Chapter 3
Measurable Functions

3.1 Definition

This chapter lays the groundwork for integration applied to a large class of real-valued functions. First we note a useful fact that is independent of the real line.

Theorem 3.1.1. *Fix a σ-algebra $\mathscr{A}$ in a set X and a function f with domain $A \in \mathscr{A}$ and range in a set Y. The collection $\mathscr{B}$ of all sets $B \subseteq Y$ such that $f^{-1}[B] \in \mathscr{A}$ contains $\varnothing$ and is stable with respect to complementation and the operation of taking countable unions. That is, $\mathscr{B}$ is a σ-algebra in Y.*

Proof. The result follows from the fact that $f^{-1}[Y \setminus B] = A \setminus f^{-1}[B]$, and if the sequence $\langle B_n : n \in \mathbb{N} \rangle$ is in $\mathscr{B}$, then

$$f^{-1}\left[\bigcup_{n=1}^{\infty} B_n\right] = \bigcup_{n=1}^{\infty} f^{-1}[B_n].$$

We will use Lebesgue measure and other measures on the real line to extend Riemann integration. The extended integral will apply to functions having an appropriate structure in terms of the family of measurable sets. The definitions and results in this chapter hold for any integrator on $\mathbb{R}$. The resulting general measure is denoted by m, but when m is just Lebesgue measure, we write λ. We denote the class of measurable sets by $\mathscr{M}$. This is the class of sets measurable with respect to a given outer measure. If that outer measure is specifically Lebesgue outer measure, then we say "Lebesgue measurable."

It is convenient to follow the convention of probability theory and write $\{S(f)\}$ instead of $\{x \in A : S(f)(x)\}$ for a function f with domain A that is understood, and a property S involving f. For example, $\{\sin > 0\}$ denotes the set $\{x \in \mathbb{R} : \sin(x) > 0\}$.

As previously noted, an extended-real valued function is one taking values in the set $\mathbb{R} \cup \{+\infty, -\infty\}$. The hyphen indicates it is $\mathbb{R}$ that is extended to a larger set;

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the function is not extended to a larger domain. In working with extended-real valued functions, we exclude certain combinations. We do not allow the addition of $+\infty$ to $-\infty$. Similarly, we do not allow multiplication of 0 with either infinity. One reason for the latter prohibition is that a product of sequences $x_n \searrow 0$ and $y_n \nearrow +\infty$ can have any nonnegative limiting result, depending on the choice of sequences. When the operation is allowed, we have $+\infty + a = +\infty$, $-\infty + a = -\infty$, and $+\infty \cdot a = +\infty$ if $a > 0$, $+\infty \cdot a = -\infty$ if $a < 0$, $-\infty \cdot a = -\infty$ if $a > 0$, and $-\infty \cdot a = +\infty$ if $a < 0$.

Definition 3.1.1. An extended-real valued function f with measurable domain A is a **measurable function** if for every $\alpha \in \mathbb{R}$, the set $\{f > \alpha\}$ is in $\mathscr{M}$. If the class $\mathscr{M}$ consists of the Lebesgue measurable sets, we say that f is a **Lebesgue measurable function**.

Definition 3.1.2. A set A is **dense** in a set B if $A \subseteq B$ and the closure of A contains the set B.

Note that if A is dense in B, the closure of A may be larger than B. For applications of the following result, we note that $\mathbb{R}$ itself is dense in $\mathbb{R}$, and the rational numbers are dense in $\mathbb{R}$. Any dense subset D of $\mathbb{R}$ contains a countable dense subset of $\mathbb{R}$ since for each rational number r and each $n \in \mathbb{N}$, there is a point $s \in D$ with $|r - s| < 1/n$, so (using the Axiom of Choice) we can choose one point $s \in D$ for each pair (r, n).

Proposition 3.1.1. *For an extended-real valued function f with measurable domain A, the following are equivalent:*

1) f is measurable.
2) $\forall \alpha$ in a dense subset of $\mathbb{R}$, $\{f > \alpha\} \in \mathscr{M}$.
3) $\forall \alpha$ in a dense subset of $\mathbb{R}$, $\{f \geq \alpha\} \in \mathscr{M}$.
4) $\forall \alpha$ in a dense subset of $\mathbb{R}$, $\{f < \alpha\} \in \mathscr{M}$.
5) $\forall \alpha$ in a dense subset of $\mathbb{R}$, $\{f \leq \alpha\} \in \mathscr{M}$.

