Chapter 7
Emergence of Oscillations in Networks of Time-Delay Coupled Inert Systems

Erik Steur and Alexander Pogromsky

Abstract We discuss the emergence of oscillations in networks of single-input–single-output systems that interact via linear time-delay coupling functions. Although the systems itself are inert, that is, their solutions converge to a globally stable equilibrium, in the presence of coupling, the network of systems exhibits ongoing oscillatory activity. We address the problem of emergence of oscillations by deriving conditions for; 1. solutions of the time-delay coupled systems to be bounded, 2. the network equilibrium to be unique, and 3. the network equilibrium to be unstable. If these conditions are all satisfied, the time-delay coupled inert systems have a nontrivial oscillatory solution. In addition, we show that a necessary condition for the emergence of oscillations in such networks is that the considered systems are at least of second order.

7.1 Introduction

This chapter is concerned with networks of identical single-input-single-output systems that interact via linear time-delay coupling functions. A little bit more precise, the coupling for a system in a network is defined to be the weighted difference of the time-delayed output of its neighbors and its own, non-delayed output. The delay models in this case the time it takes a signal to propagate from its source to its

E. Steur (✉)
Institute for Complex Molecular Systems and Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB
Eindhoven, The Netherlands
e-mail: e.steur@tue.nl

A. Pogromsky
Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands
e-mail: a.pogromsky@tue.nl

A. Pogromsky
Department of Control Systems and Informatics, Saint-Petersburg National Research University of Information Technologies Mechanics and Optics, Saint Petersburg, Russia

N. van de Wouw et al. (eds.), *Nonlinear Systems*, Lecture Notes
in Control and Information Sciences 470, DOI 10.1007/978-3-319-30357-4_7

destination, and therefore it is reasonable to assume that the systems have immediate access to their own outputs. We consider the case that the systems are inert, that is, in absence of coupling each system has a globally asymptotically stable equilibrium. We address the problem that, nevertheless, oscillations emerge in network of the time-delay coupled systems.

The problem of emergence of oscillations in coupled inert systems goes back to the early fifties of the previous century, starting with Alan Turing's work on morphogenesis [25]. About twenty years later, Steven Smale, being inspired by the work of Turing, proposed a fourth-order model of chemical kinetics that, even though the model is inert or "dead", two identical copies of them in diffusive interaction become "alive", in the sense that they start to oscillate for an infinite amount of time [18]. According to Smale there is a paradoxical aspect to the model:

> One has two dead (mathematically dead) cells interacting by a diffusion process, which has a tendency in itself to equalize the concentrations. Yet in interaction, a state continues to pulse indefinitely.

Because of the importance of the class of equations coupled via diffusion in many fields of science, Smale posed the sharp problem to "axiomatize" the necessary conditions for diffusion-driven oscillations. A partial solution to his problem was proposed in [23],[1] where the dynamics of two Lur'e systems in diffusive interaction was studied using frequency methods. In that paper, it was shown that diffusion-driven oscillations are possible with third-order systems. It was proved in [14] that diffusion-driven oscillations cannot emerge from a unique equilibrium in case the systems are of order lower than three. In that same paper, constructive conditions were presented for the emergence of diffusion-driven oscillations. It is worth mentioning that oscillations may emerge in networks of diffusively coupled systems of order two; In that case the oscillations are born after a secondary bifurcation of equilibria [1].

The above-mentioned studies all considered diffusive coupling, which is (typically) symmetric and delay-free. We introduce a time-delay in the coupling terms. Such time-delay coupling functions appear, among others, in network of neurons [6], electrical circuits [16], and networked control systems [17].

We present conditions for emergence of oscillations in networks of time-delay coupled inert systems. In particular, we present conditions for the solutions of the time-delay coupled systems to be bounded, we discuss when the network equilibrium is unique, and we derive a condition (at the level of the dynamics of the systems that comprise the network) for the network equilibrium to be unstable. If all these conditions are satisfied the coupled system is oscillatory. Our results imply immediately that only if the dimension of the systems is at least two, then in time-delayed interaction one may have oscillatory activity in the network. The results we present in this chapter extend our previous results reported in [22] in the sense that we remove the restriction to undirected networks.

We remark that we will only consider the case that the coupled systems can *not* only be oscillatory for zero time-delay. The reason for this is that the results

[1] A minor flaw in that paper was corrected in [24].

of [14, 15], which consider the delay-free case, remain true for sufficiently small time-delays. A proof of this claim follows almost immediately from Rouché's theorem, cf. [5].

7.2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real numbers and complex numbers, respectively. $\mathbb{R}_+$ is the set of positive real numbers and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$ is the set of the nonnegative real numbers. For a number $x = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and i being the imaginary unit, $i^2 = -1$, we denote $\Re(x) = a$ and $\Im(x) = b$. Let $\mathbb{C}_+ := \{x \in \mathbb{C} \mid \Re(x) \in \mathbb{R}_+\}$ and $\overline{\mathbb{C}}_+ := \{x \in \mathbb{C} \mid \Re(x) \in \overline{\mathbb{R}}_+\}$. Given positive integers p, q, and r, for $\mathscr{X} \subset \mathbb{R}^p$ and $\mathscr{Y} \subset \mathbb{R}^q$ we denote by $\mathscr{C}^r(\mathscr{X}, \mathscr{Y})$ the space of continuous functions from $\mathscr{X}$ into $\mathscr{Y}$ that are at least r-times continuously differentiable. If $r = 0$ we simply write $\mathscr{C}(\mathscr{X}, \mathscr{Y})$ instead of $\mathscr{C}^0(\mathscr{X}, \mathscr{Y})$. We denote $\mathscr{C} := \mathscr{C}([-\tau, 0], \mathbb{R}^{Nn})$ and we let this space be equipped with the norm

$$\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|, \quad \phi \in \mathscr{C}.$$

Here $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{Nn}$, $|x| = \sqrt{x^\top x}$, where $^\top$ denotes transposition. For a positive integer k we let I_k denote the $k \times k$ identity matrix and $\mathbf{1}_k$ denotes the column vector of length k with all entries equal to 1.

Let $\xi \in \mathscr{C}([0, \infty), \mathbb{R})$ be bounded on the whole interval of definition. Such a function is *oscillatory* (in the sense of Yakubovich) if $\lim_{t\to\infty} \xi(t)$ does not exist. In that spirit we say that a system is oscillatory if it admits the following properties: 1. the solutions of the system are *uniformly (ultimately) bounded* (such that solutions are defined on $[0, \infty)$) and, 2. the system has *a finite number of hyperbolically unstable equilibria*.[2] In other words, if the initial data are not an equilibrium solution or do not belong to a stable manifold of an equilibrium, then at least one state variable of an oscillatory system is an oscillatory function of time.

