Discrete Groups and Surface Automorphisms: A Theorem of A.M. Macbeath

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Abstract This short article re-examines the interaction between group actions in hyperbolic geometry and low-dimensional topology, focussing in particular on some contributions of Murray Macbeath to the study of Riemann surface automorphisms. A brief account is included of a potential extension to hyperbolic 3-manifolds.

1 Introduction

The classical results of Klein, Hurwitz and others on automorphisms of Riemann surfaces were based on the theory of projective algebraic curves. This contrasts somewhat with the approach used today, which makes essential use of the uniformisation theorem, covering spaces and the geometry of non-Euclidean crystallographic groups. Behind all this stands the rigorous theory of uniformisation which was worked out in the years before 1910 via Dirichlet's Principle by Hilbert and Courant and completed by Koebe and (using other methods) by Poincarë, thus establishing a firm basis for a systematic geometric account of surface topology. Group actions in the hyperbolic plane were analysed by Dehn and Nielsen, while the 2-volume book of Fricke and Klein [5] explored at length the immense range of discrete hyperbolic plane groups involved in this theory, formulating a classification of Fuchsian groups into distinct parameter spaces associated with each *signature* (or geometric type). At the same time, the formulation of an abstract notion of manifold, signalled by Weyl's ground-breaking book on Riemann surfaces [19], now just over a hundred years old, heralded an upsurge of interest in geometric topology generally and low dimensional manifolds in particular.

It is worth noting that something of a hiatus in the systematic development of discrete group actions began in the late 1920s. Thus, after Fricke's construction of parameter spaces for Fuchsian groups and the work of Dehn and Nielsen on surface topology, the problem of moduli for Riemann surfaces remained unresolved until

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the theory of complex analytic deformations was established, first in outline by Teichmüller from 1938 to 1943, and then in rigorous detail by the school of Lars Ahlfors and Lipman Bers in the late 1950s. The latter developments will not concern us here; that material can now be found in many sources, including the collected works of these two authors, [1] and [4].

2 Hurwitz's Theorem Revisited

A brief paper of Siegel from 1945 [17] led Murray Macbeath to formulate a systematic new approach to the study of Riemann surface automorphisms in the late 1950s. In 1893, A. Hurwitz showed that, for values of the genus $g \geq 2$, the maximum number of automorphisms of a surface is $84(g-1)$, a bound attained by the famous Klein quartic curve of genus 3, with automorphism group the simple group $PSL(2, \mathbb{F}_7)$ of order 168. This finite group action had been discovered by Klein (1879) in the appropriate setting of non-Euclidean plane geometry; for more details of that fascinating story and some contemporary developments, see [9]. Soon after, Poincaré began his own study of the discrete subgroups of the Lie group $G = PSL(2, \mathbb{R})$ (which he called Fuchsian, much to Klein's annoyance), motivated by his sudden realisation that the groups emerging from his study of uniformisation by differential equations are these same groups of isometries of the hyperbolic plane,

$$\mathscr{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}, \quad \text{with the Poincaré metric} \quad ds_h = \frac{|ds|}{2y}.$$

The (sense-preserving) isometry group of $\mathscr{H}^2$ is isomorphic to the group $G = PSL(2, \mathbb{R})$ acting transitively by fractional linear transformations: if A is a 2×2 real matrix with $\det A \neq 0$, the corresponding mapping is

$$T_A : z \mapsto \frac{az+b}{cz+d}, \quad \text{when} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The invariant Haar measure of G induces an invariant notion of area μ in the homogeneous space $\mathscr{H}^2 \cong G/PSO(2)$, known as the *Gauss-Bonnet area measure*; it coincides (up to a multiplicative constant) with the hyperbolic area element induced by the Poincaré metric ds_h.

For a Fuchsian group, a discrete subgroup Γ of G, the action on $\mathscr{H}^2$ is properly discontinuous and there is a *fundamental domain*, which we take here to mean a closed (Borel-measurable) subset F, with interior F^0, satisfying two characteristic properties:-

(i) $F^0 \cap \gamma F^0 = \emptyset$ for $\gamma \in \Gamma$ with $\gamma \neq \mathrm{Id}$;
(ii) $\bigcup_{\gamma \in \Gamma} \gamma F = \mathscr{H}^2$.

The Dirichlet region with a chosen centre point p_0 not fixed by any group element is a convenient construction, giving a (convex hyperbolic) polygonal fundamental domain for each Fuchsian group: one takes the subset of those points of $\mathscr{H}^2$ for which the distance $d_h(p, p_0)$ from p to p_0 is minimal among the points in the orbit $\Gamma \cdot p$. For full details of these basic concepts of hyperbolic geometry, see for instance Beardon's book [2, 12].

For Γ co-compact and torsion-free, i.e. such that the quotient orbit space $S = \mathscr{H}^2/\Gamma$ is a compact Riemann surface, the hyperbolic area $\mu(F)$ of any fundamental domain for Γ is a positive number, independent of the choice of fundamental set. In particular, by the Gauss-Bonnet Theorem, the area of a fundamental set for a group which has quotient a genus g closed surface is $4\pi(g-1) = (-2\pi\chi(S)$. Siegel showed in [17] that, within the range of all possible Fuchsian groups in G, there is a unique group (up to conjugacy) with the smallest positive value for the area.

