BSDE Approach for Dynkin Game and American Game Option

El Hassan Essaky and M. Hassani

Abstract Consider a Dynkin game with payoff

$$J(\lambda, \sigma) = F\Big[U_\lambda 1_{\{\lambda<\sigma\}} + L_\sigma 1_{\{\lambda>\sigma\}} + Q_\sigma 1_{\{\sigma=\lambda<T\}} + \xi 1_{\{\sigma=\lambda=T\}}\Big],$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous nondecreasing function and λ, σ are stopping times valued in $[0, T]$. We show the existence of a value as well as a saddle-point for this game using the theory of BSDE with double reflecting barriers. An American game option pricing problem is also discussed.

Keywords Backward stochastic differential equations with double reflecting barriers · Dynkin game · Saddle-point · American game option

Mathematical Subject Classification 2010 60H10 · 60H20 · 60H30

1 Introduction

Stochastic game was first introduced by Dynkin and Yushkevich [6] and later studied, in different contexts, by several authors, including Neveu [18], Bensoussan and Friedman [2], Bismut [3], Morimoto [17], Alario-Nazaret, Lepeltier and Marchal [1], Lepeltier and Maingueneau [16], Cvitanic and Karatzas [4], Touzi and Vieille [19] and others, such stochastic games are known as Dynkin games.

E.H. Essaky (✉) · M. Hassani
Département de Mathématiques et d'Informatique,
Laboratoire de Modélisation et Combinatoire, Faculté Poly-disciplinaire,
Université Cadi Ayyad, B.P. 4162, Safi, Morocco
e-mail: essaky@uca.ma

M. Eddahbi et al. (eds.), *Statistical Methods and Applications in Insurance and Finance*, Springer Proceedings in Mathematics & Statistics 158,
DOI 10.1007/978-3-319-30417-5_9

Considerable attention has been devoted to studying the association between backward stochastic differential equations (BSDEs for short) and stochastic differential games. Among others, Cvitanic and Karatzas showed in [4] existence and uniqueness of the solution to the BSDE with double reflecting barriers, and associated their equation to a stochastic games. Hamadène [9] and Hamadène and Hassani [11] studied the mixed zero-sum stochastic differential game problem using the notion of a local solution of BSDEs with double reflecting barriers. Hamadène and Lepeltier [10] added controls to the Dynkin game studied by Cvitanic and Karatzas in [4]. Karatzas and Li [14] studied a non-zero-sum game with features of both stochastic control and optimal stopping, for a process of diffusion type via the BSDE approach. Dumitrescu et al. [5] introduced a generalized Dynkin game problem associated with a BSDE with jumps.

Consider the Dynkin game, associated with processes L, U, ξ and Q, with payoff:

$$J(\lambda, \sigma) = F\left[U_\lambda 1_{\{\lambda<\sigma\}} + L_\sigma 1_{\{\lambda>\sigma\}} + Q_\sigma 1_{\{\sigma=\lambda<T\}} + \xi 1_{\{\sigma=\lambda=T\}}\right],$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous nondecreasing function and λ, σ are stopping times valued in $[0, T]$. In the direction of connection between BSDE with two reflecting barriers and Dynkin games, in order to prove that this game has a saddle point, which is a pair of stopping (λ^*, σ^*) such that for any stopping times λ and σ one has

$$\mathbb{E}\Big(J(\lambda^*, \sigma)\Big) \leq \mathbb{E}\Big(J(\lambda^*, \sigma^*)\Big) \leq \mathbb{E}\Big(J(\lambda, \sigma^*)\Big),$$

all the works [4, 9–11, 14] have considered the case of bounded or square integrable processes $F(\xi)$, $F(Q)$, $F(L)$ and $F(U)$. Moreover, they have assumed that the barriers $F(L)$ and $F(U)$ have to satisfy one of the conditions:

1. The so-called Mokobodski condition which turns out into the existence of a difference of nonnegative supermartingales between $F(L)$ and $F(U)$.
2. The complete separation i.e. $F(L) < F(U)$.

One of the main objective of this work is to weaken the assumptions assumed on the data $F(\xi)$, $F(Q)$, $F(L)$ and $F(U)$ in the case of association between BSDE with two reflecting barriers and Dynkin games. Yet, checking Mokobodski's condition appears as a difficult question. So, instead of assuming the Mokobodski's condition on the barriers $F(L)$ and $F(U)$, we suppose only that there exists a semimartingale between them. It should be also noted here that if the barriers are completely separated this implies that there exists a semimartingale between them (see [8]). Actually, if we assume the following conditions:

1. There exists a semimartingale between L and U and for every semimartingale S such that $L \leq S \leq U$, $F(S)$ is a also a semimartingale.
2. $\mathbb{E}[F(L_\sigma)^-] < +\infty$, for all stopping time $0 \leq \sigma \leq T$, where $F(L)^- = \sup(-F(L), 0)$.