7.3 Problem Setting

We consider networks consisting of N single-input-single-output systems of the form

$$\begin{cases} \dot{x}^j(t) = f(x^j(t)) + Bu^j(t) \\ y^j(t) = Cx^j(t) \end{cases} \tag{7.1}$$

---

[2] An equilibrium solution of a delay differential equation is called hyperbolic if the roots of its associated characteristic equation have nonzero real part, cf. [9].

with $j = 1, \ldots, N$, states $x^j(t) \in \mathbb{R}^n$, inputs $u^j(t) \in \mathbb{R}$, outputs $y^j(t) \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth function and matrices B, C of appropriate dimension with CB a positive constant. We shall assume that:

C1. the system (7.1) with $u^j \equiv 0$ has a unique equilibrium x_0, i.e., $f(x_0) = 0$, which is globally asymptotically stable and locally exponentially stable.

Note that local exponential stability of the equilibrium is equivalent to all eigenvalues of the matrix

$$J_0 = J(x_0),$$

with $J(x) = \frac{\partial f}{\partial x^j}(x)$ being the Jacobian matrix of f at x, having strictly negative real part, i.e., J_0 is Hurwitz.

Systems (7.1) interact via linear time-delay coupling functions of the form

$$u^j(t) = \sigma \sum_{\ell} a_{j\ell}[y^{\ell}(t - \tau) - y^j(t)] \tag{7.2}$$

where positive constant σ is the coupling strength, positive constant τ is the (propagation) delay, and nonnegative constants $a_{j\ell}$ are the interconnection weights. In particular, a_{jl} is positive if and only if there is a connection *from* system ℓ *to* system j. Define the $N \times N$ matrix $A = \left(a_{j\ell}\right)$. Matrix A is the (weighted) adjacency matrix of the graph that specifies the interaction structure. Note that we allow the graph to be *directed*. We shall assume that the matrix A is irreducible and has zero diagonal entries. This is equivalent to saying that the graph is *simple*, i.e., there is at most one edge from node j to node ℓ and self-connections are absent, and *strongly connected*, i.e., every pair of systems can be joined by a sequence of directed edges. In addition, we assume that

C2. each row-sum of A equals 1.

The latter assumption is not strictly necessary but it simplifies notation significantly. Moreover, this assumption ensures that the synchronous (oscillatory) state exists, cf. [12, 19]. We remark that **C2** implies, by the Gershgorin Disc Theorem, that all eigenvalues of A are located in the closed unit disc in $\mathbb{C}$.

7.4 Conditions for Oscillation

Given that **C1** and **C2** hold true we establish conditions for

1. the solutions of the coupled system (7.1), (7.2) to be uniformly bounded and uniformly ultimately bounded;
2. the network equilibrium

$$X_0 = \mathbf{1}_N \otimes x_0$$

to be the unique, but unstable equilibrium.

Clearly, if both points hold true, the coupled system is oscillatory. Uniqueness of the network equilibrium is not necessary for the existence of oscillations. However, the stability properties of additional equilibria are difficult to assess as the locations of these additional equilibrium solutions depend on σ. In addition, it is worth mentioning that for a unique equilibrium the state of the coupled system can be oscillatory only if one of its outputs is an oscillatory function of time. Indeed, in case none of the outputs is an oscillatory function the value of each coupling function is (or converges to) zero such that, by **C1**, the system is not oscillatory.

7.4.1 Bounded Solutions

Consider a single system (7.1) and let $u^j(\cdot)$ be a piece-wise continuous input function being defined on $[0, T)$, $T \in \mathbb{R}_+$, and taking values in a compact set $\mathscr{U} \subset \mathbb{R}$. Let $x^j(\cdot) = x^j(\cdot; x_0^j, u^j[0, T))$ be the solution of system (7.1) corresponding to input $u^j(\cdot)$ being defined on $[0, T]$ and coinciding with x_0^j at $t = 0$. Then we define a (strictly) $\mathscr{C}^r$-*semipassive* system as follows.

Definition 7.1 Suppose that there is a function $S \in \mathscr{C}^r(\mathbb{R}^n, \overline{\mathbb{R}}_+)$, called the *storage function*, such that

$$S(x^j(t)) - S(x^j(0)) \leq \int_0^t \left[(y^j u^j)(s) - H(x^j(s))\right] \mathrm{d}s \tag{7.3}$$

with $H \in \mathscr{C}(\mathbb{R}^n, \mathbb{R})$ and $t \in (0, T]$. If there is a constant $R > 0$ and a nonnegative nondecreasing function $h : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that

$$H(s) \geq h(|s|) \tag{7.4}$$

for all $|s| \geq R$, then system (7.1) is called $\mathscr{C}^r$-*semipassive*. If (7.4) holds for all $|s| \geq R$ with a function h that is strictly increasing and such that $h(s) \to \infty$ as $s \to \infty$, then system (7.1) is called *strictly* $\mathscr{C}^r$-*semipassive*.

Remark 7.1 In case the storage function S is continuously differentiable, i.e., $r \geq 1$, then (7.3) can be replaced by the differential inequality

$$\dot{S}_{(7.1)}(x^j(t)) \leq (y^j u^j)(t) - H(x^j(t)),$$

where the subscript (7.1) means that the derivative of S is taken along solutions of (7.1) for given input $u^j(\cdot)$.

Lemma 7.1 (Boundedness) *Let* $w_0, w_1 : [0, \infty) \to [0, \infty)$ *be strictly increasing functions that satisfy* $w_0(0) = w_1(0) = 0$ *and* $w_0(s), w_1(s) \to \infty$ *as* $s \to \infty$. *Suppose that each system* (7.1) *is strictly* $\mathscr{C}^1$*-semipassive with storage function* S *that satisfies*

$$w_0(|x^j(t)|) \leq S(x^j(t)) \leq w_1(|x^j(t)|).$$

Then for each fixed σ and fixed τ the solutions of the coupled systems (7.1)*,* (7.2) *are uniformly bounded and uniformly ultimately bounded.*

