Decomposition of the Pricing Formula for Stochastic Volatility Models Based on Malliavin-Skorohod Type Calculus

Josep Vives

Abstract The goal of this survey article is to present in detail a method that, for a financial derivative under a certain stochastic volatility model, allows to obtain a decomposition of its pricing formula that distinguishes clearly the impact of correlation and jumps. This decomposed pricing formula, usually called Hull and White type formula, can be potentially useful for model selection and calibration. The method is based on the obtention of an ad-hoc anticipating Itô formula.

Keywords Hull and White type formula · Malliavin-Skorohod calculus · Stochastic volatility jump-diffusion models · Derivative pricing · Quantitative finance

Mathematical Subject Classification 60H07 · 60H30 · 91G80 · 91G20

1 Introduction

The decomposition method presented in this paper is based on a series of works developed during the last ten years. In [1], E. Alòs obtained a decomposition of the pricing formula, usually called Hull and White type formula, for a plain vanilla call under a correlated stochastic volatility model, with minor hypothesis on the volatility process related with its Malliavin derivability. The decomposition was obtained applying an ad-hoc extension of the anticipative Itô formula given in [2]. The obtained formula showed clearly the impact on prices of adding correlation between price and volatility in stochastic volatility models.

In [3] the same type of formula was obtained adding also finite activity jumps in the price process. A new term appeared, showing the impact of jumps. In [5] the for-

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mula was extended to the case of assuming jumps also on the volatility process. Still a new term appeared in the formula. Finally, on [9], the result in [5] was extended for free to the case of infinite activity and finite variation jumps, and with a certain restriction in the interpretation of the formula, to the case of infinite activity and infinite variation jumps. The very general model considered in this last paper covers almost all stochastic volatility models with and without jumps, treated in the literature.

As we see in the paper, the presence of correlation and jumps in stochastic volatility models is relevant. Additional terms in the pricing formula appears from correlation, from jumps in the price process and from jumps in the volatility. Malliavin-Skorohod calculus and the decomposition method allow to obtain these pricing formulas that clearly distinguish the effect of correlation than the effect of jumps, for different types of jump models. If the stochastic volatility is correlated only with the continuous part of the price process, only Gaussian Malliavin-Skorohod calculus is needed. If the stochastic volatility is also correlated with price jumps, Lévy Malliavin-Skorohod calculus is needed.

Section 2 is devoted to the Brownian (no jump) case and Sect. 3 treats the Lévy case.

2 Decomposition of the Pricing Formula Under a General Brownian Stochastic Volatility Model

The main reference of the theory presented in this section is [1].

2.1 The Model

Let $T > 0$ be a finite horizon, $S = \{S_t, t \in [0, T]\}$ a price process, $X_t = \log S_t$ the corresponding log price process and $r > 0$ the fixed interest rate. We assume the following exponential model with stochastic volatility for the dynamics of the log-price, under a market chosen risk-neutral probability:

$$X_t = x + rt - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s (\rho dW_s + \sqrt{1-\rho^2} dB_s)$$

where x is the current log-price, W and B are independent standard Brownian motions and $\rho \in (-1, 1)$.

We denote by $\mathscr{F}^W$ and $\mathscr{F}^B$ the filtrations generated by the independent processes W and B. Moreover, we define $\mathscr{F}$, the filtration associated to S, by $\mathscr{F} := \mathscr{F}^W \vee \mathscr{F}^B$. We consider our price model defined on the product of the canonical spaces of processes W and B.

The volatility process σ is assumed to be a square-integrable stochastic process, adapted to $\mathscr{F}^W$ and with strictly positive and càdlàg trajectories.

Note that this is a very general stochastic volatility model. In this sense, recall the following facts:

- The model is a generalization of Heston model or other classical correlated stochastic volatility models in the sense that we do not assume a concrete dynamics for the volatility process σ.
- If $\rho = 0$ we have a generalization in the same sense as before of different non correlated stochastic volatility models as Hull-White, Scott, Stein-Stein or Ball-Roma.
- If σ is deterministic or constant we have the classical Osborne-Samuelson-Black-Scholes model.

For information about correlated and non-correlated stochastic volatility models, a good reference is [8].

Stochastic volatility models pursue the goal to replicate price surfaces of plain vanilla options (depending on time to maturity and strike) given by derivative markets or vanilla desks. The stochastic volatility σ is a process not directly observable, so it is not easy to model. This is a justification for trying to assume minimal conditions on it.

Let H_T be the payoff of a financial derivative. Assume it is a $\mathscr{F}_T$−measurable functional. Its price is given by $V_t = e^{-r(T-t)}\mathbb{E}_t(H_T)$ where $\mathbb{E}_t := \mathbb{E}(\cdot|\mathscr{F}_t)$. To fix ideas we will concentrate on the case of a plain vanilla call, that is, $H_T = (S_T - K)^+$. So, our goal is to obtain a decomposition of

$$V_t = e^{-r(T-t)}\mathbb{E}_t((S_T - K)^+)$$

under our risk neutral model, in order to clarify the effect of correlation in the price.

2.2 *Fast Summary of Brownian Malliavin-Skorohod Calculus*

Here we simply recall some basic definitions and facts necessary for our purpose. See for example [10] for a complete presentation of the theory.

Let W and $(\Omega^W, \mathscr{F}^W, \mathbb{P}_W)$ be the canonical Wiener process and its canonical space, respectively. Recall that $\Omega^W := C_0([0, T])$ is the space of continuous functions on $[0, T]$, null at the origin. Denote by $\mathbb{E}_W$ the expectation with respect to $\mathbb{P}_W$.

Consider the family of smooth functionals of type

$$F = f(W_{t_1}, \ldots, W_{t_n})$$

for any $n \geq 0$, $t_1, \ldots t_n \in [0, T]$ and $f \in C_b^\infty(\mathbb{R}^n)$.

