11. Qualitative Inductive Generalization and Confirmation

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Inductive generalization is a defeasible type of inference which we use to reason from the particular to the universal. First, a number of systems are presented that provide different ways of implementing this inference pattern within first-order logic. These systems are defined within the adaptive logics framework for modeling defeasible reasoning. Next, the logics are re-interpreted as criteria of confirmation. It is argued that they withstand the comparison with two qualitative theories of confirmation, Hempel's satisfaction criterion and hypothetico-deductive confirmation.

Logics of induction are tools for evaluating the strength of arguments which are not deductively valid. There are many kinds of argument the conclusion of which is not guaranteed to follow from its premises, and there are many ways to evaluate the strength of such arguments. This chapter focusses on one particular kind of non-deductive argument, and on one particular method of implementation. The type of argument under consideration here is that of inductive generalization, as when we reason from the particular to the universal. A number of logics are discussed which permit us, given a set of objects sharing or not sharing a number of properties, to infer generalizations of the form *All x are P*, or *All x with property P share property Q*. Inductive generalization is a common practice which has proven its use in scientific endeavor. For instance, given the fact that the relatively few electrons measured so far carry a charge of -1.6×10^{-19} Coulombs, we believe that all electrons have this charge [11.1].

11.1 Adaptive Logics for Inductive Generalization

The methods used here for formalizing practices of inductive generalization stem from the adaptive logics framework. Adaptive logics are tools developed for modeling defeasible reasoning, equipped with a proof theory that nicely captures the dynamics of non-monotonic – in this case, inductive – inference. In proofs for adaptive logics for inductive generalization, the conditional introduction of generalizations is allowed. The proof theory is also equipped with a mechanism taking care that conditionally introduced generalizations get retracted in case their condition is violated, for instance when the generalization in question is falsified by the premises.

In Sect. 11.2 and 11.3 the general framework of adaptive logics is introduced, and a number of existing adaptive logics for inductive generalization are defined. The differences between these logics arise from different choices made along one of two dimensions. A first dimension concerns the specific condition required for introducing generalizations in an adaptive proof. A very permissive approach allows for their free introduction, without taking into account the specifics of the premises. This is the idea behind the logic **LI**. A more economical approach is to permit the introduction of a generalization on the condition that at least one instance of it it present. This is the rationale behind a second logic, **IL**. In an **IL**-proof a generalization *All P are Q* can be introduced only if the premise set contains at least one object which is either *not-P* or *Q*. More economical still is the rationale behind a third logic, **G**, which aims to capture the requirement of knowing at least one *positive* instance of a generalization before introducing it in a proof. That is, in a **G**-proof a generalization *All P are Q* can be introduced if the premise set contains at least one object which is *both P* and *Q*.

The second dimension along which different consequence relations are generated concerns the specific mechanism used for retracting generalizations introduced in adaptive proofs. It is often not sufficient to demand retraction just in case a generalization is falsified by the premises. For instance, if the consequence sets of our logics are to be closed under classical logic, jointly incompatible generalizations should not be derivable, even though none of them is falsified by our premise set. Within the adaptive logics framework, various strategies are available for retracting conditional moves in an adaptive proof. Two such strategies are presented in this chapter: the reliability strategy and the minimal abnormality strategy.

Combining both dimensions, a family of six adaptive logics for inductive generalization is obtained (it contains the systems **LI**, **IL**, and **G**, each of which can be defined using either the reliability or the minimal abnormality strategy). These logics have all been presented elsewhere (for **LI**, see [11.2–4]. For **IL** and **G**, see [11.5]). The original contribution of this chapter consists in a study comparing these systems to some existing qualitative criteria of confirmation. There is an overlap between the fields of inductive logic and confirmation theory. In 1943 already, Hempel noted that the development of a logical theory of confirmation might be regarded as a contribution to the field of inductive logic [11.6, p. 123]. In Sect. 11.4 the logics from Sect. 11.2 and 11.3 are re-interpreted as qualitative criteria of confirmation, and are related to other qualitative models of confirmation: Hempel's satisfaction criterion (Sect. 11.4.1) and the hypothetico-deductive model (Sect. 11.4.2). Section 11.4 ends with some remarks on the heuristic guidance that adaptive logics for inductive generalization can provide in the derivation and subsequent confirmation of additional generalizations (Sect. 11.4.3).

The following notational conventions are used throughout the chapter. The formal language used is that of first-order logic without identity. A primitive functional formula of rank 1 is an open formula that does not contain any logical symbols ($\exists, \forall, \neg, \vee, \wedge, \supset, \equiv$), sentential letters, or individual constants, and that contains only predicate letters of rank 1. The set of functional atoms of rank 1, denoted $\mathcal{A}^{f1}$, comprises the primitive functional formulas of rank 1 and their negations. A *generalization* is the universal closure of a disjunction of members of $\mathcal{A}^{f1}$. That is, the set of generalizations in this technical sense is the set $\{\forall(A_1 \vee \ldots \vee A_n) \mid A_1, \ldots, A_n \in \mathcal{A}^{f1}; n \geq 1\}$, where $\forall$ denotes the universal closure of the subsequent formula. Occasionally the term generalization is also used for formulas equivalent to a member of this set, e.g., $\forall x(Px \supset Qx)$. It is easily checked that generalizations $\forall(A_1 \vee \ldots \vee A_n)$ can be rewritten as formulas of the general form $\forall((B_1 \wedge \ldots \wedge B_j) \supset (C_1 \vee \ldots \vee C_k))$, and vice versa, where all B_i and C_j belong to $\mathcal{A}^{f1}$.

11.2 A First Logic for Inductive Generalization

In this section the standard format (SF) for adaptive logics is introduced and explained. Its features are illustrated by means of the logic **LI** from [11.3, 4], chronologically the first adaptive logic for inductive generalization. A general characterization of the SF is provided, and its proof theory is explained. For a more comprehensive introduction, including the semantics and generic meta-theory of the SF, see, e.g., [11.7, 8].

11.2.1 General Characterization of the Standard Format

An adaptive logic (AL) within the SF is defined as a triple, consisting of:

(i) A *lower limit logic* (LLL), a logic that has static proofs and contains classical disjunction
(ii) A *set of abnormalities*, a set of formulas that share a (possibly) restricted logical form, or a union of such sets
(iii) An *adaptive strategy*.

The LLL is the stable part of the AL: anything derivable by means of the LLL is derivable by means of the AL. Explaining the notion of static proofs is beyond the scope of this chapter. For a full account, see [11.9]. (Alternatively, the static proofs requirement can be replaced by the requirement that the lower limit logic has a reflexive, monotonic, transitive, and compact consequence relation [11.8].) In any case, it suffices to know that the first-order fragment of Classical Logic (**CL**) meets this requirement, as we work almost exclusively with **CL** as a LLL. The lower limit logic of **LI** is **CL**.

Typically, an AL enables one to derive, for most premise sets, some extra consequences on top of those that are LLL-derivable. These supplementary consequences are obtained by interpreting a premise set *as normally as possible*, or, equivalently, by supposing abnormalities to be false *unless and until proven otherwise*. What it means to interpret a premise set *as normally as possible* is disambiguated by the strategy, element (iii).

The normality assumption made by the logics to be defined in this chapter amounts to supposing that the world is in some sense uniform. *Normal* situations are those in which it is safe to derive generalizations. *Abnormal* situations are those in which generalizations are falsified. In fact, the set of **LI**-abnormalities, denoted $\Omega_{\mathbf{LI}}$, is just the set of falsified generalizations (the definitions are those from [11.5]; in [11.10, Sect. 4.2.2] it is shown that the same logic is obtained if $\Omega_{\mathbf{LI}}$ is defined as the set of formulas of the form $\neg\forall x A(x)$, where A contains no quantifiers, free variables, or constants)

$$\Omega_{\mathbf{LI}} =_{\mathrm{df}} \{\neg\forall(A_1 \vee \ldots \vee A_n) \mid A_1, \ldots, A_n \in \mathcal{A}^{f1}; \; n \geq 1\} \; . \tag{11.1}$$

In adaptive proofs, it is possible to make conditional inferences assuming that one or more abnormalities are false. Whether or not such assumptions can be upheld in the continuation of the proof is determined by the adaptive strategy. The SF incorporates two adaptive strategies, the *reliability* strategy and the *minimal abnormality* strategy. In the generic proof theory of the SF, adaptive strategies come with a *marking definition*, which takes care of the withdrawal of certain conditional inferences in dynamic proofs. It will be easier to explain the intuitions behind these strategies after defining the generic proof theory for ALs. For now, just note that in the remainder **LI** is ambiguous between $\mathbf{LI^r}$ and $\mathbf{LI^m}$, where the subscripts *r* and *m* denote the reliability strategy, respectively the minimal abnormality strategy. Analogously for the other logics defined below.

11.2.2 Proof Theory

Adaptive proofs are *dynamic* in the sense that lines derived at a certain stage of a proof may be withdrawn at a later stage. Moreover, lines withdrawn at a certain stage can become derivable again at an even later stage, and so on. (A stage of a proof is a sequence of lines and a proof is a sequence of stages. Every proof starts off with stage 1. Adding a line to a proof by applying one of the rules of inference brings the proof to its next stage, which is the sequence of all lines written so far.)

A line in an adaptive proof consists of four elements: a line number, a formula, a justification and a *condition*. For instance, a line

$$j \quad A \quad i_1, \ldots, i_n; R \quad \Delta \, ,$$

reads: at line j, the formula A is derived from lines $i_1 - i_n$ by rule R on the condition Δ. The fourth element, the condition, is what permits the dynamics. Intuitively, the condition of a line in a proof corresponds to an assumption made at that line. In the example above, A was derived on the assumption that the formulas in Δ are false. If, later on in the proof, it turns out that this assumption was too bold, the line in question is withdrawn from the proof by a marking mechanism corresponding to an adaptive strategy. Importantly, only members of the set of abnormalities are allowed as elements of the condition of a line in an adaptive proof. Thus, assumptions always correspond to the falsity of one or more abnormalities, or, equivalently, to the truth of one or more generalizations.

