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A Robust Optimization Approach to Pension Fund Management

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Introduction

Pension plans in the United States come in two varieties. Defined contribution pension plans specify the contribution of the corporation. The employees have the right to invest the corporation's contribution and their own contribution in a limited set of funds. The participants in a defined contribution pension plan are responsible for making all the investment decisions and bear all the risks associated with these decisions; thus, the benefit to the participants is uncertain. In contrast,

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defined benefit pension plans specify the benefits due to plan participants. The plan sponsor, that is the corporation, makes all the investment decisions in a defined benefit pension plan and bears all the investment risk. Defined benefit plans have been in the news in the past few years because some firms face the prospect of bankruptcy over severely underfunded pension plans. Consequently, there is a need to develop models that account for uncertainty in future market conditions and plan accordingly.

Pension fund management is an instance of the asset-liability management problem (see, for example, Consigli and Dempster, 1998; Klaassen, 1998; Drijver *et al,* 2000; Sodhi, 2005) in which the goal of the decision maker is to manage the capital invested into a set of assets in order to meet obligations at the minimum possible cost. The typical modeling paradigm adopted in the literature is to model the uncertainty in market conditions as random variables with a known distribution, formulate the asset-liability management problem (and, hence, also the specific case of the pension fund management problem) as a stochastic program, and solve the problem by sampling the market conditions from the given distributions. All sampling-based methods suffer from the curse-of-dimensionality and become intractable as the number of decisions increases, that is either the number of assets in the portfolio or the number of decision epoch increases. In this article, we propose a robust optimization-based approach as an alternative to the stochastic programming based-methods.

Robust optimization is a methodology for explicitly incorporating the effect of parameter uncertainty in optimization problems (Ben-Tal *et al,* 2000; Ben-Tal and Nemirovski, 2001). In this approach, the parameter values are assumed to belong to known and bounded uncertainty sets, and the solution is computed assuming the worst-case behavior of the parameters. Thus, robust solutions are conservative. This is particularly appropriate for pension fund management. Typically, the uncertainty sets correspond to confidence regions around point estimates of the parameters; consequently, one is able to provide probabilistic guarantees on the performance of the robust solution. For a very large class of uncertainty sets, the computational effort required to solve the robust optimization problem is polynomial in the size of the problem (Ben-Tal and Nemirovski, 2001; Goldfarb and Iyengar, 2003) – in contrast, the computational complexity of the stochastic programming-based methods is exponential in the problem size. Consequently, robust methods are likely to become a computationally tractable alternative to stochastic programming-based methods.

A pension fund management problem involves optimizing a given objective, for example minimizing the discounted value of all contributions, while ensuring that the fund is always able to meet its liabilities. In addition, the fund's holdings must also satisfy regulatory requirements. We assume that the parameters of the financial markets of relevance to pension fund management, for example the yield curve, the expected return and volatility on an equity index and so on, are described by factors that evolve according to a stochastic differential equation. In this setting, we show that the pension fund management problem can be formulated as a chance-constrained optimization problem. However, the random variables in the chance constraints are nonlinear functions of the underlying factors. We use the Itô-Taylor expansion to linearize the nonlinear chance constraints and show that the linearized chance constraints can be approximated by second-order cone (SOC) constraints. Thus, the pension fund management problem can be approximated by a second-order cone program (SOCP). This implies that very large-scale problems can be solved efficiently both in theory (Alizadeh and Goldfarb, 2003) and in practice (Andersen and Andersen, 2006). Moreover, as a number of commercial solvers, such as MOSEK, CPLEX and Frontline System (supplier of EXCEL SOLVER), provide the capability for solving SOCPs in a numerically robust manner, we expect the robust approach to become the method of choice for solving large-scale pension fund problems.

The rest of the article is organized as follows. In the section 'Robust pension fund management', we show how to use linearization and robust optimization techniques to formulate general pension fund management problems as a SOCPs. In the section 'Numerical example', we report the results of our numerical experiments with a frozen fund and illustrate the robustness of the robust optimization solution. In the 'Concluding remarks' section, we include some concluding remarks.

Robust pension fund management

In this section, we present a robust optimization-based framework for pension fund management. As pension funds evaluate and re-balance their portfolio holdings at best on a quarterly basis, we work with a discrete time model. In this section, we discuss a general framework for approximating the typical constraints and objectives by second-order constraints; we consider a concrete example in the 'Numerical example' section.

Constraints

At each decision epoch $t \in \{0,1,\ldots, T\}$, the pension manager has to make two decisions: select a new portfolio of traded assets and decide the amount of fresh capital to be injected into the fund. Let $\mathbf{x}_t$ denote the number of shares of the traded assets held by the pension fund from time t to time $t + 1$, that is over period t, let w_t denote the fresh capital injected into the fund at time t, and let $\tilde{l}_t$ denote the random liability of the pension fund at time t. Then, assuming that the trading costs are negligible, we must have:

$$\tilde{\mathbf{p}}_t^T(\mathbf{x}_{t-1} - \mathbf{x}_t) + w_t - \tilde{l}_t \geq 0, \tag{1}$$

where $\tilde{\mathbf{p}}_t$ denotes the random prices for the traded assets at time t. As the price $\tilde{\mathbf{p}}_t$ is random, and typically has support on the entire positive orthant, one has to ascribe a proper meaning to the uncertain constraint (1). In this article, we approximate the uncertain liability constraint (1) at time t by the chance constraint

$$\mathrm{P}(\tilde{\mathbf{p}}_t^T(\mathbf{x}_{t-1} - \mathbf{x}_t) + w_t - \tilde{l}_t \geq 0) \geq 1 - \varepsilon, \tag{2}$$

where P denotes the probability measure conditioned on all available information and $\varepsilon > 0$ is the constraint violation probability. Note that we are implicitly assuming that when the event $\tilde{\mathbf{p}}_t^T(\mathbf{x}_{t-1} - \mathbf{x}_t) + w_t < \tilde{l}_t$ occurs, the fund sponsor is able to meet the shortfall using earnings or raising debt. We discuss this in greater detail in the next section on pension fund objectives.

In addition to the budget constraint (1), pension fund holding must also satisfy some regulatory requirements. These requirements typically impose constraints of the form

$$\tilde{\mathbf{p}}_t^T \mathbf{x}_t \geq \beta \left(\sum_{\tau > t} \frac{\tilde{l}_\tau}{(1+\tilde{d})^\tau} \right),$$

where $\tilde{d}$ denotes the (possibly stochastic) nominal interest rate set by the regulatory body and β is a specified funding level. We approximate this uncertain constraint by the chance constraint

$$\mathrm{P}\left(\tilde{\mathbf{p}}_t^T \mathbf{x}_t \geq \beta \left(\sum_{\tau > t} \frac{\tilde{l}_\tau}{(1+\tilde{d})^\tau} \right) \right) \geq 1 - \varepsilon.$$

Thus, the generic chance constraint encountered in the pension fund management problem is of the form

$$\mathrm{P}(\tilde{\mathbf{a}}_t^T \mathbf{y}_t \geq d_t) \geq 1 - \varepsilon, \tag{3}$$

where $\tilde{\mathbf{a}}_t$ denotes stochastic parameters such as the prices of assets, liabilities, discount factors and so on, and $\mathbf{y}_t$ and d_t are affine functions of the decision variables $\{(\mathbf{x}_t, w_t)\}_{t=1}^T$.