Proof. Let D be a dense subset of $\mathbb{R}$. We may assume that D is countable. The result is a consequence of the following equalities, which hold for any $\alpha \in \mathbb{R}$:

$$\{f < \alpha\} = A \setminus \{f \geq \alpha\}, \quad \{f > \alpha\} = A \setminus \{f \leq \alpha\}$$
$$\{f \geq \alpha\} = \bigcap_{\substack{\gamma \in D \\ \gamma < \alpha}} \{f > \gamma\}, \quad \{f < \alpha\} = \bigcup_{\substack{\gamma \in D \\ \gamma < \alpha}} \{f < \gamma\}$$
$$\{f \leq \alpha\} = \bigcap_{\substack{\gamma \in D \\ \gamma > \alpha}} \{f < \gamma\}, \quad \{f > \alpha\} = \bigcup_{\substack{\gamma \in D \\ \gamma > \alpha}} \{f > \gamma\}.$$

Corollary 3.1.1. *If an extended-real valued function f is measurable, then for any $\alpha \in \mathbb{R} \cup \{+\infty, -\infty\}$, the set $\{f = \alpha\}$ is measurable.*

Remark 3.1.1. Even for a function f that can take the value $+\infty$ or $-\infty$, measurability only depends on values α in $D \subseteq \mathbb{R}$.

We will need sets such as $\{f \geq \alpha\}$ to be measurable in order to define an integral. We will say that f is **measurable on** B if B is measurable and the restriction of f to B is measurable. Note that measurability of a function involves only measurable sets; it does not involve a measure.

Proposition 3.1.2. *The restriction of a measurable function f with measurable domain A to a measurable subset $B \subset A$ is measurable on B. Conversely, if A is the union of a finite or countably infinite number of measurable sets on which f is measurable, then f is measurable on A.*

Proof. Exercise 3.3.

In general integration theory, one speaks about a function with measurable domain A that is measurable with respect to σ-algebras $\mathscr{A}$ and $\mathscr{B}$. That is, the inverse image of each set $B \in \mathscr{B}$ is in $\mathscr{A}$. The usual definition of measurability of an extended-real valued function uses, as is the case here, the inverse images of semi-infinite open intervals in the extended real line. That definition, however, is equivalent to the following definition in terms of the inverse image of Borel sets.

Theorem 3.1.2. *An extended-real valued function f with measurable domain A is measurable if and only if the inverse image of every Borel set in $\mathbb{R}$ is measurable and also $f^{-1}[+\infty]$ and $f^{-1}[-\infty]$ are measurable.*

Proof. We have seen that if f is measurable, then the inverse image of every semi-infinite interval in the extended real line is measurable. Also, $f^{-1}[+\infty] = \cap_{n\in\mathbb{N}} f^{-1}[(n,+\infty]]$ and $f^{-1}[-\infty] = \cap_{n\in\mathbb{N}} f^{-1}[[-\infty,-n)]$ are measurable. It now follows that the inverse image of every finite open interval is measurable, and therefore the inverse image of every open subset of the real line is measurable. Since the family of Borel sets is the smallest σ-algebra containing all open sets, it follows from Theorem 3.1.1 that the inverse image of every Borel subset of the real line is measurable. The converse is clear.

Proposition 3.1.3. *A continuous real-valued function is measurable on any measurable subset B of its domain.*

Proof. If f is continuous, then $\{x \in B : f(x) > \alpha\}$ is the intersection of B with an open set.

Definition 3.1.3. If A is a measurable subset of $\mathbb{R}$, $M(A)$ denotes the collection of measurable real-valued functions with domain A.

Recall that for functions f and g, the functions $f \vee g$ and $f \wedge g$ are defined pointwise by setting $(f \vee g)(x) := \max(f(x), g(x))$ and $(f \wedge g)(x) := \min(f(x), g(x))$. When adding or multiplying measurable functions, we will often set an arbitrary value for the sum or product on the set where the original operation is not defined. Usually that value is 0, and the set where this happens will have measure 0.

Theorem 3.1.3. *If A is a measurable set, $M(A)$ forms a vector space over $\mathbb{R}$, and $M(A)$ is stable with respect to pointwise multiplication and the operations $\vee$ and $\wedge$. Given the collection of measurable extended-real valued functions on A, for each of the following operations, there is a measurable subset that depends on the functions involved and is the set where the operation is defined; moreover, the operation yields a measurable result on that subset. The operations are: Pointwise multiplication, multiplication by any real number, pointwise addition, and the operations $\vee$ and $\wedge$.*

Proof. Fix f, g in $M(A)$ and c and α in $\mathbb{R}$. If $c = 0$, cf is constant. Otherwise,

$$\{cf > \alpha\} = \{f > \alpha/c\} \text{ if } c > 0 \text{ and } \{cf > \alpha\} = \{f < \alpha/c\} \text{ if } c < 0.$$

In either case, $cf \in M(A)$. If $f(x) + g(x) < \alpha$, then since the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$, there is an $r \in \mathbb{Q}$ with $f(x) < r < \alpha - g(x)$, whence $g(x) < \alpha - r$. It follows that

$$\{f + g < \alpha\} = \bigcup_{r \in \mathbb{Q}} [\{f < r\} \cap \{g < \alpha - r\}] \in \mathscr{M}.$$

