with the shortest path distance (in the graph-theoretic sense), in this context it is called the *word distance*. See [89, 95] for relations between embeddability of graphs as vertex sets and as geodesic metric spaces.

It is worth mentioning that although different finite generating sets S_1 and S_2 in G lead to different word distances on G, the resulting metric spaces are bilipschitz equivalent: the identity map $(G, d_{S_1}) \to (G, d_{S_2})$ is bilipschitz, where d_{S_1} is the word distance corresponding to S_1 and d_{S_2} is the word distance corresponding to S_2.

We also need the following definitions used in [22]. A map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called a *quasi-isometric embedding* if there are $a_1, a_2 > 0$ and $b \geq 0$, such that

$$a_1 d_X(u, v) - b \leq d_Y(f(u), f(v)) \leq a_2 d_X(u, v) + b \tag{5}$$

for all $u, v \in X$. By a *binary tree* the authors of [22] mean an infinite tree in which each vertex has degree 3. By a *product of trees*, denoted $(\oplus_{i=1}^{n} T(i))_1$, we mean their Cartesian product with the ℓ_1-metric, that is,

$$d(\{u_i\}, \{v_i\}) = \sum_{i=1}^{n} d_{T(i)}(u_i, v_i). \tag{6}$$

Observation 2.38. *The binary tree defined as an infinite tree in which each vertex has degree* 3 *is isometric to a subset of the product in the sense of* (6) *of three copies of* T_∞.

Therefore we may replace the infinite binary tree by T_∞ in the statement of Theorem 2.34. Hence the Buyalo–Dranishnikov–Schroeder Theorem 2.34 in combination with the Baudier Theorem 2.33 implies the existence of a quasi-isometric embedding of any Gromov hyperbolic group, which is embeddable into product of n copies of T_∞, into any Banach space containing an isomorphic copy of a direct sum of n nonsuperreflexive spaces. The fact that Buyalo–Dranishnikov–Schroeder consider quasi-isometric embeddings (which are weaker than bilipschitz) is not a problem. One can easily prove the following lemma. Recall that a metric space is called *locally finite* if all balls of finite radius in it have finite cardinality (a detailed proof of Lemma 2.39 can be found in [92, Lemma 2.3]).

Lemma 2.39. *If a locally finite metric space M admits a quasi-isometric embedding into an infinite-dimensional Banach space X, then M admits a bilipschitz embedding into X.*

Remark 2.40. One can easily construct a counterexample to a similar statement for general metric spaces.

Back to Embeddings However, we know from results of Gowers–Maurey [45] that there exist nonsuperreflexive spaces which do not contain isomorphically direct sums of any two infinite-dimensional Banach spaces, so we do not get immediately bilipschitz embeddability of hyperbolic groups into nonsuperreflexive Banach spaces.

Possibly this obstacle can be overcome by modifying Baudier's proof of Theorem 2.33 for the case of a product of several trees, but at this point a more general result is available. It can be stated as:

Embeddability of locally finite spaces into Banach spaces is finitely determined.

We need the following version of the result on finite determination (this statement also explains what we mean by "finite determination"):

Theorem 2.41 ([88]). *Let A be a locally finite metric space whose finite subsets admit bilipschitz embeddings into a Banach space X with uniformly bounded distortions. Then A admits a bilipschitz embedding into X.*

Remark 2.42. This result and its version for coarse embeddings have many predecessors: Baudier [4, 5], Baudier–Lancien [6], Brown–Guentner [19], and Ostrovskii [85, 86].

Now we return to embeddings of hyperbolic groups into nonsuperreflexive spaces. Recall that we consider finitely generated groups. It is easy to see that in this case $\mathrm{Cay}(G,S)$ is a locally finite metric space (recall that we consider $\mathrm{Cay}(G,S)$ as a countable set G with its word distance). By finite determination, it suffices to show only how to embed products of n finite binary trees into an arbitrary nonsuperreflexive Banach space with uniformly bounded distortions (the distortions are allowed to grow if we increase n, since for a fixed hyperbolic group the number n is fixed). This can be done using the embedding of a finite binary tree suggested by Bourgain (Theorem 2.6) and the standard techniques for constructions of basic sequences and finite-dimensional decompositions. These techniques (going back to Mazur) allow to show that for each n and N there exists a sequence of finite-dimensional spaces X_i such that X_i contains a 2-bilipschitz image of T_N and the direct sum $(\oplus_{i=1}^n X_i)_1$ is $C(n)$-isomorphic to their linear span in X ($C(n)$ is constant which depends on n, but not on N). See [92, pp. 157–158] for a detailed argument.

So we have proved that each Gromov hyperbolic group admits a bilipschitz embedding into any nonsuperreflexive Banach space. This proves the corresponding part of the following theorem:

Theorem 2.43. *Let G be a Gromov hyperbolic group which does not contain a cyclic group of finite index. Then the Cayley graph of G is a test-space for superreflexivity.*

The other direction follows by a combination of results of Bourgain [13], Benjamini and Schramm [10], and some basic theory of hyperbolic groups [18, 82], see [92, Remark 2.5] for details.

I find the following open problem interesting:

Problem 2.44. *Characterize finitely generated infinite groups whose Cayley graphs are test-spaces for superreflexivity.*

Possibly Problem 2.44 is very far from its solution and we should rather do the following. Given a group whose structure is reasonably well understood, check

(1) Whether it admits a bilipschitz embedding into an arbitrary nonsuperreflexive Banach space?

(2) Whether it admits bilipschitz embeddings into some superreflexive Banach spaces?

Remark 2.45. There are groups which do not admit bilipschitz embeddings into some nonsuperreflexive spaces, such as L_1. Examples which I know:

- Heisenberg group (Cheeger–Kleiner [28]).
- Gromov's random groups [48] containing expanders weakly are not even coarsely embeddable into L_1.
- Recently constructed groups of Osajda [83] with even stronger properties.