We assume that the stochastic parameters are described by a factor model:

$$\tilde{\mathbf{a}}_t = f(\mathbf{Z}_t), \tag{4}$$

where f is a sufficiently smooth function mapping the m stochastic factors $\mathbf{Z}_t \in \Re^m$ into the random coefficients $\tilde{\mathbf{a}}_t$, and the m-dimensional vector of factors $\mathbf{Z}_t \in \Re^m$ evolves according to the stochastic differential equation

$$\mathrm{d}\mathbf{Z}_t = \boldsymbol{\mu}(t, \mathbf{Z}_t)\mathrm{d}t + \boldsymbol{\Sigma}(t, \mathbf{Z}t)\mathrm{d}\mathbf{W}_t, \tag{5}$$

where $\boldsymbol{\mu}(t, \mathbf{Z}_t) \in \Re^{m\times n}$, and $\boldsymbol{\Sigma}(t, \mathbf{Z}t) \in \Re^{m\times n}$, and n denotes the length of the vector of standard Brownian motions $\mathbf{W}_t$. Most popular financial models in the literature satisfy (4)–(5). For example, it is easy to show that when the universe of assets is a set of treasury bonds and the equity index, the short rates are given by the Hull–White model (Hull and White, 1990), and the equity index evolves according to a geometric Brownian motion, then the price process $\tilde{\mathbf{p}}_t$ for the asset satisfies (4)–(5).

Objective

The most obvious objective for managing a pension fund is to minimize the net present value of all the future contributions:

$$\min \sum_t w_t B_{0,t}, \tag{6}$$

where $B_{0,t}$ denotes the price at time 0 of a zero-coupon bond with face value $F = 1$ maturing at time t. Defined benefit pension funds most often use this objective.

The objective (6) does not account for the impact of the pension contributions on the fund's sponsor. There is evidence that pension contributions w_t have a serious impact on the stock price of the sponsor (Jin *et al,* 2006). We next discuss an objective that explicitly accounts for the impact of the pension fund on the sponsor. The Myers and Majluf pecking order hypothesis (Myers and Majluf, 1984) suggests that the sponsor would first use earnings, and then use debt to finance the pension contributions $\{w_t\}$. We assume that the firm will not be able to issue equity for the purpose of meeting its pension obligations. Suppose w_t^e denotes the portion of the pension fund contribution w_t that is financed directly from the firm's earnings C_t before interest and tax (EBIT). We assume that the earnings C_0 at time $t = 0$ are known and the portion of the earnings invested in the firm grows at a rate r_e. Thus,

$$C_{t+1} = (C_t - w_t^e)(1 + r_e). \tag{7}$$

We also impose the additional constraint that $w_t^e \leq uC_t$, where $u \in [0, 1]$ indicates the maximum fraction of the earnings that can be used for funding pension obligations.

Let w_t^d denote the amount raised in the debt market at time t. We assume that this debt has maturity $D = 1$. Thus, at time $t + 1$, the firm has to repay $(1 + (s_{t,1} + P))w_d$, where $s_{t,1}$ denotes the spot risk-free interest rate at time t for maturity $D = 1$ and P denotes the spread over the risk-free rate that the sponsoring firm needs to pay to raise capital. As interest payments are tax deductible, the effective cost incurred by the firm at time $t + 1$ is $(1 + (1 - \alpha_T)(s_{t,1} + P))w_d$, where α_T denotes the marginal tax rate of the firm. Thus, the discounted cost $c_t^d(P)$ of raising an amount w_t^d in the debt market is given by

$$\begin{aligned} c_t^d(P) &= (1 + (1 - \alpha_T)(s_{t,1} + P))B_{0,t+1}w_t^d \\ &= ((1 - \alpha_T)(1 + s_{t,1})B_{0,t+1} + ((1 - \alpha_T)P + \alpha_T)B_{0,t+1})w_t^d \\ &= ((1 - \alpha_T)B_{0,t} + ((1 - \alpha_T) \times P + \alpha_T)B_{0,t+1})w_t^d, \end{aligned} \tag{8}$$

where we have used the identify $B_{0,t+1}(1 + s_{t,1}) = B_{0,t}$.

The spread P is not a constant – it is a function of the credit rating of the sponsoring firm. Therefore, in order to use $c_t^d(P)$ to model the cost of debt, we have to ensure that the credit rating of the firm remains above a certain level. We assume that the credit rating of the firm is a function of the interest coverage (IC), and a firm has a credit rating Q provided $IC \in [\alpha(Q), \beta(Q)]$ and in this case the spread is given by $P(Q)$

(Damodaran, 2004). We also assume that the function mapping interest coverage IC to the credit rating Q is fixed over time. As we assume that each debt offering has a duration $D = 1$, it follows that the interest coverage IC_t is given by

$$IC_t = \frac{C_t}{(s_{t,1} + P)w_t^d}.$$

Suppose the firm maintains a debt rating $Q \geq \underline{Q}$, then the spread $P \leq P(\underline{Q})$, and we can use $c_t^d(P(\underline{Q}))$ to estimate the cost of debt. The chance constraint

$$\mathrm{P}(\alpha(\underline{Q})(s_{t,1}(\mathbf{Z}_\mathrm{t} + (P(\underline{Q}))w_t^d \leq C_t) \geq 1 - \varepsilon, \tag{9}$$

where we write $s_{t,1}(\mathbf{Z}_t)$ to emphasize that $s_{t,1}$ is a function of the factors $\mathbf{Z}_t$, ensures that $Q \geq \underline{Q}$ with high probability and we can use $c_t^d(P(\underline{Q}))$ to approximate the cost of debt. The constraint (9) also belongs to the general class of chance constraints described in (3).

We adopt $c_t^d(P(\underline{Q}))$ defined in (8) as the objective. Thus, the pension fund management optimization problem is given by the chance-constrained problem

$$\min \sum_t c_t^d(P(\underline{Q}))$$

s.t.

$$\mathrm{P}(\tilde{\mathbf{a}}_t^T \mathbf{y}_t \geq d_t,\ t = 1,\ldots, T) \geq 1 - \varepsilon, \tag{10}$$

In general, chance-constrained optimization problems are difficult to solve. In most cases, the problem is non-convex. Except for a few special cases, one has to resort to sampling to solve chance-constrained problems. Consequently, the complexity of solving chance-constrained problems is exponential in the problem dimension. In the next section, we construct a tractable approximation to (10).