Before explaining how the marking mechanism works, the generic inference rules of the SF must be introduced. There are three of them: a premise introduction rule (Prem), an unconditional rule (RU), and a conditional rule (RC). For adaptive logics with **CL** as their LLL, they are defined as follows

$$\begin{array}{lll}
\text{Prem} & \text{If } A \in \Gamma: & \begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array} \\[2ex]
\text{RU} & \text{If } A_1, \ldots, A_n \vdash_{\mathbf{CL}} B: & \begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array} \\[2ex]
\text{RC} & \text{If } A_1, \ldots, A_n \vdash_{\mathbf{CL}} B \vee \mathrm{Dab}(\Theta): & \begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}
\end{array} \; .$$

Where Γ is the premise set, Prem permits the introduction of premises on the empty condition at any time in the proof. Remember that conditions, at the intuitive level, correspond to assumptions, so Prem stipulates that premises can be introduced at any time without making any further assumptions.

Since ALs strengthen their LLL, one or more rules are needed to incorporate LLL-inferences in AL-proofs. In the proof theory of the SF, this is taken care of by the generic rule RU. This rule stipulates that whenever B is a **CL**-consequence of $A_1, \ldots, A_n$, and all of $A_1, \ldots, A_n$ have been derived in a proof, then B is derivable, provided that the conditions attached to the lines at which $A_1, \ldots, A_n$ were derived are carried over. Intuitively, if $A_1, \ldots, A_n$ are derivable assuming that the members of $\Delta_1, \ldots, \Delta_n$ are false, and if B is a **CL**-consequence of $A_1, \ldots, A_n$, then B is derivable, still assuming that all members of $\Delta_1, \ldots, \Delta_n$ are false.

Before turning to RC, here is an example illustrating the use of the rules Prem and RU. Let $\Gamma_1 = \{Pa \wedge Qa, Pb, \neg Qc\}$. Suppose we start an **LI**-proof for Γ_1 as follows

1	$Pa \wedge Qa$	Prem	$\emptyset$
2	Pb	Prem	$\emptyset$
3	$\neg Qc$	Prem	$\emptyset$
4	Pa	1; RU	$\emptyset$
5	Qa	1; RU	$\emptyset$

Let Θ be a finite set of **LI**-abnormalities, that is, $\Theta \subset \Omega_{\mathbf{LI}}$. Then Dab($\Theta$) refers to the classical disjunction of the members of Θ (Dab abbreviates *disjunction of abnormalities*; in the remainder, such disjunctions are sometimes referred to as Dab-formulas). RC stipulates that, whenever B is **CL**-derivable from $A_1, \ldots, A_n$ in disjunction with one or more abnormalities, then B can be inferred assuming that these abnormalities are false, i. e., we can derive B and add the abnormalities in question to the condition set, together with assumptions made at the lines at which $A_1, \ldots, A_n$ were derived.

For instance, (11.2) is **CL**-valid

$$\forall x(Px \vee Qx) \vee \neg\forall x(Px \vee Qx) \tag{11.2}$$

Note that the second disjunct of (11.2) is a member of $\Omega_{\mathbf{LI}}$. In the context of inductive generalization the assumption that the world is as *normal* as possible corresponds to an assumption about the uniformity of the world. In adaptive proofs, such assumptions are made explicit by applications of the conditional rule. Concretely, if a formula like (11.2) is derived in an **LI**-proof, RC can be used to derive the first disjunct on the condition that the second disjunct is false. In fact, since (11.2) is a **CL**-theorem, the generalization $\forall x(Px \vee Qx)$ can be inroduced right away, taking its negation to be false (lines 1–5 are not repeated)

6	$\forall x(Px \vee Qx)$	RC	$\{\neg\forall x(Px \vee Qx)\}$

In a similar fashion, RC can be used to derive other generalizations

7	$\forall xPx$	RC	$\{\neg\forall xPx\}$
8	$\forall xQx$	RC	$\{\neg\forall xQx\}$
9	$\forall x(\neg Px \vee Qx)$	RC	$\{\neg\forall x(\neg Px \vee Qx)\}$
10	$\forall x(Px \vee \neg Qx)$	RC	$\{\neg\forall x(Px \vee \neg Qx)\}$
11	$\forall x(\neg Px \vee \neg Qx)$	RC	$\{\neg\forall x(\neg Px \vee \neg Qx)\}$

Each generalization is derivable assuming that its corresponding condition is false. However, some of these assumptions clearly cannot be upheld. We know, for instance, that the generalizations derived at lines 8 and 11 are falsified by the premises at lines 3 and 1 respectively. So we need a way of distinguishing between *good* and *bad* inferred generalizations. This is where the adaptive strategy comes in. Since distinguishing *good* from *bad* generalizations can be done in different ways, there are different strategies available to us for making the distinction hard. First, the reliability strategy and its corresponding marking definition are introduced. The latter definition takes care of the retraction of *bad* generalizations.

Marking definitions proceed in terms of the minimal inferred Dab-formulas derived at a stage of a proof. A Dab-formula that is derived at a proof stage by RU at a line with condition $\emptyset$ is called an *inferred* Dab-*formula* of the proof stage.

Definition 11.1 Minimal inferred Dab*-formula*

Dab(Δ) is a *minimal inferred* Dab-*formula* at stage s of a proof iff Dab(Δ) is an inferred Dab-formula at stage s and there is no $\Delta' \subset \Delta$ such that Dab(Δ') is an inferred Dab-formula at stage s.

Where Dab(Δ_1), ..., Dab(Δ_n) are the minimal inferred Dab-formulas derived at stage s, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ is the set of formulas that are *unreliable* at stage s.

Definition 11.2 Marking for reliability

Where Δ is the condition of line i, line i is marked at stage s iff $\Delta \cap U_s(\Gamma) \neq \emptyset$.

To illustrate the marking mechanism, consider the following extension of the $\mathbf{LI}^{\mathbf{r}}$-proof for Γ_1 (marked lines are indicated by a ✓-sign; lines 1–5 are not repeated in the proof)

6	$\forall x(Px \vee Qx)$	RC	
	$\{\neg\forall x(Px \vee Qx)\}$✓		
7	$\forall xPx$	RC	
	$\{\neg\forall xPx\}$✓		
8	$\forall xQx$	RC	
	$\{\neg\forall xQx\}$✓		

9	$\forall x(\neg Px \vee Qx)$ $\{\neg\forall x(\neg Px \vee Qx)\}$✓	RC
10	$\forall x(Px \vee \neg Qx)$ $\{\neg\forall x(Px \vee \neg Qx)\}$	RC
11	$\forall x(\neg Px \vee \neg Qx)$ $\{\neg\forall x(\neg Px \vee \neg Qx)\}$✓	RC
12	$\neg\forall x Qx$ $\emptyset$	3; RU
13	$\neg\forall x(\neg Px \vee \neg Qx)$ $\emptyset$	1; RU
14	$\neg\forall x Px \vee \neg\forall x(\neg Px \vee Qx)$ $\emptyset$	3; RU
15	$\neg\forall x(Px \vee Qx) \vee \neg\forall x(\neg Px \vee Qx)$ $\emptyset$	3; RU

As remarked above, the generalizations derived at lines 8 and 11 are falsified by the premises, so it makes good sense to mark them and thereby consider them not derived anymore. As soon as we derive the negations of these generalizations (lines 12 and 13) Definition 11.2 takes care that lines 8 and 11 are marked. The generalizations derived at lines 6, 7, and 9 are not falsified by the data, yet they are marked according to Definition 11.2, due to the derivability of the minimal inferred Dab-disjunctions at lines 14 and 15. We know, for instance, that the generalizations derived at lines 7 and 9 cannot be upheld together: at line 14 we inferred that they are jointly incompatible in view of the premises. Definition 11.2 takes care that *both* lines 7 and 9 are marked at stage 15, since

$$U_{15}(\Gamma_1) = \{\neg\forall x Px, \neg\forall x Qx, \neg\forall x(Px \vee Qx), \neg\forall x(\neg Px \vee Qx), \neg\forall x(\neg Px \vee \neg Qx)\}\,. \tag{11.3}$$

The only inferred generalization left unmarked at stage 15 is $\forall x(Px \vee \neg Qx)$, derived at line 10.

Due to the dynamics of adaptive proofs, we cannot just take a formula to be an AL-consequence of some premise set Γ once we derived it at some stage on an unmarked line in a proof for Γ, for it may be that there are extensions of the proof in which the line in question gets marked. Likewise, we need to take into account the fact that lines marked at a stage of a proof may become unmarked at a later stage. This is taken care of by using the concept of *final derivability*:

Definition 11.3 Final derivability

A is *finally derived* from Γ at line i of a finite proof stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.

Definition 11.4 Logical consequence for $\mathbf{LI^r}$

$\Gamma \vdash_{\mathbf{LI^r}} A$ (A is finally $\mathbf{LI^r}$-derivable from Γ) iff A is finally derived at a line of an $\mathbf{LI^r}$-proof from Γ.