Remark 2.46. At the moment the only groups known to admit bilipschitz embeddings into superreflexive spaces are groups containing $\mathbb{Z}^n$ as a subgroup of finite index. de Cornulier–Tessera–Valette [35] conjectured that such groups are the only groups admitting a bilipschitz embedding into ℓ_2. This conjecture is still open. I asked about the superreflexive version of this conjecture on MathOverflow [97] (August 19, 2014) and de Cornulier commented on it as: "In the main two cases for which the conjecture is known to hold in the Hilbert case, the same argument also works for arbitrary uniformly convex Banach spaces."

Remark 2.47. Groups which are test-spaces for superreflexivity do not have to be hyperbolic. In fact, one can show that a direct product of finitely many hyperbolic groups is a test-space for superreflexivity provided at least one of them does not have a cyclic group as a subgroup of finite index. It is easy to check (using the definition) that such products are not Gromov hyperbolic unless all-except-one groups in the product are finite (the reason is that $\mathbb{Z}^2$ is not Gromov hyperbolic).

Now I would like to return to the title of this section: "Characterization of superreflexivity in terms of one test-space." This can actually be done using either the Bourgain or the Johnson–Schechtman characterization and the following elementary proposition (I published it [92], but I am sure that it was known to interested people):

Proposition 2.48 ([92, Sect. 5]).

(a) *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of finite test-spaces for some class $\mathcal{P}$ of Banach spaces containing all finite-dimensional Banach spaces. Then there is a metric space S which is a test-space for $\mathcal{P}$.*

(b) *If $\{S_n\}_{n=1}^{\infty}$ are*

- *unweighted graphs,*
- *trees,*
- *graphs with uniformly bounded degrees,*

then S also can be required to have the same property.

Sketch of the Proof. In all of the cases the constructed space S contains subspaces isometric to each of $\{S_n\}_{n=1}^{\infty}$. Therefore the only implication which is nontrivial is that the embeddability of $\{S_n\}_{n=1}^{\infty}$ implies the embeddability of S.

Each finite metric space can be considered as a weighted graph with its shortest path distance. We construct the space S as an infinite graph by joining S_n with S_{n+1} with a path P_n whose length is $\geq \max\{\mathrm{diam} S_n, \mathrm{diam} S_{n+1}\}$. To be more specific, we pick in each S_n a vertex O_n and let P_n be a path joining O_n with O_{n+1}. We endow the infinite graph S with its shortest path distance. It is clear that $\{S_n\}_{n=1}^{\infty}$ embed isometrically into S and all of the conditions in (b) are satisfied. It remains only to show that each infinite-dimensional Banach space X which admits bilipschitz embeddings of $\{S_n\}_{n=1}^{\infty}$ with uniformly bounded distortions, admits a bilipschitz embedding of S. This is done by embedding S_n into any hyperplane of X with uniformly bounded distortions. This is possible because the sets are finite, the space is infinite-dimensional, and all hyperplanes in a Banach space are isomorphic with the Banach–Mazur distances being $\leq$ some universal constant.

Now we consider in X parallel hyperplanes $\{H_n\}$ with the distance between H_n and H_{n+1} equal to the length of P_n and embed everything in the corresponding way. All computations are straightforward (see [92] for details). □

Non-local Properties

One can try to find metric characterizations of classes of Banach spaces which are not local. (We say that a class $\mathcal{P}$ of Banach spaces is *not local* if the conditions (1) $X \in \mathcal{P}$ and (2) Y is finitely representable in X, do not necessarily imply that $Y \in \mathcal{P}$). Apparently this study should not be considered as a part of the Ribe program, and this direction has developed much more slowly than the directions related to the Ribe program. It is clear that even if we restrict our attention to properties which are hereditary (inherited by closed subspaces) and isomorphic invariant, the class of non-local properties which have been already studied in the literature is huge. I found in the literature only four properties for which the problem of metric characterization was ever considered asymptotic uniform convexity and smoothness, Radon-Nykodým property, reflexivity, infinite tree property. The goal of this section is to survey the corresponding results.

Asymptotic Uniform Convexity and Smoothness

One of the first results of the described type is the following result of Baudier–Kalton–Lancien, where by T_{∞}^{∞} we denote the tree defined similarly to the tree T_{∞}, but now we consider all possible finite sequences with terms in $\mathbb{N}$, and so degrees of all vertices of T_{∞}^{∞} are infinite.

Theorem 3.1 ([7]). *Let X be a reflexive Banach space. The following assertions are equivalent:*

- T_∞^∞ *admits a bilipschitz embedding into X.*
- *X does not admit any equivalent asymptotically uniformly smooth norm or X does not admit any equivalent asymptotically uniformly convex norm.*
- *The Szlenk index of X is* $> \omega$ *or the Szlenk index of* X^* *is* $> \omega$*, where* ω *is the first infinite ordinal.*

It is worth mentioning that Dilworth et al. [37] found an interesting geometric description of the class of Banach spaces whose metric characterization is provided by Theorem 3.1.

Radon–Nikodým Property

The *Radon–Nikodým property* (RNP) is one of the most important isomorphic invariants of Banach spaces. This class also plays an important role in the theory of metric embeddings, this role is partially explained by the fact that for this class one can use differentiability to prove non-embeddability results.

There are many expository works presenting results on the RNP, we recommend the readers (depending on the taste and purpose) one of the following sources [11, Chap. 5], [12, 17, 36, 103, 113].