Therefore, $M(A)$ is a vector space. To see that $fg \in M(A)$, we note that the function $f^2 \in M(A)$ since if $\alpha < 0$, $A = \{f^2 > \alpha\}$ and for $\beta \geq 0$ and $\alpha = \beta^2$,

$$\{f^2 > \alpha\} = \{f < -\beta\} \cup \{f > \beta\} \in \mathscr{M}.$$

Therefore,

$$fg = (1/2)[(f+g)^2 - f^2 - g^2] \in M(A).$$

Since

$$\{f \vee g > \alpha\} = \{f > \alpha\} \cup \{g > \alpha\},\ \{f \wedge g > \alpha\} = \{f > \alpha\} \cap \{g > \alpha\},$$

we have $f \vee g$ and $f \wedge g \in M(A)$. The result for extended-real valued functions is left as Exercise 3.4(A).

3.2 Limits and Special Functions

Recall that for a sequence $\langle f_n : n \in \mathbb{N} \rangle$, the value of the function $\sup_n f_n$ at x is $\sup_n f_n(x)$; a similar definition holds for $\inf_n f_n$. Also, $\limsup_n f_n := \inf_n(\sup_{k \geq n} f_k)$, and $\liminf_n f_n := \sup_n(\inf_{k \geq n} f_k)$.

Theorem 3.2.1. *If $\langle f_n : n \in \mathbb{N} \rangle$ is a sequence of measurable extended-real valued functions on a measurable set A, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, and $\liminf_n f_n$ are also measurable on A.*

Proof. For any $\alpha \in \mathbb{R}$,

$$\{\sup_n f_n > \alpha\} = \cup_n \{f_n > \alpha\}, \quad \{\inf_n f_n < \alpha\} = \cup_n \{f_n < \alpha\}.$$

The rest is clear.

Definition 3.2.1. A measure space (see Definition 2.4.2) is **complete** if every subset of a set of measure 0 is measurable. In this case, the measure is also called complete. If μ is a non-complete measure on a σ-algebra $\mathscr{A}$, then the family of sets

$$\{A \cup B : A \in \mathscr{A}, B \subseteq C \text{ for some } C \in \mathscr{A} \text{ with } \mu(C) = 0\}$$

is a σ-algebra on which the extension of μ is a complete measure. The extension of μ takes the value $\mu(A)$ for all sets $A \cup B$ with $A \in \mathscr{A}$ and $B \subseteq C$ for some $C \in \mathscr{A}$ with $\mu(C) = 0$. The enlarged σ-algebra together with the extension of μ is called the **completion** of the measure space (A, μ).

Proposition 3.2.1. *The completion of a measure space is a complete measure space.*

Proof. Exercise 3.10.

In the case of a complete measure, the value of the measure on each subset of a set of measure 0 is 0. The measures we have defined using integrators are all complete. One may, however, want to consider incomplete measures such as Lebesgue measure restricted to the Borel sets. When dealing with several measures at the same time, one often cannot complete all of them, since a set of measure 0 for one measure may not be measurable for another.

When the measure m is understood, we will say that something is true **almost everywhere** (a.e.) if it is true in the complement of a set of measure 0. For example, $f = g$ a.e. on A if there is a subset B of A with $m(B) = 0$ such that $f(x) = g(x)$ for every $x \in A \setminus B$. A sequence $\langle f_n : n \in \mathbb{N}\rangle$ converges to f a.e. on A if there is a subset $B \subseteq A$ of measure 0 such that $f_n(x) \to f(x)$ for each $x \in A \setminus B$. We don't know what happens on B. This definition is useful in dealing with measures that are not complete. For example, a function f may be identically 0 except for a non-Borel subset B of a Borel set of measure 0. For integration, the value taken by f on B is not important, and so it is useful to say that $f = 0$ almost everywhere.

Proposition 3.2.2. *For a complete measure, such as Lebesgue measure, if f is measurable on A and $f = g$ almost everywhere on A, then g is measurable on A.*

Proof. Let B be the set of measure 0 outside of which $f = g$. Then g is measurable on $A \setminus B$, and since any subset of B is measurable, g is measurable on B.

Proposition 3.2.3. *If m is a complete measure and $\langle f_n : n \in \mathbb{N}\rangle$ is a sequence of measurable functions on a measurable set A such that $f_n \to f$ a.e. on A, then f is measurable on A.*

Proof. The result follows from the equality $f = \limsup_n f_n = \liminf_n f_n$ a.e.

Definition 3.2.2. A **step function** is a real-valued function g defined on an interval $[a,b]$ such that for some finite set $\{x_i : 0 \le i \le n\}$ with $a = x_0 < \cdots < x_n = b$, g is constant on each of the open intervals (x_{i-1}, x_i).

Definition 3.2.3. A **characteristic function** is a function that takes only the values 0 and 1. The set on which it takes the value 1 is the **associated set** A, and the function is called the **characteristic function** of A. We will write χ_A for this function. Another common notation for the function is 1_A. The term **indicator function** is also used.