Theorem 1 *For all co-compact Fuchsian groups, the minimum value of the invariant area μ is $\pi/21$.*

This value corresponds to the triangle group $\Gamma_0 = \langle x, y : x^2 = y^3 = (xy)^7 = 1\rangle$.

Now we let $K < \Gamma$ be a subgroup of finite index in a Fuchsian group with compact quotient space. Choosing a finite set of coset representatives, $\gamma_1, \ldots, \gamma_n$, we see that the union of these translates $\Cup_{j=1}^{n} \gamma_j F$ of a given fundamental domain F for Γ forms a fundamental domain F_K for the subgroup K, and invariance of the measure implies that

Theorem 2 *The index of the subgroup K, $[\Gamma : K]$, is equal to the quotient $\mu(F_K)/\mu(F)$.*

This is the *Gauss-Bonnet Index Theorem* for Fuchsian groups. A simple consequence of this and the result of Siegel is the *Hurwitz Theorem*: choose a Fuchsian cocompact group $K \cong \pi_1(S)$ and let Γ denote the group of all possible lifts to the universal covering of S of the automorphisms of S. Then Γ contains K as a normal subgroup, and since $|\mathrm{Aut}(S)| = [\Gamma : K]$, we obtain at once the following result.

Corollary 3 *The order of an automorphism group acting on a genus g Riemann surface is at most* $84(g-1)$.

Of course, this is not the end of the story: the same line of reasoning produces the Riemann-Hurwitz branching formula, involving the genus of $\mathscr{H}^2/\Gamma$ and the orders of the maximal periodic generators of Γ, and there is a natural extension to subgroups of finite index in arbitrary non-Euclidean crystallographic groups, leading to a vast catalogue of results analysing the patterns of conformal and anti-conformal group actions on hyperbolic surfaces.

Macbeath's paper [10] fills in the details of the above proof and unveils a method (the *'Macbeath trick'* mentioned by Marston Conder in his conference talk at Malvern) for constructing, from a single finite index torsion-free normal subgroup $K \lhd \Gamma$, an infinite sequence K_n, $n \in \mathbb{N}$ of finite index characteristic normal subgroups of Γ. For each of these subgroups, the corresponding quotient surface S_n is of course a smooth covering of $S_0 = \mathscr{H}^2/K$ and the finite groups Γ/K_n are automorphism groups of the surfaces, with orders determined by multiplying by the index just like the Euler characteristic. This proves the following result, *Macbeath's Theorem*.

Theorem 4 *If there is a Riemann surface S of genus $g \geq 2$ with a group of $h(g-1)$ automorphisms, where h is a rational number with denominator dividing $g-1$, then for infinitely many values of the integer k there is a k-sheeted covering surface S_k of genus g_k with $h(g_k - 1)$ automorphisms.*

The sequence of characteristic subgroups employed in [10] is defined as the set of product groups $K_n = [K, K].\{K^n\}$, where $\{K^n\}$ denotes the Burnside n-kernel generated by all n-th powers in K and $[K, K]$ is the commutator subgroup. It is then an easy exercise to show that the index of K_n in the surface group K is n^{2g}, so that we know from Euler characteristic considerations that S_n has genus $g_n = n^{2g}(g-1)+1$.

If the over-group concerned is Γ_0, then the induced finite groups are all (by Siegel's result) *Hurwitz groups* acting on the surfaces S_n, that is, we have produced an infinite family of surfaces with automorphism groups for which the Hurwitz bound on their order is attained.

The original exposition of this approach to Fuchsian groups and surface automorphisms was presented in a widely circulated set of lecture notes, [11] from the Summer School in Topology at Dundee in 1961; they can be obtained in pdf format by email request to this author (bill.harvey@kcl.ac.uk). A very pleasant account by Macbeath of the whole story, in the context of Klein's study of his quartic and the genus 3 action of the simple group of order 168, can be found in [13].

3 Automorphisms and Geometry in Three Dimensional Hyperbolic Space

Poincaré also initiated the study of hyperbolic 3-space with its analogous metric structure closely linked to the matrix group $PSL(2, \mathbb{C})$ and conformal geometry on the boundary 2-sphere, but progress in understanding the topological structure of 3-manifolds was slow. After the revolutionary geometric ideas, results and conjectures worked out by W.P. Thurston in the mid-1970s, a furious concentration of research effort ensued which has swept away most of the topological and group-theoretic difficulties which confronted 3-manifold topology at that time. In the process, two crucial facts emerged, the first largely due to the efforts of G. Perelman.

- (Geometrisation.) All compact 3-manifolds possess a natural geometric structure, modelled on one of the eight geometries that Thurston described.
- (Hyperbolic structure predominates.) By far the majority of compact 3-manifolds are hyperbolic.

We can now formulate a very simple topological characterisation of the class of compact hyperbolic 3-manifolds, thanks to recent breakthrough work by Jeremy Kahn and Vlad Markovic with some essential further topological and group-theoretic input from I. Agol and from D. Wise.