Given the premise set Γ_1, there are no extensions of the proof above in which any of the marked lines become unmarked, nor are there extensions in which line 10 is marked and cannot be unmarked again in a further extension of the proof. Hence, by Definitions 11.3 and 11.4

$$\Gamma_1 \nvdash_{\mathbf{LI^r}} \forall x Px\,, \tag{11.4}$$
$$\Gamma_1 \nvdash_{\mathbf{LI^r}} \forall x Qx\,, \tag{11.5}$$
$$\Gamma_1 \nvdash_{\mathbf{LI^r}} \forall x(Px \vee Qx)\,, \tag{11.6}$$
$$\Gamma_1 \vdash_{\mathbf{LI^r}} \forall x(Px \vee \neg Qx)\,, \tag{11.7}$$
$$\Gamma_1 \nvdash_{\mathbf{LI^r}} \forall x(\neg Px \vee Qx)\,, \tag{11.8}$$
$$\Gamma_1 \nvdash_{\mathbf{LI^r}} \forall x(\neg Px \vee \neg Qx)\,. \tag{11.9}$$

The logic $\mathbf{LI^r}$ is non-monotonic: adding new premises may block the derivation of generalizations that were finally derivable from the original premise set. For instance, suppose that we add the premise $\neg Pd \wedge Qd$ to Γ_1. Since the extra premise provides a counter-instance to the generalization $\forall x(Px \vee \neg Qx)$, the latter should no longer be $\mathbf{LI^r}$-derivable from the new premise set. The following proof illustrates that this is indeed the case

1	$Pa \wedge Qa$ $\emptyset$	Prem
2	Pb $\emptyset$	Prem
3	$\neg Qc$ $\emptyset$	Prem
4	$\neg Pd \wedge Qd$ $\emptyset$	Prem
5	$\forall x(Px \vee Qx)$ $\{\neg\forall x(Px \vee Qx)\}$✓	RC
6	$\forall x Px$ $\{\neg\forall x Px\}$✓	RC
7	$\forall x Qx$ $\{\neg\forall x Qx\}$✓	RC
8	$\forall x(\neg Px \vee Qx)$ $\{\neg\forall x(\neg Px \vee Qx)\}$✓	RC

9	$\forall x(Px \vee \neg Qx)$ $\{\neg\forall x(Px \vee \neg Qx)\}\checkmark$	RC
10	$\forall x(\neg Px \vee \neg Qx)$ $\{\neg\forall x(\neg Px \vee \neg Qx)\}\checkmark$	RC
11	$\neg\forall xPx$ $\emptyset$	4; RU
12	$\neg\forall xQx$ $\emptyset$	3; RU
13	$\neg\forall x(\neg Px \vee \neg Qx)$ $\emptyset$	1; RU
14	$\neg\forall x(Px \vee Qx) \vee \neg\forall x(\neg Px \vee Qx)$ $\emptyset$	3; RU
15	$\neg\forall x(Px \vee \neg Qx)$ $\emptyset$	4; RU

Line 9 is marked in view of the Dab-formula derived at line 15. There is no way to extend this proof in such a way that the line in question gets unmarked. Hence, $\Gamma_1 \cup \{\neg Pd \wedge Qd\} \nvdash_{\mathbf{LI}^r} \forall x(Px \vee \neg Qx)$. In fact, no nontautological generalizations whatsoever are $\mathbf{LI}^r$-derivable from the extended premise set $\Gamma_1 \cup \{\neg Pd \wedge Qd\}$.

11.2.3 Minimal Abnormality

Different interpretations of the same set of data may lead to different views concerning which generalizations should or should not be derivable. Each such view may be driven by its own rationale, and choosing one such rationale over the other is not a matter of pure logic. For that reason, different strategies are available to adaptive logicians, each interpreting a set of data in their own sensible way, depending on the context. The reliability strategy was defined already. The minimal abnormality strategy is slightly less skeptical. Consequently, for some premise sets, generalizations may be $\mathbf{LI}^m$-derivable, but not $\mathbf{LI}^r$-derivable.

Like reliability, the minimal abnormality strategy comes with its marking definition. Let a *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ be a set that contains one element out of each member of Σ. A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ. Where $\mathrm{Dab}(\Delta_1), \mathrm{Dab}(\Delta_2), \ldots$ are the minimal inferred Dab-formulas derived from a premise set Γ at stage s of a proof, $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \Delta_2, \ldots\}$.

Definition 11.5 Marking for minimal abnormality
Where A is the formula and Δ the condition of line i, line i is marked at stage s iff (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

An example will clarify matters. Let $\Gamma_2 = \{Pa \wedge Qa \wedge Ra, \neg Rb \wedge (\neg Pb \vee \neg Qb), \neg Pc \wedge \neg Qc \wedge Rc\}$.

1	$Pa \wedge Qa \wedge Ra$ $\emptyset$	Prem
2	$\neg Rb \wedge (\neg Pb \vee \neg Qb)$ $\emptyset$	Prem
3	$\neg Pc \wedge \neg Qc \wedge Rc$ $\emptyset$	Prem
4	$\forall x(Px \vee Qx)$ $\{\neg\forall x(Px \vee Qx)\}\checkmark$	RC
5	$\forall x(Px \vee Rx)$ $\{\neg\forall x(Px \vee Rx)\}\checkmark$	RC
6	$\forall x(\neg Px \vee Rx)$ $\{\neg\forall x(\neg Px \vee Rx)\}\checkmark$	RC
7	$\neg\forall x(Px \vee Qx)$ $\emptyset$	3; RU
8	$\neg\forall x(Px \vee Rx) \vee \neg\forall x(\neg Px \vee Rx)$ $\emptyset$	2; RU
9	$\forall x(Px \vee Rx) \vee \forall x(\neg Px \vee Rx)$ $\{\neg\forall x(Px \vee Rx)\}$	5; RU
10	$\forall x(Px \vee Rx) \vee \forall x(\neg Px \vee Rx)$ $\{\neg\forall x(\neg Px \vee Rx)\}$	6; RU

To see what is happening in this proof, we need to understand the markings. Note that there are two minimal choice sets at stage 10

$$\Phi_{10}(\Gamma_2) = \{\{\neg\forall x(Px \vee Qx), \neg\forall x(Px \vee Rx)\}, \{\neg\forall x(Px \vee Qx), \neg\forall x(\neg Px \vee Rx)\}\} . \quad (11.10)$$

Line 4 is marked in view of clause (i) in Definition 11.5, since its condition intersects with each minimal choice set in $\Phi_{10}(\Gamma_2)$. Lines 5 and 6 are marked in view of clause (ii) in Definition 11.5. For the minimal choice set $\{\neg\forall x(Px \vee Qx), \neg\forall x(Px \vee Rx)\}$, there is no line at which $\forall x(Px \vee Rx)$ was derived on a condition that does not intersect with this set. Hence line 5 is marked. Analogously, line 6 is marked because, for the minimal choice set $\{\neg\forall x(Px \vee Qx), \neg\forall x(\neg Px \vee Rx)\}$, there is no line at which $\forall x(\neg Px \vee Rx)$ was derived on a condition that does not intersect with this set.

Things change, however, when we turn to lines 9 and 10. In these cases, none of clauses (i) or (ii) of Definition 11.5 apply: for each of these lines, there is a minimal choice set in $\Phi_{10}(\Gamma_2)$ which does not intersect with the line's condition; and for each of the sets in $\Phi_{10}(\Gamma_2)$, we have derived the formula $\forall x(Px \vee Rx) \vee \forall x(\neg Px \vee Rx)$ on a condition that does not intersect with it. Hence, these lines remain unmarked at stage 10 of the proof.

Things would have been different if we made use of the reliability strategy, since

$$U_{10}(\Gamma_2) = \{\neg\forall x(Px \vee Qx), \neg\forall x(Px \vee Rx), \neg\forall x(\neg Px \vee Rx)\} . \quad (11.11)$$

In view of $U_{10}(\Gamma_2)$ and Definition 11.2, all of lines 4–6 and 9–10 would be marked if the above proof were a **LI**$^{\mathbf{r}}$-proof.

As with the reliability strategy, logical consequence for the minimal abnormality strategy is defined in terms of final derivability (Definition 11.3). A consequence relation for **LI**$^{\mathbf{m}}$ is defined simply by replacing all occurrences of **LI**$^{\mathbf{r}}$ in Definition 11.4 with **LI**$^{\mathbf{m}}$. Although the proof above can be extended in many interesting ways, showing the (non-)derivability of many more generalizations than those currently occurring in the proof, nothing will change in terms of final derivability with respect to the formulas derived at stage 10

$$\Gamma_2 \nvdash_{\mathbf{LI^m}} \forall x(Px \vee Qx) , \quad (11.12)$$
$$\Gamma_2 \nvdash_{\mathbf{LI^m}} \forall x(Px \vee Rx) , \quad (11.13)$$
$$\Gamma_2 \nvdash_{\mathbf{LI^m}} \forall x(Px \vee \neg Rx) , \quad (11.14)$$
$$\Gamma_2 \vdash_{\mathbf{LI^m}} \forall x(Px \vee Rx) \vee \forall x(\neg Px \vee Rx) , \quad (11.15)$$
$$\Gamma_2 \nvdash_{\mathbf{LI^r}} \forall x(Px \vee Qx) , \quad (11.16)$$
$$\Gamma_2 \nvdash_{\mathbf{LI^r}} \forall x(Px \vee Rx) , \quad (11.17)$$
$$\Gamma_2 \nvdash_{\mathbf{LI^r}} \forall x(Px \vee \neg Rx) , \quad (11.18)$$
$$\Gamma_2 \nvdash_{\mathbf{LI^r}} \forall x(Px \vee Rx) \vee \forall x(\neg Px \vee Rx) . \quad (11.19)$$

At the beginning of Sect. 11.2.3 it was mentioned that the rationale underlying the reliability strategy is slightly more skeptical than that underlying the minimal abnormality strategy. The point is illustrated by the proof for Γ_2. As we saw, the formula $\forall x(Px \vee Rx) \vee \forall x(\neg Px \vee Rx)$ is **LI**$^{\mathbf{m}}$-derivable from Γ_2, but not **LI**$^{\mathbf{r}}$-derivable from Γ_2.

11.3 More Adaptive Logics for Inductive Generalization

LI interprets the world as *uniform* by taking as normal those situations in which a generalization is true, and as abnormal those situations in which a generalization is false. But of course, if uniformity is identified with the truth of *every* generalization in this way, the world can never be completely uniform (for the simple fact that many generalizations are incompatible and cannot be jointly true). Perhaps a more natural way to interpret the uniformity of the world is to take all objects to have the same properties: as soon as one object has property P, we try to infer that all objects have property P. This is the rationale behind the logic **IL** from [11.5].

Roughly, the idea behind **IL** is to generalize from instances. Given an instance, the derivation of a generalization is permitted on the condition that no counter-instances are derivable. So abnormal situations are those in which both an instance and a counter-instance of a generalization are present. This is the formal definition of the set of **IL**-abnormalities

$$\Omega_{\mathbf{IL}} =_{\text{df}} \{\exists(A_1 \vee \ldots \vee A_n) \wedge \exists\neg(A_1 \vee \ldots \vee A_n) \mid A_1, \ldots, A_n \in \mathcal{A}^{f1}; n \geq 1\} . \quad (11.20)$$

The logic **IL** is defined by the lower limit logic **CL**, the set of abnormalities $\Omega_{\mathbf{IL}}$, and the adaptive strategy reliability (**IL**$^{\mathbf{r}}$) or minimal abnormality (**IL**$^{\mathbf{m}}$).