Linearization and robust constraints

Let $\mathbf{f} = (f_1,\ldots,f_l)$: $\Re^m \rightarrow \Re^l$ denote the function that defines the stochastic parameters $\mathbf{a}_t$ in terms of the factors $\mathbf{Z}_t$ at time t. By Itô's lemma (see Chang (2004) for example),

$$\mathrm{d}\mathbf{f}_t(\mathbf{Z}) = \boldsymbol{\mu}^f(t, \mathbf{Z})\mathrm{d}t + \Sigma^f(t, \mathbf{Z})\mathrm{d}\mathbf{W}_t, \tag{11}$$

where

$$\begin{aligned}\mu^f(t,\mathbf{Z}) &= \mathbf{J}_f(\mathbf{Z})\mu(t,\mathbf{Z}) + \frac{1}{2}\eta_f, \Sigma^f(t,\mathbf{Z}) \\ &= \mathbf{J}_f(\mathbf{Z})\Sigma(t,\mathbf{Z})\mathrm{d}\mathbf{W}_t, \\ \eta_f &= [\mathrm{tr}(\Sigma(t,\mathbf{Z})\Sigma(t,\mathbf{Z})^T\mathbf{H}_1(\mathbf{Z})),\ldots, \\ &\quad \mathrm{tr}(\Sigma(t,\mathbf{Z})\Sigma(t,\mathbf{Z})^T\mathbf{H}_l(\mathbf{Z}))]^T,\end{aligned}$$

$\mathbf{J}_f(\mathbf{Z})$ denotes the Jacobian matrix of $\mathbf{f}$, $\mathbf{H}_i(\mathbf{Z})$ denotes the Hessian matrix of f_i with respect to the factors, and $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. We approximate

$$\mathbf{f}_t \approx \mathbf{f}_0 + \mu_0^f t + \Sigma_0^f \mathbf{W}_t, \tag{12}$$

where

$$\begin{aligned}\mu_0^f &= \mathbf{J}_f(\mathbf{Z})\mu(t,\mathbf{Z}) + \frac{1}{2}\eta_f\bigg|_{\mathbf{Z}=\mathbf{Z}_0}, \\ \Sigma_0^f &= \mathbf{J}_f(\mathbf{Z})\Sigma(t,\mathbf{Z})\Big|_{\mathbf{Z}=\mathbf{Z}_0},\end{aligned}$$

that is, we evaluate the coefficients at time $t = 0$ and then let $\mathbf{f}_t$ evolve according to a Gaussian process. Thus, $\mathbf{f}_t \sim N(\mathbf{f}_0 + t\mu_0^f, t\mathbf{\Sigma}_0^f)$. We discuss the impact of this approximation in the section 'Numerical example'.

We can now approximate the generic chance constraint (3) by

$$\mathrm{P}((\mathbf{f}_0 + \mu_0^f t + \mathbf{\Sigma}_0^f \mathbf{W} t)^T \mathbf{y}_t \geq d_t) \geq 1-\varepsilon. \tag{13}$$

Let $\mathbf{\Phi}(\cdot)$ denote the cumulative density function of the standard normal random variable. Then $\mathrm{P}(\|\mathbf{W}_t\| \leq t\mathbf{\Phi}^{-1}(1-\varepsilon)) = 1-\varepsilon$, and it follows that (13) holds if

$$(\mathbf{f}_0 + \mu_0^f t + \Sigma_0^f \mathbf{w})^T \mathbf{y}_t \geq d_t \text{ for all } \|\mathbf{w}\| \leq \sqrt{t}\Phi^{-1}(1-\varepsilon). \tag{14}$$

A constraint of the form (14) is called a robust constraint (Ben-Tal and Nemirovski, 2002). Note that the robust constraint (14) is a conservative approximation for the chance constraint. Using the Cauchy–Schwarz inequality, (14) can be written as

$$(\mathbf{f}_0 + \mu_0^f t)^T \mathbf{y}_t - d_t \geq \sqrt{t}\Phi^{-1}(1-\varepsilon) \times \| \Sigma_0^f \mathbf{y}_t \|_2, \tag{15}$$

where $||\mathbf{x}||_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$ denote the L_2-norm. The constraint (15) is of the form

$$||\mathbf{B}\mathbf{x} - \mathbf{a}||_2 \leq \mathbf{d}^T\mathbf{x} + c,$$

where $\mathbf{B}$, $\mathbf{a}$, $\mathbf{d}$, and c are constants and $\mathbf{x}$ is the decision variable. Constraints of this form are called SOC constraints.

SOC programming approximation for pension fund management

In a pension fund management problem, we have at least one constraint of the form (3) at each decision epoch t. Suppose we have K chance constraints in total. We want to guarantee that all the chance constraints hold with probability at least η. We set $\varepsilon = \eta/K$ for each chance constraint of the form $\mathrm{P}(\mathbf{C}_i) \geq 1 - \varepsilon$, $i = 1, ..., K$. The Bonferroni inequality (see for example Boros and Prékopa, 1989) implies that

$$\begin{aligned} \mathrm{P}\left(\bigcap_{i=1}^{K} \mathbf{C}_i\right) &\geq 1 - \sum_{i=1}^{K}(1 - P(\mathbf{C}_\mathrm{i})) \\ &\geq 1 - \sum_{i=1}^{K}\frac{\eta}{K} = 1 - \eta, \end{aligned} \tag{16}$$

that is, by setting a more conservative target for each chance constraint, the Bonferroni inequality guarantees that all the chance constraints hold simultaneously. We use $\varepsilon = \eta/K$ in constraints of the form (15) to approximate each chance constraint by an SOC constraint. Thus, the resulting optimization problem is of the form

$$\min \sum_t c_t^d(P(\underline{Q}))$$

s. t.

$$||\mathbf{B}_i\mathbf{y} - \mathbf{a}_i||_2 \leq \mathbf{d}_i^T\mathbf{x} + c_i,\ i = 1, ..., K, \tag{17}$$

that is, it has one linear objective and several SOC constraints. Such an optimization problem is called an SOCP.

Very large-scale SOCPs can be solved efficiently both in theory (Alizadeh and Goldfarb, 2003) and in practice (Andersen and Andersen, 2006). Moreover, a number of commercial solvers, such as MOSEK, CPLEX and Frontline System (supplier of EXCEL SOLVER), provide the capability for solving SOCPs in a numerically robust manner. As the approximation (12) implies that the pension fund management problem can be approximated by an SOCP, the approach proposed in this article can be used to solve very large-scale pension fund management problems.

Numerical example

In this section, we consider a specific example and formulate the optimization problem that computes the optimal contribution schedule and portfolio holdings for a frozen pension fund using the general framework described in the section 'Robust pension fund management'. A frozen fund is a fund in which all the liabilities l_t are fixed; therefore, there is no actuarial risk and the only risk in the problem is financial risk.