3. $\liminf_{r\to+\infty} r\, P\left[\sup_{s\le T} F(U_s)^+ > r\right] = 0$, where $F(U)^+ = \sup(F(U), 0)$.
4. $\liminf_{r\to+\infty} r\, P\left[\sup_{s\le T} F(L_s)^- > r\right] = 0$,

then the pair of stopping times (λ^*, σ^*) defined by

$$\lambda^* = \inf\{s \ge 0 \ :\ Y_s = F(U_s)\} \wedge T \quad \text{and} \quad \sigma^* = \inf\{s \ge 0 \ :\ Y_s = F(L_s)\} \wedge T,$$

is a saddle-point for the game, where Y is the solution of the following BSDE with double reflecting barriers $F(L)$ and $F(U)$ (see Definition 2):

$$\begin{cases} (i) \quad Y_t = F(\xi) + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s\,, t \le T, \\ (ii) \quad Y \text{ between } F(L) \text{ and } F(U),\ \ i.e.\ \forall t \le T,\ \ F(L_t) \le Y_t \le F(U_t), \\ (iii) \ \ \text{the Skorohod conditions hold:} \\ \qquad \int_0^T (Y_t - F(L_t))dK_t^+ = \int_0^T (F(U_t) - Y_t)dK_t^- = 0,\ \text{ a.s..} \end{cases}$$

We should mention here that if $F(L)$ and $F(U)$ are L^1—integrable, i.e. $\mathbb{E}\sup_{t\le T}(|F(U_t)| + |F(L_t)|) < +\infty$, then the above assumptions 2–4 are satisfied and then the Dynkin game has a saddle point. This corresponds to the main assumption assumed in the general context of Dynkin games.

An American option is a contract which enables its buyer (holder) to exercise it at any time up to the maturity. An American game option gives additionally the right to the option seller (writer, issuer) to cancel it early paying for this a prescribed penalty. American game option was first introduced by Kifer [15] and studied later by several authors, see for example Hamadène [9], Hamadène and Zhang [13] and the references therein. The second aim of this work is to prove, under weaker conditions than the square integrability assumed on the data in [9], that the value of the option at any time $t \in [0, T]$ is given by $e^{rt}Y_t$, where Y is the solution of some BSDE with two reflecting barriers. Moreover, we also show that a hedge after t, against the game option, exists.

2 Preliminaries

2.1 *Notations and Assumptions*

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\le T}, P)$ be a stochastic basis on which is defined a Brownian motion $(B_t)_{t\le T}$ such that $(\mathscr{F}_t)_{t\le T}$ is the natural filtration of $(B_t)_{t\le T}$ and $\mathscr{F}_0$ contains all P-null sets of $\mathscr{F}$. Note that $(\mathscr{F}_t)_{t\le T}$ satisfies the usual conditions, i.e. it is right continuous and complete.

Let us now introduce the following notations:

- $\mathscr{P}$ the sigma algebra of $\mathscr{F}_t$-progressively measurable sets on $\Omega \times [0, T]$.
- $\mathscr{C}$ the set of $\mathbb{R}$-valued $\mathscr{P}$-measurable continuous processes $(Y_t)_{t \leq T}$.
- $\mathscr{L}^{2,d}$ the set of $\mathbb{R}^d$-valued and $\mathscr{P}$-measurable processes $(Z_t)_{t \leq T}$ such that

$$\int_0^T |Z_s|^2 ds < \infty, \ P - a.s.$$

- $\mathscr{K}$ the set of $\mathscr{P}$-measurable continuous nondecreasing processes $(K_t)_{t \leq T}$ such that $K_0 = 0$ and $K_T < +\infty, \ P-$ a.s.

Throughout the paper, we introduce the following data:

- $L := \{L_t, \ 0 \leq t \leq T\}$ and $U := \{U_t, \ 0 \leq t \leq T\}$ are two real valued barriers which are $\mathscr{P}$-measurable and continuous processes such that $L_t \leq U_t, \ \forall t \in [0, T]$.
- Q be a process such that, $\forall t \in [0, T]$ $L_t \leq Q_t \leq U_t, \ P - a.s.$
- ξ is an $\mathscr{F}_T$-measurable one dimensional random variable such that

$$L_T \leq \xi \leq U_T.$$

- $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous nondecreasing function.

We assume the following assumptions:

(**A**.**1**) There exists a continuous semimartingale $S_{.} = S_0 + V_{.}^{+} - V_{.}^{-} + \int_0^{.} \alpha_s dB_s$, with $S_0 \in \mathbb{R}$, $V^+, V^- \in \mathscr{K}$ and $\alpha \in \mathscr{L}^{2,d}$, such that

$$L_t \leq S_t \leq U_t, \ \forall t \in [0, T].$$

(**A**.**2**) For every semimartingale S such that $L \leq S \leq U$, $F(S)$ is a also a semimartingale.

2.2 Existence of Solution for BSDE with Double Reflecting Barriers

In view of clarifying this issue, we recall some results concerning BSDEs with double reflecting barriers with two continuous barriers (see Essaky and Hassani [8] for more details). Let us recall first the following definition of two singular measures.

Definition 1 Let K^1 and K^2 be two processes in $\mathscr{K}$. We say that:

K^1 and K^2 are singular if and only if there exists a set $D \in \mathscr{P}$ such that

$$\mathbb{E}\int_0^T 1_D(s,\omega)dK_s^1(\omega) = \mathbb{E}\int_0^T 1_{D^c}(s,\omega)dK_s^2(\omega) = 0.$$

This is denoted by $dK^1 \perp dK^2$.

Let us now introduce the definition of a BSDE with double reflecting obstacles L and U.

Definition 2 1. We call $(Y, Z, K^+, K^-) := (Y_t, Z_t, K_t^+, K_t^-)_{t\le T}$ a solution of the GBSDE with two reflecting barriers L and U associated with a terminal value ξ if the following hold:

$$\begin{cases} (i) & Y_t = \xi + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s\,, t \le T, \\ (ii) & Y \text{ between } L \text{ and } U,\ i.e.\ \forall t \le T,\ \ L_t \le Y_t \le U_t, \\ (iii) & \text{the Skorohod conditions hold:} \\ & \int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (U_t - Y_t)dK_t^- = 0,\ \text{ a.s.,} \\ (iv) & Y \in \mathscr{C} \quad K^+, K^- \in \mathscr{K} \quad Z \in \mathscr{L}^{2,d}, \\ (v) & dK^+ \perp dK^-. \end{cases} \tag{1}$$

2. We say that the BSDE (1) has a maximal (resp. minimal) solution (Y, Z, K^+, K^-) if for any other solution (Y', Z', K'^+, K'^-) of (1) we have for all $t \le T$, $Y_t' \le Y_t$, $P-$a.s. (resp. $Y_t' \ge Y_t$, $P-$a.s.).