Given a smooth functional F we define its Malliavin derivative $D^W F$ as the element of $L^2(\Omega^W \times [0,T])$ given by

$$D_t F = \sum_{i=1}^{n} \partial_i f(W_{t_1}, \dots, W_{t_n}) 1\!\!1_{[0,t_i]}(t).$$

The operator D^W is closed and densely defined in $L^2(\Omega^W)$, and its domain $Dom\ D^W$ is the closure of the smooth functionals with respect the norm

$$||F||_{Dom\ D^W} := \left(\mathbb{E}_W(|F|^2) + \mathbb{E}_W \int_0^T |D_t^W F|^2 dt\right)^{\frac{1}{2}}.$$

We define δ^W as the dual operator of D^W. Given $u \in L^2(\Omega^W \times [0,T])$, $\delta^W(u)$ is the element of $L^2(\Omega^W)$ characterized by

$$\mathbb{E}_W(F\delta^W(u)) = \mathbb{E}_W \int_0^T u_t D_t^W F dt$$

for any $F \in Dom\ D^W$. Note that taking $F \equiv 1$ we obtain

$$\mathbb{E}_W(\delta^W(u)) = 0.$$

The following results will be helpful:

- If F, G and $F \cdot G$ belong to $Dom\ D^W$ we have

$$D^W(F \cdot G) = F D^W G + G D^W F.$$

- If $F \in Dom\ D^W$, $u \in Dom\ \delta^W$ and $F \cdot u \in Dom\ \delta^W$ then

$$\delta^W(F \cdot u) = F\delta^W(u) - \int_0^T u_t D_t^W F dt.$$

- It is well known that D^W can be interpreted as a directional derivative on the Wiener space and δ^W is an extension of the classical Itô integral.

We define the space $\mathbb{L}_W^{1,2} := L^2([0,T]; Dom\ D^W)$, that is the space of processes $u \in L^2([0,T] \times \Omega^W)$ such that $u_t \in Dom\ D^W$ for almost all t and $Du \in L^2(\Omega^W \times [0,T]^2)$. It can be proved that $\mathbb{L}_W^{1,2} \subseteq Dom\ \delta^W$ and

$$\mathbb{E}_W(\delta^W(u)^2) \leq ||u||^2_{\mathbb{L}_W^{1,2}} := \mathbb{E}_W(||u||^2_{L^2([0,T])}) + \mathbb{E}_W(||D^W u||^2_{L^2([0,T]^2)}).$$

Finally, we will denote $\delta_t^W(u) := \delta^W(u 1\!\!1_{[0,t]})$.

2.3 The Hull and White Formula

If we assume constant volatility we have the well known geometric Brownian model. In this case, the price V_t is given by the well known Black-Scholes formula:

$$V_t = BS(t, X_t, \sigma) = e^x \Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$

where

$$d_\pm = \frac{X_t - \log K + r(T-t)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$$

and Φ is the cumulative probability function of the standard normal law.

If we allow $\sigma = \sigma(t)$ be a deterministic function, it is easy to see, that

$$X_T - X_t \sim \mathscr{N}((r - \frac{1}{2}\bar{\sigma}_t^2)(T-t), \bar{\sigma}_t^2(T-t)),$$

where

$$\bar{\sigma}_t := \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(s)ds}$$

is the so called future average volatility. Define $\bar{\sigma}_T$ as the limit of $\bar{\sigma}_t$ when $t \uparrow T$.

So, in this case, the pricing formula is exactly the Black-Scholes formula changing σ by $\bar{\sigma}_t$, that is, $V_t = BS(t, X_t, \bar{\sigma}_t)$. This suggests that it is the future average volatility and not the volatility the really relevant quantity in pricing. Black-Scholes formula would be nothing more than the particular case of constant future average volatility.

If σ is a stochastic process uncorrelated with price, that is, $\rho = 0$ in our model, we have, following for example [7]:

$$V_t = \mathbb{E}_t(BS(t, X_t, \bar{\sigma}_t)).$$

This is the classical Hull and White formula and covers non correlated stochastic volatility models as the cases of Hull-White, Scott, Stein-Stein, Ball-Roma and others. The proof is immediate, conditioning first by $\mathscr{F}_t \vee \mathscr{F}_T^W$.

Note that the future average volatility $\bar{\sigma}_t$ is an anticipative process. This suggest the use of Malliavin-Skorohod calculus as a natural tool to deal with this type of processes.

In the correlated case we have the following theorem:

Theorem 1 *Assume*

- *(A1):* $\sigma^2 \in \mathbb{L}_W^{1,2}$.
- *(A2):* $\sigma \in \mathbb{L}_W^{1,2}$.

Then we have,

$$V_t = \mathbb{E}_t[BS(t, X_t, \bar{\sigma}_t)] + \frac{\rho}{2}\mathbb{E}_t[\int_t^T e^{-r(s-t)}(\partial_x^3 - \partial_x^2)BS(s, X_s, \bar{\sigma}_s)\Lambda_s ds]$$

where

$$\Lambda_s := (\int_s^T D_s^W \sigma_r^2 dr)\sigma_s.$$

Proof The proof is based on the so called *decomposition method*.

Recall that

$$V_T = (e^{X_T} - K)^+ = BS(T, X_T, \bar{\sigma}_T)$$

and so

$$e^{-rt}V_t = \mathbb{E}_t(e^{-rT}BS(T, X_T, \bar{\sigma}_T)).$$

The idea of the proof consists in applying an ad-hoc anticipative Itô formula to the process

$$e^{-rs}BS(s, X_s, \bar{\sigma}_s)$$

between t and T, take conditional expectations $\mathbb{E}_t$ and multiply by e^{rt}. This gives the expansion for V_t.

The ad-hoc Itô formula is an adaptation to our case of the anticipative Itô formula proved in [2]. Define

$$Y_t := (T-t)\bar{\sigma}_t^2 = \int_t^T \sigma_r^2 dr.$$

Thanks to (A1), we are under the conditions of Theorem 1 in [1], and so, for any $F \in C_b^{1,2,2}([0,T] \times \mathbb{R} \times [0,\infty))$, we have

$$\begin{aligned} F(t, X_t, Y_t) = F(0, X_0, Y_0) &+ \int_0^t \partial_s F(s, X_s, Y_s)ds + \delta_t^{W,B}(\partial_x F(\cdot, X_\cdot, Y_\cdot)\sigma_\cdot) \\ &+ \int_0^t \partial_x F(s, X_s, Y_s)(r - \frac{\sigma_s^2}{2})ds - \int_0^t \partial_y F(s, X_s, Y_s)\sigma_s^2 ds \\ &+ \rho \int_0^t \partial_{xy} F(s, X_s, Y_s)\Lambda_s ds + \frac{1}{2}\int_0^t \partial_x^2 F(s, X_s, Y_s)\sigma_s^2 ds. \end{aligned}$$

Now we want to apply this result to

$$F(s, x, y) := e^{-rs}BS(s, x, \sqrt{\frac{y}{T-s}}),$$

but this function doesn't satisfy the required conditions of the previous Itô formula because the derivatives are not bounded, so we need to use a mollifier argument.

For $n \geq 1$ and $\delta > 0$, we consider the approximation,

$$F_{n,\delta}(s, x, y) := e^{-rs} BS(s, x, \sqrt{\frac{y+\delta}{T-s}})\phi(\frac{x}{n}),$$

where $\phi \in C_b^2(\mathbb{R})$, such that $\phi(z) = 1$ if $|z| \leq 1$, $\phi(z) \in [0, 1]$ if $|z| \in [1, 2]$ and $\phi(z) = 0$ if $|z| > 2$.