In an **IL**-proof generalizations cannot be conditionally introduced from scratch, since an instance is required. In this respect, **IL** is more demanding than **LI**. However, it does not follow that for this reason **IL** is a weaker logic, since it is also more difficult to derive (disjunctions of) abnormalities in **IL**. A simple example will illustrate that, for many premise sets, **IL** is in fact stronger than **LI**. Consider the following **IL**-proof from $\Gamma_3 = \{Pa, \neg Pb \vee Qb\}$

1	Pa	$\emptyset$	Prem
2	$\neg Pb \vee Qb$	$\emptyset$	Prem
3	$\forall x Px$	$\{\exists x Px \wedge \exists x\neg Px\}$	1; RC
4	Qb	$\{\exists x Px \wedge \exists x\neg Px\}$	2, 3; RU
5	$\forall x Qx$	$\{\exists x Px \wedge \exists x\neg Px, \exists x Qx \wedge \exists x\neg Qx\}$	4; RC

In view of $Pa \vdash_{\mathbf{CL}} \forall x Px \vee (\exists x Px \wedge \exists x\neg Px)$, we applied RC to line 1 and conditionally inferred $\forall x Px$ at line 3. Next, we used RU to infer Qb from this newly obtained generalization together with the premise at line 2. We now have an instance of $\forall xQx$, so we can conditionally infer the latter generalization, taking over the condition of line 4. Importantly, not a single disjunction of members of $\Omega_{\mathbf{IL}}$ is **CL**-derivable from Γ_3. This means that there is no way to mark any of lines 3–5 in any extension of this proof, independently of which strategy we use.

Consequence relations for $\mathbf{IL^r}$ and $\mathbf{IL^m}$ are again definable in terms of final derivability (Definition 11.3). All we need to do is replace all occurrences of $\mathbf{LI^r}$ in Definition 11.4 with $\mathbf{IL^r}$, respectively $\mathbf{IL^m}$. Hence

$$\Gamma_3 \vdash_{\mathbf{IL}} \forall xPx\,, \tag{11.21}$$

$$\Gamma_3 \vdash_{\mathbf{IL}} \forall xQx\,. \tag{11.22}$$

Compare the **IL**-proof above with the following **LI**-proof from Γ_3

1	Pa $\emptyset$	Prem
2	$\neg Pb \vee Qb$ $\emptyset$	Prem
3	$\forall xPx$ $\{\neg\forall xPx\}\checkmark$	RC
4	Qb $\{\neg\forall xPx\}\checkmark$	2, 3; RU
5	$\forall xQx$ $\{\neg\forall xQx\}\checkmark$	RC
6	$\neg\forall xPx \vee \neg\forall x\neg Qx$ $\emptyset$	1, 2; RU
7	$\neg\forall xQx \vee \neg\forall x(\neg Px \vee \neg Qx)$ $\emptyset$	1, 2; RU

Independently of the adaptive strategy used (reliability or minimal abnormality), there are no extensions of this **LI**-proof in which any of lines 3–5 become unmarked. Therefore

$$\Gamma_3 \nvdash_{\mathbf{LI}} \forall xPx\,, \tag{11.23}$$

$$\Gamma_3 \nvdash_{\mathbf{LI}} \forall xQx\,. \tag{11.24}$$

The premise set Γ_3 not only serves to show that **IL** is not strictly weaker than **LI** in terms of derivable generalizations. It also illustrates that, although in an **IL**-proof we generalize on the basis of instances, such an instance need not always be **CL**-derivable from the premise set. In the proof from Γ_3, we derived the generalization $\forall xQx$ even though no instance of this generalization is **CL**-derivable from Γ_3. Instead, we first derived $\forall xPx$ (of which Γ_3 *does* provide us with an instance), and then used this generalization to *infer* an instance of $\forall xQx$. This is perfectly in line with the intuition behind **IL**: If deriving a generalization on the basis of an instance leads us to more instances of other generalizations, then, assuming the world to be as uniform as possible, we take the world to be uniform with respect to these other generalizations as well.

When discussing inductive generalization, confirmation theorists often use the more fine-grained distinction between mere instances of a generalization, positive instances, and negative instances. For example, given a generalization $\forall x(Px \supset Qx)$, any a such that $Pa \supset Qa$ is an *instance* of $\forall x(Px \supset Qx)$; any a such that $Pa \wedge Qa$ is a *positive instance* of $\forall x(Px \supset Qx)$; and any a such that $Pa \wedge \neg Qa$ is a *negative instance* of $\forall x(Px \supset Qx)$. Instead of requiring a mere instance before introducing a generalization, some confirmation theorists have suggested the stronger requirement for a positive instance, that is, a negative instance of the contrary generalization (Sect. 11.4.3). According to this idea, interpreting the world as uniform as possible amounts to generalizing whenever a positive instance is available to us. Abnormal situations, then, are those in which both a positive and a negative instance of a generalization are available to us. There is a corresponding variant of **IL** that hard-codes this idea in its set of abnormalities: the logic **G** from [11.5]. The latter is defined by the lower limit logic **CL**, the set of abnormalities $\Omega_{\mathbf{G}}$ and either the reliability strategy ($\mathbf{G^r}$) or the minimal abnormality strategy ($\mathbf{G^m}$).

$$\begin{aligned}&\Omega_{\mathbf{G}} =_{\text{df}}\\ &\{\exists(A_1 \wedge \ldots \wedge A_n \wedge A_0) \wedge \exists(A_1 \wedge \ldots \wedge A_n \wedge \neg A_0) \mid\\ &\quad A_0, A_1, \ldots, A_n \in \mathcal{A}^{f1}; n \geq 0\}\,.\end{aligned} \tag{11.25}$$

In proofs to follow $\exists(A_1 \wedge \ldots \wedge A_n \wedge A_0) \wedge \exists(A_1 \wedge \ldots \wedge A_n \wedge \neg A_0)$ is abbreviated as $A_1 \wedge \ldots \wedge A_n \wedge \pm A_0$ (where again $A_0, A_1, \ldots, A_n \in \mathcal{A}^{f1}$). As an illustration of the workings of **G**, consider the following **G**-proof from $\Gamma_4 = \{Pa \wedge Qa, \neg Qb, \neg Pc\}$

1	$Pa \wedge Qa$ $\emptyset$	Prem
2	$\neg Qb$ $\emptyset$	Prem
3	$\neg Pc$ $\emptyset$	Prem
4	$\forall x(Px \supset Qx)$ $\{Px \wedge \pm Qx\}$	1; RC

5	$\forall x(Qx \supset Px)$	1; RC
	$\{Qx \wedge \pm Px\}$	
6	$\forall x(Px \equiv Qx)$	4, 5; RU
	$\{Px \wedge \pm Qx, Qx \wedge \pm Px\}$	
7	$\exists x Px \wedge \exists x \neg Px$	1, 3; RU
	$\emptyset$	
8	$\exists x Qx \wedge \exists x \neg Qx$	1, 2; RU
	$\emptyset$	

The formulas derived at lines 4–6 are finally **G**-derivable in the proof. Since **G**-consequence too is defined in terms of final derivability, it follows, independently of the strategy used, that

$$\Gamma_4 \vdash_{\mathbf{G}} \forall x(Px \supset Qx)\ , \quad (11.26)$$

$$\Gamma_4 \vdash_{\mathbf{G}} \forall x(Qx \supset Px)\ , \quad (11.27)$$

$$\Gamma_4 \vdash_{\mathbf{G}} \forall x(Px \equiv Qx)\ . \quad (11.28)$$

Now consider the following **IL**-proof from Γ_4 (where $A_1, \ldots, A_n \in \mathcal{A}^{f1}$, $!(A_1 \vee \ldots \vee A_n)$ abbreviates $\exists(A_1 \vee \ldots \vee A_n) \wedge \exists\neg(A_1 \vee \ldots \vee A_n)$)

1	$Pa \wedge Qa$	Prem
	$\emptyset$	
2	$\neg Qb$	Prem
	$\emptyset$	
3	$\neg Pc$	Prem
	$\emptyset$	
4	$\forall x(Px \supset Qx)$	1; RC
	$\{!(\neg Px \vee Qx)\}$✓	
5	$\forall x(Qx \supset Px)$	1; RC
	$\{!(\neg Qx \vee Px)\}$✓	
6	$\forall x(Px \equiv Qx)$	4, 4; RU
	$\{!(\neg Px \vee Qx), !(\neg Qx \vee Px)\}$✓	
7	$!Px$	1, 3; RU
	$\emptyset$	
8	$!Qx$	1, 2; RU
	$\emptyset$	
9	$!(Px \vee Qx) \vee !(\neg Px \vee Qx)$	1, 2; RU
	$\emptyset$	
10	$!(\neg Qx \vee Px) \vee !(Px \vee Qx)$	1, 3; RU
	$\emptyset$	
11	$!(\neg Px \vee \neg Qx)$	1, 2; RU
	$\emptyset$	

The minimal inferred Dab-formulas inferred at lines 7–11 will remain minimal in any extension of this proof (none of the disjuncts of any of the formulas derived at lines 9 or 10 is separately derivable). Accordingly, the marks in this proof will not change. Hence, independently of the strategy used

$$\Gamma_4 \nvdash_{\mathbf{IL}} \forall x(Px \supset Qx)\ , \quad (11.29)$$

$$\Gamma_4 \nvdash_{\mathbf{IL}} \forall x(Qx \supset Px)\ , \quad (11.30)$$

$$\Gamma_4 \nvdash_{\mathbf{IL}} \forall x(Px \equiv Qx)\ . \quad (11.31)$$

Two more remarks are in order. First, the example above suggests that **G** is in general stronger than **IL**. This is correct for the minimal abnormality strategy, but false for the reliability strategy. An illustration is provided by the premise set $\Gamma_5 = \{Pa, Qb, Rb, Qc, \neg Rc\}$. The generalization $\forall x(\neg Px \supset Qx)$ cannot be inferred on the condition $\neg Px \wedge \pm Qx$, since we lack a positive instance. It *can* be inferred on the conditions $\pm Qx$ or $\pm Px$ in view of $\forall x Qx \vdash_{\mathbf{CL}} \forall x(\neg Px \supset Qx)$ and $\forall x Px \vdash_{\mathbf{CL}} \forall x(\neg Px \supset Qx)$, but none of these conditions are reliable in view of the derivability of minimal Dab-formulas like $\pm Px \vee (Px \wedge \pm Rx)$ and $\pm Qx \vee (Qx \wedge \pm Px) \vee (Px \wedge \pm Rx)$.