Equivalent Definitions of RNP

One of the reasons for the importance of the RNP is the possibility to characterize (define) the RNP in many different ways. I would like to remind some of them:

- Measure-theoretic definition (it gives the name to this property) $X \in \text{RNP} \Leftrightarrow$ The following analogue of the Radon–Nikodým theorem holds for X-valued measures.
  - Let (Ω, Σ, μ) be a positive finite real-valued measure, and (Ω, Σ, τ) be an X-valued measure on the same σ-algebra which is absolutely continuous with respect to μ (this means $\mu(A) = 0 \Rightarrow \tau(A) = 0$) and satisfies the condition $\tau(A)/\mu(A)$ is a uniformly bounded set of vectors over all $A \in \Sigma$ with $\mu(A) \neq 0$. Then there is an $f \in L_1(\mu, X)$ such that

$$\forall A \in \Sigma \quad \tau(A) = \int_A f(\omega) d\mu(\omega).$$

- Definition in terms of differentiability (goes back to Clarkson [31] and Gelfand [41]) $X \in \text{RNP} \Leftrightarrow X$-valued Lipschitz functions on $\mathbb{R}$ are differentiable almost everywhere.

- Probabilistic definition [24] $X \in \text{RNP} \Leftrightarrow$ Bounded X-valued martingales converge.
  - In more detail: A Banach space X has the RNP if and only if each X-valued martingale $\{f_n\}$ on some probability space (Ω, Σ, μ), for which $\{||f_n(\omega)|| : n \in \mathbb{N}, \omega \in \Omega\}$ is a bounded set, converges in $L_1(\Omega, \Sigma, \mu, X)$.
- Geometric definition. $X \in \text{RNP} \Leftrightarrow$ Each bounded closed convex set in X is dentable in the following sense:
  - A bounded closed convex subset C in a Banach space X is called *dentable* if for each $\varepsilon > 0$ there is a continuous linear functional f on X and $\alpha > 0$ such that the set

$$\{y \in C : f(y) \geq \sup\{f(x) : x \in C\} - \alpha\}$$

    has diameter $< \varepsilon$.
- **Examples** (these lists are far from being exhaustive):
  - RNP: Reflexive (for example, L_p, $1 < p < \infty$), separable dual spaces (for example, ℓ_1).
  - non-RNP: c_0, $L_1(0, 1)$, nonseparable duals of separable Banach spaces.

RNP and Metric Embeddings

Cheeger–Kleiner [26] and Lee–Naor [66] noticed that the observation of Semmes [109] on the result of Pansu [98] can be generalized to maps of the Heisenberg group into Banach spaces with the RNP. This implies that Heisenberg group with its sub-Riemannian metric (see Definition 3.9) does not admit a bilipschitz embedding into any space with the RNP.

Cheeger–Kleiner [27] generalized some part of differentiability theory of Cheeger [25] (see also [59, 60]) to maps of metric spaces into Banach spaces with the RNP. This theory implies some non-embeddability results, for example, it implies that the Laakso space does not admit a bilipschitz embedding into a Banach space with the RNP.

Metric Characterization of RNP

In 2009 Johnson [112, Problem 1.1] suggested the problem: Find a purely metric characterization of the Radon–Nikodým property (that is, find a characterization of the RNP which does not refer to the linear structure of the space). The main goal of the rest of section "Radon–Nikodým Property" is to present such characterization.

It turns out that the RNP can be characterized in terms of *thick families of geodesics* defined as follows (different versions of this definition appeared in [91, 93, 94], the following seems to be the most suitable definition).

Definition 3.2 ([91, 94]). A family T of uv-geodesics is called *thick* if there is $\alpha > 0$ such that for every $g \in T$ and for every finite collection of points $r_1, \dots, r_n$ in the image of g, there is another uv-geodesic $\tilde{g} \in T$ satisfying the conditions:

(i) The image of $\tilde{g}$ also contains $r_1, \dots, r_n$ (we call these points *control points*).
(ii) Possibly there are some more common points of g and $\tilde{g}$.
(iii) There is a sequence $0 = q_0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m = d_M(u, v)$, such that $g(q_i) = \tilde{g}(q_i)$ $(i = 0, \dots, m)$ are common points containing $r_1, \dots, r_n$; and $\sum_{i=1}^{m} d_M(g(s_i), \tilde{g}(s_i)) \geq \alpha$.
(iv) Furthermore, each geodesic which on some intervals between the points $0 = q_0 < q_1 < q_2 < \cdots < q_m = d_M(u, v)$ coincides with g and on others with $\tilde{g}$ is also in T.

Example 3.3. Interesting and important examples of spaces having thick families of geodesics are the infinite diamond D_ω and the Laakso space L_ω, but now we consider them not as unions of finite sets, but as unions of geodesic metric spaces obtained from weighted $\{D_n\}_{n=0}^{\infty}$ and $\{L_n\}_{n=0}^{\infty}$ in which edges are identified with line segments of lengths $\{2^{-n}\}_{n=0}^{\infty}$ and $\{4^{-n}\}_{n=0}^{\infty}$, respectively. Observe that for such graphs there are also natural (although non-unique) isometric embeddings of D_n into D_{n+1} and L_n into L_{n+1}, and therefore the unions are well-defined. It is easy to check that the families of all geodesics in D_ω and L_ω joining the vertices of D_0 and L_0, respectively, are thick.

Theorem 3.4 ([91, 94]). *A Banach space X does not have the RNP if and only if there exists a metric space M_X containing a thick family T_X of geodesics which admits a bilipschitz embedding into X.*

Remark 3.5. Theorem 3.4 implies the result of Cheeger and Kleiner [27] on nonexistence of bilipschitz embeddings of the Laakso space into Banach spaces with the RNP.

It turns out that the metric space M_X whose existence is established in Theorem 3.4 cannot be chosen independently of X, because the following result holds:

Theorem 3.6 ([91]). *For each metric space M containing a thick family of geodesics there exists a Banach space X which does not have the RNP and does not admit a bilipschitz embedding of M.*

Because of Theorem 3.6 the following is an open problem:

Problem 3.7. *Can we characterize the RNP using test-spaces?*

Also I would like to mention the problem of the metric characterization of the RNP can have many different (correct) answers, so it is natural to try to find metric characterizations of the RNP in some other terms.