Clearly, $\chi_\varnothing$ is the constant 0, while the characteristic function of the set in which one is working is the constant 1. A characteristic function is measurable if and only if the associated set is a measurable set (all with respect to some fixed σ-algebra). It is easy to see that,

$$\chi_{A\cap B} = \chi_A \cdot \chi_B, \quad \chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B, \quad \chi_{\tilde{A}} = 1 - \chi_A.$$

Definition 3.2.4. A **simple function** is a measurable function with range equal to a finite subset of $\mathbb{R}$.

Any finite linear combination of measurable characteristic functions is a simple function. Such a representation is not unique. For example, the characteristic function of the union of two disjoint sets is the sum of their characteristic functions. Conversely, if $\alpha_1, \cdots, \alpha_n$ are the distinct nonzero values in the range of a simple function φ that is not identically equal to 0, then $\varphi = \sum_{i=1}^n \alpha_i \cdot \chi_{\{\varphi=\alpha_i\}}$. This is the simplest such combination that gives φ. A step function is a finite linear combination of characteristic functions of intervals. Some intervals may be degenerate intervals of the form $[c] := \{c\}$.

Recall that the family of Borel sets in $\mathbb{R}$ is the smallest σ-algebra containing the open subsets of $\mathbb{R}$.

Definition 3.2.5. A real-valued function is **Borel measurable** if the inverse image of each open subset of $\mathbb{R}$ is a Borel set.

Proposition 3.2.4. *The Borel measurable real-valued functions defined on a fixed Borel subset of* $\mathbb{R}$ *form a vector space over* $\mathbb{R}$*; that vector space contains the continuous functions and is stable with respect to pointwise multiplication and the operations* $\vee$ *and* $\wedge$.

Proof. Exercise 3.12.

Proposition 3.2.5. *Let f be a measurable real-valued function with Borel measurable range, and let h be a Borel measurable real-valued function with Borel measurable range. If g is a Borel measurable real-valued function defined on the range of f, then $g \circ f$ is measurable. If g is a Borel measurable real-valued function defined on the range of h, then $g \circ h$ is Borel measurable.*

Proof. We know that for any open set O, $f^{-1}[O]$ is a measurable set, $h^{-1}[O]$ is a Borel set, and $g^{-1}[O]$ is a Borel set. By Theorem 3.1.1, the collection of sets with measurable inverse images forms a σ -algebra. It follows that for any Borel set B, $f^{-1}[B]$ is a measurable set, $h^{-1}[B]$ is a Borel set, and $g^{-1}[B]$ is a Borel set. The rest is clear.

Example 3.2.1. The following example shows that there are Lebesgue measurable sets that are not Borel sets. Let f_1 be the Cantor-Lebesgue function on $[0,1]$. Recall that f_1 is an increasing continuous function mapping $[0,1]$ onto $[0,1]$. It is constant on each of the intervals that is removed to form the Cantor set C. For example, on $(1/3,2/3)$, f_1 takes the value $1/2$. Let f be defined on $[0,1]$ by setting $f(x) = f_1(x)+x$ for all $x \in [0,1]$. Since f_1 is an increasing continuous function and $x \mapsto x$ is a strictly increasing continuous function, f is strictly increasing and continuous; the range is $[0,2]$. Since f is a continuous bijection of the compact set $[0,1]$ onto $[0,2]$, it is a homeomorphism. (See Problem 1.36.) Each of the open intervals (a,b) removed from $[0,1]$ to form C has image $(f_1(a)+a, f_1(a)+b)$, so $f[[0,1]\setminus C]$ has the same Lebesgue measure in $[0,2]$ as has the set $[0,1]\setminus C$ in $[0,1]$. Therefore, $\lambda(f[C]) = 1$. By Problem 2.32, there is a non-Lebesgue measurable set $A \subset f[C]$. The function $g := f^{-1}$ is a homeomorphism of $[0,2]$ onto $[0,1]$, and $g[A]$ is a Lebesgue measurable subset of C since $\lambda(C) = 0$. While $g[A]$ is a Lebesgue measurable set, it is not a Borel set since g is continuous, and therefore Borel measurable, but $g^{-1}[g[A]] = f[g[A]] = A$ is not even Lebesgue measurable. Also note that the restriction of the continuous function f to $g[A]$ does not have a measurable range.

3.3 Approximations and Theorems of Lusin and Egoroff

In this section, we show that a set of finite measure has nice properties once a set of small measure, appropriate for the property, is removed. This heuristic principle, i.e., "sets of finite measure are nearly good", is essentially due to Littlewood [25]. We start with an operation used to indicate the difference of two sets. Recall that for two sets A and B, the **symmetric difference** is $A\Delta B := (A\setminus B)\cup(B\setminus A)$. We apply the symmetric difference to obtain an approximation for a measurable set of finite measure in $\mathbb{R}$.