Then, applying the previous ad-hoc Itô formula to $F_{n,\delta}(s, X_s, Y_s)$, taking the conditional expectation $\mathbb{E}_t$, using the fact that Skorohod type integrals have zero expectation and multiplying by e^{rt} we obtain

$$\begin{aligned}
& \mathbb{E}_t(e^{-r(T-t)} BS(T, X_T, \bar{\sigma}_T^\delta)\phi(\frac{X_T}{n})) \\
= \; & \mathbb{E}_t(BS(t, X_t, \bar{\sigma}_t^\delta)\phi(\frac{X_t}{n})) \\
& + \mathbb{E}_t(\int_t^T e^{-r(s-t)} A_n(s) ds) \\
& + \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)} (\partial_x^3 - \partial_x^2) BS(s, X_s, \bar{\sigma}_s^\delta)\phi_n(\frac{X_s}{n})\Lambda_s ds \\
& + \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)} (\partial_x^2 - \partial_x) BS(s, X_s, \bar{\sigma}_s^\delta)\frac{1}{n}\phi^{'}(\frac{X_s}{n}))\Lambda_s ds
\end{aligned}$$

where

$$\bar{\sigma}_s^\delta := \sqrt{\frac{Y_s+\delta}{T-s}}$$

and

$$\begin{aligned}
A_n(s) : = \; & \frac{\sigma_s^2}{n}\partial_x BS(s, X_s, \bar{\sigma}_s^\delta)\phi^{'}(\frac{X_s}{n}) \\
& + \frac{\sigma_s^2}{2n} BS(s, X_s, \bar{\sigma}_s^\delta)(\frac{1}{n}\phi^{''}(\frac{X_s}{n}) - \phi^{'}(\frac{X_s}{n})) \\
& + \frac{r}{n} BS(s, X_s, \bar{\sigma}_s^\delta)\phi^{'}(\frac{X_s}{n}).
\end{aligned}$$

The details can be found in [9] (erasing there the terms depending on jumps, that will be treated later in this paper).

Finally, the result follows from the dominated convergence theorem taking limits first on $n \uparrow \infty$ and then on $\delta \downarrow 0$. The dominated convergence runs thanks to the properties of Black-Scholes function and (A2). For the left hand side and the two first terms on the right hand side we use the fact that function $BS(t, x, \sigma)$ is bounded by $e^x + K$ and its derivative $(\partial_x BS)(t, x, \sigma)$ is bounded by e^x. For the last two terms on the right hand side we use Lemma 2 in [3] that says that for any $n \geq 0$,

$$|\mathbb{E}[\partial_x^n(\partial_x^2-\partial_x)BS(s,X_s,\bar{\sigma}_s)|\mathscr{F}_t\vee\mathscr{F}_T^W]|\le C_n(\rho)(\int_t^T\sigma_s^2ds)^{-\frac{n+1}{2}},$$

for a certain constant $C_n(\rho)$ that depends only on n and ρ.

For example, for the third term on the right hand side, we have

$$\begin{aligned}&\mathbb{E}_t(\int_t^T e^{-r(s-t)}(\partial_x^3-\partial_x^2)BS(s,X_s,\bar{\sigma}_s^\delta)\phi_n(\frac{X_s}{n})\Lambda_s ds)\\ =&\ \mathbb{E}_t(\int_t^T e^{-r(s-t)}\mathbb{E}[(\partial_x^3-\partial_x^2)BS(s,X_s,\bar{\sigma}_s^\delta)|\mathscr{F}_t\vee\mathscr{F}_T^W]\phi_n(\frac{X_s}{n})\Lambda_s ds).\end{aligned}$$

And, applying the lemma,

$$|\mathbb{E}[(\partial_x^3-\partial_x^2)BS(s,X_s,v_s^\delta)|\mathscr{F}_t\vee\mathscr{F}_T^W]\phi_n(\frac{X_s}{n})\Lambda_s|\le C_1(\rho)\frac{|\Lambda_s|}{\int_t^T\sigma_s^2ds}.$$

Using the chain rule for D^W, the problem reduces to show

$$\mathbb{E}_t(\frac{\int_t^T(\int_t^T|D_s^W\sigma_u|\sigma_u dr)\sigma_s ds}{\int_t^T\sigma_r^2dr})<\infty,$$

and applying Cauchy-Schwarz inequality twice, we can bound this expression by

$$\begin{aligned}&\mathbb{E}_t(\frac{(\int_t^T(\int_t^T|D_s^W\sigma_u|\sigma_u du)^2ds)^{\frac12}}{(\int_t^T\sigma_u^2du)^{\frac12}})\\ \le&\ \mathbb{E}_t(\frac{(\int_t^T(\int_t^T|D_s^W\sigma_u|^2dr)(\int_t^T\sigma_u^2du)ds)^{\frac12}}{(\int_t^T\sigma_u^2du)^{\frac12}})\\ \le&\ \mathbb{E}_t((\int_t^T\int_t^T|D_s^W\sigma_u|^2duds)^{\frac12})\\ \le&\ (\mathbb{E}_t\int_t^T\int_t^T|D_s^W\sigma_u|^2duds)^{\frac12}.\end{aligned}$$

So, (A2) proves that this expression is finite.

For the fourth term in the right hand side, applying the lemma and using C as a generic constant, we have

$$|\mathbb{E}[(\partial_x^2-\partial_x)BS(s,X_s,\bar{\sigma}_s^\delta)|\mathscr{F}_t\vee\mathscr{F}_T^W]\frac{1}{n}\phi_n'(\frac{X_s}{n})\Lambda_s|\le\frac{C}{n}\frac{|\Lambda_s|}{(\int_t^T\sigma_s^2ds)^{\frac12}}.$$

So, we have to show

$$\mathbb{E}_t\left(\frac{\int_t^T \sigma_s(\int_s^T |D_s^W \sigma_u|\sigma_u du)ds}{(\int_t^T \sigma_u^2 du)^{\frac{1}{2}}}\right) < \infty,$$

that follows applying Cauchy-Schwarz inequality, similarly to the previous case.

Remark 1 Note that hypothesis (A2) can be changed by the following alternative hypothesis of uniform ellipticity (A2′): The process σ^2 defined on $[0, T]$ is uniformly bounded below by a constant $a > 0$. In fact (A1) and (A2′), jointly, imply (A2).

3 The Lévy Case

The main references for this section are [3, 5, 9].