The situation is different in an $\mathbf{IL^r}$-proof, where deriving $\forall x(\neg Px \supset Qx)$ on the condition $!(Px \vee Qx)$ in a proof from Γ_5 is both possible and final. That is, for every derivable Dab-formula in which $!(Px \vee Qx)$ occurs, we can derive a shorter (minimal) disjunction of abnormalities in which it no longer occurs. Summing up

$$\Gamma_5 \nvdash_{\mathbf{G^r}} \forall x(\neg Px \supset Qx)\ , \quad (11.32)$$

$$\Gamma_5 \vdash_{\mathbf{IL^r}} \forall x(\neg Px \supset Qx)\ . \quad (11.33)$$

The second remark is that the requirement for a *positive* instance before generalizing in a **G**-proof is still insufficient to guarantee that for every **G**-derivable generalization a positive instance is **CL**-derivable from the premises. The following proof from Pa illustrates the point

1	Pa	Prem	$\emptyset$
2	$\forall x Px$	1; RC	$\{\pm Px\}$
3	$\forall x(Qx \supset Px)$	2; RU	$\{\pm Px\}$

Independently of the strategy used, no means are available to mark line 3, hence $Pa \vdash_{\mathbf{G}} \forall x(Qx \supset Px)$, even though no positive instance of $\forall x(Qx \supset Px)$ is available. More on this point below (see the discussion on Hempel's raven paradox in Sect. 11.4.1 and in the Appendix).

A total of six logics have been presented so far: the logics $\mathbf{LI^r}$, $\mathbf{LI^m}$, $\mathbf{IL^r}$, $\mathbf{IL^m}$, $\mathbf{G^r}$, and $\mathbf{G^m}$. Each of these systems interprets the claim that the world is uniform in a slightly different way, leading to slightly different logics. Importantly, there is no Carnapian embarrassment of riches here: each of the systems has a clear intuition behind it.

The systems presented here can be combined so as to implement *Popper*'s suggestion that more general hypotheses should be given precedence over less general ones [11.11]. For instance, if two generalizations $\forall x(Px \supset Qx)$ and $\forall x((Rx \wedge Sx) \supset Tx)$ are jointly incompatible with the premises, a combined system gives precedence to the more general hypothesis and delivers only $\forall x(Px \supset Qx)$ as a consequence. There are various ways to hard-code this idea, resulting in various new combined adaptive logics for inductive generalization, each slightly different from the others. These combinations are not fully spelled out here. For a brief synopsis, see [11.5, Sect. 5].

11.4 Qualitative Inductive Generalization and Confirmation

Inductive logic and confirmation theory overlap to some extent. As early as 1943, *Hempel* noted that the development of a logical theory of confirmation might be regarded as a contribution to the field of inductive logic [11.6, p. 123]. Following Carnap and Popper's influential work on inductive logic and corroboration respectively, many of the existing criteria of confirmation are quantitative in nature, measuring the *degree* of confirmation of a hypothesis by the evidence, possibly taking into account auxiliary hypotheses and background knowledge. Here, the logics defined in the previous two sections are presented as *qualitative* criteria of confirmation, and are related to other qualitative models of confirmation. Quantitative criteria of confirmation are not considered. For *Carnap*'s views on inductive logic, see [11.12]. For *Popper*'s, see [11.11]. For introductions to inductive logic and probabilistic measures of confirmation, see, e.g., [11.13–16].

Let **I** be any adaptive logic for inductive generalization defined in one of the previous sections. (All remarks on **I**-confirmation readily generalize to the combined systems from [11.5, Sect. 5].) Where H is the hypothesis and Γ contains the evidence, **I**-confirmation is defined in terms of **I**-consequence:

Definition 11.6 **I**-*confirmation*
Γ **I**-*confirms* H iff $\Gamma \vdash_{\mathbf{I}} H$.
Γ **I**-*disconfirms* H iff $\Gamma \vdash_{\mathbf{I}} \neg H$.
Γ is **I**-*neutral* with respect to H iff $\Gamma \nvdash_{\mathbf{I}} H$ and $\Gamma \nvdash_{\mathbf{I}} \neg H$.

This definition of **I**-confirmation has the virtue of simplicity and formal precision. The two main qualitative alternatives to **I**-confirmation are Hempel's satisfaction criterion and the hypothetico-deductive model of confirmation. In Sect. 11.4.1, **I**-confirmation is compared to Hempel's adequacy conditions, which serve as a basis for his satisfaction criterion. In Sect. 11.4.2, **I**-confirmation is compared to hypothetico-deductive confirmation. Section 11.4.3 concerns the use of the criteria from Definition 11.6 as heuristic tools for hypothesis generation and confirmation.

11.4.1 I-Confirmation and Hempel's Adequacy Conditions

Let an *observation report* consist of a set of molecular sentences (sentences containing no free variables or quantifiers). According to Hempel, the following conditions should be satisfied by any adequate criterion for confirmation [11.17]:

(1) *Entailment condition*: Any sentence which is entailed by an observation report is confirmed by it.
(2) *Consequence condition*: If an observation report confirms every one of a class K of sentences, then it also confirms any sentence which is a logical consequence of K:
    (a) *Special consequence condition*: If an observation report confirms a hypothesis H, then it also confirms every consequence of H.
    (b) *Equivalence condition*: If an observation report confirms a hypothesis H, then it also confirms every hypothesis which is logically equivalent to H.
(3) *Consistency condition*: Every logically consistent observation report is logically compatible with the class of all the hypotheses which it confirms.

If *logical consequence* is taken to be **CL**-consequence, as Hempel did, then **I**-confirmation satisfies conditions (1)-(3) no matter which adaptive logic for inductive generalization is used, due to **I**'s closure under **CL**. So all of the resulting criteria of confirmation meet Hempel's adequacy conditions. (For (3) the further property of *smoothness* or *reassurance* is required, from which it follows that the **I**-consequence set of consistent premise sets is consistent as well [11.7, Sect. 6].)

The definition of Hempel's own criterion requires some preparation (the formal presentation of Hempel's criterion is taken from [11.18]). An atomic formula A

is *relevant* to a formula B iff there is some model M of A such that: if M' differs from M only in the value assigned to B, M' is not a model of A. The *domain* of a formula A is the set of individual constants that occur in the atomic formulas that are relevant for A. The *development* of a universally quantified formula A for another formula B is the restriction of A to the domain of B, that is, the truth value of A is evaluated with respect to the domain of B. For instance, the domain of $Pa \wedge (Pb \vee Qc)$ is $\{a, b, c\}$ whereas the domain of $Pa \wedge Qa$ is $\{a\}$; and the development of $\forall x(Px \supset Qx)$ for $Pa \wedge \neg Qb$ is $(Pa \supset Qa) \wedge (Pb \supset Qb)$.

Definition 11.7 Hempel's satisfaction criterion

An observation report E *directly confirms* a hypothesis H if E entails the development of H for E.
An observation report E *confirms* a hypothesis H if H is entailed by a class of sentences each of which is directly confirmed by E.
An observation report E *disconfirms* a hypothesis H if it confirms the denial of H.
An observation report E is *neutral* with respect to a hypothesis H if E neither confirms nor disconfirms H.

There are two reasons for arguing that Hempel's satisfaction criterion is too restrictive, and two reasons for arguing that it is too liberal. Each of these is discussed in turn. First, in order for the evidence to confirm a hypothesis H according to Hempel's criterion, *all* objects in the development of H must be known to be instances of H. This is a very strong requirement. **I**-confirmation is different in this respect. For instance,

$$Pa, Qa, \neg Pb, \neg Qb, Pc \vdash_{\mathbf{I}} \forall x(Px \supset Qx)\ . \qquad (11.34)$$

In (11.34) it is unknown whether c instantiates the hypothesis $\forall x(Px \supset Qx)$, since the premises do not tell us whether $Pc \supset Qc$. The development of $\forall x(Px \supset Qx)$ entails $Pc \supset Qc$, whereas the premise set of (11.34) does not. So the hypothesis $\forall x(Px \supset Qx)$ is not directly confirmed by these premises according to the satisfaction criterion, nor is it entailed by one or more sentences which are directly confirmed by them. Therefore the satisfaction criterion judges the premises to be neutral with respect to the hypothesis $\forall x(Px \supset Qx)$, whereas (11.34) illustrates that $\forall x(Px \supset Qx)$ is **I**-confirmed by these premises.

Second, given the law $\forall x(Px \supset Rx)$, the report $\{Pa, Qa, Pb, Qb\}$, does not confirm the hypothesis $\forall x(Rx \supset Qx)$ according to Hempel's original formulation of the satisfaction criterion. The reason is that *auxiliary hypotheses* like $\forall x(Px \supset Rx)$ contain quantifiers and therefore cannot be elements of observation reports. (The original formulation of Hempel's criterion can, however, be adjusted so as to take into account background knowledge [11.19, 20].) For problems related to auxiliary hypotheses, see also Sect. 11.4.2. For now, it suffices to note that the criteria from Definition 11.6 do not face this problem, as quantified formulas are perfectly allowed to occur in premise sets. For instance, the set $\{Pa, Qa, Pb, Qb, \forall x(Px \supset Rx)\}$ **I**-confirms the hypothesis $\forall x(Rx \supset Qx)$

$$Pa, Qa, Pb, Qb, \forall x(Px \supset Rx) \vdash_{\mathbf{I}} \forall x(Rx \supset Qx)\ . \qquad (11.35)$$

It seems, then, that **I**-confirmation is not too restrictive a criterion for confirmation. However, there are two senses in which **I**-confirmation, like Hempelian confirmation, can be said to be too liberal. The first has to do with *Goodman*'s well-known *new riddle of induction* [11.21]. The family of adaptive logics for inductive generalization makes no distinction between regularities that are *projectible* and regularities that are not. Using Goodman's famous example, let an emerald be *grue* if it is green before January 1st 2020, and blue thereafter. Then the fact that all hitherto observed emeralds are grue confirms the hypothesis that all emeralds are grue. The latter regularity is not projectible into the future, as we do not seriously believe that in 2020 we will start observing blue emeralds. Nonetheless, it is perfectly fine to define a predicate denoting the property of being grue, just as it is perfectly fine to define a predicate denoting the property of being green. Yet the hypothesis *all emeralds are green is* projectible, whereas *all emeralds are grue* is not.