Theorem 3.3.1. *Let A be a set of finite measure in $\mathbb{R}$. Given $\delta > 0$, there is a compact subset K of A for which $m(A\setminus K) < \delta$. Given $\varepsilon > 0$, there is a finite collection of disjoint open intervals I_i such that the measure of the symmetric difference $m(A\Delta(\cup_i I_i)) < \varepsilon$.*

Proof. We may assume $m(A) > 0$. As $n \to \infty$, $A \cap [-n,n] \nearrow A$, so we may choose a positive integer n so that $m(A\setminus[-n,n]) < \delta/2$. By Theorem 2.5.1, there is a closed, and therefore compact, set $K \subseteq A\cap[-n,n]$ so that $m((A\cap[-n,n])\setminus K) < \delta/2$. Therefore, $K \subseteq A$ and $m(A\setminus K) < \delta$. Choose a compact set $K \subseteq A$ with

$m(A \setminus K) < \varepsilon/2$. We may cover K with a bounded open set O so that $m(O \setminus K) < \varepsilon/2$. Since O is the countable union of pairwise disjoint, finite open intervals, those intervals form an open cover of K. We may discard all but a finite number of them and still cover K with the union $\cup_i I_i$. This is the desired approximation since

$$A\Delta(\cup_i I_i) = (A \setminus (\cup_i I_i)) \cup ((\cup_i I_i) \setminus A) \subseteq (A \setminus K) \cup (O \setminus K).$$

Proposition 3.3.1. *Let f be a measurable extended-real valued function that is finite almost everywhere on its domain A. If $m(A) < +\infty$, then for each $\varepsilon > 0$, there is a measurable subset $B \subseteq A$ with $m(B) < \varepsilon$ such that for some $M \in \mathbb{N}$, $|f(x)| \leq M$ for all $x \in A \setminus B$.*

Proof. Fix $\varepsilon > 0$. If f is already bounded, set $B = \varnothing$. Otherwise, for each $n \in \mathbb{N}$, set $E_n = \{|f| > n\}$, and let $E = \{f = +\infty\} \cup \{f = -\infty\}$. Since $m(E_1) < +\infty$ and $E_n \searrow E$, while $m(E) = 0$, there is an $n \in \mathbb{N}$ with $m(E_n) < \varepsilon$. Set $B = E_n$ and $M = n$.

Example 3.3.1. The function given by $f(x) = x$ is finite everywhere on $\mathbb{R}$, but it is unbounded on sets of infinite Lebesgue measure.

It follows from Proposition 3.3.1 that, if we are given an unbounded measurable function f that is finite almost everywhere on an interval $[a,b]$, then for any $\delta > 0$ and some $M \in \mathbb{N}$, we may apply our next result to the function $-M \vee f \wedge M$, which equals f outside of a set of measure less than δ.

Theorem 3.3.2. *Let f be a bounded measurable function on a closed, non-degenerate interval $[a,b]$. Let $s = \inf_{x\in[a,b]} f(x)$ and $S = \sup_{x\in[a,b]} f(x)$. Fix $\varepsilon > 0$.*

a) *There is a simple function φ defined on $[a,b]$ such that $s \leq \varphi$ and $f(x) - \varepsilon \leq \varphi(x) \leq f(x)$ for all $x \in [a,b]$, whence $|f - \varphi| \leq \varepsilon$ on $[a,b]$.*
b) *If $m(\{x\}) = 0$ for each singleton set $\{x\} \subset [a,b]$, then there is a subset $B_1 \subseteq [a,b]$ with $m(B_1) < \varepsilon/2$ and a step function g defined on $[a,b]$ such that $s \leq g \leq S$ and $g(x) = \varphi(x)$ for all $x \in [a,b] \setminus B_1$.*
c) *If $m(\{x\}) = 0$ for each singleton set $\{x\} \subset [a,b]$, then there is a subset $B_2 \subseteq [a,b]$ with $m(B_2) < \varepsilon/2$ and a continuous function h defined on $[a,b]$ such that $s \leq h \leq S$ and $h(x) = g(x)$ for all $x \in [a,b] \setminus B_2$. In this case, $|f(x) - g(x)| \leq \varepsilon$ and $|f(x) - h(x)| \leq \varepsilon$ for all $x \in [a,b] \setminus (B_1 \cup B_2)$.*

Proof. **a)** Partition $[s,S]$ with a finite number of points $s = y_0 < y_1 < \cdots < y_k = S$, so that for each i, $y_i - y_{i-1} < \varepsilon$. Of course, $f^{-1}[[y_{i-1}, y_i)]$ may be empty for some values of i. Let

$$\varphi = \left(\sum_{i=1}^{k-1} y_{i-1} \cdot \chi_{f^{-1}[[y_{i-1},y_i)]}\right) + y_{k-1} \cdot \chi_{f^{-1}[[y_{k-1},y_k]]}.$$