Proof (*Sketch, a full proof is found in* [20]) Let $\phi \in \mathscr{C}$,

$$\phi(\theta) = \begin{pmatrix} \phi^1(\theta) \\ \vdots \\ \phi^N(\theta) \end{pmatrix}, \quad \phi^1(\theta), \ldots, \phi^N(\theta) \in \mathbb{R}^n, \quad \theta \in [-\tau, 0],$$

and consider the functional

$$\begin{aligned} V(\phi) &= \nu_1 S(\phi^1(0)) + \nu_2 S(\phi^2(0)) + \cdots + \nu_N S(\phi^N(0)) \\ &\quad + \frac{\sigma}{2} \sum_j \nu_j a_{j\ell} \int_{-\tau}^{0} \left(\phi^{\ell\top}(s) C^\top C \phi^\ell(s) \right) \mathrm{d}s. \end{aligned}$$

Here ν_i are positive constants such that

$$\begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_N \end{pmatrix} (I_N - A) = \nu^\top (I_N - A) = 0.$$

The existence of the positive vector $\nu \in \mathbb{R}^N$ is implied by the Perron–Frobenius theorem for irreducible matrices, cf. [10]. Note that the matrix $I_N - A$ is irreducible as A is assumed to be irreducible. Then, invoking the strict semipassivity property and after some simple algebraic manipulations, we find that

$$\dot{V}(\phi) \leq -\nu_1 H(\phi^1(0)) - \nu_2 H(\phi^2(0)) - \cdots - \nu_N H(\phi^N(0)) \leq -W(|\phi(0)|) + M$$

for some strictly increasing function $W : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ and positive constant M. An application of Theorem 4.2.10 of [3] completes the proof. □

7.4.2 Uniqueness and Instability of the Network Equilibrium

We shall start with establishing conditions for instability of the network equilibrium. Using **C2** we can write the coupled system dynamics as

$$\dot{x}(t) = F(x(t)) + \sigma \left[(A \otimes BC) x(t - \tau) - (I_N \otimes BC) x(t) \right] \tag{7.5}$$

where

$$x(t) = \begin{pmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^N(t) \end{pmatrix}, \quad F(x(t)) = \begin{pmatrix} f(x^1(t)) \\ f(x^2(t)) \\ \vdots \\ f(x^N(t)) \end{pmatrix},$$

and $\otimes$ denotes the Kronecker (tensor) product. A linearization of (7.5) around the network equilibrium X_0 yields the dynamics

$$\dot{\tilde{x}}(t) = [I_N \otimes (J_0 - \sigma BC)]\tilde{x}(t) + (\sigma A \otimes BC)\tilde{x}(t-\tau). \tag{7.6}$$

It is well known that the zero solution of the linear system (7.6) is unstable for some $\sigma > 0$ and $\tau > 0$ if (and only if) its associated characteristic equation

$$\Delta(\lambda; \sigma, \tau) = 0 \tag{7.7}$$

with

$$\Delta(\lambda; \sigma, \tau) := \det\left(\lambda I_{Nn} - I_N \otimes (J_0 - \sigma BC) - (\sigma A \otimes BC)\exp(-\lambda\tau)\right)$$

has a root in $\mathbb{C}_+$ for that σ and τ, cf. [9, 11]. However, computing the roots of the characteristic equation (7.7) for a large number of points in the (σ, τ)-parameter space may be cumbersome. As a solution, we will present (sufficient) conditions for instability of the network equilibrium at the level of the dynamics of the system (7.1). For that purpose we denote

$$\mathscr{H}(s) = C(sI_n - J_0)^{-1}B = \frac{p(s)}{q(s)}$$

the linear transfer function from u^j to y^j of the system (7.1) at its equilibrium. Here $p(s)$ is a polynomial of degree $n-1$ and $q(s)$ is a polynomial of degree n.[3] It is assumed that p and q are co-prime.

Lemma 7.2 (Instability) *Suppose that* **C2** *holds true. Let*

$$\eta = \inf_{\omega > 0} \Re\left(\mathscr{H}(i\omega)\right).$$

If $\eta < 0$, then for each $\sigma \geq \frac{-1}{2\eta}$ there exists a $\tau > 0$ such that the characteristic equation (7.7) has a root in $\mathbb{C}_+$.

The proof of the lemma is provided in the Appendix.

It is important to note that the condition for instability in Lemma 7.2 is *delay-dependent*. As we have remarked already in the introduction, we focus in this chapter only on delay-dependent conditions for oscillations.

We continue with conditions for uniqueness of the network equilibrium.

Lemma 7.3 (Uniqueness of the network equilibrium) *Let* **C1** *hold true and denote the eigenvalues of A by $\bar{\lambda}_j$, $j = 1, \ldots, N$. Let λ^* be the smallest real-valued eigenvalue of A. Choose $\bar{\sigma} \in (0, \infty]$ as the largest number for which the matrix*

[3] As $CB > 0$ the system (7.1) has relative degree one.

$$J(\xi) - \sigma(1 - \lambda^*)BC$$

is nonsingular for all $\xi \in \mathbb{R}^n$ *and all* $\sigma \in [0, \bar{\sigma})$. *Then the network equilibrium solution* $X_0 = \mathbf{1}_N \otimes x_0$ *is the unique equilibrium solution of* (7.5) *for* $\sigma \in [0, \bar{\sigma})$.

The proof of the lemma is provided in the Appendix.

7.4.3 Oscillations in Networks of Inert Systems

Lemmata 7.1, 7.2, and 7.3 provide conditions for the coupled systems to have bounded solutions, the network equilibrium to be unique, and the existence of a time-delay $\tau > 0$ for which this equilibrium is unstable. The following theorem summarizes these results.

Theorem 7.1 (Conditions for oscillation) *Consider the coupled system (7.1), (7.2) and suppose that* **C1** *and* **C2** *hold true. Suppose in addition that*

- *the systems (7.1) are strictly* $\mathscr{C}^1$*-semipassive with a storage function that satisfies the conditions of Lemma 7.1;*
- *the matrix*
$$J(\xi) - \sigma(1 - \lambda^*)BC$$
is nonsingular for all $\xi \in \mathbb{R}^n$ *and all* $\sigma \in [0, \bar{\sigma})$*, where* λ^* *is the smallest real-valued eigenvalue of A;*
- $\eta = \inf_{\omega > 0} \Re\left(\mathscr{H}(i\omega)\right) < 0$ *and*
$$\frac{-1}{2\eta} < \bar{\sigma}.$$

Then for each $\sigma \in \left[\frac{-1}{2\eta},\ \bar{\sigma}\right)$ *there exists a* $\tau > 0$ *for which the coupled system is oscillatory.*

Using the second and third condition of the theorem (see also Lemmas 7.2 and 7.3) one easily determines a (range of) coupling strength(s) for which there exist τ such that oscillations emerge. In particular, for any $\sigma \in \left[\frac{-1}{2\eta},\ \bar{\sigma}\right)$ one can use bifurcation software such as DDE-Biftool [7] for finding the values of τ for which the characteristic equation (7.7) has a root in $\mathbb{C}_+$. A viable strategy for computing the bifurcation diagram in the (σ, τ)-parameter space is to start with some $\sigma_H \in \left[\frac{-1}{2\eta},\ \bar{\sigma}\right)$ and $\tau = 0$. Then increase τ until at $\tau = \tau_H$ a Hopf bifurcation is detected. (We remark that the bifurcation that causes instability of the network equilibrium is necessarily a Hopf bifurcation because otherwise the condition of Lemma 7.3 would be violated.) A curve of Hopf bifurcation points can then be computed using a continuation algorithm starting from (σ_H, τ_H). See, for instance, [12] for an example.