3.1 A Very General Stochastic Volatility Lévy Model

Assume now the following exponential Lévy model with stochastic volatility for the dynamics of the log-price, under a market chosen risk-neutral probability:

$$X_t = x + rt - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) + L_t^0$$

where L^0 is a pure jump Lévy process with possibly infinitely many jumps with triplet $(\gamma_0, 0, \nu)$ and independent of W and B. Now, the volatility process σ is assumed to be adapted to the filtration generated by W and L^0.

Due to the well known Lévy-Itô decomposition we can write

$$L_t^0 = \gamma_0 t + \int_0^t \int_{\{|y|>1\}} yN(ds, dy) + \lim_{\varepsilon \downarrow 0} \int_0^t \int_{\{\varepsilon<|y|\leq 1\}} y\widetilde{N}(ds, dy)$$

where N denotes the Poisson measure associated to Lévy process, $\widetilde{N}(ds, dy) := N(ds, dy) - \nu(dy)ds$ is the compensated Poisson measure and the limit is a.s. and uniformly on compacts.

For the integers $i \geq 0$, we consider the following constants, provided they exist:

$$c_i := \sum_{k=i}^{\infty} \int_{\mathbb{R}} \frac{y^k}{k!}\nu(dy).$$

Observe that in particular

$$c_0 = \int_{\mathbb{R}} e^y \nu(dy),$$
$$c_1 = \int_{\mathbb{R}} (e^y - 1)\nu(dy),$$
$$c_2 = \int_{\mathbb{R}} (e^y - 1 - y)\nu(dy).$$

In order to $e^{-rt}e^{X_t}$ be a martingale, see for example [6], we must assume

$$\int_{|y|\geq 1} e^y \nu(dy) < \infty \text{ and } \gamma_0 = -\int_{\mathbb{R}} (e^y - 1 - y\mathbb{1}_{\{|y|<1\}})\nu(dy).$$

These conditions guarantee that ν has moments of all orders greater or equal than 2 and that we can write

$$L_t^0 = \int_0^t \int_{\mathbb{R}} y\widetilde{N}(ds, dy) - c_2 t.$$

So, in the following we will assume, without loosing generality, the model

$$X_t = x + (r - c_2)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) + J_t$$

with

$$J_t := \int_0^t \int_{\mathbb{R}} y\widetilde{N}(ds, dy).$$

Recall that if $\int_{\mathbb{R}} |y|\nu(dy) = \infty$ we say that the process has infinite activity and infinite variation. In this case $c_1 := \int_{\mathbb{R}}(e^y - 1)\nu(dy)$ and $c_0 := \int_{\mathbb{R}} e^y \nu(dy)$ are infinite or not defined. If ν has first order moment, that is $\int_{\mathbb{R}} |y|\nu(dy) < \infty$, we say the model has infinite activity but finite variation and c_1 is finite. In this case we can consider $c_2 = c_1 - \int_{\mathbb{R}} y\nu(dy)$ and rewrite

$$\int_0^t \int_{\mathbb{R}} y\widetilde{N}(ds, dy) - c_2 t = \int_0^t \int_{\mathbb{R}} yN(ds, dy) - c_1 t.$$

Finally, if ν is finite, the model has finite activity and so, it is a Compound Poisson process with $\nu = \lambda Q$ for a certain probability measure Q and a certain constant $\lambda = \nu(\mathbb{R}) > 0$. In this case,

$$c_1 = \int_{\mathbb{R}} (e^y - 1)\nu(dy) = c_0 - \lambda$$

and

$$\int_0^t \int_{\mathbb{R}} yN(ds, dy) = \sum_{i=1}^{N_t} V_i,$$

where N is a $\lambda-$Poisson process and V_i are i.i.d. random variables with law Q, the law that produce the jumps.

Let $\mathscr{F}^J$ be the filtration generated by J. Note that this filtration is the same as the filtration generated by L^0 because the difference of these two processes is deterministic. Define now $\mathscr{F}$, the filtration associated to S, by

$$\mathscr{F} := \mathscr{F}^W \vee \mathscr{F}^B \vee \mathscr{F}^J.$$

We will consider our price model defined on the product of the canonical spaces of processes W, B and J. This means that

$$\omega := (\omega^W, \omega^B, \omega^J) \in \Omega := \Omega^W \times \Omega^B \times \Omega^J$$

and in the rest of the paper, any hypothesis on one of the spaces will mean that the property is true almost surely with respect to the other spaces.

Note that this is a very general stochastic volatility model because, being σ adapted to $\mathscr{F}^W \vee \mathscr{F}^J$, we are allowing jumps both in price and volatility. Recall the following facts:

- If we assume no jumps, that is $\nu = 0$, we have a generalization of correlated and non correlated stochastic volatility models in the sense that we do not assume a concrete dynamics for the volatility. This is the case treated in Sect. 2.
- If we restrict our model to the case σ adapted only to $\mathscr{F}^W$ we have a generalization of the Bates model in a double sense; on one hand we do not assume any concrete dynamics for the stochastic volatility and on other hand we are not assuming finite activity nor finite variation on ν.
- If we assume no correlation but presence of jumps we cover for example Heston-Kou model or any uncorrelated model with the addition of Lévy jumps in the price process with any Lévy measure ν.
- If $\sigma = 0$ but we have jumps, we cover the so called exponential Lévy models.

3.2 Malliavin-Skorohod Type Calculus for Lévy Processes

The literature on Malliavin calculus for Lévy processes is more recent and less extended. Here we follow closely [11] and [4]. A survey of this results can be found in [12]. We refer to these references for proofs of next results.

Let us denote $\mathbb{R}_0 := \mathbb{R} - \{0\}$. Consider the canonical version of the pure jump Lévy process J. It is defined on the space Ω^N given by the finite or infinite sequences of pairs $(t_i, x_i) \in (0, T] \times \mathbb{R}_0$ such that for every $\varepsilon > 0$ there is only a finite number of (t_i, x_i) with $|x_i| > \varepsilon$. Of course, t_i denotes a jump instant and x_i a jump size.

Consider $\omega^N \in \Omega^N$. Given $(t, x) \in [0, T] \times \mathbb{R}_0$ we can introduce a jump of size x at instant t to ω^N and call the new element

$$\omega^N_{t,x} := ((t, x), (t_1, x_1)(t_2, x_s), ...).$$

For a random variable $F \in L^2(\Omega^N)$, we define

$$T_{t,x} F(\omega^N) = F(\omega^N_{t,x})$$

and

$$D^N_{t,x} F = \frac{T_{t,x} F(\omega^N) - F(\omega^N)}{x}.$$

The operator D^N is closed and densely defined in $L^2(\Omega^N)$ and its domain $Dom\ D^N$ can be characterized by the fact that

$$F \in Dom\ D^N \Longleftrightarrow D^N F \in L^2(\Omega \times [0, T] \times \mathbb{R}_0, P \otimes ds \otimes x^2 \nu(dx)).$$

On other hand we define δ^N as the dual operator of D^N.