Now φ is a simple function with $s \leq \varphi$, and $f(x) - \varepsilon \leq \varphi(x) \leq f(x)$ for all $x \in [a,b]$, whence $|f - \varphi| \leq \varepsilon$ on $[a,b]$.

b) Let $\alpha_1, \cdots, \alpha_n$ be the n distinct values taken by φ, and let these be taken on n pairwise disjoint, measurable subsets $A_1, \cdots, A_n$ of $[a,b]$. By Theorem 3.3.1, for each set A_k, $1 \le k \le n$, there is a finite, pairwise disjoint collection of open intervals $I_1^k, \cdots, I_{l_k}^k$, with each contained in (a,b), such that $m(A_k \Delta (\cup_i I_i^k)) < \varepsilon/(2n)$. Set $B_1 := \cup_{k=1}^n (A_k \Delta (\cup_i I_i^k))$, and note that $m(B_1) < \varepsilon/2$. Let $\mathscr{J}$ be the collection of all of the intervals involved; that is, $\mathscr{J} = \cup_{k=1}^n \left\{I_1^k, \cdots, I_{l_k}^k\right\}$. Let P be the finite collection of endpoints of the intervals in $\mathscr{J}$. Add the points of the null set P to B_1. For each $I \in \mathscr{J}$, if $I \cap P \neq \varnothing$, replace I with the open intervals in $I \diagdown P$. Removing duplication, this yields a finite collection $\mathscr{I}$ of pairwise disjoint open intervals such that each interval in $\mathscr{I}$ is contained in at least one interval of $\mathscr{J}$. Consider an interval $J \in \mathscr{I}$ such that for $p \neq q$ and some i_0 and j_0, $J \subseteq I_{i_0}^p \cap I_{j_0}^q$. We now show that since $A_p \cap A_q = \varnothing$, $J \subseteq B_1$. That is, fix $x \in J$. If $x \notin A_p$, then since $x \in I_{i_0}^p$, $x \in A_p \Delta (\cup_i I_i^p) \subseteq B_1$. If $x \in A_p$, then $x \notin A_q$, but x is also in $I_{j_0}^q$, so $x \in A_q \Delta \left(\cup_j I_j^q\right) \subseteq B_1$. In either case, $x \in B_1$. Thus, $J \subseteq B_1$. Discard from the collection $\mathscr{I}$ all such intervals contained in B_1. Each of the remaining intervals in $\mathscr{I}$ corresponds to a unique A_k; set the value of g equal to the appropriate α_k for each such interval. At all other points of $[a,b]$, set g equal to $(s+S)/2$. The function g is a step function such that $s \le g \le S$ and $g(x) = \varphi(x)$ for all $x \in [a,b] \setminus B_1$.

c) Given the step function g formed in Part b), center an open interval at each member of the finite collection of points consisting of the endpoints of $[a,b]$ together with the points inside $[a,b]$ where g changes values. The intervals should be pairwise disjoint forming a set B_2 of total length $< \varepsilon/2$. It follows that $[a,b] \setminus B_2$ is the disjoint union of closed intervals on each of which g is constant. Use linear interpolation to obtain a continuous function h on $[a,b]$ such that $s \le h \le S$, and $g(x) = h(x)$ for all $x \in [a,b] \setminus B_2$. It is now the case that $\varphi(x) = h(x)$ for all $x \in [a,b] \setminus (B_1 \cup B_2)$.

We have shown that if f is a measurable real-valued function on an interval $[a,b]$ where points have 0 measure, then outside of a set of small measure we may uniformly approximate f with a continuous function h. The values of h, however, are only near the values of f. An important result due to Lusin [36] states that for a measurable real-valued function f on a set A of finite measure, there is a compact subset K of A having most of the measure of A such that the values taken by f on K are equal to the values taken by a continuous real-valued function g defined on the real line. In this sense, f is "nearly" continuous on A; that is, f deviates from the continuous function g on A only on the set of small measure $A \setminus K$. Recall that by Proposition 1.11.3, once it is shown that the restriction of f to a compact subset K of $\mathbb{R}$ is continuous, there is a continuous function g defined on the whole real line such that $g = f$ on K. Moreover, $\sup_{\mathbb{R}} g = \max_K f$, and $\inf_{\mathbb{R}} g = \min_K f$.

