Turning Points of Nonlinear Circuits

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Abstract Bifurcation theory plays a key role in the qualitative analysis of dynamical systems. In nonlinear circuit theory, bifurcations of equilibria describe qualitative changes in the local phase portrait near an operating point, and are important from both an analytical and a numerical point of view. This work is focused on quadratic turning points, which, in certain circumstances, yield saddle-node bifurcations. Algebraic conditions guaranteeing the existence of this kind of points are well-known in the context of explicit ordinary differential equations (ODEs). We transfer these conditions to semiexplicit differential-algebraic equations (DAEs), in order to impose them to branch-oriented models of nonlinear circuits. This way, we obtain a description of the conditions characterizing these turning points in terms of the underlying circuit digraph and the devices' characteristics.

1 Introduction

The context of the present work is the study of bifurcation phenomena in nonlinear circuits. We have focused on quadratic turning points, which are related to certain local bifurcations in dynamical systems, in particular to the saddle-node bifurcation. With terminological abuse, we will often use the expression "turning point" to mean a "quadratic turning point". We are interested in the analysis of turning points in the equations governing nonlinear circuits, which have the structure of a semiexplicit DAE. Therefore, our first efforts are directed to adequate the classical conditions characterizing turning points in ODEs to a semiexplicit index-one DAE context (Sect. 2). Afterwards, in Sect. 3, we will analyze these reformulated conditions in terms of the circuit topology and the devices' characteristics. Finally, Sect. 4 briefly compiles some concluding remarks.

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Turning points in explicit ODEs Let us recall the algebraic conditions defining quadratic turning points in ODEs. Consider the ordinary differential equation

$$x' = f(x, \mu), \tag{1}$$

with $x \in \mathbb{R}^n$, and f sufficiently smooth and depending on a parameter $\mu \in \mathbb{R}$. Provided that $f(x^*, \mu^*) = 0$, (x^*, μ^*) is called a *quadratic turning point* of (1) if the conditions 1–3 below are satisfied [4].

1. $\text{rk} f_x(x^*, \mu^*) = n - 1$;
2. $w^T f_\mu(x^*, \mu^*) \neq 0$;
3. $w^T f_{xx}(x^*, \mu^*)(v, v) \neq 0$.

Here v (resp. w) denotes a right (resp. left) eigenvector of the zero eigenvalue of the matrix of partial derivatives $f_x(x^*, \mu^*)$. Such turning points are important e.g. in numerical continuation theory [1]. If, additionally,

4. the algebraic multiplicity of the null eigenvalue of $f_x(x^*, \mu^*)$ is one; and
5. the remaining eigenvalues of $f_x(x^*, \mu^*)$ have non-zero real parts,

then (x^*, μ^*) is called a *saddle-node bifurcation point*, because the system undergoes a saddle-node bifurcation as μ crosses μ^* [5, 7, 10]. Near (x^*, μ^*) we will observe that when $\mu < \mu^*$ (resp. when $\mu > \mu^*$) there are no equilibria, whereas for $\mu > \mu^*$ (resp. $\mu < \mu^*$) there are two hyperbolic equilibrium points. These two equilibria differ in the sign of one real eigenvalue, being in particular a saddle and a node when $x \in \mathbb{R}^2$.

2 Turning Points in Semiexplicit DAEs

Our purpose is to characterize the existence of turning points and saddle-node bifurcations in electrical circuit models and, specifically, in branch-oriented models. These models have the structure of a semiexplicit DAE [3, 8], that is,

$$y' = h(y, z, \mu) \tag{2a}$$

$$0 = g(y, z, \mu), \tag{2b}$$

where $y \in \mathbb{R}^r$, $z \in \mathbb{R}^p$, $\mu \in \mathbb{R}$, and h and g are sufficiently smooth. We will group together y and z into a single variable $x = (y, z) \in \mathbb{R}^n$, with $n = r + p$. For later use let us also define the matrices

$$M = \begin{pmatrix} h_y & h_z \\ g_y & g_z \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} M \\ (\det M)_x \end{pmatrix}. \tag{3}$$