- Proof of Theorem 3.4 (in both directions) is based on the characterization of the RNP in terms of martingales. It will be presented in section "Proof of Theorem 3.4".
- It is not true that each Banach space without RNP contains a thick family of geodesics, because Banach spaces without RNP can have the uniqueness of geodesics property (consider a strictly convex renorming of a separable Banach space without RNP), so the words 'bilipschitz embedding' in Theorem 3.4 cannot be replaced by 'isometric embedding.'
- Proof of Theorem 3.6 is based on the construction of Bourgain and Rosenthal [15] of 'small' subspaces of $L_1(0,1)$ which still do not have the Radon–Nikodým property.

- Studying metric characterizations of the RNP, it would be much more useful and interesting to get a characterization of all metric spaces which do not admit bilipschitz embeddings into Banach spaces with the RNP.
- In view of Theorem 3.4 it is natural to ask: whether the presence of bilipschitz images of thick families of geodesics characterizes metric spaces which do not admit bilipschitz embeddings into Banach spaces with the RNP?
- It is clear that the answer to this question in full generality is negative: we may just consider a dense subset of a Banach space without the RNP which does not contain any continuous curves.
- So we restrict our attention to spaces containing sufficiently large collections of continuous curves. Our next result is a negative answer even in the case of geodesic metric spaces. Recall a metric space is called *geodesic* if any two points in it are joined by a geodesic.

Theorem 3.8 ([93]). *There exist geodesic metric spaces which satisfy the following two conditions simultaneously:*

- *Do not contain bilipschitz images of thick families of geodesics.*
- *Do not admit bilipschitz embeddings into Banach spaces with the Radon–Nikodým property.*

In [93] it was shown that the Heisenberg group with its sub-Riemannian metric is an example of such metric space. Let us recall the corresponding definitions.

Definition 3.9. The *Heisenberg group* $\mathbb{H}(\mathbb{R})$ can be defined as the group of real upper-triangular matrices with 1's on the diagonal:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

One of the ways to introduce the *sub-Riemannian metric on* $\mathbb{H}(\mathbb{R})$ is to find the tangent vectors of the curves produced by left translations in x and in y directions, that is,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \begin{bmatrix} 1 & \varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

All elements of $\mathbb{H}(\mathbb{R})$ can be regarded as elements of $\mathbb{R}^3$ with coordinates x, y, z. For $u, v \in \mathbb{H}(\mathbb{R})$ we consider the set of all differentiable curves joining u and v with the restriction that the tangent vector at each point of the curve is a linear combination of the two tangent vectors computed above. Finally we introduce the distance between u and v as the infimum of lengths (in the usual Euclidean sense) of projections onto xy-plane of all such curves.

This metric has been systematically studied (see [23, 47, 80]), it has very interesting geometric properties. The Heisenberg group $\mathbb{H}(\mathbb{R})$ with its subriemannian metric is a very important example for Metric Geometry and its applications to Computer Science. One of the reasons for this is its poor embeddability into many classes of Banach spaces. As we already mentioned, Cheeger–Kleiner [26] and Lee–Naor [66] proved that the Heisenberg group does not admit a bilipschitz embedding into a Banach space with the RNP. It remains to show that it does not admit a bilipschitz embedding of a thick family of geodesics.

Remark 3.10. It is not needed for our argument, but is worth mentioning that

- Cheeger–Kleiner [28] proved that $\mathbb{H}(\mathbb{R})$ does not admit a bilipschitz embedding into $L_1(0, 1)$.
- Cheeger–Kleiner–Naor [29] found quantitative versions of the previous result for embeddings of finite subsets of $\mathbb{H}(\mathbb{R})$ into $L_1(0, 1)$. These quantitative results are important for Theoretical Computer Science.

We finish the proof of Theorem 3.8 by using the notion of Markov convexity (Definition 2.26), proving

Theorem 3.11 ([93]). *A metric space with a thick family of geodesics is not Markov p-convex for any $p \in (0, \infty)$.*

and combining it with the following result:

Theorem 3.12 ([68, 69]). *The Heisenberg group $\mathbb{H}(\mathbb{R})$ is Markov 4-convex.*

Remark 3.13. Since the infinite diamond D_ω and the Laakso space L_ω contain thick families of geodesics, they are not Markov p-convex for any $p \in (0, \infty)$. Since the unions of $\{D_n\}_{n=0}^{\infty}$ and $\{L_n\}_{n=0}^{\infty}$ (considered as finite sets) are dense in D_ω and L_ω, respectively; we conclude that Markov p-convexity constants of diamond graphs and Laakso graphs are not uniformly bounded for any $p \in (0, \infty)$.

Remark 3.14. It is worth mentioning that the discrete Heisenberg group $\mathbb{H}(\mathbb{Z})$ embeds into a Banach space with the RNP. Since $\mathbb{H}(\mathbb{Z})$ is locally finite, this follows by combining the well-known observation of Fréchet on isometric embeddability of any n-element set into ℓ_∞^n (see [90, p. 6]) with the finite determination (Theorem 2.41, actually the earlier result of [6] suffices here). In fact, these results imply bilipschitz embeddability of $\mathbb{H}(\mathbb{Z})$ into the direct sum $(\oplus_{n=1}^{\infty}\ell_\infty^n)_2$, which has the RNP because it is reflexive.

Proof of Theorem 3.4

First we prove: **No RNP $\Rightarrow$ bilipschitz embeddability of a thick family of geodesics.**

We need to define a more general structure than that of a δ-tree (see Definition 2.22), in which each element is not a midpoint of a line segment, but a convex combination.