Theorem 3.3.3 (Lusin). *Fix a measurable set $A \subseteq \mathbb{R}$ with $m(A) < +\infty$, and let f be a real-valued measurable function with domain A. For any $\varepsilon > 0$, there is a compact set $K \subseteq A$ with $m(A \setminus K) < \varepsilon$ such that the restriction of f to K is continuous.*

Proof. Let $\langle V_n : n \in \mathbb{N}\rangle$ be an enumeration of the open intervals with rational endpoints in $\mathbb{R}$. By Theorem 3.3.1, we may fix compact sets $K_n \subseteq f^{-1}[V_n]$ and $K'_n \subseteq A \setminus f^{-1}[V_n]$ for each n so that $m(A \setminus (K_n \cup K'_n)) < \varepsilon/2^n$. Now, for the compact set $K := \bigcap_{n\in\mathbb{N}}(K_n \cup K'_n)$, $m(A \setminus K) < \varepsilon$. Given $x \in K$ and an open interval I containing $f(x)$, for some $n \in \mathbb{N}$, $f(x) \in V_n \subseteq I$. Now $x \in O := \complement K'_n$, and

$$f[O \cap K] \subseteq f\left[\complement K'_n \cap \left(K_n \cup K'_n\right)\right] = f[K_n] \subseteq V_n.$$

Remark 3.3.1. This simple proof of Lusin's theorem was first published by the text's author and Erik Talvila in 2004 [34]. Lusin's theorem holds in quite general settings, where it is usually stated just for a Borel measurable function f. The domain of f should have the property that sets of finite measure can be approximated from the inside by compact sets, and the target set or range of f should have a countable collection of open sets V_n such that for each open set O and each $y \in O$, there is an n with $y \in V_n \subseteq O$. (Later, we will call this property the second axiom of countability.)

Lusin's theorem is often established as a corollary of the following approximation theorem of Egoroff [18]. That important theorem states that on a set of finite measure, almost everywhere convergence of measurable functions to a finite limit is actually uniform convergence off of a set of small measure. That is, almost everywhere convergence on a set of finite measure is "nearly" the same as uniform convergence.

Lemma 3.3.1. *On a set $A \subseteq \mathbb{R}$ of finite measure, let $\langle f_n : n \in \mathbb{N}\rangle$ be a sequence of measurable functions converging a.e. to a function f. Suppose that f is finite a.e. on A. Then for any $\delta > 0$, there is an $N \in \mathbb{N}$ and a measurable $B \subseteq A$ with $m(B) < \delta$ such that*

$$\forall x \in A \setminus B, \quad \forall n \geq N, \quad |f_n(x) - f(x)| < \delta.$$

Proof. Let D be the set where either f is not finite-valued or the convergence fails. Since $m(D) = 0$, we may set each f_n and f equal to 0 on D and work with the modified functions without loss of generality. Fix $\delta > 0$, and let $S_n = \{|f_n - f| \geq \delta\}$. Now, $\limsup\ S_n = \varnothing$, so $\lim_{k\to\infty} m(\cup_{n\geq k} S_n) = 0$. Choose $N \in \mathbb{N}$ so that $m(\cup_{n\geq N} S_n) < \delta$, and let $B = \cup_{n\geq N} S_n$.

Theorem 3.3.4 (Egoroff). *On a set $A \subseteq \mathbb{R}$ of finite measure, let $\langle f_n : n \in \mathbb{N}\rangle$ be a sequence of measurable functions converging a.e. to a function f. Suppose f is finite a.e. on A. For any $\varepsilon > 0$, there is a measurable set $B \subseteq A$ with $m(B) < \varepsilon$ such that f_n converges uniformly to f on $A \setminus B$.*

Proof. Fix $\varepsilon > 0$. For each $k \in \mathbb{N}$, it follows from Lemma 3.3.1 with $\delta = \varepsilon/2^k$ that there is an $N_k \in \mathbb{N}$ and a measurable set $B_k \subset A$ with $m(B_k) < \varepsilon/2^k$ such that

$$\forall n \geq N_k, \ |f_n - f| < \varepsilon/2^k \text{ on } A \setminus B_k.$$

Let $B = \cup_k B_k$, so $m(B) < \varepsilon$. The functions f_n converge uniformly to f on $A \setminus B$ since for all $n \geq N_k$, $|f_n - f| < \varepsilon/2^k$ on $A \setminus B_k \supseteq A \setminus B$.

3.4 Problems

Problem 3.1. Prove Corollary 3.1.1.

Problem 3.2. Let $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be measurable functions and $g(x) \neq 0$ at any $x \in \mathbb{R}$. Show that the function f/g is measurable.

Problem 3.3. Prove Proposition 3.1.2.

Problem 3.4 (A). Finish the proof of Proposition 3.1.3 for extended-real valued functions.

Problem 3.5. Let $\langle f_n \rangle$ be a sequence of real-valued measurable functions on $\mathbb{R}$. Show that the set $\{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x)$ exists and is finite$\}$ is measurable.

Problem 3.6. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a differentiable function. Show that the derivative f' is a measurable function.

Problem 3.7. Prove or give a counterexample: The supremum of an uncountable family of measurable functions is always measurable.

Problem 3.8. Given $f : [0,1] \mapsto \mathbb{R}$, suppose the set $\{x \in [0,1] : f(x) = r\}$ is measurable for every $r \in \mathbb{R}$. Does it then follows that f is measurable?