The problem of formulating precise rules for determining which regularities are projectible and which are not is difficult and important, but it is an epistemological problem that cannot be solved by purely logical means. Consequently, it falls outside the scope of this article. See [11.21] for *Goodman*'s formulation and proposed solution of the problem, and [11.22] for a collection of essays on the projectibility of regularities.

Finally, one may argue that **I**-confirmation is too liberal on the basis of Hempel's own *raven paradox*. Where Ra abbreviates that a is a raven, and Ba abbreviates that a is black, a non-black non-raven **I**-confirms the hypothesis that all ravens are black

$$\neg Ba, \neg Ra \vdash_{\mathbf{I}} \forall x(Rx \supset Bx)\ . \qquad (11.36)$$

Even the logic **G** does not block this inference. The reason is that we are given a positive instance of the generalization $\forall x(\neg Bx \supset \neg Rx)$, so we can derive this generalization on the condition $\exists x(\neg Bx \wedge \neg Rx) \wedge \exists x(\neg Bx \wedge Rx)$. As the generalization $\forall x(\neg Bx \supset \neg Rx)$

is **G**-derivable from the premises, so is the logically equivalent hypothesis that all ravens are black, $\forall x(Rx \supset Bx)$ (remember that **G**, like all logics defined in the previous section, is closed under **CL**).

Hempel's own reaction to the raven paradox was to bite the bullet and accept its conclusion [11.23]. According to Hempel, a non-black non-raven indeed confirms the raven hypothesis in case we did not know beforehand that the bird in question is not a raven. For example, if we observe a grey bird resembling a raven, then finding out that it was a crow confirms the raven hypothesis [11.18]. But as pointed out in [11.19] this defense is insufficient. Even in cases in which it is *known* that a non-black bird is not a raven, the bird in question, although irrelevant to the raven hypothesis, still confirms it.

If – like Hempel – one accepts its conclusion, the raven paradox poses no further problems for **I**-confirmation. Those who disagree are referred to the Appendix, where a relatively simple adaptive alternative to **G**-confirmation is defined which blocks the paradox by means of a non-material conditional invalidating the inference from *all non-black objects are non-ravens* to *all ravens are black*.

11.4.2 I-Confirmation and the Hypothetico-Deductive Model

If a hypothesis predicts an event which is observed at a later time, or if it subsumes a given observation report as a consequence of one of its postulates, then this counts as evidence in favor of the hypothesis. The hypothetico-deductive model of confirmation (HD confirmation) is an attempt to formalize this basic intuition according to which a piece of evidence confirms a hypothesis if the latter entails the evidence.

In its standard formulation, HD confirmation also takes into account auxiliary hypotheses. Where Δ is a set of background information distinct from the evidence E,

Definition 11.8 HD-confirmation
E *HD-confirms* H relative to Δ iff:

(i) $\{H\} \cup \Delta$ is consistent,
(ii) $\{H\} \cup \Delta$ entails E ($\{H\} \cup \Delta \vdash E$),
(iii) Δ alone does not entail E ($\Delta \nvdash E$).

The intuitive difference conveyed by HD confirmation and Hempelian confirmation becomes concrete if HD confirmation is compared with Hempel's adequacy criteria from Sect. 11.4.1. Let H abbreviate *Black swans exist*, let E consist of a black swan, and let Δ be the empty set. Then, according to Hempel's entailment condition, H is confirmed by E, since $E \vdash H$. Not so according to HD confirmation, for condition (ii) of Definition 11.8 is violated ($H \nvdash E$) [11.24]. The same example illustrates how HD confirmation violates the following condition, which holds for the satisfaction criterion in view of Definition 11.7 [11.25]:

(4) Complementarity condition: E confirms H iff E disconfirms $\neg H$.

The consequence condition too is clearly invalid for HD confirmation. For instance, $Ra \supset Ba$ HD confirms $\forall x(Rx \supset Bx)$, but it does not HD confirm the weaker hypothesis $\forall x(Rx \supset (Bx \vee Cx))$, since $\forall x(Rx \supset (Bx \vee Cx)) \nvdash Ra \supset Ba$.

An advantage of HD confirmation is that it fares better with the raven paradox. The observation of a black raven (Ra, Ba) is not deducible from the raven hypothesis $\forall x(Rx \supset Bx)$, so black ravens do not in general confirm the raven hypothesis. But birds that are known to be ravens *do* confirm the raven hypothesis once it is established that they are black. For once it is known that an object is a raven, the observation that it is black is entailed by this knowledge together with the hypothesis ($\forall x(Rx \supset Bx), Ra \vdash Ba$). Likewise, a non-black non-raven does not generally confirm the raven hypothesis. Only objects that are known to be non-black can confirm the hypothesis by establishing that they are not ravens. In formulas: $\forall x(Rx \supset Bx), \neg Ba \vdash \neg Ra$.

HD confirmation faces a number of standard objections, of which three are discussed here. The first is the problem of irrelevant conjunctions and disjunctions. In view of Definition 11.8 it is easily checked that whenever a hypothesis H confirms E relative to Δ, so does $H' = H \wedge K$ for any arbitrary K consistent with Δ. Thus adding arbitrary conjuncts to confirmed hypotheses preserves confirmation. Dually, adding arbitrary disjuncts to the data likewise preserves confirmation. That is, whenever H confirms E relative to Δ, H also confirms E' relative to Δ, where $E' = E \vee F$ for any arbitrary F.

Various solutions have been proposed for dealing with such problems of irrelevancy, but as so often the devil is in the details (see [11.20] for a nice overview and further references). For present purposes, it suffices to say that **I**-confirmation is not threatened by problems of irrelevance. Clearly, if the evidence E **I**-confirms a hypothesis H, it does not follow that it **I**-confirms $H \wedge K$ for some arbitrary K consistent with Δ, since from $\{E\} \cup \Delta \vdash_{\mathbf{I}} H$ it need not follow that $\{E\} \cup \Delta \vdash_{\mathbf{I}} H \wedge K$. Nor does it follow that $E \vee F$ confirms H relative to Δ, since from $\{E\} \cup \Delta \vdash_{\mathbf{I}} H$ it need not follow that $\{E \vee F\} \cup \Delta \vdash_{\mathbf{I}} H$.

A second objection against HD confirmation concerns the inclusion of background information in Definition 11.8. In general, this inclusion is an advantage, since evidence often does not (dis)confirm a hypothesis simpliciter. Rather, evidence (dis)confirms hypotheses with respect to a set of auxiliary (background) assumptions or theories. The vocabulary of a theory often extends beyond what is directly observable. Notwithstanding Hempel's conviction to the contrary, nowadays philosophers largely agree that the use of purely theoretical terms is both intelligible and necessary in science [11.26]. Making the confirmation relation relative to a set of auxiliaries allows for the inclusion of bridging principles connecting observation terms with theoretical terms, permitting purely theoretical hypotheses to be confirmed by pure observation statements [11.27]. However, making confirmation relative to background assumptions makes HD vulnerable to a type of objection often traced back to *Duhem* [11.28] and *Quine* [11.29]. Suppose that a hypothesis H entails an observation E relative to Δ, and that E is found to be false. Then either (a) H is false or (b) a member of Δ is false. But the evidence does not tell us which of (a) or (b) is the case, so we always have the option to retain H and blame some auxiliary hypothesis in the background information. More generally, one may object that what gets (dis)confirmed by observations is not a hypothesis taken by itself, but the conjunction of a hypothesis and a set of background assumptions or theories.

With *Elliott Sober*, we can counter such holistic objections by pointing to the different epistemic status of hypotheses *under* test and auxiliary hypotheses (or hypotheses *used* in a test). Auxiliaries are independently testable, and when used in an experiment we already have good reasons to think of these hypotheses as true. Moreover, they are epistemically independent of the test outcome. So if a hypothesis is disconfirmed by the HD criterion, we can, in the vast majority of cases, maintain that it is the hypothesis we need to retract, and not one of the background assumptions [11.30].

A parallel point can be made concerning **I**-confirmation. Here too, we can add to the premises a set Δ of auxiliary or background assumptions. And here too, we can use Sober's defence against objections from evidential holism. A nice feature of **I**-confirmation is that in adaptive proofs the weaker epistemic status of hypotheses inferred from an observation report in conjunction with a set of auxiliaries is reflected by their non-empty condition. Whereas auxiliaries are introduced as premises on the empty condition, inductively generated hypotheses are derived conditionally and may be retracted at a later stage of the proof. For a more fine-grained treatment of background information in adaptive logics for inductive generalization, see [11.5, Sect. 6].

The third objection against HD confirmation dates back to *Hempel*'s [11.17], in which he argued that a variant of HD confirmation (which he calls the *prediction criterion* of confirmation) is circular. The problem is that in HD confirmation the hypothesis to be confirmed functions as a premise from which we derive the evidence, and that it is unclear where this premise comes from. The hypothesis is not generated, but given in advance, so HD confirmation presupposes the prior attainment – by inductive reasoning – of a hypothesis. This inductive move, Hempel argues, already presupposes the idea of confirmation, making the HD account circular.

The weak step in Hempel's argument consists in his assumption that the inductive jump to the original attainment of a hypothesis already presupposes the confirmation of this hypothesis. In testing or generating a hypothesis we need not yet *believe* or *accept* it. Typically, belief and acceptance come only after confirming the hypothesis. Indeed, in probabilistic notions of confirmation the idea is often exactly this: confirming a hypothesis amounts to increasing our degree of belief in it. Hempel's circularity objection, it seems, confuses hypothesis generation and hypothesis confirmation.

Hempel's circularity objection does not undermine HD confirmation, but it points to the wider scope of the adaptive account as compared to HD confirmation. In an **I**-proof, the conditional rule allows us to *generate* hypotheses. Hypotheses are not given in advance but are computable by the logic itself. Moreover, a clear distinction can be made between hypothesis generation and hypothesis confirmation. Hypotheses generated in an **I**-proof may be derivable at some stage of the proof, but the central question is whether they can be retained – whether they are *finally* derivable. **I**-confirmation, then, amounts to final derivability in an **I**-proof whereas the inductive step of hypothesis generation is represented by retractable applications of RC.