Specifically, we will work in a local index-one context [3, 8]; this means that the matrix of partial derivatives $g_z(y^*, z^*, \mu^*)$ is non-singular. By the implicit function theorem this implies that there is a local map $\psi(y, \mu)$ such that $0 = g(y, z, \mu)$ if and only if $z = \psi(y, \mu)$, with $\psi_y = -(g_z)^{-1} g_y$. This, together with (2a), enables one to express the local dynamics of the DAE (2) in terms of the reduced ODE

$$y' = \eta(y, \mu), \tag{4}$$

with $\eta(y, \mu) = h(y, \psi(y, \mu), \mu)$. In turn, this makes it possible to define an equilibrium (y^*, z^*, μ^*) of the semiexplicit index-one DAE (2) as a (quadratic) turning point (resp. a saddle-node bifurcation point) if the reduction (4) satisfies the conditions 1–3 (respectively 1–5) stated in Sect. 1.

Theorem 1 provides conditions for system (2) to have a turning point. Additional conditions for the existence of a saddle-node point will be formulated in terms of the reduction (4); this point of view will be exploited in Sect. 3.

Theorem 1 *Consider the semiexplicit DAE* (2) *and assume there exists a point* (x^*, μ^*) *such that* $h(x^*, \mu^*) = 0$ *and* $g(x^*, \mu^*) = 0$*, with* $g_z(x^*, \mu^*)$ *non-singular. Then* (x^*, μ^*) *is a quadratic turning point if the following conditions are satisfied:*

1. $\mathrm{rk}\, M(x^*, \mu^*) = n - 1$*;*
2. $\begin{pmatrix} h_\mu \\ g_\mu \end{pmatrix} (x^*, \mu^*) \notin \mathrm{im}\, M\, (x^*, \mu^*)$*;*
3. $\mathrm{rk}\, \tilde{M}(x^*, \mu^*) = n$*.*

Proof Write $x^* = (y^*, z^*)$ and note that (y^*, μ^*) is an equilibrium point of (4), because $\eta(y^*, \mu^*) = h(y^*, z^*, \mu^*) = 0$. We check below that conditions 1–3 in Sect. 1 hold for the reduction (4) at (y^*, μ^*).

1. If we compute η_y in terms of the maps h and g, we obtain

$$\eta_y = \begin{pmatrix} h_y & h_z \end{pmatrix} \begin{pmatrix} I \\ -(g_z)^{-1} g_y \end{pmatrix} = h_y - h_z (g_z)^{-1} g_y,$$

which is the Schur complement of g_z in M [6]. The corank of a matrix and the corank of its Schur reduction are equal, therefore $\mathrm{rk}\, M(x^*, \mu^*) = n - 1$ implies $\mathrm{rk}\, \eta_y(y^*, \mu^*) = r - 1$.

2. The second condition is $w^T \eta_\mu(y^*, \mu^*) \neq 0$, where w is an eigenvector associated to the zero eigenvalue of the matrix A^T with $A = \eta_y(y^*, \mu^*)$; note that $w^T A = 0 \Leftrightarrow w^T \perp \mathrm{im} A$. Therefore, $w^T \eta_\mu(y^*, \mu^*) \neq 0 \Leftrightarrow \eta_\mu(y^*, \mu^*) \notin \mathrm{im} A$, that is, $(h_\mu - h_z g_z^{-1} g_\mu)(x^*, \mu^*) \notin \mathrm{im}\, (h_y - h_z (g_z)^{-1} g_y)(x^*, \mu^*)$ which is equivalent to

$$\begin{pmatrix} h_\mu \\ g_\mu \end{pmatrix} (x^*, \mu^*) \notin \mathrm{im} \begin{pmatrix} h_y & h_z \\ g_y & g_z \end{pmatrix} (x^*, \mu^*).$$