Assets, liabilities and dynamics

We assume that a pension fund invests in an equity index and zero-coupon bonds with face value 1 and maturities up to M years. Thus, the holdings of the fund at time t can be described by the vector

$$\mathbf{x}_t = \begin{bmatrix} \text{Number of shares of 1} - \text{year bond} \\ \vdots \\ \text{Number of shares of M} - \text{year bond} \\ \text{Number of shares of equity} \end{bmatrix} \in \Re^{M+1}.$$

Note that, if the equity investment is specified as a broad market index, we can use the index to denote price even if it is not possible to invest in the market index directly. As long as the index is used consistently over time, the investment returns can still be correctly calculated in the model. At time $t + 1$, all the bonds in the portfolio have a maturity that is 1 year shorter (the bond with 1-year maturity is now available as cash). Thus, the holding $\mathbf{x}_t + 1$ before any trading at time $t + 1$ is given by

$$\hat{\mathbf{x}}_{t+1} = \mathbf{D}\mathbf{x}_t,$$

and $\mathbf{d}^T\boldsymbol{x}_t$ is available as cash, where

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{d} = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The value of the portfolio $\mathbf{x}_t$ at time $t + 1$ is given by $\mathbf{p}_{t+1}^T\mathbf{D}\mathbf{x}_t + \mathbf{d}^T\mathbf{x}_t$.

The liability of the pension fund at time t is denoted by l_t and time $t = 0,1, \ldots, T$, that is, the time horizon for the pension fund problem is T. We assume that at time $t = 0$, all the future payments l_t, $t = 0,1, \ldots, T$, are deterministic as in the case of frozen pension funds, that is, the uncertainty in the model is only from the changing financial conditions.

Bond prices and the yield curve

We follow Nelson and Siegel (1987) and assume that the short rates

$$
\begin{aligned}
s_{t,j} = {} & Z_t^1 + Z_t^2\left[\frac{1-\exp(-j/\tau)}{j/\tau}\right] \\
& + Z_t^3\left[\frac{1-\exp(-j/\tau)}{j/\tau} - \exp(-j/\tau)\right],
\end{aligned}
\tag{18}
$$

where the factors Z_t^1, Z_t^2 and Z_t^3 refer, respectively, to level, slope and curvature of the yield curve and τ is a constant. We use the Nelson–Siegel model because this model ensures non-negative spot rates $s_{t,j}$ for large $t \gg 1$. This is necessary in our setting as we need to discount liabilities with very long durations.

In the Nelson–Siegel model, the price $B_{t,j}$ at time t of a zero-coupon bond with face value $F = 1$ and maturing at time $t + j$ is given by

$$
\begin{aligned}
B_{t,j} &= \frac{1}{(1+s_{t,j})^j} \\
&= \left(1 + Z_t^1 + Z_t^2\left[\frac{1-\exp(-j/\tau)}{j/\tau}\right]\right. \\
&\qquad \left. + Z_t^3\left[\frac{1-\exp(-j/\tau)}{j/\tau} - \exp(-j/\tau)\right]\right)^{-j}.
\end{aligned}
\tag{19}
$$

Thus, $B_{t,j}$ is a highly nonlinear function of the factors $\mathbf{Z}$. We chose the Nelson–Siegel model to illustrate our framework because a highly nonlinear yield curve is a good test for the linearization technique introduced in the section 'Robust pension fund management'.

We denote the value of the equity index by q_t. We assume that the equity index q_t and the factors $\{Z_t^i : i = 1,\ldots,3\}$ driving the yield curve (18) evolve according to the stochastic differential equation

$$\begin{bmatrix} \mathrm{d}Z_t^1 \\ \mathrm{d}Z_t^2 \\ \mathrm{d}Z_t^3 \\ \dfrac{\mathrm{d}q_t}{q_t} \end{bmatrix} = \begin{bmatrix} (m_1 - Z_t^1) \\ (m_2 - Z_t^2) \\ (m_3 - Z_t^3) \\ \mu \end{bmatrix} \mathrm{d}t + \mathbf{A}\mathrm{d}\mathbf{W}_t, \tag{20}$$

where $\mathbf{W}_t = (W_t^1, W_t^2, W_t^3, W_t^4)^T$, $\{W_t^i\}_{t\geq 0}$, $i = 1, 2, 3, 4$, are independent standard Brownian motions, and the lower triangular matrix $\mathbf{A} \in \Re^{4\times 4}$ denotes the Cholesky decomposition of the covariance matrix $\mathbf{V} \in \Re^{4\times 4}$ of the vector $(Z_t^1, \ldots, Z_t^3, q_t)$. The dynamics in (20) imply that each of the factors Z_t^i is an Ornstein–Uhlenbeck process and the equity index q_t is a geometric Brownian motion. The yield curve dynamics given by (20) is similar to the one considered in Fabozzi *et al* (2005). With the above definitions, the price vector is given by

$$\mathbf{p}_t = (B_{t,1}, \ldots, B_{t,M}, q_t)^T.$$

Note that the price vector and the stochastic differential equations (20) conform to the general framework described in the section 'Robust pension fund management'.

Optimization problem

We assume that at time $t = 0$, we determine the contribution w_t and the portfolio $\mathbf{x}_t$ for $t = 0, \ldots, \bar{T} \leq T$. We expect that the pension fund problem will be solved on a rolling-horizon basis, that is, at time $t = 1$, we will recompute the optimal portfolio for the horizon $t = 1, \ldots, \bar{T} + 1$. The horizon $\bar{T}$ is chosen to be long enough so that the impact of the liabilities l_t, $t > \bar{T}$, is minimal.

Let ψ denote the initial holdings of the fund, that is the holdings before rebalancing at time 0. We require that the portfolio $\mathbf{x}_0$ must satisfy

$$\mathbf{p}_0^T \psi + w_0 - l_0 = \mathbf{p}_0^T \mathbf{x}_0, \tag{21}$$

that is, the total value of the portfolio $\mathbf{x}_0$ must equal the difference between the available capital $(\mathbf{p}_0^T \psi + w_0)$ and the liability l_0. Note that (21) implicitly assumes that rebalancing does not incur any transaction costs. Therefore, we can assume, without loss of generality, that the portfolio ψ is held in cash.

The constraint for time $t \geq 1$ is

$$\begin{aligned} &\mathrm{P}(\mathbf{p}_t^T \mathbf{D} \mathbf{x}_{t-1} + \mathbf{d}^T \mathbf{x}_{t-1} + w_t - l_t \geq \mathbf{p}_t^T \mathbf{x}_t) \geq 1 - \varepsilon, \\ &t = 1, \ldots, \bar{T} - 1, \end{aligned} \tag{22}$$

where P denotes the probability measure conditioned on the information available at time t = 0. We also require the following target funding level constraint

$$\mathrm{P}(\mathbf{p}_{\bar{T}}^T \mathbf{D} \mathbf{x}_{\bar{T}-1} + \mathbf{d}^T \mathbf{x}_{\bar{T}-1} + w_{\bar{T}} \geq l_{\bar{T}} + \beta L_{\bar{T}}) \geq 1 - \varepsilon, \tag{23}$$

to set the target funding level at time $\bar{T}$ to be a fraction β of the future liabilities, where L_t denote the net present value at time t of the entire set of future liability at a fixed discount rate d, that is

$$L_t = \sum_{\tau=t+1}^{T} \frac{l_\tau}{(1+d)^{\tau-t}},$$

and the discount rate d is chosen by the plan sponsor subject to some regulatory constraints. The funding level of a pension fund at time t is defined to be the ratio of the total spot value $\mathbf{p}_t^T \mathbf{x}_t$ of the assets of fund to L_t.