The following theorem has already been proved in [8].

Theorem 1 *Let assumption* (**A.1**) *holds true. Then there exists a maximal (resp. minimal) solution for BSDE with double reflecting barriers* (1).

3 Dynkin Game

Our purpose in this section is to show that the existence of a solution (Y, Z, K^+, K^-) to the BSDE (1) implies that Y is the value of a certain stochastic game of stopping.

Consider the payoff

$$J(\lambda,\sigma) = F\left(U_\lambda 1_{\{\lambda<\sigma\}} + L_\sigma 1_{\{\lambda>\sigma\}} + Q_\sigma 1_{\{\sigma=\lambda<T\}} + \xi 1_{\{\sigma=\lambda=T\}}\right).$$

The setting of our problem of Dynkin game is the following. There are two players labeled player 1 and player 2. Player 1 chooses the stopping time λ, player 2 chooses the stopping time σ, and $J(\lambda, \sigma)$ represents the amount paid by player 1 to player 2. It is the conditional expectation $\mathbb{E}\Big(J(\lambda, \sigma)\Big)$ of this random payoff that player 1 tries to minimize and player 2 tries to maximize. The game stops when one player decides to stop, that is, at the stopping time $\lambda \wedge \sigma$ before time T, the payoff is then equals

$$J(\lambda, \sigma) = \begin{cases} F(U_\lambda) & \text{if player 1 stops the game first} \\ F(L_\sigma) & \text{if player 2 stops the game first} \\ F(Q_\sigma) & \text{if players stop the game simultaneously before time T} \\ F(\xi) & \text{if neither have exercised until the expiry time T.} \end{cases}$$

It is then natural to define the lower and upper values of the game:

$$\underline{V} := \sup_{\sigma \in \mathscr{M}_{t,T}} \inf_{\lambda \in \mathscr{M}_{t,T}} \mathbb{E}\Big[J(\lambda, \sigma)\Big] \le \overline{V} := \inf_{\lambda \in \mathscr{M}_{t,T}} \sup_{\sigma \in \mathscr{M}_{t,T}} \mathbb{E}\Big[J(\lambda, \sigma)\Big],$$

where $\mathscr{M}_{t,T}$ is the set of stopping times valued between t and T. If it happens that $\underline{V} = \overline{V}$, then the above Dynkin game is said to have a value. A pair $(\lambda_0^*, \sigma_0^*)$ is called a saddle point if

$$\mathbb{E}\Big(J(\lambda_0^*, \sigma)\Big) \le \mathbb{E}\Big(J(\lambda_0^*, \sigma_0^*)\Big) \le \mathbb{E}\Big(J(\lambda, \sigma_0^*)\Big).$$

Our objective is to show the existence of a saddle-point for the game and to characterize it. This implies that this game has a value.

Let assumptions (**A.1**) and (**A.2**) hold true. Let (Y, Z, K^+, K^-) be the solution, which is exists according to Theorem 1, of the following BSDE with double reflecting barriers:

$$\begin{cases} (i) \;\; Y_t = F(\xi) + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dB_s, t \le T, \\ (ii) \;\; \forall t \le T, \;\; F(L_t) \le Y_t \le F(U_t), \\ \qquad \int_0^T (Y_t - F(L_t))dK_t^+ = \int_0^T (F(U_t) - Y_t)dK_t^- = 0, \;\; \text{a.s.,} \\ (iv) \;\; Y \in \mathscr{C} \quad K^+, K^- \in \mathscr{K} \quad Z \in \mathscr{L}^{2,d}, \\ (v) \;\; dK^+ \perp dK^-. \end{cases} \tag{2}$$

Let λ_t^* and σ_t^* be the stopping times defined as follows:

$$\lambda_t^* = \inf\{s \ge t \;:\; Y_s = F(U_s)\} \wedge T \quad \text{and} \quad \sigma_t^* = \inf\{s \ge t \;:\; Y_s = F(L_s)\} \wedge T.$$

The main result of this section is the following.

Theorem 2 *Assume the following assumptions:*

1. $\mathbb{E}F(L_\sigma)^- < +\infty$, *for all stopping time* $0 \le \sigma \le T$, *where* $F(L)^- = \sup(-F(L), 0)$.
2. $\liminf_{r \to +\infty} r\, P\left[\sup_{s \le T} F(U_s)^+ > r\right] = 0$, *where* $F(U)^+ = \sup(F(U), 0)$.
3. $\liminf_{r \to +\infty} r\, P\left[\sup_{s \le T} F(L_s)^- > r\right] = 0$.

Then

$$\begin{aligned} Y_t &= \mathbb{E}\Big[J(\lambda_t^*, \sigma_t^*) \mid \mathscr{F}_t\Big] \\ &= \sup_{\sigma \in \mathscr{M}_{t,T}} \mathbb{E}\Big[J(\lambda_t^*, \sigma) \mid \mathscr{F}_t\Big] = \inf_{\lambda \in \mathscr{M}_{t,T}} \mathbb{E}\Big[J(\lambda, \sigma_t^*) \mid \mathscr{F}_t\Big] \\ &= \inf_{\lambda \in \mathscr{M}_{t,T}} \sup_{\sigma \in \mathscr{M}_{t,T}} \mathbb{E}\Big[J(\lambda, \sigma) \mid \mathscr{F}_t\Big] = \sup_{\sigma \in \mathscr{M}_{t,T}} \inf_{\lambda \in \mathscr{T}_t} \mathbb{E}\Big[J(\lambda, \sigma) \mid \mathscr{F}_t\Big], \end{aligned} \tag{3}$$

where $\mathscr{M}_{t,T}$ *is the set of stopping times valued between* t *and* T. Y_0 *can be interpreted as the value of the game and* $(\lambda_0^*, \sigma_0^*)$ *as the fair strategy for the two players (or a saddle point for the game).*