Definition 3.15. Let Z be a Banach space and let $\delta > 0$. A set of vectors $\{z_{n,j}\}_{n=0,j=1}^{\infty\ \ m_n}$ in Z is called a δ-*bush* if $m_0 = 1$ and for every $n \geq 1$ there is a partition $\{A_k^n\}_{k=1}^{m_{n-1}}$ of $\{1, \ldots, m_n\}$ such that

$$||z_{n,j} - z_{n-1,k}|| \geq \delta \tag{7}$$

for every $n \geq 1$ and for every $j \in A_k^n$, and

$$z_{n-1,k} = \sum_{j \in A_k^n} \lambda_{n,j} z_{n,j} \tag{8}$$

for some $\lambda_{n,j} \geq 0$, $\sum_{j\in A_k^n} \lambda_{n,j} = 1$.

Theorem 3.16. *A Banach space Z does not have the RNP if and only if it contains a bounded δ-bush for some $\delta > 0$.*

Remark 3.17. Theorem 3.16 can be derived from Chatterji's result [24]. Apparently Theorem 3.16 was first proved by James, possibly even before Chatterji, see [57].

- We construct a suitable thick family of geodesics using a bounded δ-bush in a Banach space without the RNP.
- It is not difficult to see (for example, using the Clarkson–Gelfand characterization) that a subspace of codimension 1 (hyperplane) in a Banach space without the RNP does not have the RNP.
- Let X be a non-RNP Banach space. We pick a norm-one vector $x \in X$, then a norm-one functional x^* on X satisfying $x^*(x) = 1$. Then (by the previous remark) we find a bounded δ-bush (for some $\delta > 0$) in the kernel $\ker x^*$. We shift this bush adding x to each of its elements, and get a (still bounded) δ-bush $\{x_{i,j}\}$ satisfying the condition $x^*(x_{i,j}) = 1$ for each i and j.

- Now we change the norm of X to equivalent. The purpose of this step is to get a norm for which we are able to construct the thick family of geodesics in X, and there will be no need in a bilipschitz embedding. (One can easily see that this would be sufficient to prove the theorem.)
- The unit ball of the norm $||\cdot||_1$ is defined as the closed convex hull of the unit ball in the original norm and the set of vectors $\{\pm x_{i,j}\}$ (recall that $\{x_{i,j}\}$ form a bush in the hyperplane $\{x:\ x^*(x)=1\}$). It is easy to check that in this new norm the set $\{x_{i,j}\}$ is a bounded δ-bush (possibly with a somewhat smaller $\delta>0$, but we keep the same notation). Also in the new norm we have $||x_{i,j}||_1=1$ for all i and j. For simplicity of notation we shall use $||\cdot||$ to denote the new norm.
- We are going to use this δ-bush to construct a thick family T_X of geodesics in X joining 0 and $x_{0,1}$. First we construct a subset of the desired set of geodesics, this subset will be constructed as the set of limits of certain broken lines in X joining 0 and $x_{0,1}$. The constructed broken lines are also geodesics (but they do not necessarily belong to the family T_X).
- The mentioned above broken lines will be constructed using representations of the form $x_{0,1}=\sum_{i=1}^m z_i$, where z_i are such that $||x_{0,1}||=\sum_{i=1}^m ||z_i||$. The broken line represented by such finite sequence $z_1,\dots,z_m$ is obtained by letting $z_0=0$ and joining $\sum_{i=0}^k z_i$ with $\sum_{i=0}^{k+1} z_i$ with a line segment for $k=0,1,\dots,m-1$. Vectors $\sum_{i=0}^k z_i$, $k=0,1,\dots,m$ will be called *vertices* of the broken line.
- The infinite set of broken lines which we construct is labelled by vertices of the infinite binary tree T_∞ in which each vertex is represented by a finite (possibly empty) sequence of 0 and 1.
- The broken line corresponding to the empty sequence $\emptyset$ is represented by the one-element sequence $x_{0,1}$, so it is just a line segment joining 0 and $x_{0,1}$.
- We have

$$x_{0,1}=\lambda_{1,1}x_{1,1}+\cdots+\lambda_{1,m_1}x_{1,m_1},$$

where $||x_{1,j}-x_{0,1}||\ge\delta$. We introduce the vectors

$$y_{1,j}=\frac{1}{2}(x_{1,j}+x_{0,1}).$$

- For these vectors we have

$$x_{0,1}=\lambda_{1,1}y_{1,1}+\cdots+\lambda_{1,m_1}y_{1,m_1},$$

$||y_{1,j}-x_{1,j}||=||y_{1,j}-x_{0,1}||\ge\frac{\delta}{2}$, and $||y_{1,j}||=1$.

- As a preliminary step to the construction of the broken lines corresponding to one-element sequences (0) and (1) we form a broken line represented by the points

$$\lambda_{1,1}y_{1,1},\dots,\lambda_{1,m_1}y_{1,m_1}. \tag{9}$$

We label the broken line represented by (9) by $\overline{\emptyset}$.

- The broken line corresponding to the one-element sequence (0) is represented by the sequence obtained from (9) if we replace each term $\lambda_{1,j}y_{1,j}$ by a two-element sequence

$$\frac{\lambda_{1,j}}{2}x_{0,1}, \frac{\lambda_{1,j}}{2}x_{1,j}. \tag{10}$$

- The broken line corresponding to the one-element sequence (1) is represented by the sequence obtained from (9) if we replace each term $\lambda_{1,j}y_{1,j}$ by a two-element sequence

$$\frac{\lambda_{1,j}}{2}x_{1,j}, \frac{\lambda_{1,j}}{2}x_{0,1}. \tag{11}$$

- At this point one can see where are we going to get the thickness property from.
- In fact, one of the inequalities above is $||x_{1,j} - x_{0,1}|| \geq \delta$. Therefore

$$\left\| \frac{\lambda_{1,j}}{2}x_{1,j} - \frac{\lambda_{1,j}}{2}x_{0,1} \right\| \geq \frac{\lambda_{1,j}}{2}\,\delta.$$