In addition, one may have to impose other constraints that meet regulatory requirements. For example, in the US, pension funds need to maintain a funding level of $\gamma = 90$ per cent and the sponsor is required to contribute if the funding level drops below γ. Such a regularity requirement can be met by imposing constraints of the form:

$$\mathbf{p}_0^T \mathbf{x}_0 \geq \gamma L_0, \tag{24}$$

and

$$\mathrm{P}(\mathbf{p}_t^T \mathbf{x}_t \geq \gamma L_t) \geq 1 - \varepsilon, \quad t = 1, \ldots, \bar{T} - 1 \tag{25}$$

See Fabozzi *et al* (2004) for a summary of regulations on pension funds in different countries.

Collecting together all the constraints and using the objective incorporating the corporate structure of the plan sponsor given as an example in the section 'Linearization and robust constraints', we solve the following optimization problem

$$\begin{aligned}&\min\sum_{t=0}^{\bar{T}}((1-\alpha_T)B_{0,t}+((1-\alpha_T)P(\underline{Q})+\alpha_T)B_{0,t+1})w_t^d\\&\text{subject to (21), (22),(23),(24),(25),(7) and (9).}\end{aligned}\tag{26}$$

In the Appendix section, we discuss how to use the general results in section 'Robust pension fund management' to reformulate (26) into an SOCP.

Discussion

Typically, the pension fund manager only chooses capital allocation to asset classes. The tactical decisions of the particular assets to purchase within each asset class are left to asset managers who are specialists in a particular asset classes. We consider two asset classes – equity and treasury bonds. The solution of the pension fund problem (26) guides the fraction of capital that should be allocated to an asset manager specializing in equity market for tactical asset allocation, and the fraction that should be given to an asset manager specializing in fixed income market. Therefore, the bond portfolio is only a proxy for total fixed income holdings.

We want our robust optimization-based approach to produce conservative portfolios. In constructing (26), we linearize the nonlinear factor dynamics, but then we use Bonferroni's inequality (see (16)), to impose a very conservative chance constraint. It is not immediately clear that the net outcome is a conservative portfolio. We show in the section 'Stationary portfolio selection' that the robust solution is indeed conservative when the risk is measured by the Value-at-Risk (VaR) and the Conditional Value-at-Risk (cVaR).

Problem parameters

Following Fabozzi *et al* (2005) (see also Barrett *et al*, 1995), we set $\tau = 3$. The other parameters used in the example are:

$$\begin{gathered}Z_0^1 = 4.5794,\ Z_0^2 = -0.3443,\\ Z_0^3 = -0.2767,\ q_0, = 1248.29,\\ \mu = 0.0783,\ m_1 = 6.1694,\\ m_2 = -2.4183,\ m_3 = 0.4244,\end{gathered}$$

the covariance matrix

$$\mathbf{V} = \begin{bmatrix} 2.1775 & -4.5778 & 19.3399 & -0.1201 \\ -4.5778 & 15.6181 & -43.6039 & 0.2679 \\ 19.3399 & -43.6039 & 179.7153 & -1.0094 \\ -0.1201 & 0.2679 & -1.0094 & 0.0078 \end{bmatrix},$$

and the correlation matrix

$$\boldsymbol{\rho} = \begin{bmatrix} 1.0000 & -0.2178 & 0.5685 & -0.4008 \\ -0.2178 & 1.0000 & -0.4452 & 0.0945 \\ 0.5685 & -0.4452 & 1.0000 & -0.1149 \\ -0.4008 & 0.0945 & -0.1149 & 1.0000 \end{bmatrix}.$$

Thus, the Cholesky decomposition **A** of **V** is given by

$$\mathbf{A} = \begin{bmatrix} 1.4756 & 0 & 0 & 0 \\ -3.1023 & 2.4482 & 0 & 0 \\ 13.1063 & -1.2027 & 2.5485 & 0 \\ -0.0814 & 0.0063 & 0.0255 & 0.0212 \end{bmatrix}$$

These parameter estimates result in the current yield curve displayed in Figure 14.1. The number of maturities M is set to $M = 10$ in our numerical experiments.

The liability stream used in our numerical experiments is shown in Figure 14.2. The liability stream ends in year $T = 85$. We obtained these data for a frozen pension fund from Goldman Sachs. We set the value of initial holding

$$\mathbf{p}_0^T \psi = 0.8(\mathrm{L}_0 + l_0),$$

Other parameters for this numerical example are set as follows:

(i) We consider the optimal plan for the first 4 years, that is $\bar{T} = 4$.
(ii) The regulation mandated minimum funding level γ is set to $\gamma = 0.9$. Thus, the fund is *underfunded* at time $t = 0$.
(iii) The target funding level β that controls the influence of liabilities beyond $\bar{T}$ is set to $\beta = 0.9$.
(iv) The liabilities are discounted at a nominal discount rate $d = 6$ per cent.
(v) The violation probability $\eta = 1$ per cent (see (16)), that is, all chance constraints in (26) are satisfied with $1–\eta = 99$ per cent probability.