Given $u \in L^2(\Omega^W \times [0, T] \times \mathbb{R}, P \otimes ds \otimes x^2 \nu(dx))$, $\delta^N(u)$ is the element of $L^2(\Omega^N)$ characterized by

$$\mathbb{E}_N(F \delta^N(u)) = \mathbb{E}_N(\int_0^T \int_{\mathbb{R}} u_{t,x} D^N_{t,x} F x^2 \nu(dx) dt)$$

for any $F \in Dom\ D^N$. In particular $\mathbb{E}_N(\delta^N(u)) = 0$.

Let us denote $\delta^N_t(u) := \delta^N(u \mathbb{1}_{[0,t]})$.

As we have seen, D^N is an increment quotient operator and it is also known that δ^N_t is an extension of Itô integral in the sense that

$$\delta^N_t(u \mathbb{1}_{\mathbb{R}_0}) = \int_0^t \int_{\mathbb{R}} u(s, x) x \tilde{N}(ds, dx)$$

for predictable integrands u.

The following formulas will be helpful:

- If F, G and $F \cdot G$ belong to $Dom\ D^N$ we have

$$D^N(F \cdot G) = F D^N G + G D^N F + x D^N F D^N G.$$

- If $F \in Dom\, D^N$, $u \in Dom\, \delta^N$ and $u \cdot T_{t,x} F \in Dom\, \delta^N$ then

$$\delta^N(F \cdot u) = F\delta^N(u) - \int_0^T \int_{\mathbb{R}} u_{t,x} D^N_{t,x} F x^2 \nu(dx)dt - \delta^N(x \cdot u \cdot D^N F).$$

As in the Wiener case we define the space

$$\mathbb{L}^{1,2}_N := L^2([0,T] \times \mathbb{R}, Dom\, D^N),$$

that is the space of processes $u \in L^2([0,T] \times \mathbb{R} \times \Omega^N)$ such that $u_{t,x} \in Dom\, D^N$ for almost all (t,x) and $Du \in L^2(\Omega^N \times ([0,T] \times \mathbb{R})^2)$.

It can be proved that $\mathbb{L}^{1,2}_N \subseteq Dom\, \delta^N$ and

$$E_N(\delta^N(u)^2) \leq ||u||^2_{\mathbb{L}^{1,2}_N} := E_N(||u||^2_{L^2([0,T]\times\mathbb{R})}) + E_N(||D^N u||^2_{L^2(([0,T]\times\mathbb{R})^2)}).$$

Moreover we introduce the space $\mathbb{L}^{1,2}_{N,-}$ as the subspace of $\mathbb{L}^{1,2}_N$ of processes u such that the left-limits

$$u(s-,y) := \lim_{r\uparrow s, x\uparrow y} u(r,x)$$

and

$$D^{N,-}_{s,y} u(s-,y) := \lim_{r\uparrow s, x\uparrow y} D^N_{s,y} u(r,x)$$

exists $\mathbb{P}_N \otimes ds \otimes x^2\nu(dx)-$a.s. and belong to $L^2(\Omega^N \times [0,T] \times \mathbb{R})$.

Assume $u \in \mathbb{L}^{1,2}_{N,-}$ and $\int_0^T \int_{\mathbb{R}_0} |u(s-,y)||y| N(ds,dy) \in L^2(\Omega^N)$. Then, for any $t \in [0,T]$,

$$T^-_{s,y} u(s-,y) := u(s-,y) + y D^{N,-}_{s,y} u(s-,y) \in Dom\, \delta^N_t$$

and

$$\int_0^t \int_{\mathbb{R}} u(s-,y) y \tilde{N}(ds,dy) = \delta^N_t(T^-_{s,y} u(s-,y) \mathbb{1}_{\mathbb{R}_0}) + \int_0^t \int_{\mathbb{R}} D^{N,-}_{s,y} u(s-,y) y^2 \nu(dy) ds.$$

If u is predictable we have $D^{N,-}_{s,y} u(s-,y) = 0$. Hence, in this case,

$$\int_0^t \int_{\mathbb{R}} u(s-,y) y \tilde{N}(ds,dy) = \delta^N_t(u(s-,y) \mathbb{1}_{\mathbb{R}_0}).$$

3.3 The Hull and White Formula in the Lévy Case

Consider the following definitions in order to shorten the notation, for a suitable function F :

- $\Delta_x F(s, X_s, Y_s) := F(s, X_s + x, Y_s) - F(s, X_s, Y_s)$.
- $\Delta_{xx} F(s, X_s, Y_s) := F(s, X_s + x, Y_s) - F(s, X_s, Y_s) - x(\partial_x F)(s, X_s, Y_s)$.
- $\Delta F(s, X_s, Y_s) = F(s, X_s + x, Y_s) - F(s, X_s, Y_s) - (e^x - 1)(\partial_x F)(s, X_s, Y_s)$.

Then, we have the following decomposition of the price formula:

Theorem 2 *Assume*

- *(B1):* $\sigma^2 \in \mathbb{L}^{1,2}_{N,-} \cap \mathbb{L}^{1,2}_W$.
- *(B2):* $\sigma \in \mathbb{L}^{1,2}_W$.
- *(B3): For any* $t \in [0, T]$, $\int_t^T \mathbb{E}_t((\int_t^s \sigma_u^2 du)^{-2}) ds < \infty$.

Then we have

$$
\begin{aligned}
V_t &= \mathbb{E}_t(BS(t, X_t, v_t)) \\
&+ \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)}(\partial_x^3 - \partial_x^2) BS(s, X_s, \bar{\sigma}_s)\Lambda_s ds) \\
&+ \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} \Delta BS(s, X_{s-}, \bar{\sigma}_s)\nu(dy)ds) \\
&+ \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} D^{N,-}_{s,y} \Delta_y BS(s, X_{s-}, \bar{\sigma}_s) y\nu(dy)ds).
\end{aligned}
$$

Remark 2 We can consider the following particular cases:

1. Observe that we cannot split the third term in two terms because in the general case

$$
\mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} \Delta_y BS(s, X_{s^-}, \bar{\sigma}_s)\nu(dy)ds)
$$

   and

$$
\mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} (e^y - 1)\partial_x BS(s, X_{s^-}, \bar{\sigma}_s)\nu(dy)ds)
$$

   are not necessarily convergent.
2. Observe that if in the previous theorem we assume $\int_{\mathbb{R}} |y|\nu(dy) < \infty$, that is, finite variation, we obtain

$$
\begin{aligned}
V_t &= \mathbb{E}_t(BS(t, X_t, v_t)) \\
&\quad + \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)}(\partial_x^3 - \partial_x^2)BS(s, X_s, \bar{\sigma}_s)\Lambda_s ds) \\
&\quad - \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)}(e^y - 1)\partial_x BS(s, X_s, \bar{\sigma}_s)\nu(dy)ds) \\
&\quad + \mathbb{E}_t(\int_t^T \int_{\mathbb{R}} e^{-r(s-t)} T_{s,y}^{-} \Delta_y BS(s, X_{s^-}, \bar{\sigma}_s)\nu(dy)ds),
\end{aligned}
$$

that is exactly the formula obtained in [5] for the finite activity case, that in fact is valid in the finite variation case.