Theorem 7.1 also provides almost immediately a necessary condition on the dimension n of the systems (7.1) for oscillations to emerge.

Corollary 7.1 *If* **C1** *and* **C2** *hold true, then a necessary condition for a network of inert systems (7.1) that interact via coupling functions (7.2) to be oscillatory is that* $n \geq 2$.

Proof An example with $n = 2$ is provided in the next section and examples of systems of order larger than two can be easily constructed. We complete the proof by showing that the equilibrium of a network of coupled inert systems with $n = 1$ is always stable. Let U be a nonsingular matrix such that

$$U^{-1}AU = \bar{\Lambda}$$

with $\bar{\Lambda}$ the *Jordan normal form* of A. Denote by $\bar{\lambda}_j$, $j = 1, \ldots, N$ the eigenvalues of A. After pre-multiplication of (7.7) by $\det(U^{-1} \otimes I_n)$ and post-multiplication of (7.7) by $\det(U \otimes I_n)$ it is straightforward to see that the characteristic equation (7.7) can have a root $\lambda \in \mathbb{C}_+$ only if there is a $j \in \{1, 2, \ldots, N\}$ such that

$$\lambda - J_0 + \sigma[1 - \bar{\lambda}_j \exp(-\lambda\tau)] = 0 \tag{7.8}$$

for some $\tau > 0$. (See also the proof of Lemma 7.2 in the Appendix.) However, as J_0 is a negative constant by **C1** and $|\bar{\lambda}_j| \leq 1$ for all j by **C2** there exists no $\lambda \in \mathbb{C}_+$ that solves (7.8). □

7.5 Example

We shall illustrate our results in networks of inert FitzHugh–Nagumo (FHN) model neurons [8]. The dynamics of this model neuron are given by the following equations:

$$\begin{cases} \dot{x}_1^j(t) = 0.08(x_2^j(t) - 0.8x_1^j(t)) \\ \dot{x}_2^j(t) = x_2^j(t) - \frac{1}{3}(x_2^j(t))^3 - x_1^j(t) - 0.559 + u^j(t) \end{cases}$$

with output $y^j(t) = x_2^j(t)$. One can easily verify that the isolated FHN model neuron has a locally exponentially stable equilibrium at $x_0 = \begin{pmatrix} -1.225 & -0.980 \end{pmatrix}^\top$. Moreover, it is shown in [21] that the FHN model neuron is strictly $\mathscr{C}^\infty$-semipassive with a quadratic storage function. Hence, we conclude that the solutions of any network of FHN model neurons are uniformly (ultimately) bounded for any nonnegative σ and τ. To check whether we can have oscillations in a network of FHN model neurons, we determine the transfer function $\mathscr{H}(s)$:

$$\mathscr{H}(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \left(sI - \begin{pmatrix} -0.064 & 0.08 \\ -1 & 1 - (-0.980)^2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

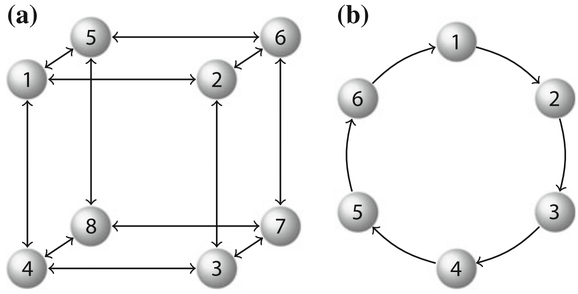

Fig. 7.1 **a** The cube network with uniform bidirectional interactions, and **b** the ring network with uniform unidirectional interactions

We find that $\eta = \inf_{\omega>0} \Re(\mathscr{H}(i\omega)) = -0.205$, which is attained at $\omega = \omega^* = 0.417$. Thus for coupling strengths

$$\sigma > \underline{\sigma} = \frac{0.5}{0.205},$$

there exist $\tau > 0$ for which the zero solution of the linearized system is unstable, hence the network equilibrium is unstable. In addition, because

$$\det(J(\xi) - \sigma(1 - \bar{\lambda}_j)BC) = (\xi_2)^2 + \tfrac{1}{4} + \sigma(1 - \bar{\lambda}_j)$$

is positive for any $\xi = \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix}^\top \in \mathbb{R}^2$, any $\sigma \geq 0$ and any real-valued $\bar{\lambda}_j \in [-1, 1]$, we conclude that the network equilibrium $X_0 = \mathbf{1}_k \otimes x_0$ is unique. Thus if $\sigma > \underline{\sigma}$ there exist values of τ for which the coupled FHN model neurons are oscillatory.

We have performed a numerical analyses with a cube network and a ring network, which are shown in Fig. 7.1a and b, respectively. The example with the cube network has been taken from [22]. For both networks, we have determined the regions of instability in the (σ, τ)-parameter space with $0 \leq \sigma \leq 8$ and $0 \leq \tau \leq 20$. These regions, which are computed with DDE-Biftool [7] using the strategy explained in the previous section, are shown in Fig. 7.2a for the cube network and Fig. 7.2b for the ring network. In these plots the areas shown in gray correspond to the regions of hyperbolic instability of the network equilibrium. The thick black curves are the stability crossing curves; At a stability crossing curve the characteristic equation (7.7) has a purely imaginary root.

In addition, we present the results of a number of numerical simulations, which are performed with Matlab using the DDE23 solver. For each simulation we have used constant initial data on the interval $[-\tau, 0]$. This initial data is chosen to be a normally distributed perturbation of the network equilibrium, with mean and variance of the perturbation being set to 0 and 0.05, respectively.