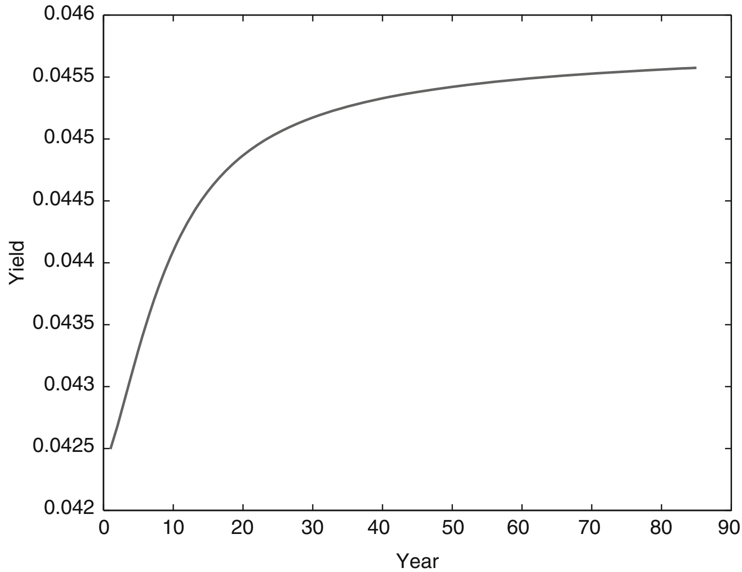

Figure 14.1 Current yield curve

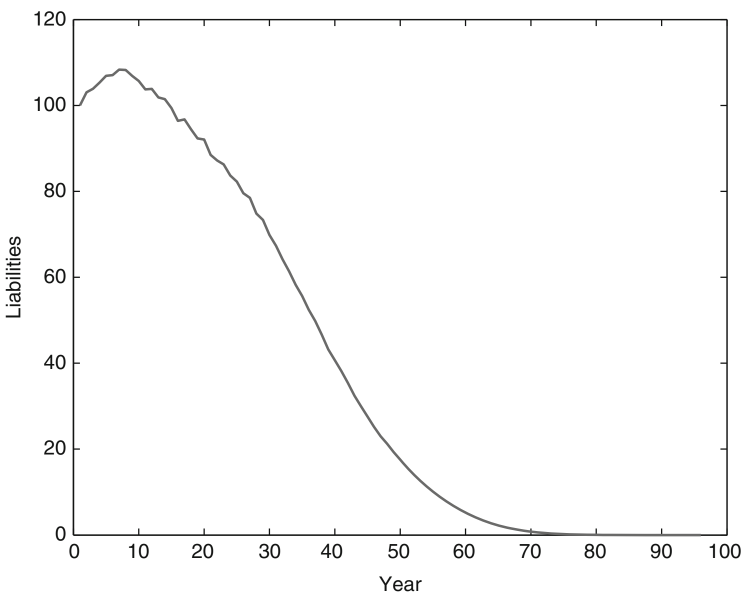

Figure 14.2 Liability as a function of time

(vi) The earnings $C_0 = 500$ and $u = 0.2$, that is, we impose a limit that at most 20 per cent of the earnings can be used to fund the pension plan. We set $r_e = 0.05$.

(vii) The marginal tax rate $\alpha_T = 0.35$ and we assume that the company wants to maintain a credit rating Q = 'A+', that is, $\alpha(\underline{Q}) = 5.5$ and $P(\underline{Q}) = 0.008$ (Damodaran, 2004).

We summarize the values for the parameters as follows.

Parameter	**Value**
$\bar{T}$	4
γ	0.9
β	0.9
d	6%
η	1%
C_0	500
u	0.2
α_T	0.35
$\alpha(\underline{Q})$	5.5
$\rho(\underline{Q})$	0.008

Stationary portfolio selection

We consider optimal portfolio selection over $\bar{T} = 4$ for a liability stream with time horizon $T = 85$. We consider this setting for simpler presentation and evaluation of the solution. As $\bar{T} \ll T$, we require that portfolio $\mathbf{x}_t$, $t = 1, \ldots, \bar{T}$ be stationary, that is, $\mathbf{x}_0 = \mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3$. In order to investigate the impact of the equity ratio, that is the fraction of the total capital of the fund that is invested in equity, we impose the constraint

$$(B_{1,1}, \ldots, B_{1,M})(x_0(1), \ldots, x_0(M))^T = \rho q' x_0(M+1)$$

that sets the equity ratio of the initial portfolio $\mathbf{x}_0$ to $1/(1 + \rho)$. We compute $\mathbf{x}_0$ and $\{w_k\}_{k=0}^{\bar{T}}$ by solving

$$\begin{aligned}
&\min \sum_{t=0}^{\bar{T}} ((1-\alpha_T)B_{0,t} + ((1-\alpha_T)P(\underline{Q}) + \alpha_T)B_{0,t+1})w_t^d \\
&\text{subject to } \mathbf{x}_0 = \mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3, \\
&(B_{1,1}, \ldots, B_{1,M})(x_0(1), \ldots, x_0(M))^T = \rho q' x_0(M+1), \\
&\quad (7), (21), (24), (34), (35), (36), \text{ and } (37).
\end{aligned} \tag{27}$$

Table 14.1 and Figure 14.3 shows the worst-case payments as a function of the equity ratio $1/(1+\rho)$ with the probability of constraint satisfaction fixed at $1-\eta = 0.99$. The contribution w_0 increases with increasing equity ratio while the contributions (w_1, w_2, w_3) all decrease with the increase in the equity ratio. The total discounted payment, however, increases with the increase in the equity ratio.

In Table 14.2 we display the worst-case payments as a function of the probability of constraint satisfaction with the equity ratio $1/(1+p)$ fixed at 0.4. As expected, the worst-case contribution decreases with a decrease in constraint satisfaction.

Table 14.1 Worst-case contribution as a function of equity ratio

Equity ratio	w_0	w_1	w_2	w_3	$\sum_{t=0}^{\bar{T}} B_{0,t} w_t$
0.2	239.90	125.00	155.60	180.41	683.69
0.4	283.96	120.04	143.91	163.44	697.36
0.6	348.05	114.88	131.76	145.81	729.92

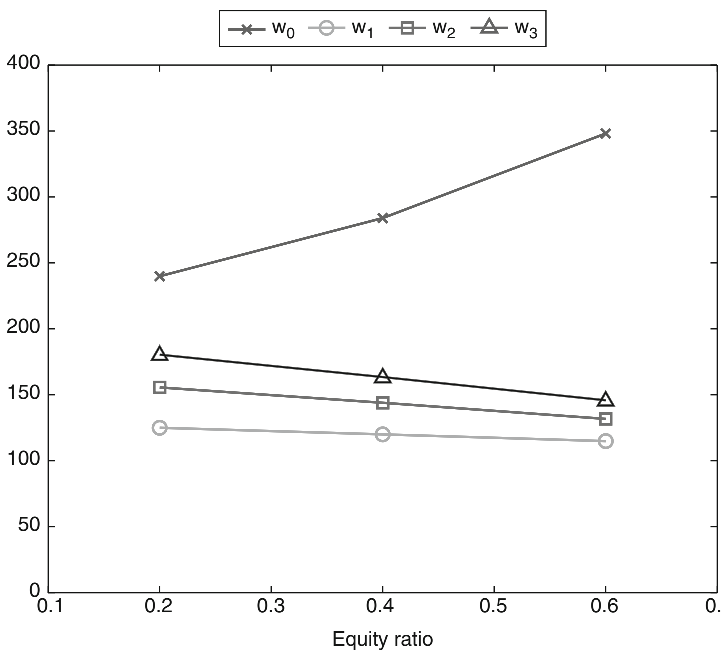

Figure 14.3 Worst-case contribution as a function of equity ratio

Table 14.2 Worst-case contribution as a function of time

Probability	w_0	w_1	w_2	w_3	$\sum_{t=0}^{\bar{T}} B_{0,t} w_t$
0.99	283.96	120.04	143.91	163.44	697.36
0.95	244.27	111.20	130.97	147.34	622.91
0.90	230.03	107.11	125.01	139.95	592.79

Conditional VaR

In this section, we test the effect of linearizing the dynamics by stress–testing the pension fund portfolio using the VaR and CVaR measures.