3. Equation $w^T\eta_{yy}(y^*,\mu^*)(v,v) \neq 0$ can be recast as $\eta_{yy}(y^*,\mu^*)(v,v) \notin \mathrm{im}\,\eta_y(y^*,\mu^*)$. The fact that for a C^2 map $f:\mathbb{R}^n \to \mathbb{R}^n$ satisfying $\mathrm{cork} f'(x^*) = 1$ we have $(\det f'(x))'v \neq 0 \Leftrightarrow f''(x)(v,v) \notin \mathrm{im} f'(x)$, where v is a non-null vector belonging to $\mathrm{Ker} f'(x)$, allows us to transform this condition into $(\det(\eta_y))_y(y^*,\mu^*)v \neq 0$, where $v \in \mathrm{Ker}\eta_y(y^*,\mu^*)$. Additionally, because η_y is the Schur complement of g_z in M, we have $\det\,\eta_y = \det\left(g_z^{-1}\right)\det M$, and then

$$(\det(\eta_y))_y = (\det\left(g_z^{-1}\right))_x \det M \begin{pmatrix} I \\ -(g_z)^{-1}g_y \end{pmatrix} + \det\left(g_z^{-1}\right)(\det M)_x \begin{pmatrix} I \\ -(g_z)^{-1}g_y \end{pmatrix}.$$

Condition 1 states that $\mathrm{rk}\,M(x^*,\mu^*) = n-1$, thus $\det(M(x^*,\mu^*)) = 0$. Additionally, $\det\left(g_z^{-1}\right)(x^*,\mu^*) \neq 0$; therefore condition 3 is satisfied if and only if

$$\left((\det M)_y \;\; (\det M)_z\right)\begin{pmatrix} I \\ -(g_z)^{-1}g_y \end{pmatrix}(x^*,\mu^*)v \neq 0,$$

for some (hence any) non-vanishing vector v belonging to $\mathrm{Ker}\eta_y(y^*,\mu^*)$. Because of the identity $\mathrm{Ker}\eta_y(y^*,\mu^*) = \mathrm{Ker}(h_y - h_z(g_z)^{-1}g_y)(x^*,\mu^*)$, condition 3 is then equivalent to the requirement that the system

$$(h_y - h_z(g_z)^{-1}g_y)(x^*,\mu^*)v = 0 \tag{5a}$$

$$((\det M)_y - (\det M)_z g_z^{-1}g_y)(x^*,\mu^*)v = 0 \tag{5b}$$

only possesses the trivial solution. Equivalently, the matrix of coefficients of (5),

$$M_1 = \begin{pmatrix} h_y - h_z(g_z)^{-1}g_y \\ \det M_y - \det M_z g_z^{-1} g_y \end{pmatrix}(x^*,\mu^*),$$

must have maximum column rank. But M_1 is the Schur complement of g_z in the matrix $\tilde{M}(x^*,\mu^*)$ arising in the statement of condition 3 of Theorem 1; hence, the maximum column rank condition on M_1 is transferred to $\tilde{M}(x^*,\mu^*)$. This means that condition 3 in Sect. 1 holds for (4) at (y^*,μ^*) and the proof is complete.

3 Nonlinear Circuits Exhibiting Turning Points

In this section, we characterize the existence of turning points and saddle-node bifurcations for nonlinear circuits, under certain restrictions to be specified later. For this purpose, we use branch-oriented circuit models [8] defined by:

$$C(v_c)v_c' = i_c \tag{6a}$$

$$L(i_l)i_l' = v_l \tag{6b}$$

$$0 = B_c v_c + B_l v_l + B_g v_g + B_n v_n + B_j v_j + B_v V \tag{6c}$$

$$0 = Q_c i_c + Q_l i_l + Q_g \gamma_1(v_g) + Q_n \gamma_2(v_n) + Q_j \mu + Q_v i_v, \tag{6d}$$

where we denote the branch voltages by v, the currents by i, and use the subscripts c, l, g, n, j and v to denote capacitors, inductors, passive resistors, non-passive resistors, current sources and voltage sources, respectively. All devices may be nonlinear, often without explicit mention. We assume that there exists only one non-passive resistor and a unique DC current source, whose current $i_j = \mu$ is the parameter of the system. The reader may think of a tunnel diode as an example of a (locally) non-passive resistor. We also assume that there exists an equilibrium point that we will denote by $(x^*, \mu^*) = (v_c^*, i_l^*, i_c^*, v_l^*, v_g^*, v_n^*, v_j^*, i_v^*, \mu^*)$. The incremental capacitance and inductance matrices, C and L, are both non-singular at (x^*, μ^*) and, finally, V is the vector of voltages in the DC voltage sources.