Proof Let $(a_n^+)_n$ and $(a_n^-)_n$ be two nondecreasing sequences such that

$$\liminf_{n \to +\infty} a_n^+ P\left[\sup_{s \le T} F(U_s)^+ > a_n^+\right] = 0, \quad \liminf_{n \to +\infty} a_n^- P\left[\sup_{s \le T} F(L_s)^- > a_n^-\right] = 0. \tag{4}$$

Let also $(\alpha_i)_{i \ge 0}$ and $(\upsilon_i^\pm)_{i \ge 0}$ be families of stopping times defined by

$$\alpha_i = \inf\{s \ge t : \int_t^s |Z_r|^2\, dr \ge i\} \wedge T, \quad \upsilon_i^\pm = \inf\{s \ge t : Y_s^\pm > a_i^\pm\} \wedge T.$$

It follows from Eq. (2) that for every stopping time $\sigma \in \mathscr{M}_{t,T}$

$$\begin{aligned} Y_t =& Y_{\lambda_t^* \wedge \sigma \wedge \alpha_i \wedge \upsilon_n^+ \wedge \upsilon_m^-} + \int_t^{\lambda_t^* \wedge \sigma \wedge \alpha_i \wedge \upsilon_n^+ \wedge \upsilon_m^-} dK_s^+ - \underbrace{\int_t^{\lambda_t^* \wedge \sigma \wedge \alpha_i \wedge \upsilon_n^+ \wedge \upsilon_m^-} dK_s^-}_{=0} \\ & - \int_t^{\lambda_t^* \wedge \sigma \wedge \alpha_i \wedge \upsilon_n^+ \wedge \upsilon_m^-} Z_s dB_s. \end{aligned}$$

Then for every stopping time $\sigma \in \mathscr{M}_{t,T}$

$$
\begin{aligned}
Y_t &\geq \mathbb{E}\left(Y_{\lambda_t^* \wedge \sigma \wedge \alpha_i \wedge \upsilon_n^+ \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) \\
&= \mathbb{E}\left(Y^+_{\lambda_t^* \wedge \sigma \wedge \alpha_i \wedge \upsilon_n^+ \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) - \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma \wedge \alpha_i \wedge \upsilon_n^+ \wedge \upsilon_m^-} \mid \mathscr{F}_t \right).
\end{aligned}
$$

In view of passing to the limit on i and n respectively and using Fatou's lemma for Y^+ and dominated convergence theorem for Y^- since it is bounded, we have

$$
Y_t \geq \mathbb{E}\left(Y^+_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) - \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right).
$$

Now taking the upper limit on m we get

$$
\begin{aligned}
Y_t &\geq \limsup_m \left[\mathbb{E}\left(Y^+_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) - \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) \right] \\
&= \limsup_m \left[\mathbb{E}\left(Y^+_{\lambda_t^* \wedge \sigma} 1_{\lambda_t^* \wedge \sigma \leq \upsilon_m^-} \mid \mathscr{F}_t \right) + \mathbb{E}\left(Y^+_{\upsilon_m^-} 1_{\lambda_t^* \wedge \sigma > \upsilon_m^-} \mid \mathscr{F}_t \right) \right. \\
&\quad \left. - \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) \right] \\
&\geq \limsup_m \left[\mathbb{E}\left(Y^+_{\lambda_t^* \wedge \sigma} 1_{\lambda_t^* \wedge \sigma \leq \upsilon_m^-} \mid \mathscr{F}_t \right) - \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) \right] \\
&= \mathbb{E}\left(Y^+_{\lambda_t^* \wedge \sigma} \mid \mathscr{F}_t \right) - \liminf_m \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right).
\end{aligned}
$$

In view of using the limit appearing in (4), we obtain

$$
\begin{aligned}
&\liminf_m \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma \wedge \upsilon_m^-} \mid \mathscr{F}_t \right) \\
&\leq \liminf_m \left[\mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma} 1_{\lambda_t^* \wedge \sigma \leq \upsilon_m^-} \mid \mathscr{F}_t \right) + a_m^- \mathbb{E}\left(1_{\lambda_t^* \wedge \sigma > \upsilon_m^-} \mid \mathscr{F}_t \right) \right] \\
&= \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma} \mid \mathscr{F}_t \right) + \liminf_{m \to +\infty} a_m^- \mathbb{E}\left(1_{\lambda_t^* \wedge \sigma > \upsilon_m^-} \mid \mathscr{F}_t \right) \\
&\leq \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma} \mid \mathscr{F}_t \right) + \liminf_{m \to +\infty} a_m^- \mathbb{E}\left(1_{\{\sup_{s \leq T} F(L_s)^- > a_m^-\}} \mid \mathscr{F}_t \right) \\
&= \mathbb{E}\left(Y^-_{\lambda_t^* \wedge \sigma} \mid \mathscr{F}_t \right),
\end{aligned}
$$

it follows then that for all stopping time $\sigma \in \mathscr{M}_{t,T}$,

$$
\begin{aligned}
Y_t &\geq \mathbb{E}\Big(Y^+_{\lambda^*_t\wedge\sigma} \mid \mathscr{F}_t\Big) - \mathbb{E}\Big(Y^-_{\lambda^*_t\wedge\sigma} \mid \mathscr{F}_t\Big) = \mathbb{E}\Big(Y_{\lambda^*_t\wedge\sigma} \mid \mathscr{F}_t\Big) \\
&\geq \mathbb{E}\Big(F(U_{\lambda^*_t})1_{\{\lambda^*_t<\sigma\}} + F(L_\sigma)1_{\{\lambda^*_t>\sigma\}} + F(Q_\sigma)1_{\{\sigma=\lambda^*_t<T\}} + F(\xi)1_{\{\sigma=\lambda^*_t=T\}} \mid \mathscr{F}_t\Big) \\
&= \mathbb{E}\Big(J(\lambda^*_t,\sigma) \mid \mathscr{F}_t\Big).
\end{aligned} \tag{5}
$$