Figure 7.3 show the results of numerical simulation for the cube network with $\sigma = 3$ and either $\tau = 8$ or $\tau = 15$. The left plots show the birth of oscillatory activity from the network equilibrium. The eight smaller plots show 100 time units of steady state oscillatory activity. In case of $\sigma = 3$ and $\tau = 8$ we observe an oscillation where neurons 1, 3, 6 and 8 oscillate synchronously, neurons 2, 4, 5 and 7 oscillate

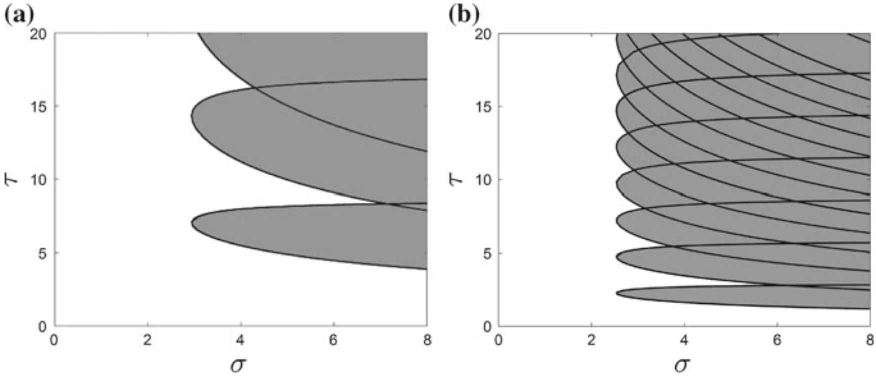

Fig. 7.2 The regions of stability (*white*) and instability (*grey*) of the network equilibrium in the (σ, τ)-parameter space for **a** the cube network, and **b** the ring network

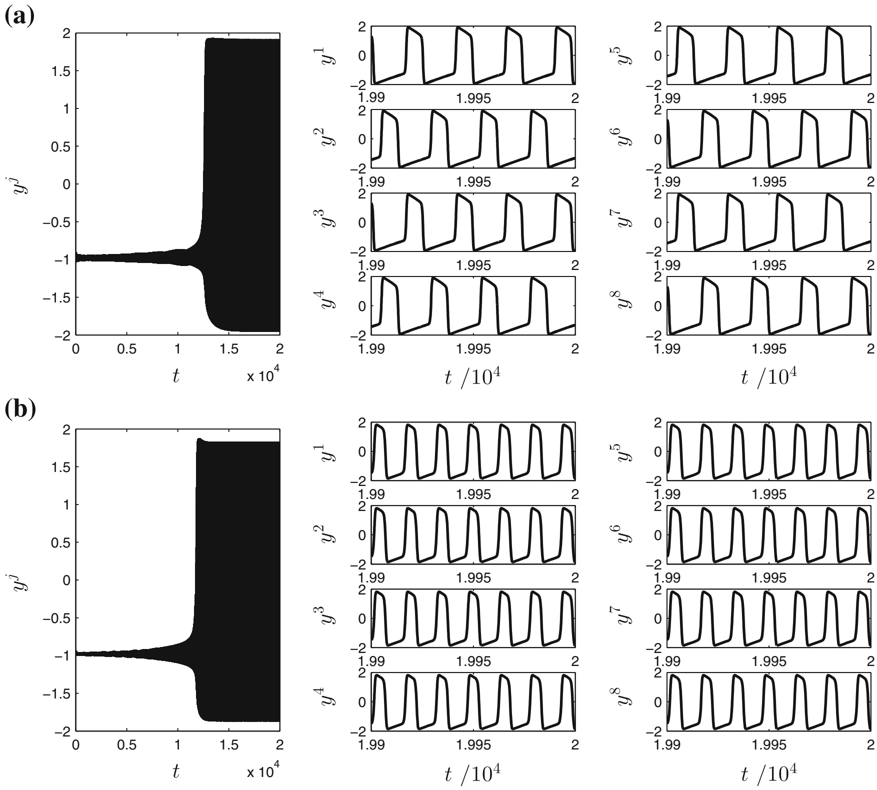

Fig. 7.3 Results of numerical simulation with the cube network for **a** $\sigma = 3$ and $\tau = 8$, and **b** $\sigma = 3$ and $\tau = 15$