11.4.3 Interdependent Abnormalities and Heuristic Guidance

For any of the adaptive logics for inductive generalization defined in this chapter, at most one positive instance is needed to try and derive and, subsequently, confirm a generalization for a given set of premises. This is a feature that **I**-confirmation shares with the other qualitative criteria of confirmation. As a simple illustration, note that an observation report consisting of a single observation Pa confirms the hypothesis $\forall x Px$ according to all qualitative criteria discussed in this chapter. Proponents of quantitative approaches to confirmation may object that this is insufficient; that a stronger criterion is needed which requires *more* than one instance for a hypothesis to be confirmed. Against this view, one can uphold that confirmation is mainly falsification-driven. Rather than confirming hypotheses by heaping up positive instances, we try and test them by searching for negative instances. In the remainder of this section, it is argued by means of a number of examples that **I**-confirmation is sufficiently selective as a criterion for confirming generated hypotheses. The examples moreover allow for the illustration of an additional feature of **I**-confirmation: its use as a heuristic guide for provoking further tests in generating and confirming additional hypotheses.

Simple examples like the one given in the previous paragraph may suggest that, in the absence of falsifying instances, a single instance usually suffices to **I**-confirm a hypothesis. This is far from the truth. Consider the simple premise set $\Gamma_6 = \{\neg Pa \vee Qa, \neg Qb, Pc\}$. This premise set contains instances of all of the generalizations $\forall x Px$, $\forall x \neg Qx$, and $\forall x(Px \supset Qx)$. Not a single one of these is **IL**-confirmed, however, due to the derivability of the following disjunctions of abnormalities

$$!Px \vee !Qx \,, \tag{11.37}$$

$$!Px \vee !(\neg Px \vee Qx) \,, \tag{11.38}$$

$$!(Px \vee Qx) \vee !(\neg Px \vee Qx) \,, \tag{11.39}$$

$$!Qx \vee !(\neg Px \vee Qx) \,, \tag{11.40}$$

$$!(\neg Px \vee Qx) \vee !(\neg Px \vee \neg Qx) \,. \tag{11.41}$$

Note that Γ_6 contains positive instances of both $\forall x Px$ and $\forall x \neg Qx$, so not even a positive instance suffices for a generalization to be finally **IL**-derivable in the absence of falsifying instances. The same is true if we switch from **IL** to **G**. None of $\forall x Px$, $\forall x \neg Qx$, or $\forall x(Px \supset Qx)$ is **G**-confirmed, due to the derivability of the following disjunctions of abnormalities

$$\pm Px \vee \pm Qx \,, \tag{11.42}$$

$$\pm Px \vee (Px \wedge \pm Qx) \,, \tag{11.43}$$

$$\pm Qx \vee (Qx \wedge \pm Px) \,. \tag{11.44}$$

The reason for the non-confirmation of generalizations like $\forall x Px$, $\forall x \neg Qx$, or $\forall x(Px \supset Qx)$ in this example has to do with the dependencies that exist between abnormalities. Even if a generalization is not falsified by the data, it is often the case that this generalization is not compatible with a different generalization left unfalsified by the data. As a further illustration, consider the premise set $\Gamma_7 = \{\neg Ra, \neg Ba, Rb\}$. Again, although no falsifying instance is present, the generalization $\forall x(Rx \supset Bx)$ is not **IL**-derivable. The reason is the derivability of the following minimal disjunction of abnormalities

$$!(\neg Rx \vee Bx) \vee !(\neg Rx \vee \neg Bx) \,. \tag{11.45}$$

Examples like these illustrate that **I**-confirmation is not too liberal a criterion of confirmation. They also serve to illustrate a different point. Minimal Dab-formulas like (11.45) evoke questions. Which of the two abnormalities is the case? For this particular premise set, establishing which of Bb or $\neg Bb$ is the case would settle the matter. For if Bb were the case, then the second disjunct of (11.45) would be derivable, and (11.45) would no longer be minimal. Consequently, the abnormality $\exists x(\neg Rx \vee Bx) \wedge \exists x \neg(\neg Rx \vee Bx)$ would no longer be part of a minimal disjunction of abnormalities, and the generalization $\forall x(Rx \supset Bx)$ would become finally derivable. Analogously, if $\neg Bb$ were the case, then the first disjunct of (11.45) would become derivable, and, by the same reasoning, the generalization $\forall x(Rx \supset \neg Bx)$ would become finally derivable. Thus

$$\Gamma_7 \cup \{Bb\} \vdash_{\mathbf{IL}} \forall x(Rx \supset Bx) \,, \tag{11.46}$$

$$\Gamma_7 \cup \{\neg Bb\} \vdash_{\mathbf{IL}} \forall x(Rx \supset \neg Bx) \,. \tag{11.47}$$

Two more comments are in order here. First, this example illustrates that confirming a hypothesis often involves the disconfirmation of the contrary hypothesis. We saw that if we use Hempel's criterion a nonblack non-raven confirms the raven hypothesis. But as *Goodman* pointed out "the prospects for indoor ornithology vanish when we notice that under these same conditions, the contrary hypothesis that no ravens are black is equally well confirmed" [11.21, p. 71]. Thus, according to Goodman, confirming the raven hypothesis $\forall x(Rx \supset Bx)$ requires disconfirming its contrary $\forall x(Rx \supset \neg Bx)$. This is exactly what happens in the example: in order to **IL**-derive $\forall x(Rx \supset Bx)$, a falsifying instance for its contrary is needed, as (11.46) illustrates. Goodman's suggestion that the confirmation of a hypothesis requires the falsification/disconfirmation of its contrary was picked up by *Israel Scheffler*, who developed it further in his [11.31]. Note that falsifying the contrary of the raven hypothesis amounts to finding a positive instance of the raven hypothesis. Thus, in demanding a positive instance before permitting generalization in a **G**-proof, the latter system goes further than **IL** in implementing Goodman's idea. As we saw, however, not even **G** goes all the way: a generalization may be **G**-derivable even in the absence of a positive instance.

Second, if empirical (observational or experimental) means are available to answer questions like $?\{Bb, \neg Bb\}$ in the foregoing example, these questions may be called *tests* [11.2]. Adaptive logics for inductive generalization provide heuristic guidance in the sense that interdependencies between abnormalities evoke such tests. Importantly, further tests may lead to the derivability of new generalizations. In the example, deciding the question $?\{Bb, \neg Bb\}$ in favor of Bb leads to the confirmation of $\forall x(Rx \supset Bx)$ and to the disconfirmation of $\forall x(Rx \supset \neg Bx)$, while deciding it in favor of $\neg Bb$ leads to the confirmation of $\forall x(Rx \supset \neg Bx)$ and to the disconfirmation of $\forall x(Rx \supset Bx)$. This is an important practical advantage of **I**-confirmation over other qualitative criteria: adaptive logics for inductive generalization evoke tests for increasing the number of confirmed generalizations.

The illustrations so far may suggest that this heuristic guidance provided by **I**-confirmation only applies to hypotheses that are logically related or closely connected, like the raven hypothesis and its contrary. But the point is more general, as the following example illustrates.

Consider the premise set

$$\Gamma_8 = \{Pa, Qa, \neg Ra, \neg Pb, \neg Qb, Rb, Pc, Rc, Qd, \neg Pe\} .$$

Despite the fact that Γ_8 contains positive instances of the generalizations $\forall x(Px \supset Qx)$ and $\forall x(Rx \supset \neg Qx)$, and despite the fact that these generalizations are not falsified by Γ_8, none of them is **IL**-derivable due to the derivability of the disjunction

$$!(\neg Px \vee Qx) \vee !(\neg Rx \vee \neg Qx) . \quad (11.48)$$

By the same reasoning as in the previous illustration, Γ_8 evokes the question $?\{Qc, \neg Qc\}$. If this question is a test (if it can be answered by empirical means), the answer will confirm one of the generalizations $\forall x(Px \supset Qx)$ and $\forall x(Rx \supset \neg Qx)$, and will disconfirm the other generalization [11.2].

The example generalizes. In **LI** and **G** too, the derivability of $\forall x(Px \supset Qx)$ and $\forall x(Rx \supset \neg Qx)$ is blocked due to the **CL**-derivability of the **LI**-minimal Dab-formula (11.49), respectively the **G**-minimal Dab-formula (11.50)

$$\neg\forall x(Px \supset Qx) \vee \neg\forall x(Rx \supset \neg Qx) , \quad (11.49)$$

$$(Px \wedge \pm Qx) \vee (Rx \wedge \pm Qx) . \quad (11.50)$$

Here too, deciding the question $?\{Qc, \neg Qc\}$ resolves the matter. Thus, where $\mathbf{I} \in \{\mathbf{LI}, \mathbf{IL}, \mathbf{G}\}$

$$\Gamma_8 \nvdash_{\mathbf{I}} \forall x(Px \supset Qx) , \quad (11.51)$$

$$\Gamma_8 \nvdash_{\mathbf{I}} \forall x(Rx \supset \neg Qx) , \quad (11.52)$$

$$\Gamma_8 \cup \{Qc\} \vdash_{\mathbf{I}} \forall x(Px \supset Qx) , \quad (11.53)$$

$$\Gamma_8 \cup \{Qc\} \nvdash_{\mathbf{I}} \forall x(Rx \supset \neg Qx) , \quad (11.54)$$

$$\Gamma_8 \cup \{\neg Qc\} \nvdash_{\mathbf{I}} \forall x(Px \supset Qx) , \quad (11.55)$$

$$\Gamma_8 \cup \{\neg Qc\} \vdash_{\mathbf{I}} \forall x(Rx \supset \neg Qx) . \quad (11.56)$$

For some concrete heuristic rules applicable to the logic **LI**, see [11.3].

11.5 Conclusions

A number of adaptive logics for inductive generalization were presented each of which, it was argued, can be re-interpreted as a criterion of confirmation. The logics in question can be classified along two dimensions. The first dimension concerns when it is permitted to introduce a generalization in an adaptive proof. The logic **LI** permits the free introduction of generalizations. **IL** and **G** require instances of a generalization before introducing it in a proof. Interestingly, these stronger requirements do not result in stronger logics.