Now it follows from Eq. (2) that for every stopping time $\lambda \in \mathscr{M}_{t,T}$

$$
\begin{aligned}
Y_t &\leq \mathbb{E}\Big(Y_{\lambda\wedge\sigma^*_t\wedge\alpha_i\wedge\upsilon^-_m\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) \\
&= \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t\wedge\alpha_i\wedge\upsilon^-_m\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) - \mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t\wedge\alpha_i\wedge\upsilon^-_m\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big).
\end{aligned}
$$

In view of passing to the limit on i and m respectively and using dominated convergence theorem for Y^+ since it is bounded, we have

$$
\begin{aligned}
&Y_t \\
&\leq \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) - \limsup_m \mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t\wedge\upsilon^-_m\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) \\
&= \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) - \limsup_m \Big[\mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} 1_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n\leq\upsilon^-_m} \mid \mathscr{F}_t\Big) \\
&\quad + \mathbb{E}\Big(Y^-_{\upsilon^-_m} 1_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n>\upsilon^-_m} \mid \mathscr{F}_t\Big)\Big] \\
&\leq \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) - \mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) - \limsup_m \mathbb{E}\Big(Y^-_{\upsilon^-_m} 1_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n>\upsilon^-_m} \mid \mathscr{F}_t\Big) \\
&\leq \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) - \mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big).
\end{aligned}
$$

By using Fatou's lemma and assumption 1. Of Theorem 2 we get

$$
\begin{aligned}
Y_t + \mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t} \mid \mathscr{F}_t\Big) &\leq Y_t + \liminf_n \mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) \\
&\leq \liminf_n \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t\wedge\upsilon^+_n} \mid \mathscr{F}_t\Big) \\
&\leq \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t} \mid \mathscr{F}_t) + \liminf_{n\to+\infty} a^+_n \mathbb{E}\Big(1_{\lambda\wedge\sigma^*_t>\upsilon^+_n} \mid \mathscr{F}_t\Big) \\
&\leq \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t} \mid \mathscr{F}_t) + \liminf_{n\to+\infty} a^+_n \mathbb{E}\Big(1_{\{\sup_{s\leq T} F(U_s)^+>a^+_n\}} \mid \mathscr{F}_t\Big) \\
&\leq \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t} \mid \mathscr{F}_t\Big),
\end{aligned}
$$

where we have used the limit appeared in (4).

It follows that for every stopping time $\lambda \in \mathcal{M}_{t,T}$

$$\begin{aligned} Y_t &\le \mathbb{E}\Big(Y^+_{\lambda\wedge\sigma^*_t} \mid \mathcal{F}_t\Big) - \mathbb{E}\Big(Y^-_{\lambda\wedge\sigma^*_t} \mid \mathcal{F}_t\Big) \\ &= \mathbb{E}\Big(Y_{\lambda\wedge\sigma^*_t} \mid \mathcal{F}_t\Big). \\ &\le \mathbb{E}\Big(F(U_\lambda)1_{\{\lambda<\sigma^*_t\}} + F(L_{\sigma^*})1_{\{\lambda>\sigma^*_t\}} + F(Q_{\sigma^*})1_{\{\sigma^*_t=\lambda<T\}} + F(\xi)1_{\{\sigma^*_t=\lambda=T\}} \mid \mathcal{F}_t\Big) \\ &= \mathbb{E}\Big(J(\lambda,\sigma^*_t) \mid \mathcal{F}_t\Big). \end{aligned} \tag{6}$$

In force of inequalities (5) and (6) we obtain that for all $\sigma, \lambda \in \mathcal{M}_{t,T}$

$$\mathbb{E}\Big(J(\lambda^*_t,\sigma) \mid \mathcal{F}_t\Big) \le Y_t = \mathbb{E}\Big[J(\lambda^*_t,\sigma^*_t) \mid \mathcal{F}_t\Big] \le \mathbb{E}\Big(J(\lambda,\sigma^*_t) \mid \mathcal{F}_t\Big).$$

Then it is immediately checked that

$$\begin{aligned} \inf_{\lambda\in\mathcal{M}_{t,T}} \sup_{\sigma\in\mathcal{T}_t} \mathbb{E}\Big(J(\lambda,\sigma) \mid \mathcal{F}_t\Big) &\le \sup_{\sigma\in\mathcal{T}_t} \mathbb{E}\Big(J(\lambda^*_t,\sigma) \mid \mathcal{F}_t\Big) \\ &\le Y_t = \mathbb{E}\Big[J(\lambda^*_t,\sigma^*_t) \mid \mathcal{F}_t\Big] \\ &\le \inf_{\lambda\in\mathcal{M}_{t,T}} \mathbb{E}\Big(J(\lambda,\sigma^*_t) \mid \mathcal{F}_t\Big) \\ &\le \sup_{\sigma\in\mathcal{M}_{t,T}} \inf_{\lambda\in\mathcal{M}_{t,T}} \mathbb{E}\Big(J(\lambda,\sigma) \mid \mathcal{F}_t\Big). \end{aligned}$$