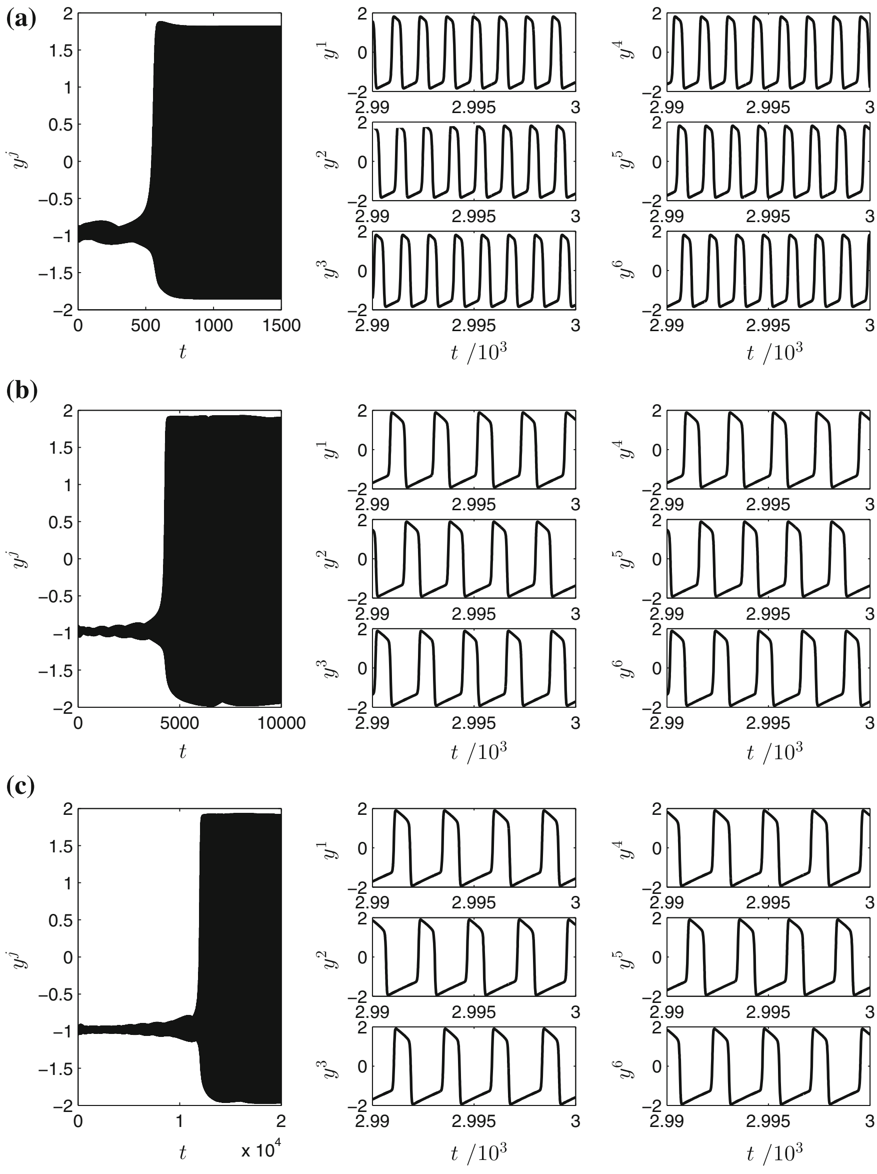

Fig. 7.4 Results of numerical simulation with the ring network for **a** $\sigma = 5$ and $\tau = 2$, **b** $\sigma = 5$ and $\tau = 7$, and **c**: $\sigma = 5$ and $\tau = 12$

synchronously, but the oscillations of the two synchronized clusters alternate. An increase of the time-delay to $\tau = 15$ results in completely synchronous oscillatory activity. For more details about the (prediction of) resulting oscillatory activity in this network we refer to [22].

Figure 7.4 show the results of numerical simulation for the ring network with $\sigma = 5$ and either $\tau = 2$, $\tau = 7$ or $\tau = 12$. Again the left plots show the onset of oscillation and the other plots show 100 time units of steady state oscillatory activity. In all cases, we observe oscillatory activity in the form of persistent propagating waves. In case of $\sigma = 5$ and $\tau = 2$ stable *rotating wave* oscillations have emerged. Indeed, the steady state oscillations are periodic and there is a constant time-shift between the oscillations of any two adjacent neurons. For time-delay $\tau = 12$ we observe the emergence of stable *standing wave* steady state activity, which is characterized by the synchronous activity of neurons 1, 3 and 5 that alternates with synchronous activity of neurons 2, 4 and 6. A somewhat intermediate oscillatory behavior is found for $\tau = 7$. In this case, neurons 1 and 4 oscillate synchronously, neurons 2 and 5 are synchronized, and the steady state oscillations of neurons 3 and 6 are completely identical. However, the oscillations of neurons 1, 2 and 3, hence those of neurons 4, 5 and 6, take the form of a rotating wave. The emerged oscillatory activity in the ring network can be analyzed and predicted using the theory presented in [12].

7.6 Conclusions

We have considered the problem of emergence of oscillatory activity in networks of identical inert systems that interact via linear time-delay coupling. We have presented conditions for

- the solutions of the coupled systems to be uniformly (ultimately) bounded;
- the network equilibrium, which is exponentially stable in absence of coupling, to become unstable in the presence of coupling;
- the network equilibrium to be unique.

If all three points are satisfied the network of time-delay coupled system will be oscillatory. Our conditions for the first two points above to hold true are expressed at the level of the systems. In particular, a strict semipassivity property of the systems ensures that the whole network has bounded solutions, and conditions for instability of the network equilibrium can be verified by evaluating the transfer function (from u^j to y^j) of the uncoupled system in equilibrium. As a corollary to these results, we have shown that a network of inert systems (7.1) with time-delay coupling (7.2) can be oscillatory only if the systems are at least of second order.

We have illustrated our results with two networks, a cube and a ring, with FHN model neurons as systems. For both networks we have determined the values of the coupling strength σ and time-delay τ for which oscillations emerge. Trajectories of the coupled systems are obtained for several values of the coupling strength and time-delay by numerical integration of the governing equations. It is shown that interesting patterns of oscillatory activity may emerge from a network equilibrium.

Afterword

This book chapter is written for the occasion of the 60th birthday of Henk Nijmeijer. Both authors have shared many ideas, thoughts and papers with Henk on the collective

behavior of coupled dynamical systems. We are certain to continue working together with Henk on this fascinating topic for many more years.

Appendix

Proof of Lemma 7.2

As mentioned in [4], the characteristic equation $\Delta(\lambda; \sigma, \tau)$ can have a root in $\mathbb{C}_+$ (for some $\tau > 0$) only if (at least) one of the following conditions is violated:

- $I_N \otimes (J_0 - \sigma BC)$ is a stable matrix;
- $I_N \otimes (J_0 - \sigma BC) + \sigma A \otimes BC$ is a stable matrix;
- the spectral radius ρ[4] of the frequency dependent matrix

$$[I_N \otimes (J_0 - \sigma BC)]^{-1} [\sigma A \otimes BC]$$

is strictly smaller than one for all frequencies:

$$\rho \left([I_N \otimes (J_0 - \sigma BC)]^{-1} [\sigma A \otimes BC] \right) < 1 \quad \forall \omega > 0.$$

As already remarked in the introduction, we restrict ourselves to the case where the network equilibrium is stable in case of zero time-delay. This implies that the first two conditions are satisfied such that instability of the network equilibrium in presence of time-delay requires the third condition to be violated. We show that the condition of Lemma 7.2 implies this to be the case.

Using some elementary properties of the Kronecker product, cf. [2], we obtain that

$$\rho \left([i\omega I_{nm} - I_N \otimes (J_0 - \sigma BC)]^{-1} [\sigma (A \otimes BC)] \right) = \rho \left(\sigma A \otimes [i\omega I_n - J_0 + \sigma BC)]^{-1} BC \right).$$

Condition **C2** implies that A has an eigenvalue equal to 1 (with right eigenvector in span $\{\mathbf{1}_N\}$). Then employing the fact that the eigenvalues of $\sigma A \otimes [i\omega I_n - J_0 + \sigma BC)]^{-1} BC$ are the product of all eigenvalues of σA and all eigenvalues of $[i\omega I_n - J_0 + \sigma BC)]^{-1} BC$, cf. [2], we find that

$$\rho \left(\sigma [i\omega I_n - J_0 + \sigma BC)]^{-1} BC \right) > 1 \quad \Rightarrow \quad \rho \left([i\omega I_{nm} - I_N \otimes (J_0 - \sigma BC)]^{-1} [\sigma (A \otimes BC)] \right) > 1.$$

[4] The spectral radius of a square (complex) matrix is the largest eigenvalue in absolute value of that matrix.