The second dimension along which the logics defined in this chapter can be classified concerns their adaptive strategy. Here, no surprises arise. A logic defined using the reliability strategy is in general weaker than its counterpart logic defined using the minimal abnormality strategy (this was shown to be the case for all adaptive logics defined within the standard format [11.7, Theorem 11]).

When re-interpreted as criteria of confirmation, the logics defined here withstand the comparison with their main rivals, i. e., Hempel's satisfaction criterion and the hypothetico-deductive model of confirmation. In conclusion, the adaptive confirmation criteria defined in Sect. 11.4 offer an interesting alternative perspective on (qualitative) confirmation theory.

11.A Appendix: Blocking the Raven Paradox?

If a formalism defined in terms of **CL** behaves overly permissive, a good strategy to remedy this problem is to add further criteria of validity or relevance. For instance, in order to avoid problems of irrelevant conjunctions and disjunctions, hypothetico-deductivists may impose further demands on HD confirmation [11.32–35].

A similar strategy could be adopted with respect to **I**-confirmation and the raven paradox. In this appendix, an alternative adaptive logic of induction, **IC**, is defined, as is a corresponding criterion of confirmation which is slightly less permissive than the criteria from Sect. 11.4. **IC** makes use of a non-classical conditional resembling a number of conditionals originally defined in order to avoid the so-called paradoxes of material implication. First, an extension of **CL** is introduced, including this new conditional connective. Next, the adaptive logic **IC** is defined.

The new conditional, $\rightarrow$, is fully characterized by the following rules and axiom schema's

$$\frac{A,\ (A \rightarrow B)}{B}\ , \qquad \text{(MP)}$$

$$\frac{A \equiv B}{(A \rightarrow C) \equiv (B \rightarrow C)}\ , \qquad \text{(RCEA)}$$

$$\frac{A \equiv B}{(C \rightarrow A) \equiv (C \rightarrow B)}\ , \qquad \text{(RCEC)}$$

$$(A \rightarrow (B \wedge C)) \equiv ((A \rightarrow B) \wedge (A \rightarrow C))\ , \qquad (\mathrm{D}\wedge)$$

$$((A \vee B) \rightarrow C) \equiv ((A \rightarrow C) \wedge (B \rightarrow C))\ , \qquad (\mathrm{D}\vee)$$

((RCEA), (RCEC), and (D$\wedge$) fully characterize the conditional of Chellas's logic **CR** from [11.36]. The latter was also used for capturing explanatory conditionals in [11.37]. See also [11.38, Chap. 5] for some closely related conditional logics, including an extension of Chellas's systems that validates (MP).)

Let $\mathbf{CL}^{\rightarrow}$ be the logic resulting from adding $\rightarrow$ to the language of **CL**, and from adding (MP)-(D$\vee$) to the list of rules and axioms of **CL**. Note that the conditional $\rightarrow$ is strictly stronger than $\supset$

$$(A \rightarrow B) \supset (A \supset B)\ . \qquad (11.57)$$

(By (MP), $A, (A \rightarrow B) \vdash_{\mathbf{CL}^{\rightarrow}} B$. By the deduction theorem for $\supset$, $A \rightarrow B \vdash_{\mathbf{CL}^{\rightarrow}} A \supset B$. By the deduction theorem again, $\vdash_{\mathbf{CL}^{\rightarrow}} (A \rightarrow B) \supset (A \supset B)$.)

In view of this bridging principle between both conditionals it is easily seen that counter-instances to a formula of the form $\forall x(A(x) \supset B(x))$ form counter-instances to $\forall x(A(x) \rightarrow B(x))$, and falsify the latter formula as well. For instance, if $Pa \wedge \neg Qa$, then, by **CL**, $\neg\forall x(Px \supset Qx)$, and, by (11.57), $\neg\forall x(Px \rightarrow Qx)$.

The adaptive logic **IC** is fully characterized by the lower limit logic $\mathbf{CL}^{\rightarrow}$, the set of abnormalities

$$\begin{aligned}\Omega_{\mathbf{IC}} =_{\mathrm{df}} \{&\exists(A_1 \wedge \ldots \wedge A_n \wedge A_0) \\ &\wedge \neg\forall((A_1 \wedge \ldots \wedge A_n) \rightarrow A_0) \mid \\ &A_0, A_1, \ldots, A_n \in \mathcal{A}^{f1}; n \geq 0\}\ ,\end{aligned} \qquad (11.58)$$

and the adaptive strategy reliability ($\mathbf{IC}^{\mathbf{r}}$) or minimal abnormality ($\mathbf{IC}^{\mathbf{m}}$). **IC** is defined within the SF. All rules and definitions for its proof theory are as for the other logics defined in this chapter, except that in the definition of RU and RC, **CL** is replaced with $\mathbf{CL}^{\rightarrow}$.

The following proof illustrates how formulas are derived conditionally in **IC**

1	$\neg Ra$	Prem
	$\emptyset$	
2	$\neg Ba$	Prem
	$\emptyset$	
3	$\forall x(\neg Bx \rightarrow \neg Rx)$	1, 2; RC
	$\{\exists x(\neg Bx \wedge \neg Rx) \wedge \neg\forall x(\neg Bx \rightarrow \neg Rx)\}$	

Given only the premises $\neg Ra$ and $\neg Ba$, there is no possible extension of this proof in which line 3 gets marked. Hence

$$\neg Ra, \neg Ba \vdash_{\mathbf{IC}} \forall x(\neg Bx \rightarrow \neg Rx)\ . \qquad (11.59)$$

However, contraposition is invalid for the new conditional $\rightarrow$, hence we cannot derive the raven hypothesis from the formula derived at line 3. Note also that, in view of (11.60), we cannot use the conditional rule RC to derive $\forall x(Rx \rightarrow Bx)$ on the condition $\{\exists x(Rx \wedge Bx) \wedge \neg\forall x(Rx \rightarrow Bx)\}$ in an **IC**-proof, since

$$\begin{aligned}\neg Ra, \neg Ba \nvdash_{\mathbf{CL}^{\rightarrow}} &\ \forall x(Rx \rightarrow Bx) \\ &\vee (\exists x(Rx \wedge Bx) \wedge \neg\forall x(Rx \rightarrow Bx))\ .\end{aligned} \qquad (11.60)$$

Therefore

$$\neg Ra, \neg Ba \nvdash_{\mathbf{IC}} \forall x(Rx \rightarrow Bx)\ . \qquad (11.61)$$

Thus, if conditional statements of the form *for all x, if* $A(x)$ *then* $B(x)$ are taken to be **IC**-confirmed only if the conditional in question is an arrow ($\rightarrow$) instead of a material implication, then the raven paradox, in its original formulation, is blocked.

An additional property of **IC** is that *strengthening the antecedent* fails for $\rightarrow$. In Sect. 11.3, for instance, we saw that

$$Pa \vdash_{\mathbf{G}} \forall x(Qx \supset Px)\ . \qquad (11.62)$$

In **IC**, (11.62) still holds for the material implication, but not for the new conditional. In an **IC**-proof from Pa we can still derive $\forall x Px$ on the condition $\{\exists x Px \wedge \exists x \neg Px\}$, and since **IC** extends **CL** it still follows that $\forall x(Px \supset Qx)$

$$Pa \vdash_{\mathbf{IC}} \forall x Px\,, \quad (11.63)$$

$$Pa \vdash_{\mathbf{IC}} \forall x(Qx \supset Px)\,. \quad (11.64)$$

However, since $\forall x Px \nvdash_{\mathbf{CL}^{\rightarrow}} \forall x(Qx \rightarrow Px)$, and since we do not have any further means to conditionally derive the formula $\forall x(Qx \rightarrow Px)$ in an **IC**-proof

$$Pa \nvdash_{\mathbf{IC}} \forall x(Qx \rightarrow Px)\,. \quad (11.65)$$

Originally, the logics in the **G**-family were constructed as logics requiring a *positive instance* before we are allowed to apply RC. This is reflected in the definition of the set of **G**-abnormalities. In order to derive a formula like $\forall x(Px \supset Qx)$ on its corresponding condition, a positive instance, e.g., $Pa \wedge Qa$, is needed. Examples like (11.36) and (11.62) show, however, that such a positive instance is not always required in order to **G**-derive a generalization. The logic **IC**, it seems, does much better in this respect. However, it still does not fully live up to the requirement for a positive instance before generalizing, as the following **IC**-proof from $\Gamma_9 = \{\neg Ra \wedge \neg Ba, Rb, Bc\}$ illustrates (where $A_0, A_1, \ldots, A_n \in \mathcal{A}^{f1}$, $\dagger((A_1 \wedge \ldots \wedge A_n) \rightarrow A_0)$ abbreviates $\exists(A_1 \wedge \ldots \wedge A_n \wedge A_0) \wedge \neg\forall((A_1 \wedge \ldots \wedge A_n) \rightarrow A_0)$).

1	$\neg Ra \wedge \neg Ba$	Prem
	$\emptyset$	
2	Rb	Prem
	$\emptyset$	
3	Bc	Prem
	$\emptyset$	
4	$\forall x(\neg Bx \rightarrow \neg Rx)$	1; RC
	$\{\dagger(\neg Bx \rightarrow \neg Rx)\}$	
5	Bb	2, 4; RU
	$\{\dagger(\neg Bx \rightarrow \neg Rx)\}$	
6	$\forall x(Rx \rightarrow Bx)$	2, 5; RC
	$\{\dagger(Rx \rightarrow Bx), \dagger(\neg Bx \rightarrow \neg Rx)\}$	

The key step in this proof is the derivation of Bb at line 5, which together with Rb provides us with a positive instance of the raven hypothesis. Bb is derivable from lines 2 and 4 in view of **CL** and (11.57). Except for the formulas $\exists x Rx \wedge \exists x \neg Rx$ and $\exists x Bx \wedge \exists x \neg Bx$, no minimal Dab-formulas are $\mathbf{CL}^{\rightarrow}$-derivable from Γ_9. Therefore

$$\Gamma_9 \vdash_{\mathbf{IC}} \forall x(Rx \rightarrow Bx)\,. \quad (11.66)$$

As (11.61) illustrates the logic **IC** avoids the raven paradox in its original formulation. A possible drawback of **IC** is that it does not fully meet the demand for a positive instance when confirming a hypothesis (Sect. 11.4.3). It is left open whether it is possible and desirable to further extend **IC** so as to fully meet this demand.

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