We simulate the asset prices using the dynamics described by (18)–(20) (that is, we do not linearize the dynamics) and compute the real (as opposed to the worst-case) payments $\bar{w}_t$ required to finance the portfolio strategy. From the constraints (21), (22), (23), (24) and (25), it follows that

$$\bar{w}_t = \begin{cases} \max(l_t + \mathbf{p}_t^T \mathbf{x}_t - \alpha_{t-1}(\mathbf{p}_t^T \mathbf{D}\mathbf{x}_{t-1} \\ \quad + \mathbf{d}^T \mathbf{x}_{t-1}), \gamma L_t - \mathbf{p}_t^T \mathbf{x}_t, 0) \\ \quad \text{if } 1 \le t \le \bar{\mathrm{T}}, \\ \max(l_t + \beta L_t - \alpha_{t-1}(\mathbf{p}_t^T \mathbf{D}\mathbf{x}_{t-1} \\ \quad + \mathbf{d}^T \mathbf{x}_{t-1}), 0) \\ \quad \text{if t} = \bar{\mathrm{T}}, \end{cases} \tag{28}$$

where

$$\alpha_t = \max\left(\frac{\gamma L_t}{\mathbf{p}_t^T \mathbf{x}_t}, 1\right). \tag{29}$$

The variable α_t keeps track of whether the payment $\bar{w}_t$ is needed to maintain the regulation requirement $\gamma L_t \,/\, \mathbf{p}_t^T \mathbf{x}_t \le 1$, and the value of the portfolio in the next period will increase or remain unchanged accordingly. Note that, in our numerical experiments, $\mathbf{x}_t$ is fixed over time.

We generated $K = 100\,000$ independent sample paths and set the shortfall probability

$$\bar{\eta} = \frac{\sum_{k=1}^{K} \max_{0 \le t \le 3} \mathbf{1}(w_t < \bar{w}_t^{(k)})}{K},$$

where $\{\bar{w}_t^{(k)}\}$ denotes the real payments on the k-th simulation run and $\mathbf{1}(\cdot)$ is the indicator function that takes the value 1 when the argument is true and 0 otherwise. Thus, $\bar{\eta}$ is the empirical probability that the real payment $\bar{w}_t$ is larger than the worst case payment w_t. The expected net shortfall $\bar{W}$ was defined as follows.

$$\bar{W} = \frac{\sum_{k=0}^{K}\sum_{t=0}^{\bar{T}} B_{0,t}(\bar{w}_t^{(k)} - w_t)^+}{\sum_{k=0}^{K}\mathbf{1}\left(\sum_{t=0}^{\bar{T}} B_{0,t}(\bar{w}_t^{(k)} - w_t)^+ > 0\right)}, \tag{30}$$

that is $\bar{W}$ is the expected shortfall conditioned on their being a shortfall. We define the Value-at-Risk (VaR_p) at probability p of the discounted total real payment as

$$\mathrm{VaR}_p = \sup_{x\geq 0}\left\{\sum_k \mathbf{1}\left(\sum_{t=0}^{\bar{T}} B_{0,t}\bar{w}_t^{(k)} \geq x\right) \geq (1-p)K\right\}$$

and Conditional Value-at-Risk (CVaR_p) of the discounted total real payment $\sum_{t=0}^{I} B_{0,t}\bar{w}_t$ as

$$\begin{aligned}\mathrm{CvaR}_p = \frac{1}{1-p}&\left(\sum_{k=1}^{K}\left(\sum_{t=0}^{\bar{T}} B_{0,t}\bar{w}_t^{(k)}\right)\right.\\ &\left.\times\mathbf{1}\left(\sum_{t=0}^{\bar{T}} B_{0,t}\bar{w}_t^{(k)} \geq \mathrm{VaR}_p\right)\right).\end{aligned}$$

Table 14.3 plots the shortfall probability $\bar{p}$, the expected shortfall $\bar{W}$, the VaR and CVaR as a function of the probability p. From the numerical results, we can conclude that the linearized robust problem (27) does produce a conservative solution for the true nonlinear problem (note that, this is not guaranteed). In all cases, the empirical shortfall probability is at least an order of magnitude lower than that guaranteed by the robust problem. This result confirms our initial hypothesis that linearizing the dynamics should not result in a significant deterioration in performance.

For a fixed p, let $\tilde{p}$ denote the probability such that the corresponding shortfall probability $\bar{\tilde{p}} \approx 1-p$. For example, for $p = 0.98$, $\tilde{p} = 0.85$ as the corresponding shortfall probability $\bar{\tilde{p}} = 0.0182 \approx 1 - p = 0.02$.

Table 14.3 Simulation results

p	$\bar{p}$	$\bar{w}$	$\sum_{t=0}^{\bar{T}} B_{0,t} w_t$	VaR	CVaR
0.99	0.0014	5.98	697.36	537.40	546.96
0.98	0.0027	5.07	667.48	511.58	521.73
0.97	0.0039	5.41	644.63	496.18	506.83
0.96	0.0050	5.34	632.27	487.12	498.22
0.95	0.0063	5.43	622.91	479.91	491.22
0.90	0.0126	5.64	592.79	456.23	496.00
0.85	0.0182	5.99	574.24	441.90	455.63
0.80	0.0235	6.08	560.48	430.38	445.12

Another such pair is $(p, \tilde{p}) = (0.99, 0.90)$. Then total discounted worst-case payment corresponding to p is approximately equal to the CVaR_p – note that, this is in spite of the fact that the robust problem does *not* minimize the total discounted payment.

Computational efficiency

All numerical computations reported in this work were conducted using Matlab 6.5 and MOSEK 4.0 (Andersen and Andersen, 2006). We used a Windows/32-X86 platform with Intel-PM. A typical portfolio problem had less than 100 constraints and 100 variables and it took no longer than a second for MOSEK to solve the portfolio problem.

Concluding remarks

In this article, we introduce a robust optimization framework for pension fund management that minimizes the worst-case pension contributions of the sponsoring firm. The illustrated model is able to account for some aspects of the corporate structure of the firm, for example cost of debt. The optimal pension plan from the proposed framework is computed by solving an SOCP and is, therefore, very efficient both in theory and in practice. In addition, we show that the framework is very versatile in that it allows us to compute both the optimal plan and also stress test any existing pension plans. The solution to the pension fund management problem is shown to be robust and conservative in the stress testing result.

There are fundamental differences between the robust approach and the stochastic programming approach. In the stochastic programming approach, the evolution of the stochastic parameters is approximated by a tree and one computes an optimal portfolio for each node in the tree taking the evolving information into account. As the tree can be constructed for any stochastic model, the stochastic programming approach is extremely versatile. However, a tree has zero probability and the stochastic programming approach is not able to provide any worst-case guarantees. Moreover, the complexity of the associated optimization problem is exponential in the time horizon and number of assets. In the robust optimization approach, one is able to provide a worst-case probabilistic guarantee; however, the portfolio selection cannot take advantage of evolving information (adjustable robust optimization somewhat mitigates this objection (Ben-Tal *et al*, 2004)). The computational complexity of the robust approach is polynomial in the time horizon and the number of assets. Both of these approaches cannot be implemented in an open-loop manner, and a new optimization problem has to be solved at each decision epoch. In summary, neither of these two approaches are clear winners; however, robust methods are very well suited for solving large-scale pension fund management problems.