  Summing over all j, we get that the total sum of deviations is $\geq \frac{\delta}{2}$.
- In the obtained broken lines each line segment corresponds either to a multiple of $x_{0,1}$ or to a multiple of some $x_{1,j}$. In the next step we replace each such line segment by a broken line. Now we describe how we do this.
- Broken lines corresponding to 2-element sequences are also formed in two steps. To get the broken lines labelled by $(0, 0)$ and $(0, 1)$ we apply the described procedure to the geodesic labelled (0), to get the broken lines labelled by $(1, 0)$ and $(1, 1)$ we apply the described procedure to the geodesic labelled (1).
- In the first step we replace each term of the form $\frac{\lambda_{1,k}}{2}x_{0,1}$ by a multiplied by $\frac{\lambda_{1,k}}{2}$ sequence $\lambda_{1,1}y_{1,1}, \dots, \lambda_{1,m_1}y_{1,m_1}$, and we replace a term of the form $\frac{\lambda_{1,k}}{2}x_{1,k}$ by the multiplied by $\frac{\lambda_{1,k}}{2}$ sequence

$$\{\lambda_{2,j}y_{2,j}\}_{j \in A_k^2}, \tag{12}$$

  ordered arbitrarily, where $y_{2,j} = \frac{x_{1,k}+x_{2,j}}{2}$ and $\lambda_{2,j}$, $x_{2,j}$, and A_k^2 are as in the definition of the δ-bush (it is easy to check that in the new norm we have $||y_{2,j}|| = 1$). We label the obtained broken lines by $\overline{(0)}$ and $\overline{(1)}$, respectively.

- To get the sequence representing the broken line labelled by $(0, 0)$ we do the following operation with the preliminary sequence labelled $\overline{(0)}$.

  – Replace each multiple $\lambda y_{1,j}$ present in the sequence by the two-element sequence

$$\lambda\frac{x_{0,1}}{2}, \lambda\frac{x_{1,j}}{2}. \tag{13}$$

– Replace each multiple $\lambda y_{2,j}$, with $j \in A_k^2$, present in the sequence by the two-element sequence

$$\lambda \frac{x_{1,k}}{2}, \lambda \frac{x_{2,j}}{2}. \tag{14}$$

- To get the sequence representing the broken line labelled by $(0, 1)$ we do the same but changing the order of terms in (13) and (14). To get the sequences representing the broken lines labelled by $(1, 0)$ and $(1, 1)$, we apply the same procedure to the broken line labelled $\overline{(1)}$.

- We continue in an "obvious" way and get broken lines for all vertices of the infinite binary tree T_∞. It is not difficult to see that vertices of a broken line corresponding to some vertex $(\theta_1, \dots, \theta_n)$ are contained in the broken line corresponding to any extension $(\theta_1, \dots, \theta_m)$ of $(\theta_1, \dots, \theta_n)$ $(m > n)$.
- This implies that broken lines corresponding to any ray (that is, a path infinite in one direction) in T_∞ has a limit (which is not necessarily a broken line, but is a geodesic), and limits corresponding to two different infinite paths have common points according to the number of common $(\theta_1, \dots, \theta_n)$ in the vertices of those paths.
- A thick family of geodesics is obtained by pasting pieces of these geodesics in all "reasonable" ways. All verifications are straightforward; see the details in [94].

It remains to prove:

Bilipschitz embeddability of a thick family of geodesics ⇒ No RNP.

Proof. We assume that a metric space (M, d) with a thick family of geodesics admits a bilipschitz embedding $f : M \to X$ into a Banach space X and show that there exists a bounded divergent martingale $\{M_i\}_{i=0}^\infty$ on $(0, 1]$ with values in X. We assume that

$$\ell d(x, y) \le ||f(x) - f(y)||_X \le d(x, y) \tag{15}$$

for some $\ell > 0$. We assume that the thick family consists of uv-geodesics for some $u, v \in M$ and that $d(u, v) = 1$ (dividing all distances in M by $d(u, v)$, if necessary).

Each function in the martingale $\{M_i\}_{i=0}^\infty$ will be obtained in the following way. We consider some finite sequence $V = \{v_i\}_{i=0}^m$ of points on any uv-geodesic, satisfying $v_0 = u$, $v_m = v$ and $d(u, v_{k+1}) \ge d(u, v_k)$. We define M_V as the function on $(0, 1]$ whose value on the interval $(d(u, v_k), d(u, v_{k+1})]$ is equal to

$$\frac{f(v_{k+1}) - f(v_k)}{d(v_k, v_{k+1})}.$$

It is clear that the bilipschitz condition (15) implies that $||M_V(t)|| \le 1$ for any collection V and any $t \in (0, 1]$. Since $\{v_i\}$ are on a geodesic, is clear that an infinite collection of such functions $\{M_{V(k)}\}_{k=0}^\infty$ forms a martingale if for each $k \in \mathbb{N}$ the sequence $V(k)$ contains $V(k-1)$ as a subsequence. So it remains to find a collection of sequences $\{V(k)\}_{k=0}^\infty$ for which the martingale $\{M_{V(k)}\}_{k=0}^\infty$ diverges. We denote $M_{V(k)}$ by M_k.

Now we describe some of the ideas of the construction.

- It suffices to have differences $||M_k - M_{k-1}||$ to be bounded away from zero for some infinite set of values of k.
- On steps for which we achieve such estimates from below we add exactly one new point z'_j into $V(k)$ between any two consequent points w_{j-1} and w_j of $V(k-1)$. In such a case it suffices to make the choice of points in such a way that the values of M_k on the intervals corresponding to pairs (w_{j-1}, z'_j) and (z'_j, w_j) are 'far' from each other, and thus from the value of M_{k-1} corresponding to (w_{j-1}, w_j). Actually we do not need this condition for each pair (w_{j-1}, w_j), but only "on average."
- Using the definition of a thick family of geodesics and the bilipschitz condition, we can achieve this goal. A detailed description follows.