3. If the volatility process is independent from price jumps, we have

$$D_{s,y}^{N,-}\Delta_y BS(s, X_{s-}, \bar{\sigma}_s) = 0$$

and we obtain

$$
\begin{aligned}
V_t &= \mathbb{E}_t(BS(t, X_t, \bar{\sigma}_t)) \\
&\quad + \frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)}(\partial_x^3 - \partial_x^2)BS(s, X_s, \bar{\sigma}_s)\Lambda_s ds) \\
&\quad + \mathbb{E}_t \int_t^T \int_{\mathbb{R}} e^{-r(s-t)} \Delta BS(s, X_{s^-}, \bar{\sigma}_s)\nu(dy)ds,
\end{aligned}
$$

that generalizes the formula in [3]. As in the previous remark, only in the finite variation case we recuperate exactly the formula in [3]. This formula covers Bates model and any correlated model with any type of Lévy jumps in the price process.

4. If moreover, the volatility process is independent from the price process, that is, $\rho = 0$, we obtain

$$V_t = \mathbb{E}_t(BS(t, X_t, v_t)) + \mathbb{E}_t \int_t^T \int_{\mathbb{R}} e^{-r(s-t)} \Delta BS(s, X_{s^-}, \bar{\sigma}_s)\nu(dy)ds.$$

This covers all the so called uncorrelated models plus jumps (Heston-Kou model for example) and in the particular case of constant volatility, the so called exponential Lévy models. In the jump part we can consider infinite activity jumps as CGMY model (for $Y \geq 0$) or Meixner model for example.

Proof We follow the same general idea of Theorem 1. The necessary ad-hoc Itô formula, see [9], is now

$$
\begin{aligned}
F(t, X_t, Y_t) = {} & F(0, X_0, Y_0) + \int_0^t \partial_s F(s, X_s, Y_s) ds + \delta_t^{W,B}(\partial_x F(\cdot, X_\cdot, Y_\cdot)\sigma_\cdot) \\
& + \int_0^t \partial_x F(s, X_s, Y_s)(r - c_2 - \frac{\sigma_s^2}{2}) ds - \int_0^t \partial_y F(s, X_s, Y_s)\sigma_s^2 ds \\
& + \rho \int_0^t \partial_{xy} F(s, X_s, Y_s)\Lambda_s ds + \frac{1}{2}\int_0^t \partial_x^2 F(s, X_s, Y_s)\sigma_s^2 ds \\
& + \int_0^t \int_{\mathbb{R}_0} (\Delta_y F(s, X_{s-}, Y_s) - y(\partial_y F)(s, X_{s-}, Y_s))\nu(dy) ds \\
& + \delta_t^N(\frac{\Delta_y F(s, X_{s-}, Y_s)}{y} \mathbb{1}_{\mathbb{R}_0}(y)) + \delta_t^N(D_{s,y}^{N,-}\Delta_y(s, X_{s-}, Y_s)) \\
& + \int_0^t \int_{\mathbb{R}} D_{s,y}^{N,-}\Delta_y F(s, X_{s-}, Y_s) y \nu(dy) ds.
\end{aligned}
$$

To prove it, fix first $\varepsilon > 0$, and consider the process

$$
\begin{aligned}
X_t^\varepsilon := {} & x + (r - c_2)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2} dB_s) \\
& + \int_0^t \int_{|x|>\varepsilon} x \tilde{N}(ds, dx).
\end{aligned}
$$

This process has a finite number of jumps and converges a.s. and in L^2 sense to X_t.

Denote by T_i^ε the jump instants, and write $T_0^\varepsilon := 0$. Then

$$
\begin{aligned}
& F(T_{i+1}^\varepsilon, X_{T_{i+1}^\varepsilon}^\varepsilon, Y_{T_{i+1}^\varepsilon}) - F(T_i^\varepsilon, X_{T_i^\varepsilon}^\varepsilon, Y_{T_i^\varepsilon}) = \int_{T_i^\varepsilon}^{T_{i+1}^\varepsilon -} dF(s, X_s^\varepsilon, Y_s) \\
& + F(T_{i+1}^\varepsilon, X_{T_{i+1}^\varepsilon}^\varepsilon, Y_{T_{i+1}^\varepsilon}) - F(T_{i+1}^\varepsilon, X_{T_{i+1}^\varepsilon -}^\varepsilon, Y_{T_{i+1}^\varepsilon}).
\end{aligned}
$$

On the stochastic interval $[T_j^\varepsilon, T_{j+1}^\varepsilon[$ we can apply the anticipative Itô formula for continuous process presented in Sect. 2. Then we have that

$$
\partial_x F(s, X_{s^-}, Y_s)\sigma_s \mathbb{1}_{[0,t]}(s) \in Dom\, \delta^{W,B}
$$

and

$$
\begin{aligned}
F(t, X_t^\varepsilon, Y_t) = {} & F(0, X_0, Y_0) + \int_0^t \partial_s F(s, X_s^\varepsilon, Y_s) ds \\
& + \int_0^t \partial_x F(s, X_s^\varepsilon, Y_s)(r - \frac{\sigma_s^2}{2} - c_2) ds + \delta_t^{W,B}\left(\partial_x F(s, X_{s^-}^\varepsilon, Y_s)\sigma_s\right) \\
& - \int_0^t \int_{\{|x|>\varepsilon\}} \partial_x F(s, X_s^\varepsilon, Y_s) x \nu(dx) ds - \int_0^t \partial_y F(s, X_s^\varepsilon, Y_s)\sigma_s^2 ds
\end{aligned}
$$

$$+ \rho \int_0^t \partial_{xy} F(s, X_s^\varepsilon, Y_s) \Lambda_s ds + \frac{1}{2} \int_0^t \partial_x^2 F(s, X_s^\varepsilon, Y_s) \sigma_s^2 ds$$
$$+ \sum_i [F(T_i^\varepsilon, X_{T_i^\varepsilon}^\varepsilon, Y_{T_i^\varepsilon}) - F(T_i^\varepsilon, X_{T_i^\varepsilon -}^\varepsilon, Y_{T_i^\varepsilon})].$$