System (6) has the semiexplicit DAE structure displayed in (2) with $y = (v_c, i_l)$ and $z = (i_c, v_l, v_g, v_n, v_j, i_v)$. Note that Eqs. (6a) and (6b) stand for the constitutive relations of capacitors and inductors, whereas Eqs. (6c) and (6d) are the expression of Kirchhoff laws. In (6d) we have eliminated the resistors currents using the constitutive relations γ_1 and γ_2. In the formulation of Kirchhoff laws we have made use of the so-called loop and cutset matrices B, Q, which are well-known in digraph theory and whose main properties are compiled in Lemma 1 [2, 9].

Lemma 1 *The loop and cutset matrices B, Q of a digraph verify the following.*

1. *B_K (resp. Q_K) has full column rank if and only if the branches specified by K do not contain any cutset (resp. loop).*
2. *The loop and cutset spaces are orthogonal to each other, that is, if columns of Q and B are arranged in the same order, then $QB^T = 0$.*
3. *Suppose the branches of a given digraph are split in four disjoint sets K_1, K_2, K_3 and K_4, and denote by B_i and Q_i the submatrices of the loop and cutset matrices defined by K_i; assume additionally that P is a positive definite matrix. Then*

$$\text{Ker}\begin{pmatrix} B_1 & 0 & B_3 \\ 0 & Q_2 & Q_3 P \end{pmatrix} = \text{Ker}B_1 \times \text{Ker}Q_2 \times \{0\}.$$

These properties allow us to prove Theorem 2, which characterizes turning points and saddle-node bifurcations for the circuit model (6). By a K-loop (resp. K-cutset) we mean a loop (resp. cutset) defined only by elements of K; this way, for instance a JCN-cutset is a cutset defined only by current sources, capacitors and/or non-passive resistors. JLN-cutsets, VC-loops, etc. are defined analogously.

Theorem 2 *In the setting defined above, assume that $\gamma_2'(v_n^*) = 0$, $\gamma_2''(v_n^*) \neq 0$ at the equilibrium point (x^*, μ^*). This equilibrium is then a turning point of (6) if*

- *there is a unique JCN-cutset, which includes the current source, the non-passive resistor and at least one capacitor; and*
- *there are no JLN-cutsets, VC-loops or JVL-loops.*

If, additionally, L and C are symmetric positive definite and there are no VCL-loops, then the turning point yields a saddle-node bifurcation.

Proof The matrices $g_z(x^*, \mu^*)$ and $M(x^*, \mu^*)$ read for system (6) as:

$$g_z(x^*, \mu^*) = \begin{pmatrix} 0 & B_l & B_g & B_n & B_j & 0 \\ Q_c & 0 & Q_g G & 0 & 0 & Q_v \end{pmatrix}, \; M(x^*, \mu^*) = \begin{pmatrix} 0 & 0 & C^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L^{-1} & 0 & 0 & 0 & 0 \\ B_c & 0 & 0 & B_l & B_g & B_n & B_j & 0 \\ 0 & Q_l & Q_c & 0 & Q_g G & 0 & 0 & Q_v \end{pmatrix},$$

where $G = \gamma_1'(v_g^*)$ is the incremental conductance matrix of passive resistors, which is positive definite. In light of item 3 in Lemma 1, non-trivial entries in $\mathrm{Ker} g_z(x^*, \mu^*)$ must come either from $\mathrm{Ker}\left(B_l \; B_n \; B_j\right)$ or from $\mathrm{Ker}\left(Q_c \; Q_v\right)$. Since there are neither JLN-cutsets nor VC-loops, we conclude that $g_z(x^*, \mu^*)$ is non-singular.