Since $\sup_{\sigma\in\mathcal{M}_{t,T}} \inf_{\lambda\in\mathcal{M}_{t,T}} \mathbb{E}\Big(J(\lambda,\sigma) \mid \mathcal{F}_t\Big) \le \inf_{\lambda\in\mathcal{M}_{t,T}} \sup_{\sigma\in\mathcal{M}_{t,T}} \mathbb{E}\Big(J(\lambda,\sigma) \mid \mathcal{F}_t\Big)$, we have

$$\begin{aligned} Y_t &= \mathbb{E}\Big[J(\lambda^*_t,\sigma^*_t) \mid \mathcal{F}_t\Big] \\ &= \sup_{\sigma\in\mathcal{M}_{t,T}} \mathbb{E}\Big[J(\lambda^*_t,\sigma) \mid \mathcal{F}_t\Big] = \inf_{\lambda\in\mathcal{M}_{t,T}} \mathbb{E}\Big[J(\lambda,\sigma^*_t) \mid \mathcal{F}_t\Big] \\ &= \inf_{\lambda\in\mathcal{M}_{t,T}} \sup_{\sigma\in\mathcal{M}_{t,T}} \mathbb{E}\Big[J(\lambda,\sigma) \mid \mathcal{F}_t\Big] = \sup_{\sigma\in\mathcal{M}_{t,T}} \inf_{\lambda\in\mathcal{T}_t} \mathbb{E}\Big[J(\lambda,\sigma) \mid \mathcal{F}_t\Big], \end{aligned}$$

Theorem 2 is then proved. □

Remark 1 We should remark here that:

1. If $F(L)$ and $F(U)$ are L^1-integrable, i.e. $\mathbb{E}\sup_{t\le T}(|F(U_t)| + |F(L_t)|) < +\infty$, then the assumption of Theorem 2 are satisfied.

2. If we suppose that $F(x) = e^{\theta x}$ (or $F(x) = -e^{-\theta x}$), $\theta > 0$, we have an utility function which is of exponential type and then our result can give, in particular, a solution to the existence a saddle point for the risk-sensitive problem (see [7] for more details).

4 American Game Option

4.1 Problem Formulation

We deal with American game option or a game contingent claim which is a contract between a seller A and a buyer B at time $t = 0$ such that both have the right to exercise at any stopping time before the maturity time T . If the buyer exercises at time t he receives the amount $L_t \geq 0$ from the seller and if the seller exercises at time t before the buyer he must pay to the buyer the amount $U_t \geq L_t$ so that $U_t - L_t$ is viewed as a penalty imposed on the seller for cancellation of the contract. If both exercise at the same time t before the maturity time T then the buyer may claim Q_t and if neither have exercised until the expiry time T then the buyer may claim the amount ξ. In short, if the the seller decides to exercise at a stopping time $\lambda \leq T$ and the buyer exercises at a stopping time $\sigma \leq T$ then the former pays to the latter the amount:

$$J^1(\lambda, \sigma) = U_\lambda 1_{\{\lambda<\sigma\}} + L_\sigma 1_{\{\lambda>\sigma\}} + Q_\sigma 1_{\{\sigma=\lambda<T\}} + \xi 1_{\{\sigma=\lambda=T\}}.$$

Such game option is considered in a standard securities market consisting of a non-random component S_t^0 representing the value of a savings account at time t with an interest rate r and of a random component S_t representing the stock price at time t. More precisely and following the same idea as in Hamadène [9], we consider a security market $\mathscr{M}$ that contains, say, one bond and one stock and we suppose that their prices are subject to the following system of stochastic differential equations:

$$\begin{cases} dS_t^0 = rS_t^0 dt, & S_0^0 > 0 \\ dS_t = S_t(bdt + \delta dB_t), & S_0 > 0. \end{cases}$$

Let X be an $\mathscr{F}_t$-measurable random variable such that $X \geq 0$. The classical approach suggests that valuation of options should be based on the notions of a self-financing portfolio and on hedging. For this reason, we give the following definitions.

Definition 3 A self-financing portfolio after t with endowment at time t is X, is a $\mathscr{P}$-measurable process $\pi = (\beta_s, \gamma_s)_{t\leq s\leq T}$ with values in $\mathbb{R}^2$ such that:
(i) $\int_t^T (|\beta_s| + (\gamma_s S_s)^2) ds < \infty$.
(ii) If $\Delta_s^{\pi,X} = \beta_s S_s^0 + \gamma_s S_s, \quad s \leq T,$ then $\Delta_s^{\pi,X} = X + \int_t^s \beta_u dS_u^0 + \int_t^s \gamma_u dS_u,$ $\forall s \leq T$.

Definition 4 A hedge against the game with payoff

$$J^1(s,\lambda) := U_\lambda 1_{\{\lambda<s\}} + L_s 1_{\{s<\lambda\}} + Q_s 1_{\{s=\lambda<T\}} + \xi 1_{\{s=\lambda=T\}},$$

after t whose endowment at t is X is a pair (π,λ), where π is self-financing portfolio after t whose endowment at t is X and a stopping time $\lambda \in \mathcal{M}_{t,T}$, satisfying: P-a.s. $\forall s \in [t,T]$,

$$\Delta^{\pi,X}_{s\wedge\lambda} \geq J^1(s,\lambda).$$

Definition 5 The fair price of a contingent claim game is the infimum of capitals X for which the hedging strategy exists. It is defined by

$$V_t := \inf\{X \geq 0,\ \exists(\pi,\lambda) \text{ such that } \Delta^{\pi,X}_{s\wedge\lambda} \geq J^1(s,\lambda),\ \forall t \leq s \leq T,\ \ P - a.s.\}.$$

4.2 Fair Price of the Game as a Solution of BSDE with Two Reflecting Barriers

Now, let P^* be the probability on $(\Omega,\mathcal{F})$ under which the actualized price of the asset is a martingale, i.e.

$$\frac{dP^*}{dP} := exp\left(-\delta^{-1}(b-r)B_t - \frac{1}{2}(\delta^{-1}(b-r))^2 t\right), \quad t \leq T.$$

Hence the process $W_t = B_t + \delta^{-1}(b-r)t$ is an $(\mathcal{F}_t, P^*)$-Brownian motion.