We let $V(0) = \{u, v\}$ and so M_0 is a constant function on $(0, 1]$ taking value $f(v)-f(u)$. In the next step we apply the condition of the definition of a thick family to control points $\{u, v\}$ and any geodesic g of the family. We get another geodesic $\tilde{g}$, the corresponding sequence of common points $\{w_i\}_{i=0}^m$ and the corresponding pair of sufficiently well-separated sequences $\{z_i, \tilde{z}_i\}_{i=1}^m$ on the geodesics g and $\tilde{g}$. The separation condition is $\sum_{i=1}^m d(z_i, \tilde{z}_i) \geq \alpha$.

We let $V(1) = \{w_i\}_{i=0}^m$. Observe that in this step we cannot claim any nontrivial estimates for $||M_1 - M_0||_{L_1(X)}$ from below because we have not made any nontrivial assumptions on this step of the construction (it can even happen that $M_1 = M_0$). Lower estimates for martingale differences in our argument are obtained only for differences of the form $||M_{2k} - M_{2k-1}||_{L_1(X)}$.

We choose $V(2)$ to be of the form

$$w_0, z'_1, w_1, z'_2, w_n, \ldots, z'_m, w_m, \tag{16}$$

where each z'_i is either z_i or $\tilde{z}_i$ depending on the behavior of the mapping f. We describe this dependence below. Observe that since z_i or $\tilde{z}_i$ are images of the same point in $[0, 1]$, the corresponding partition of the interval $(0, 1]$ does not depend on our choice.

To make the choice of z'_i we consider the quadruple $w_{i-1}, z_i, w_i, \tilde{z}_i$. The bilipschitz condition (15) implies $||f(z_i) - f(\tilde{z}_i)|| \geq \ell d(z_i, \tilde{z}_i)$. Consider two pairs of vectors corresponding to two different choices of z'_i:

Pair 1: $f(w_i) - f(z_i), f(z_i) - f(w_{i-1})$. **Pair 2:** $f(w_i) - f(\tilde{z}_i), f(\tilde{z}_i) - f(w_{i-1})$.

The inequality $||f(z_i) - f(\tilde{z}_i)|| \geq \ell d(z_i, \tilde{z}_i)$ implies that at least one of the following is true:

$$\left\|\frac{f(w_i) - f(z_i)}{d(w_i, z_i)} - \frac{f(z_i) - f(w_{i-1})}{d(z_i, w_{i-1})}\right\| \geq \frac{\ell}{2} d(z_i, \tilde{z}_i)\left(\frac{1}{d(w_i, z_i)} + \frac{1}{d(z_i, w_{i-1})}\right) \tag{17}$$

or

$$\left\|\frac{f(w_i) - f(\tilde{z}_i)}{d(w_i, \tilde{z}_i)} - \frac{f(\tilde{z}_i) - f(w_{i-1})}{d(\tilde{z}_i, w_{i-1})}\right\| \geq \frac{\ell}{2} d(z_i, \tilde{z}_i)\left(\frac{1}{d(w_i, \tilde{z}_i)} + \frac{1}{d(\tilde{z}_i, w_{i-1})}\right). \tag{18}$$

Since in the definition of a thick family of geodesics we have $\sum_i d(z_i, \tilde{z}_i) \geq \alpha$, these inequalities show that if we choose z'_i to be the one of z_i and $\tilde{z}_i$, for which

$\frac{f(w_i)-f(z_i')}{d(w_i,z_i')}$ and $\frac{f(z_i')-f(w_{i-1})}{d(z_i',w_{i-1})}$ are more distant from each other, we have a chance to get the desired condition. (This is what we verify below.)

We pick z_i' to be z_i if the left-hand side of (17) is larger than the left-hand side of (18), and pick $z_i' = \tilde{z}_i$ otherwise.

Let us estimate $||M_2 - M_1||_1$. First we estimate the part of this difference corresponding to the interval $(d(w_0, w_{i-1}), d(w_0, w_i)]$. Since the restriction of M_2 to the interval $(d(w_0, w_{i-1}), d(w_0, w_i)]$ is a two-valued function, and M_1 is constant on the interval, the integral

$$\int_{d(w_0,w_{i-1})}^{d(w_0,w_i)} ||M_2 - M_1||dt \tag{19}$$

can be estimated from below in the following way. Denote the value of M_2 on the first part of the interval by x, the value on the second by y, the value of M_1 on the whole interval by z, the length of the first interval by A and of the second by B. We have: the desired integral is equal to $A||x - z|| + B||y - z||$ and therefore can be estimated in the following way:

$$\begin{aligned} A||x-z|| + B||y-z|| &\geq \max\{||x-z||, ||y-z||\} \cdot \min\{A, B\} \\ &\geq \frac{1}{2}||x-y|| \min\{A, B\}. \end{aligned}$$

Therefore, assuming without loss of generality that the left-hand side of (17) is larger than the left-hand side of (18), we can estimate the integral in (19) from below by

$$\begin{aligned} &\frac{1}{2}\left\| \frac{(f(w_i)-f(z_i))}{d(w_i,z_i)} - \frac{(f(z_i)-f(w_{i-1}))}{d(z_i,w_{i-1})} \right\| \cdot \min\{d(w_i,z_i), d(z_i,w_{i-1})\} \\ &\quad \geq \frac{1}{4}\,\ell\, d(z_i,\tilde{z}_i)\left(\frac{1}{d(w_i,z_i)} + \frac{1}{d(z_i,w_{i-1})}\right) \cdot \min\{d(w_i,z_i), d(z_i,w_{i-1})\} \\ &\quad \geq \frac{1}{4}\,\ell\, d(z_i,\tilde{z}_i). \end{aligned}$$

Summing over all intervals and using the condition $\sum_{i=1}^{m} d(z_i, \tilde{z}_i) \geq \alpha$, we get $||M_2 - M_1|| \geq \frac{1}{4}\ell\alpha$.