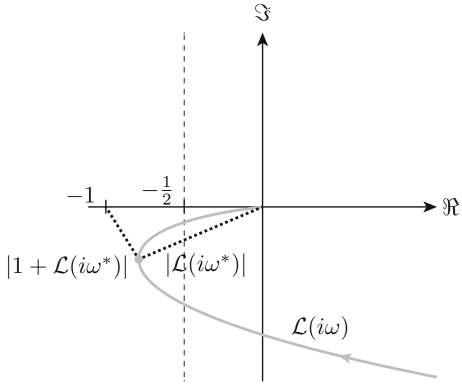

Fig. 7.5 If the Nyquist plot of $\mathscr{L}(i\omega) = \sigma\mathscr{H}(i\omega))$ intersects with $\{s \in \mathbb{C} \mid \Re(s) < -\frac{1}{2}\}$, then $|\mathscr{L}(i\omega^*)| > |1 + \mathscr{L}(i\omega^*)|$, where $\omega^* = \arg\min_{\omega>0}\Re\left(\mathscr{L}(i\omega)\right)$

Some straightforward manipulations (that involve some theory about the inverse of the sum of two matrices, cf. [13]) show that

$$\begin{aligned}\bar{\rho}(\omega;\sigma) &:= \rho\left(\sigma[i\omega I_n - J_0 + \sigma BC)]^{-1}BC\right)\\ &= \frac{|\sigma p(i\omega)|}{|q(i\omega)+\sigma p(i\omega)|} = \left|\frac{\sigma\mathscr{H}(i\omega)}{1+\sigma\mathscr{H}(i\omega)}\right|.\end{aligned}$$

It follows from Fig. 7.5 that if the Nyquist plot of $\mathscr{L}(i\omega) = \sigma\mathscr{H}(i\omega)$ intersects with $\{s \in \mathbb{C} \mid \Re(s) < \frac{1}{2}\}$, then there exists $\omega^* = \arg\min_{\omega>0}\Re\left(\mathscr{L}(i\omega)\right) > 0$ such that

$$|\mathscr{L}(i\omega^*)| > |1 + \mathscr{L}(i\omega^*)| \Rightarrow \bar{\rho}(\omega^*;\sigma) > 1.$$

In other words, if $\eta = \inf_{\omega>0}\Re\left(\mathscr{H}(i\omega)\right) = \Re\left(\mathscr{H}(i\omega^*)\right) < 0$, then for each $\sigma \geq \frac{-1}{2\eta}$,

$$\bar{\rho}(\omega^*;\sigma) > 1 \quad \Rightarrow \quad \rho\left(\left[i\omega^* I_{nm} - I_N \otimes (J_0 - \sigma BC)\right]^{-1}[\sigma(A \otimes BC)]\right) > 1.$$

Fix $\sigma^* \geq \frac{-1}{2\eta}$. We now show that $\bar{\rho}(\omega^*;\sigma^*) > 1$ implies (7.7) to have a root in $\mathbb{C}_+$. Define

$$\beta(\lambda;\sigma^*,\tau) = 1 - \alpha(\lambda;\sigma^*)\exp(-\lambda\tau)$$

with

$$\alpha(\lambda;\sigma^*) = \frac{\sigma^*\mathscr{H}(\lambda)}{1+\sigma^*\mathscr{H}(\lambda)}.$$

Note that $\bar{\rho}(\omega;\sigma^*) = |\alpha(i\omega;\sigma^*)|$. Consider the function $\kappa : \mathbb{R}_+ \to \mathbb{R}$, $\kappa(\omega) = 1 - |\alpha(i\omega;\sigma^*)|^2 = 1 - \bar{\rho}^2(\omega;\sigma^*)$. Note that $\lim_{\omega\to\infty}\kappa(\omega) = 1$ as $\lim_{\omega\to\infty}\bar{\rho}(\omega;\sigma^*) = 0$. Because $\bar{\rho}(\omega^*;\sigma^*) > 1$ there exist a $\omega_0 > 0$ such that $\bar{\rho}(\omega_0;\sigma^*) = 1$. Let us choose, without loss of generality, this ω_0 such that for any small number $\delta > 0$ we have $\bar{\rho}(\omega_0 - \delta;\sigma^*) > 1$ and $\bar{\rho}(\omega_0 + \delta;\sigma^*) < 1$, i.e., $\kappa' = \frac{d\kappa}{d\omega} > 0$ at $\omega = \omega_0$. In addition, there is a $\tau_0 > 0$ for which $\beta(i\omega_0; \sigma^*, \tau_0) = 0$. Following [11], pp. 95, if we differentiate $\beta(\lambda; \sigma, \tau) = 0$ at $\lambda = i\omega_0$ with respect to τ, we find

$$\Re\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{1}{2\omega_0}\kappa'(\omega_0) < 0.$$

This implies the existence of some $\tau^* < \tau_0$ for which $\beta(\lambda; \sigma^*, \tau^*)$ has a root in $\mathbb{C}_+$.

Let U be a nonsingular matrix such that

$$U^{-1}AU = \bar{\Lambda}$$

with $\bar{\Lambda}$ the *Jordan normal form* of A and let $\bar{\lambda}_j$, $j = 1, 2, \ldots, N$, be the eigenvalues of A. After pre-multiplication of (7.7) by $\det(U^{-1} \otimes I_n)$ and post-multiplication of (7.7) by $\det(U \otimes I_n)$, we conclude that the roots of (7.7) (of course, for $\sigma = \sigma^*$) are identical to the roots of

$$\prod_{j=1}^{N} \Delta_j(\lambda; \sigma^*, \tau)$$

with

$$\Delta_j(\lambda; \sigma^*, \tau) = \det\left(\lambda I_n - (J_0 - \sigma^* BC) - \sigma^* \bar{\lambda}_j BC \exp(-\lambda\tau)\right).$$

By **C2** there is always an eigenvalue of A equal to 1. Without loss of generality, we let $\bar{\lambda}_1 = 1$ such that

$$\Delta_1(\lambda; \sigma^*, \tau) = \det\left(\lambda I_n - (J_0 - \sigma^* BC) - \sigma^* BC \exp(-\lambda\tau)\right).$$

It is straightforward to verify that $\Delta_1(\lambda; \sigma^*, \tau)$ and $\beta(\lambda; \sigma^*, \tau)$ have the same roots. Thus $\Delta_1(\lambda; \sigma^*, \tau)$ has a root in $\mathbb{C}_+$ for some $\tau^* < \tau_0$, which implies that (7.7) has a root in $\mathbb{C}_+$ for $\sigma = \sigma^*$ and $\tau = \tau^*$. □