1. The non-singularity of C, L allows us to study the rank of the matrix $M(x^*, \mu^*)$ in terms of

$$\begin{pmatrix} B_c & 0 & B_g & B_n & B_j & 0 \\ 0 & Q_l & Q_g G & 0 & 0 & Q_v \end{pmatrix}.$$

   By applying item 3 of Lemma 1, non-zero entries of $\mathrm{Ker} M(x^*, \mu^*)$ must come either from $\mathrm{Ker}\left(B_c \; B_n \; B_j\right)$ or from $\mathrm{Ker}\left(Q_l \; Q_v\right)$. Since there is a unique JCN-cutset and no JVL-loops, we have $\mathrm{null}\left(B_c \; B_n \; B_j\right) = 1$, where null stands for the nullity, that is, the dimension of the kernel, $\mathrm{null}\left(Q_l \; Q_v\right) = 0$ and therefore $\mathrm{null}\, M(x^*, \mu^*) = 1$, that is, $\mathrm{rk}\, M(x^*, \mu^*) = n - 1$, which is condition 1 in Theorem 1.
2. The 2nd condition in Theorem 1 may be restated as $\mathrm{null}\, M(x^*, \mu^*) = \mathrm{null}\, \hat{M}(x^*, \mu^*)$, with

$$\hat{M}(x^*, \mu^*) = \begin{pmatrix} 0 & 0 & C^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L^{-1} & 0 & 0 & 0 & 0 & 0 \\ B_c & 0 & 0 & B_l & B_g & B_n & B_j & 0 & 0 \\ 0 & Q_l & Q_c & 0 & Q_g G & 0 & 0 & Q_v & Q_j \end{pmatrix}.$$

   Proceeding as above, we observe that non-trivial entries in $\mathrm{Ker}\hat{M}(x^*, \mu^*)$ must be due to those in $\mathrm{Ker}\left(B_c \; B_n \; B_j\right)$ or in $\mathrm{Ker}\left(Q_l \; Q_v \; Q_j\right)$. The absence of JVL-loops implies $\mathrm{null}\, \hat{M}(x^*, \mu^*) = \mathrm{null}\left(B_c \; B_n \; B_j\right)$ and therefore $\mathrm{null}\, M = \mathrm{null}\, \hat{M}$.
3. The third condition in Theorem 1 says that the matrix $\tilde{M}$ (cf. (3)) has full column rank or, equivalently, $\mathrm{rk}\, \tilde{M} = n$. Provided that $\mathrm{null}\, M = 1$, requiring $\tilde{M}$ to have full column rank is equivalent to $(\det M)_x v \neq 0$, where v is any vector that spans $\mathrm{Ker} M$. For any point $\hat{x} = (v_c, i_l, i_c, v_l, v_g, v_n^*, v_j, i_v)$, $M(\hat{x}, \mu)$ is a singular matrix

because $\gamma_2'(v_n^*) = 0$. Thus, $(\det M)_x = \big(0\ 0\ 0\ 0\ 0\ a\ 0\ 0\big)$ and $a \neq 0$ because $\gamma_2''(v_n^*) \neq 0$.

The absence of VL-loops and the existence of a JCN-cutset imply that vectors belonging to Ker$M$ have the form of v where $v^T = (v_1, 0, 0, 0, 0, v_6, v_7, 0)$. Additionally, the fact that there are no JC-cutsets implies $v_6 \neq 0$; it follows that the multiplication of $(\det M)_x$ by vectors of Ker$M$ does not vanish.

4. To complete the proof it remains to show that the absence of VCL-loops leads to a saddle-node bifurcation. To do this we make use of conditions 4 and 5 in Sect. 1.

In order to prove that the zero eigenvalue is simple, we will show that the intersection of the kernel and the image of $\eta_y = (h_y - h_z(g_z)^{-1}g_y)$ at (x^*, μ^*) only contains the null vector. First, a vector u belongs to im η_y if and only if $\hat{u}$ belongs to im M, with $\hat{u}^T = \big(u^T\ 0\big)$, that is, if and only if there exists a vector v satisfying

$$u_1 = C^{-1}v_3 \tag{7a}$$

$$u_2 = L^{-1}v_4 \tag{7b}$$

$$0 = B_c v_1 + B_l v_4 + B_g v_5 + B_n v_6 + B_j v_7 \tag{7c}$$

$$0 = Q_l v_2 + Q_c v_3 + Q_g G v_5 + Q_v v_8. \tag{7d}$$

On the other hand, a vector u belongs to Kerη_y if and only if

$$\begin{pmatrix} C^{-1} & 0 & 0\,0\,0\,0 \\ 0 & L^{-1} & 0\,0\,0\,0 \end{pmatrix} (g_z^{-1}) \begin{pmatrix} B_c & 0 \\ 0 & Q_l \end{pmatrix} u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order to satisfy this relation there must be a vector y such that