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Appendix

Derivation

The Itô-Taylor expansion applied to (19) at time 0 using (18) and (20) implies that

$$B_{t,j} \approx B_{0,j} + \left(\sum_{i=1}^{3} (m_i - Z_0^i) \frac{\partial B_{s,j}}{\partial Z_s^i}\bigg|_{s=0} + \frac{1}{2} \sum_{i,l=1}^{3} \rho_{il} \frac{\partial^2 B_{s,j}}{\partial (Z_s^i) \partial (Z_s^l)}\bigg|_{s=0} \right) t + \sum_{i=1}^{3} \frac{\partial B_{s,j}}{\partial Z_s^i}\bigg|_{s=0} \sum_{k=1}^{4} v_{ik} W_t^k, \tag{A.1}$$

where $\rho_{il} = \sum_{k=1}^{4} v_{ik} v_{kl}$ and $\mathbf{A} = [v_{ij}]$ is the covariance matrix of the factor vector $(W_t^1, ..., W_t^4)$. Similarly, for the equity index q_t, we have

$$q_t \approx q_0 + q_0 \mu t + q_0 \sum_{k=1}^{4} v_{4k} W_t^k. \tag{A.2}$$

Thus,

$$\mathbf{p}_t \approx \mathbf{p}_0 + \mu_0^p t + \sum\nolimits_0^p W_t, \tag{A.3}$$

where

$$\mu_0^p = \begin{bmatrix} \sum_{i=1}^{3} (m_i - Z_0^i) \left.\dfrac{\partial B_{s,1}}{\partial Z_s^i}\right|_{s=0} + \dfrac{1}{2} \sum_{i,l=1}^{3} \rho_{il} \left.\dfrac{\partial^2 B_{s,1}}{\partial(Z_s^i)\partial(Z_s^l)}\right|_{s=0} \\ \vdots \\ \sum_{i=1}^{3} (m_i - Z_0^i) \left.\dfrac{\partial B_{s,M}}{\partial Z_s^i}\right|_{s=0} + \dfrac{1}{2} \sum_{i,l=1}^{3} \rho_{il} \left.\dfrac{\partial^2 B_{s,M}}{\partial(Z_s^i)\partial(Z_s^l)}\right|_{s=0} \\ q_{0\mu} \end{bmatrix} \in \Re^{M+1},$$

and

$$\Sigma_0^p = \begin{bmatrix} \left.\dfrac{\partial B_{s,1}}{\partial Z_s^1}\right|_{s=0} & \left.\dfrac{\partial B_{s,1}}{\partial Z_s^2}\right|_{s=0} & \left.\dfrac{\partial B_{s,1}}{\partial Z_s^3}\right|_{s=0} & \left.\dfrac{\partial B_{s,1}}{\partial q_s}\right|_{s=0} \\ \vdots & \vdots & \vdots & \vdots \\ \left.\dfrac{\partial B_{s,M}}{\partial Z_s^1}\right|_{s=0} & \left.\dfrac{\partial B_{s,M}}{\partial Z_s^2}\right|_{s=0} & \left.\dfrac{\partial B_{s,M}}{\partial Z_s^3}\right|_{s=0} & \left.\dfrac{\partial B_{s,M}}{\partial q_s}\right|_{s=0} \\ 0 & 0 & 0 & q_0 \end{bmatrix} \in \Re^{(\mathrm{M}+1)\times 4}.$$

It then follows that for all $t = 1, ..., \overline{T} - 1$,

$$\begin{aligned} &\mathrm{P}(\mathbf{p}_t^T \mathbf{D}\mathbf{x}_{t-1} + \mathbf{d}^T \mathbf{x}_{t-1} + w_t - l_t \geq \mathbf{p}_t^T \mathbf{x}_t) \\ &\quad = \mathrm{P}(\mathbf{p}_t^T (\mathbf{D}\mathbf{x}_{t-1} - \mathbf{x}_t) + \mathbf{d}^T \mathbf{x}_{t-1} + w_t - l_t \geq 0) \\ &\quad = \mathrm{P}((\mathbf{p}_0 + \mu_0^p t)^T (\mathbf{D}\mathbf{x}_{t-1} - \mathbf{x}_t) + \mathbf{d}^T \mathbf{x}_{t-1} t \\ &\qquad + w_t - l_t \geq (\mathbf{D}\mathbf{x}_{t-1} - \mathbf{x}_t)^T \Sigma_0^p W_t) \end{aligned}$$

Since $-(\mathbf{Dx}_{t-1} - \mathbf{x}_t)^T \Sigma_0^p W_t \sim N \times (0, \|(\mathbf{Dx}_{t-1} - \mathbf{x}_t)^T \Sigma_0^p\|_2^2\, t)$, if $\varepsilon < 0.5$, we have

$$(\mathbf{p}_0 + \mu_0^p t)^T(\mathbf{Dx}_{t-1} - \mathbf{x}_t) + \mathbf{d}^T\mathbf{x}_{t-1} + w_t - l_t$$
$$\geq \sqrt{t}\Phi^{-1}(1-\varepsilon)\,\|(\mathbf{Dx}_{t-1} - \mathbf{x}_t)^T\Sigma_0^p\|_2$$
$$\Downarrow$$
$$\mathrm{P}(\mathbf{p}_t^T\mathbf{Dx}_{t-1} + \mathbf{d}^T\mathbf{x}_{t-1} + w_t - l_t \geq \mathbf{p}_t^T\mathbf{x}_t) \geq 1-\varepsilon, \quad \text{(A.4)}$$

where $\Phi(\cdot)$ denotes cumulative density function of the standard normal random variable.

Using an analysis similar to the one employed above, the constraint (23) can be reformulated as the SOC constraint

$$(\mathbf{p}_0 + \mu_0^p\bar{T})^T\mathbf{Dx}_{\bar{T}-1} + \mathbf{d}^T\mathbf{x}_{\bar{T}-1} + w_{\bar{T}} - l_{\bar{T}} - \beta L_{\bar{T}}$$
$$\geq \sqrt{\bar{T}}\Phi^{-1}(1-\varepsilon)\,\|(\mathbf{Dx}_{\bar{T}-1})^T\Sigma_0^p\|_2, \quad \text{(A.5)}$$

and the regulation constraint (25) can be reformulated as the SOC constraint

$$(\mathbf{p}_0 + \mu_0^p t)^T\mathbf{x}_t - \gamma L_t \geq \Phi^{-1}(1-\varepsilon)\sqrt{t}\,\|x_t^T\Sigma_0^p\|_2 \quad \text{(A.6)}$$

As the short rates $s_{t,1}$ are described by a Ornstein–Uhlenbeck process whose marginal distribution is normal, it follows that the interest-coverage constraint (9) is equivalent to the linear constraint

$$\alpha(\underline{Q})\left(E(s_{t,1}) + \sqrt{\mathrm{var}[s_{t,1}]}\Phi^{-1}(1-\varepsilon) + P(\underline{Q})\right)w_t^d \leq C_t,$$
$$t = 1,2,\ldots,\bar{T}, \quad \text{(A.7)}$$

where $\mathrm{var}[s_{t,1}]$ denotes the variance of $s_{t,1}$.

Finally, we can solve the following SOCP

$$\min\sum_{t=0}^{\bar{T}}((1-\alpha_T)B_{0,t} + ((1-\alpha_T)\times P(\underline{Q}) + \alpha_T)B_{0,t+1})w_t^d$$
$$\text{subject to (21), (34), (35), (24), and (36), (7) and (37).} \quad \text{(A.8)}$$