Of course we can write

$$\sum_i [F(T_i^\varepsilon, X_{T_i^\varepsilon}^\varepsilon, Y_{T_i^\varepsilon}) - F(T_i^\varepsilon, X_{T_i^\varepsilon -}^\varepsilon, Y_{T_i^\varepsilon})] = \int_0^t \int_{|x|>\varepsilon} \Delta_x F(s, X_{s-}, Y_s) N(ds, dx).$$

Then,

$$\sum_i [F(T_i^\varepsilon, X_{T_i^\varepsilon}^\varepsilon, Y_{T_i^\varepsilon}) - F(T_i^\varepsilon, X_{T_i^\varepsilon -}^\varepsilon, Y_{T_i^\varepsilon})] - \int_0^t \int_{|x|>\varepsilon} \partial_x F(s, X_{s-}^\varepsilon, Y_s) x \nu(dx) ds$$

$$= \int_0^t \int_{|x|>\varepsilon} \Delta_x F(s, X_{s-}^\varepsilon, Y_s) \tilde{N}(ds, dx) + \int_0^t \int_{|x|>\varepsilon} \Delta_{xx} F(s, X_{s-}^\varepsilon, Y_s) \nu(dx) ds.$$

Observe that this equality is the crucial step of the proof. Only introducing $\Delta_{xx} F(s, X_{s-}^\varepsilon, Y_s)$ we become able to apply succesfully the dominated convergence theorem, even if Y has no jumps.

Using the relation between δ^N and the integral with respect to $\tilde{N}$ we have

$$\int_0^t \int_{|x|>\varepsilon} \Delta_x F(s, X_{s-}^\varepsilon, Y_s) \tilde{N}(ds, dx)$$
$$= \delta_t^N (T_{s,x}^- \frac{\Delta_x F(s, X_{s-}^\varepsilon, Y_s)}{x} \mathbb{1}_{\{|x|>\varepsilon\}})$$
$$+ \int_0^t \int_{|x|>\varepsilon} D_{s,x}^{N,-} \frac{\Delta_x F(s, X_{s-}^\varepsilon, Y_s)}{x} x^2 \nu(dx) ds.$$

And using mean value theorem and the fact that first and second derivatives of F are bounded we have

$$|T_{s,x}^- \frac{\Delta_x F(s, X_{s-}^\varepsilon, Y_s)}{x}| = |\frac{\Delta_x F(s, X_{s-}^\varepsilon, T_{s,x}^- Y_s)}{x}| \le C,$$

$$|D_{r,y}^{N,-} \frac{\Delta_x F(s, X_{s-}^\varepsilon, T_{s,x}^- Y_s)}{x}| \le C |D_{r,y}^{N,-} T_{s,x}^- Y_s| = C \int_s^T |D_{r,y}^{N,-} T_{s,x}^- \sigma_u^2| du$$

and

$$|D_{s,x}^{N,-} \frac{\Delta_x F(s, X_{s-}^\varepsilon, Y_s)}{x}| \le C |D_{s,x}^{N,-} Y_s| = C \int_s^T |D_{s,x}^{N,-} \sigma_r^2| dr,$$

for a generic constant C.

Finally, using (B1) and the dominated convergence theorem the right hand side of the previous equality converges when ε goes to 0. The other terms converge also by the dominated convergence theorem, and the Itô formula follows.

Then, following the same steps of the proof of Theorem 1, after applying this last ad-hoc Itô formula, taking conditional expectations, using the fact that Skorohod type integrals have zero expectation and multiplying by e^{rt} we obtain

$$
\begin{aligned}
&\mathbb{E}_t(e^{-r(T-t)}BS(T,X_T,\bar{\sigma}_T^\delta)\phi(\frac{X_T}{n})) \qquad (1)\\
&= \mathbb{E}_t(BS(t,X_t,\bar{\sigma}_t^\delta)\phi(\frac{X_t}{n}))\\
&+\mathbb{E}_t(\int_t^T e^{-r(s-t)}A_n(s)ds)\\
&+\frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)}(\partial_x^3-\partial_x^2)BS(s,X_s,\bar{\sigma}_s^\delta)\phi_n(\frac{X_s}{n})\Lambda_s ds)\\
&+\frac{\rho}{2}\mathbb{E}_t(\int_t^T e^{-r(s-t)}(\partial_x^2-\partial_x)BS(s,X_s,\bar{\sigma}_s^\delta)\frac{1}{n}\phi'(\frac{X_s}{n})\Lambda_s ds)\\
&-c_2\mathbb{E}_t(\int_t^T e^{-r(s-t)}\partial_x BS(s,X_s,\bar{\sigma}_s^\delta)\phi(\frac{X_s}{n})ds)\\
&+\mathbb{E}_t(\int_t^T\int_{\mathbb{R}} e^{-r(s-t)}\Delta_{yy}BS(s,X_{s^-},\bar{\sigma}_s^\delta)\phi(\frac{X_{s-}}{n})\nu(dy)ds)\\
&+\mathbb{E}_t(\int_t^T\int_{\mathbb{R}} e^{-r(s-t)}D_{s,y}^{N,-}\frac{\Delta_y BS(s,X_{s^-},\bar{\sigma}_s^\delta)\phi(\frac{X_{s-}}{n})}{y}y^2\nu(dy)ds).
\end{aligned}
$$

And as in Theorem 1, applying the dominated convergence theorem, letting first $n\uparrow\infty$ and then $\delta\downarrow 0$ we obtain the result.

To assure the dominated convergence, we have to treat the last three terms of (1) as a unique term and separate it in two integrals, one on $|y|\le 1$ and the other on $|y|>1$.