Problem 3.9. Suppose that $f : [0,1] \mapsto \mathbb{R}$ is a function with the property that for any $\varepsilon > 0$, there is a continuous function $f_\varepsilon : [0,1] \mapsto \mathbb{R}$ such that $f = f_\varepsilon$ on A_ε, where $m([0,1] \setminus A_\varepsilon) < \varepsilon$. Show that f is measurable.

Problem 3.10. Prove Proposition 3.2.1. **Hint:** Given subsets A, B, and C of a set X with $B \subseteq C$,

$$X \setminus (A \cup B) = (X \setminus (A \cup C)) \cup (C \setminus (A \cup B)).$$

Problem 3.11. Let f be a real-valued function defined on $\mathbb{R}$ such that for each $\alpha \in \mathbb{R}$, $f^{-1}[(\alpha, +\infty)]$ is a Borel set.

a) Show that for each open subset O of $\mathbb{R}$, $f^{-1}[O]$ is a Borel set.
b) Show that for each Borel set $E \subseteq \mathbb{R}$, $f^{-1}[E]$ is a Borel set.
c) Show that if f is actually continuous on $\mathbb{R}$, then for each Borel set $E \subseteq \mathbb{R}$, $f^{-1}[E]$ is a Borel set.

Problem 3.12. Prove Proposition 3.2.4.

Problem 3.13. Let $\{I_\alpha : \alpha \in A\}$ be an uncountable collection of open intervals in the real line such that the measure of the union, $m(\cup_{\alpha\in A} I_\alpha)$, is a finite number $r > 0$. Given an arbitrary $\varepsilon > 0$, show that there is a finite subcollection $\{I_1, I_2, \cdots, I_n\}$ of the collection $\{I_\alpha : \alpha \in A\}$ such that $\sum_{i=1}^{n} m(I_i) > r - \varepsilon$.

Problem 3.14. a) Show that there does not exist a simple function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ such that $|x^2 - \varphi(x)| \leq 1$ for all $x \in \mathbb{R}$.

b) Prove or give a counterexample: For every Lebesgue measurable set $E \subset \mathbb{R}$ of finite measure, there exists a simple function $\psi : \mathbb{R} \mapsto \mathbb{R}$ such that $|x^2 - \psi(x)| \leq 1$ for all $x \in E$.

Problem 3.15. **a)** Show that if f is a measurable real-valued function with measurable range and g a continuous real-valued function defined on the real line, then $g \circ f$ is measurable.
b) Show that a continuous function with measurable range followed by a measurable function need not be measurable. **Hint:** See Example 3.2.1.

Problem 3.16. Show that an increasing real-valued function on the interval $[0,1]$ can have only a finite or countably infinite number of jumps.

Problem 3.17. Define $f : (0,1) \longmapsto \mathbb{R}$ as follows: For each $k \in \mathbb{N}$, set $f(x) = \left(\frac{1}{k} - x\right)^{-1}$ for all $x \in \left[\frac{1}{k+1}, \frac{1}{k}\right)$. For example, for $x \in [\frac{1}{2}, 1)$, $f(x) = \frac{1}{1-x}$. For each $n \in \mathbb{N}$, set $f_n(x) := \frac{1}{n} f(x)$ for all $x \in (0,1)$. Note that f_n converges pointwise to 0, but not uniformly to 0 on $(0,1)$.

a) Show that f_n is a measurable function on $(0,1)$ for each $n \in \mathbb{N}$.
b) Fix $\varepsilon > 0$. Construct a Lebesgue measurable set E such that $\lambda(E) < \varepsilon$ and f_n converges uniformly to 0 on $(0,1) \setminus E$.

Problem 3.18 (A). Given an increasing real-valued function f on an interval I, show that f is measurable. **Hint:** First consider the strictly increasing function for some $n \in \mathbb{N}$, $x \longmapsto f(x) + x/n$.

Problem 3.19. Let f be a continuous real-valued function on $\mathbb{R}$. Show that if A is an F_σ subset of $\mathbb{R}$, then $f[A]$ is an F_σ set.

Problem 3.20. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a **Lipschitz function**; that is, there is an $M > 0$ such that $|f(x) - f(y)| \leq M \cdot |x - y|$ for all $x, y \in \mathbb{R}$. Show that for any Lebesgue measurable set E, $f[E]$ is a Lebesgue measurable set. **Hint:** Recall Corollary 2.5.1 and Problem 3.19.

Problem 3.21. Let $E \subseteq \mathbb{R}$ be a measurable set of finite measure, and let f be a real-valued measurable function on E. Show that f is the a.e. limit of a sequence of continuous functions.

Problem 3.22. Let f be a real-valued function with domain $\mathbb{R}$ such that the inverse image of every closed subset of $\mathbb{R}$ is an open subset of $\mathbb{R}$. Show that for some value $a \in \mathbb{R}$, $f(x) \equiv a$ on $\mathbb{R}$. **Hint:** Recall Problem 1.22.