Now we recall that the last condition of the definition of a thick family of geodesics implies that

$$w_0, z_1', w_1, z_2', w_2, \ldots, z_m', w_m, \tag{20}$$

where each z_i' is either z_i or $\tilde{z}_i$ depending on the choice made above belongs to some geodesic in the family.

We use all of points in (20) as control points and find new sequence $\{w_i^2\}_{i=0}^{m_2}$ of common points and a new sequence of pairs $\{z_i^2, \tilde{z}_i^2\}_{i=1}^{m_2}$ with substantial separation: $\sum_{i=1}^{m_2} d(z_i^2, \tilde{z}_i^2) \geq \alpha$.

We use $\{w_i^2\}_{i=0}^{m_2}$ to construct M_3 and the suitably selected sequence

$$w_0^2, z_1'^2, w_1^2, z_2'^2, w_2^2, \dots, z_{m_2}'^2, w_{m_2}^2$$

to construct M_4. We continue in an obvious way. The inequalities $||M_{2k} - M_{2k-1}|| \geq \frac{1}{4}\ell\alpha$ imply that the martingale is divergent. □

Reflexivity

Problem 3.18. *Is it possible to characterize the class of reflexive spaces using test-spaces?*

Some comments on this problem:

Remark 3.19. It is worth mentioning that a metric space (or spaces) characterizing in the described sense reflexivity or the Radon–Nikodým property cannot be uniformly discrete (that is, cannot satisfy $\inf_{u \neq v} d(u, v) > 0$). This statement follows by combining the example of Ribe [107] of Banach spaces belonging to these classes which are uniformly homeomorphic to Banach spaces which do not belong to the classes, and the well-known fact (Corson–Klee [32]) that uniformly continuous maps are Lipschitz for (nontrivially) "large" distances.

I noticed that combining two of the well-known characterizations of reflexivity (Pták [104]—Singer [110]—Pełczyński [99]—James [54]—Milman and Milman [79]) and some differentiation theory (Mankiewicz [72]—Christensen [30]—Aronszajn [2], see also presentation in [11]) we get a purely metric characterization of reflexivity. This characterization can be described as a submetric test-space characterization of reflexivity:

Definition 3.20. A *submetric test-space* for a class $\mathcal{P}$ of Banach spaces is defined as a metric space T with a marked subset $S \subset T \times T$ such that the following conditions are equivalent for a Banach space X:

1. $X \notin \mathcal{P}$.
2. There exist a constant $0 < C < \infty$ and an embedding $f : T \to X$ satisfying the condition

$$\forall (x, y) \in S \quad d_T(x, y) \leq ||f(x) - f(y)|| \leq C d_T(x, y). \tag{21}$$

An embedding satisfying (21) is called a *partially bilipschitz* embedding. Pairs (x, y) belonging to S are called *active*.

Let $\Delta \geq 1$. The submetric space X_Δ is the space ℓ_1 with its usual metric. The only thing which makes it different from ℓ_1 is the set of active pairs S_Δ: A pair $(x, y) \in X_\Delta \times X_\Delta$ is active if and only if

$$||x-y||_1 \leq \Delta ||x-y||_s, \tag{22}$$

where $||\cdot||_s$ is the summing norm, that is,

$$||\{a_i\}_{i=1}^{\infty}||_s = \sup_k \left| \sum_{i=1}^{k} a_i \right|.$$

Theorem 3.21 ([91]). *X_Δ, $\Delta \geq 2$ is a submetric test-space for reflexivity.*

The proof goes as follows. Let Z be a nonreflexive space. If you know the characterization of reflexivity which I meant, you see immediately that it implies that the space ℓ_1 admits a partially bilipschitz embedding into Z with the set of active pairs described as above.

The other direction. If ℓ_1 admits a partially bilipschitz embedding with the described set of active pairs, then the embedding is Lipschitz on ℓ_1, because each vector in ℓ_1 is a difference of two positive vectors.

Now, if Z does not have the Radon–Nikodým property (RNP), we are done (Z is nonreflexive). If Z has the RNP, we use the result of Mankiewicz–Christensen–Aronszajn and find a point of Gâteaux differentiability of this embedding. The Gâteaux derivative is a bounded linear operator which is "bounded below in certain directions." Using this we can get a sequence in Z which, after application of the non-reflexivity criterion (due to Pták—Singer—Pełczyński—James—D.&V. Milman), implies non-reflexivity of Z; see [91] for details.

Infinite Tree Property

See Definition 2.22 for the definition of the infinite tree property. Using a bounded δ-tree in a Banach space X one can easily construct a bounded divergent X-valued martingale. Hence the infinite tree property implies non-RNP. For some time it was an open problem whether the infinite tree property coincides with non-RNP. A counterexample was constructed by Bourgain and Rosenthal [15] in the paper mentioned above. The infinite tree property admits the following metric characterization.

Theorem 3.22 ([91]). *The class of Banach spaces with the infinite tree property admits a submetric characterization in terms of the metric space D_ω with the set S_ω of active pairs defined as follows: a pair is active if and only if it is a pair of vertices of a quadrilateral introduced in one of the steps.*

It would be interesting to answer the following open problem:

Problem 3.23. *Whether the infinite diamond D_ω is a test-space for the infinite tree property?*

Remark 3.24. It is worth mentioning that if we restrict our attention to **dual Banach spaces**, the following three properties are equivalent:

(1) Non-RNP.
(2) Infinite tree property.
(3) Bilipschitz embeddability of D_ω.

The implication (1) $\Rightarrow$ (2) is due to Stegall [111]. The implication (2) $\Rightarrow$ (1) follows from Chatterji [24]. The equivalence of (1) and (3) was proved in [91].

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