In the case $|y|>1$, things simplify and we obtain

$$
\begin{aligned}
&-\mathbb{E}_t(\int_t^T\int_{|y|>1} e^{-r(s-t)}\partial_x BS(s,X_s,\bar{\sigma}_s^\delta)\phi(\frac{X_s}{n})(e^y-1)\nu(dy)ds)\\
&+\mathbb{E}_t(\int_t^T\int_{|y|>1} e^{-r(s-t)}T_{s,y}^{N,-}\Delta_y BS(s,X_{s^-},\bar{\sigma}_s^\delta)\phi(\frac{X_{s-}}{n})\nu(dy)ds).
\end{aligned}
$$

For the first term we use that $\partial_x BS(s,X_s,\bar{\sigma}_s^\delta)$ is bounded by e^{X_s} and for the second term we use the fact that

$$
|T_{s,y}^{N,-}\Delta_y BS(s,X_{s^-},\bar{\sigma}_s^\delta)|\le 2K+e^{X_{s-}+y}+e^{X_{s-}}
$$

and that

$$\int_{\{|y|>1\}} e^y \nu(dy) < \infty.$$

In the case $|y| \leq 1$, the fifth term in the right hand side of (1) is bounded because $\partial_x BS$ is bounded. The sixth term can be written as

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}_t(\int_t^T \int_{|y|\leq 1} e^{-r(s-t)} \partial_x^2 BS(s, X_s + \alpha, \bar{\sigma}_s^\delta)\phi(\frac{X_{s-}}{n}) y^2 \nu(dy) ds) \\
&= \frac{1}{2}\mathbb{E}_t(\int_t^T \int_{|y|\leq 1} e^{-r(s-t)} (\partial_x^2 - \partial_x) BS(s, X_s + \alpha, \bar{\sigma}_s^\delta)\phi(\frac{X_{s-}}{n}) y^2 \nu(dy) ds) \\
&+ \frac{1}{2}\mathbb{E}_t(\int_t^T \int_{|y|\leq 1} e^{-r(s-t)} \partial_x BS(s, X_s + \alpha, \bar{\sigma}_s^\delta)\phi(\frac{X_{s-}}{n}) y^2 \nu(dy) ds)
\end{aligned}$$

where $|\alpha| \leq |y|$.

The second term on the right hand side of this last expression is bounded because $\partial_x BS$ is bounded. For the first term we use Lemma 2 in [3] as in Theorem 1 and we bound it by

$$C\mathbb{E}_t(\int_t^T \int_{|y|\leq 1} \sqrt{\frac{1}{Y_t}} y^2 \nu(dy) ds),$$

for a certain constant C. Hypothesis (B3) guarantees the convergence of this integral, because

$$\mathbb{E}_t(\int_t^T \int_{|y|\leq 1} \sqrt{\frac{1}{Y_t}} y^2 \nu(dy) ds) = C(T-t)\mathbb{E}_t(\sqrt{\frac{1}{Y_t}})$$

and

$$\mathbb{E}_t(\sqrt{\frac{1}{Y_t}}) \leq (\mathbb{E}_t(\frac{1}{Y_t^2}))^{\frac{1}{4}} \leq (\mathbb{E}_t((\int_t^s \sigma_u^2 du)^{-2}))^{\frac{1}{4}}$$

and so, the term is bounded by (B3).

Finally, the last term of (1) can be bounded by

$$\begin{aligned}
&\mathbb{E}_t(\int_t^T \int_{|y|\leq 1} |D_{s,y}^{N,-}(\partial_x BS)(s, X_{s^-} + \alpha, \bar{\sigma}_s^\delta)| y^2 \nu(dy) ds) \\
&= \mathbb{E}_t(\int_t^T \int_{|y|\leq 1} |D_{s,y}^{N,-}(\partial_x BS)(s, X_{s^-} + \alpha, \sqrt{\frac{Y_s + \delta}{T-s}})| y^2 \nu(dy) ds) \\
&= \mathbb{E}_t(\int_t^T \int_{|y|\leq 1} |(\partial_{x\sigma} BS)(s, X_{s^-} + \alpha, \sqrt{\frac{\theta_{s,y} + \delta}{T-s}})| \frac{|D_{s,y}^{N,-} Y_s|}{2(T-s)\sqrt{\frac{\theta_{s,y}+\delta}{T-s}}} y^2 \nu(dy) ds)
\end{aligned}$$

$$= \frac{1}{2}\mathbb{E}_t(\int_t^T \int_{|y|\le 1} |(\partial_x(\partial_x^2 - \partial_x)BS)(s, X_{s^-} + \alpha, \sqrt{\frac{\theta_{s,y} + \delta}{T-s}})||D_{s,y}^{N,-}Y_s|y^2\nu(dy)ds), \tag{2}$$

where $|\alpha| \le |y|$ and $\theta_{s,y}$ is a quantity between Y_s and $T_{s,y}^{N,-}Y_s$.

Now, we cannot apply directly Lemma 2 in [3], but mimicking the proof we have that the last integral is less or equal than

$$C\mathbb{E}_t(\int_t^T \int_{|y|\le 1} \frac{|D_{s,y}^{N,-}Y_s|}{\int_t^s \sigma_u^2 du} y^2\nu(dy)ds).$$

Then, applying Cauchy-Schwarz inequality, this expression is bounded by

$$C(\mathbb{E}_t(\int_t^T \int_{|y|\le 1} \frac{1}{(\int_t^s \sigma_u^2 du)^2} y^2\nu(dy)ds))^{\frac{1}{2}}(\mathbb{E}_t(\int_t^T \int_{|y|\le 1} |D_{s,y}^{N,-}Y_s|^2 y^2\nu(dy)ds))^{\frac{1}{2}}.$$

The first term of this product is bounded by (B3) and the second one by (B1)

Remark 3

1. As in the case of Theorem 1, (B2) can be changed by (A2′).
2. If σ not depends on jumps, (B3) reduces to $\mathbb{E}_t((\int_t^T \sigma_u^2 du)^{-\frac{1}{2}}) < \infty$, that it is weaker than (A2′).
3. In the case of finite variation, (B3) is not necessary.
4. In the complete general case, but only in this case, (B1) and (A2′) are not enough. An alternative treatment of (2), using (A2′), is to bound directly

$$|(\partial_x(\partial_x^2 - \partial_x)BS)(s, X_{s^-} + \alpha, \sqrt{\frac{\theta_{s,y} + \delta}{T-s}})| \le e^{X_{s^-}+\alpha}(\frac{1}{\sqrt{Y_s}} + \frac{1}{Y_s}).$$

   So, we can decompose this term in two new terms. The term with $Y_s^{-\frac{1}{2}}$ can be treated easily and it is bounded with no other requirements than (B1) and (A2′). But the term with Y_s^{-1} requires to assume, alternatively to (B3), the following hypothesis,

$$\mathbb{E}_t(\int_t^T \int_{|y|\le 1} \frac{|D_{s,y}^{N,-}Y_s|}{Y_s} y^2\nu(dy)ds) < \infty,$$

   that using (A2′) is equivalent to assume

$$(B4) : \mathbb{E}_t(\int_t^T \int_{|y|\le 1} \frac{|D_{s,y}^{N,-}Y_s|}{T-s} y^2\nu(dy)ds) < \infty.$$

   Note that this last hypothesis is stronger than (B1). So, we need (B1), (A2′) and (B4) to guarantee the complete general case.

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