$$(g_z^{-1}) \begin{pmatrix} B_c u_1 \\ Q_l u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \Leftrightarrow \begin{pmatrix} B_c u_1 \\ Q_l u_2 \end{pmatrix} = g_z \begin{pmatrix} 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},$$

that is,

$$B_c u_1 = B_g y_1 + B_n y_2 + B_j y_3 \tag{8a}$$

$$Q_l u_2 = Q_g G y_1 + Q_v y_4. \tag{8b}$$

Using the orthogonality of the cutset and loop spaces, namely, the fact that Ker$B$ and Ker$Q$ are orthogonal to one another (cf. [2]), it is not difficult to obtain from (8) the relation $y_1^T G y_1 = 0$; y_1 must then vanish because G is positive definite. Making use of (7a) and (7b), Eqs. (8a) and (8b) then read as

$$0 = B_c C^{-1} v_3 - B_n y_2 - B_j y_3 \tag{9a}$$

$$0 = Q_l L^{-1} v_4 - Q_v y_4. \tag{9b}$$

Therefore if $u \in \mathrm{Ker}\eta_y \cap \mathrm{im}\,\eta_y$, then (7c), (7d) and (9) must hold. Applying the aforementioned orthogonality property to (7c) and (9b) we obtain that $v_4^T L^{-1} v_4 = 0$ and from Eqs. (7d) and (9a), $v_3^T C^{-1} v_3 = 0$. Altogether this yields $u = 0$.

5. It remains to be proved that if there are no VCL-loops then $\eta_y(x^*, \mu^*)$ has no purely imaginary eigenvalues. A complex number λ is an eigenvalue of $\eta_y(x^*, \mu^*)$ if and only if

$$\lambda \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} h_y & h_z \\ g_y & g_z \end{pmatrix} (x^*, \mu^*)$$

is singular or, equivalently, there exists non-trivial solutions to

$$0 = \lambda^{-1} B_c C^{-1} i_c + \lambda B_l L i_l + B_g v_g + B_n v_n + B_i v_i \tag{10a}$$

$$0 = Q_c i_c + Q_l i_l + Q_g G v_g + Q_v i_v. \tag{10b}$$

The orthogonality of the cutset and cycle spaces implies that if $Qp = 0$ and $Bq = 0$ then $p^T q = 0$. Applying this result to the conjugate of (10b) in conjunction with (10a), we obtain

$$0 = \lambda^{-1} i_c^* C^{-1} i_c + \lambda i_l^* L i_l + v_g^* G v_g, \tag{11}$$

where * stands for the Hermitian (conjugate transpose). If we take the sum of (11) and its Hermitian, we obtain:

$$0 = 2\mathrm{Re}(\lambda^{-1}) i_c^* C^{-1} i_c + 2\mathrm{Re}(\lambda) i_l^* L i_l + v_g^* (G + G^T) v_g. \tag{12}$$

For purely imaginary eigenvalues, $\mathrm{Re}(\lambda^{-1}) = \mathrm{Re}(\lambda) = 0$ and therefore we must have $v_g = 0$ for (12) to hold. System (10) can be then simplified to:

$$0 = \lambda^{-1} B_c C^{-1} i_c + \lambda B_l L i_l + B_n v_n + B_i v_i \tag{13a}$$

$$0 = Q_c i_c + Q_l i_l + Q_v i_v. \tag{13b}$$

Since there are no VCL-loops, $\begin{pmatrix} Q_c & Q_l & Q_v \end{pmatrix}$ has full column rank and, consequently, $i_c = i_l = i_v = 0$ must hold to satisfy (13b). The absence of JLN-cutsets then yields $v_n = v_i = 0$ from (13a). This means that (10) only has the trivial solution, and this rules out purely imaginary eigenvalues. The proof of Theorem 2 is then complete.

4 Concluding Remarks

We have performed a circuit-theoretic analysis of the existence of turning points and saddle-node bifurcations in nonlinear circuits. The analysis of these phenomena in broader contexts, including e.g. other non-passive devices, higher-index configurations or parameters with other roles, as well as the study of other related bifurcations in similar terms, are in the scope of future research.

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