Let ξ, L, U and Q be as in the beginning such that: $0 \leq L \leq U$. Assume moreover that assumption (**A.1**) holds true and consider, on the probability space $(\Omega,\mathcal{F},P^*)$, the following BSDE with two reflecting barriers whose solution exists according to Theorem 1

$$\begin{cases} (i)\ \ Y_t = e^{-rT}\xi + \int_t^T dK_s^+ - \int_t^T dK_s^- - \int_t^T Z_s dW_s, t \leq T, \\ (ii)\ \ \forall t \leq T,\ \ e^{-rt}L_t \leq Y_t \leq e^{-rt}U_t, \\ \qquad \int_0^T (Y_t - e^{-rt}L_t)dK_t^+ = \int_0^T (e^{-rt}U_t - Y_t)dK_t^- = 0, \ \text{a.s.}, \\ (iv)\ Y \in \mathcal{C} \quad K^+, K^- \in \mathcal{K} \quad Z \in \mathcal{L}^{2,d}, \\ (v)\ \ dK^+ \perp dK^-. \end{cases} \tag{7}$$

Let ϱ_t^* and ϑ_t^* be the stopping times defined as follows:

$$\varrho_t^* = \inf\{s \geq t\ :\ Y_s = e^{-rt}U_s\} \wedge T \quad \text{and} \quad \vartheta_t^* = \inf\{s \geq t\ :\ Y_s = e^{-rt}L_s\} \wedge T.$$

If we suppose that $\liminf_{r\to+\infty} rP^*(\sup_{s\leq T} U_s > r) = 0$, it follows then from Theorem 2, since $L \geq 0$, that for all $\sigma, \lambda \in \mathcal{M}_{t,T}$, Y_t solution of BSDE (7) is given by

$$
\begin{aligned}
Y_t &= \mathbb{E}^*\left[\overline{J}(\varrho_t^*, \vartheta_t^*) \mid \mathscr{F}_t\right] \\
&= \inf_{\lambda \in \mathscr{M}_{t,T}} \sup_{\sigma \in \mathscr{M}_{t,T}} \mathbb{E}^*\left[\overline{J}(\lambda, \sigma) \mid \mathscr{F}_t\right] = \sup_{\sigma \in \mathscr{M}_{t,T}} \inf_{\lambda \in \mathscr{T}_t} \mathbb{E}^*\left[\overline{J}(\lambda, \sigma) \mid \mathscr{F}_t\right],
\end{aligned} \tag{8}
$$

where

$$
\overline{J}(\lambda, \sigma) = e^{-r\lambda} U_\lambda 1_{\{\lambda<\sigma\}} + e^{-r\sigma} L_\sigma 1_{\{\lambda>\sigma\}} + e^{-r\lambda} Q_\sigma 1_{\{\sigma=\lambda<T\}} + e^{-rT} \xi 1_{\{\sigma=\lambda=T\}}.
$$

The main result of this section is the following.

Theorem 3 *Assume that* $\liminf_{r\to+\infty} r P^*(\sup_{s\le T} U_s > r) = 0$. *Then, the fair price of our game is given by* $V_t = e^{rt} Y_t$, *for any* $t \le T$. *Moreover, a hedge after* t *against the option exists and it is given by:*

$$
\gamma_s = \frac{e^{rs} Z_s}{\delta S_s} 1_{\{s \le \vartheta_t^*\}} \text{ and } \beta_s = \left(e^{rs}(Y_t + \int_t^s Z_u dW_u) - \gamma_s S_s\right)(S_s^0)^{-1}, \ \forall s \in [t, T].
$$

Proof Let (π, λ) a hedge after t against the option. Therefore $\lambda \in \mathscr{M}_{t,T}$ and $\pi = (\beta_s, \gamma_s)_{t\le t\le T}$ is a self-financing portfolio whose value at t is X satisfying $\Delta_{s\wedge\lambda}^{\pi,X} \ge J^1(s, \lambda), \ \forall t \le s \le T$. But

$$
e^{-r(s\wedge\lambda)} \Delta_{s\wedge\lambda}^{\pi,X} = e^{-rt} X + \delta \int_t^{s\wedge\lambda} \gamma_u S_u e^{-ru} dW_u \ge e^{-r(s\wedge\lambda)} J^1(s, \lambda), \ \forall t \le s \le T.
$$

Let $\sigma \ge t$ be a stopping time. Putting $s = \sigma$ and taking the conditional expectation we obtain

$$
e^{-rt} X \ge \mathbb{E}^*\left(e^{-r(\sigma\wedge\lambda)} J^1(\sigma, \lambda) \mid \mathscr{F}_t\right).
$$

In view of relation (8) we have

$$
\begin{aligned}
e^{-rt} X &\ge \sup_{\sigma \in \mathscr{M}_{t,T}} \mathbb{E}^*\left(e^{-r(\sigma\wedge\lambda)} J^1(\sigma, \lambda) \mid \mathscr{F}_t\right) \\
&\ge \inf_{\lambda \in \mathscr{M}_{t,T}} \sup_{\sigma \in \mathscr{M}_{t,T}} \mathbb{E}^*\left(e^{-r(\sigma\wedge\lambda)} J^1(\sigma, \lambda) \mid \mathscr{F}_t\right) \\
&= \inf_{\lambda \in \mathscr{M}_{t,T}} \sup_{\sigma \in \mathscr{M}_{t,T}} \mathbb{E}^*\left(\overline{J}(\sigma, \lambda) \mid \mathscr{F}_t\right) \\
&= Y_t.
\end{aligned}
$$