Theorem 5 *A compact 3-manifold has a hyperbolic structure if and only if it has a finite covering which fibers over the circle with fibre a compact surface of genus at least 2 and with pseudo-Anosov holonomy.*

An accessible summary treatment of these developments which brings out well the range of ideas and work involved can be found in a recent Bourbaki Seminar report [3]. The key result which drives them appears in [7]. It confirms a remarkable string of conjectures made by Thurston [18], following his proof that both Haken manifolds and pseudo-Anosov surface bundles over the circle carry hyperbolic structures.

In the present context, it is natural to look for a parallel approach to study automorphism groups of (compact) hyperbolic 3 manifolds via the structure of 3D hyperbolic orbifolds. This turns out to be possible in principle, but the combinatorial patterns which exist have not yet been completely understood. However, for the privileged class of surface bundles, we can reason as follows.

A compact hyperbolic surface bundle induced by a pseudo-Anosov map $\varphi : S \to S$ is, by definition, a 3-manifold obtained from the product of a surface S of genus at least 2 with a closed interval by identifying the two end surfaces using φ:

$$M_\varphi = S \times \{0 \leq t \leq 1\}/\{x \times 0 \sim \varphi(x) \times 1 \quad \text{for each} \quad x \in S\}.$$

Now any automorphism of M_φ must preserve the fibration structure since, by Macbeath's method of lifting automorphisms to the universal cover, it is induced by conjugation with a hyperbolic isometry—equally this follows by Mostow rigidity. But such an isometry must induce an automorphism of a typical fibre surface S preserving the holomorphic quadratic form (Teichmüller differential) on S which determines the hyperbolic axis in $\mathscr{H}^3$ up to conjugacy. Note that any hyperbolic isometry f of $\mathscr{H}^3$ preserving an axis must be contained in the stabiliser of the axis and, furthermore, lies in the normaliser of $\pi_1(M_\varphi)$, a discrete subgroup of $PSL(2, \mathbb{C})$ which means that f lies in a discrete subgroup of that stabiliser. We can assume for convenience that this is the vertical axis I at O, the origin in the horizontal plane $\mathbb{C}$, and it follows that the group of automorphisms of a hyperbolic surface bundle is very restricted.

Theorem 6 *Let $M = M_\varphi$ be a smooth hyperbolic surface bundle over the circle induced by a pseudo-Anosov homeomorphism $\varphi : S \to S$ of a genus g surface. The automorphism group of the fibered hyperbolic 3-manifold M is either cyclic or dihedral, with order bounded above by a linear function $(2g - 1)$ of the genus of S.*

Proof (Sketch.) The stabiliser in $PSL(2, \mathbb{C})$ of an axis A is isomorphic to $G(A) = \mathbb{Z}/2 \times \mathbb{C}^*$ and the intersection of this stabiliser with $K = \pi_1(M)$ is cyclic, generated by some loxodromic element. Note that any discrete subgroup of $G(A)$ is dihedral or cyclic. The automorphisms of M, when lifted to the universal covering, generate a discrete overgroup $\Gamma > K$, the normaliser of $\pi_1(M)$, just as in the Riemann surface case. But in contrast to the case of surfaces, where the $(2, 3, 7)$—triangle group Γ_0 gives the Hurwitz upper bound, the orbifold fibre surface must in this case be *sufficiently large*, in the sense that it contains an essential closed loop and admits a pseudo-Anosov automorphism. This implies that the fibre subgroup Γ, which must contain the surface subgroup $\pi_1(S)$ with cyclic or dihedral quotient automorphism group, has at least 4 generators, and restrictions on the orders of torsion generators coming from the so-called lcm condition (see [6]) imply that the smallest area Γ (in terms of Euler-Poincaré characteristic) is a $(2, 2, n, m)$—group for suitable periods n, m dividing the index. Hence the index, if the quotient is cyclic, is at most $2g - 1$ by a short argument using Theorem 2.2. The argument in the dihedral case is similar but a little more complicated.

A more detailed discussion and complete proof will be published elsewhere. Notice that this result does not hold if the fibration is not smooth: an example of a hyperbolic orbifold fibering in three mutually orthogonal ways, which goes back to Sullivan (and probably Thurston), is described briefly in Otal's text [16].

We note, finally, that Macbeath's result on a sequence of characteristic subgroups applies here for any given example as in the theorem, to produce an infinite family of fibered 3-manifolds which cover it and enjoy the same symmetry property.

More general results are known about automorphisms of hyperbolic manifolds in dimension 3 which chime with dimension 2; for instance a finite volume or compact hyperbolic 3-manifold may have any finite group as automorphism group. See, for instance, [8] or [14]. At present, however, a normal form for crystallographic groups in hyperbolic 3-space is unknown and no general direct analysis of automorphism groups analogous to the 2D case seems possible. A good account of the basic facts about 3D hyperbolic volumes can be found in Milnor's paper [15], and more recently the smallest volume manifolds and orbifolds have been determined, both compact and cusped. Clearly, much remains to be done in analysing the combinatorial structure of hyperbolic 3-manifolds and their automorphisms.

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