Proof of Lemma 7.3

The proof follows from arguments given first in [15]. First we show that the conditions of the lemma imply that the Jacobian matrix of

$$F(x) - \sigma[(I_N - A) \otimes BC)]x \tag{7.9}$$

is nonsingular at all $x \in \mathbb{R}^{Nn}$. Let again U be a nonsingular matrix such that

$$U^{-1}AU = \bar{\Lambda}$$

with $\bar{\Lambda}$ the Jordan normal form of A. Then the Jacobian matrix of (7.9) is nonsingular if and only if

$$\begin{pmatrix} J(\tilde{x}^1) & & \\ & \ddots & \\ & & J(\tilde{x}^N) \end{pmatrix} - \sigma(I_N - \bar{\Lambda}) \otimes BC$$

is nonsingular for all $\tilde{x}^j \in \mathbb{R}^n$, $j = 1, \ldots, N$. Due to the triangular structure of $\bar{\Lambda}$ the above matrix is singular if and only if (at least) one of the matrices

$$J(\xi) - \sigma(1 - \bar{\lambda}_j)BC, \quad j = 1, \ldots, N, \quad \xi \in \mathbb{R}^n$$

is singular. By construction the matrix BC has one positive diagonal entry and all other entries equal zero. Thus $J(\xi) - \sigma(1 - \lambda_j)BC$ can only be singular if $\bar{\lambda}_j$ is real valued. It follows that the conditions of the lemma imply that the Jacobian matrix is nonsingular for all $\sigma \in [0, \ \bar{\sigma})$.

Now we consider an auxiliary coupled system (7.5) with σ replaced by $\epsilon\sigma$ with parameter $\epsilon \in [0, 1]$. By the conditions of the lemma the Jacobian matrix of this auxiliary coupled system is, like the original coupled system, nonsingular at each point in $\mathbb{R}^{Nn}$. Now suppose that for this auxiliary coupled system there is some $\epsilon = \epsilon^* \in (0, 1)$ for which there exists an equilibrium X_0^* other than the network equilibrium X_0. Then, due to the implicit function theorem, this additional equilibrium point is determined by an equation of the form $X_0^* = \mathscr{F}(\epsilon^*)$. Decreasing ϵ from ϵ^* to zero implies the existence of an equilibrium other than X_0 for $\epsilon = 0$. This contradicts **C1**, which states that the isolated system has a globally asymptotically stable (hence unique) equilibrium. □

References

1. Alexander, J.C.: Spontaneous oscillations in two 2-component cells coupled by diffusion. J. Math. Biol. **23**, 205–219 (1986)
2. Brewer, J.W.: Kronecker products and matrix calculus in system theory. IEEE Trans. Circuits Syst. **25**(9), 772–781 (1978)
3. Burton, T.A.: Stability and Periodic Solutions of Ordinary and Functional Differential Equations. Academic, New York (1985)
4. Chen, J., Latchman, H.A.: Frequency sweeping tests for stability independent of delay. IEEE Trans. Autom. Control **40**, 1640–1645 (1995)
5. Churchill, R.V., Brown, J.W.: Complex Variables and Applications, 5th edn. McGraw-Hill, Inc., New York (1990)
6. Dhamala, M., Jirsa, V.K., Ding, M.: Enhancement of neural synchrony by time delay. Phys. Rev. Lett. **92**, 074104 (2004)
7. Engelborghs, K., Luzyanina, T., Samaey, G.: DDE-BIFTOOL v. 2.00: a Matlab package for bifurcation analysis of delay differential equations. Tw report 330, Dept. Computer Science, KU Leuven (2001)
8. FitzHugh, R.: Impulses and physiological states in theoretic models of nerve membrane. Biophys. J. **1**, 445–466 (1961)

9. Hale, J.K., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations. Applied Mathematical Sciences, vol. 99. Springer, Heidelberg (1993)
10. Horn, R.A., Johnson, C.R.: Matrix Analysis, 6th edn. Cambridge University Press, Cambridge (1999)
11. Michiels, W., Niculescu, S.-I.: Stability and Stabilization of Time-Delay Systems. Advances in Design and Control. SIAM (2007)
12. Michiels, W., Nijmeijer, H.: Synchronization of delay-coupled nonlinear oscillators: an approach based on the stability analysis of synchronized equilibria. Chaos **19**, 033110 (2009)
13. Miller, K.S.: On the inverse of the sum of matrices. Math. Mag. **5**(2), 67–72 (1981)
14. Pogromsky, A., Glad, T., Nijmeijer, H.: On diffusion driven oscillations in coupled dynamical systems. Int. J. Bifurc. Chaos **9**(4), 629–644 (1999)
15. Pogromsky, A., Kuznetsov, N., Leonov, G.: Pattern generation in diffusive networks; How do those brainless centipedes walk? In: 50th IEEE Conference on Decision and Control, pp. 7849–7854 (2011)
16. Ramana Reddy, D.V., Sen, A., Johnston, G.L.: Experimental evidence of time-delay-induced death in coupled limit-cycle oscillators. Phys. Rev. Lett. **85**(16), 3381–3384 (2000)
17. Sipahi, R., Niculescu, S., Abdallah, C.T., Michiels, W., Gu, K.: Stability and stabilization of systems with time delay. IEEE Control Syst. **31**(1), 38–65 (2011)
18. Smale, S.: A mathematical model of two cells via Turing's equation. The Hopf Bifurcation and Its Applications, vol. 19. Springer, New York (1976)
19. Steur, E., Nijmeijer, H.: Synchronization in networks of diffusively time-delay coupled (semi-)passive systems. IEEE Trans. Circuits Syst. I **58**(6), 1358–1371 (2011)
20. Steur, E., Oguchi, T., van Leeuwen, C., Nijmeijer, H.: Partial synchronization in diffusively time-delay coupled oscillator networks. Chaos **22**, 043144 (2012)
21. Steur, E., Tyukin, I., Nijmeijer, H.: Semi-passivity and synchronization of diffusively coupled neuronal oscillators. Physica D **238**, 2119–2128 (2009)
22. Steur, E., van Leeuwen, C., Pogromsky, A.Y.: Synchronous oscillations in networks of time-delay coupled inert systems. In: Proceedings of the fourth IFAC Conference on Analysis and Control of Chaotic Systems (IFAC CHAOS 2015), Tokyo, Japan (2015)
23. Tomberg, E.A., Yakubovich, V.A.: Conditions for auto-oscillations in nonlinear systems. Sibirsk. Mat. Zh. **30**(4), 641–653 (1989)
24. Tomberg, E.A., Yakubovich, V.A.: On one problem of Smale. Sibirsk. Mat. Zh. **41**(4), 926–928 (2000)
25. Turing, A.: The chemical basis of morphogenesis. Phylos. Trans. Royal Soc. **B.237**, 37–72 (1952)