Henceforth $V_t \ge e^{rt} Y_t$. Let us now prove the converse inequality. It follows that for every $t \le s \le T$,

$$
\begin{aligned}
& Y_t + \int_t^{s\wedge\vartheta_t^*} Z_u dW_u \\
& \leq Y_{s\wedge\vartheta_t^*} - \int_t^{s\wedge\vartheta_t^*} dK_u^- \\
& \leq Y_{s\wedge\vartheta_t^*} \\
& \leq e^{-rs} U_s 1_{\{s<\vartheta_t^*\}} + e^{-r\vartheta_t^*} L_{\vartheta_t^*} 1_{\{s>\vartheta_t^*\}} + e^{-rs} Q_{\vartheta_t^*} 1_{\{\vartheta_t^*=s<T\}} + e^{-rT}\xi 1_{\{\vartheta_t^*=s=T\}} \\
& = e^{-r(s\wedge\vartheta_t^*)} J^1(s, \vartheta_t^*).
\end{aligned}
$$

Hence for every $t \leq s \leq T$,

$$J^1(s, \vartheta_t^*) \geq e^{r(s\wedge\vartheta_t^*)}(Y_t + \int_t^{s\wedge\vartheta_t^*} Z_u dW_u).$$

Now if we put for all $s \in [t, T]$, $\gamma_s = \frac{e^{rs} Z_s}{\delta S_s} 1_{\{s\leq\vartheta_t^*\}}$ and $\beta_s = \Big(e^{rs}(Y_t + \int_t^s Z_u dW_u) - \gamma_s S_s\Big)(S_s^0)^{-1}$.

Hence $(\beta_s, \gamma_s)_{t\leq s\leq T}$ is a self-financing portfolio whose value at t is $e^{rt} Y_t$. On other hand we have

$$e^{r(s\wedge\varrho_t^*)}(Y_t + \int_t^{s\wedge\varrho_t^*} Z_u dW_u) \geq J^1(s, \varrho_t^*), \ \ \forall s \in [t, T].$$

Hence $((\beta_s, \gamma_s)_{t\leq t\leq T}, \varrho_t^*)$ is a hedge against the game option. Then $e^{rt} Y_t \geq V_t$. Henceforth $e^{rt} Y_t = V_t$. The proof of Theorem 3 is then achieved. □

References

1. Alario-Nazaret, M., Lepeltier, J.P., Marchal, B.: Dynkin Games. Lecture Notes in Control and Information Sciences, vol. 43, pp. 23–42. Springer, Berlin (1982)
2. Bensoussan, A., Friedman, A.: Non-linear variational inequalities and differential games with stopping times. J. Funct. Anal. **16**, 305–352 (1974)
3. Bismut, J.M.: Sur un problème de Dynkin. Z. Wahrsch. Verw. Gebiete **39**, 31–53 (1977)
4. Cvitanic, J., Karatzas, I.: Backward SDEs with reflection and Dynkin games. Ann. Prob. **24**(4), 2024–2056 (1996)
5. Dumitrescu, R., Quenez, M.C., Sulem, A.: Generalized Dynkin games and doubly reflected BSDEs with jumps. arXiv:1310.2764v2 (2014)
6. Dynkin, E.B., Yushkevich, A.A.: Theorems and Problems in Markov Processes. Plenum Press, New York (1968)
7. El-Karoui, N., Hamadène, S.: BSDEs and risk-sensitive control, zero-sumand nonzero-sum game problems of stochastic functional differential equations. Stoch. Process. Appl. **107**, 145–169 (2003)
8. Essaky, E.H., Hassani, M.: Generalized BSDE with 2-reflecting barriers and stochastic quadratic growth. J. Differ. Equ. **254**(3), 1500–1528 (2013)
9. Hamadène., S.: Mixed zero-sum stochastic differental game and American game options. SIAM J. Optim. **45**(2), 496–518 (2007)
10. Hamadène, S., Lepeltier, J.P.: Reflected BSDEs and mixed game problem. Stoch. Process. Appl. **85**, 177–188 (2000)

11. Hamadène, S., Hassani, M.: BSDEs with two reflecting barriers: the general result. Prob. Theory Relat. Fields **132**, 237–264 (2005)
12. Hamadène, S., Hassani, M.: BSDEs with two reflecting barriers driven by a Brownian and a Poisson noise and related Dynkin game. Electron. J. Prob. **11**(5), 121–145 (2006)
13. Hamadéne, S., Zhang, J.: The continuous time nonzero-sum Dynkin game and application in game options. SIAM J. Control Optim. **48**(5), 3659–3669 (2010)
14. Karatzas, I., Li, Q.: BSDE approach to non-zero-sum stochastic differential games of control and stopping. Submitted (2011)
15. Kifer, Y.: Game options. Finance Stoch. **4**, 443–463 (2000)
16. Lepeltier, J.P., Maingueneau, M.A.: Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. Stochastics **13**, 25–44 (1984)
17. Morimoto, H.: Dynkin games and martingale methods. Stochastics **13**, 213–228 (1984)
18. Neveu, J.: Discrete-Parameter Martingales. North-Holland, Amsterdam (1975)
19. Touzi, N., Vieille, N.: Continuous-time-Dynkin games with mixed strategies. SIAM J. Control Optim. **41**(4), 1073–1088 (2002)